Problem Set 1 Solution

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Exercise 1: The Transportation Paradox

1.1

Given:

- n: number of students on a route.
- **Route ACB**: Travel time is $1 + \frac{\alpha n}{2}$.
- **Route ADB**: Travel time is $1 + \frac{\alpha n}{2}$.

Possible scenarios:

1. **Both students choose route ACB:**

$$\text{Travel time} = 1 + \alpha$$

2. **One student on ACB and the other on ADB:**

$$\text{Travel time for each student} = 1 + \frac{\alpha}{2} + 1 = 2 + \frac{\alpha}{2}$$

Since $\alpha < 1$, we observe:

$$1 + \alpha < 2 + \frac{\alpha}{2}$$

Thus, the only pure-strategy Nash equilibrium occurs when both students take either route ACB or route ADB.

1.2

With the new super-fast route CD (taking 0 time), two new route options are introduced: ACDB and ADCB. Now, the travel times are:

- **ACB**: $1 + \frac{\alpha n}{2}$
- **ADB**: $1 + \frac{\alpha n}{2}$

- **ACDB and ADCB**: $1 + 0 + 1 = 2$

Since ACDB has a constant travel time of 2 hours, it is strictly dominated by ACB or ADB (which can be faster when only one student uses the route). Thus, we reduce the game to three choices: ACB, ADB, and ACDB for each student.

1.3

Define:

$$T(\text{ACB}) = 1 + \frac{\alpha n}{2}, \quad T(\text{ADB}) = 1 + \frac{\alpha n}{2}, \quad T(\text{ACDB}) = 2$$

1. For values of α: - If $\alpha < 1$: Both students choosing ACB or ADB (resulting in travel times < 2) forms a Nash equilibrium. - If $\alpha \geq 1$: Route ACDB, with a fixed time of 2 hours, becomes an equilibrium as both ACB and ADB take at least 2 hours.

Therefore, when $\alpha < 1$, the Nash equilibria are ACB and ADB. As α increases, ACDB becomes the preferred equilibrium.

Exercise 2

Considering the given game:

	L	R
U	(1, 9)	(4, 7)
M	(3, 10)	(2, 5)
D	(2, 5)	(6, 8)

Here, the payoff pairs are given in the form $(\text{Player 1}, \text{Player 2})$.

2.1

To find the pure-strategy Nash equilibria, we look for cells where each player's choice is a best response to the other's action.

- For Player 1: - If Player 2 plays L, Player 1's best response is M (payoff 3). - If Player 2 plays R, Player 1's best response is D (payoff 6).

- For Player 2: - If Player 1 plays U, Player 2's best response is L (payoff 9). - If Player 1 plays M, Player 2's best response is L (payoff 10). - If Player 1 plays D, Player 2's best response is R (payoff 8).

Thus, the two pure-strategy Nash equilibria are:

$$(M, L) \quad \text{and} \quad (D, R)$$

2.2

Suppose Player 2 plays a mixed strategy of the form $(p_L^2, 1 - p_L^2)$, where p_L^2 is the probability of playing L.

To make Player 1 indifferent between actions M and D, we equate the expected payoffs of M and D for Player 1.

$$\text{Expected payoff of } M = p_L^2 \cdot 3 + (1 - p_L^2) \cdot 2$$

$$\text{Expected payoff of } D = p_L^2 \cdot 2 + (1 - p_L^2) \cdot 6$$

Setting these equal:

$$p_L^2 \cdot 3 + (1 - p_L^2) \cdot 2 = p_L^2 \cdot 2 + (1 - p_L^2) \cdot 6$$

Solving this equation:

$$3p_L^2 + 2 - 2p_L^2 = 2p_L^2 + 6 - 6p_L^2$$

$$p_L^2 = \frac{4}{5}$$

Thus, $p_L^2 = \frac{4}{5}$.

2.3

Suppose Player 1 plays a mixed strategy of the form $(0, p_M^1, 1 - p_M^1)$, where p_M^1 is the probability of playing M.

To make Player 2 indifferent between actions L and R, we equate the expected payoffs of L and R for Player 2.

$$\text{Expected payoff of } L = p_M^1 \cdot 10 + (1 - p_M^1) \cdot 5$$

$$\text{Expected payoff of } R = p_M^1 \cdot 5 + (1 - p_M^1) \cdot 8$$

Setting these equal:

$$10p_M^1 + 5(1 - p_M^1) = 5p_M^1 + 8(1 - p_M^1)$$

Solving this equation:

$$10p_M^1 + 5 - 5p_M^1 = 5p_M^1 + 8 - 8p_M^1$$

$$p_M^1 = \frac{3}{8}$$

Thus, $p_M^1 = \frac{3}{8}$.

2.4

From the above, the mixed-strategy Nash equilibrium occurs when:

$$p_L^2 = \frac{4}{5} \quad \text{and} \quad p_M^1 = \frac{3}{8}$$

In this equilibrium, each player is indifferent between their actions, resulting in a stable strategy profile where no player has an incentive to deviate. This indifference between choices confirms it as a Nash equilibrium.

Exercise 3

Considering the following game:

	L	R
T	(4, 2)	(3, 2)
M	(0, 0)	(1, 1)
B	(1, 1)	(0, 0)

The payoffs are given in the form (Player 1, Player 2).

3.1

A strictly dominated strategy is one where another strategy yields a higher payoff, regardless of the opponent's choice.

- **Player 1:** - Strategy M is strictly dominated by B, as B yields a higher payoff in both cases (1 vs 0 when Player 2 plays L, and 1 vs 0 when Player 2 plays R).

- **Player 2:** - There are no strictly dominated strategies for Player 2.

3.2

A weakly dominated strategy is one where another strategy yields a higher or equal payoff in every case and a strictly higher payoff in at least one case.

- **Player 1**: - Strategy T weakly dominates M since T yields a higher or equal payoff for both actions of Player 2 (4 vs 0 for L, and 3 vs 1 for R).

- **Player 2:** - Strategy L weakly dominates R since L yields an equal or better payoff in both cases (2 vs 2 when Player 1 plays T, and 1 vs 1 when Player 1 plays M, and 1 vs 0 when Player 1 plays B).

3.3

Removing strictly dominated strategies:

1. Eliminate M for Player 1 (strictly dominated by B). 2. The resulting reduced game is:

	L	R
T	(4, 2)	(3, 2)
B	(1, 1)	(0, 0)

3. In this reduced game, the Nash equilibrium is (T, L) with payoffs $(4, 2)$.

3.4

Now we consider eliminating both strictly and weakly dominated strategies. There are two approaches:

Approach 1: 1. Remove M (strictly dominated by B). 2. Remove R (weakly dominated by L). 3. The resulting game is:

	L
T	(4, 2)
B	(1, 1)

The Nash equilibrium is (T, L) with payoffs $(4, 2)$.

Approach 2: 1. Remove T (weakly dominates M). 2. Remove R (weakly dominated by L). 3. The resulting game is:

	L
M	(0, 0)
B	(1, 1)

The Nash equilibrium in this game is (B, L) with payoffs $(1, 1)$.

Thus, using different combinations of strictly and weakly dominated strategies, we arrive at two distinct Nash equilibria: (T, L) with payoffs $(4, 2)$ and (B, L) with payoffs $(1, 1)$.

Exercise 4

4.1

Let the players declare values v_1 and v_2 for the artifact, where $3 \leq v_1, v_2 \leq 100$.

- If $v_1 = v_2$, both players receive v_1.
- If $v_1 < v_2$, player 1 receives $v_1 + 2$ and player 2 receives $v_1 - 2$.
- If $v_2 < v_1$, player 2 receives $v_2 + 2$ and player 1 receives $v_2 - 2$.

To find the Nash equilibrium, consider the following reasoning:

1. If both players declare the maximum value ($100), then either player can increase their payoff by declaring a slightly lower value, say $99, to get a higher reimbursement. Therefore, declaring the maximum value is not stable. 2. This

logic applies down the entire range of values until reaching the minimum allowed value, \$3, where no player has an incentive to deviate further down.

Thus, the unique Nash equilibrium is when both players declare $v_1 = v_2 = 3$, where neither can gain by unilaterally changing their strategy.

Nash Equilibrium: $v_1 = v_2 = 3$.

4.2

The Nash equilibrium $v_1 = v_2 = 3$ is unsatisfactory and unrealistic. In practice, both travelers would likely expect a higher reimbursement closer to the artifact's true value. This equilibrium is paradoxical because the players are driven to declare the lowest possible value, despite potentially higher reimbursements being available, reflecting a counterintuitive outcome for both rational players.

4.3

Consider now an alternative plan where:

- The player who declared the lower value v_L gets $v_L + 1$.
- The other player receives $v_L - 1$.

In this new setup, the players' payoffs change as follows:

- If $v_1 = v_2$, both players receive v_1.
- If $v_1 < v_2$, player 1 receives $v_1 + 1$ and player 2 receives $v_1 - 1$.
- If $v_2 < v_1$, player 2 receives $v_2 + 1$ and player 1 receives $v_2 - 1$.

Finding the Nash Equilibria: Using similar reasoning, each player is incentivized to undercut the other's declaration to achieve a slightly higher payoff. By iterating downward, both players are again incentivized to declare the minimum allowed value, $v_1 = v_2 = 3$, as any higher declaration would lead to a lower payoff if the other player declares a lower value.

New Nash Equilibrium: $v_1 = v_2 = 3$.

Exercise 5

5.1

In this case:

- If $s_1 + s_2 \leq 10$, both players receive the amount they demanded.
- If $s_1 + s_2 > 10$, both players receive 0.

Nash Equilibrium: The unique Nash equilibrium occurs when both players demand $s_1 = s_2 = 5$. This way, $s_1 + s_2 = 10$, ensuring both players receive their demands, as any increase would result in both receiving 0.

5.2

In this case:

- If $s_1 + s_2 \leq 10$, both players receive their demands.
- If $s_1 + s_2 > 10$ and $s_1 \neq s_2$, the player with the lower demand s_L receives s_L, and the other player receives $10 - s_L$.
- If $s_1 = s_2$ and $s_1 + s_2 > 10$, both players receive 5.

Nash Equilibrium: The Nash equilibrium in this case occurs when $s_1 = s_2 = 5$, as any deviation from this would lead to either player receiving less if the total exceeds 10.

5.3 Bonus:

Now, s_i can be any real number between 0 and 10. Following the same rules as in part 2:

- If $s_1 + s_2 \leq 10$, both players receive their demands.
- If $s_1 + s_2 > 10$ and $s_1 \neq s_2$, the player with the lower demand s_L receives s_L, and the other player receives $10 - s_L$.
- If $s_1 = s_2$ and $s_1 + s_2 > 10$, both players receive 5.

Nash Equilibria: In this case, there is a continuum of Nash equilibria where $s_1 + s_2 = 10$ and $s_1 \neq s_2$. For example, $s_1 = 4.6721$ and $s_2 = 5.3279$ forms a Nash equilibrium, as neither player can increase her payoff by unilaterally changing her demand.

Solution to Exercise 6

6.1

- Player 1 has an initial choice between A and B, and if Player 2 chooses f, Player 1 can further choose G or H. - Player 2 has four pure strategies based on Player 1's initial choice:

- If Player 1 chooses A: c or d.
- If Player 1 chooses B: e or f.

Thus, the pure strategies are:

$$\text{Player 1: } \{A, B \to G, B \to H\}, \quad \text{Player 2: } \{c, d, e, f\}.$$

6.2

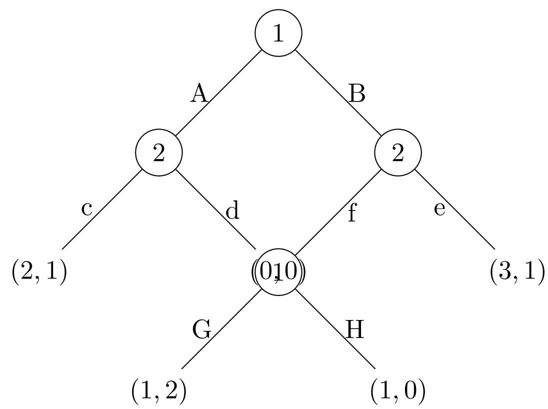
The game in normal form is represented by the following payoff matrix:

	Player 2: c	d	e	f
A	$(2,1)$	$(0,0)$	$-$	$-$
$B \to G$	$-$	$-$	$(3,1)$	$(1,2)$
$B \to H$	$-$	$-$	$(3,1)$	$(1,0)$

Nash Equilibria: 1. $(B \to G, e) = (3,1)$ 2. $(B \to G, f) = (1,2)$

6.3

Using backward induction on the game tree:



Step 1: Analyzing the Subgame After f

- If Player 2 chooses f, Player 1 then chooses between G (with payoff $(1,2)$) and H (with payoff $(1,0)$). - Player 1 prefers G over H since $(1,2) > (1,0)$. - Therefore, if Player 2 chooses f, Player 1 would choose G.

Step 2: Analyzing Player 2's Choices After Player 1 Chooses B

- Player 2 compares the payoffs from choosing e and f:

- Choosing e results in payoff $(3,1)$.
- Choosing f (knowing Player 1 will choose G) results in payoff $(1,2)$.

- Since $(3,1) > (1,2)$, Player 2 prefers to choose e over f if Player 1 chooses B.

Step 3: Analyzing Player 1's Initial Choice

- Player 1 compares the payoffs from choosing A and B:

- Choosing A allows Player 2 to choose c or d, resulting in:
  - (A, c): $(2, 1)$
  - (A, d): $(0, 0)$
- Choosing B leads to Player 2 choosing e, resulting in payoff $(3, 1)$.

- Since $(3, 1) > (2, 1)$, Player 1 prefers to choose B at the root.

Conclusion: Subgame-Perfect Equilibrium (SPE)

The SPE path, determined by backward induction, is:

$$(B, e) \text{ with payoff } (3, 1)$$

Thus, the only pure-strategy subgame-perfect equilibrium is:

$$(B, e) = (3, 1)$$

6.4

Another NE is $(B \to G, f) = (1, 2)$. However, it is not subgame-perfect because it involves a non-credible threat. Player 2 would not rationally choose f, as it gives a lower payoff compared to e in the subgame.