MGT300: Problem Set 3

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Exercise 1 (17 pts): Application of the Revenue-Equivalence Theorem

Part 1:

From the revenue equivalence theorem, we know that the expected payment for a bidder in a fourth-price auction is the same as in other standard auctions (e.g., first- or second-price). For N bidders with private values $v_i \sim U[0,1]$ (uniform distribution), the expected payment for bidder i with value v_i is:

$$\text{Expected Payment} = \mathbb{E}[\text{Payment}] = \int_0^{v_i} x \cdot (N-1)(1-x)^{N-2}\,dx.$$

Simplifying this integral:

$$\text{Expected Payment} = v_i \cdot \frac{N-1}{N}.$$

This result follows directly from the probability that bidder i's value exceeds the critical threshold set by other bidders, scaled by the uniform distribution's properties.

Part 2:

We assume the bidding strategy is of the form:

$$b(v) = av$$

where a is a constant to be determined. In a fourth-price auction, the price paid is determined by the fourth-highest bid. Let the private values be $v_1, v_2, \ldots, v_N$, where $N \gg 4$. The fourth-highest bid corresponds to the fourth-order statistic $X_{(4)}$ of N independent and identically distributed values from $U[0,1]$.

The expected value of the k-th order statistic is given by:

$$\mathbb{E}[X_{(k)}] = \int_0^1 x \cdot \frac{N!}{(k-1)!(N-k)!} x^{k-1}(1-x)^{N-k}\,dx.$$

For $k = 4$, the expected value becomes:

$$\mathbb{E}[X_{(4)}] = \int_0^1 x \cdot \frac{N!}{3!(N-4)!} x^3(1-x)^{N-4}\,dx.$$

Solving this integral for large N gives:

$$\mathbb{E}[X_{(4)}] \approx \frac{4}{N+1}.$$

The equilibrium condition for the optimal bidding strategy requires that a bidder's expected utility is maximized when bidding according to $b(v)$. Substituting $b(v) = av$ and solving for a, we find:

$$a = \frac{4}{5}.$$

Thus, the optimal bidding strategy is:

$$b(v) = \frac{4}{5}v.$$

Conclusion

1. The expected payment for a bidder with private value v_i in a fourth-price auction is:

$$\mathbb{E}[\text{Payment}] = v_i \cdot \frac{N-1}{N}.$$

2. The optimal bidding strategy for this auction is:

$$b(v) = \frac{4}{5}v.$$

Exercise 2 (16 pts): Twice-Repeated Game

Solution

Part 1:

A pair of strategies (s_1, s_2) is a Nash equilibrium if: 1. Given s_2, Player 1 has no incentive to deviate from s_1. 2. Given s_1, Player 2 has no incentive to deviate from s_2.

From the pay-off matrix:

1. **Case 1: Both players choose L:** - Payoff: $(1, 1)$. - No player can improve their payoff unilaterally. (L, L) is a Nash equilibrium.

2. **Case 2: Both players choose M:** - Payoff: $(4, 4)$. - No player can improve their payoff unilaterally. (M, M) is a Nash equilibrium.

3. **Case 3: Both players choose R:** - Payoff: $(3, 3)$. - No player can improve their payoff unilaterally. (R, R) is a Nash equilibrium.

Thus, the pure-strategy Nash equilibria are:

$$(L, L), (M, M), (R, R).$$

Part 2: The twice-repeated game has two stages, and the discount factor is 1. To find SPEs, we will use backward induction:

Stage 2 Analysis:

In the second stage, players play a Nash equilibrium of the stage game. Possible payoffs are:

$$(1, 1), (4, 4), (3, 3).$$

Stage 1 Analysis:

Players can use strategies in Stage 1 to incentivize cooperation by threatening to punish deviations in Stage 2.

Example strategies:

- **Trigger Strategy:** Play M in Stage 1. If someone deviates, play L in stage 2 (punishment).
- **Always Cooperate:** Play M in both stages regardless of history.

SPE 1: Always play M:

- Stage 1: Both play M.
- Stage 2: Both play M.
- Payoff: $(4 + 4, 4 + 4) = (8, 8)$.

SPE 2: Play M in stage 1; punish deviation with L in Stage 2:

- Stage 1: Both play M.
- Stage 2: Play L if anyone deviates; otherwise, play M.
- Payoff: $(4 + 1, 4 + 1) = (5, 5)$ if punishment is triggered.

SPE 3: Play R in both stages:

- Stage 1: Both play R.
- Stage 2: Both play R.
- Payoff: $(3 + 3, 3 + 3) = (6, 6)$.

Exercise 3 (17 pts): Price Competition in Duopoly

Solution

Part 1:

A pair of strategies (s_1, s_2) is a Nash equilibrium if neither firm can improve its payoff by unilaterally deviating.

From the payoff matrix:

- **Case 1: Both firms choose Low (L, L):**
  - Payoff: $(5, 5)$.
  - If Firm 1 switches to High, payoff becomes $(0, 20)$.
  - If Firm 2 switches to High, payoff becomes $(20, 0)$.
  - Neither firm has an incentive to deviate. (L, L) is a Nash equilibrium.
- **Case 2: Both firms choose High (H, H):**
  - Payoff: $(10, 10)$.
  - If Firm 1 switches to Low, payoff becomes $(20, 0)$.

- If Firm 2 switches to Low, payoff becomes $(0, 20)$.
- Neither firm has an incentive to deviate. (H, H) is a Nash equilibrium.

Thus, the stage game has two Nash equilibria:

$$(L, L), (H, H).$$

Part 2:
To sustain (H, H) as an SPE in the infinitely-repeated game, firms must have no incentive to deviate from the collusive outcome. We use a **trigger strategy**: - Both firms play High (H) as long as no one deviates. - If any firm deviates, revert to playing Low (L) forever.

Payoffs

- **Collude (Always play High):** - Per-period payoff: 10 for each firm. - Total discounted payoff (infinite sum):

$$\text{Payoff}_{\text{Collude}} = 10 + 10\delta + 10\delta^2 + \cdots = \frac{10}{1-\delta}.$$

- **Deviate (Play Low for One Period):** - Immediate payoff from deviation: 20. - After deviation, punishment: 5 per period forever. - Total discounted payoff:

$$\text{Payoff}_{\text{Deviate}} = 20 + 5\delta + 5\delta^2 + \cdots = 20 + \frac{5\delta}{1-\delta}.$$

Incentive Compatibility

To sustain (H, H) as an SPE, the collusion payoff must be at least as large as the deviation payoff:

$$\text{Payoff}_{\text{Collude}} \geq \text{Payoff}_{\text{Deviate}}.$$

Substitute the expressions:

$$\frac{10}{1-\delta} \geq 20 + \frac{5\delta}{1-\delta}.$$

Simplify:

$$10 \geq 20(1-\delta) + 5\delta.$$

$$10 \geq 20 - 20\delta + 5\delta.$$

$$10 \geq 20 - 15\delta.$$

$$15\delta \geq 10.$$

$$\delta \geq \frac{2}{3}.$$

Conclusion

- If $\delta \geq \frac{2}{3}$, firms can sustain (H, H) as an SPE. - The discount factor δ reflects the value firms place on future profits. A higher discount factor ensures collusion is stable.