Problem Set 2 submitted by Mayur Agrawal

Solution to Exercise 1

1. Strategies Available to Players

- **Player 1 (P1)**: Player 1 observes the card and decides:

- If the card is **good**: Quit (Q) or Raise (R).
- If the card is **bad**: Quit (Q) or Raise (R).

Thus, Player 1's strategies are:

$$\{(Q, Q), (Q, R), (R, Q), (R, R)\},$$

where the first action corresponds to a good card and the second to a bad card.

- **Player 2 (P2)**: Player 2 does not observe Player 1's card and decides:

- Quit (Q): Give up the pot.
- See (S): Invest 5 CHF to see Player 1's card.

Thus, Player 2's strategies are:

$$\{Q, S\}.$$

2. Number of Subgames

There are **four subgames**, corresponding to:

1. Player 1 has a **good card** and chooses **Raise**.
2. Player 1 has a **bad card** and chooses **Raise**.
3. Player 1 has a **good card** and chooses **Quit**.
4. Player 1 has a **bad card** and chooses **Quit**.

3. Pure-Strategy Bayesian Nash Equilibria (BNE)

Using the **Harsanyi Transformation**, we treat Player 1's card as nature's move:

- p: Probability of Player 1 having a good card.
- $1 - p$: Probability of Player 1 having a bad card.

Payoff Analysis:

1. If Player 1 **raises**:
   - If Player 2 quits: Player 1 wins the pot (+1).
   - If Player 2 sees:
     - If Player 1 has a good card: Player 1 wins (+6), Player 2 loses (−6).
     - If Player 1 has a bad card: Player 1 loses (−6), Player 2 wins (+6).
2. If Player 1 **quits**: Player 2 wins the pot (+1), Player 1 loses (−1).

Conclusion: No pure-strategy Nash equilibrium exists, so we analyze mixed strategies.

4. Mixed-Strategy Bayesian Nash Equilibria (BNE)

Let:

- x_g: Probability of Player 1 raising with a **good card**.
- x_b: Probability of Player 1 raising with a **bad card**.
- y: Probability of Player 2 quitting after Player 1 raises.

Indifference Conditions:

1. Player 1 (Good Card): Indifference between quitting and raising:

$$-1 = x_g \cdot [6 \cdot (1 - y) - 6 \cdot y] .$$

2. Player 1 (Bad Card): Indifference between quitting and raising:

$$-1 = x_b \cdot [-6 \cdot (1 - y) + 6 \cdot y] .$$

3. Player 2: Indifference between quitting and seeing:

$$1 = y \cdot [-6 \cdot p + 6 \cdot (1 - p)] .$$

Solution: Solving these equations, we get:

$$x_g = \frac{-(2p-1)}{4(3p-1)}, \quad x_b = \frac{(2p-1)}{4(3p-1)}, \quad y = \frac{-1}{6(2p-1)}.$$

5. Perfect Bayesian Equilibrium (PBE)

To find the Perfect Bayesian Equilibrium (PBE), we specify:

1. Strategies of Player 1 and Player 2.
2. Beliefs Player 2 holds about Player 1's card after observing a raise.
3. Ensure strategies are sequentially rational, i.e., best responses given updated beliefs.

Step 1: Belief Updates

Player 2 updates their belief about Player 1's card after observing a raise using Bayes' Rule:

$$\text{Belief (Good Card)} = \frac{\Pr(\text{good card}) \cdot \Pr(\text{raise — good card})}{\Pr(\text{raise})},$$

where:

- $\Pr(\text{good card}) = p$,
- $\Pr(\text{raise — good card}) = x_g$,
- $\Pr(\text{raise — bad card}) = x_b$,

- $\Pr(\text{raise}) = p \cdot x_g + (1-p) \cdot x_b$.

  Thus:
  $$\text{Belief (Good Card)} = \frac{p \cdot x_g}{p \cdot x_g + (1-p) \cdot x_b}.$$
  Belief that Player 1 has a **bad card**:
  $$\text{Belief (Bad Card)} = 1 - \text{Belief (Good Card)}.$$

Step 2: Sequential Rationality of Player 2

Player 2 compares the expected payoffs of **Quit (Q)** and **See (S)** after observing a raise:

- Payoff from Quit:
  $$\text{Payoff (Quit)} = 1.$$
- Payoff from See:
  $$\text{Payoff (See)} = \text{Belief (Good Card)} \cdot (-6) + \text{Belief (Bad Card)} \cdot 6.$$

  Substituting the updated beliefs:
  $$\text{Payoff (See)} = \left(\frac{p \cdot x_g}{p \cdot x_g + (1-p) \cdot x_b}\right) \cdot (-6) + \left(\frac{(1-p) \cdot x_b}{p \cdot x_g + (1-p) \cdot x_b}\right) \cdot 6.$$
  Player 2's indifference condition is:
  $$1 = \text{Payoff (See)}.$$

Step 3: Sequential Rationality of Player 1

1. **Player 1 with a Good Card:**
   - Payoff from Quit:
     $$\text{Payoff (Quit)} = -1.$$
   - Payoff from Raise:
     $$\text{Payoff (Raise)} = 6 \cdot (1-y) - 6 \cdot y = 6(1-2y).$$

Indifference condition:

$$-1 = 6(1 - 2y).$$

Solving for y:

$$y = \frac{1}{2}.$$

2. **Player 1 with a Bad Card:**

   - Payoff from Quit:
   $$\text{Payoff (Quit)} = -1.$$
   - Payoff from Raise:
   $$\text{Payoff (Raise)} = -6 \cdot (1 - y) + 6 \cdot y = -6(1 - 2y).$$

   Indifference condition:

   $$-1 = -6(1 - 2y).$$

   Solving for y:

   $$y = \frac{1}{2}.$$

Step 4: PBE Strategies

1. **Player 1's Mixed Strategy:**

$$x_g = \frac{-(2p - 1)}{4(3p - 1)}, \quad x_b = \frac{(2p - 1)}{4(3p - 1)}.$$

2. **Player 2's Mixed Strategy:**

$$y = \frac{1}{2}.$$

3. **Player 2's Beliefs:**

$$\text{Belief (Good Card)} = \frac{p \cdot x_g}{p \cdot x_g + (1 - p) \cdot x_b}.$$

Conclusion

The Perfect Bayesian Equilibrium (PBE) is:

- Player 1 raises with probabilities:

$$x_g = \frac{-(2p-1)}{4(3p-1)}, \quad x_b = \frac{(2p-1)}{4(3p-1)}.$$

- Player 2 quits with probability:

$$y = \frac{1}{2}.$$

- Player 2 updates their beliefs using Bayes' Rule:

$$\text{Belief (Good Card)} = \frac{p \cdot x_g}{p \cdot x_g + (1-p) \cdot x_b}.$$

Solution to Exercise 2

1. Symmetric Bayesian Nash Equilibrium (BNE)

Step 1: Hypothesis of a threshold strategy Each player adopts a threshold strategy where they work if and only if:

$$v_i \geq v^*,$$

where v^* is the threshold to be determined.

Step 2: Payoffs for working and not working If a player decides to work, the public good is guaranteed to be produced, and their payoff is:

$$\text{Payoff for working} = v_i - c.$$

If a player decides not to work, the public good is produced only if at least one of the other $n-1$ players works. Let $F(v^*)$ be the probability that a player does not work, which for a uniform distribution on $[0,1]$ is:

$$F(v^*) = v^*.$$

The probability that all other $n-1$ players do not work is:

$$(F(v^*))^{n-1} = (v^*)^{n-1}.$$

Thus, the probability that at least one of the other players works is:

$$1 - (v^*)^{n-1}.$$

The expected payoff for not working is therefore:

$$\text{Payoff for not working} = v_i \cdot \left(1 - (v^*)^{n-1}\right).$$

Step 3: Indifference condition at v^* At equilibrium, the player must be indifferent between working and not working when $v_i = v^*$. Thus:

$$v^* - c = v^* \cdot \left(1 - (v^*)^{n-1}\right).$$

Simplify:

$$v^* - c = v^* - v^* \cdot (v^*)^{n-1}.$$

Cancel v^* from both sides (as $v^* > 0$):

$$c = v^* \cdot (v^*)^{n-1}.$$

$$c = (v^*)^n.$$

Solve for v^*:

$$v^* = c^{1/n}.$$

2. Probability that the public good is produced as a function of n

The public good is produced if at least one player works. The probability of a single player not working is:

$$P(\text{Not working}) = F(v^*) = v^* = c^{1/n}.$$

The probability that all n players do not work is:

$$P(\text{None work}) = (v^*)^n = \left(c^{1/n}\right)^n = c.$$

Thus, the probability that at least one player works (i.e., the good is produced) is:

$$P(\text{Good produced}) = 1 - P(\text{None work}) = 1 - c.$$

Explicit dependence on n: Although c is constant, the threshold $v^* = c^{1/n}$ decreases as n increases, which implies that the probability of each individual player working increases as n grows. However, the total probability of the good being produced remains:

$$P(\text{Good produced}) = 1 - c,$$

and does not explicitly depend on n for fixed c.

Final Answers

1. The symmetric Bayesian Nash equilibrium (BNE) threshold is:

$$v^* = c^{1/n}.$$

2. The probability that the public good is produced is:

$$P(\text{Good produced}) = 1 - c.$$

Solution to Exercise 3

Part 1: Bayesian Nash Equilibrium (BNE)

Payoff for keeping the bag: If the player keeps their bag, the payoff is simply:

$$\text{Payoff(Keep)} = x.$$

Payoff for switching: If the player switches their bag, the possible outcomes are:

- Gain $2x$ coins with probability $1/2$,
- Gain $x/2$ coins with probability $1/2$.

The expected payoff for switching is:

$$\text{Payoff(Switch)} = \frac{1}{2}(2x) + \frac{1}{2}\left(\frac{x}{2}\right) = x + \frac{x}{4} = \frac{5x}{4}.$$

Decision Rule: Switching is beneficial if:

$$\text{Payoff(Switch)} > \text{Payoff(Keep)} \implies \frac{5x}{4} > x.$$

Simplify:

$$\frac{5x}{4} > x \implies x < 80.$$

Thus, the optimal strategy is:

- **Switch if** $x < 80$,
- **Do not switch if** $x \geq 80$.

Analysis of the Hint

The problem asks, ”Would you switch if your bag contains 160 coins? And 80?”

Case 1: Bag contains 160 coins If your bag contains 160 coins, switching would mean:

- Gain 80 coins with probability 1/2,
- Gain 320 coins with probability 1/2.

The expected value of switching is:

$$\text{EV(Switching)} = \frac{1}{2}(80) + \frac{1}{2}(320) = 200.$$

Since $200 < 160$, you would **not switch**.

Case 2: Bag contains 80 coins If your bag contains 80 coins, switching would mean:

- Gain 40 coins with probability 1/2,
- Gain 160 coins with probability 1/2.

The expected value of switching is:

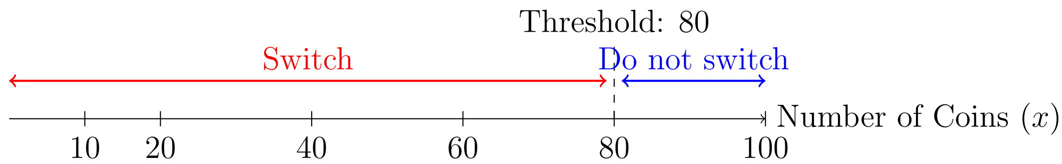
$$\text{EV(Switching)} = \frac{1}{2}(40) + \frac{1}{2}(160) = 100.$$

Since $100 > 80$, you would **switch**.

Conclusion: The hint is consistent with the BNE strategy:

- Do not switch if $x = 160$,
- Switch if $x = 80$.

BNE Strategy



2. Evaluating the Friend's Argument

Case 1: Bag contains 20 coins The possible pairs are:

$$(10, 20) \quad \text{or} \quad (20, 40),$$

with equal probability.

If you switch:

- Gain 10 coins with probability 1/2,
- Gain 40 coins with probability 1/2.

The expected value of switching is:

$$\text{Expected value} = \frac{1}{2}(10) + \frac{1}{2}(40) = 25.$$

Case 2: Bag contains 40 coins The possible pairs are:

$$(20, 40) \quad \text{or} \quad (40, 80),$$

with equal probability.

If you switch:

- Gain 20 coins with probability 1/2,
- Gain 80 coins with probability 1/2.

The expected value of switching is:

$$\text{Expected value} = \frac{1}{2}(20) + \frac{1}{2}(80) = 50.$$

Conclusion: Your friend's argument is **correct**. The expected value of switching is 25 for 20 coins and 50 for 40 coins.

Solution to Exercise 4

Part 1:

This is a **common-value auction**, as the value of the object (x_1+x_2) is the same for both players. However, each player has private information about the other's contribution (x_1 or x_2).

Part 2:

This is a **first-price auction**, as the highest bidder pays their bid (b) to win the auction.

Part 3:

Player 1 uses the bidding strategy:

$$b_1 = a^{(1)}x_1 + c^{(1)}.$$

Player 2 bids b_2. For Player 2 to win the auction, we need:

$$b_2 > b_1 \implies b_2 > a^{(1)}x_1 + c^{(1)} \implies x_1 < \frac{b_2 - c^{(1)}}{a^{(1)}}.$$

The variable x_1 is uniformly distributed over $[0, 50]$. Thus, the probability density function (PDF) is:

$$f_{x_1}(x_1) = \frac{1}{50}, \quad x_1 \in [0, 50].$$

The probability that $x_1 < \frac{b_2-c^{(1)}}{a^{(1)}}$ is:

$$\mathbb{P}(x_1 < \frac{b_2 - c^{(1)}}{a^{(1)}}) = \int_0^{\frac{b_2-c^{(1)}}{a^{(1)}}} \frac{1}{50}\, dx = \frac{\frac{b_2-c^{(1)}}{a^{(1)}}}{50}.$$

If $\frac{b_2-c^{(1)}}{a^{(1)}} \notin [0, 50]$, the probability is 0 (if the value is negative) or 1 (if the value exceeds 50).

Part 4:

If Player 2 wins the auction, their surplus is:

$$\text{Surplus} = x_1 + x_2 - b_2.$$

The expectation of x_1, conditional on $x_1 < \frac{b_2 - c^{(1)}}{a^{(1)}}$, is:

$$\mathbb{E}[x_1 \mid x_1 < \frac{b_2 - c^{(1)}}{a^{(1)}}] = \frac{\frac{b_2 - c^{(1)}}{a^{(1)}}}{2}.$$

Thus, the expected surplus, conditional on winning, is:

$$\mathbb{E}[\text{Surplus} \mid b_2 \text{ wins}] = \mathbb{E}[x_1 \mid x_1 < \frac{b_2 - c^{(1)}}{a^{(1)}}] + x_2 - b_2 = \frac{\frac{b_2 - c^{(1)}}{a^{(1)}}}{2} + x_2 - b_2.$$

Part 5:

The expected payout of Player 2 is:

$$\mathbb{E}[\text{Payout}] = \mathbb{P}(\text{Winning}) \cdot \mathbb{E}[\text{Surplus} \mid \text{Winning}].$$

Substituting the expressions for probability of winning and expected surplus:

$$\mathbb{E}[\text{Payout}] = \frac{\frac{b_2 - c^{(1)}}{a^{(1)}}}{50} \cdot \left(\frac{\frac{b_2 - c^{(1)}}{a^{(1)}}}{2} + x_2 - b_2 \right).$$

Simplifying:

$$\mathbb{E}[\text{Payout}] = \frac{(b_2 - c^{(1)})}{50a^{(1)}} \cdot \left(\frac{b_2 - c^{(1)}}{2a^{(1)}} + x_2 - b_2 \right).$$

Optimization for symmetric Nash equilibrium

For symmetric equilibrium, let $a^{(1)} = a^{(2)} = a$ and $c^{(1)} = c^{(2)} = c$. Substituting:

$$\mathbb{E}[\text{Payout}] = \frac{(b_2 - c)}{50a} \cdot \left(\frac{b_2 - c}{2a} + x_2 - b_2 \right).$$

To find the optimal bid b_2, we maximize $\mathbb{E}[\text{Payout}]$ by solving:

$$\frac{\partial \mathbb{E}[\text{Payout}]}{\partial b_2} = 0.$$

After differentiating and simplifying, the optimal bid is:

$$b_2 = \frac{ac + ax_2 - c}{2a - 1}.$$

Final Answer

The symmetric Nash equilibrium bidding strategy for Player 2 is:

$$b_2 = \frac{a^{(2)}c^{(2)} + a^{(2)}x_2 - c^{(2)}}{2a^{(2)} - 1}.$$

Solution to Exercise 5

Part 1:

Step 1: Payout if you do not pay the fee

If you do not pay the entry fee, your payout is:

$$\text{Payout} = 0.$$

Step 2: Payout if you pay the fee

If you pay the fee, your net expected payout depends on:

- The probability of winning the auction.
- The surplus when you win.

The net expected payout is:

$$\mathbb{E}[\text{Net Payout}] = \mathbb{E}[\text{Surplus}] - E,$$

where:

$$\mathbb{E}[\text{Surplus}] = \mathbb{P}(\text{Win}) \cdot (v - \mathbb{E}[v_{\text{second}}]).$$

Step 3: Probability of winning

You win if your private value v is greater than the values of all N other participants. Since the private values of others are i.i.d. and uniformly distributed on $[0, 1]$, the cumulative distribution function (CDF) of the maximum value among the N participants is:

$$F_{\max}(x) = x^N.$$

The probability density function (PDF) is:

$$f_{\max}(x) = Nx^{N-1}.$$

The probability of winning is:

$$\mathbb{P}(\text{Win}) = \mathbb{P}(v > \max\{v_1, v_2, \ldots, v_N\}) = 1 - F_{\max}(v) = 1 - v^N.$$

Step 4: Expected value of the second-highest bid

The second-highest value, denoted v_{second}, determines the payment if you win. The expected value of v_{second} is:

$$\mathbb{E}[v_{\text{second}}] = \frac{N-1}{N} \cdot \frac{v}{2}.$$

Step 5: Expected surplus

The surplus when you win is:

$$\text{Surplus} = v - v_{\text{second}}.$$

The expected surplus is:

$$\mathbb{E}[\text{Surplus}] = v - \mathbb{E}[v_{\text{second}}].$$

Substituting:

$$\mathbb{E}[\text{Surplus}] = v - \frac{N-1}{N} \cdot \frac{v}{2}.$$

Simplify:

$$\mathbb{E}[\text{Surplus}] = v\left(1 - \frac{N-1}{2N}\right).$$

Step 6: Net expected payout

The net expected payout is:

$$\mathbb{E}[\text{Net Payout}] = \mathbb{P}(\text{Win}) \cdot \mathbb{E}[\text{Surplus}] - E.$$

Substituting:

$$\mathbb{E}[\text{Net Payout}] = v^N \cdot v\left(1 - \frac{N-1}{2N}\right) - E.$$

Simplify:

$$\mathbb{E}[\text{Net Payout}] = v^{N+1} \cdot \frac{1}{2N} - E.$$

Step 7: Condition for paying the fee

You pay the fee if:

$$\mathbb{E}[\text{Net Payout}] > 0.$$

Substitute the expression for net payout:

$$v^{N+1} \cdot \frac{1}{2N} > E.$$

Simplify:

$$v^{N+1} > 2NE.$$

Thus, the condition for paying the fee is:

$$v^{N+1} > 2NE.$$

Part 2:

In a second-price auction, the optimal strategy is to bid your true value v. This is because:

- If you bid higher than v, you might overpay and incur a negative surplus.
- If you bid lower than v, you might lose the auction even when the second-highest bid is less than your true value.

Thus, the optimal bidding strategy is:

$$b = v.$$