UNIT 07

MATRICES AND DETERMINANTS

Introduction

The of traces of the beginnings of matrices and determinants could be found in the fourth century BC. However, it goes back to the second century BC although. The subject area of matrices and determinants clearly appeared near the end of the 17th Century.

It is believed that the beginnings of matrices and determinants emerge through the study of systems of linear equations. The Babylonians studied problems which lead to simultaneous linear equations and some of these are preserved in clay tablets which survive. In the period, between 200 BC and 100 BC the Chinese came much closer to matrices than the Babylonians.

Matrices form an important tool in the study of various fields of Mathematics. This is useful in engineering applications. Our aim in this unit is to consider in some detail the elementary notions in the theory of matrices.
This unit begins with the very basic concepts of matrices and then discusses some properties and algebraic operations of matrices.

Session 24

Basic concepts of Matrices

Introduction

Matrices is one of the most important and powerful tools in mathematics. Applications of matrices can be found various fields like Engineering, Statistics, Economics, and Science etc. In this study session, we shall confine ourselves to the study of basic laws of Matrix Algebra.

24.1 Matrix

A set of m times n numbers arranged in a rectangular array of m number of horizontal lines and n number of vertical lines, enclosed by [] or ( ) is called a matrix of size m by n. It is written as $m \times n$ matrix.

The numbers that are in a matrix are called entries or elements of the matrix. They may be numbers or any other objects.

The horizontal lines and the vertical lines of a matrix are called rows and columns of the matrix respectively.

Matrices are denoted by English capital letters such as $\underline{A, B, C, D, \ldots}$ etc. In general, the elements of matrix are denoted by simple English letters.

Example 1

$\underline{A} = \begin{bmatrix} 2 & 3 \\ 6 & 7 \end{bmatrix}_{2\times 2}$ $\quad$ $\underline{B} = \begin{bmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \end{bmatrix}_{2\times 3}$

$\underline{C} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}_{3\times 3}$ $\quad$ $\underline{D} = [1\ \ 2\ \ 3\ \ 4]_{1\times 4}$

$\underline{E} = \begin{bmatrix} 6 \\ 7 \\ 8 \end{bmatrix}_{3\times 1}$

Notation of a general matrix

The element in the i^{th} row and the j^{th} column of a matrix $\underline{A}$ is denoted by a_{ij}. The first suffix (i) indicates the row and the second suffix (j) indicates the column of which that element belongs.

With this notation a matrix $\underline{A}$ of size $m \times n$ can be written as follows.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & \cdots & a_{1j} & \cdots & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & & & a_{2j} & & & a_{2n} \\ a_{31} & a_{32} & a_{33} & & & a_{3j} & & & a_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{i1} & a_{i2} & a_{i3} & \cdots & \cdots & a_{ij} & \cdots & \cdots & a_{in} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n-1\,1} & a_{n-1\,2} & a_{n-2\,3} & & & a_{n-1\,j} & & a_{n-1\,n-1} & a_{n-1\,n} \\ a_{n1} & a_{n2} & a_{n3} & & & a_{n\,j} & & a_{n\,n-1} & a_{nn} \end{bmatrix}$$

The above matrix is denoted by $[a_{ij}]_{m\times n}$ where $i = 1,2,\cdots,m$ and $j = 1,2,\cdots,n$ or simply $\underline{A = [a_{ij}]_{m\times n}}$.

Therefore, A in the Example 1.1.1 is a matrix having 2 rows and 2 columns. It's size is $\mathbf{2 \times 2}$ *and it has 4 elements.* $\boldsymbol{a_{11} = 2, a_{12} = 3, a_{21} = 6, a_{22} = 7}$.

B is a matrix having 2 rows and 3 columns. Its order is 2×3 and it has 6 elements.

$$a_{11} = 1, a_{12} = 2, a_{13} = 3, a_{21} = 5, a_{22} = 6, a_{23} = 7$$

Example 2

Construct the matrices of size 4×3 whose elements are defined as follows.

(i) $a_{ij} = i - j$ (ii) $a_{ij} = i \times (j+1)$ (iii) $a_{ij} = \frac{j+1}{2}$

Solution:

Let $A = [a_{ij}]_{4\times 3}$ $i = 1,\ 2, 3, 4$ and $j = 1, 2, 3$

(i) $a_{ij} = i - j$

$a_{11} = 1 - 1 = 0$ $a_{12} = 1 - 2 = -1$ $a_{13} = 1 - 3 = -2$

$a_{21} = 2 - 1 = 1$ $a_{22} = 2 - 2 = 0$ $a_{23} = 2 - 3 = -1$

$a_{31} = 3 - 1 = 2$ $a_{32} = 3 - 2 = 1$ $a_{33} = 3 - 3 = 0$

$a_{41} = 4 - 1 = 3$ $a_{42} = 4 - 2 = 2$ $a_{43} = 4 - 3 = 1$

$$\therefore A = \begin{bmatrix} 0 & 1 & -2 \\ 1 & 0 & -1 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$$

(ii) $a_{ij} = i \times (j+1)$

$a_{11} = 1.2 = 2$ $a_{12} = 1.3 = -3$ $a_{13} = 1.4 = 4$

$a_{21} = 2.2 = 4$ $a_{22} = 2.3 = 6$ $a_{23} = 2.4 = 8$

$a_{31} = 3.2 = 6$ $a_{32} = 3.3 = 9$ $a_{33} = 3.4 = 12$

$a_{41} = 4.2 = 4$ $a_{42} = 4.3 = 12$ $a_{43} = 4.4 = 16$

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 4 & 6 & 8 \\ 6 & 9 & 12 \\ 8 & 12 & 16 \end{bmatrix}$$

(iii) $a_{ij} = \frac{j+1}{2}$

$a_{11} = {}^{2}/_{1} = 2$ $\quad a_{12} = {}^{3}/_{1} = 3$ $\quad a_{13} = {}^{4}/_{1} = 4$

$a_{21} = {}^{2}/_{2} = 1$ $\quad a_{22} = {}^{3}/_{2}$ $\quad a_{23} = {}^{4}/_{2} = 2$

$a_{31} = {}^{2}/_{3}$ $\quad a_{32} = {}^{3}/_{3} = 1$ $\quad a_{33} = {}^{4}/_{3}$

$a_{41} = {}^{2}/_{4} = {}^{1}/_{2}$ $\quad a_{42} = {}^{3}/_{4}$ $\quad a_{43} = {}^{4}/_{4} = 1$

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 1 & {}^{3}/_{2} & 2 \\ {}^{2}/_{3} & 1 & {}^{4}/_{3} \\ {}^{1}/_{2} & {}^{3}/_{4} & 1 \end{bmatrix}$$

Example 3

Construct the size3 × 3 matrices whose elements are defined as follows.

(i) $a_{ij} = |2i - 3j|$ (ii) $a_{ij} = -\frac{(i-2j)^2}{3}$

(iii) $a_{ij} = \begin{cases} i + 2j & i > j \\ i \times 2j & i = j \\ i - 2j & i < j \end{cases}$

Solutions

(i) $a_{ij} = |2i - 3j| \quad i = 1,2,3 \quad j = 1,2,3$

$a_{11} = |2 - 3| = 1$ $\quad a_{12} = |2 - 6| = 4$ $\quad a_{13} = |2 - 9| = 7$

$a_{21} = |4 - 3| = 1$ $\quad a_{22} = |4 - 6| = 2$ $\quad a_{23} = |4 - 9| = 5$

$a_{31} = |6 - 3| = 3$ $\quad a_{32} = |6 - 6| = 0$ $\quad a_{33} = |6 - 9| = 3$

$$A = \begin{bmatrix} 1 & 4 & 7 \\ 1 & 2 & 5 \\ 3 & 0 & 3 \end{bmatrix}$$

(ii) $a_{ij} = -\frac{(i-2j)^2}{3} \quad i = 1,2,3 \quad j = 1,2,3$

$a_{11} = -\frac{(1-2)^2}{3} = -{}^{1}/_{3}$ $\quad a_{12} = -\frac{(1-4)^2}{3} = -3$ $\quad a_{13} = -\frac{(1-6)^2}{3} = -{}^{2}/_{3}$

$a_{21} = -\frac{(2-2)^2}{3} = 0 \qquad a_{22} = -\frac{(2-4)^2}{3} = -{}^4/_3 \quad a_{23} = -\frac{(2-6)^2}{3} = -{}^{16}/_3$

$a_{31} = -\frac{(3-2)^2}{3} = -{}^1/_3 \quad a_{32} = -\frac{(3-4)^2}{3} = -{}^1/_3 \quad a_{33} = -\frac{(3-6)^2}{3} = -3$

$$A = \begin{bmatrix} -{}^1/_3 & -3 & -{}^{25}/_3 \\ 0 & -{}^4/_3 & -{}^{16}/_3 \\ -{}^1/_3 & -{}^1/_3 & -3 \end{bmatrix}$$

(iii)

$$a_{ij} = \begin{cases} i + 2j & \text{for} \quad i > j \\ i \times 2j & \text{for} \quad i = j \\ i - 2j & \text{for} \quad i < j \end{cases} \qquad i = 1, 2, 3 \quad j = 1, 2, 3$$

$a_{11} = 1 \times 2 = 2 \qquad a_{12} = 1 - 2.2 = -3 \qquad a_{13} = 1 - 2 \times 3 = -5$

$a_{21} = 2 + 2 = 4 \qquad a_{22} = 2 \times 2 \times 2 = 8 \qquad a_{23} = 2 - 2 \times 3 = -4$

$a_{31} = 3 + 2 = 5 \qquad a_{32} = 3 + 2 \times 2 = 7 \qquad a_{33} = 3 \times 6 = 18$

$$A = \begin{bmatrix} 2 & -3 & -5 \\ 4 & 8 & -4 \\ 5 & 7 & 18 \end{bmatrix}$$

Activity 1

(1) Construct the matrix of size 3×4 whose elements are $a_{ij} = \frac{i}{j}$

(2) Construct the matrix of size 2×3 matrix whose elements are

(i) $a_{ij} = i + ij - 1$

(ii) $a_{ij} = \frac{(2i-j)^2}{2}$

(3) Construct a 4×3 matrix whose elements are $a_{ij} = \begin{cases} i + j & i < j \\ i \times j & i = j \\ i - j & i > j \end{cases}$

Equality of Matrices

Two Matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are said to be equal if

(i) They have the same size. In other words the number of rows and the number of columns in both the matrices are equal,

(ii) Their corresponding elements are equal. According to the notations $a_{ij} = b_{ij}$ for each i and j.

Thus, two matrices are equal if and only if one is a duplicate of the other.

Example 4

$$A = \begin{bmatrix} sin^2\theta + cos^2\theta & sec^2\theta \\ cosec^2\theta & sec^2\theta - tan^2\theta \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 1 + tan^2\theta \\ 1 + cot^2\theta & 1 \end{bmatrix}$$

Activity 2

Find the values of x, y, a and b such that the matrices $\begin{bmatrix} 3x + 4y & x - 2y \\ a + b & 2a - b \end{bmatrix}$ and $\begin{bmatrix} 2 & 4 \\ 5 & -5 \end{bmatrix}$ are equal.

24.2 Types of Matrices

Row Matrix

A matrix having exactly one row and any number of column is called a row matrix.

Example 5

$[2 \quad 3 \quad 5 \quad 6]$ is a 1×4 row matrix.

Column Matrix

A matrix having exactly one column and any number of rows is called a column matrix.

Example 6

$\begin{bmatrix} 6 \\ 7 \\ 8 \end{bmatrix}$ is a 3×1 column matrix.

Null (or zero) Matrix

A matrix having all the elements zero is called a null matrix or zero matrix. The zero matrix of size $m \times n$ denoted by $\underline{O}_{m\times n}$

Example 7

$\underline{O}_{1\times 3} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$ $\underline{O}_{2\times 2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ $\underline{O}_{2\times 3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ are zero matrices.

Square Matrix

A matrix having the same number of rows and columns is called a square matrix. The number of rows or columns of square matrix is called the order of that square matrix.

Example 8

$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ is a 2×2 square matrix of order 2.

$\begin{bmatrix} 1 & 2 & 4 \\ 5 & 6 & 7 \\ 3 & 8 & 9 \end{bmatrix}$ is a square matrix of order 3.

In general, a square matrix can be indicated as follows.

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1i} & \cdots & \cdots & \cdots & \cdots & a_{1n} \\ a_{21} & a_{22} & & & a_{2i} & & & & a_{2n} \\ a_{31} & a_{32} & & & & a_{3i} & & & a_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{i1} & a_{i2} & a_{i3} & \cdots & \cdots & \cdots & \cdots & \cdots & a_{in} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n-1\,1} & a_{n-1\,2} & a_{n-2\,3} & & & & & a_{n-1\,n-1} & a_{n-1\,n} \\ a_{n1} & a_{n2} & a_{n3} & & & & & a_{n\,n-1} & a_{nn} \end{bmatrix}$$

A is a square matrix of order n. The elements $a_{11}, a_{22}, a_{33}, \cdots, a_{ii}, \cdots,$ $a_{n-1\,n-1}, a_{nn}$ are called its principal diagonal elements and the diagonal along which these elements lie is called the principal diagonal or main diagonal

Thus for diagonal elements $i = j$

For elements above the principal diagonal $i < j$

For elements below the principal diagonal $i > j$

For non-diagonal elements $i \neq j$

Diagonal matrix

A square matrix is called a diagonal matrix if all its non-diagonal elements are zero. Thus square matrix $[a_{ij}]$ is a diagonal matrix if $a_{ij} = 0$ for $i \neq j$

Example 9

$\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ are diagonal matrices.

You can see that the diagonal elements in a diagonal matrix can be zero or can be same numerical value.

Scalar Matrix

A square matrix is called a scalar matrix if its non-diagonal elements are zero and diagonal elements are equal each other.

Thus the square matrix $[a_{ij}]_{n\times n}$ is a scalar matrix if

$$a_{ij} = \begin{cases} c & \text{for } i = j \\ 0 & \text{for } i \neq j \end{cases}$$

Example 10

$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ are scalar matrices.

You can see that every scalar matrix is a diagonal matrix.

Unit Matrix or Identity Matrix

A square matrix is called a unit matrix if it's all non-diagonal elements are zero and diagonal elements are all equal to one (unity).

Thus, the square matrix $[a_{ij}]_{n\times n}$ is a unit matrix if

$$a_{ij} = \begin{cases} 0 & \text{for } i \neq j \\ 1 & \text{for } i = j \end{cases}$$

Example 11

$\underline{I_3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is a unit matrix of order 3 and is denoted by I_3.

$\underline{I_4} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ is a unit matrix of order 4 and is denoted by I_4.

You can see that the every unit matrix is a scalar matrix.

Triangular Matrices

(a) Upper Triangular Matrix

A square matrix is called an upper triangular matrix if all the elements below the principal diagonal are zero.

Example 12

$A = \begin{bmatrix} 3 & 4 & 2 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$ are upper triangular matrices.

If A is an upper triangular matrix and $A = [a_{ij}]_{n\times n}$

Then $a_{ij} = 0$ when for all $i > j$.

(b) Lower Triangular Matrix

A square matrix is called a lower triangular matrix if it's all the elements above the principal diagonal are zero.

Example 13

$A = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 3 & 0 \end{bmatrix}$ $\qquad A = \begin{bmatrix} 3 & 0 & 0 \\ 2 & 4 & 0 \\ 1 & 2 & 1 \end{bmatrix}$ are lower triangular matrices.

If A is a lower triangular matrix $A = [a_{ij}]_{n\times n}$ then $a_{ij} = 0$ for all $i < j$.

A square matrix is called a triangular matrix if it is either an upper-triangular matrix or a lower triangular matrix.

Thus the square matrix $[a_{ij}]_{n\times n}$ is a triangular matrix if $a_{ij} = 0$ for all $i > j$ (or all $i < j$).

Symmetric Matrix

A square matrix $A = [a_{ij}]_{n\times n}$ is said to be symmetric if $a_{ij} = a_{ji}$ for all i and j.

Example 14

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix} \qquad A = \begin{bmatrix} 2 & 3 & 7 & 9 \\ 3 & 4 & 5 & 6 \\ 7 & 5 & 8 & 9 \\ 9 & 6 & 9 & 10 \end{bmatrix}$$ are symmetric matrices.

Skew Symmetric Matrix

A square matrix $A = [a_{ij}]_{n\times n}$ is said to be skew symmetric if $a_{ij} = -a_{ji}$ for all $i \neq j$ and $a_{ij} = 0$ for all $i = j$.

Example 15

$$A = \begin{bmatrix} 0 & 3 & 7 \\ -3 & 0 & 8 \\ -7 & -8 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & -1 & 8 & -3 \\ 1 & 0 & 3 & 9 \\ -8 & -3 & 0 & 4 \\ 3 & -9 & -4 & 0 \end{bmatrix}$$ are skew symmetric matrices.

24.3 Algebra of Matrices

Addition of Matrices

If $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{m\times n}$ are two matrices. The addition of $\underline{A}$ and $\underline{B}$ is defined to be the matrix $A + B = (a_{ij} + b_{ij})_{m\times n}$.

Also, the subtraction of A and B is defined to be the matrix $A - B = (a_{ij} - b_{ij})_{m\times n}$

Note

The addition and subtraction of matrices are defined only for the matrices are in the same size.

When$A$ and B are matrices are in the same size, the addition of A and B is denoted by $A + B$.

The subtraction of A and B is denoted by $A - B$.

Example 16

$A = \begin{bmatrix} 1 & 4 & 9 \\ 2 & 3 & 6 \end{bmatrix}$ $\quad B = \begin{bmatrix} 3 & 1 & 2 \\ 4 & 6 & 9 \end{bmatrix}$

$$\therefore A + B = \begin{bmatrix} 1+3 & 4+1 & 9+2 \\ 2+4 & 3+6 & 6+9 \end{bmatrix} = \begin{bmatrix} 4 & 5 & 11 \\ 6 & 9 & 15 \end{bmatrix}$$

$$A - B = \begin{bmatrix} 1-3 & 4-1 & 9-2 \\ 2-4 & 3-6 & 6-9 \end{bmatrix} = \begin{bmatrix} -2 & 3 & 7 \\ -2 & -3 & -3 \end{bmatrix}$$

Properties of Matrix Addition

(1) Matrix addition is commutative if A and B be two matrices of the same size, then
$A + B$=$B + A$.

(2) Matrix addition is associative if A, B and C be three matrices of the same size then

$$(A + B) + C = A + (B + C)$$

(3) Existence of additive identity if $A = [a_{ij}]_{m\times n}$ and $\underline{0} = [0]_{m\times n}$ then
$A + \underline{0} = A$

(4) Existence of additive inverse if $A = [a_{ij}]_{m\times n}$ then there exists a unique matrix $-A = [-a_{ij}]_{m\times n}$ such that $A + (-A) = [a_{ij} + (-a_{ij})]_{m\times n} = [0]_{m\times n} = \underline{0}$

Hence $A + (-A) = \underline{0}$ the additive identity

Therefore $-A$ is the additive inverse of A.

(5) Cancellation laws

If A, B and C be three matrices of the same order then

$A + B = A + C$ implies that $B = C$ (Left cancellation law)

$B + A = C + A$implies that $B = C$ (Right cancellation law)

(6) Scalar multiplication

Let $A = [a_{ij}]_{m\times n}$ be matrix and λ be any real number, the λA is the matrix obtained by multiplying each element by λ, i.e. $\lambda A = \lambda[a_{ij}]_{m\times n} = [\lambda a_{ij}]_{m\times n}$

The number λ is called scalar.

Example 17

If $A = \begin{bmatrix} 4 & 3 \\ 9 & 1 \\ 8 & 2 \end{bmatrix}$,then $3A = \begin{bmatrix} 12 & 9 \\ 27 & 3 \\ 24 & 6 \end{bmatrix}$

If $A = \begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix}$,then $\frac{1}{2}A = \begin{bmatrix} 2 & 3/2 \\ 1 & 5/2 \end{bmatrix}$

If $A = \begin{bmatrix} 2 & 3 & 6 \\ 7 & 2 & 1 \end{bmatrix}$, then $-3A = \begin{bmatrix} -6 & -9 & -18 \\ -21 & -6 & -3 \end{bmatrix}$

Properties of Scalar Multiplication

If $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{m\times n}$ be two matrices of the same size, and for any scalars λ_1, λ_2 and λ, then

(i) $(\lambda_1 + \lambda_2)A = \lambda_1 A + \lambda_2 A$

(ii) $\lambda(A + B) = \lambda A + \lambda B$

(iii) $\lambda_1(\lambda_2 A) = (\lambda_1\lambda_2)A$

(iv) $0A = \underline{0}$

Example 18

If $B = \begin{bmatrix} 2 & 3 & 0 \\ 1 & -1 & 5 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & -2 & 3 \\ -1 & 0 & 2 \end{bmatrix}$

Find $3B - 4\,C$

Solution

$$3B - 4\,C = 3\begin{bmatrix} 2 & 3 & 0 \\ 1 & -1 & 5 \end{bmatrix} + (-4)\begin{bmatrix} 1 & -2 & 3 \\ -1 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 9 & 0 \\ 3 & -3 & 15 \end{bmatrix} + \begin{bmatrix} -4 & 8 & -12 \\ 4 & 0 & -8 \end{bmatrix}$$

$$= \begin{bmatrix} 6-4 & 9+8 & 0-12 \\ 3+4 & -3+0 & 15-8 \end{bmatrix}$$

$$3B - 4\,C = \begin{bmatrix} 2 & 17 & 2 \\ 7 & -3 & 7 \end{bmatrix}$$

Multiplication of matrices

If A be $m \times n$ matrix and B be $n \times p$ matrix, we say that A and B conformable to define AB. In other words, the matrix multiplication $AB$is conformable if the number of column of the matrix A must be equal to the number of rows B.

Then their product AB is an $m \times p$ matrix whose the element in the place (i, j) is obtained by multiplying the elements of i^{th} row of A with the corresponding elements of j^{th} column of B and then adding the products.

Symbolically

$$AB = C$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{i1} & a_{i2} & a_{i3} & \cdots & a_{in} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}_{m\times n} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1j} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2j} & \cdots & b_{2p} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ b_{n1} & b_{n2} & \cdots & b_{nj} & \cdots & b_{np} \end{bmatrix}_{n\times p}$$

Let $C = (C_{ij})_{m\times p}$

$$\therefore c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + \cdots + a_{in}b_{nj}$$

$$c_{11} = a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} + \cdots + a_{1n}b_{n1}$$

$$c_{12} = a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} + \cdots + a_{1n}b_{n1}$$

$$c_{mp} = a_{m1}b_{1n} + a_{m2}b_{2n} + a_{m3}b_{3n} + \cdots + a_{mn}b_{np}$$

Example 19

$$A = \begin{bmatrix} 3 & 2 & 1 \\ 4 & 0 & 6 \\ 7 & 8 & 9 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 0 \\ 3 & 6 \\ 1 & 2 \end{bmatrix}$$

Find AB

Solution

$$AB = \begin{bmatrix} 3 & 2 & 1 \\ 4 & 0 & 6 \\ 7 & 8 & 9 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 3 & 6 \\ 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 3\times 2 + 2\times 3 + 1\times 1 & 3\times 0 + 2\times 6 + 1\times 2 \\ 4\times 2 + 0\times 3 + 6\times 1 & 4\times 0 + 0\times 6 + 6\times 2 \\ 7\times 2 + 8\times 3 + 9\times 1 & 7\times 0 + 8\times 6 + 9\times 2 \end{bmatrix}$$

$$= \begin{bmatrix} 13 & 14 \\ 14 & 12 \\ 47 & 66 \end{bmatrix}$$

Example 20

$$A = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 6 & 8 \\ 7 & 5 & 6 \end{bmatrix} \qquad B = \begin{bmatrix} 4 & 3 & 5 \\ 2 & 9 & 1 \\ 6 & 7 & 8 \end{bmatrix}$$

Find AB and BA

Solution

$$AB = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 6 & 8 \\ 7 & 5 & 6 \end{bmatrix}\begin{bmatrix} 4 & 3 & 5 \\ 2 & 9 & 1 \\ 6 & 7 & 8 \end{bmatrix}$$

$$= \begin{bmatrix} 1\times 4 + 3\times 2 + 4\times 6 & 1\times 3 + 3\times 9 + 4\times 7 & 1\times 5 + 3\times 1 + 4\times 8 \\ 2\times 4 + 6\times 2 + 8\times 6 & 2\times 3 + 6\times 9 + 8\times 7 & 2\times 5 + 6\times 1 + 8\times 8 \\ 7\times 4 + 5\times 2 + 6\times 6 & 7\times 3 + 5\times 9 + 6\times 7 & 7\times 5 + 5\times 1 + 6\times 8 \end{bmatrix}$$

$$= \begin{bmatrix} 24 & 58 & 40 \\ 48 & 116 & 80 \\ 74 & 29 & 88 \end{bmatrix}$$

$$BA = \begin{bmatrix} 4 & 3 & 5 \\ 2 & 9 & 1 \\ 6 & 7 & 8 \end{bmatrix}\begin{bmatrix} 1 & 3 & 4 \\ 2 & 6 & 8 \\ 7 & 5 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 4\times1+3\times2+5\times7 & 4\times3+3\times6+5\times5 & 4\times4+3\times8+5\times6 \\ 2\times1+9\times2+1\times7 & 2\times3+9\times6+1\times5 & 2\times4+9\times8+1\times6 \\ 6\times1+7\times2+8\times7 & 6\times3+7\times6+8\times5 & 6\times4+7\times8+8\times6 \end{bmatrix}$$

$$BA = \begin{bmatrix} 45 & 55 & 70 \\ 27 & 65 & 86 \\ 76 & 100 & 128 \end{bmatrix}$$

From this result, you can see that if the products AB and BA both exist and thy have the same order. It is not necessary that they should be equal.

Properties of a Matrix Multiplication

(a) The matrix multiplication is not always commutative.

Example 21

$A = \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$

Then,

$$AB = \begin{bmatrix} 4 & 1 \\ -2 & -1 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 & 4 \\ 1 & 2 \end{bmatrix}$$

In this case $AB \neq BA$ and the matrix multiplication is not commutative.

But $A = \begin{bmatrix} 2 & 0 \\ 0 & 6 \end{bmatrix} \quad B = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$

Then $AB = \begin{bmatrix} 6 & 0 \\ 0 & 12 \end{bmatrix}$

$$BA = \begin{bmatrix} 6 & 0 \\ 0 & 12 \end{bmatrix}$$

In this case $AB = BA$ commutative.

The matrix multiplication is not always commutative.

(b) Matrix multiplication is associative

ie if A, B, C are matrices of the order $m \times n$,$n \times p$ and $p \times z$ respectively,

then $(AB)C = A(BC)$

(c) Multiplication of matrices is distributive with respect to addition of matrices i.e. if A is a matrix of the size $m \times n$ and B, C of the same size $n \times p$ then $A(B + C) = AB + AC$

(d) Multiplicative identity if A be any $m \times n$ matrix, then $I_m . A = AI_n$

24.4 Transpose of a Matrix

Let $A = [a_{ij}]_{m\times n}$ then the transpose of the matrix A is defined as $[a_{ji}]_{n\times m}$. The transpose of A is denoted by A^T.

The matrix obtained from the given matrix A, by interchanging its rows and columns called the transpose of A.

Example 22

If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 0 & -1 \end{bmatrix}$,then find A^T, AA^T and $A^T A$.

Solution

$$\boldsymbol{A} = \begin{bmatrix} \mathbf{1} & \mathbf{2} \\ \mathbf{3} & \mathbf{4} \\ \mathbf{0} & \mathbf{-1} \end{bmatrix} \quad \therefore \boldsymbol{A^T} = \begin{bmatrix} \mathbf{1} & \mathbf{3} & \mathbf{0} \\ \mathbf{2} & \mathbf{4} & \mathbf{-1} \end{bmatrix}$$

$$AA^{\mathrm{T}} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 0 \\ 2 & 4 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \times 1 + 2 \times 2 & 1 \times 3 + 2 \times 4 & 1 \times 0 + 2 \times -1 \\ 3 \times 1 + 4 \times 2 & 3 \times 3 + 4 \times 4 & 3 \times 0 + 4 \times -1 \\ 0 \times 1 - 1 \times 2 & 0 \times 3 - 1 \times 4 & 0 \times 0 - 1 \times -1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 11 & -2 \\ 11 & 25 & -4 \\ -2 & -4 & 1 \end{bmatrix}$$

$$\boldsymbol{A^T A} = \begin{bmatrix} \mathbf{1} & \mathbf{3} & \mathbf{0} \\ \mathbf{2} & \mathbf{4} & \mathbf{-1} \end{bmatrix} \begin{bmatrix} \mathbf{1} & \mathbf{2} \\ \mathbf{3} & \mathbf{4} \\ \mathbf{0} & \mathbf{-1} \end{bmatrix}$$

$$= \begin{bmatrix} 1 \times 1 + 3 \times 3 + 0 \times 0 & 1 \times 2 + 3 \times 4 + 0 \times -1 \\ 2 \times 1 + 4 \times 3 - 1 \times 0 & 2 \times 2 + 4 \times 4 + -1 \times -1 \end{bmatrix}$$

$$= \begin{bmatrix} 10 & 14 \\ 14 & 21 \end{bmatrix}$$

$$\therefore A^{\mathrm{T}}\mathrm{A} \neq AA^{\mathrm{T}}$$

Properties of Transpose

(1) **Let** $\boldsymbol{A} = [\boldsymbol{a_{ij}}]_{\boldsymbol{m \times n}}$ $\boldsymbol{B} = [\boldsymbol{b_{ij}}]_{\boldsymbol{m \times n}}$

(i) $A^{\mathrm{T}} = \mathrm{A}$

(ii) $(A + B)^{\mathrm{T}} = A^{\mathrm{T}} + B^{\mathrm{T}}$

(iii) $\left(k\mathrm{A}\_\_\right)^{T} = k\underline{\mathrm{A}}\mathrm{A}^{\mathrm{T}}$

(2) If $A_1, A_2, \cdots, A_n$ are matrices such that the product $A_1 A_2 \cdots A_n$ can be defined, then $(A_1 A_2 \dots A_n)^T = {A_n}^T {A_{n-1}}^T \dots {A_2}^T {A_1}^T$

Example 23

$$(AB)^T = B^T A^T$$

$$(ABC)^T = C^T B^T A^T$$

Example 24

IfA= $\begin{bmatrix} 1 & 2 & 3 \\ -4 & 2 & 0 \\ 3 & 4 & 0 \end{bmatrix}$, find $A + A^T$ and $A - A^T$ and what can say about the matrices $A + A^T$ and $A - A$

Solution

$$A = \begin{bmatrix} 1 & 2 & 3 \\ -4 & 2 & 0 \\ 3 & 4 & 0 \end{bmatrix} \therefore A^T = \begin{bmatrix} 1 & -4 & 3 \\ 2 & 2 & 4 \\ 3 & 0 & 0 \end{bmatrix}$$

$$\therefore A + A^T = \begin{bmatrix} 2 & -2 & 6 \\ -2 & 4 & 4 \\ 6 & 4 & 0 \end{bmatrix}$$

$$A - A = \begin{bmatrix} 0 & 6 & 0 \\ -6 & 0 & -4 \\ 0 & 4 & 0 \end{bmatrix}$$

We can see that $A + A^T$ is symmetric matrix and $A - A^T$ is skew symmetric matrix.

- ❖ Every square matrix can be uniquely expressed as a sum of symmetric matrix and skew symmetric matrix.

From the above example

$$A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$$

$$\begin{bmatrix} 1 & 2 & 3 \\ -4 & 2 & 0 \\ 3 & 4 & 0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2 & -2 & 6 \\ -2 & 4 & 4 \\ 6 & 4 & 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & 6 & 0 \\ -6 & 0 & -4 \\ 0 & 4 & 0 \end{bmatrix}$$

Example 25

(1) For the matrices $A = \begin{bmatrix} 1 & -4 \\ 0 & 5 \\ 6 & 7 \end{bmatrix}_{3\times 2}$, $B = \begin{bmatrix} 2 & 3 & -1 \\ 1 & 0 & 7 \end{bmatrix}_{2\times 3}$

Verify that

$$(AB)^T = B^T A^T$$

$$AB = \begin{bmatrix} 1 & -4 \\ 0 & 5 \\ 6 & 7 \end{bmatrix}\begin{bmatrix} 2 & 3 & -1 \\ 1 & 0 & 7 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1\times 2 - 4\times 1 & 1\times 3 - 4\times 0 & 1\times -1 - 4\times -7 \\ 0\times 2 + 5\times 1 & 0\times 3 + 5\times 0 & 0\times -1 + 5\times -7 \\ 6\times 2 + 7\times 1 & 6\times 3 + 7\times 0 & 6\times -1 + 7\times -7 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 3 & 27 \\ 5 & 0 & -35 \\ 19 & 18 & -55 \end{bmatrix}$$

$$(AB)^T = \begin{bmatrix} -2 & 5 & 19 \\ 3 & 0 & 18 \\ 27 & -35 & -55 \end{bmatrix}$$

$$B^T A^T = \begin{bmatrix} 2 & 1 \\ 3 & 0 \\ -1 & -7 \end{bmatrix}\begin{bmatrix} 1 & 0 & 6 \\ -4 & 5 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 2\times 1 + 1\times -4 & 2\times 0 + 1\times 5 & 2\times 6 + 1\times 7 \\ 3\times 1 + 0\times -4 & 3\times 0 + 0\times 5 & 3\times 6 + 0\times 7 \\ -1\times 1 + -7\times -4 & -1\times 0 + -7\times 5 & -1\times 6 - 7\times 7 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 5 & 19 \\ 3 & 0 & 18 \\ 29 & -35 & 55 \end{bmatrix}$$

$$\therefore (AB)^T = B^T A^T$$

(2) $A = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ x & 2 & y \end{bmatrix}$

$$A^T = \frac{1}{3}\begin{bmatrix} 1 & 2 & x \\ 2 & 1 & 2 \\ 2 & -2 & y \end{bmatrix}$$

$$AA^T = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ x & 2 & y \end{bmatrix}\frac{1}{3}\begin{bmatrix} 1 & 2 & x \\ 2 & 1 & 2 \\ 2 & -2 & y \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix} 9 & 0 & x+4+2y \\ 0 & 9 & 2x+2-2y \\ x+4+2y & 2x+2-2y & x^2+4+y^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\therefore x + 4 + 2y = 0 \rightarrow (1)$$

$$2x + 2 - 2y = 0 \rightarrow (2)$$

$$3x + 6 = 0$$

$x = -2$

$$y = -1$$

$\therefore\ x = -2\ ,\ y = -1$

(3) $A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 3 & -1 \\ -4 & 9 & 5 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 & -1 \\ -2 & -1 & 2 \\ 1 & -3 & -1 \end{bmatrix}$

$$A + 2B = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 3 & -1 \\ -4 & 9 & 5 \end{bmatrix} + \begin{bmatrix} 2 & 4 & -2 \\ -4 & -2 & 4 \\ 2 & -6 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 2 & 1 \\ -2 & 1 & 3 \\ -2 & 3 & 3 \end{bmatrix}$$

$$(A + 2B)^2 = \begin{bmatrix} 3 & 2 & 1 \\ -2 & 1 & 3 \\ -2 & 3 & 3 \end{bmatrix}\begin{bmatrix} 3 & 2 & 1 \\ -2 & 1 & 3 \\ -2 & 3 & 3 \end{bmatrix}$$

$$= \begin{pmatrix} 9-4-2 & 6+2+3 & 3+6+3 \\ -6-2-6 & -4+1+9 & -2+3+9 \\ -6-6-6 & -4+3+9 & -2+9+9 \end{pmatrix}$$

$$= \begin{bmatrix} 3 & 11 & 12 \\ -14 & 6 & 10 \\ -18 & 8 & 16 \end{bmatrix}$$

(4) $\begin{bmatrix} 5 & 4 \\ 1 & 1 \end{bmatrix} X = \begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix}$

Let, $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$\begin{bmatrix} 5 & 4 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 5a+4d & 5b+4d \\ a+c & b+d \end{bmatrix}$$

$5a + 4c = 1 \rightarrow (1)\ 5b + 4d = -2 \rightarrow (2)$

$a + c = 1 \rightarrow (3)\ \ b + d = 3 \rightarrow (4)$

$$(1) - (3) \times 4; a = -3$$

$$c = 4$$

$(2) - (4) \times 4$; d=-14

$$b = 17$$

$$\therefore X = \begin{bmatrix} -3 & 17 \\ 4 & -14 \end{bmatrix}$$

(5) $A = \begin{bmatrix} 2 & 3 \\ 3 & 10 \end{bmatrix}$

$$(2I - A) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} - \begin{bmatrix} 2 & 3 \\ 3 & 10 \end{bmatrix} = \begin{bmatrix} 0 & -3 \\ -3 & -8 \end{bmatrix}$$

$$(10I - A) = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix} - \begin{bmatrix} 2 & 3 \\ 3 & 10 \end{bmatrix} = \begin{bmatrix} 8 & -3 \\ -3 & 0 \end{bmatrix}$$

$$(2I - A)(10I - A) = \begin{bmatrix} 0 & -3 \\ -3 & -8 \end{bmatrix}\begin{bmatrix} 8 & -3 \\ -3 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix}$$

$$= 9I$$

Activity 1.3

1. If $A = \begin{bmatrix} -2 & 3 & -1 \\ -1 & 2 & -1 \\ -6 & 9 & -4 \end{bmatrix}$and $B = \begin{bmatrix} 1 & 3 & -1 \\ 2 & 2 & -1 \\ 3 & 0 & -1 \end{bmatrix}$, then how that A and B commutative on multiplication.

2. Find the matrix $\begin{bmatrix} x & y \\ z & u \end{bmatrix}$ such that $\begin{bmatrix} 5 & -7 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} x & y \\ z & u \end{bmatrix} = \begin{bmatrix} -16 & -6 \\ 7 & 2 \end{bmatrix}$

3. If $A = \begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix}$ and $B = \begin{bmatrix} a^2 & ab & ac \\ ab & b^2 & bc \\ ac & bc & c^2 \end{bmatrix}$, then how that $AB = 0$

4. Given that $A = \begin{bmatrix} 0 & -3 & 1 \\ 2 & -1 & 1 \\ 2 & -1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 3 & -3 \end{bmatrix}$

   Compute AB and hence show that $AB = \underline{0}$ even when $A \neq \underline{0}\,, B \neq \underline{0}$

5. Find the values x, y, z if the matrix $A = \begin{bmatrix} 0 & 2y & z \\ x & y & -z \\ x & -y & z \end{bmatrix}$ obeys the law $A^T A = \underline{I}$

6. Express the matrix $A = \begin{bmatrix} 1 & 4 & 5 \\ 2 & 2 & 3 \\ 3 & 1 & 0 \end{bmatrix}$ as the sum of a symmetric and skew symmetric matrices.

Solutions of Activities

Activity 1

(1) $\begin{bmatrix} 1 & {}^{1}/_{2} & {}^{1}/_{3} & {}^{1}/_{4} \\ 2 & 1 & {}^{2}/_{3} & {}^{1}/_{2} \\ 3 & {}^{3}/_{2} & 1 & {}^{3}/_{4} \end{bmatrix}$

(2) (i) $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 5 & 7 \end{bmatrix}$ (ii) $\begin{bmatrix} {}^{1}/_{2} & 0 & {}^{1}/_{2} \\ {}^{9}/_{2} & 2 & {}^{1}/_{2} \end{bmatrix}$

(3) $\begin{bmatrix} 1 & 3 & 4 \\ 1 & 4 & 5 \\ 2 & 1 & 9 \\ 3 & 2 & 1 \end{bmatrix}$

Activity 2

$$x = 2, y = -1, a = 0, b = 5$$

Activity 3

(2) $\begin{bmatrix} \mathbf{1} & \mathbf{-4} \\ \mathbf{3} & \mathbf{-2} \end{bmatrix}$

(5) $x = \pm\frac{1}{\sqrt{2}}$ $y = \pm\frac{1}{\sqrt{6}}$ $z = \pm\frac{1}{\sqrt{3}}$

(6) $\begin{bmatrix} 1 & 3 & 4 \\ 3 & 2 & 2 \\ 4 & 2 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix}$

Summary

A matrix is defined as a rectangular array of mn number of numbers that belong to a field. The numbers are called the element of the matrix.

Two matrices $A = [a_{ij}]_{m\times n}$, $B = [b_{ij}]_{p\times q}$ are said to be equal if they have the same size and the corresponding elements are equal.

A matrix having exactly one row and any number of column is called a row matrix.

A matrix having exactly one column and any number of rows is called a column matrix.

A matrix having the same number of rows and columns is called a square matrix.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & \cdots & a_{1n} \\ a_{21} & a_{22} & & & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & & & a_{nn} \end{bmatrix}_{n\times n}$$

$a_{11}\ a_{22} \cdots a_{nn}$ are called the principal diagonal of the matrix.

$A = [a_{ij}]_{m\times n}$, $B = [b_{ij}]_{m\times n}$

Then $C = A + B = [c_{ij}]_{m\times n}$

Where $[c_{ij}]_{m\times n} = a_{ij} + b_{ij}$

$$D = A - B = [d_{ij}]_{m\times n}$$

$[d_{ij}]_{m\times n} = a_{ij} - b_{ij}$

If $A = [a_{ij}]_{m\times n}$, $B = [b_{ij}]_{n\times p}$, $E = [e_{ij}]_{m\times p}$

$$AB = E$$

Where

$$e_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$$

If $A = [a_{ij}]_{m\times n}$ then $A^T = [a_{ji}]_{n\times m}$

$\underline{A}^T$ is the transpose of the matrix A.

Learning Outcomes

At the end of this session you should be able to

- Define a matrix.
- Recognize the various types of Matrices.
- Define and apply the algebraic operation of the matrix
- Define and recognize the properties of the transpose of a matrix.

Session 25

Determinant

Introduction

In this study session, the determinant of a square matrix is defined, and the properties of determinant are discussed.

The determinants are very useful for obtaining the solutions of system of linear equations.

25.1 Determinant

A determinant is a number associated with a square matrix. Corresponding to each square matrix,

$$A = \begin{bmatrix} a_{11} & a_{12} & . & . & . & a_{1n} \\ a_{21} & a_{22} & . & . & . & a_{2n} \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ a_{n1} & a_{n2} & . & . & . & a_{nn} \end{bmatrix}$$

there is associated expression, called the determinant of A, denoted by $detA$ or $|A|$ written as

$$detA = |A| = \begin{vmatrix} a_{11} & a_{12} & . & . & . & a_{1n} \\ a_{21} & a_{22} & . & . & . & a_{2n} \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ a_{n1} & a_{n2} & . & . & . & a_{nn} \end{vmatrix}$$

A matrix is an arrangement of numbers and it has no fixed value, but a determinant is a number and it has fixed value. A determinant having n rows and n columns is called a determinant of order$n$.

Determinant of a square matrix of order 1.

Let $A = [a_{11}]$ be a 1×1 matrix, then the determinant of A is the number a_{11} itself

i.e. $|a_{11}| = a_{11}$

Determinant of a square matrix of order 2.

Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ be a 2×2 matrix then

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

ie. The determinant of a 2×2 matrix is obtained by taking the product of the entries on the main diagonal and subtracting from it the product of the entries in the diagonal.

Example

$$\begin{vmatrix} 4 & 3 \\ 6 & 1 \end{vmatrix} = 4 \times 1 - 3 \times 6 = 4 - 18 = -14$$

$$\begin{vmatrix} x-1 & x+1 \\ x^2 - x + 1 & x+1 \end{vmatrix} = (x-1)(x+1) - (x+1)(x^2 - x + 1)$$

$$= (x+1)[x - 1 - x^2 + x - 1]$$

$$= (x+1)[-x^2 + 2x - 2]$$

Determinant of a square matrix of order 3

If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{22} \end{bmatrix}$ is a square matrix of order 3, then the expression $a_{11}\ a_{22}\ a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}$ is defined as the determinant of $|A|$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{22} \end{bmatrix}$$

$$= a_{11}\ (a_{22}\ a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31})$$
$$+ a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

$$= a_{11} \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12} \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13} \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

$$= (-1)^{1+1} \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} + (-1)^{1+2} \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} +$$
$$(-1)^{1+3} \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

Working Rule to expand a determinant of order 3

1. Write the elements of the first row with alternatively positive and negative sign. The first element always has positive sign before it.
2. Multiply each signed element by the determinant of second order obtained after deleting the row and the column in which that element occurs.

Example 1

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{22} \end{vmatrix}$$

$$= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

The value of determinant is same when expanded by any row or any column.

$$|A| = \begin{vmatrix} 2 & 4 & 5 \\ 6 & 7 & 9 \\ 8 & 10 & 11 \end{vmatrix}$$

Expanded by the first row

$$|A| = 2 \begin{bmatrix} 7 & 9 \\ 10 & 11 \end{bmatrix} - 4 \begin{bmatrix} 6 & 9 \\ 8 & 11 \end{bmatrix} + 5 \begin{bmatrix} 6 & 7 \\ 8 & 10 \end{bmatrix}$$

$$= 2(77 - 90) - 4(66 - 72) + 5(60 - 56)$$

$$= -26 + 24 + 20 = -18$$

Expanded by the second row.

$$(-1)^{2+1} = -1$$

$$|A| = -6 \begin{bmatrix} 4 & 5 \\ 10 & 11 \end{bmatrix} + 7 \begin{bmatrix} 2 & 5 \\ 8 & 11 \end{bmatrix} - 9 \begin{bmatrix} 2 & 4 \\ 8 & 10 \end{bmatrix}$$

$$= -6(44 - 50) + 7(22 - 40) - 9(20 - 32)$$

$$= 36 - 126 + 108 = 144 - 126 = 18$$

Expanded by the first column

$$|A| = 2\begin{bmatrix} 7 & 9 \\ 10 & 11 \end{bmatrix} - 6\begin{bmatrix} 4 & 5 \\ 10 & 11 \end{bmatrix} + 8\begin{bmatrix} 4 & 5 \\ 7 & 9 \end{bmatrix}$$

$$= 2(77 - 90) - 6(44 - 50) + 8(36 - 35)$$

$$= -26 + 36 + 8$$

$$= 18$$

Expanded by the third column

$$|A| = 5\begin{bmatrix} 6 & 7 \\ 8 & 10 \end{bmatrix} - 9\begin{bmatrix} 2 & 4 \\ 8 & 10 \end{bmatrix} + 11\begin{bmatrix} 2 & 4 \\ 6 & 7 \end{bmatrix}$$

$$= 5(60 - 56) - 9(20 - 32) + 11(14 - 24)$$

$$= 20 + 108 - 110 = 18$$

$\therefore$ We can see that the value of determinant is the same, when it is expanded by any row any column.

Sarrus Diagram for Expansion of determinant of order 3

The following diagram called Sarrus diagram, enables us to obtain the value of the determinant of order 3.

Let $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Write the elements as shown in the adjoining diagram.

$$\begin{matrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{matrix} \qquad\qquad \begin{matrix} a_1 & b_1 & c_1 & a_1 & b_1 \\ a_2 & b_2 & c_2 & a_2 & b_2 \\ a_3 & b_3 & c_3 & a_3 & b_3 \end{matrix}$$

$$\Delta = a_1b_2c_3 + b_1c_2a_3 + c_1a_2b_3 - (a_3b_2c_1 + b_3c_2a_1 + c_3a_2b_1)$$

$$= a_1b_2c_3 + b_1c_2a_3 + c_1a_2b_3 - (a_3b_2c_1 + b_3c_2a_1 + c_3a_2b_1)$$

Example 2

$$|A| = \begin{vmatrix} 3 & -4 & 3 \\ 5 & 7 & 9 \\ 4 & 2 & 6 \end{vmatrix}$$

$$\Delta = \begin{matrix} 3 & -4 & 3 & 3 & -4 \\ 5 & 7 & 9 & 5 & 7 \\ 4 & 2 & 6 & 4 & 2 \end{matrix}$$

$= 3 \times 7 \times 6 + (-4 \times 9 \times 4) + (3 \times 5 \times 2) - (4 \times 7 \times 3 + 2 \times 9 \times 3 + 6 \times 5 \times -4)$

$= 126 - 144 + 30 - (84 + 54 - 120) \;= -6$

Notes;

If a row or a column of a determinant consists of all zeros, then the value of the determinant is zero.

If each element above or below the main diagonal of a determinant is zero, then the value of the determinant is the product of elements along the main diagonal.

Example 3

Find the value of $\begin{vmatrix} 3 & 4 \\ 9 & -f \end{vmatrix}$.

$\begin{vmatrix} 3 & 4 \\ 9 & -7 \end{vmatrix} = (3 \times -7) - (9 \times 4)$

Example 4

Find the value of $\begin{vmatrix} -1 & 6 & -2 \\ 2 & 1 & 1 \\ 4 & 1 & -3 \end{vmatrix}$.

$$\begin{vmatrix} -1 & 6 & -2 \\ 2 & 1 & 1 \\ 4 & 1 & -3 \end{vmatrix}$$

Expand by the 1st raw.

$\Delta = -\begin{vmatrix} 1 & 1 \\ 1 & -3 \end{vmatrix} - 6\begin{vmatrix} 2 & 1 \\ 4 & -3 \end{vmatrix} - 2\begin{vmatrix} 2 & 1 \\ 4 & 1 \end{vmatrix}$

$\Delta = 4 + 60 + 4$

Example 5

Find the value of $\begin{vmatrix} 1^2 & 2^2 & 3^2 \\ 2^2 & 3^2 & 4^2 \\ 3^2 & 4^2 & 5^2 \end{vmatrix}$.

$\begin{vmatrix} 1^2 & 2^2 & 3^2 \\ 2^2 & 3^2 & 4^2 \\ 3^2 & 4^2 & 5^2 \end{vmatrix}$ By expanding along the 1st row

$$\Delta = 1^2(3^2 5^2 - 4^2 4^2) - 2^2(2^2 5^2 - 4^2 3^2) + 3^2(2^2 4^2 - 3^2 3^2)$$

$$= (225 - 256) - 4(100 - 144) + 9(64 - 81)$$

$$= -31 + 176 - 153$$

$$= -184 + 176$$

$$= -8$$

Activity 1

1. If $\begin{vmatrix} 3 & x \\ 4 & 5 \end{vmatrix} = 3$ find x,

2. Solve for x, where $\begin{vmatrix} x & 2 & -1 \\ 2 & 5 & x \\ -1 & 2 & x \end{vmatrix} = 0$

3. Evaluate the following determinants

i. $\begin{vmatrix} 2 & 4 & 7 \\ 3 & 6 & 9 \\ 4 & 8 & 11 \end{vmatrix}$

ii. $\begin{vmatrix} 5 & 1 & 0 \\ 2 & 3 & -1 \\ -3 & 2 & 0 \end{vmatrix}$

iii. $\begin{vmatrix} 0 & 2 & 6 \\ 1 & 5 & 6 \\ 3 & 7 & 1 \end{vmatrix}$

25.2 Properties of Determinant

We now discuss some useful properties of determinants of order three only. However, these properties hold for determinants of any order. These properties help a good deal with the evaluation of a given determinant by converting it into an equal determinant which is easier to evaluate.

Property I

The value of a determinant remains unaltered if the rows and columns are interchanged.

Proof:- $|A| = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$

Expanding by 1st row we have

$a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$

$$= a(ei - hf) - b(di - fg) + c(hd - eg)$$
$$= aei + bfg + chd - (ahf + bdi + ceg)$$
$$= aei + bfg + cdh - (afh + bdi + ceg)$$

Let $A' = \begin{vmatrix} a & d & g \\ b & e & h \\ c & f & i \end{vmatrix}$ be the determinant obtained from $|A|$ by interchanging rows and columns.

Expanding by 1st row, we get

$$|A'| = a\begin{vmatrix} e & h \\ f & i \end{vmatrix} - d\begin{vmatrix} b & h \\ c & i \end{vmatrix} + g\begin{vmatrix} b & e \\ c & f \end{vmatrix}$$
$$= a[ei - hf] - d[bi - ch] + g[bf - ce]$$
$$= (aei + cdh + bfg) - (afh + bdi + ceg)$$

$\therefore |A| = |A'|$

Property II.

If two rows (or column) of a determinant are interchanged, then the value of the determinant is multiplied by -1;

Proof

Let $|A| = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$ be any determinant of order 3.

We have $|A| = aei + bfg + cdh - (afh + bdi + ceg)$ from the above property.

Let $|B| = \begin{vmatrix} g & h & i \\ d & e & f \\ a & b & c \end{vmatrix}$ be the determinant obtained from $|A|$ by interchanging first and third.

$|B| = g\begin{vmatrix} e & f \\ b & c \end{vmatrix} - h\begin{vmatrix} d & f \\ a & c \end{vmatrix} + i\begin{vmatrix} d & e \\ a & b \end{vmatrix}$

$= g(ec - gf) - h(cd - af) + i(bd - ae)$

$= (ceg + afh + bdi) - (bfg + cdh + aei)$

$\therefore |B| = -|A|$

Property III.

The sign of determinant is either changed or not changed according as the number of interchanges of two adjacent rows (or columns) is odd or even.

Property IV.

If any two rows (or columns) of a determinant are identical, then the value of determinant is zero.

Proof

$|A| = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$ be any determinant in which second and third rows identical expanding by 1st row we get

$$|A| = a\begin{vmatrix} e & f \\ e & f \end{vmatrix} - b\begin{vmatrix} d & f \\ d & f \end{vmatrix} + c\begin{vmatrix} d & e \\ d & e \end{vmatrix} = 0$$

Property V

If the elements of a row (or column) of a determinant are multiplied by a scalar, then the value of the new determinant is equal to same scalar times the values of the original determinant.

Proof

Let $|A| = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$

$|A| = aei + bfg + cdh - (afh + bdi + ceg)$

Let $|B| = \begin{vmatrix} ka & b & c \\ kd & e & f \\ kg & h & i \end{vmatrix}$

$$\left|\underline{B}\right| = ka \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} kd & f \\ kg & i \end{vmatrix} + c \begin{vmatrix} kd & e \\ kg & h \end{vmatrix}$$

$$= ka(ei - hf) - b(kdi - kfg) + c|kdh - keg|$$

$$= k[aei + bfg + cdh - (afh + bdi + ceg)]$$

$$|B| = k|A|$$

Property VI

If each element of any row (or column) of a determinant is the sum of two number, then the determinant is expressible as the sum of two determinant of the same order.

Let $|D| = \begin{vmatrix} a+\lambda & b+\eta & c+\gamma \\ d & e & f \\ g & h & i \end{vmatrix}$

Expanding $|D|$ by first row

$$|D| = (a+\lambda) \begin{vmatrix} e & f \\ h & i \end{vmatrix} - (b+\eta) \begin{vmatrix} d & f \\ g & i \end{vmatrix} + (c+\gamma) \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

$$= a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix} + \lambda \begin{vmatrix} e & f \\ h & i \end{vmatrix} - \eta \begin{vmatrix} d & f \\ g & i \end{vmatrix} + \gamma \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

$$= \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} + \begin{vmatrix} \lambda & \eta & \gamma \\ d & e & f \\ g & h & i \end{vmatrix}$$

Property VII

If to each element of a row (or column) of a determinant be added equi--multiples of the corresponding element of one or more rows (or column), then the value of the determinant is not changed.

Proof

Let $|A| = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$

$$= (aei + bfg + cdh) - (afh + bdi + ceg)$$

Let $|B|$ be the determinant obtained from $|A|$ by adding λ times the elements of second column and η times corresponding elements of the first column.

$$|B| = \begin{vmatrix} a+\lambda b+\eta c & b & c \\ d+\lambda e+\eta f & e & f \\ g+\lambda h+\eta i & h & i \end{vmatrix}$$

$$= \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} + \lambda \begin{vmatrix} b & b & c \\ e & e & f \\ h & h & i \end{vmatrix} + \eta \begin{vmatrix} c & b & c \\ f & e & f \\ i & h & i \end{vmatrix}$$

$= |A| + \lambda 0 + \eta 0$ 1st and 3rd columns are identical.

1st and 2nd columns are identical.

$= |A|$

Example 6

Using the properties of determinants, prove that the values of the following determinants are zero.

i. $\begin{vmatrix} 3 & 1 & 6 \\ 5 & 2 & 10 \\ 7 & 4 & 14 \end{vmatrix}$ ii. $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$

(i) $\Delta = \begin{vmatrix} 3 & 1 & 6 \\ 5 & 2 & 10 \\ 7 & 4 & 14 \end{vmatrix} = \begin{vmatrix} 3 & 1 & 2\times 3 \\ 5 & 2 & 2\times 5 \\ 7 & 4 & 2\times 7 \end{vmatrix}$

$$= 2\begin{vmatrix} 3 & 1 & 3 \\ 5 & 2 & 5 \\ 7 & 4 & 7 \end{vmatrix}$$

$= 0 \quad \because c_1$ and c_3 are identical.

(ii) $\Delta = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} \xrightarrow[R_2 \to R_2 - R_1]{R_3 \to R_3 - R_2} \begin{vmatrix} 1 & 2 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{vmatrix}$

$= 0$

The 2nd and 3rd rows are identical.

Example 7

Prove that

i. $$\begin{vmatrix} a-b & b-c & c-a \\ x-y & y-z & z-x \\ p-q & q-r & r-p \end{vmatrix} = 0$$

ii. $$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ b+c & c+a & a+b \end{vmatrix} = 0$$

Solution

The 2nd and 3rd rows are identical.

(i) $$\begin{vmatrix} a-b & b-c & c-a \\ x-y & y-z & z-x \\ p-q & q-r & r-p \end{vmatrix} \xrightarrow{c_1 \rightarrow c_1+c_2+c_3} \begin{vmatrix} 0 & b-c & c-a \\ 0 & y-z & z-x \\ 0 & q-r & r-p \end{vmatrix} = 0$$

(ii) $$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ b+c & c+a & a+b \end{vmatrix} \xrightarrow{R_3 \rightarrow R_2+R_3} \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a+b+c & a+b+c & a+b+c \end{vmatrix}$$

$$\longrightarrow (a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix} = 0$$ [Since 1st and 3rd rows are identical]

Example 8

i. $$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$$

ii. $$\begin{vmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix} = (b-c)(c-a)(a-b)(ab+bc+ca)$$

Solution

i. $$\Delta = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} \xrightarrow[R_2 \rightarrow R_2-R_3]{R_1 \rightarrow R_1-R_3} \begin{vmatrix} 0 & (a-c) & (a^2-c^2) \\ 0 & (b-c) & (b^2-c^2) \\ 1 & c & c^2 \end{vmatrix}$$

$$(a-c)(b-c)\begin{vmatrix}0 & 1 & a+c\\ 0 & 1 & b+c\\ 1 & c & c^2\end{vmatrix}$$

$R_1 \to R_1 - R_2$

$$(a-c)(b-c)\begin{vmatrix}0 & 0 & a-b\\ 0 & 1 & b+c\\ 1 & c & c^2\end{vmatrix}$$

$$(a-c)(b-c)(a-b)\begin{vmatrix}0 & 0 & 1\\ 0 & 1 & b+c\\ 1 & c & c^2\end{vmatrix}$$

Expanding by 1st row

$$= (a-c)(b-c)(a-b)\left[1\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\right]$$

$$= -(a-c)(b-c)(a-b)$$

$$= (a-b)(b-c)(c-a)$$

ii. $$\begin{vmatrix}1 & 1 & 1\\ a^2 & b^2 & c^2\\ a^3 & b^3 & c^3\end{vmatrix} \xrightarrow[C_3 \to C_3 - C_1]{C_2 \to C_2 - C_1} \begin{vmatrix}1 & 1 & 1\\ a^2 & b^2-a^2 & c^2-a^2\\ a^3 & b^3-a^3 & c^3-a^3\end{vmatrix}$$

$$(b-a)(c-a)\begin{vmatrix}1 & 0 & 0\\ a^2 & b+a & c+a\\ a^3 & b^2+ab+a^2 & c^2+ac+a^2\end{vmatrix}$$

$C_2 \to C_2 - C_3$

$$(b-a)(c-a)\begin{vmatrix} 1 & 0 & 0 \\ a^2 & b-c & c+a \\ a^3 & (b^2-c^2)+a(b-c) & c^2+ac+a^2 \end{vmatrix}$$

$$(b-a)(c-a)(b-c)\begin{vmatrix} 1 & 0 & 0 \\ a^2 & 1 & c+a \\ a^3 & a+b+c & c^2+ac+a^2 \end{vmatrix}$$

$$(a-b)(c-a)(b-c)[c^2+ac+a^2-(c+a)(a+b+c)]$$

$$(a-b)(c-a)(b-c)[c^2+ac+a^2-ac-a^2-bc-ab-c^2-ac]$$

$$=-(a-b)(c-a)(b-c)[-bc-ab-ac]$$

$$=(a-b)(c-a)(b-c)(ab+bc+ca)$$

Activity 2

1. Prove that

a) $\begin{vmatrix} x+4 & 2x & 2x \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix} = (5x+4)(4-x)^2$

b) $\begin{vmatrix} x+a & b & c \\ a & x+b & c \\ a & b & x+c \end{vmatrix} = x^2(x+a+b+c)$

c) $\begin{vmatrix} a+b+c & -c & -b \\ -c & a+b+c & -a \\ -b & -a & a+b+c \end{vmatrix} = 2(a+b)(b+c)(c+a)$

4. Using properties of determinants, evaluate each of the following determinant.

i. $\begin{vmatrix} 43 & 1 & 6 \\ 35 & 7 & 4 \\ 17 & 3 & 2 \end{vmatrix}$

ii. $\begin{vmatrix} 5 & 15 & -25 \\ 7 & 21 & 30 \\ 8 & 24 & 42 \end{vmatrix}$

iii. $\begin{vmatrix} 1 & 3 & 5 \\ 2 & 6 & 10 \\ 11 & 13 & 15 \end{vmatrix}$

Solutions of Activities

Activity 1

1. If $\begin{vmatrix} 3 & x \\ 4 & 5 \end{vmatrix} = 3$, then find x.

Answer $x = 3$

2. Solve for x; $\begin{vmatrix} x & 2 & -1 \\ 2 & 5 & x \\ -1 & 2 & x \end{vmatrix} = 0$

Answer $x = 3$ or -1

3. Evaluate the following determinants

i. $\begin{vmatrix} 2 & 4 & 7 \\ 3 & 6 & 9 \\ 4 & 8 & 11 \end{vmatrix}$

ii. $\begin{vmatrix} 5 & 1 & 0 \\ 2 & 3 & -1 \\ -3 & 2 & 0 \end{vmatrix}$

iii. $\begin{vmatrix} 0 & 2 & 6 \\ 1 & 5 & 6 \\ 3 & 7 & 1 \end{vmatrix}$

Answer (i).0 (ii). 13 (iii).-50

Activity 2

Using properties of determinants evaluate each of the following determinant. .

i. $\begin{vmatrix} 43 & 1 & 6 \\ 35 & 7 & 4 \\ 17 & 3 & 2 \end{vmatrix}$

ii. $\begin{vmatrix} 5 & 15 & -25 \\ 7 & 21 & 30 \\ 8 & 24 & 42 \end{vmatrix}$

iii. $\begin{vmatrix} 1 & 3 & 5 \\ 2 & 6 & 10 \\ 11 & 13 & 15 \end{vmatrix}$

Answer (i) 0 (ii) 0 (iii) 0

Summary

If $\left|\underline{A}\right| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

$$|A| = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$

$$|A| = a\begin{vmatrix} e & f \\ e & f \end{vmatrix} - b\begin{vmatrix} d & f \\ d & f \end{vmatrix} + c\begin{vmatrix} d & e \\ d & e \end{vmatrix}$$

$$|A| = aei + bfg + cdh - (afh + bdi + ceg)$$

Learning Outcomes

At the end of this session the student should be able

- To define a determinant of a square matrix.
- Simply the determinant using the definition.
- To evaluate the values of determinants by using properties of determinants.

Session 26

Inverse of a Matrix

Introduction

In this session the concept of the inverse matrix is introduced in detail. Inverse of matrix is very useful for solving systems of linear equations. In the set of real numbers, we know that for each non-zero real number x, there exists a real number x^{-1} such that $xx^{-1} = x^{-1}x = 1$ and x^{-1} is called the multiplication inverse of x. Is the same true for the set of $n \times n$ square matrices?

26.1 Inverse of a Matrix

Let A be a square matrix with order n. If there exists a matrix B such that $AB = BA = I_n$, then B is defined as the inverse matrix of A and it is denoted by A^{-1}.

i.e. $\underline{A}A^{-1}$ is a matrix such that __.$AA^{-1} = A^{-1}A = I_n$. For the products AB and BA to be both define and equal it is necessary that, B are both square matrix of the same order. Hence, the inverse matrices cannot be defined for non-square matrices. A square matrix is said to be singular if it has no inverse. and every non-singular has an inverse. If the matrix A has the inverse, then we say that the matrix A is invertible.

26.2 Properties of the Inverse of Matrix

a) If a matrix has an inverse, then it is unique.

Let A be a square matrix and it has inverse matrices B and C.

Then $\quad AB = BA = \underline{I}$

and $\quad AC = CA = \underline{I}$

$$AB = \underline{I} \qquad (1)$$

$C.I \quad C(AB) = CI$

$$IB = C$$

$$B = C$$

The inverse matrix of a matrix is unique.

b) A square matrix is invertible if and only if it is non-singular.

c) Let A be a square matrix. The inverse matrix of A exists. $\det A \neq 0$

d) The inverse of the is the original matrix itself.ie $(A^{-1})^{-1} = A$

e) The inverse of transpose of a matrix is the transpose of its inverse$(A^T)^{-1} = (A^{-1})^T$

f) (Reversed law) If A and B are invertible matrices of the same order, then AB is also invertible and moreover

i.e. $(AB)^{-1} = B^{-1}A^{-1}$

Proof $\quad (AB)^{-1} = B^{-1}A^{-1}$

$(AB)(AB)^{-1}$

$= AB\,B^{-1}A^{-1}$

$= A\,\underline{I}\,A^{-1}$

$= AA^{-1}$

$= \underline{I}$

$\therefore (AB)^{-1} = B^{-1}A^{-1}$

g) If A, B and C are invertible matrices then

Then $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$

Proof $(ABC)(ABC)^{-1}$

$$ABCC^{-1}B^{-1}A^{-1}$$

$ABI\underline{B}B^{-1}A^{-1}$

$ABB^{-1}A^{-1}$

$A\ \underline{I}\ A^{-1}$

$A\ \ A^{-1}$

$= \underline{I}$

$\therefore\ (ABC)(ABC)^{-1} = \underline{I}$

$\therefore\ (ABC)^{-1} = (C^{-1}B^{-1}A^{-1})$

Example 1

Show that $A = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$ is the inverse of $B = \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}$

Solution

$$A = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}$$

$$BA = \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix} \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

$$BA = \begin{bmatrix} 3-2+0 & 6+6-12 & -6+0+6 \\ 1-1+0 & 2+3-4 & -2+0+2 \\ 2-2+0 & 4+6-10 & -4+0+5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$BA = \underline{I}$

$$\therefore B = A^{-1}$$

Example 2

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ find the inverse of A and discuss the existence of A^{-1}

Hence, find the inverse of the following 2×2 matrices.

i. $\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$

ii. $\begin{bmatrix} 3 & 6 \\ 2 & 4 \end{bmatrix}$

iii. $\begin{bmatrix} 2 & 5 \\ -3 & 1 \end{bmatrix}$

Solution

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ Let assume that $A^{-1} = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$

$$A\,A^{-1} = \underline{I} \Rightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} p & q \\ r & s \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\therefore ap + br = 1$ ——— (1)

$aq + bs = 0$ ——— (2)

$cp + dr = 0$ ——— (3)

$cq + ds = 1$ ——— (4)

From (3) $r = -\frac{c}{d}p$ from (2) $s = -\frac{aq}{b}$

From (1) $ap - \frac{bc}{d}p = 1$ from (4) $cq - \frac{ad}{b}q = 1$

$(ad - bc)p = d$ $(bc - ad)q = b$

$p = \frac{d}{ad-bc}$ $q = \frac{-b}{ad-bc}$

$r = \frac{-c}{ad-bc}$ $s = \frac{a}{ad-bc}$

$$\therefore A^{-1} = \begin{bmatrix} \dfrac{d}{ad-bc} & \dfrac{-b}{ad-bc} \\ \dfrac{-c}{ad-bc} & \dfrac{a}{ad-bc} \end{bmatrix}$$

$$\therefore \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Change the sign

$$\therefore \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Change the position

If $ad - bc = 0$ A^{-1} does not exists

We have $ad - bc$ is the $det \begin{vmatrix} a & b \\ c & d \end{vmatrix}$

$\therefore$ If det $A = 0$ then A is a singular matrix.

i. $\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$

$$\therefore A^{-1} = \frac{1}{2\times 3-(1\times -1)}\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{7}\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$$

ii. $\begin{bmatrix} 3 & 6 \\ 2 & 4 \end{bmatrix}$

det $A = 12 - 12 = 0$

$\therefore$ A^{-1} doesn't exists.

iii. $\begin{bmatrix} 2 & 5 \\ -3 & 1 \end{bmatrix}$

$$\therefore A^{-1} = \frac{1}{2\times 1-(5\times -3)}\begin{bmatrix} 1 & -5 \\ 3 & 2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{17}\begin{bmatrix} 1 & -5 \\ 3 & 2 \end{bmatrix}$$

Example 3

Let $A = \begin{pmatrix} 5 & 4 \\ 4 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} -14 & 20 \\ -12 & 17 \end{pmatrix}$ and $D = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ write down $\underline{A}^{-1}$verify the result

$B = AD\,A^{-1}$ and deduce from this result that $B^3 = A\,D\,^3A^{-1}$.

Solution

$$A = \begin{pmatrix} 5 & 4 \\ 4 & 3 \end{pmatrix}$$

$$B = \begin{pmatrix} -14 & 20 \\ -12 & 17 \end{pmatrix}$$

$$D = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$$

$\therefore\ A^{-1} = \frac{1}{5\times3-4\times4}\begin{bmatrix} 3 & -4 \\ -4 & 5 \end{bmatrix} = -1\begin{bmatrix} 3 & -4 \\ -4 & 5 \end{bmatrix} = \begin{bmatrix} -3 & 4 \\ 4 & -5 \end{bmatrix}$

$AD\ A^{-1} = \begin{pmatrix} 5 & 4 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -3 & 4 \\ 4 & -5 \end{pmatrix}$

$= \begin{pmatrix} 5\times2 & 4\times1 \\ 4\times2 & 3\times1 \end{pmatrix}\begin{pmatrix} -3 & 4 \\ 4 & -5 \end{pmatrix}$

$= \begin{pmatrix} 10 & 4 \\ 8 & 3 \end{pmatrix}\begin{pmatrix} -3 & 4 \\ 4 & -5 \end{pmatrix}$

$= \begin{pmatrix} -30+16 & 40-20 \\ -24+12 & 32-15 \end{pmatrix}$

$= \begin{pmatrix} -14 & 20 \\ -12 & 17 \end{pmatrix}$

$$AD\ A^{-1} = B$$

$\underline{B}^3 = (AD\ A^{-1})(AD\ A^{-1})(AD\ A^{-1})$

$= (AD\ A^{-1}AD\ A^{-1})(AD\ A^{-1})$

$= (A)(AD\ A^{-1})$

$= A\ D^{\ 2}\ A^{-1}AD\ A^{-1}$

$= A\ D^{\ 2}\underline{I}\ D\ A^{-1}$

$= D\ D^{\ 2}\ D\ A^{-1}$

$= A\ D^{\ 3}\ A^{-1}$

Example 4

The matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ has the property $A^T A = \underline{I}_2$ where A^T is the transpose of A. Prove that

i. If $a = 0$ then $d = 0$ and $b^2 = c^2 = 1$

ii. If $a \neq 0$ then $d = \mp a$ and $b = \mp c$

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $\qquad\qquad$ $\underline{A}^T = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$

Given that $A^T\ A = \underline{I}_2$

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\therefore\ a^2 + c^2 = 1$ —— (1) $\qquad ab + cd = 0$ —— (3)

$ab + cd = 0$ —— (2) $\qquad b^2 + d^2 = 1$ —— (4)

$\therefore$

i. If $a = 0$ from (1) $\quad c^2 = 1 \Rightarrow c = \mp 1$

from (2) $\quad cd = 0$

$\therefore d = 0 \qquad \because c \neq 0$

from $\quad b^2 = 1 \Rightarrow b = \mp 1$

ii. If $a \neq 0$ from (1) $\quad a^2 = 1 - c^2$

We have also $ad - bc \neq 0$ $\quad$ [A is non- ingular matrix]

(1) and (3) $a^2 + c^2 = b^2 + d^2$ —— (4)

$(a^2 - d^2) + (c^2 - b^2) = 0$

But from (2) $\quad ab = -cd \qquad a^2\ b^2 = c^2 d^2$

$\therefore (a^2 - \frac{a^2\ b^2}{c^2}) + (c^2 - b^2) = 0$

$a^2(c^2 - b^2) + c^2(c^2 - b^2) = 0$

$(a^2 + c^2)(c^2 - b^2) = 0$

$(a^2 + c^2) \neq 0 \qquad (c^2 - b^2) = 0$

$c = \mp b$

$b = \mp c$

again (4) $\quad a^2 = d^2 \qquad \because\ b^2 = c^2$

$d = \mp a$

Example 4

If $A = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$ write down A^{-1}. Verify that if $M = \begin{pmatrix} -1 & -2 \\ 6 & 6 \end{pmatrix}$ and

$D = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$ then $M = AD\ A^{-1}$ Show further that if $B = \begin{pmatrix} 2\lambda & -m \\ -3\lambda & 2m \end{pmatrix}$

where $\lambda \neq 0, m \neq 0$ then $M\ \underline{B}^{-1}$

Solution

$A = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$ $\qquad \therefore A^{-1} = \frac{1}{4-3}\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$

$$A^{-1} = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$$

$$AD\ A^{-1} = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}$$

$$= \begin{bmatrix} 4 & -3 \\ -6 & 6 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 8-9 & 4-6 \\ -12+18 & -6+12 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -2 \\ 6 & 6 \end{bmatrix}$$

$\therefore AD\ A^{-1} = \underline{M}$

If $B = \begin{bmatrix} 2\lambda & -m \\ -3\lambda & 2m \end{bmatrix}$ $\qquad \therefore B^{-1} = \frac{1}{4\lambda m - 3\lambda m}\begin{bmatrix} 2m & m \\ 3\lambda & 2\lambda \end{bmatrix}$

$$B^{-1} = \frac{1}{\lambda m}\begin{bmatrix} 2m & m \\ 3\lambda & 2\lambda \end{bmatrix}$$

$$BD\ B^{-1} = \begin{bmatrix} 2\lambda & -m \\ -3\lambda & 2m \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}\frac{1}{\lambda m}\begin{bmatrix} 2m & m \\ 3\lambda & 2\lambda \end{bmatrix}$$

$$= \frac{1}{\lambda m}\begin{bmatrix} 4\lambda & -3m \\ -6\lambda & 6m \end{bmatrix}\begin{bmatrix} 2m & m \\ 3\lambda & 2\lambda \end{bmatrix} \qquad \lambda m \neq 0$$

$$= \frac{1}{\lambda m}\begin{bmatrix} -\lambda m & -2\lambda m \\ 6\lambda m & 6\lambda m \end{bmatrix} = \begin{bmatrix} -1 & -2 \\ 6 & 6 \end{bmatrix} = M$$

$$M = \begin{bmatrix} -1 & -2 \\ 6 & 6 \end{bmatrix}$$

Example 5

Let $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$ prove that $A^2 - 4A - 5\underline{I} = \underline{0}$ Hence obtain A^{-1}.

Solution

$$\text{Let } A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix}$$

$$A^2 - 4A - 5\underline{I} = \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix} - \begin{bmatrix} 4 & 8 & 8 \\ 8 & 4 & 8 \\ 8 & 8 & 4 \end{bmatrix} - \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$= \underline{0}$$

$$A^2 - 4A - 5\underline{I} = \underline{0} \quad (1)$$

$(1)\times A^{-1}$

$$A^2A^{-1} - 4AA^{-1} - 5\underline{I}A^{-1} = \underline{0}\, A^{-1}$$

$$AAA^{-1} - 4\underline{I} - 5A^{-1} = \underline{0}$$

$$A - 4\underline{I} - 5A^{-1} = \underline{0}$$

$$A^{-1} = \frac{1}{5}\left[A - 4\underline{I}\right] = \frac{1}{5}\left\{\begin{vmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{vmatrix} - \begin{vmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{vmatrix}\right\}$$

$$= \frac{1}{5}\begin{vmatrix} -3 & 2 & 2 \\ 2 & -3 & 2 \\ 2 & 2 & -3 \end{vmatrix}$$

Example 6

Let $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$ verify that $A^3 - 6A^2 + 9A - 4\underline{I} = \underline{0}$ and hence find A^{-1}.

Solution

$$A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix}$$

$$A^3 - 6A^2 + 9A - 4\underline{I}$$

$$= \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix} - 6\begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} + 9\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$- 4\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$A^3 - 6A^2 + 9A - 4\underline{I} = \underline{0}$ (1)

(1)$\times A^{-1}$

$$A^3A^{-1} - 6A^2A^{-1} + 9AA^{-1} - 4\underline{I}\,A^{-1} = \underline{0}\,\underline{A}^{-1}$$

$$\underline{A}^2 - 6\underline{A}^2 + 9\underline{I} - 4\underline{A}^{-1} = \underline{0}$$

$$4A^{-1} = A^2 - 6A + 9\underline{I}$$

$$A^{-1} = \frac{1}{4}\left\{\begin{vmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{vmatrix} - 6\begin{vmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{vmatrix} + 9\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}\right\}$$

$$A^{-1} = \frac{1}{4}\begin{vmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{vmatrix}$$

Example 7

Show that if $A = \begin{bmatrix} 1 & 3 & 4 \\ 1 & 4 & 5 \\ 2 & 5 & 7 \end{bmatrix}$ then A^{-1} doesn't exist.

Solution

$$A = \begin{bmatrix} 1 & 3 & 4 \\ 1 & 4 & 5 \\ 2 & 5 & 7 \end{bmatrix}$$

$$\det\ A = 1\begin{vmatrix} 4 & 5 \\ 5 & 7 \end{vmatrix} - 3\begin{vmatrix} 1 & 5 \\ 2 & 7 \end{vmatrix} + 4\begin{vmatrix} 1 & 4 \\ 2 & 5 \end{vmatrix}$$

$$= 3 - 3(-3) + 4 - 12$$

$$= 0$$

$\therefore \det A = 0$ $\qquad\qquad$ $\therefore A^{-1}$ doesn't exist

Example 8

If $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$ verify that $A^3 = A^{-1}$.

Solution

1. $A = \begin{pmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{pmatrix}$

$$A^2 = \begin{pmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & -4 & 4 \\ 0 & -1 & 0 \\ -2 & 2 & -3 \end{pmatrix}$$

$$A^3 = \begin{pmatrix} 3 & -4 & 4 \\ 0 & -1 & 0 \\ -2 & 2 & -3 \end{pmatrix}\begin{pmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{pmatrix}$$

$$A^3 A = \begin{pmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{pmatrix}\begin{pmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$A^3 A = \underline{I}$ $\qquad\qquad$ $\therefore\ A^{-1} = A^3$

Example 9

Find a 2×2 matrix N such that

a) $N = \begin{bmatrix} -7 & 4 \\ 5 & -8 \end{bmatrix} = \begin{bmatrix} 36 & 0 \\ 0 & 36 \end{bmatrix}$

b) $\begin{bmatrix} 5 & 4 \\ 1 & 1 \end{bmatrix} N = \begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix}$

c) $\begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix} N \begin{bmatrix} -1 & 1 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 0 & 4 \end{bmatrix}$

Solutions

a) $N \begin{bmatrix} -7 & 4 \\ 5 & -8 \end{bmatrix} = \begin{bmatrix} 36 & 0 \\ 0 & 36 \end{bmatrix}$

Let $A = \begin{bmatrix} -7 & 4 \\ 5 & -8 \end{bmatrix}$

$\therefore\ A^{-1} = \frac{1}{56-20} \begin{bmatrix} -8 & -4 \\ -5 & -7 \end{bmatrix}$

$= \frac{1}{36} \begin{bmatrix} -8 & -5 \\ -4 & -7 \end{bmatrix}$

$\therefore N\ A = \begin{bmatrix} 36 & 0 \\ 0 & 36 \end{bmatrix}$ ——— (1)

(1)$\times\ A^{-1}$

$N\ A\ A^{-1} = \begin{bmatrix} 36 & 0 \\ 0 & 36 \end{bmatrix} \begin{bmatrix} \frac{-8}{36} & \frac{-4}{36} \\ \frac{-5}{36} & \frac{-7}{36} \end{bmatrix}$

$N\ \underline{I} = \begin{bmatrix} -8 & -4 \\ -5 & -7 \end{bmatrix}$

$N\ = \begin{bmatrix} -8 & -4 \\ -5 & -7 \end{bmatrix}$

b) $\begin{bmatrix} 5 & 4 \\ 1 & 1 \end{bmatrix} N = \begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix}$

$A = \begin{bmatrix} 5 & 4 \\ 1 & 1 \end{bmatrix}$ $\qquad A^{-1} = \frac{1}{5-4} \begin{bmatrix} 1 & -4 \\ -1 & 5 \end{bmatrix} = \begin{bmatrix} 1 & -4 \\ -1 & 5 \end{bmatrix}$

$$A\,N = \begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix} \qquad (1)$$

$A^{-1} \times (1)$

$$A^{-1}A\,N = \begin{bmatrix} 1 & -4 \\ -1 & 5 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix}$$

$$\underline{I}\,N = \begin{bmatrix} 1 & -4 \\ -1 & 5 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix}$$

$$N = \begin{bmatrix} -3 & -14 \\ 4 & 17 \end{bmatrix}$$

c) $\begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix} N \begin{bmatrix} -1 & 1 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 0 & 4 \end{bmatrix}$

$$A = \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix} \qquad A^{-1} = \frac{1}{15-14}\begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix} = \begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix}$$

$$B = \begin{bmatrix} -1 & 1 \\ -2 & 1 \end{bmatrix} \qquad B^{-1} = \frac{1}{-1+2}\begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}$$

$$ABN = \begin{bmatrix} 2 & -1 \\ 0 & 4 \end{bmatrix} \qquad (1)$$

$$A^{-1} \times (1) \times B^{-1}$$

$$A^{-1}A\,N\,B\,B^{-1} = \begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}$$

$$IN\,I = \begin{bmatrix} 10 & -13 \\ -14 & 19 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}$$

$$N = \begin{bmatrix} -16 & 3 \\ -24 & -5 \end{bmatrix}$$

Activity 1

1. Show that $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$ is the inverse of

   $B = \begin{bmatrix} \frac{-3}{11} & \frac{4}{11} & \frac{5}{11} \\ \frac{9}{11} & \frac{-1}{11} & \frac{-4}{11} \\ \frac{5}{11} & \frac{-3}{11} & \frac{-1}{11} \end{bmatrix}$

2. Find the inverse of the matrix $A = \begin{bmatrix} a & b \\ c & \frac{1+bc}{a} \end{bmatrix}$

   and show that

$aA^{-1} = (a^2 + bc + 1)I - aA$

Hence deduce that A^{-1},where $A = \begin{bmatrix} 2 & 3 \\ 4 & \frac{13}{2} \end{bmatrix}$

3. If $A = \begin{pmatrix} 4 & 7 \\ 1 & 2 \end{pmatrix}$, $B = \begin{pmatrix} -5 & -14 \\ 4 & 10 \end{pmatrix}$ and $D = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$ show that $B = A^{-1}D\,\underline{A}$; Show also that $B^4 = A^{-1}D^4A$;

If $P = \begin{pmatrix} 4\lambda & 7m \\ \lambda & 2m \end{pmatrix}$ where $\lambda \neq 0$.Prove that $B = P^{-1}D^4P$.

4. Find A^2 where $A = \begin{pmatrix} 2 & -1 \\ -5 & 3 \end{pmatrix}$ Find the values of x and y such that $A^2 + xA + y\underline{\ \ }I_2 = \underline{0}$. Use this relation to obtain A^{-1} and A^3
5. If $A = \frac{1}{9}\begin{vmatrix} -8 & 1 & 4 \\ 4 & 4 & 7 \\ 1 & -8 & 4 \end{vmatrix}$ prove that $A^{-1} = A^T$
6. Show that the matrix $A = \begin{bmatrix} 1 & 0 & -2 \\ -2 & -1 & 2 \\ 3 & 4 & 1 \end{bmatrix}$

   satisfies the equation $A^3 - A^2 - 3A - I = \underline{0}$.

   Hence find A^{-1}
7. Find the matrix X satisfing the matrix equation

$\begin{bmatrix} 2 & 5 \\ -3 & 1 \end{bmatrix} X \begin{bmatrix} 2 & -2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

8. Find a 2×2 matrix A such that $A = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$

9. If $A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ show that $A^2 = A^{-1}$

Solutions of Activities

Activity 1

4.

$$A^2 = \begin{pmatrix} 9 & -5 \\ -25 & 14 \end{pmatrix} \qquad x = -5 \qquad y = 1$$

$$A^{-1} = \begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix} \qquad A^3 = \begin{pmatrix} 43 & -24 \\ -120 & 67 \end{pmatrix}$$

6.

$$\begin{bmatrix} -9 & -8 & -2 \\ 8 & 7 & 2 \\ -5 & -4 & -1 \end{bmatrix}$$

7.

$$X = \frac{1}{136}\begin{bmatrix} 8 & -8 \\ 7 & 10 \end{bmatrix}$$

8.

$$X = \begin{bmatrix} -16 & -6 \\ 7 & 2 \end{bmatrix}$$

Summary

Let A be a square matrix such that $\det A = \underline{0}$, If there exists a matrix B such that $AB = BA = I$ then $\underline{B}$ is defined as the inverse matrix of A

Learning Outcomes

At the end of this session you will be able to

- define on Inverse matrix
- Determini whether a given matrix has an inverse matrix or not
- Find the inverse matrix of a given matrix which has an inverse matrix

Session 27

Application of Determinant and Matrices

Introduction

In this session, we discuss the applications of determinants and Matrices in finding the area of a triangle and in solving a system of linear equations.

27.1 Area of a Triangle

Let PQR is a triangle whose vertices are $\equiv (x_1, y_1)$, $Q \equiv (x_2, y_2)$, $R \equiv (x_3, y_3)$ is given by area

$$\Delta = \frac{1}{2}\left[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\right]$$

The above expression is the expansion of determinant

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Thus, the area of the triangle with vertices

$P \equiv (x_1, y_1)$, $Q \equiv (x_2, y_2)$, and $R \equiv (x_3, y_3)$ is

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Note that we always take the absolute value of the above determinant for the area since the area is always a positive quantity.

Condition of Co-linearity of three points

Let $P \equiv (x_1, y_1)$ $Q \equiv (x_2, y_2)$, and $R \equiv (x_3, y_3)$ be three points. Then P, Q, R are co-linear and hence the area of the triangle $PQR = 0$

$$\therefore \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

$$\begin{pmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{pmatrix} = 0$$

Example 1

1)

a) Find the area of the triangle whose vertices are $A \equiv (2,3)$ $B \equiv (-3,-2)$, and $C \equiv (1,8)$

b) For what value of , the points $K \equiv (\lambda, 1)$ $L \equiv (3,-4)$, and $N \equiv (4,-5)$ are collinear?

Solution

a) The area $ABC = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$

$$= \frac{1}{2}\begin{vmatrix} 2 & 3 & 1 \\ -3 & -2 & 1 \\ 1 & 8 & 1 \end{vmatrix}$$

$$= \frac{1}{2}|2(-2-8) + 3(3-8) + 1(3+2)|$$

$$= \frac{1}{2}|-20 - 15 + 5|$$

$$= \frac{1}{2}[-30]$$

$$= 15\ sq\ units$$

b) If KLM are co-linear then

$$\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

$$\frac{1}{2}\begin{vmatrix} \lambda & 1 & 1 \\ 3 & -4 & 1 \\ 4 & -5 & 1 \end{vmatrix} = 0$$

$$\lambda(-4+5) - 3(1+5) + 4(1+4) = 0$$

$$\lambda = 18 - 20 = -2$$

$$\therefore \lambda = -2$$

Activity 1

a) Find the area of the triangle whose vertices are $X \equiv (3,8),\ Y \equiv (-4,2)$ and $Z \equiv (5,-1)$

b) Show the following three points are co-linear $A \equiv (1,1),\ B \equiv (-5,-7)$ and C$\equiv (-8,-11)$

c) For what values of m the points $C \equiv (3,0),\ D \equiv (-2,5)$ and $E \equiv (-3,m)$ are co-linear.

d) If the points $A \equiv (a,b)$, $B \equiv (c,d)$ and $C \equiv (a-c, b-d)$ are co-linear, then show that $ad = cb$

27.2 Cramer's Rule

Let us solve the following equations

$$a_1x + b_1y + c_1z = d_1$$

$$a_2x + b_2y + c_2z = d_2$$

$$a_3x + b_3y + c_3z = d_3$$

Let $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} D_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$

$$D_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix} D_3 = \begin{vmatrix} a_1 & b_1 & d \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

∴ By Cramer's Rule $x = \frac{D_1}{D}$

$y = \frac{D_2}{D}$

$x = \frac{D_3}{D}$

Without proof we can use this result.

Cramer's Rule can be applied for a system of linear equations which has n number of equation containing n number of unknowns.

Example 2

Solve the following equation system $2x - 3y + 4z = 9$

$$-3x + 4y + 2z = 124x - 2y - 3z = 3$$

Solution

$$2x - 3y + 4z = 9$$

$$-3x + 4y + 2z = 12$$

$$4x - 2y - 3z = 3$$

$$\begin{bmatrix} 2 & -3 & 4 \\ -3 & 4 & 2 \\ 4 & -2 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 12 \\ 3 \end{bmatrix}$$

$$D = \begin{vmatrix} 2 & -3 & 4 \\ -3 & 4 & 2 \\ 4 & -2 & -3 \end{vmatrix} = 2\begin{vmatrix} 4 & 2 \\ -2 & -3 \end{vmatrix} + 3\begin{vmatrix} -3 & 2 \\ 4 & -3 \end{vmatrix} + 4\begin{vmatrix} -3 & 4 \\ 4 & -2 \end{vmatrix}$$

$$= 2[-12 + 4] + 3[9 - 8] + 4[6 - 16]$$

$$= 2 \times -8 + 3 - 40$$

$$= -16 + 3 - 40 = -53$$

$$D_1 = \begin{vmatrix} 9 & -3 & 4 \\ 12 & 4 & 2 \\ 3 & -2 & -3 \end{vmatrix} = 9\begin{vmatrix} 4 & 2 \\ -2 & -3 \end{vmatrix} + 3\begin{vmatrix} 12 & 2 \\ 3 & -3 \end{vmatrix} + 4\begin{vmatrix} 12 & 4 \\ 3 & -2 \end{vmatrix}$$

$$= 9(-8) + 3(-42) + 4(-36)$$

$$= -[72 + 126 + 144] = -342$$

$$D_2 = \begin{vmatrix} 2 & 9 & 4 \\ -3 & 12 & 2 \\ 4 & 3 & -3 \end{vmatrix} = 2\begin{vmatrix} 12 & 2 \\ 3 & -3 \end{vmatrix} - 9\begin{vmatrix} -3 & 2 \\ 4 & -3 \end{vmatrix} + 4\begin{vmatrix} -3 & 12 \\ 4 & 3 \end{vmatrix}$$

$$= 2(-42) - 9(1) + 4(-57)$$

$$= -321$$

$$D_3 = \begin{vmatrix} 2 & -3 & 9 \\ -3 & 4 & 12 \\ 4 & -2 & 3 \end{vmatrix} = 2\begin{vmatrix} 4 & 12 \\ -2 & 3 \end{vmatrix} + 3\begin{vmatrix} -3 & 12 \\ 4 & 3 \end{vmatrix} + 9\begin{vmatrix} -3 & 4 \\ 4 & -2 \end{vmatrix}$$

$$= 2.36 + 3(-57) + 9(-10) = -189$$

$$\therefore\ x = \frac{D_1}{D} = \frac{-342}{-53} = \frac{342}{53}$$

$$y = \frac{D_2}{D} = \frac{-321}{-53} = \frac{321}{53}$$

$$x = \frac{D_3}{D} = \frac{-189}{-53} = \frac{189}{53}$$

Activity 2

Using Cramer's Rule, solve the following system of equation

i. $3x - 4y = 1$

$-2x + 5y = -3$

ii. $6x + y - 32 = 5$

$x + 3y - 2z = 5$

$2x + y + 4z = 8$

iii. $x + y = 5$

$y + z = 3$

$x + z = 4$

Example 3

Using determinants, show that the following system of equation is inconsistent

$3x - y + 2z = 3$

$2x + y + 3z = 5$

$$x - 2y - z = 1$$

Solution

$3x - y + 2z = 3$

$2x + y + 3z = 5$

$x - 2y - z = 1$

$$D = \begin{vmatrix} 3 & -1 & 2 \\ 2 & 1 & 3 \\ 1 & -2 & -1 \end{vmatrix} = 3(5) + 1(-5) + 2(-5) = 0$$

$$D_1 = \begin{vmatrix} 3 & -1 & 2 \\ 5 & 1 & 3 \\ 1 & -2 & -1 \end{vmatrix} = 3(-1 + 6) + 1(-5 - 3) + 2(-10 - 1)$$

$$= 15 - 8 - 22 = 15$$

$D_1 \neq 0$ But $D = 0$

$\therefore$ The given system of equation is inconsistent.

Example 4.2.3

Solve the following system of equations by Cramer's Rule

$$x - y + 3z = 6$$
$$x + 3y - 3z = -4$$
$$5x + 3y + 3z = 10$$

Solution

$$x - y + 3z = 6$$
$$x + 3y - 3z = -4$$
$$5x + 3y + 3z = 10$$

$$D = \begin{vmatrix} 1 & -1 & 3 \\ 1 & 3 & -3 \\ 5 & 3 & 3 \end{vmatrix} = 1(18) + 1(18) + 3(-12) = 0$$

$$D_1 = \begin{vmatrix} 6 & -1 & 3 \\ -4 & 3 & -3 \\ 10 & 3 & 3 \end{vmatrix} = 6(18) + 1(18) + 3(-42) = 0$$

$$D_2 = \begin{vmatrix} 1 & 6 & 3 \\ 1 & -4 & -3 \\ 5 & 10 & 3 \end{vmatrix} = 1(18) - 6(18) + 3(30) = 0$$

$$D_3 = \begin{vmatrix} 1 & -1 & 6 \\ 1 & 3 & -4 \\ 5 & 3 & 10 \end{vmatrix} = 1(42) + 1(30) + 6(-12) = 0$$

$$D = D_1 = D_2 = D_3 = 0$$

Thus, the given system of linear equation is consistent and has infinite solution.

Putting $z = \lambda$ and considering first two equations we get $x - y = 6 - 3\lambda x + 3y = -4 + 3\lambda$

$$D = \begin{vmatrix} 1 & -1 \\ 1 & 3 \end{vmatrix} = 4$$

$$D_1 = \begin{vmatrix} 6 - 3\lambda & -1 \\ -4 + 3\lambda & 3 \end{vmatrix} = 18 - 9\lambda - 4 + 3\lambda = 14 - 6\lambda$$

$$D_2 = \begin{vmatrix} 1 & 6 - 3\lambda \\ 1 & -4 + 3\lambda \end{vmatrix} = -4 + 3\lambda - 6 + 3\lambda = -10 + 6\lambda$$

$$\therefore x = \frac{D_1}{D} = \frac{14 - 6\lambda}{4} = \frac{7 - 3\lambda}{2}$$

$$y = \frac{D_2}{D} = \frac{-10 + 6\lambda}{4} = \frac{-5 + 3\lambda}{2}$$

By giving arbitrary values to λ, we find that the given system has infinite number of solutions.

Activity 3

Classify the following system of equations as consistent or inconsistent. If consistent, then solve them

1) $x + y + 3z = 6$
   $x - 3y + 3z = -4$
   $5x - 3y + 3z = 8$
2) $2x - y + z = 4$
   $x + 3y + 2z = 12$
   $3x + 2y + 3z = 10$
3) $x - y + 3z = 6$
   $x + 3y - 3z = -4$

$5x + 3y + 3z = 14$

4) $2x + 5y - z = 9$

$3x - 3y + 2z = 7$

$2x - 4y + 3z = 1$

5) $2x - y + 2 = 4$

$x + 3y + z = 12$

$3x + 2y + 3z = 10$

27.3 System of Homogeneous Linear Equation

In the system of equations

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_3z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

If we consider $d_1 = d_2 = d_3 = 0$

$$a_1x + b_1y + c_1z = 0$$
$$a_2x + b_2y + c_3z = 0$$
$$a_3x + b_3y + c_3z = 0$$

This system of equations is said to be 'Homogeneous'. In this if $D \neq 0$ then its solution is $x = y = z = 0$ called the trivial solution. If $D = 0$, then it has a non-trivial solution [non zero solution].

Example 4

Solve the following systems of homogeneous linear equations.

i. $4x + 3y - 2z = 0$

$$6x + 3y + z = 0$$
$$x + y + z = 0$$

ii. $5x + 2y - z = 0$

$$3x + y + z = 0$$
$$13x + 7y - z = 0$$

Solution

i. $4x + 3y - 2z = 0$

$$6x + 3y + z = 0$$

$$x + y + z = 0$$

$$D = \begin{vmatrix} 4 & 3 & -2 \\ 6 & 3 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 4\begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix} - 3\begin{vmatrix} 6 & 1 \\ 1 & 1 \end{vmatrix} - 2\begin{vmatrix} 6 & 3 \\ 1 & 1 \end{vmatrix}$$

$$= 4(3-1) - 3(6-1) - 2(6-3)$$

$$= 8 - 15 - 6$$

$$= -13 \therefore D \neq 0$$

The system has trivial solution $x = y = z = 0$.

iii. $5x + 2y - z = 0$

$$3x + y + z = 0$$

$$13x + 7y - z = 0$$

$$D = \begin{vmatrix} 5 & 2 & -1 \\ 3 & 1 & 1 \\ 13 & 7 & -1 \end{vmatrix} = 5\begin{vmatrix} 1 & 1 \\ 7 & -1 \end{vmatrix} - 2\begin{vmatrix} 3 & 1 \\ 13 & -1 \end{vmatrix} - 1\begin{vmatrix} 3 & 1 \\ 13 & 7 \end{vmatrix}$$

$$= -40 + 32 + 8 = 0$$

$$D = 0$$

The given system has infinitely many nontrivial solutions.

Now, Let$z = \lambda$ be any real number

Then the first two equations

$$\frac{5x}{z} + \frac{2y}{z} = 1$$

$$\frac{3x}{z} + \frac{y}{z} = 1$$

$$\frac{5x}{\lambda} + \frac{2y}{\lambda} = 1$$

$$\frac{3x}{\lambda} + \frac{y}{\lambda} = 1$$

$$D = \begin{bmatrix} \frac{5}{\lambda} & \frac{2}{\lambda} \\ \frac{3}{\lambda} & \frac{1}{\lambda} \end{bmatrix} = \frac{5}{\lambda}.\frac{1}{\lambda} - \frac{2}{\lambda}.\frac{3}{\lambda} = -\frac{1}{\lambda^2}$$

$D \neq 0$

$$D_1 = \begin{bmatrix} 1 & \frac{2}{\lambda} \\ 1 & \frac{1}{\lambda} \end{bmatrix} = \frac{1}{\lambda} - \frac{2}{\lambda} = -\frac{1}{\lambda}$$

$$D_2 = \begin{bmatrix} \frac{5}{\lambda} & 1 \\ \frac{3}{\lambda} & 1 \end{bmatrix} = \frac{5}{\lambda} - \frac{3}{\lambda} = \frac{2}{\lambda}$$

$$x = \frac{D_1}{D} = \frac{-\frac{1}{\lambda}}{-\frac{1}{\lambda^2}} = \lambda$$

$$y = \frac{D_2}{D} = \frac{\frac{2}{\lambda}}{-\frac{1}{\lambda^2}} = -2\lambda$$

$\therefore$

$x = \lambda\,, y = -2\lambda, z = \lambda$ where λ is any real number

Activity 4

i. If the following system of equations has a non-trivial solution, then find the value of λ.

i.

$x + 2y + 3z = 0$

$3x - 2y + z = 0$

$\lambda x - 14y + 15z = 0$

ii. Solve the following systems of equations by using Cramer's Rule

a) $x - 3y = 0$

$2x - 3y - z = 0$

$x + 3y - 2z = 0$

b) $3x - 4y + 52 = 0$

$x + y - 2z = 0$

$2x + 3y + z = 0$

27.4 Solution of the Linear Equation System by using the method of Matrices inverse

Consider a system of a linear equation

$$\begin{array}{cccccc} a_{11}x_1 & a_{12}x_2 + & . & . & . & a_{1n}x_n = & b_1 \\ a_{21}x_2 & a_{22}x_2 & . & . & . & a_{2n}x_n = & b_2 \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ a_{n1}x_1 & a_{n2}x_2 & . & . & . & a_{nn}x_n = & b_n \end{array}$$

We can express the above system of linear equations as the following matrix form.

$$\begin{bmatrix} a_{11} & a_{12} + & . & . & . & a_{1n} \\ a_{21} & a_{22} & . & . & . & a_{2n} \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ a_{n1} & a_{n2} & . & . & . & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ . \\ . \\ . \\ b_n \end{bmatrix}$$

$\underline{AX} = \underline{B}$

Let $|\underline{A}| \neq 0$ so that $\underline{A}^{-1}$exists.

Multiplying both side of $\underline{AX} = \underline{B}$ by $\underline{A}^{-1}$

$$\therefore \underline{A}^{-1}\underline{AX} = \underline{A}^{-1}\underline{B}$$

$I\underline{X} = \underline{A}^{-1}\underline{B}$

$\underline{X} = \underline{A}^{-1}\underline{B}$

Where $\underline{A} = \begin{bmatrix} a_{11} & a_{12} + & . & . & . & a_{1n} \\ a_{21} & a_{22} & . & . & . & a_{2n} \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ a_{n1} & a_{n2} & . & . & . & a_{nn} \end{bmatrix}$

$\underline{B} = \begin{bmatrix} b_1 \\ b_2 \\ . \\ . \\ . \\ b_n \end{bmatrix}$ and $\underline{X} = \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_n \end{bmatrix}$

Hence, $\underline{X} = \underline{A}^{-1}\underline{B}$ is the unique solution of the system$\underline{AX} = \underline{B}; |\underline{A}| \neq 0$

Criterion of Consistency

Let $\underline{AX} = \underline{B}$ be a system of a linear equations in *n* variables.

i. If $|\underline{A}| \neq 0$, then the system of equations is consistent and has a unique solution given by $\underline{X} = \underline{A}^{-1}\underline{B}$.

ii. If $|\underline{A}| = 0$ then there are two cases.

   a) The system of equations is consistent and has infinitely many solutions.

   b) The system of equation is inconsistent and it has no solutions.

Example 5

Solve the following system of equations by finding the inverse of the matrix

a) $3x + 2y = 7$
   $5x - 3y = 1$

b) $4x + 2y = 3$
   $3x - 4y = 5$

c) $3x - y - 2z = 3$
   $2x + 3y + 3z = 5$
   $5x + 2y + z = 8$

d) $3x + 2y - 5z = 2$
   $5x + y + z = 4$
   $8x + 3y - 4z = 8$

Solution

a) $3x + 2y = 7$
   $5x - 3y = 1$

$$\underline{A} = \begin{bmatrix} 3 & 2 \\ 5 & -3 \end{bmatrix} \therefore \begin{bmatrix} 3 & 2 \\ 5 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$$

$\underline{AX} = \underline{B} \longleftarrow (1)$ where $\underline{B} = \begin{bmatrix} 7 \\ 1 \end{bmatrix} \underline{X} = \begin{bmatrix} x \\ y \end{bmatrix}$

$$\underline{A}^{-1} = \frac{1}{(3 \times -3 - 5 \times 2)} \begin{bmatrix} -3 & -2 \\ -5 & 3 \end{bmatrix} = \begin{bmatrix} \frac{3}{19} & \frac{2}{19} \\ \frac{5}{19} & \frac{-3}{19} \end{bmatrix}$$

$\underline{A}^{-1}$exists.

$\underline{AX} = \underline{B}$

$\underline{A}^{-1}\underline{AX} = \underline{A}^{-1}\underline{B}$

$\underline{IX} = \underline{A}^{-1}\underline{B}$

$$\underline{X} = \begin{bmatrix} \frac{3}{19} & \frac{2}{19} \\ \frac{5}{19} & \frac{-3}{19} \end{bmatrix} \begin{bmatrix} 7 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{21}{19} + \frac{8}{19} \\ \frac{35}{19} + \frac{-3}{19} \end{bmatrix} = \begin{bmatrix} \frac{29}{19} \\ \frac{32}{19} \end{bmatrix}$$

$x = \frac{29}{19}, \quad y = \frac{32}{19}$

This is the unique solution of the system.

b) $4x + 2y = 3$

$3x - 4y = 5$

$$A = \begin{bmatrix} 4 & 2 \\ 3 & -4 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix} \underline{X} = \begin{bmatrix} x \\ y \end{bmatrix} \underline{B} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

$$\underline{AX} = \underline{B} \longleftarrow (1)$$

$$\underline{A} = \begin{bmatrix} 4 & 2 \\ 3 & -4 \end{bmatrix} \therefore \underline{A}^{-1} = \frac{1}{4 \times -4 - 3 \times 2} \begin{bmatrix} -4 & -2 \\ -3 & 4 \end{bmatrix}$$

$$\underline{A}^{-1} = \frac{-1}{20} \begin{bmatrix} -4 & -2 \\ -3 & 4 \end{bmatrix}$$

$$\underline{A}^{-1} = \begin{bmatrix} -\frac{1}{5} & \frac{1}{10} \\ \frac{3}{20} & \frac{-1}{5} \end{bmatrix}$$

$\underline{A}^{-1}.(1)$

$\underline{A}^{-1}\underline{AX} = \underline{A}^{-1}\underline{B}$

$\underline{IX} = \underline{A}^{-1}\underline{B}$

$$\underline{X} = \begin{bmatrix} -\frac{1}{5} & \frac{1}{10} \\ \frac{3}{20} & \frac{-1}{5} \end{bmatrix} \begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} \frac{-3}{5} + \frac{5}{10} \\ \frac{9}{20} - 1 \end{bmatrix}$$

$$\underline{X} = \begin{bmatrix} \frac{-1}{10} \\ \frac{11}{20} \end{bmatrix}$$

$x = \frac{-1}{10}$, $y = \frac{11}{20}$ is the unique solution.

c) $3x - y - 2z = 3$

$2x + 3y + 3z = 5$

$5x + 2y + z = 8$

$$\underline{A} = \begin{bmatrix} 3 & -1 & -2 \\ 2 & 3 & 3 \\ 5 & 2 & 1 \end{bmatrix} \underline{X} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \underline{B} = \begin{bmatrix} 3 \\ 5 \\ 8 \end{bmatrix}$$

$\underline{AX} = \underline{B}$

$$\underline{A} = \begin{bmatrix} 3 & -1 & -2 \\ 2 & 3 & 3 \\ 5 & 2 & 1 \end{bmatrix}$$

$$\det \underline{A} = 3\begin{vmatrix} 3 & 3 \\ 2 & 1 \end{vmatrix} + 1\begin{vmatrix} 2 & 3 \\ 5 & 1 \end{vmatrix} - 2\begin{vmatrix} 2 & 3 \\ 5 & 2 \end{vmatrix}$$

$$= 3(3-6) + (2-15) - 2(4-15)$$

$$= -9 - 13 + 22 = 0$$

$\det \underline{A} = 0$

According to the properties of the determinant, we have the one of the row of the determinant some combination of the other two.

We have $R_3 \equiv R_1 + R_2$

$\therefore$ If the system has infinite solution the right hand of the third equation has the above combination.

$8 = 3 + 5$

$\therefore$ Therefore this system has only two independent equations and third one is some combination of the other two.

$3x - y - 2z = 3$

$2x + 3y + 3z = 5$

When $z = \lambda$

$3x - y = 3 + 2\lambda$

$2x + 3y = 5 - 3\lambda$

Now $\underline{A} = \begin{bmatrix} 3 & -1 \\ 2 & 3 \end{bmatrix} \underline{B} = \begin{bmatrix} 3 + 2\lambda \\ 5 - 3\lambda \end{bmatrix}$

$\begin{bmatrix} 3 & -1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 + 2\lambda \\ 5 - 3\lambda \end{bmatrix}$

$$\underline{A} = \begin{bmatrix} 3 & -1 \\ 2 & 3 \end{bmatrix} \therefore \underline{A}^{-1} = \frac{1}{3 \times 3 - 2 \times -1} \begin{bmatrix} 3 & 1 \\ -2 & 3 \end{bmatrix}$$

$$\underline{A}^{-1} = \begin{bmatrix} \frac{3}{11} & \frac{1}{11} \\ \frac{-2}{11} & \frac{3}{11} \end{bmatrix}$$

$$\underline{AX} = \underline{B} \Longrightarrow \underline{X} = \underline{A}^{-1}\underline{B} = \begin{bmatrix} \frac{3}{11} & \frac{1}{11} \\ \frac{-2}{11} & \frac{3}{11} \end{bmatrix} = \begin{bmatrix} 3 + 2\lambda \\ 5 - 3\lambda \end{bmatrix}$$

$$\underline{X} = \begin{bmatrix} \frac{3}{11}(3 + 2\lambda) + \frac{1}{11}(5 - 3\lambda) \\ \frac{-2}{11}(3 + 2\lambda) + \frac{3}{11}(5 - 3\lambda) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{(4 + 3\lambda)}{11} \\ \frac{(9 - 13\lambda)}{11} \end{bmatrix}$$

$$x = \frac{(4 + 3\lambda)}{11}$$

$$y = \frac{(9 - 13\lambda)}{11}$$

In the above solutions, λ is any real number

d) $3x + 2y - 5z = 2$

$5x + y + z = 4$

$8x + 3y - 4z = 8$

$$\underline{A} = \begin{bmatrix} 3 & 2 & -5 \\ 5 & 1 & 1 \\ 8 & 3 & -4 \end{bmatrix} \underline{X} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \underline{B} = \begin{bmatrix} 2 \\ 4 \\ 8 \end{bmatrix}$$

$\underline{AX} = \underline{B}$

$$\det \underline{A} = \begin{vmatrix} 3 & 2 & -5 \\ 5 & 1 & 1 \\ 8 & 3 & -4 \end{vmatrix} = 3\begin{vmatrix} 1 & 1 \\ 3 & -4 \end{vmatrix} - 2\begin{vmatrix} 5 & 1 \\ 8 & -4 \end{vmatrix} - 5\begin{vmatrix} 5 & 1 \\ 8 & 3 \end{vmatrix}$$

$$= 3(-7) - 2(-28) - 5 \times 7$$

$$= 0$$

Since $\det \underline{A} = 0$

Then we know that one of row of $\det \underline{A}$ is some combination of the other two.

$$R_3 = R_1 + R_2$$

But right hand of the equation $8 \neq 2 + 4$

∴ This system has no solution.

Example 6

Find $\underline{AB}$ where $\underline{A} = \begin{bmatrix} -5 & 1 & 3 \\ 7 & 1 & -5 \\ 1 & -1 & 1 \end{bmatrix} \underline{B} = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix}$

Hence solve

$$x + y - 2z = 1$$

$3x + 2y + z = 1$

$2x + y + 3z = 2$

Solution

$$\underline{A} = \begin{bmatrix} -5 & 1 & 3 \\ 7 & 1 & -5 \\ 1 & -1 & 1 \end{bmatrix} \underline{B} = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix}$$

$$\underline{AB} = \begin{bmatrix} -5 & 1 & 3 \\ 7 & 1 & -5 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} -5+3+6 & -5+2+3 & -10+1+9 \\ 7+3-10 & 7+2-5 & 14+1-15 \\ 1-3+2 & 1-2+1 & 2-1+3 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} = 4\underline{I}$$

$$\frac{1}{4}\underline{A}.\underline{B} = \underline{I}$$

$$\therefore \underline{B}^{-1} = \frac{1}{4}\underline{A} = \begin{bmatrix} \frac{-5}{4} & \frac{1}{4} & \frac{3}{4} \\ \frac{7}{4} & \frac{1}{4} & -\frac{5}{4} \\ \frac{1}{4} & \frac{-1}{4} & \frac{1}{4} \end{bmatrix}$$

$$x + y - 2z = 1$$

$$3x + 2y + z = 1$$

$$2x + y + 3z = 2$$

$$\begin{bmatrix} 1 & 1 & -2 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

$$\underline{BX} = \underline{C}$$

$$\underline{B}^{-1}\underline{BX} = \underline{B}^{-1}\underline{C}$$

$$\underline{X} = \begin{bmatrix} \frac{-5}{4} & \frac{1}{4} & \frac{3}{4} \\ \frac{7}{4} & \frac{1}{4} & -\frac{5}{4} \\ \frac{1}{4} & \frac{-1}{4} & \frac{1}{4} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 8 \\ 4 \\ -4 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$$

$$x = 2, y = 1\,, z = -1$$

Activity 5

(1) Which of the following equations are consistent, find the solutions.

(i) $3x + 2y = 2$

$2x + 3y = 5$

(ii) $3x + 6y = 15$

$x + 2y = 5$

(iii) $4x - 2y = 7$

$2x - y = 5$

(2). Find the solutions of the following system of equations.

(i) $x + 2y - 2z = -1$

$2x + y - z = 2$

$x - y + z = 3$

(ii) $3x + 2y + 3z = 13$

$x + y + z = 6$

$5x + 4y + 5z = 25$

(3).Solve the following system of equations by using matrix method

(i) $7x + 3y = 5$

$5x + 2y = 4$

(ii) $2x - y = -2$

$3x + 4y = 3$

(iii) $3x - 5y = 7$

$4x - 3y = 3$

(iv) $3\mathrm{x} + 2\mathrm{y} = 5$

$3x + 2y = 3$

(v) $2x + y = 5$

$-5x + 2y = -8$

(4).

If $A = \begin{bmatrix} 1 & -2 & 0 \\ 2 & 1 & 3 \\ 0 & -2 & 1 \end{bmatrix} B = \begin{bmatrix} 7 & 2 & -6 \\ -2 & 1 & -3 \\ -4 & 2 & 5 \end{bmatrix}$

Find $\underline{A}.\underline{B}$ and deduce A^{-1}

And hence solve $x - 2y = 10$

$2x + y + 3z = 8$

$-2y + z = 7$

(5).Find the product of matrices $\underset{\sim}{A} = \begin{bmatrix} 2 & 3 & 4 \\ 5 & 4 & -6 \\ 3 & -2 & -2 \end{bmatrix}$ $\underset{\sim}{B} = \begin{bmatrix} 20 & 2 & 34 \\ 8 & 16 & -32 \\ 22 & -13 & 7 \end{bmatrix}$ and hence solve the the following system of equations.

$$\frac{2}{x} + \frac{3}{y} + \frac{4}{z} = -3$$

$$\frac{5}{x} + \frac{4}{y} + \frac{6}{z} = 4$$

$$\frac{3}{x} - \frac{2}{y} - \frac{2}{z} = 6$$

(6).

If $\underset{\sim}{A} = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

Show that $\underset{\sim}{A}^3 - 6\underset{\sim}{A}^2 + 9\underset{\sim}{A} - 4\underset{\sim}{I} = \underset{\sim}{0}$

And hence find A^{-1}.

Solve the following system of linear equations

$$2x - y + z = 6$$

$$-x + 2y - z = -5$$

$$x - y + 2z = 5$$

(7).

Find the product of matrices $\begin{bmatrix} 2 & 3 & 10 \\ 4 & -6 & 5 \\ 6 & 9 & -20 \end{bmatrix}$ and $\begin{bmatrix} 75 & 150 & 75 \\ 110 & -100 & 30 \\ 72 & 0 & -24 \end{bmatrix}$. Using the above product solve the system of equation

$$\frac{2}{x} + \frac{3}{y} + \frac{10}{z} = 4$$

$$\frac{4}{x} - \frac{6}{y} + \frac{5}{z} = 1$$

$$\frac{6}{x} + \frac{9}{y} - \frac{20}{z} = 2$$

Solutions of Activities

Activity 1

(a) $\frac{75}{2}$ sq units

(c) 6

Activity 2

(i) $x = -1,\ y = -1$

(ii) $x = 1,\ y = 2,\ z = 1$

(iii) $x = 3, y = 2,\ z = 1$

Activity 3

(1) Inconsistent

(2) Consistent $z = \lambda, x = \frac{24-5\lambda}{7}, y = \frac{20-3\lambda}{7}$

(3) Consistent $z = \lambda, x = \frac{7-8\lambda}{2}, y = \frac{3\lambda-5}{2}$

(4) Consistent $x = \frac{79}{21}, y = \frac{-4}{21}, z = \frac{-n}{7}$

(5) Inconsistent

Activity 4

(i) $\lambda = 29$

(ii) (a)$x = \lambda, y = \frac{\lambda}{3}, z = \lambda$

λ is any real number.

(b) $x = y = z = 0$

Activity 5

(1)

(i) Consistent, $x = {}^{-4}/_{5}, y = {}^{11}/_{5}$

(ii) Consistent, has infinite number of solution, $y = \lambda, x = 5 - 2\lambda$

(iii) Inconsistent

(2) (i) The solution has infinite number of solutions, $x = {}^{5}/_{3}, y = \lambda - {}^{5}/_{3}, z = \lambda$ for any real number λ.

(iii) The system is inconsistent, no solution.

(3) (i) $x = 2, y = -3$ (ii) $x = {}^{-5}/_{11}, y = {}^{12}/_{11}$

(iv) $x = {}^{-6}/_{5}, y = {}^{-19}/_{11}$ (iv) $x = -1, y = 4$

(v) $x = 2, y = 1$

Summary

The area of the triangle ABC, where $A \equiv (x_a, y_a), B \equiv (x_b, y_b)$ and $C \equiv (x_c, y_c)$ is given by $\frac{1}{2}\begin{vmatrix} x_a & y_a & 1 \\ x_b & y_b & 1 \\ x_c & y_c & 1 \end{vmatrix}$

Cramer's rule for solving a system of simultaneous linear equations

Consider the system of linear equations

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \cdots \\ b_n \end{bmatrix}$$

Let $\underline{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \underline{X} = \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix} \underline{B} = \begin{bmatrix} b_1 \\ b_2 \\ \cdots \\ b_n \end{bmatrix}$

Here $\underline{A}$ is $n \times n$ matrix and suppose that $\det \underline{A} \neq 0$

Let $\underline{A}^k$ is the matrix obtained by replacing the k^{th} column of the matrix $\underline{A}$ by the column vector $\underline{B}$.

The solution of x_k is given by

$x_k = \frac{\det \underline{A}^{(k)}}{\det \underline{A}}$ for $k = 1,2,\ldots,n$.

If $\det \underline{A} = \det \underline{A}^1 = \det \underline{A}^2 = \cdots = \det \underline{A}^n = 0$

Thus the given system of equations is consistence and has infinite number of solution.

If $\det \underline{A} = \det \underline{A}^1 = \det \underline{A}^2 = \cdots = \det \underline{A}^n$ are not zero, Then the given system is consistent.

If $b_1 = b_2 = \cdots b_n = 0$

Then the above $\underline{AX} = \underline{B}\ \rightarrow \underline{AX} = \underline{0}$is called homogeneous linear equation.

If $\det \underline{A} \neq 0$, then its only solution is $x = y = z = 0$ called the trivial solution.

If $\det \underline{A} = 0$ then it has a non trivial solution, (non zero solution)

Again of the system of equation

$$\underline{AX} = \underline{B}$$

Let $\det \underline{A} \neq 0$ then $\underline{A}^{-1}$exists uniquely.

Solution $\underline{X} = \underline{A}^{-1}\underline{B}$

If $|\underline{A}| = 0$ then there are two cases;

(a) The system of equations is consistence and has infinitely many solutions.
(b) The system of equations is inconsistent it has no solution.

Learning Outcomes

At the end of this study session you should be able to

- Find the area of the triangle ABO whose vertices are $A \equiv (x_a, y_a) B \equiv (x_b, y_b)$, $C = (x_c, y_c)$ by using determinants.
- Solve a system of linear equations by using Cramer's rule.
- Solve the system of linear equation by using the inverse of matrix.