Matrix

A $m \times n$ matrix with entries from a field F is a rectangular array of the form

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

Let $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ and $B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$

Then $A + B = \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13} \\ a_{21}+b_{21} & a_{22}+b_{22} & a_{23}+b_{23} \\ a_{31}+b_{31} & a_{32}+b_{32} & a_{33}+b_{33} \end{pmatrix}$

$$AB = \begin{pmatrix} a_{11}b_{11}+a_{12}b_{21}+a_{13}b_{31} & a_{11}b_{12}+a_{12}b_{22}+a_{13}b_{32} & a_{11}b_{13}+a_{12}b_{23}+a_{13}b_{33} \\ * & * & * \\ * & * & * \end{pmatrix}$$

Matrix

We define elementary operations. There are two type of elementary matrix operations — row operations and column operation. Of these, row operations is more useful. They arise from three operations that can be used to ~~elim~~ eliminate variables in a system of linear equations.

<u>Definition</u>: Let A be an $m \times n$ matrix. Any one of the following three operations on the rows (columns) of A is called an <u>elementary row (column) operation</u>.

(i) Interchanging any two rows (columns) of A.

(ii) ~~Mul~~ Multiplying any row (column) of A by a non-zero scalar.

(iii) ~~a~~ Adding any scalar multiple of a row of A to another row (column).

Elimentary operations are of <u>type 1</u>, <u>type 2</u> or <u>type 3</u> depending on whether they are obtained by (i), (ii), or (iii).

<u>Example 1</u>: $$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & -1 & 3 \\ 4 & 0 & 1 & 2 \end{pmatrix}$$

Interchanging 2nd row of A with 1st row (elementary operation of type 1), we obtain

$$B = \begin{pmatrix} 2 & 1 & -1 & 3 \\ 1 & 2 & 3 & 4 \\ 4 & 0 & 1 & 2 \end{pmatrix}.$$

Multiplying 2nd column of A by 2 (Elementary operation of type 2), we obtain

$$\cancel{C = \begin{pmatrix} 2 & 2 \\ 1 & 4 \\ 4 & 0 \end{pmatrix}}$$

$$C = \begin{pmatrix} 1 & 4 & 3 & 4 \\ 2 & 2 & -1 & 3 \\ 4 & 0 & 1 & 2 \end{pmatrix}.$$

Adding 2 times the third row of A to first row (Elementary operation of type 3), we obtain

$$D = \begin{pmatrix} 6 & 4 & 7 & 10 \\ 2 & 1 & -1 & 3 \\ 4 & 0 & 1 & 2 \end{pmatrix}.$$

$$I_n = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}, \text{ identity matrix of order } n.$$

Definition: An $n \times n$ elementary matrix is matrix obtained by performing an elementary operation on I_n. The elementary matrix is said to be of type 1, 2 or 3 according to whether the elementary operation performed on I_n is a type 1, 2 or 3 respectively.

Interchanging 1st row of I_3 with 2nd row, we obtain elementary matrix

$$E = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Theorem: Let $A \in M_{m \times n}(F)$, and suppose that B is obtained from A by performing an elementary row (column) operation. Then there exists an $m \times m$ $(n \times n)$ elementary matrix E such that $B = EA$. $(B = AE)$

E is obtained from I_m (I_n) by performing the same elementary row (column) operation as that which was performed on A to obtain B.

Conversely, if E is an elementary $m \times m$ $(n \times n)$ matrix, then EA (AE) is the matrix obtained from A by performing the same elementary row (column) operation as that which produces E from I_m (I_n).

Example 2: Consider the matrix A in Example 1.

$$B = \begin{pmatrix} 2 & 1 & -1 & 3 \\ 1 & 2 & 3 & 4 \\ 4 & 0 & 1 & 2 \end{pmatrix} \quad \left(\text{Interchanging 1st row of } A \text{ with 2nd row}\right)$$

Again $E = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ (Interchanging 1st row of I_3 with 2nd row)

row

$$EA = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & -1 & 3 \\ 4 & 0 & 1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 1 & -1 & 3 \\ 1 & 2 & 3 & 4 \\ 4 & 0 & 1 & 2 \end{pmatrix} = B.$$

Echelon form

Definition: A matrix is said to be in reduced row echelon form if the following three conditions are satisfied.

(a) Any row containing a non zero entry precedes any row in which all the entries are zero (if any)

(b) The first non zero entry in each row is the only nonzero entry in its column.

(c) The first non zero entry in each row is 1 and it occurs in a column to the right of the first non zero entry in the preceding row.

Example 1: The matrix $\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ is in reduced row echelon form.

Example 2: The matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$ is not in reduced row echelon form,

As the first column which contains the first non-zero entry in the first row, contains another non-zero entry.

The matrix $\begin{pmatrix} 0 & 1 & 0 & 2 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$ is not in reduced row echelon form. As the first non zero entry in the 2nd row is not to the right of the first non-zero entry of the first row.

The matrix $\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ is not in reduced row echelon form. As the first nonzero entry in the first row is not 1.

The matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is not in reduced row echelon form. As the 3rd row which is a non-zero row does not preced the zero row (2nd row)

Rank of a matrix :

The rank of a matrix A is the number of nonzero rows in the reduced row echelon form of A.

Example 1:

$$A = \begin{pmatrix} 2 & 4 & 6 & 2 & 4 \\ 1 & 2 & 3 & 1 & 1 \\ 2 & 4 & 8 & 0 & 0 \\ 3 & 6 & 7 & 5 & 9 \end{pmatrix}$$

$$A \xrightarrow{R_1 \to \frac{1}{2}R_1} \begin{pmatrix} 1 & 2 & 3 & 1 & 2 \\ 1 & 2 & 3 & 1 & \end{pmatrix}$$

Procedure to obtain reduced row echelon form of a matrix.

We show the procedure by giving an example.

Consider, the matrix
$$A = \begin{pmatrix} 3 & 2 & 3 & -2 \\ 1 & 1 & 1 & 0 \\ 1 & 2 & 1 & -1 \end{pmatrix}.$$

step 1 : In the leftmost non-zero column, create a 1 in the first row.

On A interchange 1st and 3rd row

$$\begin{pmatrix} 1 & 2 & 1 & -1 \\ 1 & 1 & 1 & 0 \\ 3 & 2 & 3 & -2 \end{pmatrix}$$

step 2 : By means of type 3 row operation, use the 1st row to obtain zeros in the remaining positions of the leftmost non-zero column.

Perform $R_2 \to R_2 - R_1$, $R_3 \longrightarrow R_3 - 3R_1$

$$\begin{pmatrix} 1 & 2 & 1 & -1 \\ 0 & -1 & 0 & 1 \\ 0 & -4 & 0 & 1 \end{pmatrix}$$

step 3 : Create a 1 in the next row in the leftmost possible column, without using previous row(s).

Perform $R_2 \longrightarrow (-1) R_2$

$$\begin{pmatrix} 1 & 2 & 1 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & -4 & 0 & 1 \end{pmatrix}$$

Step 4: Use type 3 row operations to obtain zeros below 1 created in the preceding step.

Perform $R_3 \rightarrow R_3 + 4R_2$

$$\begin{pmatrix} 1 & 2 & 1 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & -3 \end{pmatrix}$$

Step 5: Repeat steps 3 & step 4 on each succeeding row untill no nonzero rows remains.

Perform $R_3 \rightarrow -\frac{1}{3}R_3$

$$\begin{pmatrix} 1 & 2 & 1 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Step 6: Work upward beginning with the last nonzero row and add multiples of row to the rows above. This creates zeros above the first nonzero entry in each row.

Perform $R_1 \rightarrow R_1 + R_3$
$R_2 \rightarrow R_2 + R_3$

$$\begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Step 7: Repeat the process describe in step 6 for each preceding row untill it is performed with the second row. This completes the procedure.

Perform $R_1 \to R_1 - 2R_2$

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

This is the reduced row echelon form of A.

Rank of a matrix:

The rank of a matrix A is the number of non zero rows in the reduced row echelon form of A.

Example 1: $A = \begin{pmatrix} 1 & 2 & 0 & 1 & 1 \\ 2 & 4 & 1 & 3 & 0 \\ 3 & 6 & 2 & 5 & 1 \\ -4 & -8 & 1 & -3 & 1 \end{pmatrix}$

$$A \xrightarrow[R_4 \to R_4 + 4R_1]{\substack{R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1}} \begin{pmatrix} 1 & 2 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & -2 \\ 0 & 0 & 2 & 2 & -2 \\ 0 & 0 & 1 & 1 & 5 \end{pmatrix}$$

$$\Big\downarrow \begin{matrix} R_3 \to R_3 - 2R_2 \\ R_4 \to R_4 - R_2 \end{matrix}$$

$$\begin{pmatrix} 1 & 2 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 7 \end{pmatrix}$$

$$\Big\downarrow R_3 \to \tfrac{1}{2} R_3$$

$$\begin{pmatrix} 1 & 2 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 7 \end{pmatrix}$$

$\xrightarrow[R_4 \to R_4 - 7R_3]{}$ $\begin{pmatrix} 1 & 2 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

$\downarrow$ $R_1 \to R_1 - R_3$, $R_2 \to R_2 + 2R_3$

$$\begin{pmatrix} 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$\therefore$ The number of non-zero rows is 3

$\therefore$ rank $(A) = 3$.

Example 2: $A = \begin{pmatrix} 2 & 4 & 6 & 2 & 4 \\ 1 & 2 & 3 & 1 & 1 \\ 2 & 4 & 8 & 0 & 0 \\ 3 & 6 & 7 & 5 & 9 \end{pmatrix}$

$A \xrightarrow{R_1 \to \frac{1}{2}R_1} \begin{pmatrix} 1 & 2 & 3 & 1 & 2 \\ 1 & 2 & 3 & 1 & 1 \\ 2 & 4 & 8 & 0 & 0 \\ 3 & 6 & 7 & 5 & 9 \end{pmatrix}$

$\downarrow$ $R_2 \to R_2 - R_1$, $R_3 \to R_2 - 2R_1$, $R_4 \to R_4 - 3R_1$

$$\begin{pmatrix} 1 & 2 & 3 & 1 & 2 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 2 & -2 & -4 \\ 0 & 0 & -2 & 2 & 3 \end{pmatrix} \xrightarrow{R_2 \leftrightarrow R_4}$$

$$\begin{pmatrix} 1 & 2 & 3 & 1 & 2 \\ 0 & 0 & 2 & -2 & 3 \\ 0 & 0 & 2 & -2 & -4 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix} \xrightarrow{R_2 \to \frac{1}{2}R_2} \begin{pmatrix} 1 & 2 & 3 & 1 & 2 \\ 0 & 0 & 1 & -1 & \frac{3}{2} \\ 0 & 0 & 2 & -2 & -4 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

$$\Big\downarrow R_3 \to R_3 - 2R_2$$

$$\begin{pmatrix} 1 & 2 & 3 & 1 & 2 \\ 0 & 0 & 1 & -1 & 3/2 \\ 0 & 0 & 0 & 0 & -7 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

$$\Big\downarrow R_3 \to -\frac{1}{7}R_3$$

$$\begin{pmatrix} 1 & 2 & 3 & 1 & 2 \\ 0 & 0 & 1 & -1 & 3/2 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

$$\Big\downarrow R_4 \to R_4 + R_3$$

$$\begin{pmatrix} 1 & 2 & 3 & 1 & 2 \\ 0 & 0 & 1 & -1 & 3/2 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\Big\downarrow \begin{matrix} R_1 \to R_1 - 2R_3 \\ R_2 \to R_2 - \frac{3}{2}R_3 \end{matrix}$$

$$\begin{pmatrix} 1 & 2 & 3 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_1 \to R_1 - 3R_2} \begin{pmatrix} 1 & 2 & 0 & 4 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$\therefore$ The number of non-zero rows is 3

$\therefore$ rank$(A) = 3$.

Exercises: Find the rank of the following matrices

1. $\begin{pmatrix} 1 & 1 & 0 \\ 2 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

2. $\begin{pmatrix} 1 & 2 & 3 & 1 & 1 \\ 1 & 4 & 0 & 1 & 2 \\ 0 & 2 & -3 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$

3. $\begin{pmatrix} 1 & 1 & 0 & 1 \\ 2 & 2 & 0 & 2 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \end{pmatrix}$

Some properties of rank of a matrix.

1. Let A be an $n \times n$ matrix. If P and Q are invertible (inverse exist) $m \times m$ and $n \times n$ matrices then

(a) $\text{rank}(AQ) = \text{rank}(A)$

(b) $\text{rank}(PA) = \text{rank}(A)$

(c) $\text{rank}(PAQ) = \text{rank}(A)$

2. The rank of any matrix equals the maximum number of its linearly independent columns (rows). In other words rank of a matrix is the dimension of the subspace generated by its columns (rows)

~~3. The rank of any matrix equals the~~

3. $\text{rank}(A^t) = \text{rank}(A)$, A^t is transpose of A.

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}_{m \times n}$$

$$A^t = \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \cdots & \cdots & \cdots & \cdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{pmatrix}_{n \times m}$$

Determinants:

Every square matrix can be associated to an expression or a number which is known as determinant.

The determinant of a square matrix $A = (a_{ij})$ of order n is denoted ~~denoted by~~ by $|A|$ and given by

$$|A| = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

Determinant of a square matrix of order 2.

Let $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

$$|A| = a_{11}a_{22} - a_{12}a_{21}$$

Determinant of a square matrix of order 3

Let $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

$$|A| = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Properties of Determinant :

1. If any row (or column) of a matrix A is completely zero, then $|A| = 0$

2. If any two rows or two columns of a matrix are interchanged, the value of determinant is multiplied by -1.

3. If A is a $n \times n$ matrix and K be any scalar, then $|KA| = K^n |A|$

4. $|AB| = |A| \cdot |B|$ where A & B are two matrices

   Also $|A^n| = (|A|)^n$.

Minors and Co-factors :

$$\cancel{\text{Let } A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}}$$

Let $A = (a_{ij})_{n \times n}$. If delete the i-th row and the j-th column passing through a_{ij} of A, then the determinant of square sub matrix of order $(n-1)$ is called minor of a_{ij} and is denoted by M_{ij}.

Let $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

The minor of a_{11} is $\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = M_{11}$

The minor of a_{21} is $\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} = M_{21}$

and so on.

The minor M_{ij} multiplied by $(-1)^{i+j}$, is called the co-factor of the element a_{ij} and is denoted by A_{ij}.

The co-factor of $a_{11} = A_{11} = (-1)^{1+1} M_{11} = M_{11}$

$$= \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

The co-factor of $a_{21} = A_{21} = (-1)^{2+1} M_{21} = -M_{21}$

$$= - \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}$$

Adjoint of a square matrix :

Let $A = (a_{ij})_{n \times n}$. The adjoint of A is defined as the transpose of $(A_{ij})_{n \times n}$ where A_{ij} is the co-factor of a_{ij} in the determinant $|A|$.

Let $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{23} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

Co-factor of A or $Cof(A) = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{32} \end{pmatrix}$

Then $adj(A) = [Cof(A)]^t$

$$= \begin{pmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{32} \end{pmatrix}$$

Inverse of a matrix :

A square matrix A of order n is said to be invertible if there is a square matrix B of order n such that

$$AB = BA = I_n \quad \left(I_n = \text{identity matrix of order } n \right)$$

B is called the inverse of A.

and given by $A^{-1} = \dfrac{adj(A)}{|A|}$, provided $|A| \neq 0$.

(i) Inverse of a square matrix A exists if and only if A is non-singular, that is, $|A| \neq 0$.

(ii) Inverse of a square matrix if exists is unique.

(iii) A and B are inverse of each other if $AB = BA = I$.

(iv) If A and B are square matrices of same order, then AB is invertible if and only if both A & B are non-singular and $(AB)^{-1} = B^{-1}A^{-1}$

Augmented matrix: Let A and B be $m \times n$ and $n \times p$ matrices respectively. Then augmented matrix $(A|B)$ is the $m \times (n+p)$ matrix $(A\,B)$, i.e. a matrix whose first n columns are the columns of A and last p columns are the columns of B.

Example : $A = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 4 \\ 1 & 1 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 0 \\ 1 & 2 \\ 4 & 3 \end{pmatrix}$

The augmented matrix

$$(A|B) = \left(\begin{array}{ccc|cc} 1 & 2 & 3 & 2 & 0 \\ 3 & 1 & 4 & 1 & 2 \\ 1 & 1 & 2 & 4 & 3 \end{array} \right)$$

If A is an invertible $n \times n$ matrix, then it is possible to transform $(A \mid I_n)$ into the matrix $(I_n \mid A^{-1})$ by means of a finite number of elementary row operations.

<u>Example</u> : Find the inverse of $A = \begin{pmatrix} 1 & 2 & 1 \\ -1 & 1 & 2 \\ 1 & 0 & 1 \end{pmatrix}$

$$(A \mid I_3) = \left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ -1 & 1 & 2 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\Big\downarrow \begin{array}{l} R_2 \to R_1 + R_2 \\ R_3 \to R_1 - R_3 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 3 & 3 & 1 & 1 & 0 \\ 0 & 2 & 0 & 1 & 0 & -1 \end{array}\right)$$

$$\Big\downarrow R_2 \to \tfrac{1}{3} R_2$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1/3 & 1/3 & 0 \\ 0 & 2 & 0 & 1 & 0 & -1 \end{array}\right)$$

$$\Big\downarrow R_3 \to R_2 - \tfrac{1}{2} R_3$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 1 & -\frac{1}{6} & \frac{1}{3} & \frac{1}{2} \end{array}\right)$$

$$\Big\downarrow R_1 \to R_1 - R_3, \quad R_2 \to R_2 - R_3$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 7/6 & -\frac{1}{3} & -1/2 \\ 0 & 1 & 0 & 1/2 & 0 & -1/2 \\ 0 & 0 & 1 & -1/6 & 1/3 & 1/2 \end{array}\right) \xrightarrow{R_1 \to R_1 - 2R_2}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/6 & -1/3 & 1/2 \\ 0 & 1 & 0 & 1/2 & 0 & -1/2 \\ 0 & 0 & 1 & -1/6 & 1/3 & 1/2 \end{array}\right)$$

$$\therefore A^{-1} = \begin{pmatrix} 1/6 & -1/3 & 1/2 \\ 1/2 & 0 & -1/2 \\ -1/6 & 1/3 & 1/2 \end{pmatrix}$$

Exercies: Find the inverse of the following matrices

1. $\begin{pmatrix} 1 & 2 & 1 \\ 1 & 3 & 4 \\ 2 & 3 & -1 \end{pmatrix}$

2. $\begin{pmatrix} 1 & 2 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

3. $\begin{pmatrix} 1 & 2 & 1 & 0 \\ 2 & 5 & 5 & 1 \\ -2 & -3 & 0 & 3 \\ 3 & 4 & -2 & -3 \end{pmatrix}$

~~Let~~ A be a square matrix of order n. Then A^{-1} exists if and only if rank$(A) = n$.

In otherwords, if rank$(A) < n$, then A^{-1} does not exist.

System of Linear Equations :

Consider the system of equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \cdots\cdots\cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned} \qquad \text{—— (1)}$$

Where a_{ij} & b_i $(1 \le i \le m, 1 \le j \le n)$ are scalar in a field F. and $x_1, x_2, \cdots, x_n$ are n-variables.
It is a system of m linear equations in n unknowns over the field F.

The $m \times n$ matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}_{m\times n}$$

is called the coefficient matrix of the system (1).

Let $x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}_{n\times 1}$ and $b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}_{m\times 1}$

The system (1) can be rewritten as

$$\boxed{Ax = b}$$

A solution to the system (1) is an n-tuple

$$s = \begin{pmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{pmatrix} \in F^n$$

such that $As = b$

The set of all solutions to the system ① is called the solution set of the system.

The system ① is called consistent if its solution set is non.empty, otherwise it is called inconsistent

□ A system $Ax=b$ of m linear equations and n unknown variables is said to be homogeneous if $b=0$. Otherwise the system is said to be non-homoge-neous.

Here $b=0$ means $b=\begin{pmatrix}0\\0\\\vdots\\0\end{pmatrix}_{n\times 1}$.

Any homogeneous equation has at least one solution, namely the zero vector

i.e. $x=\begin{pmatrix}0\\0\\\vdots\\0\end{pmatrix}$ where $Ax=0$.

□ Let $Ax=b$ be a system of n linear equations in n unknown variables. If A is invertible, then the system has exactly one solution, namely, $x=A^{-1}b$. Conversly, if the system has exactly one solution, then A is invertible.

□ Let $Ax = b$ be a system of equations, where

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}_{m \times n}$$

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}_{n \times 1}$$

$$b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}_{m \times 1}$$

The augmented matrix $A' \ (A|b) = \left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \cdots & \cdots & \cdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right)$

**

Criteria for a system of non-homogeneous linear equations $Ax = b$:

(1) If $\text{rank}(A) \neq \text{rank}(A|b)$, then the system is inconsistent.

(2) If $\text{rank}(A) = \text{rank}(A|b) =$ number of unknown variables, then the system has unique solution.

(3) If $\text{rank}(A) = \text{rank}(A|b) <$ number of unknown variables, then the system has infinite number of solutions.

Criteria for a system of homogeneous linear equation $Ax = 0$:

(1) If rank $(A) = n$, the number of unknown variables, then the system has only the trivial solution or unique solution, namely $x = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

(2) If rank $(A) < n$, the number of unknown, then the system has an infinite number of solution.

Example 1:

$$x_1 + x_2 - x_3 + 2x_4 = 2$$
$$x_1 + x_2 + 2x_3 = 1$$
$$2x_1 + 2x_2 + x_3 + 2x_4 = 4$$

Let $Ax = b$

where $A = \begin{pmatrix} 1 & 1 & -1 & 2 \\ 1 & 1 & 2 & 0 \\ 2 & 2 & 1 & 2 \end{pmatrix}$, $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$, $b = \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix}$

$$(A|b) = \left(\begin{array}{cccc|c} 1 & 1 & -1 & 2 & 2 \\ 1 & 1 & 2 & 0 & 1 \\ 2 & 2 & 1 & 2 & 4 \end{array}\right) \xrightarrow[R_3 \to R_3 - 2R_1]{R_2 \to R_2 - R_1} \left(\begin{array}{cccc|c} 1 & 1 & -1 & 2 & 2 \\ 0 & 0 & 3 & -2 & -1 \\ 0 & 0 & 3 & -2 & 0 \end{array}\right)$$

$$\xrightarrow{R_3 \to R_3 - R_2} \left(\begin{array}{cccc|c} 1 & 1 & -1 & 2 & 2 \\ 0 & 0 & 3 & -2 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right) \xrightarrow[R_2 \to R_2 + R_1]{R_1 \to R_1 - 2R_3} \left(\begin{array}{cccc|c} 1 & 1 & -1 & 2 & 0 \\ 0 & 0 & 3 & -2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{R_2 \to \frac{1}{3}R_2} \left(\begin{array}{cccc|c} 1 & 1 & -1 & 2 & 0 \\ 0 & 0 & 1 & -2/3 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_1 \to R_1 + R_2} \left(\begin{array}{cccc|c} 1 & 1 & 0 & \frac{4}{3} & 0 \\ 0 & 0 & 1 & -2/3 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

A b

Observe that $\text{rank}(A) = 2$ but $\text{ran}(A|b) = 3$

Hence the system is inconsistent

or there is no solution to this system.

Example 2:

$$x_1 + 2x_2 + 3x_3 = 1$$
$$x_1 + x_2 - x_3 = 0$$
$$x_1 + 2x_2 + x_3 = 3$$

Let $Ax = b$

where $A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & -1 \\ 1 & 2 & 1 \end{pmatrix}$, $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$, $b = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$

$$(A|b) = \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 1 & 1 & -1 & 0 \\ 1 & 2 & 1 & 3 \end{array}\right) \xrightarrow[R_3 \to R_1 - R_3]{R_2 \to R_1 - R_2} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 2 & -2 \end{array}\right)$$

$$\Big\downarrow R_3 \to \tfrac{1}{2}R_3$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & -1 \end{array}\right) \xleftarrow[R_2 \to R_2 - 4R_3]{R_1 \to R_1 - 3R_3} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right)$$

$$\Big\downarrow R_1 \to R_1 - 2R_2$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & -6 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & -1 \end{array}\right)$$

$\text{rank}(A) = \text{rank}(A|b) = 3$

$\therefore$ The solution is unique.

Now we can write, the equations as

$$x_1 = -6$$
$$x_2 = 5$$
$$x_3 = -1$$

$\therefore$ $x_1 = -6$, $x_2 = 5$, $x_3 = -1$. is the solution.

Example 3 :

$$2x_1 + 3x_2 + x_3 + 4x_4 - 9x_5 = 17$$
$$x_1 + x_2 + x_3 + x_4 - 3x_5 = 6$$
$$x_1 + x_2 + x_3 + 2x_4 - 5x_5 = 8$$
$$2x_1 + 2x_2 + 2x_3 + 3x_4 - 8x_5 = 14$$

Let $Ax = b$

The augmented matrix
$$(A|b) = \left(\begin{array}{ccccc|c} 2 & 3 & 1 & 4 & -9 & 17 \\ 1 & 1 & 1 & 1 & -3 & 6 \\ 1 & 1 & 1 & 2 & -5 & 8 \\ 2 & 2 & 2 & 3 & -8 & 14 \end{array}\right)$$

$$\Big\downarrow R_1 \leftrightarrow R_2$$

$$\left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & -3 & 6 \\ 2 & 3 & 1 & 4 & -9 & 17 \\ 1 & 1 & 1 & 2 & -5 & 8 \\ 2 & 2 & 2 & 3 & -8 & 14 \end{array}\right)$$

$$\Big\downarrow R_2 \to R_2 - 2R_1,\ R_3 \to R_3 - R_1,\ R_4 \to R_4 - 2R_1$$

$$\left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & -3 & 6 \\ 0 & 1 & -1 & 2 & -3 & 5 \\ 0 & 0 & 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 1 & -2 & 2 \end{array}\right)$$

$$\Big\downarrow R_4 \to R_4 - R_3$$

$$\left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & -3 & 6 \\ 0 & 1 & -1 & 2 & -3 & 5 \\ 0 & 0 & 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\Big\downarrow R_1 \to R_1 - R_3,\ R_2 \to R_2 - 2R_3$$

$$\left(\begin{array}{ccccc|c} 1 & 1 & 1 & 0 & -1 & 4 \\ 0 & 1 & -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{R_1 \to R_1 - R_2}$$

$$\left(\begin{array}{ccccc|c} 1 & 0 & 2 & 0 & -2 & 3 \\ 0 & 1 & -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$\text{rank}(A) = \text{rank}(A|b) = 3 < 4$

$\therefore$ This system has infinite number of solution

We can now write the equations as

$$x_1 + 2x_3 + 2x_5 = 3$$
$$x_2 - x_3 + x_5 = 1$$
$$x_4 - 2x_5 = 2$$

We have,

$$x_4 = 2 + 2x_5, \quad x_2 = x_3 - x_5 + 1,$$
$$x_1 = 3 - 2x_3 + 2x_5$$

Let $x_3 = t_1$, $x_5 = t_2$

Then
$$x_1 = 3 - 2t_1 + 2t_2$$
$$x_2 = t_1 - t_2 + 1$$
$$x_4 = 2 + 2t_2$$

So the solution is

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 3 - 2t_1 + 2t_2 \\ t_1 - t_2 + 1 \\ t_1 \\ 2 + 2t_2 \\ t_2 \end{pmatrix}$$

where $t_1, t_2 \in \mathbb{R}$

Exercises: solve the following systems of linear equations.

(1)
$$x_1 + 2x_2 - x_3 = 1$$
$$2x_1 + x_2 + 2x_3 = 3$$
$$x_1 - 4x_2 + 7x_3 = 4$$

(2)
$$x_1 + x_2 + 3x_3 - x_4 = 0$$
$$x_1 + x_2 + x_3 + x_4 = 1$$
$$x_1 - 2x_2 + x_3 - x_4 = 1$$
$$4x_1 + x_2 + 8x_3 - x_4 = 0$$

(3)
$$x_1 + 2x_2 - x_3 + 3x_4 = 2$$
$$2x_1 + 4x_2 - x_3 + 6x_4 = 5$$
$$x_2 + 2x_4 = 3$$

1. Echelon form.

2. Finding the rank of a matrix.

3. Computing inverse of a matrix,

4. Solving system of linear equations.