Analysis of Piezoelectric Microcantilever

Key Variables

- **Geometric Parameters:**
  - w_t: Beam width.
  - l_t: Beam length.
  - t: Beam thickness.
  - electrode_length, electrode_width: Electrode dimensions.
  - w_c, l_c: Spectral dimensions of the mass block at the end of the beam.
  - d: Gap between electrodes.
- **Material Parameters:**
  - E: Young's modulus.
  - ρ: Material density.
  - ϵ_0: Vacuum permittivity.
- **Electrical and Vibrational Parameters:**
  - Q: Quality factor.
  - V: Driving voltage.
  - Vac: AC voltage amplitude (harmonic driving).
  - ϕ: Harmonic driving phase range.
  - β, α_n: Parameters related to the modal shape.

Key Calculations

Modal Shape Calculation

Modal shape describes the deformation distribution of the beam under resonance, based on Euler-Bernoulli beam theory:

$$\begin{aligned}\text{modeshape\_unnormalized} = &-\big(-\sin\big(\beta x/l_t\big) + \sinh\big(\beta x/l_t\big) \\ &+ \alpha_n\big(-\cos\big(\beta x/l_t\big) + \cosh\big(\beta x/l_t\big)\big)\big)\end{aligned}$$

Normalized modal shape:

$$\text{modeshape1} = \frac{\text{modeshape\_unnormalized}}{\max(\text{modeshape\_unnormalized})}$$

Mechanical Properties

The stiffness and mass of the beam are calculated as follows:

- Normalized stiffness coefficients:

$$\begin{aligned}k_{\text{coef\_b}} &= \int \frac{\text{second\_derivative}^2}{l_t}\,dx, \\ k_{\text{coef\_b3}} &= \int \frac{\text{first\_derivative}^2}{l_t}\,dx.\end{aligned}$$

- Actual stiffness:

$$\begin{aligned}k_{tt} &= \frac{k_{\text{coef\_b}}}{12} Et\left(\frac{w_t}{l_t}\right)^3, \\ k_{t3} &= k_{\text{coef\_b3}} Et \frac{w_t}{l_t^3}.\end{aligned}$$

- Total stiffness:

$$k_t = k_{tt}.$$

- Total mass:

$$M = \rho\left(tw_t l_t m_{\text{coef\_b}} + \text{electrode\_length} \cdot \text{electrode\_width} \cdot t + 2w_c \cdot l_c \cdot t\right).$$

- Natural angular frequency:

$$\omega_0 = \sqrt{\frac{k_t}{M}}, \quad f_0 = \frac{\omega_0}{2\pi}.$$

Electromechanical Coupling

Electromechanical effects introduce additional stiffness and non-linear terms:

- Coupling factor:

$$\text{trans\_factor} = \frac{\epsilon_0 V \cdot \text{electrode\_length} \cdot t}{d^2}.$$

- Electromechanical stiffness:

$$k_e = 2 \cdot \text{trans\_factor} \cdot \frac{V}{d},$$
$$k_{e3} = 4 \cdot \text{trans\_factor} \cdot \frac{V}{d^3}.$$

Dynamic Response Under Harmonic Driving

The dynamic response is computed by solving a nonlinear system of equations for frequency and motional current:

$$-M\omega^2 y + (k_t - k_e)y + (k_{t3} - k_{e3})\frac{3}{4}y^3 - F_{\text{ac}}\cos(\phi) = 0,$$
$$c\omega y - F_{\text{ac}}\sin(\phi) = 0.$$

Where:

$$F_{\text{ac}} = \text{Vac} \cdot \text{trans\_factor},$$
$$c = \frac{\sqrt{M k_t}}{Q}.$$

The solutions are filtered according to the following condition:

$$\frac{\omega_0}{2} < \omega < \frac{3\omega_0}{2}, \quad \omega > 0.$$

主要变量

- **几何参数:**
  - w_t: 梁的宽度。
  - l_t: 梁的长度。
  - t: 梁的厚度。
  - electrode_length, electrode_width: 电极的长度和宽度。
  - w_c, l_c: 梁末端集中质量块的宽度和长度。
  - d: 电极间隙距离。

- 材料参数:
  - E: 弹性模量。
  - ρ: 材料密度。
  - ϵ_0: 真空介电常数。
- 电学和振动条件:
  - Q: 品质因数。
  - V: 驱动电压。
  - Vac: 交流电压幅值（谐波驱动）。
  - ϕ: 谐波驱动的相位范围。
  - β, α_n: 与模态形状相关的参数。

主要计算过程

模态形状计算

模态形状描述梁在共振条件下的变形分布，基于欧拉-伯努利梁理论：

$$\begin{aligned}\text{modeshape\_unnormalized} = -\Big(& -\sin\left(\beta x/l_t\right) + \sinh\left(\beta x/l_t\right) \\ & + \alpha_n\big(-\cos\left(\beta x/l_t\right) + \cosh\left(\beta x/l_t\right)\big)\Big)\end{aligned}$$

归一化模态形状：

$$\text{modeshape1} = \frac{\text{modeshape\_unnormalized}}{\max(\text{modeshape\_unnormalized})}$$

力学特性

梁的弯曲刚度和质量计算如下：

- 刚度系数（归一化）：

$$k_{\text{coef\_b}} = \int \frac{\text{second\_derivative}^2}{l_t}\,dx,$$
$$k_{\text{coef\_b3}} = \int \frac{\text{first\_derivative}^2}{l_t}\,dx.$$

- 实际刚度：

$$k_{tt} = \frac{k_{\text{coef\_b}}}{12} Et \left(\frac{w_t}{l_t}\right)^3,$$
$$k_{t3} = k_{\text{coef\_b3}} Et \frac{w_t}{l_t^3}.$$

- 总刚度：

$$k_t = k_{tt}.$$

- 总质量：

$$M = \rho \left(t w_t l_t m_{\text{coef\_b}} + \text{electrode\_length} \cdot \text{electrode\_width} \cdot t + 2 w_c \cdot l_c \cdot t \right).$$

- 自然角频率：

$$\omega_0 = \sqrt{\frac{k_t}{M}}, \quad f_0 = \frac{\omega_0}{2\pi}.$$

电耦合效应

电场引入附加刚度和非线性效应：

- 转换因子：

$$\text{trans\_factor} = \frac{\epsilon_0 V \cdot \text{electrode\_length} \cdot t}{d^2}.$$

- 电耦合刚度：

$$k_e = 2 \cdot \text{trans\_factor} \cdot \frac{V}{d},$$
$$k_{e3} = 4 \cdot \text{trans\_factor} \cdot \frac{V}{d^3}.$$

谐波驱动下的动态响应

通过求解非线性方程组，计算频率和运动电流：

$$-M\omega^2 y + (k_t - k_e)y + (k_{t3} - k_{e3})\frac{3}{4}y^3 - F_{\text{ac}} \cos(\phi) = 0,$$
$$c\omega y - F_{\text{ac}} \sin(\phi) = 0.$$

其中：

$$F_{\text{ac}} = \text{Vac} \cdot \text{trans\_factor},$$
$$c = \frac{\sqrt{M k_t}}{Q}.$$

解的筛选条件：

$$\frac{\omega_0}{2} < \omega < \frac{3\omega_0}{2}, \quad \omega > 0.$$