# UNIT 3
# TRIGONOMETRY

**Contents:**

## Unit Introduction

The literal meaning of Trigonometry is “triangle measurement”. Trigonometry mainly involves study of triangles. The good knowledge of trigonometry is essential to you when learn Mathematics course related Engineering field.

The branch of Mathematics which treats of the relations of the sides and the angles of triangles, plane figures which the methods of the deducting from certain given parts other required is the meaning of Trigonometry. Without the knowledge of trigonometry, you cannot solve most of the problems related statics and dynamics.

In this we will learn very basic concepts in the Trigonometry.

In this unit there are two numbers of sessions

(1) Trigonometric Functions and Identities.

(2) Solution of Trigonometric Equation and Triangles.

# Session 11

# Trigonometric Functions and Identities



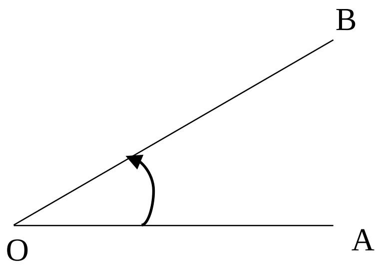
## Introduction

In this session, you can learn the basic concepts of trigonometry, namely measurement of angles. We define the trigonometric ratios for any angle as circular functions. We discuss the fundamental trigonometric identities. Trigonometric ratios of sum and difference of two angles, product formula of trigonometric ratios) Pythagorean identities are presented in detail. We provide the formulae for trigonometric value of addition of two angles and subtraction of two angles. We express the trigonometric value of a double angle in terms of trigonometric of the angle. We introduce formulae for addition and subtraction of sine values, cosine values and tangent values of any two angles. Using above formulae, we prove some trigonometric identities.

# 11.1 Measurements of Angle

## Idea of an angle

An angle is defined as being generated by rotating a half line about its end point from an initial position to terminal position. A half line is that portion of a line to one side of a fixed point on a line. We call the initial position of the half-line as the initial side, the terminal position of the half line is the terminal side, and the fixed point is the vertex.




O - Fixed Point
OA - Initial Side
OB - Terminal Side

*Figure 11.1.1*

If the rotation of the terminal side from the initial side is counter-clockwise, then the angle is said to be positive.

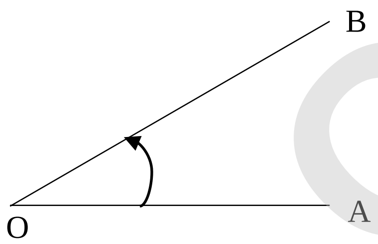



*Figure 11.1.2 Positive Angle*

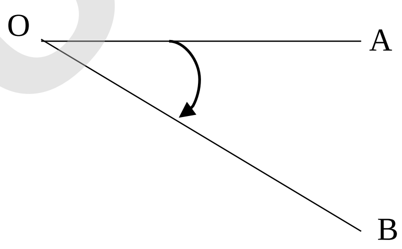



*Figure 11.1.3 Negative angle*

If the rotation of the terminal side from the initial side is clockwise then, the angle is negative.

## The measurement of degree

During the early time, mathematicians divided a circle into $360$ equal parts. The angle subtended at the center by each part was called a degree that is denoted by $1°$. Also, we can define a degree as the $\frac{1}{360}^{th}$ of a complete rotation.

The degree is divided into 60 equal parts called “minute”. The minute is divided into 60 equal parts called” second”. The symbols °, ‘ and ” used to denote degrees, minutes and seconds respectively.

*Example 1*

(a) Convert the angle $65^0 20' 25''$ to decimal of degrees form

(b) Convert the angle $68.3053^0$ to minutes, seconds form

*Solution*

(a) $\theta = 65^0\ 20'\ 25''$

$$25'' = \frac{25'}{60}$$

$$20'25'' = \left(20 + \frac{25'}{60}\right)' = \left(20 + \frac{5}{12}\right)' = \left(\frac{245}{12}\right)'$$

$$= \frac{\frac{245}{12}}{60} = \frac{245}{720} = 0.3403^0$$

$$\theta = 65.3404^0$$

(b) $\theta = 68.3053^0$

$$0.3053^0 = 0.3053 \times 60'$$

$$= 18.318'$$

$$= 18' + 0.318'$$

$$= 18' + 0.318 \times 60''$$

$$= 18' + 19.08''$$

$$\theta = 68^0 18'\ 19.08''$$

## Measurement of Radian

Consider a circle with the center $O$ andthe radius $r$ as shown in figure 11.1.4
Let $\widehat{AB}$ be an arc on the circle of length equal to $r$.
We define the magnitude of the angle $AOB$ of which the arc $\widehat{AB}$ subtends at the centre $O$ as one radian.

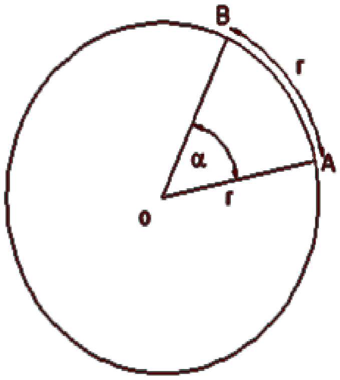



*Figure 11.1.4*

Since the circumference of a circle is equal to $2\pi r$, it subtends a center angle of $2\pi$.

$\therefore 2\pi\, rad \quad = 360^0$

$\pi\, rad \ = 180^0$

$1\, rad \ = \frac{180^0}{\pi}$

$1\, rad \ = 57.2958^0$

$1\, rad \ = 57^0 17'\, 45''$

$1^0 \quad = \frac{\pi}{180}\, rad$

$1^0 \quad = 0.01745\, rad$

## Activity 1

(1) Convert the given angles to equal angles expressed in decimal form

(a) $323^0 48'$ (b) $85^0 3' 15''$

(2) Convert the given angles to equal angles expressed to the nearest minute

(a) $235.8°$ (b) $29.75°$

(3) Complete the following table

| Angle (in degree) | Angle (in radian) |
|---|---|
| $180^0$ | $\pi\ rad$ |
| $10^0$ | $^{\pi}/_{18}\ rad$ |
| | $^{\pi}/_{6}\ rad$ |
| $45^0$ | |
| $60^0$ | |
| $67\ {^1/_2}^0$ | |
| | $^{\pi}/_{2}\ rad$ |
| | $^{5\pi}/_{8}\ rad$ |
| | $^{2\pi}/_{3}\ rad$ |
| $150^0$ | |
| $165^0$ | |
| | $^{7\pi}/_{6}\ rad$ |
| | $^{4\pi}/_{3}\ rad$ |
| $270^0$ | |
| $330^0$ | |
| $345^0$ | |
| $360^0$ | |

## 11.2 Definition of Trigonometric Ratios of an Angle

Figure 11.2.1 shows a circle of unit radius with its center at the origin on a $xy$ plane. The radius $OP$ rotates through an angle $\theta$ from the $x$-axis.

We agree that $\theta$ is positive if it is generated in the anti-clockwise sense and negative otherwise.

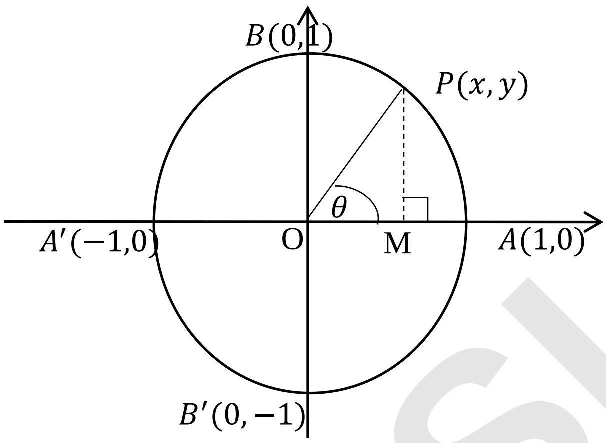



*Figure 11.2.1*

Let $(x, y)$ be the coordinates of the point $P$. Notice that $x$ and $y$ exist for every $\theta \in \mathbb{R}$, and there is only one value of $x$ and only one value of $y$ for each $\theta \in \mathbb{R}$. We define two circular functions (Ratios) "Sine" and "Cosine" of the angle $\theta$ as follows.

Sine: $\theta \to y$ where $\theta \in \mathbb{R}$

$\therefore \sin\theta = y$

Coine: $\theta \to x$ where $\theta \in \mathbb{R}$

$\therefore \cos\theta = x$

Looking at the unit circle of figure 11.2.1 it is clear that

$-1 \leq x \leq 1$ and $-1 \leq y \leq 1$

$\therefore -1 \leq \cos\theta \leq 1$ and $-1 \leq \sin\theta \leq 1$

**Tangent Function (Ratio)**

tan: $\theta \to \frac{\sin\theta}{\cos\theta}$ where $\theta \in \mathbb{R}$, and $\cos\theta \neq 0$

$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{y}{x}$, $x \neq 0$

**Secant Function (Ratio)**

$\sec\colon \theta \to \dfrac{1}{\cos\theta}$ where $\theta \in \mathbb{R}$, and $\cos\theta \neq 0$

$\sec\theta = \dfrac{1}{\cos\theta} = \dfrac{1}{x},\ x \neq 0$

**Cosecant Function (Ratio)**

$\text{cosec}\colon \theta \to \dfrac{1}{\sin\theta}$ where $\theta \in \mathbb{R}$, and $\sin\theta \neq 0$

$\text{cosec}\,\theta = \dfrac{1}{\sin\theta} = \dfrac{1}{y},\ y \neq 0$

**Cotangent Function (Ratio)**

$\cot\colon \theta \to \dfrac{1}{\tan\theta}$ where $\theta \in \mathbb{R}$, and $\tan\theta \neq 0$

$$\cot\theta = \frac{1}{\tan\theta}$$

$\cot\theta = \dfrac{\cos\theta}{\sin\theta},\ \sin\theta \neq 0$

$\cot\theta = \dfrac{x}{y},\ y \neq 0$

The signs of each trigonometric function and its reciprocal in different quadrants are shown below.

| | |
|---|---|
| 2nd Quadrant<br>${}^{\pi}/_{2} \leq \theta \leq \pi;\ 90^0 \leq \theta \leq 180^0$<br>$\sin\theta\ (+)$<br>$\text{cosec}\,\theta\ (+)$<br>Others $(-)$ | 1st Quadrant<br>$0 \leq \theta \leq {}^{\pi}/_{2};\ 0^0 \leq \theta \leq 90^0$<br>All trigonometric functions $(+)$ |
| $\tan\theta\ (+)$<br>$\cot\theta\ (+)$<br>Others $(-)$<br>$\pi \leq \theta \leq {}^{3\pi}/_{2};\ 180^0 \leq \theta \leq 270^0$<br>3rd Quadrant | $\cos\theta\ (+)$<br>$\sec\theta\ (+)$<br>Others $(-)$<br>${}^{3\pi}/_{2} \leq \theta \leq 2\pi;\ 270^0 \leq \theta \leq 360^0$<br>4th Quadrant |

Although there are tables to find the values of six trigonometric ratios, it is useful to memorize the values of these ratios of frequently encountered angles as shown in the table 11.2.1,

| Angle | | $\sin\theta$ | $\cos\theta$ | $\tan\theta$ |
|---|---|---|---|---|
| Degree | Radian | | | |
| $0^0$ | $0$ | $0$ | $1$ | $0$ |
| $30^0$ | $\pi/6$ | $1/2$ | $\sqrt{3}/2$ | $1/\sqrt{3}$ |
| $45^0$ | $\pi/4$ | $1/\sqrt{2}$ | $1/\sqrt{2}$ | $1$ |
| $60^0$ | $\pi/3$ | $\sqrt{3}/2$ | $1/2$ | $\sqrt{3}$ |
| $90^0$ | $\pi/2$ | $1$ | $0$ | (not define) |
| $120^0$ | $2\pi/3$ | $\sqrt{3}/2$ | $-1/2$ | $-\sqrt{3}$ |
| $135^0$ | $3\pi/4$ | $1/\sqrt{2}$ | $-1/\sqrt{2}$ | $-1$ |
| $150^0$ | $5\pi/6$ | $1/2$ | $-\sqrt{3}/2$ | $-1/\sqrt{3}$ |
| $180^0$ | $\pi$ | $0$ | $-1$ | $0$ |
| $210^0$ | $7\pi/6$ | $-1/2$ | $-\sqrt{3}/2$ | $1/\sqrt{3}$ |
| $225^0$ | $5\pi/4$ | $-1/\sqrt{2}$ | $-1/\sqrt{2}$ | $-1$ |
| $240^0$ | $4\pi/3$ | $-\sqrt{3}/2$ | $-1/2$ | $\sqrt{3}$ |
| $270^0$ | $3\pi/2$ | $-1$ | $0$ | (not define) |
| $300^0$ | $5\pi/3$ | $-\sqrt{3}/2$ | $1/2$ | $-\sqrt{3}$ |
| $315^0$ | $7\pi/4$ | $-1/\sqrt{2}$ | $1/\sqrt{2}$ | $-1$ |
| $330^0$ | $11\pi/6$ | $-1/2$ | $\sqrt{3}/2$ | $-1/\sqrt{3}$ |
| $360^0$ | $2\pi$ | $0$ | $1$ | $0$ |

*Table 11.2.1*

Note that the corresponding values of cotangent, secant and cosecant ratios are not included in the table as they can be easily obtained by taking respectively the reciprocals of the values of tangent, cosine and sine ratios.

## 11.3 Trigonometric Ratios for Supplementary Angle

By using the definitions of the trigonometric ratios, we can obtain the following results.

(a) Expressions for trigonometric ratios of the angle $\left(\frac{\pi}{2} - \theta\right)$ or $(90 - \theta)^0$ in terms of $\theta$

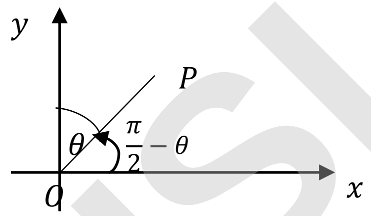



*Figure 11.3.1*

$\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$;$\sin(90^0 - \theta) = \cos\theta$

$\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$; $\cos(90^0 - \theta) = \sin\theta$

$\tan\left(\frac{\pi}{2} - \theta\right) = \cot\theta$; $\tan(90^0 - \theta) = \cot\theta$

$\text{cosec}\left(\frac{\pi}{2} - \theta\right) = \sec\theta$; $\text{cosec}(90^0 - \theta) = \sec\theta$

$\sec\left(\frac{\pi}{2} - \theta\right) = \text{cosec}\,\theta$; $\sec(90^0 - \theta) = \text{cosec}\,\theta$

$\cot\left(\frac{\pi}{2} - \theta\right) = \tan\theta$; $\cot(90 - \theta) = \tan\theta$

Example: When $\theta = {}^{\pi}/_{12}$; $\theta = 15°$

$\sin\left({}^{\pi}/_{2} - {}^{\pi}/_{12}\right) = \sin\left({}^{5\pi}/_{12}\right) = \cos\left({}^{\pi}/_{12}\right)$

$\sin(90 - 15)° = \sin 75° = \cos 15°$

$\cos\left(\pi/2 - \pi/12\right) = \cos\left(5\pi/12\right) = \sin\left(\pi/12\right)$

$\cos(90 - 15)^\circ = \cos 75^\circ = \sin 15^\circ$

$\tan\left(\pi/2 - \pi/12\right) = \tan\left(5\pi/12\right) = \cot\left(\pi/12\right)$

$\tan(90 - 15)^\circ = \tan 75^\circ = \cot 15^\circ$

$\operatorname{cosec}\left(\pi/2 - \pi/12\right) = \operatorname{cosec}\left(5\pi/12\right) = \sec\left(\pi/12\right)$

$\operatorname{cosec}(90 - 15)^\circ = \operatorname{cosec} 75^\circ = \sec 15^\circ$

$\sec\left(\pi/2 - \pi/12\right) = \sec\left(5\pi/12\right) = \operatorname{cosec}\left(\pi/12\right)$

$\sec(90 - 15)^\circ = \sec 75^\circ = \operatorname{cosec} 15^\circ$

$\cot\left(\pi/2 - \pi/12\right) = \cot\left(5\pi/12\right) = \tan\left(\pi/12\right)$

$$\cot(90 - 15)^\circ = \cot 75^\circ = \tan 15^\circ$$

(b) Expressions for trigonometric ratios of the angle $\left(\pi/2 + \theta\right)$ or $(90 + \theta)^0$ in terms of $\theta$

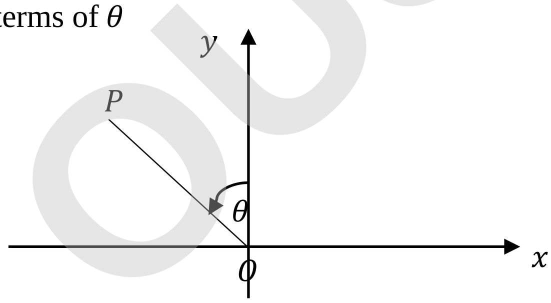



*Figure 11.3.2*

$\sin\left(\pi/2 + \theta\right) = \cos\theta$; $\quad \sin(90 + \theta) = \cos\theta$
$\cos\left(\pi/2 + \theta\right) = -\sin\theta$; $\quad \cos(90 + \theta) = -\sin\theta$
$\tan\left(\pi/2 + \theta\right) = -\cot\theta$; $\quad \tan(90 + \theta) = -\cot\theta$
$\operatorname{cosec}\left(\pi/2 + \theta\right) = \sec\theta$; $\quad \operatorname{cosec}(90 + \theta) = \sec\theta$
$\sec\left(\pi/2 + \theta\right) = -\operatorname{cosec}\theta$; $\quad \sec(90 + \theta) = -\operatorname{cosec}\theta$
$\cot\left(\pi/2 + \theta\right) = -\tan\theta$; $\quad \cot(90 + \theta) = -\tan\theta$

Example: When $\theta = {}^{\pi}/_{5}$; $\theta = 36°$

$$\sin\left({}^{\pi}/_{2} + {}^{\pi}/_{5}\right) = \sin\left({}^{7\pi}/_{10}\right) = \cos\left({}^{\pi}/_{5}\right)$$

$$\sin(90 + 36)° = \sin 126° = \cos 36°$$

$$\cos\left({}^{\pi}/_{2} + {}^{\pi}/_{5}\right) = \cos\left({}^{7\pi}/_{10}\right) = -\sin\left({}^{\pi}/_{5}\right)$$

$$\cos(90 + 36)° = \cos 126° = -\sin 36°$$

$$\tan\left({}^{\pi}/_{2} + {}^{\pi}/_{5}\right) = \tan\left({}^{7\pi}/_{10}\right) = -\cot\left({}^{\pi}/_{5}\right)$$

$$\tan(90 + 36)° = \tan 126° = -\cot 36°$$

$$\operatorname{cosec}\left({}^{\pi}/_{2} + {}^{\pi}/_{5}\right) = \operatorname{cosec}\left({}^{7\pi}/_{10}\right) = \sec\left({}^{\pi}/_{5}\right)$$

$$\operatorname{cosec}(90 + 36)° = \operatorname{cosec} 126° = \sec 36°$$

$$\sec\left({}^{\pi}/_{2} + {}^{\pi}/_{5}\right) = \sec\left({}^{7\pi}/_{10}\right) = -\operatorname{cosec}\left({}^{\pi}/_{5}\right)$$

$$\sec(90 + 36)° = \sec 126° = -\operatorname{cosec} 36°$$

$$\cot\left({}^{\pi}/_{2} + {}^{\pi}/_{5}\right) = \cot\left({}^{7\pi}/_{10}\right) = -\tan\left({}^{\pi}/_{5}\right)$$

$$\cot(90 + 36)° = \cot 126° = -\tan 36°$$



(c) Expressions for trigonometric ratios of the angle $(\pi - \theta)$or$(180 - \theta)^0$in terms of $\theta$

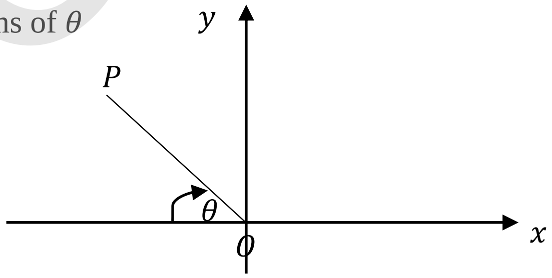



*Figure 11.3.3*

| | |
|---|---|
| $\sin(\pi - \theta) = \sin\theta$; | $\sin(180^0 - \theta) = \sin\theta$ |
| $\cos(\pi - \theta) = -\cos\theta$; | $\cos(180^0 - \theta) = -\cos\theta$ |
| $\tan(\pi - \theta) = -\tan\theta$; | $\tan(180^0 - \theta) = -\tan\theta$ |
| $\operatorname{cosec}(\pi - \theta) = \operatorname{cosec}\theta$; | $\operatorname{cosec}(180^0 - \theta) = \operatorname{cosec}\theta$ |
| $\sec(\pi - \theta) = -\sec\theta$; | $\sec(180^0 - \theta) = -\sec\theta$ |
| $\cot(\pi - \theta) = -\cot\theta$; | $\cot(180^0 - \theta) = -\cot\theta$ |

Example: When $\theta = {}^{3\pi}/_{10}$; $\theta = 54°$

$\sin\left(\pi - {}^{3\pi}/_{10}\right) = \sin\left({}^{7\pi}/_{10}\right) = \sin\left({}^{3\pi}/_{10}\right)$

$\sin(180 - 54)° = \sin 126° = \sin 54°$

$\cos\left(\pi - {}^{3\pi}/_{10}\right) = \cos\left({}^{7\pi}/_{10}\right) = -\cos\left({}^{3\pi}/_{10}\right)$

$\cos(180 - 54)° = \cos 126° = -\cos 54°$

$\tan\left(\pi - {}^{3\pi}/_{10}\right) = \tan\left({}^{7\pi}/_{10}\right) = -\tan\left({}^{3\pi}/_{10}\right)$

$\tan(180 - 54)° = \tan 126° = -\tan 54°$

$\mathrm{cosec}\left(\pi - {}^{3\pi}/_{10}\right) = \mathrm{cosec}\left({}^{7\pi}/_{10}\right) = \mathrm{cosec}\left({}^{3\pi}/_{10}\right)$

$\mathrm{cosec}(180 - 54)° = \mathrm{cosec}\, 126° = \mathrm{cosec}\, 54°$

$\sec\left(\pi - {}^{3\pi}/_{10}\right) = \sec\left({}^{7\pi}/_{10}\right) = -\sec\left({}^{3\pi}/_{10}\right)$

$\sec(180 - 54)° = \sec 126° = -\sec 54°$

$\cot\left(\pi - {}^{3\pi}/_{10}\right) = \cot\left({}^{7\pi}/_{10}\right) = -\cot\left({}^{3\pi}/_{10}\right)$

$\cot(180 - 54)° = \cot 126° = -\cot 54°$

(d) Expressions for trigonometric ratios of the angle $(\pi + \theta)$or $(180 + \theta)^0$ in terms of $\theta$

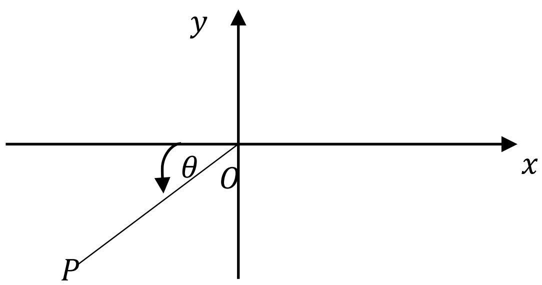



*Figure 11.3.4*

$\sin(\pi + \theta) = -\sin\theta$; $\sin(180^0 + \theta) =$
$-\sin\theta$
$\cos(\pi + \theta) = -\cos\theta$; $\cos(180^0 + \theta) =$
$-\cos\theta$
$\tan(\pi + \theta) = \tan\theta$; $\tan(180^0 + \theta) = \tan\theta$
$\mathrm{cosec}(\pi + \theta) = -\mathrm{cosec}\,\theta$; $\mathrm{cosec}(180^0 + \theta) =$
$-\mathrm{cosec}\,\theta$
$\sec(\pi + \theta) = -\sec\theta$; $\sec(180^0 + \theta) =$

Example: When $\theta = {}^{\pi}/_{10}$; $\theta = 18°$

$$\sin\left(\pi + {}^{\pi}/_{10}\right) = \sin\left({}^{11\pi}/_{10}\right) = -\sin\left({}^{\pi}/_{10}\right)$$

$$\sin(180 + 18)° = \sin 198° = -\sin 18°$$

$$\cos\left(\pi + {}^{\pi}/_{10}\right) = \cos\left({}^{11\pi}/_{10}\right) = -\cos\left({}^{\pi}/_{10}\right)$$

$$\cos(180 + 18)° = \cos 198° = -\cos 18°$$

$$\tan\left(\pi + {}^{\pi}/_{10}\right) = \tan\left({}^{11\pi}/_{10}\right) = \tan\left({}^{\pi}/_{10}\right)$$

$$\tan(180 + 18)° = \tan 198° = \tan 18°$$

$$\mathrm{cosec}\left(\pi + {}^{\pi}/_{10}\right) = \mathrm{cosec}\left({}^{11\pi}/_{10}\right) = -\mathrm{cosec}\left({}^{\pi}/_{10}\right)$$

$$\mathrm{cosec}(180 + 18)° = \mathrm{cosec}\,198° = -\mathrm{cosec}\,18°$$

$$\sec\left(\pi + {}^{\pi}/_{10}\right) = \sec\left({}^{11\pi}/_{10}\right) = -\sec\left({}^{\pi}/_{10}\right)$$

$$\sec(180 + 18)° = \sec 198° = -\sec 18°$$

$$\cot\left(\pi + {}^{\pi}/_{10}\right) = \cot\left({}^{11\pi}/_{10}\right) = \cot\left({}^{\pi}/_{10}\right)$$

$$\cot(180 + 18)° = \cot 198° = \cot 18°$$

(e) Expressions for trigonometric ratios of the angle $\left({}^{3\pi}/_{2} - \theta\right)$ or $(270 - \theta)^0$ in terms of $\theta$.

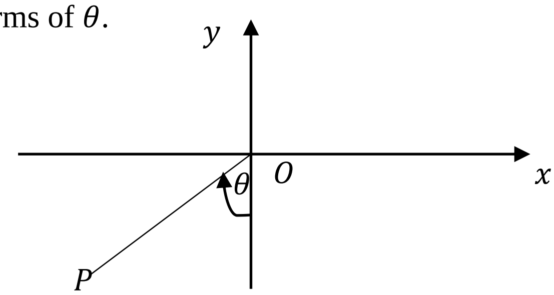



*Figure 11.3.5*

$\sin\left(\frac{3\pi}{2} - \theta\right) = -\cos\theta$; $\sin(270^0 - \theta) = -\cos\theta$

$\cos\left(\frac{3\pi}{2} - \theta\right) = -\sin\theta$; $\cos(270^0 - \theta) = -\sin\theta$

$\tan\left(\frac{3\pi}{2} - \theta\right) = \cot\theta$; $\tan(270^0 - \theta) = \cot\theta$

$\operatorname{cosec}\left(\frac{3\pi}{2} - \theta\right) = -\sec\theta$; $\operatorname{cosec}(270^0 - \theta) = -\sec\theta$

$\sec\left(\frac{3\pi}{2} - \theta\right) = -\operatorname{cosec}\theta$; $\sec(270^0 - \theta) = -\operatorname{cosec}\theta$

$\cot\left(\frac{3\pi}{2} - \theta\right) = \tan\theta$; $\cot(270^0 - \theta) = \tan\theta$

Example: When $\theta = {}^{2\pi}/_{5}$; $\theta = 72°$

$$\sin\left(\frac{3\pi}{2} - {}^{2\pi}/_{5}\right) = \sin\left({}^{11\pi}/_{10}\right) = -\cos\left({}^{2\pi}/_{5}\right)$$

$$\sin(270 - 72)° = \sin 198° = -\cos 72°$$

$$\cos\left(\frac{3\pi}{2} - {}^{2\pi}/_{5}\right) = \cos\left({}^{11\pi}/_{10}\right) = -\sin\left({}^{2\pi}/_{5}\right)$$

$$\cos(270 - 72)° = \cos 198° = -\sin 72°$$

$$\tan\left(\frac{3\pi}{2} - {}^{2\pi}/_{5}\right) = \tan\left({}^{11\pi}/_{10}\right) = \cot\left({}^{2\pi}/_{5}\right)$$

$$\tan(270 - 72)° = \tan 198° = \cot 72°$$

$$\operatorname{cosec}\left(\frac{3\pi}{2} - {}^{2\pi}/_{5}\right) = \operatorname{cosec}\left({}^{11\pi}/_{10}\right) = -\sec\left({}^{2\pi}/_{5}\right)$$

$$\operatorname{cosec}(270 - 72)° = \operatorname{cosec} 198° = -\sec 72°$$

$$\sec\left(\frac{3\pi}{2} - {}^{2\pi}/_{5}\right) = \sec\left({}^{11\pi}/_{10}\right) = -\operatorname{cosec}\left({}^{2\pi}/_{5}\right)$$

$$\sec(270 - 72)° = \sec 198° = -\operatorname{cosec} 72°$$

$$\cot\left(\frac{3\pi}{2} - {}^{2\pi}/_{5}\right) = \cot\left({}^{11\pi}/_{10}\right) = \tan\left({}^{2\pi}/_{5}\right)$$

$$\cot(270 - 72)° = \cot 198° = \tan 72°$$

(f) Expressions for trigonometric ratios of the angle $\left({}^{3\pi}/_{2} + \theta\right)$ or $(270 + \theta)^0$ in terms of $\theta$.

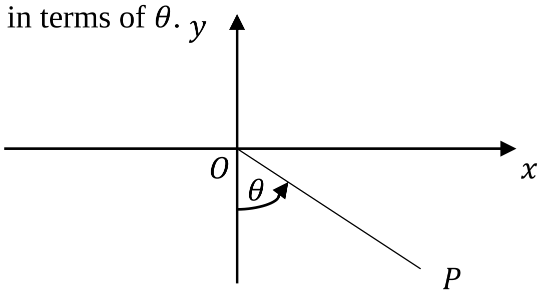



*Figure 11.3.6*

| | |
|---|---|
| $\sin\left(\frac{3\pi}{2} + \theta\right) = -\cos\theta;$ | $\sin(270^0 + \theta) = -\cos\theta$ |
| $\cos\left(\frac{3\pi}{2} + \theta\right) = \sin\theta;$ | $\cos(270^0 + \theta) = \sin\theta$ |
| $\tan\left(\frac{3\pi}{2} + \theta\right) = -\cot\theta;$ | $\tan(270^0 + \theta) = -\cot\theta$ |
| $\text{cosec}\left(\frac{3\pi}{2} + \theta\right) = -\sec\theta;$ | $\text{cosec}(270^0 + \theta) = -\sec\theta$ |
| $\sec\left(\frac{3\pi}{2} + \theta\right) = \text{cosec}\,\theta;$ | $\sec(270^0 + \theta) = \text{cosec}\,\theta$ |
| $\cot\left(\frac{3\pi}{2} + \theta\right) = -\tan\theta;$ | $\cot(270^0 + \theta) = -\tan\theta$ |

Example: When$\theta = {}^{\pi}/_{6}; \theta = 30°$

$$\sin\left(\frac{3\pi}{2} + {}^{\pi}/_{6}\right) = \sin\left({}^{5\pi}/_{3}\right) = -\cos\left({}^{\pi}/_{6}\right)$$

$$\sin(270 + 30)° = \sin 300° = -\cos 30°$$

$$\cos\left(\frac{3\pi}{2} + {}^{\pi}/_{6}\right) = \cos\left({}^{5\pi}/_{3}\right) = \sin\left({}^{\pi}/_{6}\right)$$

$$\cos(270 + 30)° = \cos 300° = \sin 30°$$

$$\tan\left(\frac{3\pi}{2} + {}^{\pi}/_{6}\right) = \tan\left({}^{5\pi}/_{3}\right) = -\cot\left({}^{\pi}/_{6}\right)$$

$$\tan(270 + 30)° = \tan 300° = -\cot 30°$$

$$\text{cosec}\left(\frac{3\pi}{2} + {}^{\pi}/_{6}\right) = \text{cosec}\left({}^{5\pi}/_{3}\right) = -\sec\left({}^{\pi}/_{6}\right)$$

$$\text{cosec}(270 + 30)° = \text{cosec}\, 300° = -\sec 30°$$

$\sec\left(\frac{3\pi}{2} + {}^{\pi}/_{6}\right) = \sec\left({}^{5\pi}/_{3}\right) = \text{cosec}\left({}^{\pi}/_{6}\right)$

$\sec(270 + 30)° = \sec 300° = \text{cosec}\, 30°$

$\cot\left(\frac{3\pi}{2} + {}^{\pi}/_{6}\right) = \cot\left({}^{5\pi}/_{3}\right) = -\tan\left({}^{\pi}/_{6}\right)$

$\cot(270 + 30)° = \cot 300° = -\tan 30°$

(g) Expressions for trigonometric ratios of the angle $(-\theta)\ rad, (-\theta°)$ or $(2\pi - \theta)rad\ (360 - \theta)^0$

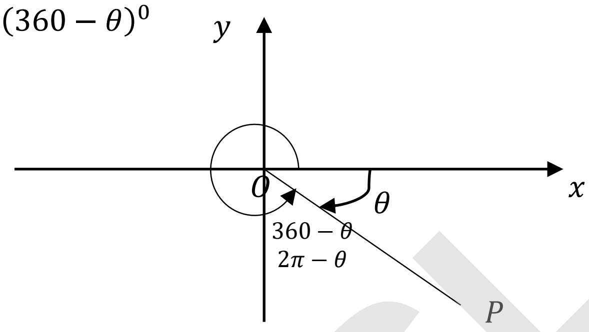



*Figure 11.3.7*

$\sin(-\theta) = \sin(2\pi - \theta) = -\sin\theta$; $\sin(-\theta) = \sin(360^0 - \theta) = -\sin\theta$
$\cos(-\theta) = \cos(2\pi - \theta) = \cos\theta$; $\cos(-\theta) = \cos(360^0 - \theta) = \cos\theta$
$\tan(-\theta) = \tan(2\pi - \theta) = -\tan\theta$; $\tan(-\theta) = \tan(360^0 - \theta) = -\tan\theta$
$\text{cosec}(-\theta) = \text{cosec}(2\pi - \theta) = -\text{cosec}\,\theta$;$\text{cosec}(-\theta) = \text{cosec}(360^0 - \theta) = -\text{cosec}\,\theta$
$\sec(-\theta) = \sec(2\pi - \theta) = \sec\theta$; $\sec(-\theta) = \sec(360^0 - \theta) = \sec\theta$
$\cot(-\theta) = \cot(2\pi - \theta) = -\cot\theta$; $\cot(-\theta) = \cot(360^0 - \theta) = -\cot\theta$

Example: When $\theta = {}^{\pi}/_{3}$; $\theta = 60°$

$\sin\left(-{}^{\pi}/_{3}\right) = \sin\left(2\pi - {}^{\pi}/_{3}\right) = \sin\left({}^{5\pi}/_{3}\right) = -\sin\left({}^{\pi}/_{3}\right)$

$\sin(-60°) = \sin(360 - 60)° = \sin 300° = -\sin 60°$

$\cos\left(-{}^{\pi}/_{3}\right) = \cos\left(2\pi - {}^{\pi}/_{3}\right) = \cos\left({}^{5\pi}/_{3}\right) = \cos\left({}^{\pi}/_{3}\right)$

$\cos(-60°) = \cos(360 - 60)° = \cos 300° = \cos 60°$

$\tan\left(-{}^{\pi}/_{3}\right) = \tan\left(2\pi - {}^{\pi}/_{3}\right) = \tan\left({}^{5\pi}/_{3}\right) = -\tan\left({}^{\pi}/_{3}\right)$

$$\tan(-60°) = \tan(360 - 60)° = \tan 300° = -\tan 60°$$

$$\operatorname{cosec}\left(-{}^{\pi}/_{3}\right) = \operatorname{cosec}\left(2\pi - {}^{\pi}/_{3}\right) = \operatorname{cosec}\left({}^{5\pi}/_{3}\right) = -\operatorname{cosec}\left({}^{\pi}/_{3}\right)$$

$$\operatorname{cosec}(-60°) = \operatorname{cosec}(360 - 60)° = \operatorname{cosec} 300° = -\operatorname{cosec} 60°$$

$$\sec\left(-{}^{\pi}/_{3}\right) = \sec\left(2\pi - {}^{\pi}/_{3}\right) = \sec\left({}^{5\pi}/_{3}\right) = \sec\left({}^{\pi}/_{3}\right)$$

$$\sec(-60°) = \sec(360 - 60)° = \sec 300° = \sec 60°$$

$$\cot\left(-{}^{\pi}/_{3}\right) = \cot\left(2\pi - {}^{\pi}/_{3}\right) = \cot\left({}^{5\pi}/_{3}\right) = -\cot\left({}^{\pi}/_{3}\right)$$

$$\cot(-60°) = \cot(360 - 60)° = \cot 300° = -\cot 60°$$

## 11.4 Trigonometric Ratios of the Sum and Difference of Two Angles

We can obtain the following identities from the basic concepts.

Let $A$ and $B$ are two angles.

$$\sin(A + B) = \sin A \cos B + \cos A \sin B$$

$$\cos(A + B) = \cos A \cos B - \sin A \sin B$$

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$\sin(A - B) = \sin A \cos B - \cos A \sin B$$

$$\cos(A - B) = \cos A \cos B + \sin A \sin B$$

$$\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

*Example 2*

Given that $\cos {}^{\pi}/_{4} = \sin {}^{\pi}/_{4} = {}^{1}/_{\sqrt{2}}$

$$\sin {}^{\pi}/_{3} = \cos {}^{\pi}/_{6} = {}^{\sqrt{3}}/_{2}$$

$$\sin {}^{\pi}/_{6} = \cos {}^{\pi}/_{3} = {}^{1}/_{2}$$

Find the values of $\sin {}^{7\pi}/_{12}$, $\cos {}^{7\pi}/_{12}$, $\sin {}^{\pi}/_{12}$, $\cos {}^{\pi}/_{12}$, $\sin {}^{5\pi}/_{12}$, $\cos {}^{5\pi}/_{12}$, $\tan {}^{\pi}/_{12}$ , $\tan {}^{5\pi}/_{12}$ , $\tan {}^{7\pi}/_{12}$

Let $A = {}^{\pi}/_{4}$, $B = {}^{\pi}/_{6}$

$$A + B = \frac{\pi}{4} + \frac{\pi}{6} = \frac{(3+2)\pi}{12} = \frac{5\pi}{12}$$

$$\sin\frac{5\pi}{12} = \sin\left(\frac{\pi}{4} + \frac{\pi}{6}\right)$$

$$= \sin\frac{\pi}{4}\cos\frac{\pi}{6} + \cos\frac{\pi}{4}\sin\frac{\pi}{6}$$

$$= \frac{1}{\sqrt{2}}.\frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}}.\frac{1}{2}$$

$$= \frac{\sqrt{3}+1}{2\sqrt{2}}$$

$$\cos\frac{5\pi}{12} = \cos\left(\frac{\pi}{4} + \frac{\pi}{6}\right)$$

$$= \cos\frac{\pi}{4}\cos\frac{\pi}{6} - \sin\frac{\pi}{4}\sin\frac{\pi}{6}$$

$$= \frac{1}{\sqrt{2}}.\frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}}.\frac{1}{2}$$

$$= \frac{\sqrt{3}-1}{2\sqrt{2}}$$

$$\tan\frac{\pi}{4} = \frac{\sin {}^{\pi}/_{4}}{\cos {}^{\pi}/_{4}} = \frac{{}^{1}/_{\sqrt{2}}}{{}^{1}/_{\sqrt{2}}} = 1$$

$$\tan\frac{\pi}{6} = \frac{\sin {}^{\pi}/_{6}}{\cos {}^{\pi}/_{6}} = \frac{{}^{1}/_{2}}{{}^{\sqrt{3}}/_{2}} = \frac{1}{\sqrt{3}}$$

$$\tan\frac{5\pi}{12} = \tan\left(\frac{\pi}{4} + \frac{\pi}{6}\right) = \frac{\tan {}^{\pi}/_{4} + \tan {}^{\pi}/_{6}}{1 - \tan {}^{\pi}/_{4}\tan {}^{\pi}/_{6}}$$

$$= \frac{1 + {}^{1}/_{\sqrt{3}}}{1 - {}^{1}/_{\sqrt{3}}} = \frac{\sqrt{3}+1}{\sqrt{3}-1} = \frac{\left(\sqrt{3}+1\right)^2}{\left(\sqrt{3}-1\right)\left(\sqrt{3}+1\right)} = \frac{4+2\sqrt{3}}{3-1} = 2 + \sqrt{3}$$

$$A - B = \frac{\pi}{4} - \frac{\pi}{6} = \frac{\pi}{12}$$

$$\sin\frac{\pi}{12} = \sin\left(\frac{\pi}{4} - \frac{\pi}{6}\right)$$

$$= \sin\frac{\pi}{4}\cos\frac{\pi}{6} - \cos\frac{\pi}{4}\sin\frac{\pi}{6}$$

$$= \frac{1}{\sqrt{2}}.\frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}}.\frac{1}{2}$$

$$= \frac{\sqrt{3}-1}{2\sqrt{2}}$$

$$\cos\frac{\pi}{12} = \cos\left(\frac{\pi}{4} - \frac{\pi}{6}\right)$$

$$= \cos\frac{\pi}{4}\cos\frac{\pi}{6} + \sin\frac{\pi}{4}\sin\frac{\pi}{6}$$

$$= \frac{1}{\sqrt{2}} . \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} . \frac{1}{2} \qquad = \frac{\sqrt{3}+1}{2\sqrt{2}}$$

$$\tan\frac{\pi}{12} = \tan\left(\frac{\pi}{4} - \frac{\pi}{6}\right) = \frac{\tan{}^{\pi}/_{4} - \tan{}^{\pi}/_{6}}{1+\tan{}^{\pi}/_{4}\tan{}^{\pi}/_{6}}$$

$$= \frac{1-{}^{1}/_{\sqrt{3}}}{1+{}^{1}/_{\sqrt{3}}} = \frac{\sqrt{3}-1}{\sqrt{3}+1} = \frac{(\sqrt{3}-1)(\sqrt{3}+1)}{(\sqrt{3}+1)^2} = \frac{3-1}{4+2\sqrt{3}} = \frac{1}{2+\sqrt{3}}$$

$$= \frac{(2-\sqrt{3})}{(2+\sqrt{3})(2-\sqrt{3})}$$

$$= 2 - \sqrt{3}$$

Let $A = {}^{\pi}/_{3}$, $B = {}^{\pi}/_{4}$

$$A + B = \frac{\pi}{3} + \frac{\pi}{4} = \frac{7\pi}{12}$$

$$\sin\frac{7\pi}{12} = \sin\left(\frac{\pi}{3} + \frac{\pi}{4}\right)$$

$$= \sin\frac{\pi}{3}\cos\frac{\pi}{4} + \cos\frac{\pi}{3}\sin\frac{\pi}{4}$$

$$= \frac{\sqrt{3}}{2} . \frac{1}{\sqrt{2}} + \frac{1}{2} . \frac{1}{\sqrt{2}}$$

$$= \frac{\sqrt{3}+1}{2\sqrt{2}}$$

$$\cos\frac{7\pi}{12} = \cos\left(\frac{\pi}{3} + \frac{\pi}{4}\right)$$

$$= \cos\frac{\pi}{3}\cos\frac{\pi}{4} - \sin\frac{\pi}{3}\sin\frac{\pi}{4}$$

$$= \frac{1}{2} . \frac{1}{\sqrt{2}} - \frac{\sqrt{3}}{2}\frac{1}{\sqrt{2}}$$

$$= -\frac{(\sqrt{3}-1)}{2\sqrt{2}}$$

$$\tan\frac{\pi}{3} = \frac{\sin {}^{\pi}/_{3}}{\cos {}^{\pi}/_{3}} = \frac{{}^{\sqrt{3}}/_{2}}{{}^{1}/_{2}} = \sqrt{3}$$

$$\tan\frac{7\pi}{12} = \tan\left(\frac{\pi}{3}+\frac{\pi}{4}\right) = \frac{\tan{}^{\pi}/_{3}+\tan{}^{\pi}/_{4}}{1-\tan{}^{\pi}/_{3}\tan{}^{\pi}/_{4}}$$

$$= \frac{\sqrt{3}+1}{1-\sqrt{3}} = \frac{\left(\sqrt{3}+1\right)^2}{-\left(\sqrt{3}-1\right)\left(\sqrt{3}+1\right)} = \frac{4+2\sqrt{3}}{-2} = -\left(2+\sqrt{3}\right)$$

## Product Formulae

We have $\sin(A+B) = \sin A\cos B + \cos A\sin B \leftarrow \boxed{1}$

$\sin(A-B) = \sin A\cos B - \cos A\sin B \leftarrow \boxed{2}$

$\boxed{1}+\boxed{2}$, $\sin(A+B)+\sin(A-B) = 2\sin A\cos B \leftarrow \boxed{3}$

$\boxed{1}-\boxed{2}$, $\sin(A+B)-\sin(A-B) = 2\cos A\sin B \leftarrow \boxed{4}$

Let $A+B = C$and $A-B = D$

$\therefore A = \frac{C+D}{2},\ B = \frac{C-D}{2}$

From $\boxed{3}$, $\boxed{\sin C+\sin D = 2\sin\frac{C+D}{2}\cos\frac{C-D}{2}}$

From $\boxed{4}$, $\boxed{\sin C-\sin D = 2\cos\frac{C+D}{2}\sin\frac{C-D}{2}}$Also,

we have $\cos(A+B) = \cos A\cos B - \sin A\sin B \leftarrow \boxed{5}$

$\cos(A-B) = \cos A\cos B + \sin A\sin B \leftarrow \boxed{6}$

$\boxed{5}+\boxed{6}$ $\cos(A+B)+\cos(A-B) = 2\cos A\cos B \leftarrow \boxed{7}$

$\boxed{6}-\boxed{5}$ $\cos(A-B)-\cos(A+B) = 2\sin A\sin B \leftarrow \boxed{8}$

Let $A+B = C$ and $A-B = D$

From $\boxed{7}$, $\boxed{\cos C+\cos D = 2\cos\frac{C+D}{2}\cos\frac{C-D}{2}}$

From $\boxed{8}$, $\boxed{\cos D-\cos C = 2\sin\frac{C+D}{2}\sin\frac{C-D}{2}}$

*Example 3*

(a) Express the following as a product of factors

I. $\cos 5\theta + \cos 3\theta$

II. $\cos 5\theta - \cos 7\theta$

III. $\sin 5\theta + \sin \theta$

IV. $\sin 12\theta - \sin 4\theta$

(b) Simplify the following expressions

I. $\frac{(\sin 5\theta - \sin 3\theta)(\cos 6\theta + \cos 2\theta)}{(\sin 6\theta - \sin 2\theta)(\cos 5\theta + \cos 3\theta)}$

II. $\frac{\cos 105° + \cos 15°}{\cos 105° - \cos 15°}$

III. $\sin 6x + \sin 8x + \sin 5x + \sin 3x$

IV. $\cos 7\theta + 2\cos 4\theta + \cos \theta$

V. $\frac{\cos 4\theta - \cos 6\theta}{\sin 4\theta + \sin 2\theta}$

VI. $\frac{\sin 8\theta - \sin 4\theta}{\cos 4\theta + \cos 2\theta}$

*Solution*

(a)

I. $\cos 5\theta + \cos 3\theta$

$$= 2\cos\frac{5\theta + 3\theta}{2}\cos\frac{5\theta - 3\theta}{2}$$

$$= 2\cos 4\theta \cos \theta$$

II. $\cos 5\theta - \cos 7\theta$

$$= 2\sin\frac{5\theta + 7\theta}{2}\sin\frac{7\theta - 5\theta}{2}$$

$$= 2\sin 6\theta \sin \theta$$

III. $\sin 5\theta + \sin \theta$

$$= 2\sin\frac{5\theta + \theta}{2}\cos\frac{5\theta - \theta}{2}$$

$$= 2\sin 3\theta \cos 2\theta$$

IV. $\sin 12\theta - \sin 4\theta$

$$= 2\cos\frac{12\theta + 4\theta}{2}\sin\frac{12\theta - 4\theta}{2}$$

$$= 2\cos 8\theta \sin 4\theta$$

(b)

I. $\frac{(\sin 5\theta - \sin 3\theta)(\cos 6\theta + \cos 2\theta)}{(\sin 6\theta - \sin 2\theta)(\cos 5\theta + \cos 3\theta)}$

$$= \frac{2\cos\frac{5\theta+3\theta}{2}\sin\frac{5\theta-3\theta}{2}.2\cos\frac{6\theta+2\theta}{2}\cos\frac{6\theta-2\theta}{2}}{2\cos\frac{6\theta+2\theta}{2}\sin\frac{6\theta-2\theta}{2}.2\cos\frac{5\theta+3\theta}{2}\cos\frac{5\theta-3\theta}{2}}$$

$$= \frac{\cos 4\theta\sin\theta.\cos 4\theta\cos 2\theta}{\cos 4\theta\sin 2\theta\cos 4\theta\cos\theta}$$

$$= \frac{\tan\theta}{\tan 2\theta}$$

$$= \tan\theta\cot 2\theta$$

II. $\frac{\cos 105° + \cos 15°}{\cos 105° - \cos 15°}$

$$= \frac{2\cos\left(\frac{105°+15°}{2}\right)\cos\left(\frac{105°-15°}{2}\right)}{2\sin\left(\frac{105°+15°}{2}\right)\sin\left(\frac{15°-105°}{2}\right)}$$

$$= \frac{2\cos 60°\cos 45°}{2\sin 60°\sin(-45°)}$$

$$= -\cot 60° = -\frac{1}{\sqrt{3}}$$

III. $\sin 6x + \sin 8x + \sin 5x + \sin 3x$

$$= 2\sin\frac{6x+8x}{2}\cos\frac{6x-8x}{2} + 2\sin\frac{5x+3x}{2}\cos\frac{5x-3x}{2}$$

$$= 2\sin 7x\cos x + 2\sin 4x\cos x$$

$$= 2\cos x(\sin 7x + \sin 4x)$$

$$= 2\cos x\left(2\sin\frac{7x+4x}{2}\cos\frac{7x-4x}{2}\right)$$

$$= 4\cos x\sin\frac{11x}{2}\cos\frac{3x}{2}$$

IV. $\cos 7\theta + 2\cos 4\theta + \cos\theta$

$$= \cos 7\theta + \cos\theta + 2\cos 4\theta$$

$$= 2\cos\frac{7\theta+\theta}{2}\cos\frac{7\theta-\theta}{2} + 2\cos 4\theta$$

$$= 2\cos 4\theta\cos 3\theta + 2\cos 4\theta$$

$$= 2\cos 4\theta\,(\cos 3\theta + 1)$$

V. $\frac{\cos 4\theta - \cos 6\theta}{\sin 4\theta + \sin 2\theta}$

$$= \frac{2\sin\frac{4\theta+6\theta}{2}\sin\frac{6\theta-4\theta}{2}}{2\sin\frac{4\theta+2\theta}{2}\cos\frac{4\theta-2\theta}{2}}$$

$$= \frac{\sin 5\theta \sin\theta}{\sin 3\theta \cos\theta}$$

VI. $\frac{\sin 8\theta - \sin 4\theta}{\cos 4\theta + \cos 2\theta}$

$$= \frac{2\cos\frac{8\theta+4\theta}{2}\sin\frac{8\theta-4\theta}{2}}{2\cos\frac{4\theta+2\theta}{2}\cos\frac{4\theta-2\theta}{2}}$$

$$= \frac{\cos 6\theta \sin 2\theta}{\cos 3\theta \cos\theta}$$

### Trigonometric Expression Double Angle

We have $\sin(A+B) = \sin A\cos B + \cos A\sin B$

$$\cos(A+B) = \cos A\cos B - \sin A\sin B$$

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$$

When $A = B = \theta$;

$$\sin 2\theta = \sin\theta\cos\theta + \cos\theta\sin\theta$$

$$\boxed{\sin 2\theta = 2\sin\theta\cos\theta}$$

$$\cos 2\theta = \cos\theta\cos\theta - \sin\theta\sin\theta$$

$$\boxed{\cos 2\theta = \cos^2\theta - \sin^2\theta}$$

$$\tan 2\theta = \frac{\tan\theta + \tan\theta}{1 - \tan\theta\tan\theta}$$

$$\boxed{\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}}$$

*Example 4*

Find the values of $\sin 2\theta, cos\ 2\theta, tan\ 2\theta$ when $cos\ \theta = \frac{7}{25}$ and $\theta$ is an acute angle.

*Solution*

$cos\,\theta = \frac{7}{25}$ $\quad sin\,\theta = \frac{24}{25}$ $\quad \tan\theta = \frac{24}{7}$

## Pythagorean identities

Let $\theta$ is a real angle.

$\sin^2\theta + \cos^2\theta \equiv 1$ is called 'Pythagorean relationship'.

(a) $\boxed{sin^2\theta + cos^2\theta \equiv 1} \leftarrow (1)$

$$(1)/_{cos^2\theta} \Rightarrow \frac{\sin^2\theta}{\cos^2\theta} + 1 \equiv \frac{1}{\cos^2\theta}$$

(b) $\boxed{tan^2\theta + 1 \equiv sec^2\theta} \leftarrow (2)$

$$(1)/_{sin^2\theta} \Rightarrow 1 + \frac{\cos^2\theta}{\sin^2\theta} \equiv \frac{1}{\sin^2\theta}$$

$$\boxed{1 + \cot^2\theta \equiv \operatorname{cosec}^2\theta}$$

*Example 5*

(a) Prove that

$$\cos 2\theta \equiv 2\cos^2\theta - 1$$

$$\cos 2\theta \equiv 1 - 2\sin^2\theta$$

$$\cos 2\theta \equiv \frac{1-\tan^2\theta}{1+\tan^2\theta}$$

$$\sin 2\theta \equiv \frac{2\tan\theta}{1+\tan^2\theta}$$

(b) Prove that

I. $\frac{\cos\theta}{1+\sin\theta} - \frac{1-\sin\theta}{\cos\theta} = 0$

II. $\frac{\tan\theta}{1+\sec\theta} - \frac{\tan\theta}{1-\sec\theta} = 2\operatorname{cosec}\theta$

III. $\cos^4 A - \sin^4 A = 2\cos^2 A - 1$

IV. $\frac{1}{\sec\theta - \tan\theta} = \sec\theta + \tan\theta$

V. $\sec^2 x + \operatorname{cosec}^2 x = \sec^2 x \operatorname{cosec}^2 x$

VI. $\sqrt{\frac{1+\cos A}{1-\cos A}} = \operatorname{cosec} A + \cot A$

VII. $\sqrt{\sec^2 A + \operatorname{cosec}^2 A} = \tan A + \cot A$

VIII. $\sin x\,(\operatorname{cosec} x - \sin x) = \cos^2 x$

*Solution*

(a) $\cos 2\theta = \cos^2\theta - \sin^2\theta$ and $\cos^2\theta + \sin^2\theta \equiv 1$

$= \cos^2\theta - (1 - \cos^2\theta)$ $\sin^2\theta \equiv 1 - \cos^2\theta$

$\cos 2\theta = 2\cos^2\theta - 1$

$\cos 2\theta = \cos^2\theta - \sin^2\theta$ and $\cos^2\theta = 1 - \sin^2\theta$

$= (1 - \sin^2\theta) - \sin^2\theta$

$\cos 2\theta = 1 - 2\sin^2\theta$

$\cos 2\theta = \cos^2\theta - \sin^2\theta$

$$= \frac{\cos^2\theta - \sin^2\theta}{\cos^2\theta + \sin^2\theta}$$

$$= \frac{1 - \sin^2\theta / \cos^2\theta}{1 + \sin^2\theta / \cos^2\theta}$$

$$\cos 2\theta = \frac{1 - \tan^2\theta}{1 + \tan^2\theta}$$

$\sin 2\theta = 2\sin\theta\cos\theta$

$$= \frac{2\sin\theta\cos\theta}{\cos^2\theta + \sin^2\theta}$$

$$= \frac{2\sin\theta\cos\theta / \cos^2\theta}{1 + \sin^2\theta / \cos^2\theta}$$

$$= \frac{2\sin\theta / \cos\theta}{1 + \left(\sin\theta / \cos\theta\right)^2}$$

$$\sin 2\theta = \frac{2\tan\theta}{1 + \tan^2\theta}$$

(b)

I. $\frac{\cos\theta}{1+\sin\theta} - \frac{1-\sin\theta}{\cos\theta} = 0$

$LHS \quad = \frac{\cos\theta}{1+\sin\theta} - \frac{1-\sin\theta}{\cos\theta}$

$= \frac{\cos^2\theta - (1-\sin\theta)(1+\sin\theta)}{\cos\theta\,(1+\sin\theta)}$

$= \frac{\cos^2\theta - (1-\sin^2\theta)}{\cos\theta(1+\sin\theta)}$

$= \frac{\cos^2\theta + \sin^2\theta - 1}{\cos\theta(1+\sin\theta)}$

$= \ 0 \qquad = RHS$

II. $\frac{\tan\theta}{1+\sec\theta} - \frac{\tan\theta}{1-\sec\theta} = 2\,\mathrm{cosec}\,\theta$

$LHS \quad = \frac{\tan\theta}{1+\sec\theta} - \frac{\tan\theta}{1-\sec\theta}$

$= \frac{\tan\theta[1-\sec\theta-(1+\sec\theta)]}{(1+\sec\theta)(1-\sec\theta)}$

$= \frac{-2\tan\theta\sec\theta}{1-\sec^2\theta} \qquad = \frac{-2\tan\theta\sec\theta}{-\tan^2\theta}$

$= \frac{2\sec\theta}{\tan\theta}$

$= \frac{2\sec\theta}{\left(\sin\theta/\cos\theta\right)}$

$= 2\,\mathrm{cosec}\,\theta$

$= RHS$

III. $\cos^4 A - \sin^4 A = 2\cos^2 A - 1$

$LHS \quad = \cos^4 A - \sin^4 A$

$= (\cos^2 A - \sin^2 A)(\cos^2 A + \sin^2 A)$

$= (\cos^2 A - \sin^2 A)$

$= \cos^2 A - (1 - \cos^2 A)$

$= 2\cos^2 A - 1$

$= RHS$

IV. $\frac{1}{\sec\theta - \tan\theta} = \sec\theta + \tan\theta$

$$LHS = \frac{1}{\sec\theta - \tan\theta} \qquad 1 + \tan^2\theta = \sec^2\theta$$

$$= \frac{\sec^2\theta - \tan^2\theta}{\sec\theta - \tan\theta}$$

$$= \frac{(\sec\theta - \tan\theta)(\sec\theta + \tan\theta)}{\sec\theta - \tan\theta}$$

$$= sec\,\theta + tan\,\theta$$

$$= RHS$$

V. $\sec^2 x + \mathrm{cosec}^2 x = \sec^2 x\,\mathrm{cosec}^2 x$

$$LHS = \sec^2 x + \mathrm{cosec}^2 x$$

$$= \frac{1}{cos^2 x} + \frac{1}{\sin^2 x}$$

$$= \frac{\sin^2 x + cos^2 x}{cos^2 x \sin^2 x}$$

$$= \frac{1}{cos^2 x \sin^2 x}$$

$$= \sec^2 x\,\mathrm{cosec}^2 x$$

$$= RHS$$

VI. $\sqrt{\frac{1+\cos A}{1-\cos A}} = \mathrm{cosec}\,A + \cot A$

$$LHS = \sqrt{\frac{1+\cos A}{1-\cos A}}$$

$$= \frac{\sqrt{1+\cos A}}{\sqrt{1-\cos A}} \times \frac{\sqrt{1+\cos A}}{\sqrt{1+\cos A}}$$

$$= \frac{1+\cos A}{\sqrt{1-\cos^2 A}}$$

$$= \frac{1+\cos A}{\sin A}$$

$$= \frac{1}{\sin A} + \frac{\cos A}{\sin A}$$

$$= \mathrm{cosec}\,A + \cot A$$

$$= RHS$$

VII. $\sqrt{\sec^2 A + \mathrm{cosec}^2 A} = \tan A + \cot A$

$$LHS = \sqrt{\sec^2 A + \mathrm{cosec}^2 A}$$

$$= \sqrt{(1+\tan^2 A) + (1+\cot^2 A)}$$

$$= \sqrt{\tan^2 A + 2 + \cot^2 A}$$

$$= \sqrt{\tan^2 A + 2\tan A \cot A + \cot^2 A}\text{ , } since\ \cot A = \frac{1}{\tan A}$$

$$= \sqrt{(\tan A + \cot A)^2}$$

$$= \tan A + \cot A$$

$$= RHS$$

VIII. $\sin x\,(\operatorname{cosec} x - \sin x) = \cos^2 x$

$LHS \quad = \sin x\,(\operatorname{cosec} x - \sin x)$

$$= \sin x \operatorname{cosec} x - \sin^2 x$$

$$= 1 - \sin^2 x$$

$$= \cos^2 x$$

$$= RHS$$

*Example 6*

Prove the following identities.

(a) $\frac{1-\tan^2(45^0-A)}{1+\tan^2(45^0-A)} \equiv \sin 2A$

(b) $\frac{\sin 2A}{1+\cos 2A} \equiv \tan A$

(c) $\cos^6\theta + \sin^6\theta \equiv 1 - \frac{3}{4}\sin^2 2\theta$

(d) $\frac{1}{\tan 3A - \tan A} - \frac{1}{\cot 3A - \cot A} = \cot 2A$

(e) $\sec 2A + \tan 2A \equiv \tan(45^0 + A)$

*Solution*

(a) $\frac{1-\tan^2(45^0-A)}{1+\tan^2(45^0-A)} \equiv \sin 2A$

$LHS \quad = \frac{1-\tan^2\left(45^0-A\right)}{1+\tan^2\left(45^0-A\right)} = \cos 2(45^0 - A)\ ;\frac{1-\tan^2\theta}{1+\tan^2\theta} = \cos 2\theta$

$$= \cos(90^0 - 2A)$$

$$= \sin 2A$$

$$= RHS$$

(b) $\frac{\sin 2A}{1+\cos 2A} \equiv \tan A$

$$LHS = \frac{\sin 2A}{1+\cos 2A}$$
$$= \frac{2\sin A \cos A}{1+(2\cos^2 A-1)}$$
$$= \frac{2\sin A \cos A}{2\cos^2 A}$$
$$= \frac{\sin A}{\cos A}$$
$$= \tan A$$
$$= RHS$$

(c) $\cos^6\theta + \sin^6\theta \equiv 1 - \frac{3}{4}\sin^2 2\theta$

$$LHS = \cos^6\theta + \sin^6\theta$$
$$= (\cos^2\theta)^3 + (\sin^2\theta)^3$$
$$= (\cos^2\theta + \sin^2\theta)(\cos^4\theta - \cos^2\theta\sin^2\theta + \sin^4\theta)$$
$$= 1(\cos^4\theta + 2\cos^2\theta\sin^2\theta + \sin^4\theta - 3\cos^2\theta\sin^2\theta)$$
$$= (\cos^2\theta + \sin^2\theta)^2 - 3\sin^2\theta\cos^2\theta$$
$$= 1 - \frac{3}{4}(2\sin\theta\ \cos\theta)^2$$
$$= 1 - \frac{3}{4}\sin^2 2\theta$$
$$= RHS$$

(d) $\frac{1}{\tan 3A - \tan A} - \frac{1}{\cot 3A - \cot A} = \cot 2A$

$$LHS = \frac{1}{\tan 3A-\tan A} - \frac{1}{\cot 3A-\cot A}$$
$$= \frac{1}{\tan 3A-\tan A} - \frac{1}{\left(\frac{1}{\tan 3A}-\frac{1}{\tan A}\right)}$$
$$= \frac{1}{\tan 3A-\tan A} - \frac{1}{\left(\frac{\tan A-\tan 3A}{\tan A\tan 3A}\right)}$$
$$= \frac{1}{\tan 3A-\tan A} + \frac{\tan A\tan 3A}{\tan 3A-\tan A}$$
$$= \frac{1+\tan A\tan 3A}{\tan 3A-\tan A}$$
$$= \frac{1}{\left(\frac{\tan 3A-\tan A}{1+\tan A\tan 3A}\right)}$$

$$= \frac{1}{\tan(3A-A)}$$

$$= \frac{1}{\tan 2A}$$

$$= \cot 2A$$

$$= RHS$$

(e) $\sec 2A + \tan 2A \equiv \tan(45 + A)$

$$LHS \quad = \sec 2A + \tan 2A$$

$$= \frac{1}{\cos 2A} + \frac{\sin 2A}{\cos 2A}$$

$$= \frac{1+\sin 2A}{\cos 2A}$$

$$= \frac{\cos^2 A + \sin^2 A + 2\sin A\cos A}{\cos^2 A - \sin^2 A}$$

$$= \frac{(\cos A + \sin A)^2}{(\cos A + \sin A)(\cos A - \sin A)}$$

$$= \frac{\cos A + \sin A}{\cos A - \sin A}$$

$$= \frac{\frac{1}{\sqrt{2}}cosA + \frac{1}{\sqrt{2}}sinA}{\frac{1}{\sqrt{2}}cosA - \frac{1}{\sqrt{2}}sinA}$$

$$= \frac{\cos 45^0 \sin A + \sin 45^0 \cos A}{\cos 45^0 \cos A - \sin 45^0 \sin A}$$

$$\because \cos 45^0 = \sin 45^0$$

$$= \frac{\sin(A+45^0)}{\cos(A+45^0)}$$

$$= \tan(A + 45^0)$$

$$= RHS$$

## Activity 2

1. By using the table 11.2.1 and 11.3 section find the values of following

(a) $\sin 135°, \sin 225°, \sin 315°, \cos 135°, \cos 225°, \tan 315°$

(b) $\sin 135° \cos 60°$ $\quad$ $\cos 120° \sin 225°$

$\tan 120° \tan 30°$ $\quad$ $\sin 30° \sin 120°$

(c) $\dfrac{\sin 315° + \cos 225°}{\cos 60° + \cos 300°}$ $\dfrac{\tan 225° + \tan 315°}{\cos 135° + \sin 225°}$

2.

(a) If $tan\,A = \frac{4}{3}$ and $cos\,B = \frac{12}{13}$ and given that $A$ and $B$ are acute angles;

Find

I. $\cos A, \sin A, \sin B, \tan B$

II. $\sin(A - B), \cos(A + B)$

III. $\tan(A + B)$

(b) If $sin\,\alpha = \frac{3}{5}$ and $cos\,\beta = \frac{9}{41}$ and $\alpha, \beta$ are acute angles,

I. Find $cos\,\alpha\,, tan\,\alpha\,, sin\,\beta\,, tan\,\beta$

II. Find values of $sin(\alpha + \beta)\,, cos(\alpha - \beta)$

III. Find $tan(\alpha - \beta)$

IV. Find $sin\,2\alpha\,, cos\,2\alpha\,, cos\,2\beta\,, sin\,2\beta$

V. Find $\sin 2(\alpha + \beta)\,, \cos 2(\alpha - \beta)$ and verify that

$$\cos 2(\alpha - \beta) = 2\cos^2(\alpha - \beta) - 1$$

3.

(a) Express the following as a product of factors.

I. $cos\,13\theta - cos\,19\theta$ II. $sin\,12\theta - sin\,6\theta$

III. $cos\,15\theta + cos\,7\theta$ IV. $sin\,8\theta + sin\,6\theta$

(b) Prove the following identities.

I. $\dfrac{\cos 6\theta - \cos 4\theta}{\sin 6\theta + \sin 4\theta} = -\,tan\,\theta$

II. $\dfrac{\sin 7A - \sin A}{\sin 8A - \sin 2A} = cos\,4A\,sec\,5A$

III. $\dfrac{\cos 2B + \cos 2A}{\cos 2B - \cos 2A} = cot(A + B)\,cot(A - B)$

IV. $\dfrac{\sin 7\theta - \sin 5\theta}{\cos 7\theta + \cos 5\theta} = tan\,\theta$

V. $\dfrac{\sin A + \sin 3A}{\cos A + \cos 3A} = tan\,2A$

VI. $\dfrac{\sin 5A - \sin 3A}{\cos 3A + \cos 5A} = tan\,A$

VII. $\dfrac{\sin 2A + \sin 2B}{\sin 2A - \sin 2B} = \dfrac{\tan(A - B)}{\tan(A + B)}$

VIII. $\dfrac{\sin A + \sin 2A}{\cos A - \cos 2A} = cot\,\dfrac{A}{2}$

IX. $\dfrac{\cos 2B - \cos 2A}{\sin 2B + \sin 2A} = tan(A - B)$

X. $\frac{\cos 3A - \cos A}{\sin 3A - \sin A} + \frac{\cos 2A - \cos 4A}{\sin 4A - \sin 2A} = \frac{\sin A}{\cos 2A \cos 3A}$

4.

(a) If $tan\, A = {}^{1}/_{2}$ and $tan\, B = {}^{1}/_{3}$ find the values of

$$tan\, 2A\, , tan\, 2B\, , tan\, 2(A+B)\, , tan\, 2(A-B)$$

(b) By using the answers of part (a) find

$$sin\, 2A\, , cos\, 2A\, , sin\, 2B\, , cos\, 2B$$

(c)

I. If $cos\, \alpha = {}^{11}/_{61}$, find $cos\, 2\alpha$

II. If $sin\, \beta = {}^{7}/_{25}$, find $cos\, 2\beta$

III. If $tan\, \gamma = {}^{41}/_{40}$, find $sin\, 2\gamma\, ,\ cos\, 2\gamma$

5. Prove the following identities

(a) $sin^4\, \alpha + cos^4\, \alpha \equiv \frac{1}{2}(1 + 3\, cos^2\, 2\alpha)$

(b) $\frac{\sin 2A}{1 - \cos 2A} = cot\, A$

(c) $cosec\, 2A + cot\, 2A = cot\, A$

(d) $\frac{1 - \cos 2A}{1 + \cos 2A} = tan^2\, A$

(e) $tan\, A + cot\, A = 2\, cosec\, 2A$

(f) $\frac{\sin 2A}{1 + \cos 2A} = tan\, A$



## Solutions of Activities

### Activity 1

1.

(a) $323.8°$ (b) $85.0542°$

2.

(a) $235°\, 48'$ (b) $29°\, 4'$

3.

Angle (in degree) Angle (in radian)

| | |
|---|---|
| $180^0$ | $\pi\ rad$ |
| $10^0$ | $\pi/_{18}\ rad$ |
| $30^0$ | $\pi/_{6}\ rad$ |
| $45^0$ | $\pi/_{4}\ rad$ |
| $60^0$ | $\pi/_{3}\ rad$ |
| $67\,{}^{1}/_{2}{}^{0}$ | $2\pi/_{8}\ rad$ |
| $90^0$ | $\pi/_{2}\ rad$ |
| $112\,{}^{1}/_{2}{}^{0}$ | $5\pi/_{8}\ rad$ |
| $120^0$ | $2\pi/_{3}\ rad$ |
| $150^0$ | $5\pi/_{6}\ rad$ |
| $165^0$ | $11\pi/_{12}\ rad$ |
| $210^0$ | $7\pi/_{6}\ rad$ |
| $240^0$ | $4\pi/_{3}\ rad$ |
| $270^0$ | $3\pi/_{2}\ rad$ |
| $330^0$ | $11\pi/_{6}\ rad$ |
| $345^0$ | $23\pi/_{12}\ rad$ |
| $360^0$ | $2\pi\ rad$ |

## Activity 2

1.

(a) $\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, -1$

(b) $\frac{1}{2\sqrt{2}}, \frac{1}{2\sqrt{2}}, -1, \frac{\sqrt{3}}{4}$

(c) $-\sqrt{2},\ 0,\ 1, \frac{1}{2}$

2.

(a)

I. ${}^{3}/_{5}, {}^{4}/_{5}, {}^{5}/_{13}, {}^{5}/_{12}$

II. ${}^{33}/_{65}, {}^{16}/_{65}$

III. ${}^{63}/_{16}$

(b)

I. ${}^{4}/_{5}, {}^{3}/_{4}, {}^{40}/_{41}, {}^{41}/_{9}$

II. ${}^{187}/_{205}, {}^{156}/_{205}$

III. ${}^{-133}/_{156}$

3.

(a)

I. $2\sin 16\theta \sin 3\theta$

II. $2\cos 9\theta \sin 3\theta$

III. $2\cos 11\theta \cos 4\theta$

IV. $2\sin 7\theta \cos\theta$

4.

(a) ${}^{4}/_{3}, {}^{3}/_{4}, 3, {}^{9}/_{13}$

(b) ${}^{24}/_{25}, {}^{-7}/_{25}, {}^{24}/_{25}, {}^{7}/_{25}$

(c)

I. ${}^{-3479}/_{3721}$

II. ${}^{527}/_{625}$

III. ${}^{3280}/_{3281}, {}^{81}/_{3281}$



## Summary

The relationship between degree and radians is $180° = \pi\, rad$

Remember the following important relations.

$\sin(90-\theta) = \cos\theta$ $\quad$ $\sin(90+\theta) = \cos\theta$

$\cos(90-\theta) = \sin\theta$ $\quad$ $\cos(90+\theta) = -\sin\theta$

$\tan(90-\theta) = \cot\theta$ $\quad$ $\tan(90+\theta) = -\cot\theta$

$\sin(180-\theta) = \sin\theta$ $\quad$ $\sin(180+\theta) = -\sin\theta$

$\cos(180-\theta) = -\cos\theta$ $\quad$ $\cos(180+\theta) = -\cos\theta$

$\tan(180-\theta) = -\tan\theta$ $\quad$ $\tan(180+\theta) = \tan\theta$

$\sin(270-\theta) = -\cos\theta$ $\quad$ $\sin(270+\theta) = -\cos\theta$

$\cos(270 - \theta) = -\sin\theta$ $\cos(270 + \theta) = \sin\theta$

$\tan(270 - \theta) = \cot\theta$ $\tan(270 + \theta) = -\cot\theta$

$\sin(-\theta) = \sin(360 - \theta) = -\sin\theta$

$\cos(-\theta) = \cos(360 - \theta) = \cos\theta$

$\tan(-\theta) = \tan(360 - \theta) = -\tan\theta$

$\sin(A + B) \equiv \sin A \cos B + \cos A \sin B$

$\sin(A - B) \equiv \sin A \cos B - \cos A \sin B$

$\cos(A + B) \equiv \cos A \cos B - \sin A \sin B$

$\cos(A - B) \equiv \cos A \cos B + \sin A \sin B$

$\sin C + \sin D \equiv 2\sin\frac{C + D}{2}\cos\frac{C - D}{2}$

$\sin C - \sin D \equiv 2\cos\frac{C + D}{2}\sin\frac{C - D}{2}$

$\cos C + \cos D \equiv 2\cos\frac{C + D}{2}\cos\frac{C - D}{2}$

$$\cos C - \cos D \equiv 2\sin\frac{C + D}{2}\sin\frac{D - C}{2}$$

$$\sin 2A \equiv 2\sin A\cos A \equiv \frac{2\tan A}{1 + \tan^2 A}$$

$$\cos 2A \equiv \cos^2 A - \sin^2 A \equiv 2\cos^2 A - 1 \equiv 1 - 2\sin^2 A \equiv \frac{1 - \tan^2 A}{1 + \tan^2 A}$$

$$\tan 2A \equiv \frac{2\tan A}{1 - \tan^2 A}$$

$\sin^2\theta + \cos^2\theta = 1$

$1 + \tan^2\theta = \sec^2\theta$

$1 + \cot^2\theta = \mathrm{cosec}^2\,\theta$

## Learning Outcomes

On completion of this study session you should be able to

1. Describe the basic concepts of trigonometry such as angle, trigonometric ratios, degree, radian and identity etc.
2. Calculate trigonometric ratios for an angle.
3. Derive basic trigonometric identities and Pythagorean identities
4. Prove trigonometric identities by using basic identities, Pythagorean identities, and formulae for multiple angles and product.

# Session 12

# Solutions of Trigonometric Equation and Triangles





## Introduction

An equation that involves in at least one trigonometric function of an unknown angle is called a trigonometric equation.

Any value of the unknown angle for which the equation is satisfied is called a solution of the equation. Solving a trigonometric equation means to find the all solutions. We have limited study of solutions in triangles to right angles. However, many triangles which require a solution do not contain a right angle. Such triangle is called an oblique triangle.

In this session, we discuss the solution of oblique triangles. There are four possible cases of application of solving a triangle.

These cases can be expressed as follows.

i. two angles and one side (In this case we know all three angles)
ii. two sides and the angle opposite to one of them
iii. two sides and the included angle

iv. three sides

We discuss "Sine rule" and "Cosine rule" that are used to solve triangles.

## 12.1 General solutions of basic trigonometric equations

General solutions of the equation $\sin\theta = a$ Let $\sin\theta = \frac{1}{2}$

Then we know that $\theta$ can be $30^0\ (\pi/6)$, $150^0\ (5\pi/6)$, $390^0\ (13\pi/6)$, $510^0\ (17\pi/6)$, $750^0\ (25\pi/6)$…

$390^0 = 360^0 + 30^0 \qquad 13\pi/6 = 2\pi + \pi/6$

$510^0 = 360^0 + 150^0 \qquad 17\pi/6 = 2\pi + 5\pi/6$

$750^0 = 2 \times 360^0 + 30^0 \qquad 25\pi/6 = 2(2\pi) + \pi/6$

$870^0 = 2 \times 360^0 + 150^0 \qquad 29\pi/6 = 2(2\pi) + 5\pi/6$

$\therefore$ All the above values are the solutions of the equation $\sin\theta = \frac{1}{2}$.

If $\sin\theta = \sin\alpha \Rightarrow \sin\theta = \sin(\pi - \alpha)$

$\therefore \theta = \alpha$ or $\theta = \pi - \alpha$

Also $\sin\theta = \sin(2\pi + \alpha) \qquad \sin\theta = \sin(2\pi + \pi - \alpha)$

$\therefore \theta = 2\pi + \alpha \qquad \theta = 2\pi + \pi - \alpha$

$\therefore$The general solution for the equation $\sin\theta = \sin\alpha$ or $\operatorname{cosec}\theta = \operatorname{cosec}\alpha$ is

$\theta = n\pi + (-1)^n\alpha \qquad \theta = 180n^0 + (-1)^n\alpha^0$ where $n$ is an integer.

When $n = 0$ we have $\theta = \alpha$

$n = 1$ we have $\theta = \pi - \alpha$ or $\theta = 180^0 - \alpha^0$

$n = 2$ we have $\theta = 2\pi + \alpha$ or $\theta = 360^0 + \alpha^0$

$n = 3$ we have $\theta = 3\pi - \alpha$ or $\theta = 540^0 - \alpha^0$

∴The general solution of $\sin\theta = \sin\alpha$; $\operatorname{cosec}\theta = \operatorname{cosec}\alpha$ is

$\theta = n\pi + (-1)^n\alpha$ or $\theta = 180n^0 + (-1)^n\alpha^0$

where, $n$ is an integer.

*Example 1*

Find the general solution of the following equations

Let (a) $\sin\theta = -\sqrt{3}/2$ (b) $\sin\theta = 1/\sqrt{2}$

*Solution*

(a) $\sin\theta = -\sqrt{3}/2$

$\sin\theta = \sin(-\pi/3)$ $\sin\theta = \sin(-60^0)$

The general solution is,

$\theta = n\pi - (-1)^n\,\pi/3,\, n \in \mathbb{Z}$ $\theta = 180n^0 - (-1)^n 60^0 n \in \mathbb{Z}$

When $n = 0$ $\theta = -\pi/3$ $\theta = -60^0$

$n = 1$ $\theta = \pi + \pi/3 = 4\pi/3$ $\theta = 180 + 60^0 = 240^0$

$n = 2$ $\theta = 2\pi - \pi/3 = 5\pi/3$ $\theta = 360^0 - 60^0 = 300^0$

$n = 3$ $\theta = 3\pi + \pi/3 = 10\pi/3$ $\theta = 540^0 + 60^0 = 600^0$

(b) $\sin\theta = \frac{1}{\sqrt{2}}$

$\therefore \sin\theta = \sin\frac{\pi}{4}$ $\sin\theta = \sin 45^0$

$\theta = n\pi + (-1)^n\frac{\pi}{4} n \in \mathbb{Z}$ $\theta = 180n^0 + (-1)^n 45^0 n \in \mathbb{Z}$

when $n = 0$ $\theta = \frac{\pi}{4}$ $\theta = 45^0$

$n = 1$ $\theta = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$ $\theta = 180^0 - 45^0 = 135^0$

$n = 2$ $\theta = 2\pi + \frac{\pi}{4} = \frac{9\pi}{4}$ $\theta = 2 \times 180^0 + 45^0 = 405^0$

∴The general solution for the equation $\sin\theta = \sin\alpha$ or $\operatorname{cosec}\theta = \operatorname{cosec}\alpha$ is

$\theta = n\pi + (-1)^n\alpha$ or $\theta = 180n^0 + (-1)^n\alpha^0$

where $n$ is an integer.

**General solution of the equation $\cos\theta = \alpha$**

Let $\cos\theta = -\frac{\sqrt{3}}{2}$

We know that $\theta$ can be expressed as $150^0\left({}^{5\pi}/_{6}\right)$, $210^0\left({}^{7\pi}/_{6}\right)$, $510^0\left({}^{17\pi}/_{6}\right)$, $570^0\left({}^{19\pi}/_{6}\right)$, $870^0\left({}^{29\pi}/_{6}\right)$, $930^0\left({}^{31\pi}/_{6}\right)$…

$510^0 = 360^0 + 150^0$ $\qquad {}^{17\pi}/_{6} = 2\pi + {}^{5\pi}/_{6}$

$570^0 = 360^0 + 210^0$ $\qquad {}^{19\pi}/_{6} = 2\pi + {}^{7\pi}/_{6}$

$870^0 = 720^0 + 150^0$ $\qquad {}^{29\pi}/_{6} = 2(2\pi) + {}^{5\pi}/_{6}$

$930^0 = 720^0 + 210^0$ $\qquad {}^{31\pi}/_{6} = 2(2\pi) + {}^{7\pi}/_{6}$

All the above values are the solutions of the equation $\cos\theta = -{}^{\sqrt{3}}/_{2}$.

If $\cos\theta = \cos\alpha$ and $\cos\theta = \cos(-\alpha)$ are satisfy the equation

$\cos\theta = \cos\alpha$ or $\sec\theta = \sec\alpha$

$\therefore \theta = \alpha$ or $\theta = -\alpha$

$\therefore$The general solution of the equation $\cos\theta = \cos\alpha$ or $\sec\theta = \sec\alpha$ is

$\theta = 2n\pi \pm \alpha$ $\qquad \theta = 360n^0 \pm \alpha^0$

Where $n$ is an integer.

When $n = 0$ $\theta = \alpha$ $\qquad \theta = -\alpha$

$n = 1$ $\theta = 2\pi + \alpha$ $\qquad \theta = 2\pi - \alpha$

$\theta = 360^0 + \alpha^0$ $\qquad \theta = 360^0 - \alpha^0$

$n = 2$ $\theta = 4\pi + \alpha$ $\qquad \theta = 4\pi - \alpha$

$\theta = 2 \times 360^0 + \alpha^0$ $\qquad \theta = 2 \times 360^0 - \alpha^0$

$\therefore$The general solution of the equation $\cos\theta = \cos\alpha$, $\sec\theta = \sec\alpha$ is

$\theta = 2n\pi \pm \alpha$ or $\qquad \theta = 360n^0 \pm \alpha^0$

where $n$ is an integer.

*Example 2*

(a) $\sec\theta = -\sqrt{2}$ (b) $\cos\theta = {}^{1}/_{2}$

(a) $\sec\theta = \sec{}^{3\pi}/_{4} = \sec{}^{5\pi}/_{4} = -\sqrt{2}$

$\sec\theta = \sec 135^0 = \sec 225^0 = -\sqrt{2}$

$\therefore$ the general solution

$\theta = 2n\pi \pm {}^{3\pi}/_{4}, \quad n \in \mathbb{Z}$ $\qquad \theta = 360n^0 \pm 135^0, n \in \mathbb{Z}$

From this general solution we can select the solution in $-360^0 \leq \theta \leq 720^0$

or $-2\pi \leq \theta \leq 4\pi$.

Thus, we can find the solutions that lie between $-2\pi$ and $4\pi$ by substituting suitable value for $n$.

When $n = 0$ $\qquad \theta = \pm{}^{3\pi}/_{4}$ $\qquad \theta = \pm 135^0$

$n = 1$ $\qquad \theta = 2\pi \pm {}^{3\pi}/_{4} = {}^{11\pi}/_{4}, {}^{5\pi}/_{4}$

$\theta = 360^0 \pm 135^0 = 495^0, 225^0$

$n = 2$ $\qquad \theta = 4\pi \pm {}^{3\pi}/_{4} = {}^{19\pi}/_{4}, {}^{13\pi}/_{4}$

$\theta = 720^0 \pm 135^0 = 855^0, 585^0$

$n = -1$ $\qquad \theta = -2\pi \pm {}^{3\pi}/_{4} = -{}^{5\pi}/_{4}, -{}^{11\pi}/_{4}$

$\theta = -360^0 \pm 135^0 = -225^0, -495^0$

$\therefore$The solutions of $\theta \in [-2\pi, 4\pi]$ or $\theta \in [-360^0, 720^0]$ are

${}^{5\pi}/_{4}, {}^{11\pi}/_{4}, {}^{13\pi}/_{4}, {}^{-5\pi}/_{4}$

$225^0, 495^0, 585^0, -225^0$

(b) $\cos\theta = {}^{1}/_{2}$

$\therefore \cos\theta = \cos{}^{\pi}/_{3}$ $\qquad \cos\theta = \cos 60^0$

$\therefore \theta = 2n\pi \pm {}^{\pi}/_{3}\, n \in \mathbb{Z};$ $\qquad \theta = 360n^0 \pm 60^0 n \in \mathbb{Z}$

From this general solution we can select the solution in the range $(-\pi, 4\pi)$ or $(-180^0, 720^0)$

$$\theta = 2n\pi \pm {}^{\pi}/_{3} \qquad \theta = 360n^0 \pm 60^0$$

$n = 0$ $\qquad \theta = \pm {}^{\pi}/_{3} \, ({}^{\pi}/_{3}, -{}^{\pi}/_{3})$

$\qquad \theta = \pm 60^0(-60^0, 60^0)$

$n = 1$ $\qquad \theta = 2\pi \pm {}^{\pi}/_{3} \left({}^{5\pi}/_{3}, {}^{7\pi}/_{3}\right)$

$\qquad \theta = 360^0 \pm 60^0(300^0, 420^0)$

$n = 2$ $\qquad \theta = 4\pi \pm {}^{\pi}/_{3} \, ({}^{11\pi}/_{3}, {}^{13\pi}/_{3})$

$\qquad \theta = 720^0 \pm 60^0(660^0, 780^0)$

$n = -1$ $\qquad \theta = -2\pi \pm {}^{\pi}/_{3} \, (-{}^{5\pi}/_{3}, {}^{-7\pi}/_{3})$

$\qquad \theta = -360^0 \pm 60^0(-300^0, -420^0)$

∴The solution of $\theta \in (-\pi, 4\pi)$ or $\theta \in (-180^0, 720^0)$ are

$$-{}^{\pi}/_{3}, {}^{\pi}/_{3}, {}^{5\pi}/_{3}, {}^{7\pi}/_{3}, {}^{11\pi}/_{3}$$

or

$$-60^0, 60^0, 300^0, 420^0, 660^0$$

**General solution of the equation** $\tan\theta = a$ Let $\tan\theta = -1$.

We know that $\theta$ can be satisfies by$135^0\left({}^{3\pi}/_{4}\right)$, $315^0\left({}^{7\pi}/_{4}\right)$, $495^0\left({}^{11\pi}/_{4}\right)$, $675^0\left({}^{15\pi}/_{4}\right)$, $855^0\left({}^{19\pi}/_{4}\right)$…

$495^0 = 360^0 + 135^0 \qquad {}^{11\pi}/_{4} = 2\pi + {}^{3\pi}/_{4}$

$675^0 = 360^0 + 315^0 \qquad {}^{15\pi}/_{4} = 2\pi + {}^{7\pi}/_{4}$

$855^0 = 720^0 + 135^0 \qquad {}^{19\pi}/_{4} = 2(2\pi) + {}^{3\pi}/_{4}$

All the above values are the solutions of the equation $\tan\theta = -1$.

If $\tan\theta = \tan\alpha$ or $\cot\theta = \cot\alpha$

$\theta = \alpha$ or $\qquad \theta = \pi + \alpha, \qquad (\theta = \alpha \text{ or } \theta = 180^0 + \alpha)$

∴The general solution of the equation $\tan\theta = \tan\alpha$ or $\cot\theta = \cot\alpha$ is

$\theta = n\pi + \alpha n \in \mathbb{Z};$ $\theta = 180n^0 + \alpha^0 n \in \mathbb{Z}$

When $n = 0,$ $\theta = \alpha$ $\theta = \alpha$

$n = 1,$ $\theta = \pi + \alpha$ $\theta = 180^0 + \alpha^0$

$n = 2,$ $\theta = 2\pi + \alpha$ $\theta = 360^0 + \alpha^0$

$n = 3,$ $\theta = 3\pi + \alpha$ $\theta = 540^0 + \alpha^0$

The general solution of the equations $\tan\theta = \tan\alpha, \cot\theta = \cot\alpha$ can be expressed as follows.

$$\theta = n\pi + \alpha n \in \mathbb{Z}; \qquad \theta = 180n^0 + \alpha^0 n \in \mathbb{Z}$$

*Example 3*

*Find the general solutions of the equations*

*(a)* $tan\,\theta = {}^{1}/_{\sqrt{3}}$ *(b)* $cot\,\theta = {}^{-1}/_{\sqrt{3}}$

*Solution*

*(a)* $tan\,\theta = {}^{1}/_{\sqrt{3}}$

$\tan\theta = \tan{}^{\pi}/_{6}$ $\tan\theta = \tan 30^0$

$\theta = n\pi + {}^{\pi}/_{6}\, n \in \mathbb{Z};$ $\theta = 180n^0 + 30^0 n \in \mathbb{Z}$

From this general solution, we can find the solution in the range $\left(-{}^{3\pi}/_{2}, {}^{5\pi}/_{2}\right)(-270^0, 450^0)$

$\theta = n\pi + {}^{\pi}/_{6}$ $\theta = 180n^0 + 30^0$

$n = 0,$ $\theta = {}^{\pi}/_{6}$ $\theta = 30^0$

$n = 1,$ $\theta = \pi + {}^{\pi}/_{6} = {}^{7\pi}/_{6}$ $\theta = 180^0 + 30^0 = 210^0$

$n = 2,$ $\theta = 2\pi + {}^{\pi}/_{6} = {}^{13\pi}/_{6}$ $\theta = 360^0 + 30^0 = 390^0$

$n = 3, \qquad \theta = 3\pi + \pi/_6 = {}^{19\pi}/_6 \qquad \theta = 480^0 + 30^0 =$

$510^0$

$n = -1, \qquad \theta = -\pi + \pi/_6 = {}^{-5\pi}/_6 \qquad \theta = -180^0 + 30^0 = -150^0$

$n = -2, \qquad \theta = -2\pi + \pi/_6 = {}^{-11\pi}/_6 \qquad \theta = -360^0 + 30^0 = -330^0$

∴The solutions of $\theta \in (-{}^{3\pi}/_2, {}^{5\pi}/_2)$ or $\theta \in (-270^0, 450^0)$

$\theta = {}^{\pi}/_6, {}^{7\pi}/_6, {}^{13\pi}/_6, {}^{-5\pi}/_6 \qquad \theta = 30^0, 210^0, 390^0, -150^0$

(b) $\cot\theta = -{}^{1}/_{\sqrt{3}}$

$\cot\theta = \cot\left(-{}^{\pi}/_3\right) \qquad \cot\theta = \cot(-60^0)$

∴General solutions

$\theta = n\pi - {}^{\pi}/_3, n \in \mathbb{Z}; \qquad \theta = 180n^0 - 60^0, n \in \mathbb{Z}$

From the above general solution, we can find the solution in the range $(-3\pi, 2\pi)$, ($-540^0$, $360^0$)

$\theta = n\pi - {}^{\pi}/_3 \qquad \theta = 180n^0 - 60^0$

$n = 0, \qquad \theta = -{}^{\pi}/_3 \qquad \theta = -60^0$

$n = -2, \qquad \theta = -2\pi - {}^{\pi}/_3 = {}^{-7\pi}/_3 \qquad \theta = -360^0 - 60^0 = -420^0$

$n = -3, \qquad \theta = -3\pi - {}^{\pi}/_3 = {}^{-10\pi}/_3 \qquad \theta = -540^0 - 60^0 = -600^0$

$n = 1, \qquad \theta = \pi - {}^{\pi}/_3 = {}^{2\pi}/_3 \qquad \theta = 180^0 - 60^0 = 120^0$

$n = 2, \qquad \theta = 2\pi - {}^{\pi}/_3 = {}^{5\pi}/_3 \qquad \theta = 360^0 - 60^0 = 300^0$

$n = 3, \qquad \theta = 3\pi - {}^{\pi}/_3 = {}^{8\pi}/_3 \qquad \theta = 540^0 - 60^0 = 480^0$

∴ solution of $\theta \in (-3\pi, 2\pi)$ ; $\theta \in (-540^0, 360^0)$ are

$\theta = -{}^{\pi}/_3, -{}^{7\pi}/_3, {}^{2\pi}/_3, {}^{5\pi}/_3$ ; $\qquad \theta = -60^0, -420^0, 120^0, 300^0$

*Example 4*

1.(a) Find the general solutions of the following equations

(i) $\sin 3\theta = {}^{1}/_{2}$ (ii) $\cos 5\theta = -{}^{1}/_{2}$

(iii) $\cot 4\theta = -1$ (iv) $\operatorname{cosec} 9\theta = -{}^{2}/_{\sqrt{3}}$

(v) $\sec 3\theta = -{}^{2}/_{\sqrt{3}}$ (vi) $\tan 3\theta = \sqrt{3}$

(b) Find the all solutions of the following equations in the range $\theta \in (-90^0, 180^0)$ $\theta \in (-{}^{\pi}/_{2}, \pi)$

(i) $\sin 5\theta = -{}^{1}/_{\sqrt{2}}$ (ii) $\operatorname{cosec} 4\theta = {}^{2}/_{\sqrt{3}}$

(iii) $\cos 6\theta = {}^{1}/_{2}$ (iv) $\sec 5\theta = -\sqrt{2}$

(v) $\tan 3\theta = \sqrt{3}$ (vi) $\cot 2\theta = -\sqrt{3}$

2.(a) Find the general solution of the following equations.

(i) $\sin 5\theta = \cos 3\theta$ (ii) $\sin 5\theta + \cos 3\theta = 0$

(iii) $\sin 5\theta = \sin 3\theta$ (iv) $\sin 5\theta + \sin 3\theta = 0$

(v) $\cos 3\theta = \sin 5\theta$ (vi) $\cos 3\theta + \sin 5\theta = 0$

(vii) $\cos 5\theta = \cos 3\theta$ (viii) $\cos 5\theta + \cos 3\theta = 0$

(ix) $\tan 3\theta = \tan 4\theta$ (x) $\tan 3\theta + \tan 4\theta = 0$

(xi) $\tan 3\theta = \cot 4\theta$ (xii) $\tan 3\theta + \cot 4\theta = 0$

(b) Find all the solutions in the range$(-180^0, 180^0)(-\pi, \pi)$in the above (a).

*Solution*

1.(a)

(i) $\sin 3\theta = {}^{1}/_{2}$

$\sin 3\theta = \sin 30^0$

$3\theta = 180n^0 + (-1)^n 30^0$

$\theta = 60n^0 + (-1)^n 10^0, n \in \mathbb{Z}$

$\sin 3\theta = \sin {}^{\pi}/_{6}$

$3\theta = n\pi + (-1)^n \pi/6$ $\qquad$ $\theta = n\pi/3 + (-1)^n \pi/18, n \in \mathbb{Z}$

(ii) $\cos 5\theta = -1/2$

$\cos 5\theta = \cos 120^0$ $\qquad$ $\cos 5\theta = \cos 2\pi/3$

$5\theta = 360n^0 \pm 120^0$ $\qquad$ $5\theta = 2n\pi \pm 2\pi/3$

$\theta = 72n^0 \pm 24^0; n \in \mathbb{Z}$ $\qquad$ $\theta = 2n\pi/5 \pm 2\pi/15; n \in \mathbb{Z}$

(iii) $\cot 4\theta = -1$

$\cot 4\theta = \cot(-45^0)$ $\qquad$ $\cot 4\theta = \cot(-\pi/4)$

$4\theta = 180n^0 - 45^0$ $\qquad$ $4\theta = n\pi - \pi/4$

$\theta = 45n^0 - 45^0/4; n \in \mathbb{Z}$ $\qquad$ $\theta = n\pi/4 - \pi/16; n \in \mathbb{Z}$

(iv) $\operatorname{cosec} 9\theta = -2/\sqrt{3}$

$\operatorname{cosec} 9\theta = \operatorname{cosec}(-60^0)$ $\qquad$ $\operatorname{cosec} 9\theta = \operatorname{cosec}(-\pi/3)$

$9\theta = 180n^0 + (-1)^n(-60^0)$ $\qquad$ $9\theta = n\pi + (-1)^n(-\pi/3)$

$\theta = 20n^0 - (-1)^n 20^0/3; n \in \mathbb{Z}$ $\qquad$ $\theta = n\pi/9 - (-1)^n \pi/27; n \in \mathbb{Z}$

(v) $\sec 3\theta = -2/\sqrt{3}$

$\sec 3\theta = \sec 150^0$ $\qquad$ $\sec 3\theta = \sec 5\pi/3$

$3\theta = 360n^0 \pm 150^0$ $\qquad$ $3\theta = 2n\pi \pm 5\pi/3$

$\theta = 120n^0 \pm 50^0; n \in \mathbb{Z}$ $\qquad$ $\theta = 2n\pi/3 \pm 5\pi/9; n \in \mathbb{Z}$

(vi) $\tan 3\theta = \sqrt{3}$

$\tan 3\theta = \tan 60^0$ $\qquad$ $\tan 3\theta = \tan \pi/3$

$3\theta = 180n^0 + 60^0$ $\qquad$ $3\theta = n\pi + \pi/3$

$\theta = 60n^0 + 20^0; n \in \mathbb{Z}$ $\qquad$ $\theta = n\pi/3 + \pi/9; n \in \mathbb{Z}$

(b) (i) $\sin 5\theta = -{}^{1}/_{\sqrt{2}}$

$\sin 5\theta = \sin -45^0$

$5\theta = 180n^0 + (-1)^n(-45^0)$

$\theta = 36n^0 - (-1)^n 9^0, n \in \mathbb{Z}$

$\sin 5\theta = \sin -{}^{\pi}/_{4}$

$5\theta = n\pi + (-1)^n\left(-{}^{\pi}/_{4}\right)$

$\theta = {}^{n\pi}/_{5} - (-1)^n\,{}^{\pi}/_{20}, n \in \mathbb{Z}$

$n = 0,$ $\theta = -9^0$

$\theta = -{}^{\pi}/_{20}$

$n = 1,$ $\theta = 36^0 - (-1)9^0 = 45^0$

$\theta = {}^{\pi}/_{5} - (-1)\,{}^{\pi}/_{20} = {}^{5\pi}/_{20} = {}^{\pi}/_{4}$

$n = 2,$ $\theta = 72^0 - (1)9^0 = 63^0$

$\theta = {}^{2\pi}/_{5} - {}^{\pi}/_{20} = {}^{7\pi}/_{20}$

$n = 3,$ $\theta = 108^0 - (-1)9^0 = 117^0$

$\theta = {}^{3\pi}/_{5} - (-1)\,{}^{\pi}/_{20} = {}^{13\pi}/_{20}$

$n = 4,$ $\theta = 144^0 - (1)9^0 = 135^0$

$\theta = {}^{4\pi}/_{5} - {}^{\pi}/_{20} = {}^{15\pi}/_{20} = {}^{3\pi}/_{4}$

$n = 5,$ $\theta = 180^0 - (-1)9^0 = 189^0$

$\theta = \pi - (-1)\,{}^{\pi}/_{20} = {}^{21\pi}/_{20}$

$\theta = 189^0$ or $\theta = {}^{21\pi}/_{20}$ not in the given range.

$n = -1,$ $\theta = -36^0 - (-1)9^0 = -27^0$

$\theta = -{}^{\pi}/_{5} - (-1)\,{}^{\pi}/_{20} = {}^{-3\pi}/_{20}$

$n = -2,$ $\theta = -72^0 - (1)9^0 = -81^0$

$\theta = -{}^{2\pi}/_{5} - (1)\,{}^{\pi}/_{20} = {}^{-9\pi}/_{20}$

$n = -3,$ $\theta = -108^0 - (-1)9^0 = -99^0$

$\theta = -{}^{3\pi}/_{5} - (-1)\,{}^{\pi}/_{20} = {}^{-11\pi}/_{20}$

$\theta = -99^0$or $\theta = -{}^{11\pi}/_{20}$ not in the given range.

$$\therefore \theta = -9^0, 45^0, 63^0, 117^0, 135^0, -27^0, -81^0$$

$$\theta = -{}^{\pi}/_{20}, {}^{\pi}/_{4}, {}^{7\pi}/_{20}, {}^{13\pi}/_{20}, {}^{3\pi}/_{4}, -{}^{3\pi}/_{20}, -{}^{9\pi}/_{20}$$

(ii) $\operatorname{cosec} 4\theta = {}^{2}/_{\sqrt{3}}$

$\operatorname{cosec} 4\theta = \operatorname{cosec} 60^0$ $\quad$ $\operatorname{cosec} 4\theta = \operatorname{cosec} {}^{\pi}/_{3}$

$4\theta = 180n^0 + (-1)^n(60^0)$ $\quad$ $4\theta = n\pi + (-1)^n({}^{\pi}/_{3})$

$\theta = 45n^0 + (-1)^n 15^0 n \in \mathbb{Z}$ $\quad$ $\theta = {}^{n\pi}/_{4} + (-1)^n\, {}^{\pi}/_{12}\, n \in \mathbb{Z}$

$n = 0,\ \theta = 15^0$ $\quad$ $\theta = {}^{\pi}/_{12}$

$n = 1,\ \theta = 45^0 - 15^0 = 30^0$ $\quad$ $\theta = {}^{\pi}/_{4} - {}^{\pi}/_{12} = {}^{\pi}/_{6}$

$n = 2,\ \theta = 90^0 + 15^0 = 105^0$ $\quad$ $\theta = {}^{2\pi}/_{4} + {}^{\pi}/_{12} = {}^{7\pi}/_{12}$

$n = 3,\ \theta = 135^0 - 15^0 = 120^0$ $\quad$ $\theta = {}^{3\pi}/_{4} - {}^{\pi}/_{12} = {}^{2\pi}/_{3}$

$n = 4,\ \theta = 180^0 + 15^0 = 195^0$ $\quad$ $\theta = \pi + {}^{\pi}/_{12} = {}^{13\pi}/_{12}$

$\theta = 195^0$or $\theta = {}^{13\pi}/_{12}$is not in the given range.

$n = -1, \theta = -45^0 - 15^0 = -60^0$ $\quad$ $\theta = -{}^{\pi}/_{4} - {}^{\pi}/_{12} = -{}^{\pi}/_{3}$

$n = -2, \theta = -90^0 + 15^0 = -75^0$ $\quad$ $\theta = -{}^{\pi}/_{2} + {}^{\pi}/_{12} = {}^{-5\pi}/_{12}$

$n = -3, \theta = -135^0 - 15^0 = -150^0$ $\quad$ $\theta = -{}^{3\pi}/_{4} - {}^{\pi}/_{12} = {}^{-5\pi}/_{6}$

$\theta = -150^0$or $\theta = -{}^{5\pi}/_{6}$is not in the given range.

$$\therefore \theta = 15^0, 30^0, 105^0, 120^0, -60^0, -75^0$$

$$\theta = {}^{\pi}/_{12}, {}^{\pi}/_{6}, {}^{7\pi}/_{12}, {}^{2\pi}/_{3}, -{}^{\pi}/_{3}, -{}^{5\pi}/_{12}$$

(iii) $\cos 6\theta = {}^{1}/_{2}$

$\cos 6\theta = \cos {}^{\pi}/_{3}$

$6\theta = 2n\pi \pm {}^{\pi}/_{3}$

$\theta = {}^{n\pi}/_{3} \pm {}^{\pi}/_{18}; n \in \mathbb{Z}$

$\cos 6\theta = \cos 60^0$

$6\theta = 360n^0 \pm 60^0$

$\theta = 60n^0 \pm 10^0; n \in \mathbb{Z}$

$n = 0,$ $\theta = \pm 10^0$ $\theta = \pm {}^{\pi}/_{18}$

$n = 1,$ $\theta = 60^0 \pm 10^0 = 70^0, 50^0$

$\theta = {}^{\pi}/_{3} \pm {}^{\pi}/_{18} = {}^{7\pi}/_{18}, {}^{5\pi}/_{18}$

$n = 2,$ $\theta = 120^0 \pm 10^0 = 130^0, 110^0$

$\theta = {}^{2\pi}/_{3} \pm {}^{\pi}/_{18} = {}^{13\pi}/_{18}, {}^{11\pi}/_{18}$

$n = 3,$ $\theta = 180^0 \pm 10^0 = 190^0, 170^0$

$\theta = \pi \pm {}^{\pi}/_{18} = {}^{19\pi}/_{18}, {}^{17\pi}/_{18}$

$\theta = 190^0$or $\theta = {}^{19\pi}/_{18}$ is not in the given range.

$n = -1,$ $\theta = -60^0 \pm 10^0 = -70^0, -50^0$

$\theta = -{}^{\pi}/_{3} \pm {}^{\pi}/_{18} = -{}^{7\pi}/_{18}, -{}^{5\pi}/_{18}$

$n = -2,$ $\theta = -120^0 \pm 10^0 = -130^0, -110^0$

$\theta = -{}^{2\pi}/_{3} \pm {}^{\pi}/_{18} = -{}^{13\pi}/_{18}, -{}^{11\pi}/_{18}$

$\theta = -130^0, -110^0$or$\theta = -{}^{13\pi}/_{18}, -{}^{11\pi}/_{18}$ are not in the given range.

$$\therefore \theta = 10^0, 50^0, 70^0, 110^0, 130^0, 170^0, -10^0, -50^0, -70^0$$

$$\theta = {}^{\pi}/_{18}, {}^{5\pi}/_{18}, {}^{7\pi}/_{18}, {}^{11\pi}/_{18}, {}^{13\pi}/_{18}, {}^{17\pi}/_{18},$$

$-{}^{\pi}/_{18}, -{}^{5\pi}/_{18}, -{}^{7\pi}/_{18}$

(iv) $\sec 5\theta = -\sqrt{2}$

$\sec 5\theta = \sec 135^0$

$5\theta = 360n^0 \pm 135^0$

$\theta = 72n^0 \pm 27^0; n \in \mathbb{Z}$

$\sec 5\theta = \sec {}^{3\pi}/_4$

$5\theta = 2n\pi \pm {}^{3\pi}/_4$

$\theta = {}^{2n\pi}/_5 \pm {}^{3\pi}/_{20}; n \in \mathbb{Z}$

$n = 0,$ $\theta = \pm 27^0$ $\theta = \pm {}^{3\pi}/_{20}$

$n = 1,$ $\theta = 72^0 \pm 27^0 = 99^0, 45^0$

$\theta = {}^{2\pi}/_5 \pm {}^{3\pi}/_{20} = {}^{11\pi}/_{20}, {}^{\pi}/_4$

$n = 2,$ $\theta = 144^0 \pm 27^0 = 171^0, 117^0$

$\theta = {}^{4\pi}/_5 \pm {}^{3\pi}/_{20} = {}^{19\pi}/_{20}, {}^{13\pi}/_{20}$

$n = 3,$ $\theta = 216^0 \pm 27^0$ (not in the given range)

$n = -1,$ $\theta = -72^0 \pm 27^0 = -45^0, -99^0$

$\theta = -{}^{2\pi}/_5 \pm {}^{3\pi}/_{20} = {}^{-\pi}/_4, -{}^{9\pi}/_{20}$

$\theta = -99^0$ or $\theta = -{}^{9\pi}/_{20}$ is not in the given range.

$\therefore \theta = 27^0, 45^0, 99^0, 117^0, 171^0, -27^0, -45^0$

$\theta = {}^{3\pi}/_{20}, {}^{\pi}/_4, {}^{11\pi}/_{20}, {}^{13\pi}/_{20}, {}^{19\pi}/_{20}, -{}^{3\pi}/_{20}, -{}^{\pi}/_4$

(v) $\tan 3\theta = \sqrt{3}$

$\tan 3\theta = \tan 60^0$

$3\theta = 180n^0 + 60^0$

$\theta = 60n^0 + 20^0; n \in \mathbb{Z}$

$\tan 3\theta = \tan {}^{\pi}/_3$

$3\theta = n\pi + {}^{\pi}/_3$

$\theta = {}^{n\pi}/_3 + {}^{\pi}/_9; n \in \mathbb{Z}$

$n = 0,\ \theta = 20^0$ $\quad\theta = {}^{\pi}/_{9}$

$n = 1,\ \theta = 60^0 \times 1 + 20^0 = 80^0$ $\quad\theta = {}^{\pi}/_{3} + {}^{\pi}/_{9} = {}^{4\pi}/_{9}$

$n = 2,\ \theta = 60^0 \times 2 + 20^0 = 140^0$ $\quad\theta = {}^{2\pi}/_{3} + {}^{\pi}/_{9} = {}^{7\pi}/_{9}$

$n = 3,\ \theta = 60^0 \times 3 + 20^0 = 200^0$ $\quad\theta = \pi + {}^{\pi}/_{9} = {}^{10\pi}/_{9}$

$\theta = 200^0$or $\theta = {}^{10\pi}/_{9}$ is not in the given range.

$n = -1, \theta = -60^0 + 20^0 = -40^0$ $\quad\theta = -{}^{\pi}/_{3} + {}^{\pi}/_{9} = {}^{-2\pi}/_{9}$

$n = -2, \theta = -120^0 + 20^0 = -100^0$ $\quad\theta = -{}^{2\pi}/_{3} + {}^{\pi}/_{9} = {}^{-5\pi}/_{9}$

$\theta = -100^0$or $\theta = -{}^{5\pi}/_{9}$ is not in the given range.

$$\therefore \theta = 20^0, 80^0, 140^0, -40^0$$

$$\theta = {}^{\pi}/_{9}, {}^{4\pi}/_{9}, {}^{7\pi}/_{9}, -{}^{2\pi}/_{9}$$



(vi) $\cot 2\theta = -\sqrt{3}$

$\cot 2\theta = \cot -30^0$ $\quad\cot 2\theta = \cot -{}^{\pi}/_{6}$

$2\theta = 180n^0 - 30^0$ $\quad 2\theta = n\pi - {}^{\pi}/_{6}$

$\theta = 90n^0 - 15^0; n \in \mathbb{Z}$ $\quad\theta = {}^{n\pi}/_{2} - {}^{\pi}/_{12}; n \in \mathbb{Z}$

$n = 0,\ \theta = -15^0$ $\quad\theta = -{}^{\pi}/_{12}$

$n = 1,\ \theta = 90^0 - 15^0 = 75^0$ $\quad\theta = {}^{\pi}/_{2} - {}^{\pi}/_{12} =$
${}^{5\pi}/_{12}$

$n = 2,\ \theta = 180^0 - 15^0 = 165^0$ $\quad\theta = \pi - {}^{\pi}/_{12} = {}^{11\pi}/_{12}$

$n = -1, \theta = -90^0 - 15^0 = -105^0$ $\quad\theta = -{}^{\pi}/_{2} - {}^{\pi}/_{12} = {}^{-7\pi}/_{12}$

$\theta = -105^0$or $\theta = -{}^{7\pi}/_{12}$ is not in the given range.

$$\therefore \theta = 75^0, 165^0, -15^0$$

$$\theta = {}^{5\pi}/_{12}, {}^{11\pi}/_{12}, -{}^{\pi}/_{12}$$

2.(a) (i) $\sin 5\theta = \cos 3\theta$

$$\sin 5\theta = \sin\left(\pi/2 - 3\theta\right)$$
$$5\theta = n\pi + (-1)^n\left(\pi/2 - 3\theta\right)$$

When $n$ is an even number $n = 2r$ $\quad (-1)^{2r} = 1$

$$5\theta = 2r\pi + (1)\left(\pi/2 - 3\theta\right)$$
$$5\theta + 3\theta = 2r\pi + \left(\pi/2\right)$$
$$8\theta = 2r\pi + \left(\pi/2\right)$$
$$\theta = r\pi/4 + \pi/16, r \in \mathbb{Z}$$

When $n$ is an odd number $n = 2r + 1$ $\quad (-1)^{2r+1} = -1$

$$5\theta = (2r + 1)\pi - \left(\pi/2 - 3\theta\right)$$
$$5\theta - 3\theta = 2r\pi + \left(\pi - \pi/2\right)$$
$$2\theta = 2r\pi + \left(\pi/2\right)$$
$$\theta = r\pi + \pi/4, r \in \mathbb{Z}$$

(ii) $\sin 5\theta + \cos 3\theta = 0$

$$\sin 5\theta = -\cos 3\theta = \cos(\pi - 3\theta)$$
$$5\theta = 2n\pi \pm (\pi - 3\theta)$$

$(+)$
$$5\theta = 2n\pi + \pi - 3\theta$$
$$8\theta = (2n + 1)\pi$$
$$\theta = (2n + 1)\,\pi/8, \qquad n \in \mathbb{Z}$$

$(-)$
$$5\theta = 2n\pi - \pi + 3\theta$$
$$2\theta = (2n - 1)\pi$$
$$\theta = (2n - 1)\,\pi/2, \qquad n \in \mathbb{Z}$$

(iii) $\sin 5\theta = \sin 3\theta$

$$5\theta = n\pi + (-1)^n(3\theta)$$

When $n$ is an even number $n = 2r$ $\quad (-1)^{2r} = 1$

$$5\theta = 2r\pi + (3\theta)$$
$$\theta = r\pi, r \in \mathbb{Z}$$

When $n$ is an odd number $n = 2r + 1$ $\quad (-1)^{2r+1} = -1$

$$5\theta = (2r + 1)\pi - (3\theta)$$
$$8\theta = (2r + 1)\pi$$
$$\theta = {(2r + 1)\pi}/_{8}, \qquad r \in \mathbb{Z}$$

(iv) $\sin 5\theta + \sin 3\theta = 0$

$$\sin 5\theta = -\sin 3\theta = \sin(-3\theta)$$
$$5\theta = n\pi + (-1)^n(-3\theta)$$

$n = 2r$ $\quad 5\theta = 2r\pi + (-3\theta) \quad ; (-1)^{2r} = 1$

$$8\theta = 2r\pi$$
$$\theta = {r\pi}/_{4}; \ r \in \mathbb{Z}$$

$n = 2r + 1$ $\quad 5\theta = (2r + 1)\pi - (-3\theta) \quad ; (-1)^{2r+1} = -1$

$$2\theta = (2r + 1)\pi$$
$$\theta = {(2r + 1)\pi}/_{2}, \qquad r \in \mathbb{Z}$$

(v) $\cos 3\theta = \sin 5\theta$

$$\cos 3\theta = \cos\left({\pi}/_{2} - 5\theta\right)$$
$$3\theta = 2n\pi \pm \left({\pi}/_{2} - 5\theta\right)$$

$(+)$ $\quad 3\theta = 2n\pi + {\pi}/_{2} - 5\theta$

$$8\theta = 2n\pi + {\pi}/_{2}$$
$$\theta = {n\pi}/_{4} + {\pi}/_{16}, \qquad n \in \mathbb{Z}$$

$(-)$ $\quad 3\theta = 2n\pi - \left({\pi}/_{2} - 5\theta\right)$

$$3\theta - 5\theta = 2n\pi - {\pi}/_{2}$$
$$-2\theta = 2n\pi - {\pi}/_{2}$$
$$\theta = -n\pi + {\pi}/_{4} \quad n \in \mathbb{Z}$$

(vi) $\cos 3\theta + \sin 5\theta = 0$

$$\cos 3\theta = -\sin 5\theta$$

$$\cos 3\theta = \cos\left({}^{\pi}/_{2} + 5\theta\right)$$

$$3\theta = 2n\pi \pm \left({}^{\pi}/_{2} + 5\theta\right)$$

$(+)$ $$3\theta = 2n\pi + {}^{\pi}/_{2} + 5\theta$$

$$3\theta - 5\theta = 2n\pi + {}^{\pi}/_{2}$$

$$-2\theta = 2n\pi + {}^{\pi}/_{2}$$

$$\theta = -n\pi - {}^{\pi}/_{4}, \qquad n \in \mathbb{Z}$$

$(-)$ $$3\theta = 2n\pi - ({}^{\pi}/_{2} + 5\theta)$$

$$8\theta = 2n\pi - {}^{\pi}/_{2}$$

$$\theta = {}^{n\pi}/_{4} - {}^{\pi}/_{16}, \qquad n \in \mathbb{Z}$$

(vii) $\cos 5\theta = \cos 3\theta$

$$5\theta = 2n\pi \pm 3\theta$$

$(+)$ $$5\theta = 2n\pi + 3\theta$$

$$2\theta = 2n\pi$$

$$\theta = n\pi \qquad n \in \mathbb{Z}$$

$(-)$ $$5\theta = 2n\pi - 3\theta$$

$$8\theta = 2n\pi$$

$$\theta = {}^{n\pi}/_{4}, \qquad n \in \mathbb{Z}$$

(viii) $\cos 5\theta + \cos 3\theta = 0$

$$\cos 5\theta = -\cos 3\theta$$

$$\cos 5\theta = \cos(\pi - 3\theta)$$

$$5\theta = 2n\pi \pm (\pi - 3\theta)$$

$(+)$ $$5\theta = 2n\pi + \pi - 3\theta$$

$$8\theta = (2n + 1)\pi$$

$$\theta = (2n + 1)\,{}^{\pi}/_{8}, \qquad n \in \mathbb{Z}$$

$(-)$ $$5\theta = 2n\pi - \pi + 3\theta$$

$$2\theta = (2n - 1)\pi$$

$$\theta = (2n - 1)\,{}^{\pi}/_{2}, \qquad n \in \mathbb{Z}$$

(ix) $\tan 3\theta = \tan 4\theta$

$$3\theta = n\pi + 4\theta$$
$$-\theta = n\pi$$
$$\theta = -n\pi \ ; \ n \in \mathbb{Z}$$

(x) $\tan 3\theta + \tan 4\theta = 0$

$$\tan 3\theta = -\tan 4\theta$$
$$= \tan(-4\theta)$$
$$3\theta = n\pi - 4\theta$$
$$7\theta = n\pi$$
$$\theta = {}^{n\pi}/_{7} \quad n \in \mathbb{Z}$$

(xi) $\tan 3\theta = \cot 4\theta$

$$\tan 3\theta = \tan\left({}^{\pi}/_{2} - 4\theta\right)$$
$$3\theta = n\pi + \left({}^{\pi}/_{2} - 4\theta\right)$$
$$4\theta + 7\theta = n\pi + {}^{\pi}/_{2}$$
$$11\theta = n\pi + {}^{\pi}/_{2}$$
$$\theta = {}^{n\pi}/_{11} + {}^{\pi}/_{22}, \qquad n \in \mathbb{Z}$$

(xii) $\tan 3\theta + \cot 4\theta = 0$

$$\tan 3\theta = -\cot 4\theta = \tan\left({}^{\pi}/_{2} + 4\theta\right)$$
$$3\theta = n\pi + \left({}^{\pi}/_{2} + 4\theta\right)$$
$$-\theta = n\pi + {}^{\pi}/_{2}$$
$$\theta = -n\pi - {}^{\pi}/_{2} \ ; \ n \in \mathbb{Z}$$



## 12.2 Solutions for trigonometric equations

We explain this method by some examples. Study the following simple examples.

*Example 5*

Solve the following equations.

(i) $\cos x + \cos 5x = \cos 3x$ for $-2\pi \leq x \leq 2\pi$

(ii) $\sin x^0 - \sin 2x^0 + \sin 3x^0 = 0$ for $0^0 \leq x \leq 360^0$

(iii) $\sin\theta + \sin 2\theta + \sin 3\theta + \sin 4\theta = 0$ for $-\pi \leq \theta \leq \pi$

(iv) $\sin\theta + \sin 3\theta = \cos\theta + \cos 3\theta$ for $0 \leq \theta \leq 2\pi$

*Solution*

(i) $\cos x + \cos 5x = \cos 3x$

$$2\cos\frac{5x+x}{2}\cos\frac{5x-x}{2} - \cos 3x = 0$$

$$2\cos 3x\cos 2x - \cos 3x = 0$$

$$\cos 3x\,(2\cos 2x - 1) = 0$$

$$\cos 3x = 0 \text{ or } 2\cos 2\mathrm{x} - 1 = 0$$

$\cos 3x = \cos {}^{\pi}/_{2}$ $\qquad\qquad$ $\cos 2x = {}^{1}/_{2} = \cos {}^{\pi}/_{3}$

$3x = 2n\pi \pm {}^{\pi}/_{2}$ $\qquad\qquad$ $2x = 2n\pi \pm {}^{\pi}/_{3}$

$x = {}^{2n\pi}/_{3} \pm {}^{\pi}/_{6}\ \ n \in \mathbb{Z}$ $\qquad\qquad$ $x = n\pi \pm {}^{\pi}/_{6}\ \ n \in \mathbb{Z}$

$n = 0,$ $\quad x = \pm {}^{\pi}/_{6}$ $\qquad\qquad$ $x = \pm {}^{\pi}/_{6}$

$n = 1,$ $\quad x = {}^{2\pi}/_{3} \pm {}^{\pi}/_{6}$ $\qquad\qquad$ $x = \pi \pm {}^{\pi}/_{6}$

$\quad = {}^{5\pi}/_{6}, {}^{\pi}/_{2}$ $\qquad\qquad$ $= {}^{7\pi}/_{6}, {}^{5\pi}/_{6}$

$n = 2,$ $\quad x = {}^{4\pi}/_{3} \pm {}^{\pi}/_{6}$ $\qquad\qquad$ $x = 2\pi \pm {}^{\pi}/_{6}$

$\quad = {}^{3\pi}/_{2}, {}^{7\pi}/_{6}$ $\qquad\qquad$ $= {}^{11\pi}/_{6}, {}^{13\pi}/_{6}$

$n = 3,$ $\quad x = 2\pi \pm {}^{\pi}/_{6} = {}^{11\pi}/_{6}, {}^{13\pi}/_{6}$

$n = -1,$ $\quad x = -{}^{2\pi}/_{3} \pm {}^{\pi}/_{6}$ $\qquad\qquad$ $x = -2\pi \pm {}^{\pi}/_{6}$

$\quad = -{}^{\pi}/_{2}, -{}^{5\pi}/_{6}$ $\qquad\qquad$ $= -{}^{11\pi}/_{6}, -{}^{13\pi}/_{6}$

$\therefore$ *the solutions of the equation in the range* $-2\pi \leq x \leq 2\pi$ *are*

$$x = \left\{-{}^{11\pi}/_{6}, -{}^{5\pi}/_{6}, -{}^{\pi}/_{2}, -{}^{\pi}/_{6}, {}^{\pi}/_{6}, {}^{\pi}/_{2}, {}^{5\pi}/_{6}, {}^{7\pi}/_{6}, {}^{3\pi}/_{2}, {}^{11\pi}/_{6}\right\}$$

(ii) $\sin x^0 - \sin 2x^0 + \sin 3x^0 = 0$

$$\sin x^0 + \sin 3x^0 - \sin 2x^0 = 0$$

$$2\sin\left(\frac{x+3x}{2}\right)^0 \cos\left(\frac{3x-x}{2}\right)^0 - \sin 2x^0 = 0$$

$$2\sin 2x^0 \cos x^0 - \sin 2x^0 = 0$$

$$\sin 2x^0\,(2\cos x^0 - 1) = 0$$

$$\sin 2x^0 = 0 \quad \text{or} \quad 2\cos \mathrm{x}^0 - 1 = 0$$

$$\sin 2x^0 = \sin 0^0 \qquad\qquad \cos x^0 = {}^1\!/_2 = \cos 60^0$$

$$2x^0 = 180n^0 \qquad\qquad x^0 = 360n^0 \pm 60^0 \; n \in \mathbb{Z}$$

$$x^0 = 90n^0 \quad n \in \mathbb{Z}$$

$n = 0 \qquad x^0 = 0^0 \qquad\qquad x^0 = \pm 60^0$

$n = 1 \qquad x^0 = 90^0 \qquad\qquad x^0 = 360^0 - 60^0 = 300^0$

$n = 2 \qquad x^0 = 180^0$

$n = 3 \qquad x^0 = 270^0$

$n = 4 \qquad x^0 = 360^0$

The solutions of the equation in the range $0^0 \le x \le 360^0$ are,

$$x = \{0^0, 60^0, 90^0, 180^0, 270^0, 300^0, 360^0\}$$

(iii) $\qquad \sin\theta + \sin 2\theta + \sin 3\theta + \sin 4\theta = 0 \qquad -\pi \le \theta \le \pi$

$$(\sin\theta + \sin 4\theta) + \sin 2\theta + \sin 3\theta = 0$$

$$2\left\{\sin\frac{(\theta+4\theta)}{2}\cos\frac{(4\theta-\theta)}{2}\right\} + 2\sin\frac{(2\theta+3\theta)}{2}\cos\frac{(3\theta-2\theta)}{2} = 0$$

$$2\sin {}^{5\theta}\!/_2 \cos {}^{3\theta}\!/_2 + 2\sin {}^{5\theta}\!/_2 \cos {}^{\theta}\!/_2 = 0$$

$$2\sin {}^{5\theta}\!/_2 \left\{\cos {}^{3\theta}\!/_2 + \cos {}^{\theta}\!/_2\right\} = 0$$

$$\sin {}^{5\theta}\!/_2 = 0 \quad \text{or} \quad \cos {}^{3\theta}\!/_2 + \cos {}^{\theta}\!/_2 = 0$$

$$\sin {}^{5\theta}\!/_2 = \sin 0$$

$$5\theta/_2 = n\pi \qquad \cos {3\theta}/_2 = -\cos {\theta}/_2 = \cos(\pi - {\theta}/_2)$$

$$\theta = {2n\pi}/_5 \qquad {3\theta}/_2 = 2n\pi \pm (\pi - {\theta}/_2)$$

$$n \in \mathbb{Z} \qquad (+) \quad {3\theta}/_2 + {\theta}/_2 = (2n+1)\pi$$

$$\theta = (2n+1)\,{\pi}/_2 \quad n \in \mathbb{Z}$$

$$(-) \quad {3\theta}/_2 - {\theta}/_2 = (2n-1)\pi$$

$$\theta = (2n-1)\pi \quad n \in \mathbb{Z}$$

$$\therefore \theta = \frac{2n\pi}{5} \qquad \theta = (2n+1)\frac{\pi}{2} \qquad \theta = (2n-1)\pi$$

| | | | |
|---|---|---|---|
| $n = 0$ | $\theta = 0$ | $\theta = {\pi}/_2$ | $\theta = -\pi$ |
| $n = 1$ | $\theta = {2\pi}/_5$ | $\theta = {3\pi}/_2$ | $\theta = +\pi$ |
| $n = 2$ | $\theta = {4\pi}/_5$ | $\theta = {5\pi}/_2$ | $\theta = 3\pi$ |
| $n = -1$ | $\theta = -{2\pi}/_5$ | $\theta = -{\pi}/_2$ | $\theta = -3\pi$ |
| $n = -2$ | $\theta = -{4\pi}/_5$ | $\theta = -{3\pi}/_2$ | $\theta = -5\pi$ |

The solutions of the equation in the range $-\pi \le \theta \le \pi$ are,

$$\theta = \{-\pi, -{4\pi}/_5, -{\pi}/_2, -{2\pi}/_5, 0, {2\pi}/_5, {\pi}/_2, {4\pi}/_5, \pi\}$$

(iv) $\sin\theta + \sin 3\theta = \cos\theta + \cos 3\theta \qquad 0 \le \theta \le 2\pi$

$$\sin\theta + \sin 3\theta - (\cos\theta + \cos 3\theta) = 0$$

$$2\sin\frac{(3\theta+\theta)}{2}\cos\frac{(3\theta-\theta)}{2} - 2\cos\frac{(3\theta+\theta)}{2}\cos\frac{(3\theta-\theta)}{2} = 0$$

$$2\sin 2\theta\cos\theta - 2\cos 2\theta\cos\theta = 0$$

$$2\cos\theta\,(\sin 2\theta - \cos 2\theta) = 0$$

$$\cos\theta = 0 \qquad \text{or} \qquad \sin 2\theta - \cos 2\theta = 0$$

$$\cos\theta = \cos{\pi}/_2$$

$$\theta = 2n\pi \pm {\pi}/_2$$

$$\theta = 2n\pi \pm {\pi}/_2 \quad n \in \mathbb{Z}$$

$$\sin 2\theta = \cos 2\theta$$

$$\tan 2\theta = 1 = \tan{\pi}/_4$$

$$2\theta = n\pi + {\pi}/_4$$

$$\theta = {n\pi}/_2 + {\pi}/_8 \quad n \in \mathbb{Z}$$

$n = 0 \quad \theta = \pm {}^{\pi}/_{2}$ $\qquad\qquad\qquad\qquad$ $\theta = {}^{\pi}/_{8}$

$n = 1 \quad \theta = 2\pi - {}^{\pi}/_{2} = {}^{3\pi}/_{2}$ $\qquad\qquad$ $\theta = {}^{\pi}/_{2} + {}^{\pi}/_{8} = {}^{5\pi}/_{8}$

$n = 2 \quad \theta = 4\pi \pm {}^{\pi}/_{2}$ (not in the range) $\qquad$ $\theta = {}^{3\pi}/_{2} + {}^{\pi}/_{8} = {}^{13\pi}/_{8}$

The solutions of the equations in the range $0 \leq \theta \leq 2\pi$ are,

$$\therefore \theta = \left\{{}^{\pi}/_{8}, {}^{\pi}/_{2}, {}^{5\pi}/_{8}, {}^{3\pi}/_{2}, {}^{13\pi}/_{8}\right\}$$

## 12.3 Solutions based on the quadratic equations

In this we are going to solve the equations in types of $aX^2 + bX + c = 0$ where $X = \sin\theta$ or $X = \cos\theta$, or $X = \tan\theta$. We know that we can solve this types of equations by the knowledge of quadratic equations.

*Example 6*

Solve the following equations.

(i) $\quad 2\sin^2 x - \sqrt{3}\cos x + 1 = 0$

(ii) $\quad \sin^2 x + \cos x - {}^{1}/_{4} = 0$

(iii) $\quad 2\cos^2 x - 3\sin x = 0$

(iv) $\quad 2\sqrt{3}\sin^2 x + \cos x = 0$

(v) $\quad \cos^2 x + 2\sin x + {}^{1}/_{4} = 0$

(vi) $\quad \tan^2 x - \left(1 - \sqrt{3}\right)\tan x - \sqrt{3} = 0$

(vii) $\tan^2\theta - \left(\sqrt{3} + \frac{1}{\sqrt{3}}\right)\tan\theta + 1 = 0$

(viii) $3(\operatorname{cosec}^2\theta + \cot^2\theta) = 5$

(ix) $\quad 2\cos\theta\cos 2\theta + \sin 2\theta = 2(3\cos^3\theta - \cos\theta)$ for $0 < \theta < 2\pi$

*Solution*

(i) $\quad 2\sin^2 x - \sqrt{3}\cos x + 1 = 0$

$$2(1 - \cos^2 x) - \sqrt{3}\cos x + 1 = 0$$

$$2\cos^2 x + \sqrt{3}\cos x - 3 = 0$$

$$\left(2\cos x - \sqrt{3}\right)\left(\cos x + \sqrt{3}\right) = 0$$

$2\cos x - \sqrt{3} = 0$ or $\left(\cos x + \sqrt{3}\right) = 0$

$\cos x = {}^{\sqrt{3}}/_{2} = \cos {}^{\pi}/_{6}$; $\quad \cos x = -\sqrt{3}$ (not defined)

$[\cos x < -1]$

$$\therefore x = 2n\pi \pm {}^{\pi}/_{6}; n \in \mathbb{Z}$$

(ii) $\sin^2 x + \cos x - {}^{1}/_{4} = 0$

$$(1 - \cos^2 x) + \cos x - {}^{1}/_{4} = 0$$

$$4\cos^2 x - 4\cos x - 3 = 0$$

$$(2\cos x - 3)(2\cos x + 1) = 0$$

$(2\cos x - 3) = 0$ or $2\cos x + 1 = 0$

$\cos x = {}^{3}/_{2}$ $is$ not defined. $\quad 2\cos x + 1 = 0$

$(\cos x > 1)$ $\quad \cos x = -{}^{1}/_{2} = \cos {}^{2\pi}/_{3}$

$x = 2n\pi \pm {}^{2\pi}/_{3}; n \in \mathbb{Z}$

(iii) $2\cos^2 x - 3\sin x = 0$

$$2(1 - \sin^2 x) - 3\sin x = 0$$

$$2\sin^2 x + 3\sin x - 2 = 0$$

$$(2\sin x - 1)(\sin x + 2) = 0$$

$2\sin x - 1 = 0$ or $\sin x + 2 = 0$

$\sin x = {}^{1}/_{2} = \sin {}^{\pi}/_{6}$ $\quad \sin x = -2$

$x = n\pi + (-1)^n\, {}^{\pi}/_{6}; n \in \mathbb{Z}$ $\quad$ is not defined, $\sin x < -1$

(iv) $2\sqrt{3}\sin^2 x + \cos x = 0$

$2\sqrt{3}(1 - \cos^2 x) + \cos x = 0$

$2\sqrt{3}\cos^2 x - \cos x - 2\sqrt{3} = 0$

$\left(2\cos x + \sqrt{3}\right)\left(\sqrt{3}\cos x - 2\right) = 0$

$2\cos x + \sqrt{3} = 0$ or $\sqrt{3}\cos x - 2 = 0$

$\cos x = -{}^{\sqrt{3}}/_{2} = \cos {}^{5\pi}/_{6}$; $\quad \cos x = {}^{2}/_{\sqrt{3}}$ not defined

(since $\cos x > 1$)

$x = 2n\pi \pm {}^{5\pi}/_{6}\,; n \in \mathbb{Z}$

(v) $\cos^2 x + 2\sin x + {}^{1}/_{4} = 0$

$$4(1 - \sin^2 x) + 8\sin x + 1 = 0$$

$$4\sin^2 x - 8\sin x - 5 = 0$$

$$(2\sin x + 1)(2\sin x - 5) = 0$$

$2\sin x + 1 = 0$ $\qquad$ $2\sin x - 5 = 0$

$\sin x = -{}^{1}/_{2}$ $\qquad$ $\sin x = {}^{5}/_{2}$ not defined

$\sin x = \sin(-{}^{\pi}/_{6})$ $\qquad$ ($\sin x > 1$)

$x = n\pi - (-1)^n\,{}^{\pi}/_{6}\,; n \in \mathbb{Z}$

(vi) $\tan^2 x - \left(1 - \sqrt{3}\right)\tan x - \sqrt{3} = 0$

$$\left(\tan x + \sqrt{3}\right)(\tan x - 1) = 0$$

$\tan x + \sqrt{3} = 0$ or $\tan x - 1 = 0$

$\tan x = -\sqrt{3} = \tan -{}^{\pi}/_{3}\,;$ $\qquad$ $\tan x = 1 = \tan{}^{\pi}/_{4}$

$x = n\pi - {}^{\pi}/_{3}\,,\ n \in \mathbb{Z}$ $\qquad$ $x = n\pi + {}^{\pi}/_{4}\,,\ n \in \mathbb{Z}$

(vii) $\tan^2\theta - \left(\sqrt{3} + \frac{1}{\sqrt{3}}\right)\tan\theta + 1 = 0$

$$\left(\tan\theta - \sqrt{3}\right)\left(\tan\theta - {}^{1}/_{\sqrt{3}}\right) = 0$$

$\tan\theta - \sqrt{3} = 0$ or $\tan\theta - {}^{1}/_{\sqrt{3}} = 0$

$\tan\theta = \sqrt{3} = \tan{}^{\pi}/_{3}$ $\qquad$ $\tan\theta = {}^{1}/_{\sqrt{3}} = \tan{}^{\pi}/_{6}$

$\theta = n\pi + {}^{\pi}/_{3}\,,\ n \in \mathbb{Z}$ $\qquad$ $\theta = n\pi + {}^{\pi}/_{6}\,,\ n \in \mathbb{Z}$

(viii) $3(\operatorname{cosec}^2\theta + \cot^2\theta) = 5$

$$3(1 + \cot^2\theta + \cot^2\theta) = 5$$

$$3 + 6\cot^2\theta = 5$$

$$6\cot^2\theta = 2$$

$$\cot^2\theta = {}^1/_3$$

$$\cot\theta = \pm\,{}^1/_{\sqrt{3}}$$

$$\cot\theta = {}^1/_{\sqrt{3}} = \cot{}^{\pi}/_3 \qquad \cot\theta = -\,{}^1/_{\sqrt{3}} = \cot(-{}^{\pi}/_3)$$

$$\theta = n\pi + {}^{\pi}/_3 \qquad \theta = n\pi - {}^{\pi}/_3$$

$$\therefore \theta = n\pi \pm {}^{\pi}/_3 \quad n \in \mathbb{Z}$$

(ix) $2\cos\theta\cos 2\theta + \sin 2\theta = 2(3\cos^3\theta - \cos\theta)$

$$2\cos\theta\,(2\cos^2\theta - 1) + 2\sin\theta\cos\theta - 6\cos^3\theta + 2\cos\theta = 0$$

$$2\cos\theta\,(2\cos^2\theta - 1 + \sin\theta - 3\cos^2\theta + 1) = 0$$

$$2\cos\theta\,(-\cos^2\theta + \sin\theta) = 0$$

$$2\cos\theta\,(-(1-\sin^2\theta) + \sin\theta) = 0$$

$$2\cos\theta\,(\sin^2\theta + \sin\theta - 1) = 0$$

$\cos\theta = 0$ or $\sin^2\theta + \sin\theta - 1 = 0$

$$\therefore \sin\theta = \frac{-1 \pm \sqrt{1+4}}{2}$$

$$\sin\theta = \frac{\sqrt{5}-1}{2}, \text{ or } \sin\theta = -\left(\frac{\sqrt{5}+1}{2}\right)$$

$$\sin\theta = 0.6180$$

$$\theta = 38.170^0 = 0.666\,rad$$

$\sin\theta = -\left(\frac{\sqrt{5}+1}{2}\right)$ is not defined.

$\cos\theta = \cos{}^{\pi}/_2 = 0$ or $\sin\theta = \sin 0.666$

$\theta = 2n\pi \pm {}^{\pi}/_2 \qquad\qquad \theta = n\pi + (-1)^n 0.666$

$$0 < \theta < 2\pi$$

$\theta = {}^{\pi}/_2, {}^{3\pi}/_2 \qquad\qquad \theta = 0.666$

$\qquad\qquad\qquad\qquad \theta = \pi - 0.666$

The solutions of the given equation in the range are,

$$\theta = \{0.666, {}^{\pi}/_2, (\pi - 0.666), {}^{3\pi}/_2\}$$

In this method most cases we have to use the identities

$\sin 2x = \frac{2t}{1+t^2}$, $\cos 2x = \frac{1-t^2}{1+t^2}$ and $\tan 2x = \frac{2t}{1-t^2}$

After substituting, these values for the given trigonometric equation it becomes equation in terms of '$t$'.

*Example 7*

Solve the following equations.

(i) $2\tan x + \sec 2x = 2\tan 2x$ where $-\pi < x < \pi$

(ii) $\sin 2x + 2\cos 2x = 1$

(iii) $\operatorname{cosec}\theta - \cot\theta = \sqrt{3}$ substitute $\tan {}^{x}/_{2} = t$

(iv) $\cos x + \sin x = 1$ Substitute $\tan {}^{x}/_{2} = t$

(v) $\sin x + \sqrt{3}\cos x = 1$ substitute $\tan {}^{x}/_{2} = t$

*Solution*

(i) $2\tan x + \sec 2x = 2\tan 2x \qquad -\pi < x < \pi$

$$2\tan x + {}^{1}/_{\cos 2x} = 2\tan 2x$$

$$\tan x = t; \quad 2t + \frac{1+t^2}{1-t^2} = \frac{2.2t}{1-t^2}$$

$$2t(1-t^2) + (1+t^2) = 4t$$

$$2t^3 - t^2 + 2t - 1 = 0$$

$$2t^3 + 2t - (t^2+1) = 0$$

$$2t(t^2+1) - (t^2+1) = 0$$

$$(t^2+1)(2t-1) = 0$$

$$t^2 + 1 \neq 0 \quad \therefore 2t - 1 = 0$$

$$t = \frac{1}{2} = 0.5$$

$$\tan x = 0.5 \quad \therefore x = 26.57^0$$

$$= 0.464\, rad$$

$$\tan x = \tan 0.464$$

$$\therefore x = n\pi + 0.464$$

$$n = 0, x = 0.464\, rad$$

$$n = -1, x = -\pi + 0.464$$

$$= -3.142 + 0.464$$
$$= -2.678\, rad$$

since $-\pi < x < \pi$ $\quad\therefore x = 0.464\, rad$

$$x = -2.678\, rad$$

(ii) $\sin 2x + 2\cos 2x = 1$

$$\frac{2t}{1+t^2} + 2.\frac{1-t^2}{1+t^2} = 1$$
$$2t + 2 - 2t^2 = 1 + t^2$$
$$3t^2 - 2t - 1 = 0$$
$$(t-1)(3t+1) = 0$$
$$t = 1 \text{ or } t = -{}^{1}\!/_{3}$$

$\tan x = 1 = \tan {}^{\pi}\!/_{4}$ $\qquad \tan x = -{}^{1}\!/_{3}$

$x = n\pi + {}^{\pi}\!/_{4}$

(iii)$\operatorname{cosec}\theta - \cot\theta = \sqrt{3}$

$$\frac{1}{\sin\theta} - \frac{\cos\theta}{\sin\theta} = \sqrt{3}$$
$$1 - \cos\theta = \sqrt{3}\sin\theta$$

$\sqrt{3}\sin\theta + \cos\theta = 1$ $\qquad \tan {}^{\theta}\!/_{2} = t$

$\sin\theta = \frac{2t}{1+t^2}$ $\cos\theta = \frac{1-t^2}{1+t^2}$

$$\frac{2\sqrt{3}t}{1+t^2} + \frac{1-t^2}{1+t^2} = 1$$
$$2\sqrt{3}t + 1 - t^2 = 1 + t^2$$
$$2t\left(t - \sqrt{3}\right) = 0$$

$t = 0$ $\qquad$ or $\qquad t - \sqrt{3} = 0$

$\tan {}^{x}\!/_{2} = 0$ $\qquad \tan {}^{x}\!/_{2} = \sqrt{3} = \tan {}^{\pi}\!/_{3}$

${}^{x}\!/_{2} = n\pi$ $\qquad {}^{x}\!/_{2} = n\pi + {}^{\pi}\!/_{3}$

$x = 2n\pi, n \in \mathbb{Z}$ $\qquad x = 2n\pi + {}^{2\pi}\!/_{3}, n \in \mathbb{Z}$

(iv)$\cos x + \sin x = 1$ $\qquad \tan {}^{x}\!/_{2} = t$

$$\frac{1-t^2}{1+t^2}+\frac{2t}{1+t^2}=1$$

$$1-t^2+2t=1+t^2$$

$$2t^2-2t=0$$

$$2t(t-1)=0$$

$$t=0\text{or}t-1=0$$

$t=\tan {}^{x}/_{2}=0$

${}^{x}/_{2}=n\pi$

$x=2n\pi, n\in\mathbb{Z}$

$t=1=\tan {}^{\pi}/_{4}$

$\tan {}^{x}/_{2}=\tan {}^{\pi}/_{4}$

${}^{x}/_{2}=n\pi+{}^{\pi}/_{4}$

$x=2n\pi+$

${}^{\pi}/_{2}, n\in\mathbb{Z}$

(v) $\sin x+\sqrt{3}\cos x=1$

$$\frac{2t}{1+t^2}+\frac{\sqrt{3}(1-t^2)}{1+t^2}=1$$

$$2t+\sqrt{3}-\sqrt{3}t^2=1+t^2$$

$$\left(1+\sqrt{3}\right)t^2-2t-\left(\sqrt{3}-1\right)=0$$

$$t=\frac{-(-2)\pm\sqrt{(-2)^2-4\left(1+\sqrt{3}\right)\times\left(-\left(\sqrt{3}-1\right)\right)}}{2\left(1+\sqrt{3}\right)}$$

$$=\frac{2\pm2\sqrt{1+(2)}}{2\left(1+\sqrt{3}\right)}=\frac{1\pm\sqrt{3}}{1+\sqrt{3}}$$

$t=\frac{1+\sqrt{3}}{1+\sqrt{3}}=1$

$\tan {}^{x}/_{2}=1=\tan {}^{\pi}/_{4}$

${}^{x}/_{2}=n\pi+{}^{\pi}/_{4}$

$x=2n\pi+{}^{\pi}/_{2}, n\in\mathbb{Z}$

$t=\frac{-(\sqrt{3}-1)}{1+\sqrt{3}}=-\frac{-(\sqrt{3}-1)^2}{3-1}=-\left(2-\sqrt{3}\right)$

$\tan {}^{x}/_{2}=\tan {}^{-\pi}/_{12}$

${}^{x}/_{2}=n\pi-{}^{\pi}/_{12}$

$x=2n\pi-{}^{\pi}/_{6}, n\in\mathbb{Z}$

## Activity 1

(1) What are the most general values of $\theta$ which satisfy the equations?

(i) $\sin\theta={}^{1}/_{2}$ (ii) $\sin\theta={}^{-\sqrt{3}}/_{2}$ (iii) $\sin\theta={}^{1}/_{\sqrt{2}}$

(iv) $\cos\theta = {}^{-1}/_{2}$ (v) $\cos\theta = {}^{\sqrt{3}}/_{2}$ (vi) $\cos\theta = {}^{-1}/_{\sqrt{2}}$ (vii) $\tan\theta = \sqrt{3}$

(viii) $\tan\theta = -1$ (ix) $\cot\theta = 1$ (x) $\sec\theta = 2$ (xi) $\operatorname{cosec}\theta = {}^{2}/_{\sqrt{3}}$

(2) Solve the following equations.

(i) $\sin 9\theta = \sin\theta$ (ii) $\sin 3\theta = \sin 2\theta$

(iii) $\sin 2\theta = \cos 3\theta$ (iv) $\cos 5\theta = \cos 4\theta$

(v) $\cot\theta = \tan 8\theta$ (vi) $\tan 2\theta = \tan {}^{\theta}/_{2}$

(vii) $\sin 2\theta = \sin 3\theta$ (viii) $\tan 3\theta = \cot\theta$

(ix) $\cos 5\theta + \cos 2\theta = 0$ (x) $\cot\theta + \tan 2\theta = 0$

(xi) $\sin\theta + \cos 2\theta = 0$ (xii) $\cos\theta + \sin 2\theta = 0$

(3) Solve the following equations.

(i) $\sin 2\theta - \cos 2\theta - \sin\theta + \cos\theta = 0$

(ii) $\sin\theta + \sin 3\theta + \sin 5\theta = 0$

(iii) $\cos\theta + \cos 2\theta + \cos 3\theta = 0$

(iv) $\sin 7\theta = \sin\theta + \sin 3\theta$

(v) $\cos\theta - \sin 3\theta = \cos 2\theta$

(vi) $\sin 4\theta - \sin 2\theta = \cos 3\theta$

(vii) $\cos\theta + \cos 3\theta = 2\cos 2\theta$

(viii) $2\cos\theta + \cos 7\theta = \cos 4\theta$

(ix) $\sin 7\theta + \sin\theta - \sin 4\theta = 0$

(x) $\sin\theta + \sin 5\theta = \sin 3\theta$

(4) Solve the following equations.

(i) $2\sqrt{3}\cos^2\theta - \sin\theta = 0$

(ii) $2\sin^2\theta + 3\cos\theta = 0$

(iii) $\cos^2\theta - \sin\theta - {}^{1}/_{4} = 0$

(iv) $\sin^2\theta - 2\cos\theta + {}^{1}/_{4} = 0$

(v) $\tan^2\theta - \left(1+\sqrt{3}\right)\tan\theta + \sqrt{3} = 0$

(vi) $\cot^2\theta + \left(\sqrt{3} + {}^{1}/_{\sqrt{3}}\right)\cot\theta + 1 = 0$

(vii) $3\sec^2\theta + 3\tan^2\theta = 5$

(viii) $2\sin^2\theta + \sqrt{3}\cos\theta + 1 = 0$

(5) Solve the following equations by substituting $\tan {}^{x}/_{2} = t$.

(i) $\sqrt{3}\cos x + \sin x = \sqrt{2}$

(ii) $\sin x + \cos x = \sqrt{2}$

(iii) $\sqrt{3}\cos x - \cos x = \sqrt{2}$

(iv) $\tan\theta + \sec\theta = \sqrt{3}$

(v) $\operatorname{cosec}\theta - \cot\theta = \sqrt{3}$

## 12.4 Relations between the sides and the angles of a triangle

### General notation

A triangle has 6 elements, three sides and three angles.

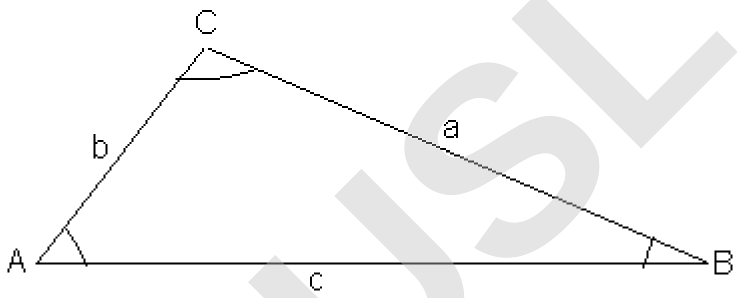



*Figure 12.4.1*

If $A$, $B$ and $C$ are used to denote the angles of the triangle, it is conventional to denote the sides opposite these angles by the corresponding small letters $a$, $b$ and $c$ respectively.

### The Sine Rule

The law of sine is a statement of proportionality between the sides of a triangle and sines of the angles opposite them. Let $ABC$ be any triangle with the sides $a$,$b$,$c$ and opposite angles $A$, $B$ and $C$ respectively.

Then the sine rule states that

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

If the triangle is an acute angle triangle or obtuse angle triangle or right angled triangle, sine rule can be applied.

*Example 8*

Prove that (i) $(b+c)\sin\frac{A}{2} = a\cos\frac{B-C}{2}$

(ii) $(a-b)\cot\frac{A-B}{2} = (a+b)\cot\frac{A+B}{2}$

(iii) $\frac{a-b}{c} = \frac{\tan\frac{A}{2}-\tan\frac{B}{2}}{\tan\frac{A}{2}+\tan\frac{B}{2}}$

*Solution*

(i) $(b+c)\sin\frac{A}{2} = a\cos\frac{B-C}{2}$

Let

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = \lambda \text{ (say)}$$

$$\frac{b+c}{a} = \frac{\lambda(\sin B + \sin C)}{\lambda \sin A} = \frac{\sin B + \sin C}{\sin A}$$

$$= \frac{2\sin\frac{B+C}{2}.\cos\frac{B-C}{2}}{2\sin\frac{A}{2}\cos\frac{A}{2}}$$

$$A + B + C = \pi$$

$$B + C = \pi - A$$

$$\frac{B+C}{2} = \left({}^{\pi}/_{2} - {}^{A}/_{2}\right)$$

$$\sin\frac{B+C}{2} = \sin\left({}^{\pi}/_{2} - {}^{A}/_{2}\right) = \cos\frac{A}{2}$$

$$\therefore \frac{b+c}{a} = \frac{\cos\frac{A}{2}\cos\frac{B-C}{2}}{\sin\frac{A}{2}\cos\frac{A}{2}}$$

$$\frac{b+c}{a} = \frac{\cos\frac{B-C}{2}}{\sin\frac{A}{2}}$$

$$\therefore (b+c)\sin\frac{A}{2} = a\cos\frac{B-C}{2}$$

(ii) $$(a-b)\cot\frac{A-B}{2}=(a+b)\cot\frac{A+B}{2}$$

Let

$$\frac{a-b}{a+b}=\frac{\lambda(\sin A-\sin B)}{\lambda(\sin A+\sin B)}$$

$$=\frac{2\cos\frac{A+B}{2}\sin\frac{A-B}{2}}{2\sin\frac{A+B}{2}\cos\frac{A-B}{2}}$$

$$=\frac{\cot\frac{A+B}{2}}{\cot\frac{A-B}{2}}$$

$$(a-b)\cot\frac{A-B}{2}=(a+b)\cot\frac{A+B}{2}$$

iii) $$\frac{a-b}{c}=\frac{\tan\frac{A}{2}-\tan\frac{B}{2}}{\tan\frac{A}{2}+\tan\frac{B}{2}}$$

$$\frac{a-b}{c}=\frac{\lambda(\sin A-\sin B)}{\lambda\sin C}$$

$$=\frac{2\cos\frac{A+B}{2}\sin\frac{A-B}{2}}{2\sin\frac{C}{2}\cos\frac{C}{2}}$$

$$=\frac{\sin\frac{C}{2}\sin\frac{A-B}{2}}{\sin\frac{C}{2}\cos\frac{C}{2}}\left(\cos\frac{A+B}{2}=\sin\frac{C}{2},\sin\frac{A+B}{2}=\cos\frac{C}{2}\right)$$

$$=\frac{\sin\frac{A-B}{2}}{\cos\frac{C}{2}}=\frac{\sin\frac{A}{2}\cos\frac{B}{2}-\cos\frac{A}{2}\sin\frac{B}{2}}{\sin\frac{A+B}{2}}$$

$$=\frac{\sin\frac{A}{2}\cos\frac{B}{2}-\cos\frac{A}{2}\sin\frac{B}{2}}{\sin\frac{A}{2}\cos\frac{B}{2}+\cos\frac{A}{2}\sin\frac{B}{2}}$$

$$=\frac{\left(\tan\frac{A}{2}-\tan\frac{B}{2}\right)\cos\frac{A}{2}\cos\frac{B}{2}}{\left(\tan\frac{A}{2}+\tan\frac{B}{2}\right)\cos\frac{A}{2}\cos\frac{B}{2}}$$

$$\frac{a-b}{c}=\frac{\tan\frac{A}{2}-\tan\frac{B}{2}}{\tan\frac{A}{2}+\tan\frac{B}{2}}$$

## The Cosine Rule

The law of cosine is a statement, relation between the length.

We know that the sine rule cannot be used if the only information given is that of combination3 (two sides and included angle) or combination (4) three sides.

Therefore, it is necessary to develop a method of finding at least one or more part of the triangle. Here we can use the law of cosine. If $ABC$ is any triangle;

Cosine rule say

$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$b^2 = c^2 + a^2 - 2ca\cos B$$

$$c^2 = a^2 + b^2 - 2ab\cos C$$

*Example 9*

(i) Obtain expressions for

(a) $\cos A$, $\cos B$ and $\cos C$

(b) $\cos\frac{A}{2}$, $\cos\frac{B}{2}$ and $\cos\frac{C}{2}$

(c) $\sin\frac{A}{2}$, $\sin\frac{B}{2}$ and $\sin\frac{C}{2}$

(d) $\sin A$, $\sin B$ and $\sin C$

(ii) In a triangle $ABC$, prove that

$$a^2 = (b-c)^2 + 4bc\sin^2\frac{A}{2}$$

and hence show that $a = (b-c)\sec\theta$ where

$$\tan\theta = \frac{2\sqrt{\mathrm{bc}}}{(\mathrm{b}-\mathrm{c})}\sin\frac{A}{2}$$

(iii)If in a triangle $ABC, ab = c^2$prove that

$$\cos(A-B) + \cos C + \cos 2C = 1$$

(iv)In a triangle $ABC$, prove that

$$\tan B \cot C = \frac{a^2 + b^2 - c^2}{a^2 - b^2 + c^2}$$

*Solution*

(i)

$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$2bc\cos A = b^2 + c^2 - a^2$$

$$\boldsymbol{\cos A = \frac{b^2 + c^2 - a^2}{2bc}} \quad \text{——} ①$$

Similarly

$$\boldsymbol{\cos B = \frac{c^2 + a^2 - b^2}{2ac}}$$

$$\boldsymbol{\cos C = \frac{a^2 + b^2 - c^2}{2ab}}$$

From ①;

$$\cos A = 2\cos^2\frac{A}{2} - 1$$

$$2\cos^2\frac{A}{2} = 1 + \cos A$$

$$\cos^2\frac{A}{2} = \frac{1+\cos A}{2} = \frac{1 + \frac{b^2+c^2-a^2}{2bc}}{2}$$

$$= \frac{2bc + b^2 + c^2 - a^2}{4bc}$$

$$\cos^2\frac{A}{2} = \frac{(b+c)^2 - a^2}{4bc} = \frac{(\mathrm{a}+\mathrm{b}+\mathrm{c})(\mathrm{b}+\mathrm{c}-\mathrm{a})}{4\mathrm{bc}}$$

In a triangle $\cos\frac{A}{2}, \cos\frac{B}{2}, \cos\frac{C}{2}, \sin\frac{A}{2}, \sin\frac{B}{2}, \sin\frac{C}{2}$ are all positive.

$$\therefore \cos\frac{A}{2} = \sqrt{\frac{(a+b+c)(b+c-a)}{4bc}} \text{——} ②$$

Similarly,

$$\cos\frac{B}{2} = \sqrt{\frac{(a+b+c)(a+c-b)}{4ac}}$$

$$\cos\frac{C}{2} = \sqrt{\frac{(a+b+c)(a+b-c)}{4ab}}$$

Again from ①,

$$\cos A = 1 - 2\sin^2\frac{A}{2}$$

$$2\sin^2\frac{A}{2} = 1 - \cos A$$

$$= 1 - \frac{b^2+c^2-a^2}{2bc}$$

$$= \frac{a^2-(b^2-2bc+c^2)}{2bc}$$

$$\sin^2\frac{A}{2} = \frac{a^2-(b-c)^2}{4bc}$$

$$= \frac{(a-(b-c))(a+b-c)}{4bc}$$

$$\sin\frac{A}{2} = \sqrt{\frac{(a+c-b)(a+b-c)}{4bc}}$$

Similarly

$$\sin\frac{B}{2} = \sqrt{\frac{(a+b-c)(b+c-a)}{4ac}}$$

$$\sin\frac{C}{2} = \sqrt{\frac{(a+c-b)(b+c-a)}{4ab}}$$

$$\sin A = 2\sin\frac{A}{2}\cos\frac{A}{2}$$

$$= 2\frac{\sqrt{(a+c-b)(a+b-c)(a+b+c)(b+c-a)}}{4bc}$$

$$= \frac{\sqrt{(a+b+c)(b+c-a)(a+c-b)(a+b-c)}}{2bc}$$

Similarly

$$\sin B = \frac{\sqrt{(a+b+c)(b+c-a)(a+c-b)(a+b-c)}}{2ac}$$

$$\sin C = \frac{\sqrt{(a+b+c)(b+c-a)(a+c-b)(a+b-c)}}{2ab}$$

$$\tan\frac{A}{2} = \frac{\sin\frac{A}{2}}{\cos\frac{A}{2}} = \frac{\sqrt{\frac{(a+b-c)(a+c-b)}{4bc}}}{\sqrt{\frac{(a+b+c)(b+c-a)}{4bc}}}$$

$$\tan\frac{A}{2} = \sqrt{\frac{(a+b-c)(a+c-b)}{(a+b+c)(b+c-a)}}$$

similarly,

$$\tan\frac{B}{2} = \sqrt{\frac{(a+b-c)(b+c-a)}{(a+b+c)(a+c-b)}}$$

$$\tan\frac{C}{2} = \sqrt{\frac{(\mathrm{a+c-b})(\mathrm{b+c-a})}{(\mathrm{a+b+c})(\mathrm{a+b-c})}}$$

If we know the values of sides of a triangle $(a, b, c)$, we can find

$\cos A$,$\cos B$ , $\cos C$ , $\cos\frac{A}{2}$, $\cos\frac{B}{2}$, $\cos\frac{C}{2}$, $\sin\frac{A}{2}$, $\sin\frac{B}{2}$, $\sin\frac{C}{2}$, $\sin A$,$\sin B$,$\sin C$

and $\tan\frac{A}{2}$, $\tan\frac{B}{2}$, $\tan\frac{C}{2}$.

(ii) From the cosine rule,

$$\begin{aligned} a^2 &= b^2 + c^2 - 2bc\cos A \\ &= b^2 + c^2 - 2bc\left(1 - 2\sin^2\frac{A}{2}\right) \\ &= b^2 - 2bc + c^2 + 4bc\sin^2\frac{A}{2} \end{aligned}$$

$$a^2 = (b-c)^2 + 4bc\sin^2\frac{A}{2} \text{——} (1)$$

given that

$$\tan\theta = \frac{2\sqrt{\mathrm{bc}}}{(\mathrm{b-c})}\sin\frac{A}{2}$$

$$\tan^2\theta = \frac{4bc}{(\mathrm{b}-\mathrm{c})^2}\sin^2\frac{A}{2} \text{——} ②$$

∴from①

$$\frac{a^2}{(\mathrm{b}-\mathrm{c})^2} = 1 + \frac{4bc\sin^2\frac{A}{2}}{(\mathrm{b}-\mathrm{c})^2}$$

$$\therefore \frac{a^2}{(\mathrm{b}-\mathrm{c})^2} = 1 + \tan^2\theta = \sec^2\theta$$

$$a^2 = (b-c)^2\sec^2\theta$$

$$a = (b-c)\sec\theta$$

(iii)

$$LHS = \cos(A-B) + \cos C + \cos 2C$$
$$= \cos(A-B) + \cos\big(\pi - (A+B)\big) + \cos 2C$$
$$= \cos(A-B) - \cos(A+B) + \cos^2 C - \sin^2 C$$
$$= 2\sin A\sin B + \cos^2 C - \sin^2 C$$
$$= 2\frac{a}{c}\sin C\,.\frac{b}{c}\sin C + \cos^2 C - \sin^2 C\,;\left(\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}\right)$$
$$= 2\frac{ab}{c^2}\sin^2 C + \cos^2 C - \sin^2 C$$
$$= 2(1)\sin^2 C + \cos^2 C - \sin^2 C\,;\text{since } ab = c^2$$
$$= \sin^2 C + \cos^2 C$$
$$= 1$$
$$\cos(A-B) + \cos C + \cos 2C = 1$$

(iv)

$$\tan B\cot C = \frac{\sin B\cos C}{\cos B\sin C} = \frac{\sin B}{\sin C}.\frac{\cos C}{\cos B}$$

From the sine rule, we have $\frac{\sin B}{\sin C} = \frac{b}{c}$

$$\cos B = \frac{c^2 + a^2 - b^2}{2ac}\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

$$\therefore \tan B\cot C = \frac{b}{c}\frac{(a^2 + b^2 - c^2)}{2ab}.\frac{2ac}{(c^2 + a^2 - b^2)} = \frac{a^2 + b^2 - c^2}{c^2 + a^2 - b^2}$$

*Example 10*

(1) Solve the triangles in which

(i) $c = 15.6m$ $B = 34^0 20'$ $C = 62^0 9'$

(ii) $a = 0.5m$ $b = 0.7m$ $A = 62^0$

(iii) $a = 0.17m$ $b = 0.11m$ $c = 0.10m$

(iv) $a = 7.0m$ $b = 3.59m$ $C = 47^0$

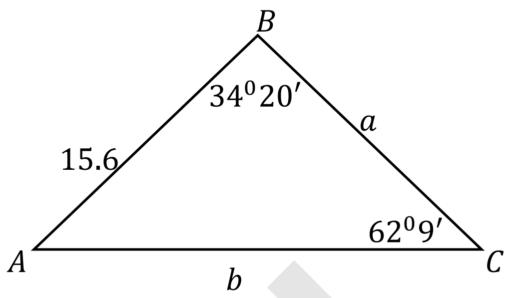



*Figure 12.5.2*

*Solution*

(i) $\hat{B} = 34^0 20'$ $\hat{C} = 62^0 9'$

$$\therefore \hat{A} = 180 - (34^0 20' + 62^0 9')$$

$$A = 83^0 31'$$

$$\frac{a}{\sin 83^0 31'} = \frac{b}{\sin 34^0 20'} = \frac{15.6}{\sin 62^0 9'}$$

$a = 15.6\ \frac{\sin 83^0 31'}{\sin 62^0 9'} = 17.5m$ $b = 15.6\ \frac{\sin 34^0 20'}{\sin 62^0 9'} = 10m$

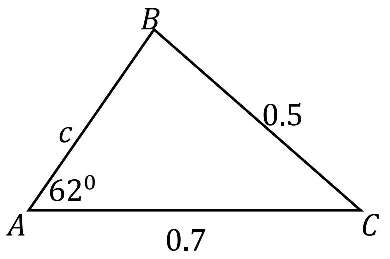



*Figure 2.5.3*

$$\frac{0.5}{\sin 62^0} = \frac{0.7}{\sin B} = \frac{c}{\sin C}$$

$$\sin B = \frac{0.7}{0.5} \sin 62^0 = 12.36$$

(There is no solution)

(iii) $a = 0.17m \quad b = 0.11m \quad c = 0.10m$

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{0.11^2 + 0.1^2 - 0.17^2}{2 \times 0.11 \times 0.10} = \frac{0.0221 - 0.0289}{2 \times 0.11 \times 0.10}$$

$$= -0.3091$$

$$\therefore A = 108^0 0' 16'' = 108^0$$

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac} = \frac{0.17^2 + 0.10^2 - 0.11^2}{2 \times 0.17 \times 0.1} = 0.7882$$

$$B = 37^0 59'$$

$$\therefore C = 180^0 - (37^0 59' + 108^0)$$

$$= 34^0 1'$$

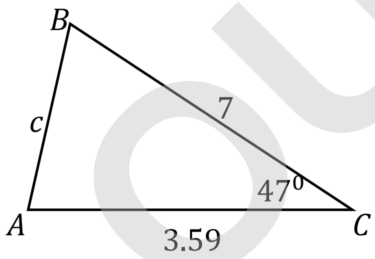



*Figure 12.5.4*

(iv)

$$c^2 = 7^2 + 3.59^2 - 2 \times 7 \times 3.59 \cos 47^0$$

$$c^2 = 7^2 + 3.59^2 - 2(7)(3.59)$$

$$c^2 = 49 + 3.59^2 - 34.2772$$

$$= 27.6105$$

$$c = 5.255m$$

$$\frac{7}{\sin A} = \frac{5.255}{\sin 47^0}$$

$$\sin A = \frac{7}{5.255}\sin 47^0 = 0.9742$$

$$A = 76^0 58'$$

$$B = 180 - (47^0 + 76^0 58')$$

$$= 56^0 2'$$

## Activity 2

1. (a) Prove that in any $ABC$ triangle

$$\frac{a+b}{c} = \frac{1+\tan\frac{A}{2}\tan\frac{B}{2}}{1-\tan\frac{A}{2}\tan\frac{B}{2}}$$

(b) Given that $\frac{b+c}{11} = \frac{c+a}{12} = \frac{a+b}{13}$

find $a{:}\,b{:}\,c$ and hence show that

$$\frac{\sin A}{7} = \frac{\sin B}{6} = \frac{\sin C}{5}$$

and

$$\frac{\cos A}{7} = \frac{\cos B}{19} = \frac{\cos C}{25}$$

(c) If $\sin\theta = \frac{2\sqrt{bc}\cos\frac{A}{2}}{b+c}$

Prove that $(b+c)\cos\theta = a$

(d) Solve the following triangles.

| | | | |
|---|---|---|---|
| (i) | $a = 723cm$ | $b = 958cm$ | $c = 158cm$ |
| (ii) | $a = 21.4cm$ | $c = 4.28cm$ | $B = 86.3^0$ |
| (iii) | $a = 20cm$ | $b = 847cm$ | $C = 158^0$ |
| (iv) | $A = 48^0$ $B = 68^0$ $a = 14.5cm$ | | |
| (v) | $a = 22.8cm$ | $B = 33.5^0$ | $C = 125.3^0$ |

# Solution to Activities

## Activity 1

(1) (i) $n\pi + (-1)^n \pi/6$ (ii) $n\pi - (-1)^n \pi/3$

(iii) $n\pi + (-1)^n \pi/4$ (iv) $2n\pi \pm 2\pi/3$

(v) $2n\pi \pm \pi/6$ (vi) $2n\pi \pm 3\pi/4$

(vii) $n\pi + \pi/3$ (viii) $n\pi + 3\pi/4$

(ix) $n\pi + \pi/4$ (x) $2n\pi \pm \pi/3$

(xi) $n\pi + (-1)^n \pi/3$

(2) (i) $n\pi/4$ or $(2n+1)\pi/10$ (ii) $2n\pi$ or $(2n+1)\pi/5$

(iii) $(2n + 1/2)\pi/5$ or $2n\pi - \pi/2$

(iv) $2n\pi$ or $2n\pi/9$ (v) $(n + 1/2)\pi/9$

(vi) $n\pi \pm \sqrt{1 + n^2\pi^2/16}$

(vii) $\theta = 2n\pi$ or $\theta = (2n+1)\pi/5$

(viii) $\theta = (2n+1)\pi/8$

(ix) $\theta = (2n+1)\pi/7$ or $\theta = (2n-1)\pi/3$

(x) $\theta = (2n+1)\pi/2$

(xi) $\theta = (4r-1)\pi/2$ or $\theta = (4r+1)\pi/2$

(xii) $\theta = (4n-1)\pi/6$ or $\theta = (4n-1)\pi/2$

(3) (i) $2n\pi$ or $\left(\frac{2n}{3} + \frac{1}{2}\right)\pi$

(ii) $n\pi/3$ or $(n \pm 1/3)\pi$

(iii) $(n + 1/2)\pi$ or $2n\pi \pm 2\pi/3$

(iv) $n\pi/3$ or $(2n \pm 1/3)\pi/4$

(v) $2n\pi/3$ or $(n + 1/4)\pi$ or $(2n - 1/2)\pi$

(vi) $(n + 1/2)\pi/3$ or $n\pi + (-1)^n \pi/6$

(vii) $(n + 1/2)\pi/2$ or $2n\pi$

(viii) $(n + 1/2)\pi/4$ or $(2n \pm 1/3)\,\pi/3$

(ix) $n\pi/4$ or $(2n\pi \pm \pi/3)$

(x) $\quad {}^{n\pi}/_{3}$ *or* $n\pi \pm {}^{\pi}/_{6}$

(4) (i) $n\pi + (-1)^n\, {}^{\pi}/_{3}$ $\quad$ (ii) $2n\pi \pm {}^{2\pi}/_{3}$

(iii) $n\pi + (-1)^n\, {}^{\pi}/_{6}$ $\quad$ (iv) $2n\pi \pm {}^{\pi}/_{3}$

(v) $n\pi + {}^{\pi}/_{4}$ *or* $n\pi + {}^{\pi}/_{3}$

(vi) $n\pi + {}^{2\pi}/_{3}$ *or* $n\pi + {}^{5\pi}/_{6}$

(vii) $n\pi \pm {}^{\pi}/_{6}$ $\quad$ (viii) $2n\pi \pm {}^{5\pi}/_{6}$

(5) (i) $2n\pi$ or $2n\pi + {}^{\pi}/_{2}$

(ii) $2n\pi + {}^{\pi}/_{12}$ or $2n\pi + {}^{5\pi}/_{12}$

(iii) $2n\pi + {}^{5\pi}/_{12}$ *or* $2n\pi - {}^{\pi}/_{12}$

(iv) $2n\pi + {}^{\pi}/_{6}$

(v) $2n\pi + {}^{2\pi}/_{3}$

## Activity 2

| | | | |
|---|---|---|---|
| (i) | $A = 137.9^0$ | $B = 33.7^0$ | $C = 84^0$ |
| (ii) | $A = 82.3^0$ | $b = 21.6cm$ | $C = 11.4^0$ |
| (iii) | $A = 6.0^0$ $B = 16.0^0$ | | $c = 1150cm$ |
| (iv) | $b = 18.1cm$ | $C = 64^0$ $c = 17.5cm$ | |
| (v) | $A = 21.2^0$ | $b = 34.8cm$ | $c = 51.5cm$ |



## Summary

General solutions for

(a) $\sin\theta = \sin\alpha$ and $\operatorname{cosec}\theta = \operatorname{cosec}\alpha$ are

$$\theta = n\pi + (-1)^n\alpha \quad n \in \mathbb{Z}$$

$$\theta = 180n^0 + (-1)^n\alpha^0 n \in \mathbb{Z}$$

(b) $\cos\theta = \cos\alpha$ and $\sec\theta = \sec\alpha$ are

$\theta = 2n\pi \pm \alpha \quad n \in \mathbb{Z}$

$\theta = 360n^0 \pm \alpha^0 n \in \mathbb{Z}$

(c) $\tan\theta = \tan\alpha$ and $\cot\theta = \cot\alpha$ are

$\theta = n\pi + \alpha \quad n \in \mathbb{Z}$

$\theta = 180n^0 + \alpha^0 n \in \mathbb{Z}$

For the solutions in some types of trigonometric equations can apply addition, subtraction and product formulae

Also, in some types of trigonometric equation can be solve by using the substitution $\tan x = t$ or $\tan {}^{x}/_{2} = t$

In the general notation we have following rules;

(i) Sine rule

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

(ii) Cosine rule

$$a^2 = b^2 + c^2 - 2bc\cos A$$
$$b^2 = c^2 + a^2 - 2ca\cos B$$
$$c^2 = a^2 + b^2 - 2ab\cos C$$



## Learning outcomes

On completion of this study session you should be able to find

- General solutions of the simple trigonometric equation
- Solve the trigonometric equations by
  - (i) Using the addition, subtraction and product formulae
  - (ii) Substituting $\tan x = t$ or $\tan {}^{x}/_{2} = t$ workout the unknown sides or angles of a triangle by using the sine and cosine rule