UNIT 4
CALCULUS

Contents:

Unit introduction

Calculus is the most important area in mathematics world. The concept of calculus are used in the technology field, economics and social studies

Calculus which is divided mainly in two branches; differential calculus and Integral calculus.

This unit consists basic idea and concept in the calculus

In this unit has five number of session

1. Limits
2. The principle of Derivative
3. Applications of Derivatives
4. Introduction to Integration
5. Integration by parts and Definite Integrals

Session 13

Limits

Contents

Introduction

In this session we discuss the concept of limits of functions. If a variable y is stated in terms of another independent variable x, such that the value "y" can be determined for all given values of x. Hence, we know that y is a function of x, we already notice that $y = f(x), y = g(x)$ or $y = h(x)$ etc. The main idea of this session we have to examine the behavior of the function $f(x)$, when x approaches to a certain value "a". By providing some examples, we explain the meaning of a variable goes to some finite value". The intuitive idea of a limit of a function is presented in this session. Giving some values near a for a variable, we observe the limiting values of a function. Further, we look at some infinite limits. We evaluate limits of some functions by using properties of limits and standard limits. We calculate the limits some functions as goes to infinity. We discuss the limits of trigonometric function using results.

13.1 The meaning of x goes to a

If a variable x can take consecutively all the values near a and x is not equal to a, we say that x goes to a . It is denoted by $x \to a$. In other words,x is very close to a, but not equal to a.

Example 1

Consider the example $x \to$

Then

x and $x < 1$	x and $x > 1$	$\lvert x - a \rvert$
0.9	1.1	0.1
0.99	1.01	0.01
0.999	1.001	0.001
0.9999	1.0001	0.0001
0.99999	1.00001	0.00001

Therefore, you can see $|x - a|$ is very small number, but this number never become zero.

13.2 Intuitive idea of a limit of a function

The meaning of $\lim_{x \to a} f(x)$ is the behavior of $f(x)$, As x approaches to "a", if the function goes to certain value, the it is called **the limit of the function $\boldsymbol{f}$ at $\boldsymbol{x = a}$**. It is denoted by $\boldsymbol{\lim_{x \to a} f(x)}$.

Example 2

Let us consider the function $f(x) = \frac{x^2-4^2}{x-4}$. We investigate the behavior of $f(x)$ near the point $x = 4$.

$f(x) = \frac{x^2-4^2}{x-4}$.

It is clear that $f(4) = \frac{0}{0}$ is meaningless. Thus $f(x)$ is undefined at $x = 4$.

Now, we calculate the values of $f(x)$ that are very close to 4.

Table 13.2.1

$x < 4$	$f(x)$	$x > 4$	$f(x)$
$x = 3.9$	7.900000	$x = 4.1$	8.100000
$x = 3.99$	7.990000	$x = 4.01$	8.010000
$x = 3.999$	7.999000	$x = 4.001$	8.001000
$x = 3.9999$	7.999900	$x = 4.0001$	8.000100
$x = 3.99999$	7.999990	$x = 4.00001$	8.000010
$x = 3.999999$	7.999999	$x = 4.000001$	8.000001

According to the above table, it is clear that as x gets the values nearer and nearer to 4, from either side of 4, $f(x)$ gets closer to 8 from each side. Then the number 8could be considered as the limit of $f(x)$, as x approaches 4 and we write,

$$\lim_{x \to 4} \frac{x^2 - 4^2}{x - 4} = 8.$$

We summarize the intuitive idea of a limit of a function as follows.

We say that $\lim_{x \to a} f(x) = l$ if the values of $f(x)$ get closer and closer to the number “l” as x gets nearer and nearer to “a”, but not equal to “a”.

Note:

We are not concerned with what happens to $f(x)$ when x equals "a", but only what happens to it when x is close to "a", we emphasize that the limit must be the same as x approaches a from the left or from right.

Example 3

Investigate

(a) $\lim_{x \to 1} \frac{1}{x-1}$

(b) $\lim_{x \to 1} \frac{1}{(x-1)^2}$

Solution

$f(x) = \frac{1}{x-1}$ we know that $x \neq 1$.

Some values of $f(x)$can be found as follows.

Table 13.2.2

$x < 1$	$f(x)$	$x > 1$	$f(x)$
0.9	-10	1.1	10
0.99	$(-10)^2$	1.01	10^2
0.999	$(-10)^3$	1.001	10^3
0.9999	$(-10)^4$	1.0001	10^4
0.99999	$(-10)^5$	1.00001	10^5
0.999999	$(-10)^6$	1.000001	10^6

We can see that if x is close to 1 from the left -hand side, then the values of $f(x)$ become very large negative number. Also, if x is close to 1 from the right hand, then the values of $f(x)$ become very large positive number. Further, you can see that these two values are different. Therefore, the limit $\lim_{x \to 1} \frac{1}{x-1}$ is undefined.

Let $g(x) = \frac{1}{(x-1)^2}, x \neq 1$.

Some values of $g(x)$ can be found as follows.

Table 13.2.3

$x < 1$	$x > 1$	$g(x)$
0.9	1.1	10^2
0.99	1.01	10^4
0.999	1.001	10^6
0.9999	1.0001	10^8
0.99999	1.00001	10^{10}
0.999999	1.000001	10^{12}

We can see that if$x$ is very close to 1 from the left hand side and the right hand side, then the values of $g(x)$ become very large positive numbers.

Therefore, we conclude that $\lim_{x\to 1} \frac{1}{(x-1)^2}$ exists and $\lim_{x\to 1} \frac{1}{(x-1)^2} = \infty$ (Infinity).

13.3 Properties of Limits

We now state some properties of limits without proofs.

If $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$exist finitely, then the following are true.

(i) $\lim_{x\to a}\{f(x) \pm g(x)\} = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$.

(ii) $\lim_{x\to a}\{f(x) \cdot g(x)\} = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$

(iii) $\lim_{x\to a} \left\{\frac{f(x)}{g(x)}\right\} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$; provided $\lim_{x\to a} g(x) \neq 0$.

(iv) $\lim_{x\to a}\{f(x)\}^n = \left\{\lim_{x\to a} f(x)\right\}^n$ where $n \in \mathbb{R}$.

(v) $\lim_{x\to a} cf(x) = c\left[\lim_{x\to a} f(x)\right]$ where $c \in \mathbb{R}$.

(vi) If $F(x) = c$ (constant function) where $c \in \mathbb{R}$ $\therefore$ for all values of $\lim_{x \to a} F(x) = c$.

(vii) $\lim_{x \to a} \{f(x)\}^{1/n} = \left\{\lim_{x \to a} f(x)\right\}^{1/n}, \ n \in \mathbb{Z}^+$

(viii) $\lim_{x \to a} x^n = a^n, \ n \in \mathbb{Z}^+$.

13.4 Evaluation of limits of functions

Theorem

$$\lim_{x \to a} \frac{x^n - a^n}{x - a} = na^{n-1}, n \in \mathbb{R}$$

(1) Evaluate the following limits

(i) $\lim_{x \to 2} \frac{x^7 - 128}{x - 2}$ (ii) $\lim_{x \to -2} \frac{x^5 + 32}{x + 2}$

(iii) $\lim_{x \to 5} \frac{x^{1/6} - 5^{1/6}}{x - 5}$ (iv) $\lim_{x \to -3} \frac{x^5 + 243}{x^3 + 27}$

(v) $\lim_{x \to 1} \frac{1 - x^{-1/3}}{1 - x^{-2/3}}$ (vi) $\lim_{x \to -1} \frac{1 + x^{1/3}}{1 + x^{1/5}}$

(vii) $\lim_{x \to 3} \frac{x^{1/5} - 3^{1/5}}{x^{1/4} - 3^{1/4}}$ (viii) $\lim_{x \to 0} \frac{(1+x)^7 - 1}{x}$

(ix) $\lim_{x \to 0} \frac{(2+x)^8 - 2^8}{(2+x)^6 - 2^6}$ (x) $\lim_{x \to a} \frac{(x+2)^{5/2} - (a+2)^{5/2}}{x - a}$

(2)

(a) If $\lim_{x \to 2} \frac{x^n - 2^n}{x - 2} = 80$ and if n is a positive integer find n.

(b) If $\lim_{x \to 1} \frac{x^4 - 1}{x - 1} = \lim_{x \to k} \frac{x^3 - k^3}{x^2 - k^2}$ find the value of k.

(c) $\lim_{x \to a} \frac{x^9 - a^9}{x - a} = 9$ find all possible values of a.

(d) Evaluate $\lim_{x \to 2} \frac{(2x+4)^{1/3} - 2}{(x-2)}$

Solution

(1)

We have $\lim_{x \to a} \frac{x^n - a^n}{x - a} = na^{n-1}$.

(i) $\lim_{x\to 2}\frac{x^7-128}{x-2}$

$= \lim_{x\to 2}\frac{x^7-2^7}{x-2} = 7(2)^6 = 448.$

(ii) $\lim_{x\to -2}\frac{x^5+32}{x+2}$

$= \lim_{x\to -2}\frac{x^5-(-2)^5}{x-(-2)} = 5(-2)^4 = 80.$

(iii) $\lim_{x\to 5}\frac{x^{1/6}-5^{1/6}}{x-5} = \frac{1}{6}5^{1/6-1} = \frac{1}{6}5^{-5/6} = \frac{1}{615^{5/6}}.$

(iv) $\lim_{x\to -3}\frac{x^5+243}{x^3+27}$

$$= \lim_{x\to -3}\left[\left\{\frac{x^5-(-3)^5}{x-(-3)}\right\}\times\frac{1}{\left\{\frac{x^3-(-3)^3}{x-(-3)}\right\}}\right]$$

$$= \frac{\lim_{x\to -3}\left\{\frac{x^5-(-3)^5}{x-(-3)}\right\}}{\lim_{x\to -3}\left\{\frac{x^3-(-3)^3}{x-(-3)}\right\}} = \frac{5(-3)^4}{3.(-3)^2}$$

$$= \frac{5}{3}(-3)^2 = 15.$$

(v) $\lim_{x\to 1}\frac{1-x^{-1/3}}{1-x^{-2/3}}$

$$= \frac{\lim_{x\to 1} -\left\{\frac{x^{-1/3}-1^{-1/3}}{x-1}\right\}}{\lim_{x\to 1} -\left\{\frac{x^{-2/3}-1^{-2/3}}{x-1}\right\}}$$

$$= \frac{-\frac{1}{3}(1)^{(-1/3-1)}}{-\frac{2}{3}(1)^{(-2/3-1)}} = \frac{1}{2}.$$

(vi) $\lim_{x\to -1}\frac{1+x^{1/3}}{1+x^{1/5}}$

$$= \lim_{x\to -m}\left\{\frac{\frac{x^{1/3}-(-1)^{1/3}}{x-(-1)}}{\frac{x^{1/5}-(-1)^{1/5}}{x-(-1)}}\right\}$$

$$= \frac{\frac{1}{3}((-1^{(1/3-1)}}{\frac{1}{5}(1)1^{(1/5-1)}}$$

$$= \frac{\frac{1}{3}(1)1^{2/3}}{\frac{1}{5}(1)1^{4/5}} = \frac{5}{3}.$$

(vii)

(viii) $\lim\limits_{x\to 0} \frac{(1+x)^7-1}{x}$

$$\therefore \lim_{x+1\to 1} \left\{\frac{(1+x)^7-1}{(x+1)-1}\right\}$$

Let $x+1=y$.

$\lim\limits_{y\to 1} \frac{y^7-1}{y-1} = 7(1)^6 = 7.$

(ix) $\lim\limits_{x\to 0} \frac{(2+x)^8-2^8}{(2+x)^6-2^6}$

$$\therefore \lim_{2+x\to 2} \frac{(2+x)^8-2^8}{(2+x)^6-2^6}$$

Let $2+x=y$.

$$= \lim_{y\to 2} \left\{\left(\frac{y^8-2^8}{y-2}\right) \times \frac{1}{\left(\frac{y^6-2^6}{y-2}\right)}\right\}$$

$$= \frac{\lim\limits_{y\to 2} \frac{y^8-2^8}{y-2}}{\lim\limits_{y\to 2} \frac{y^6-2^6}{y-2}} = \frac{8\times 2^7}{6\times 2^5} = \frac{4}{3}\times 2^2$$

$$= \frac{16}{3}.$$

(x) $\lim\limits_{x\to a} \frac{(x+2)^{5/2}-(a+2)^{5/2}}{x-a}$

$$\lim_{(x+2)\to(a+2)} \frac{(x+2)^{5/2}-(a+2)^{5/2}}{(x+2)-(a+2)}$$

Let $x+2=y$ and $a+2=b$.

$$= \lim_{y \to b} \frac{y^{5/2} - b^{5/2}}{y - b} = \frac{5}{2} b^{5/2 - 1}$$

$$= \frac{5}{2} b^{3/2} = \frac{5}{2}(a+2)^{3/2}.$$

(2)

(a) $\lim_{x \to 2} \frac{x^n - 2^n}{x - 2} = 80$

$$\therefore n(2)^{n-1} = 80$$

We can assume that n is multiple of 5.

Case No. 1 $n = 5$, LHS $5 \cdot (2)^4 = 80$. $\therefore n = 5$.

(b) $\lim_{x \to 1} \frac{x^4 - 1}{x - 1} = \lim_{x \to k} \frac{x^3 - k^3}{x^2 - k^2}$

$$= \lim_{x \to k} \left\{ \frac{x^3 - k^3}{x - k} \times \frac{1}{\frac{x^2 - k^2}{x - k}} \right\}$$

$$\lim_{x \to 1} \frac{x^4 - 1}{x - 1} = \frac{\lim_{x \to k} \frac{x^3 - k^3}{x - k}}{\lim_{x \to k} \frac{x^2 - k^2}{x - k}}$$

$$4(1)^3 = \frac{3(k)^2}{2(k)}$$

$$\therefore k = \frac{8}{3}.$$

(c) $\lim_{x \to a} \frac{x^9 - a^9}{x - a} = 9$

$$9a^8 = 9$$

$$\therefore a^8 = 1$$

The real solutions for a are $a = \pm 1$.

(d) $\lim_{x \to 2} \frac{(2x+4)^{1/3} - 2}{(x-2)}$

Let $t = 2x + 4$.

$x = \left(\frac{t-4}{2}\right)$ we have $x \to 2$, $t \to 8$.

$$x-2=\frac{t-4}{2}-2=\left(\frac{t-8}{2}\right)$$

$$\therefore \lim_{x\to 2}\frac{(2x+4)^{1/3}-2}{(x-2)}=\lim_{t\to 8}\frac{t^{1/3}-2}{\left(\frac{t-8}{2}\right)}$$

$$\lim_{x\to 2}\frac{(2x+4)^{1/3}-2}{(x-2)}=\lim_{t\to 8}\frac{t^{1/3}-8^{1/3}}{t-8}\times 2$$

$$=\tfrac{1}{3}(8)^{1/3-1}\times(2)=\tfrac{2}{3}(2^3)^{-2/3}$$

$$=\tfrac{2}{3}(2)^{-2}=\tfrac{2}{3\cdot 4}=\tfrac{1}{6}.$$

$$\therefore \lim_{x\to 2}\tfrac{(2x+4)^{1/3}-2}{(x-2)}=\tfrac{1}{6}.$$

13.5 Evaluation of algebraic limits

Method of factorization

If $f(x)$ and $g(x)$ are polynomials and $g(a)\neq 0$ then we have

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{\lim_{x\to a}f(x)}{\lim_{x\to a}g(x)}=\frac{f(a)}{g(a)}.$$

Now if $f(a)=g(a)=0$, then we know that $(x-a)$ is a factor of both $f(x)$ and $g(x)$. We can cancel this common factor $(x-a)$ since $x-a\neq 0$. Again put $x=a$ in the resulted expression. If we get a meaningful number then that number is the limit of the given function, otherwise we repeat this process till we get a meaningful answer.

We can use following results for simplification.

$$x^2-y^2\equiv(x-y)(x+y)$$

$$x^3-y^3\equiv(x-y)(x^2+xy+y^2)$$

$$x^3+y^3\equiv(x+y)(x^2-xy+y^2)$$

$$x^4-y^4\equiv(x+y)(x-y)(x^2+y^2)$$

Example 5

Evaluate the following limits.

(i) $\lim_{x\to 2}\frac{x^7-2^7}{x-2}$ (ii) $\lim_{x\to 1}\frac{x-1}{2x^2-7x+5}$

(iii) $\lim_{x\to -3}\frac{x^3+4x^2+4x+3}{x^2+2x-3}$ (iv) $\lim_{x\to 3}\frac{x^3-4x-15}{x^3+x^2-6x-18}$

(v) $\lim_{x\to 1}\frac{x^4-3x^3+2}{x^3-5x^2+3x+1}$ (vi) $\lim_{x\to 2}\frac{x^2-9}{x-3}$

(vii) $\lim_{x\to 3}\left\{(x^2-9)\left[\frac{1}{x+3}+\frac{1}{x-3}\right]\right\}$ (viii) $\lim_{x\to 3}\left\{\left[\frac{1}{x-3}-\frac{3}{x(x^2-5x+6)}\right]\right\}$

(ix) $\lim_{x\to 2}\frac{x^3-8}{x^2-4}$ (x) $\lim_{x\to -3}\frac{x^3+27}{x+3}$

Solution

Let $\lim_{x\to a}\frac{f(x)}{g(x)}$

(i) $\lim_{x\to 2}\frac{x^7-2^7}{x-2}$

$$f(x) = x^7-2^7 \therefore f(2) = 0$$

$$g(x) = x-2 \qquad g(2) = 0$$

$\therefore (x-2)$ is the common factor of $f(x)$ and $g(x)$.

$$f(x) \equiv x^7-2^7 \equiv (x-2)\{x^6+2x^5+4x^4+8x^3+16x^2+32x+64\}$$

$$\therefore \frac{f(x)}{g(x)} = \frac{(x-2)\{x^6+2x^5+4x^4+8x^3+16x^2+32x+64\}}{(x-2)}$$

$$= (x^6+2x^5+4x^4+8x^3+16x^2+32x+64)$$

$$\therefore \lim_{x\to 2}\frac{x^7-2^7}{x-2} = \lim_{x\to 2}\{x^6+2x^5+4x^4+8x^3+16x^2+32x+64\}$$

$$= 7sim$$

$$= 448.$$

Note that

$$x^n-a^n \equiv (x-a)\{x^{n-1}+ax^{n-2}+a^2x^{n-3}+a^3x^{n-4}+\cdots+a^rx^{n-r-1} + \cdots + a^{n-2}x+a^{n-1}\}$$

(ii) $\lim_{x\to 1}\frac{x-1}{2x^2-7x+5}$

$$f(x) = x - 1 \quad f(1) = 0$$

$$g(x) = 2x^2 - 7x + 5 \quad g(1) = 0$$

$\therefore$ $(x - 1)$ is a common factor of $f(x)$ and $g(x)$.

$$2x^2 - 7x + 5 = (x - 1)(2x - 5)$$

$\lim\limits_{x\to 1} \frac{x-1}{2x^2-7x+5} = \lim\limits_{x\to 1} \frac{(x-1)}{(x-1)(2x-5)} = \lim\limits_{x\to 1} \frac{1}{(2x-5)} = \frac{1}{2-5} = -\frac{1}{3}$.

(iii) $\lim\limits_{x\to -3} \frac{x^3+4x^2+4x+3}{x^2+2x-3}$

$$f(x) = x^3 + 4x^2 + 4x + 3 \qquad f(-3) = -27 + 36 - 12 + 3 = 0$$

$$g(x) = x^2 + 2x - 3 \qquad g(-3) = 9$$

$(x + 3)$ is a common factor of $x^3 + 4x^2 + 4x + 3$ and $x^2 + 2x - 3$.

$$(x^3 + 4x^2 + 4x + 3\ ) = (x + 3)\{x^2 + x + 1\}$$

$$x^2 + 2x - 3 \equiv (x + 3)(x - 1)$$

$$\therefore \lim_{x\to -3} \frac{x^3 + 4x^2 + 4x + 3}{x^2 + 2x - 3}$$

$$= \lim_{x\to -3} \frac{(x + 3)\{x^2 + x + 1\}}{(x + 3)(x - 1)} = \lim_{x\to -3} \frac{(x^2 + x + 1)}{x - 1}$$

$$= \frac{(-3)^2 + (-3) + 1}{-3 - 1}$$

$$= -3\frac{7}{4}.$$

(iv) $\lim\limits_{x\to 3} \frac{x^3-4x-15}{x^3+x^2-6x-18}$

$$f(x) = x^3 - 4x - 15 \qquad f(3) = 27$$

$$g(x) = x^3 + x^2 - 6x - 18 \qquad g(3) = 27 + 9 \qquad c$$

$\therefore$ $(x - 3)$ is a common factor of $x^3 - 4x - 15$ and $x^3 + x^2 - 6x - 18$

$$x^3 - 4x - 15 = (x - 3)\{x^2 + 3x + 5\}$$

$$x^3 + x^2 - 6x - 18 = (x - 3)\{x^2 + 4x + 6\}$$

$\lim\limits_{x\to 3} \frac{x^3-4x-15}{x^3+x^2-6x-18} = \lim\limits_{x\to 3} \frac{(x-3)\{x^2+3x+5\}}{(x-3)\{x^2+4x+6\}} = \lim\limits_{x\to 3} \frac{x^2+3x+5}{x^2+4x+6} = \frac{3^2+3^2+5}{3^2+4\cdot 3+6} = \frac{23}{27}$.

(v) $\lim_{x\to 1} \frac{x^4-3x^3+2}{x^3-5x^2+3x+1}$

$$f(x) = x^4 - 3x^3 + 2 \qquad f(1) = 0$$

$$g(x) = x^3 - 5x^2 + 3x + 1 \qquad g(1) = 0$$

$(x-1)$ is a factor of $x^4 - 3x^3 + 2$ and $x^3 - 5x^2 + 3x + 1$;

$$x^4 - 3x^3 + 2 = (x-1)\{x^3 - 2x^2 - 2x - 2\}$$

$$x^3 - 5x^2 + 3x + 1 \equiv (x-1)\{x^2 - 4x - 1\}$$

$$\therefore \lim_{x\to 1} \frac{x^4 - 3x^3 + 2}{x^3 - 5x^2 + 3x + 1} = \lim_{x\to 1} \frac{(x-1)\{x^3 - 2x^2 - 2x - 2\}}{(x-1)\{x^2 - 4x - 1\}}$$

$$= \lim_{x\to 1} \frac{x^3-2x^2-2x-2}{x^2-4x-1} = \frac{1424242}{14242} = \frac{5}{4}.$$

(vi) $\lim_{x\to 2} \frac{x^2-9}{x-3} = \lim_{x\to 2} \frac{(x-3)(x+3)}{(x-3)} = \lim_{x\to 2} x + 3 = 5.$

(vii) $\lim_{x\to 3} \left\{(x^2-9)\left[\frac{1}{x+3} + \frac{1}{x-3}\right]\right\}$

$$= (x^2-9)\left[\frac{1}{x+3} + \frac{1}{x-3}\right]$$

$$= (x^2-9)\frac{(x-3+x+3)}{(x+3)(x-3)} = \frac{(x^2-9)2x}{(x^2-9)} = 2x$$

$$\therefore \lim_{x\to 3} \left\{(x^2-9)\left[\frac{1}{x+3} + \frac{1}{x-3}\right]\right\} = \lim_{x\to 3} 2x = 2.3 = 6$$

(viii) $\lim_{x\to 3} \left\{\left[\frac{1}{x-3} - \frac{3}{x(x^2-5x+6)}\right]\right\}$

$$\frac{1}{x-3} - \frac{3}{x(x^2-5x+6)} = \frac{1}{x-3} - \frac{3}{x(x-3)(x-2)}$$

$$= \frac{x(x-2)-3}{x(x-3)(x-2)}$$

$$= \frac{x^2-2x-3}{x(x-3)(x-2)}$$

$$= \frac{(x-3)(x+1)}{x(x-3)(x-2)} = \frac{x+1}{x(x-2)}.$$

$$\therefore \lim_{x\to 3} \left\{\left[\frac{1}{x-3} - \frac{3}{x(x^2-5x+6)}\right]\right\} = \lim_{x\to 3} \frac{x+1}{x(x-2)} = \frac{3+1}{3(3-2)} = \frac{4}{3}.$$

(ix) $\lim\limits_{x \to 2} \frac{x^3 - 8}{x^2 - 4}$

$= \lim\limits_{x \to 2} \frac{x^3 - 2^3}{x^2 - 4}$

$= \lim\limits_{x \to 2} \dfrac{(x-2)(x^2+2x+4)}{(x-2)(x+2)} = \lim\limits_{x \to 2} \dfrac{x^2+2x+4}{x+2}$

$= \frac{2^2 + 2^2 + 4}{2+2} = 3.$

(x) $\lim\limits_{x \to -3} \frac{x^3 + 27}{x+3}$

$= \lim\limits_{x \to -3} \frac{(x+3)(x^2-3x+9)}{(x+3)}$

$= \lim\limits_{x \to -3} (x^2 - 3x + 9)$

$= (-3)^2 - 3(-3) + 9 = 27.$

As x goes to infinity if a function approaches some value, then it is called the limit of the function as x goes to infinity.

Note

Let c be a real number and k be a positive rational number,

then $\lim\limits_{x \to \infty} \frac{c}{x^k} = 0$

Example 6

Evaluate the following limits.

(a) $\lim\limits_{x \to \infty} \frac{2x^3 + x^2 + 2}{x^2 + x + 2}$

(b) $\lim\limits_{x \to \infty} \frac{2x^3 + x^2 + 2}{x^4 + x^3 + 2}$

(c) $\lim\limits_{x \to \infty} \frac{ax^2 + bx + c}{cx^2 + bx + a}$

(d) $\lim\limits_{x \to \infty} \frac{\sqrt{1+x^3} - \sqrt{1+x}}{\sqrt{1+x^2} - \sqrt{1+x}}$

(e) $\lim\limits_{x \to \infty} \frac{3x^2 + 4x + 5}{x+1}$

(f) $\lim\limits_{x \to \infty} \frac{3x+5}{x^2+1}$

(g) $\lim\limits_{x \to \infty} \left(\sqrt{x^2+1} - x\right)$

Solution

(a) $\lim\limits_{x\to\infty} \frac{2x^3+x^2+2}{x^2+x+2} = \lim\limits_{x\to\infty} \left\{\frac{\frac{2x^3}{x^3}+\frac{x^2}{x^3}+\frac{2}{x^3}}{\frac{x^2}{x^3}+\frac{x}{x^3}+\frac{2}{x^3}}\right\}$

$$= \lim_{x\to\infty} \left[\frac{2+\frac{1}{x}+\frac{2}{x^3}}{\frac{1}{x}+\frac{1}{x^2}+\frac{2}{x^3}}\right]$$

We know that when $\lim\limits_{x\to\infty} \frac{1}{x} = 0,\ \lim\limits_{x\to\infty} \frac{1}{x^2} = 0$ and $\lim\limits_{x\to\infty} \frac{1}{x^3} = 0$

$$\lim_{x\to\infty} \frac{2x^3+x^2+2}{x^2+x+2} = \left(\frac{2+0+0}{0+0+0}\right)$$

(b) $\lim\limits_{x\to\infty} \frac{2x^3+x^2+2}{x^4+x^3+2} = \lim\limits_{x\to\infty} \frac{2+\frac{1}{x}+\frac{2}{x^3}}{x+1+\frac{2}{x^3}}$

$$\lim_{x\to\infty} \frac{1}{x} = 0,\ \lim_{x\to\infty} \frac{1}{x^3} = 0$$

$$= \frac{2}{\infty} = 0.$$

(c) $\lim\limits_{x\to\infty} \frac{ax^2+bx+c}{cx^2+bx+a}$

$$= \lim_{x\to\infty} \frac{a+\frac{b}{x}+\frac{c}{x^2}}{c+\frac{b}{x}+\frac{a}{x^2}}$$

$\lim\limits_{x\to\infty} \frac{b}{x} = 0,\ \lim\limits_{x\to\infty} \frac{c}{x^2} = 0,\ \lim\limits_{x\to\infty} \frac{a}{x^2} = 0$

$\therefore \lim\limits_{x\to\infty} \frac{ax^2+bx+c}{cx^2+bx+a} = \frac{a}{c}.$

(d) $\lim\limits_{x\to\infty} \frac{\sqrt{1+x^3}-\sqrt{1+x}}{\sqrt{1+x^2}-\sqrt{1+x}}$

We can evaluate this question by two ways.

1st method: Since $x \to \infty$ the term containing “1” in $1+x^3$, $1+x$,

$1+x^2$ can be neglected when it is compared with x

$$\lim_{x\to\infty}\frac{\sqrt{1+x^3}-\sqrt{1+x}}{\sqrt{1+x^2}-\sqrt{1+x}}=\lim_{x\to\infty}\frac{x^{3/2}-x^{1/2}}{x-x^{1/2}}$$

$$=\lim_{x\to\infty}\frac{x^{3/2}\left(1-\frac{1}{x}\right)}{x\left(1-\frac{1}{x^{1/2}}\right)}$$

$$=\lim_{x\to\infty}\frac{x^{1/2}\left(1-\frac{1}{x}\right)}{\left(1-\frac{1}{x^{1/2}}\right)}$$

$$=\infty \text{ since } \lim_{x\to\infty}\frac{1}{x}=0,\ \lim_{x\to\infty}\frac{1}{\sqrt{x}}=0.$$

2nd method: $\lim_{x\to\infty}\frac{\sqrt{1+x^3}-\sqrt{1+x}}{\sqrt{1+x^2}-\sqrt{1+x}}=$

$$\lim_{x\to\infty}\frac{\left(\sqrt{1+x^3}-\sqrt{1+x}\right)\left(\sqrt{1+x^3}+\sqrt{1+x}\right)\left(\sqrt{1+x^2}+\sqrt{1+x}\right)}{\left(\sqrt{1+x^2}-\sqrt{1+x}\right)\left(\sqrt{1+x^2}+\sqrt{1+x}\right)\left(\sqrt{1+x^3}+\sqrt{1+x}\right)}$$

By multiplying both numerator and denominator their conjugates.

$$=\lim_{x\to\infty}\left\{\frac{(1+x^3)-(1+x)}{(1+x^2)-(1+x)}\right\}\left(\frac{\sqrt{1+x^2}+\sqrt{1+x}}{\sqrt{1+x^3}+\sqrt{1+x}}\right)$$

$$=\lim_{x\to\infty}\frac{(x^3-x)}{(x^2-x)}\frac{\left(\sqrt{1+x^2}+\sqrt{1+x}\right)}{\left(\sqrt{1+x^3}+\sqrt{1+x}\right)}$$

$$\lim_{x\to\infty}\frac{x(x-1)(x+1)}{x(x-1)}\left(\frac{\sqrt{1+x^2}+\sqrt{1+x}}{\sqrt{1+x^3}+\sqrt{1+x}}\right)$$

$$\lim_{x\to\infty}x\left(\frac{\sqrt{1+x^2}+\sqrt{1+x}}{\sqrt{1+x^3}+\sqrt{1+x}}\right)$$

$$\lim_{x\to\infty}\frac{x^2\left[\sqrt{\frac{1}{x^2}+1}+\sqrt{\frac{1}{x^2}+\frac{1}{x}}\right]}{x^{3/2}\left[\sqrt{\frac{1}{x^3}+1}+\sqrt{\frac{1}{x^3}+\frac{1}{x^2}}\right]}$$

$$\lim_{x\to\infty} x^{1/2} \frac{\left[\sqrt{\frac{1}{x^2}+1}+\sqrt{\frac{1}{x^2}+\frac{1}{x}}\right]}{\left[\sqrt{\frac{1}{x^3}+1}+\sqrt{\frac{1}{x^3}+\frac{1}{x^2}}\right]} = \infty$$

Since $\lim_{x\to\infty}\frac{1}{x} = 0,\ \lim_{x\to\infty}\frac{1}{x^2} = 0,\ \lim_{x\to\infty}\frac{1}{x^3} = 0.$

(e) $\lim_{x\to\infty}\frac{3x^2+4x+5}{x+1}$

$$= \lim_{x\to\infty} x\frac{\left[3+\frac{4}{x}+\frac{5}{x^2}\right]}{\left(1+\frac{1}{x}\right)}$$

$$= \infty$$

Since $\lim_{x\to\infty}\frac{1}{x} = 0,\ \lim_{x\to\infty}\frac{1}{x^2} = 0.$

(f) $\lim_{x\to\infty}\frac{3x+5}{x^2+1}$

$$= \lim_{x\to\infty}\frac{x\left[3+\frac{5}{x}\right]}{x^2\left(1+\frac{1}{x^2}\right)}$$

$$= \lim_{x\to\infty}\frac{1}{x}\frac{\left(3+\frac{5}{x}\right)}{\left(1+\frac{1}{x^2}\right)}$$

$$= 0 \quad \because \lim_{x\to\infty}\frac{1}{x} = 0,\ \lim_{x\to\infty}\frac{1}{x^2} = 0.$$

(g) $\lim_{x\to\infty}\left(\sqrt{x^2+1}-x\right)$

$$= \lim_{x\to\infty}\frac{\left(\sqrt{x^2+1}-x\right)\left(\sqrt{x^2+1}+x\right)}{\left(\sqrt{x^2+1}+x\right)}$$

$$= \lim_{x\to\infty}\frac{(x^2+1-x^2)}{\left(\sqrt{x^2+1}+x\right)} = \lim_{x\to\infty}\frac{1}{\left(\sqrt{x^2+1}+x\right)}$$

$$= 0.$$

Example 7

Find the limiting values of the followings.

(a) $\lim\limits_{x\to 0}\frac{\sqrt{x^4+1}-(2x^2+1)}{x^2}$

(b) $\lim\limits_{x\to 0}\frac{\sqrt{1+x}-\sqrt{1+x}}{\sqrt{1+x^2}-\sqrt{1+x^2}}$

(c) $\lim\limits_{x\to 0}\frac{x}{\sqrt{1+x}-1}$

(d) $\lim\limits_{x\to 0}\frac{1}{x}$

(e) $\lim\limits_{x\to 0}\frac{1}{x^2}$

(f) $\lim\limits_{x\to 4}\frac{1}{(x-4)}$

(g) $\lim\limits_{x\to 4}\frac{1}{(x-4)^2}$

Solution

(a) $\lim\limits_{x\to 0}\frac{\sqrt{x^4+1}-(2x^2+1)}{x^2}$

$$\lim_{x\to 0}\frac{\left[\sqrt{x^4+1}-(2x^2+1)\right]}{x^2}\frac{\left[\sqrt{x^4+1}+(2x^2+1)\right]}{\sqrt{x^4+1}+(2x^2+1)}$$

$$=\lim_{x\to 0}\frac{(x^4+1)-(2x^2+1)^2}{x^2}\frac{1}{\sqrt{x^4+1}+(2x^2+1)}$$

$$=\lim_{x\to 0}\frac{(x^4+1)-4x^4-4x^2-1}{x^2}\frac{1}{\sqrt{x^4+1}+(2x^2+1)}$$

$$=\lim_{x\to 0}\left(-\frac{[3x^4+4x^2]}{x^2}\right)\frac{1}{\sqrt{x^4+1}+(2x^2+1)}$$

$$=\lim_{x\to 0}-(3x^2+4)\frac{1}{\sqrt{x^4+1}+(2x^2+1)}$$

$$=-\frac{4}{1+1}=-2\,.$$

(b) $\lim\limits_{x\to 0}\frac{\sqrt{1+x}-\sqrt{1+x}}{\sqrt{1+x^2}-\sqrt{1+x^2}}$

$$=\lim_{x\to 0}\frac{\left(\sqrt{1+x}-\sqrt{1+x}\right)\left(\sqrt{1+x}+\sqrt{1+x}\right)\left(\sqrt{1+x^2}+\sqrt{1+x^2}\right)}{\left(\sqrt{1+x^2}-\sqrt{1+x^2}\right)\left(\sqrt{1+x^2}+\sqrt{1+x^2}\right)\left(\sqrt{1+x}+\sqrt{1+x}\right)}$$

$$=\lim_{x\to 0}\frac{[(1+x)-(1-x)]}{(1+x^2)-(1-x^2)}\left\{\frac{\sqrt{1+x^2}+\sqrt{1+x^2}}{\sqrt{1+x}+\sqrt{1+x}}\right\}$$

$$= \lim_{x \to 0} \frac{2x}{2x^2} \left(\frac{\sqrt{1+x^2} + \sqrt{1+x^2}}{\sqrt{1+x} + \sqrt{1+x}} \right)$$

$$= \lim_{x \to 0} \frac{1}{x} \left(\frac{\sqrt{1+x^2} + \sqrt{1-x^2}}{\sqrt{1+x} + \sqrt{1+x}} \right)$$

In this case we can see that if x is very close to zero from the left hand side, then the value of the given function becomes very large negative number. Also, if x is very close to zero from the right hand side, then the value of the given function becomes very large positive number.

$\lim_{x \to 0^-} \frac{1}{x} \left(\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x}+\sqrt{1+x}} \right) \to -\infty$ and $\lim_{x \to 0^+} \frac{1}{x} \left(\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x}+\sqrt{1+x}} \right) \to +\infty$

$$\lim_{x \to 0^-} \frac{1}{x} \left(\frac{\sqrt{1+x^2} + \sqrt{1-x^2}}{\sqrt{1+x} + \sqrt{1+x}} \right) \neq \lim_{x \to 0^+} \frac{1}{x} \left(\frac{\sqrt{1+x^2} + \sqrt{1-x^2}}{\sqrt{1+x} + \sqrt{1+x}} \right)$$

$\therefore \lim_{x \to 0} \frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x^2}-\sqrt{1+x^2}}$ is not defined.

Note

$\lim_{x \to a^-} f(x)$= *the left- handed limit of* $f(x)$ *as x tends to* a^-

$\lim_{x \to a^+} f(x)$= *the right- handed limit of* $f(x)$ *as x tends to* a^+

$\boldsymbol{\lim_{x \to a} f(x) = l}$ *exists if and only if* $\boldsymbol{\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = l}$

(c) $\lim_{x \to 0} \frac{x}{\sqrt{1+x}-1} = \lim_{x \to 0} \frac{x}{\sqrt{1+x}-1} \cdot \frac{(\sqrt{1+x}+1)}{(\sqrt{1+x}+1)}$

$$= \lim_{x \to 0} \frac{x\left[(1+x)^{1/2} + 1\right]}{(1+x-1)}$$

$$= \lim_{x \to 0} \left[\sqrt{1+x} + 1\right]$$

$$= 2.$$

(d) $\lim_{x \to 0} \frac{1}{x}$ is undefined.

$\lim_{x \to 0^-} \frac{1}{x} = -\infty$ and $\lim_{x \to 0^+} \frac{1}{x} = +\infty$

$$\lim_{x\to 0^-}\frac{1}{x} \neq \lim_{x\to 0^+}\frac{1}{x}$$

$\lim\limits_{x\to 0}\frac{1}{x}$ is undefined

(e) $\lim\limits_{x\to 0}\frac{1}{x^2} = \infty$.

(f) $\lim\limits_{x\to 4}\frac{1}{(x-4)}$ is undefined.

$$\lim_{x\to a}\frac{1}{(x-a)^n} = \begin{cases} undefined & \text{when } n \text{ is odd} \\ +\infty & \text{when } n \text{ is even} \end{cases}$$

(g) $\lim\limits_{x\to 4}\frac{1}{(x-4)^2} = \infty$.

Activity 1

Evaluate the following limits.

(a) $\lim\limits_{x\to 6}\frac{x^3-216}{x-6}$

(b) $\lim\limits_{x\to -2}\frac{x^5+32}{x+2}$

(c) $\lim\limits_{x\to 3}\frac{x^5-243}{x^2-9}$

(d) $\lim\limits_{x\to 2}\frac{x^{10}-1024}{x^5-32}$

(e) $\lim\limits_{x\to 27}\frac{\left(x^{1/3}+3\right)\left(x^{1/3}-3\right)}{x-27}$

(f) $\lim\limits_{x\to 0}\frac{(1+x)^5-1}{3x+5x^2}$

(g) $\lim\limits_{x\to 2}\frac{x^3-8}{x^2-4}$

(h) $\lim\limits_{x\to 3}\frac{x^{-3}-3^{-3}}{x^{-5}-3^{-5}}$

If $\lim\limits_{x\to a}\frac{x^n-3^n}{x-3} = 405$ find the integer value of n.

If $\lim\limits_{x\to a}\frac{x^3-a^3}{x^2-a^2} = 3$ find a.

If $\lim\limits_{x\to -a}\frac{x^9+a^9}{x+a} = 9$ find all possible values of n.

Evaluate the following limits.

(a) $\lim\limits_{x\to 3}\frac{x^2-4x+3}{2x^2-3x-9}$

(b) $\lim\limits_{x\to -3}\frac{2x^2+7x+3}{3x^2+8x-3}$

(c) $\lim\limits_{x\to 2}\frac{x^3-3x^2+4}{x^4-8x^2+16}$

(d) $\lim\limits_{x\to 0}\frac{(1+x)^4-1}{x}$

(e) $\lim\limits_{x\to \sqrt{3}}\frac{x^4-9}{x^2+4\sqrt{3}x-15}$

(f) $\lim\limits_{x\to 9}\frac{x^2-81}{\sqrt{x}-3}$

Find the limiting values of the following.

(a) $\lim_{x\to\infty}\frac{3x^2+5x+1}{2x^2+x+1}$

(b) $\lim_{x\to\infty}\frac{2x^3}{x^2+1}$

(c) $\lim_{x\to\infty}\frac{x^2+3x+1}{3x^3+1}$

(d) $\lim_{x\to\infty}\frac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1+x^3}-\sqrt{1+x}}$

(e) $\lim_{x\to\infty}\frac{x}{\sqrt{1+x}-1}$

(f) $\lim_{x\to\infty}\left(\sqrt{x+1}-x^2\right)$

Find the limit

(a) $\lim_{x\to 0}\frac{\sqrt{1+x^4}-\sqrt{1+x^2}}{\sqrt{1+x^6}-\sqrt{1+x^2}}$

(b) $\lim_{x\to 0}\frac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1+x^4}-\sqrt{1-x^4}}$

13.6 Evaluation of limits of trigonometric functions

Standard Results

We use following results to evaluate limits of trigonometric functions

(i) $\lim_{\theta\to 0}\sin\theta = 0$

(ii) $\lim_{\theta\to 0}\cos\theta = 1$

(iii) $\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$

(iv) $\lim_{\theta\to 0}\frac{\tan\theta}{\theta} = 1$,

Where θ is in radians.

Example 8

Evaluate the following limits.

i. $\lim_{x\to 0}\frac{\sin 5x}{6x}$

ii. $\lim_{x\to 0}\frac{\tan\left({}^{5x}/_{2}\right)}{4x}$

iii. $\lim_{x\to 0}\frac{\sin 6x}{x\cos x}$

iv. $\lim_{\theta\to 0}6\theta\csc\theta$

v. $\lim_{\theta\to 0}\frac{\sin 5\theta}{\sin 8\theta}$

vi. $\lim_{x\to 0}\frac{\sin 5x\cos 3x}{3x}$

vii. $\lim_{\theta\to 0}\frac{\tan 5\theta}{\tan 3\theta}$

viii. $\lim_{\theta\to 0}\frac{1-\cos\theta}{\theta^2}$

ix. $\lim_{x\to 0}\frac{1-\cos x}{x\sin x}$

x. $\lim_{\theta\to 0}\frac{1-\cos 4\theta}{1-\cos 6\theta}$

xi. $\lim\limits_{x\to 0}\frac{\cos 3x-\cos 5x}{x^2}$

xii. $\lim\limits_{x\to 0}\frac{\sin 3x+\sin x}{\sin 4x-\sin x}$

xiii. $\lim\limits_{x\to 0}\frac{\sin 2x+3x}{2x+\sin 3x}$

Solution

i. We have $\lim\limits_{\theta\to 0}\frac{\sin\theta}{\theta}=1.$

$$\lim_{x\to 0}\frac{\sin 5x}{6x}=\lim_{x\to 0}\frac{\sin 5x}{5x}\times\frac{5}{6}$$

$$==\frac{5}{6}.$$

ii. $\lim\limits_{x\to 0}\frac{\tan x}{x}=1.$

$$\lim_{x\to 0}\frac{\tan\left({}^{5x}/_{2}\right)}{4x}=\lim_{x\to 0}\frac{\tan\left({}^{5x}/_{2}\right)}{{}^{5x}/_{2}}\cdot\frac{{}^{5}/_{2}}{4}$$

$$=1s\,\frac{5}{8}=\frac{5}{8}.$$

iii. $\lim\limits_{x\to 0}\frac{\sin 6x}{x\cos x}=\lim\limits_{x\to 0}\left\{\left(\frac{\sin 6x}{6x}\right)\times\left(\frac{6}{\cos x}\right)\right\}$

$$=\lim_{x\to 0}\left(\frac{\sin 6x}{6x}\right)\times\lim_{x\to 0}\left(\frac{6}{\cos x}\right)$$

$=1s\,\frac{1}{6}=6.$

iv. $\lim\limits_{\theta\to 0}6\theta\csc\theta=\lim\limits_{\theta\to 0}6\,\frac{\theta}{\sin\theta}$

$=6n\,\frac{1}{1}=6.$

v. $\lim\limits_{\theta\to 0}\frac{\sin 5\theta}{\sin 8\theta}=\lim\limits_{\theta\to 0}\left\{\frac{\sin 5\theta}{5\theta}\times\frac{5}{\frac{\sin 8\theta}{8\theta}\times 8}\right\}$

$$=\frac{5}{8}\left\{\frac{\lim\limits_{\theta\to 0}\frac{\sin 5\theta}{5\theta}}{\lim\limits_{\theta\to 0}\frac{\sin 8\theta}{8\theta}}\right\}$$

$=\frac{5}{8}\times\frac{1}{1}=\frac{5}{8}.$

vi. $\lim\limits_{x\to 0}\frac{\sin 5x\cos 3x}{3x}$

$$=\lim_{x\to 0}\frac{1}{2}\frac{[\sin 8x+\sin 2x]}{3x}$$

$$= \lim_{x\to 0}\left\{\left(\frac{\sin 8x}{8x}\right)\cdot\frac{8}{6}+\frac{\sin 2x}{2x}\cdot\frac{1}{3}\right\}$$

$$= \lim_{x\to 0}\frac{8}{6}\cdot\frac{\sin 8x}{8x}+\lim_{x\to 0}\frac{1}{3}\cdot\frac{\sin 2x}{2x}$$

$= \frac{8}{6}\times x\frac{1}{3}\times x\frac{10}{6}=\frac{5}{3}.$

vii. $\lim_{\theta\to 0}\frac{\tan 5\theta}{\tan 3\theta}=\lim_{\theta\to 0}\left\{\frac{\tan 5\theta}{5\theta}\times\frac{5}{\frac{\tan 3\theta}{3\theta}\times 3}\right\}$

$= \frac{5}{3}\frac{\left[\lim_{\theta\to 0}\frac{\tan 5\theta}{5\theta}\right]}{\lim_{\theta\to 0}\frac{\tan 3\theta}{3\theta}}=\frac{5}{3}\times\frac{1}{1}=\frac{5}{3}.$

viii. $\lim_{\theta\to 0}\frac{1-\cos\theta}{\theta^2}=\lim_{\theta\to 0}\frac{2\left(\sin\frac{\theta}{2}\right)^2}{\theta^2}$

$= \lim_{\theta\to 0}\left[\frac{\sin\frac{\theta}{2}}{\frac{\theta}{2}}\right]^2\times\frac{2}{4}=1^2\times\frac{1}{2}=\frac{1}{2}.$

ix. $\lim_{x\to 0}\frac{1-\cos x}{x\sin x}$

$$\lim_{x\to 0}\frac{1-\left(1-2\left(\sin\frac{x}{2}\right)^2\right)}{x\cdot 2\sin\frac{x}{2}\cos\frac{x}{2}}=\lim_{x\to 0}\frac{2\left(\sin\frac{x}{2}\right)^2}{x\cdot 2\sin\frac{x}{2}\cos\frac{x}{2}}$$

$$= \lim_{x\to 0}\frac{\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}}{x}=\lim_{x\to 0}\left(\frac{\tan\frac{x}{2}}{\frac{x}{2}}\right)\frac{1}{2}$$

$= 1n\frac{1}{2}=\frac{1}{2}.$

x. $\lim_{\theta\to 0}\frac{1-\cos 4\theta}{1-\cos 6\theta}=\lim_{\theta\to 0}\frac{1-\left(1-2(\sin 2\theta)^2\right)}{1-(1-2(\sin 3\theta)^2)}$

$$= \lim_{\theta\to 0}\frac{2(\sin 2\theta)^2}{2(\sin 3\theta)^2}=\lim_{\theta\to 0}\left\{\left(\frac{\sin 2\theta}{2\theta}\right)^2\times\frac{4}{9}\times\frac{1}{\left(\frac{\sin 3\theta}{3\theta}\right)^2}\right\}$$

$$= \left(\lim_{\theta\to 0}\frac{\sin 2\theta}{2\theta}\right)^2\times\frac{1}{\left(\lim_{\theta\to 0}\frac{\sin 3\theta}{3\theta}\right)^2}\times\frac{4}{9}$$

$= \frac{4}{9}.$

xi. $\lim_{x\to 0}\frac{\cos 3x-\cos 5x}{x^2}=\lim_{x\to 0}\frac{2\sin 4x\sin x}{x^2}$

$$= \lim_{x\to 0}\left\{2\frac{\sin 4x}{4x}\cdot\frac{\sin x}{x}\cdot 4\right\}$$

$$= 8\left(\lim_{x\to 0}\frac{\sin 4x}{4x}\right)\lim_{x\to 0}\left(\frac{\sin x}{x}\right) = 8n0 \quad .$$

xii. $\lim_{x\to 0}\frac{\sin 3x+\sin x}{\sin 4x-\sin x}$

$$= \lim_{x\to 0}\frac{2\sin 4x\cos x}{2\cos\frac{5x}{2}\sin\frac{3x}{2}}$$

$$= \lim_{x\to 0}\left[\frac{\sin 4x}{4x}\cdot\frac{4}{\frac{3}{2}}\cdot\frac{1}{\left(\frac{\sin\frac{3x}{2}}{\frac{3x}{2}}\right)}\times\frac{\cos x}{\cos\frac{5x}{2}}\right]$$

$$= \frac{8}{3}\left\{\lim_{x\to 0}\frac{\sin 4x}{4x}\right\}\times\frac{1}{\lim_{x\to 0}\left(\frac{\sin\frac{3x}{2}}{\frac{3x}{2}}\right)}$$

$$= \frac{8}{3}\times x\frac{1}{1}\times\frac{1}{1} = \frac{8}{3}.$$

xiii. $\lim_{x\to 0}\frac{\sin 2x+3x}{2x+\sin 3x} = \lim_{x\to 0}\left\{\frac{\frac{\sin 2x}{x}+3}{2+\frac{\sin 3x}{x}}\right\}$

$$= \lim_{x\to 0}\left\{\frac{2\frac{\sin 2x}{2x}+3}{2+3\frac{\sin 3x}{3x}}\right\} = \frac{\lim_{x\to 0}\left[2\frac{\sin 2x}{2x}+3\right]}{\lim_{x\to 0}\left[2+3\frac{\sin 3x}{3x}\right]}$$

$$= \frac{2\times x3}{2+3} = \frac{5}{5} = 1.$$

Activity 2

Evaluate the following limits.

i. $\lim_{\theta\to 0}\frac{\sin 2\theta}{\sin 3\theta}$

ii. $\lim_{x\to 0}\frac{1-\cos 4x}{x^2}$

iii. $\lim_{x\to 0}\frac{\sin 3x-\sin x}{x}$

iv. $\lim_{x\to 0}\frac{\sin 3x+\sin x}{x}$

v. $\lim_{x\to 0}\frac{\csc x-\cot x}{x}$

vi. $\lim_{x\to 0}\frac{\cos 8x-\cos 2x}{\cos 12x-\cos 4x}$

vii. $\lim_{x\to 0}\frac{\tan 3x-2x}{3x-(\sin x)^2}$

viii. $\lim_{x\to 0}\frac{x^3\cot x}{1-\cos x}$

ix. $\lim_{x\to 0}\frac{\cot 2x-\csc 2x}{x}$

x. $\lim_{x\to 0}\frac{1-\cos x}{(\sin x)^2}$

Solutions to Activities

Activity 1

1.

(a) 108	(b) 18	(c) $\frac{135}{2}$	(d) 64
(e) $\frac{2}{9}$	(f) $\frac{14}{3}$	(g) 3	(h) $\frac{81}{5}$

2.

(a) $n = 5$	(b) 2	(c) ± 1

3.

(a) $\frac{2}{9}$	(b) $\frac{1}{2}$	(c) $\frac{3}{16}$
(d) 4	(e) 2	(f) 108

4.

(a) $\frac{3}{2}$	(b) ∞	(c) 0
(d) 0	(e) ∞	(f) 0

5.

(a) 1	(b) ∞

Activity 2

i. $\frac{2}{3}$	ii. 8	iii. 2	iv. 4
v. $\frac{1}{2}$	vi. $\frac{15}{32}$	vii. $\frac{1}{3}$	viii. 2
ix. -1	x. $\frac{1}{2}$		

Summary

The meaning of $x \to a$

When x gets nearer to nearer to a we can denote $x \to a$. That's means x is very very close to a, but not equal to a.

The meaning of $\lim_{x\to a} f(x)$ is the behavior of $f(x)$, when x approaches to "a".

Properties of limits.

i. $\lim_{x\to a}\{f(x) \pm g(x)\} = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$

ii. $\lim_{x\to a}\{f(x) \cdot g(x)\} = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$

iii. $\lim_{x\to a}\left\{\frac{f(x)}{g(x)}\right\} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$; provided $\lim_{x\to a} g(x) \neq 0$.

iv. $\lim_{x\to a}\{f(x)\}^n = \left\{\lim_{x\to a} f(x)\right\}^n$.

v. $\lim_{x\to a} cf(x) = c\left[\lim_{x\to a} f(x)\right]$ where c is a constant.

Standard Limits

i. $\lim_{x\to a} \frac{x^n - a^n}{x-a} = na^{n-1}, n \in \mathbb{R}$

ii. $\lim_{\theta\to 0} \sin\theta = 0$

iii. $\lim_{\theta\to 0} \cos\theta = 1$

iv. $\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$

v. $\lim_{\theta\to 0} \frac{\tan\theta}{\theta} = 1$,

Where θ is in radians.

Learning outcomes

At the end of this study session you will be able to

- Describe the intuitive idea about the limit
- Evaluate limits of a function by using the properties of the limit and the result of the standard limits.

Session 14

The Principles of Derivative

Contents

Introduction

This session is to introduce an important area in mathematics, called “Derivative” which is a method of finding the rate of change of a function with respect to the independent variable. Derivative is based on the concepts of the limits. Derivative or Differentiation has a variety of applications, including curve sketching, analyzing of rates of change and the optimization of function.

The concept of derivative has been applied to other fields as well. A biologist uses to determine the rate of growth of bacteria in a culture.

In this session we study in detail the meaning and the process of differentiation.

14.1 Differentiability of a function at a point

Let f be a real function and "a" be a real number

We say that f is differentiable or derivable at a if

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

exists finitely,

We denote this limit by $f'(a)$ and called the first derivative or differential coefficient of f with respect to x at a.

$$\therefore f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

Example 1

Find the derivative of the following functions.

a) $f(x) = x^2 + 2x$ at $x = 2$.

b) $g(x) = \frac{x^2+1}{x}$ at $x = 3$.

c) $h(x) = 3x^2 + 2x + 1$ at $x = 1$.

Solution

a) $f(x) = x^2 + 2x$; $x = 2$.

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

$a = 2; f(2 + h) = (2 + h)^2 + 2(2 + h)$

$$= 8 + 6h + h^2$$

$$f(2) = 2^2 + 2 \cdot 2 = 8$$

$$f'(2) = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h}$$

$$= \lim_{h \to 0} \frac{(8+6h+h^2)-8}{h}$$

$$= \lim_{h \to 0}(6 + h)$$

$$= 6.$$

$\therefore f^{'}(2) = 6.$

b) $g(x) = \frac{x^2+1}{x}$ at $x = 3$

$$\therefore a = 3$$

$$g(3+h) = \frac{(3+h)^2+1}{3+h} = \left(\frac{10+6h+h^2}{3+h}\right)$$

$$g(3) = \frac{10}{3}$$

$$g^{'}(3) = \lim_{h\to 0}\frac{g(3+h)-g(3)}{h}$$

$$= \lim_{h\to 0}\frac{1}{h}\left(\frac{10+6h+h^2}{3+h} - \frac{10}{3}\right)$$

$$= \lim_{h\to 0}\frac{1}{h}\left(\frac{30+18h+3h^2-30-10h}{3(3+h)h}\right)$$

$$= \lim_{h\to 0}\frac{1}{h}\frac{[8h+3h^2]}{3(3+h)h}$$

$$= \lim_{h\to 0}\frac{[8+3h]}{3(3+h)} = \frac{8}{9}.$$

$$\therefore g^{'}(3) = \frac{8}{9}.$$

c) $h(x) = 3x^2 + 2x + 1$

$$h^{'}(1) = \lim_{h\to 0}\frac{h(1+h)-h(1)}{h}$$

$$= \lim_{h\to 0}\frac{3(1+h)^2+2(1+h)+1-(3\cdot 1^2+2\cdot 1+1)}{h}$$

$$= \lim_{h\to 0}\frac{3+6h+3h^2+2+2h+1-3-2-1}{h}$$

$$= \lim_{h\to 0}\frac{(8h+3h^2)}{h} = \lim_{h\to 0}(8+3h) = 8.$$

$$\therefore h^{'}(1) = 8.$$

14.2 The derivative of a function at a general point "x"

Let $f(x)$ be a function which is defined in all value of a given interval.

Then the derivative of the function $f(x)$ at general point "x" is defined by

$$f^{'}(x) = \lim_{\delta x \to 0}\left\{\frac{f(x+\delta x) - f(x)}{\delta x}\right\}$$

Example 2

Find the derivative of the following functions $f(x) = x^2 + 2x$

i. $g(x) = \frac{x^2+1}{x}$

ii. $h(x) = 3x^2 + 2x + 1$

and hence find

i. $f^{'}(2)$

ii. $g^{'}(3)$

iii. $h^{'}(1)$

And compare with the answer in the example 1

Solution

i. $f(x) = x^2 + 2x$

$$f^{'}(x) = \lim_{\delta x \to 0}\frac{f(x+\delta x) - f(x)}{\delta x}$$

$$= \lim_{\delta x \to 0}\left\{\frac{(x+\delta x)^2 + 2(x+\delta x) - (x^2+2x)}{\delta x}\right\}$$

$$= \lim_{\delta x \to 0}\left\{\frac{x^2 + 2x\cdot\delta x + (\delta x)^2 + 2x + 2\delta x - x^2 - 2x}{\delta x}\right\}$$

$$= \lim_{\delta x \to 0}\left\{\frac{2(x+1)\cdot\delta x + (\delta x)^2}{\delta x}\right\}$$

$$= \lim_{\delta x \to 0}[2(x+1) + \delta x]$$

$$= 2(x+1)$$

$$\therefore f^{'}(2) = 2(2+1) = 6.$$

ii. $g(x) = \frac{x^2+1}{x}$

$$g^{'}(x) = \lim_{\delta x \to 0}\left[\frac{g(x+\delta x) - g(x)}{\delta x}\right]$$

$$= \lim_{\delta x \to 0}\frac{1}{\delta x}\left[\frac{(x+\delta x)^2+1}{(x+\delta x)} - \frac{(x^2+1)}{x}\right]$$

$$g^{'}(x) = \lim_{\delta x \to 0}\left[\frac{(x^2+2x\cdot\delta x+(\delta x)^2+1)x-(x^2+1)(x+\delta x)}{\delta x(x+\delta x)x}\right]$$

$$= \lim_{\delta x \to 0}\left[\frac{x^3+2x^2\cdot\delta x+x(\delta x)^2+x-x^3-x-x^2\delta x)-\delta x}{\delta x(x+\delta x)x}\right]$$

$$= \lim_{\delta x \to 0}\left[\frac{x^2\cdot\delta x+x(\delta x)^2-\delta x}{x\cdot\delta x(x+\delta x)}\right]$$

$$= \lim_{\delta x \to 0}\left[\frac{x^2-1+x\cdot\delta x}{x(x+\delta x)}\right]$$

$$= \left(\frac{x^2-1}{x^2}\right) = \left(1-\frac{1}{x^2}\right)$$

$$g^{'}(x) = \left(1-\frac{1}{x^2}\right)$$

$$\therefore g^{'}(3) = 1-\frac{1}{3^2} = 1-\frac{1}{9} = \frac{8}{9}.$$

iii. $h(x) = 3x^2+2x+1$

$$h^{'}(x) = \lim_{\delta x \to 0}\frac{h(x+\delta x)-h(x)}{\delta x}$$

$$= \lim_{\delta x \to 0}\left[\frac{3(x+\delta x)^2+2(x+\delta x)+1-(3x^2+2x+1)}{\delta x}\right]$$

$$= \lim_{\delta x \to 0}\left[\frac{3x^2+6x\cdot\delta x+3(\delta x)^2+2x+2\delta x+1-3x^2-2x-1}{\delta x}\right]$$

$$= \lim_{\delta x \to 0} \left[\frac{(6x+2)\delta x + 3(\delta x)^2}{\delta x}\right]$$

$$= \lim_{\delta x \to 0} [(6x+2) + 3\delta x] = 6x + 2$$

$$h'(x) = 6x + 2$$

$$\therefore h'(1) = 6 \times 1 + 2 = 8.$$

14.3 Fundamental theorems on derivative

a) Derivative of a constant function is zero.

b) If $\frac{d}{dx} f(x) = f'(x)$, then

$$y = F(x) = k \cdot f(x)$$

$\therefore F'(x) = k \cdot f'(x)$, where k is any constant.

c) If $\frac{d}{dx} f(x) = f'(x)$ and $\frac{d}{dx} g(x) = g'(x)$

Then $\frac{d}{dx}\{f(x) \pm g(x)\} = f'(x) \pm g'(x)$

d) If $\frac{d}{dx} f(x) = f'(x)$, Then $\frac{d}{dx} f(ax) = af'(ax)$

Example 3

In the Example 1, we have

$$f(x) = x^2 + 2x$$

$$\therefore f'(x) = 2(x+1)$$

$$g(x) = \frac{x^2+1}{x} \qquad g'(x) = 1 - \frac{1}{x^2}$$

$$h(x) = 3x^2 + 2x + 1 \qquad h'(x) = 6x + 2$$

Hence find the derivatives of the functions

a. $F(x) = 5f(x)$, $G(x) = 6g(x)$, $H(x) = \frac{1}{3} h(x)$

b. $\frac{d}{dx}[f(x) + g(x) - h(x)]$

c. $\frac{d}{dx} f(3x)$, $\frac{d}{dx} g(2x)$, $\frac{d}{dx} h\left(\frac{1}{5}x\right)$

Solution

a. $\frac{d}{dx}F(x) = \frac{d}{dx}(5f(x)) = 5 \cdot \frac{d}{dx}f(x)$

$$= 5f^{'}(x) = 5 \times (2)(x+1)$$

$$= 10(x+1).$$

$$\frac{d}{dx}G(x) = \frac{d}{dx}(6g(x)) = 6 \cdot \frac{d}{dx}g(x)$$

$$= 6g^{'}(x) = 6 \times \left(1 - \frac{1}{x^2}\right)$$

$$= 6\left(1 - \frac{1}{x^2}\right).$$

$$\frac{d}{dx}H(x) = \frac{d}{dx}\left(\frac{1}{3}h(x)\right) = \frac{1}{3} \cdot \frac{d}{dx}h(x)$$

$$= \frac{1}{3} \times (6x+2) = \frac{1}{3}(6x+2).$$

e) The Product Rule of Derivative

If $f(x)$ and $g(x)$ are differentiable functions of x, then

$$\frac{d}{dx}[f(x)g(x)] = f(x) \cdot \frac{d}{dx}g(x) + g(x) \cdot \frac{d}{dx}f(x)$$

$$\frac{d}{dx}[f(x)g(x)] = f(x) \cdot g^{'}(x) + g(x) \cdot f^{'}(x)$$

f) The Quotient Rule of Derivative

If $f(x)$ and $g(x)$ are two differentiable functions of x then

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}f(x) - f(x)\frac{d}{dx}g(x)}{\left(g(x)\right)^2}$$

$$\boxed{\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\,f^{'}(x) - f(x)\,g^{'}(x)}{\left(g(x)\right)^2}}$$

Example 4

Find $\frac{d}{dx}[(x^2+2x) \cdot (3x^2+2x+1)]$ and $\frac{d}{dx}\left[\frac{x^2+2x}{3x^2+2x+1}\right]$.

Solution

In the Example 1,

$$f(x) = x^2 + 2x \Longrightarrow f^{'}(x) = 2(x+1)$$

$$h(x) = 3x^2 + 2x + 1 \Longrightarrow h^{'}(x) = 6x + 2$$

$$\therefore \frac{d}{dx}[(x^2+2x)\cdot(3x^2+2x+1)] = \frac{d}{dx}[f(x)\cdot h(x)]$$

From the above rule

$$\frac{d}{dx}[f(x)\cdot h(x)] = (x)\cdot\frac{d}{dx}h(x) + h(x)\cdot\frac{d}{dx}f(x)$$

$$= (x^2+2x)(6x+2) + (3x^2+2x+1)(2)(x+1)$$

$$= 6x^3 + 12x^2 + 2x^2 + 4x + 6x^3 + 4x^2 + 2x + 6x^2 + 4x + 2$$

$$= 12x^3 + 24x^2 + 10x + 2.$$

$$\frac{d}{dx}\left[\frac{f(x)}{h(x)}\right] = \frac{h(x)\frac{d}{dx}f(x) - f(x)\frac{d}{dx}h(x)}{\left(h(x)\right)^2}$$

$$= \frac{(3x^2+2x+1)[2(x+1)] - (x^2+2x)(6x+2)}{(3x^2+2x+1)^2}$$

$$= \frac{6x^3+4x^2+2x+6x^2+4x+2-6x^3-12x^2-12x^2-4x}{(3x^2+2x+1)^2}$$

$$= -\frac{14x^2+2x+2}{(3x^2+2x+1)^2}.$$

14.4. Derivatives of some standard functions

i. $f(x) = x^n, n \in \mathbb{R}.$

$$f^{'}(x) = \lim_{\delta x \to 0}\frac{f(x+\delta x) - f(x)}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{(x+\delta x)^n - x^n}{\delta x}$$

$$= \lim_{x+\delta x \to x} \frac{(x+\delta x)^n - x^n}{(x+\delta x) - x} \ where\ x + \delta x$$

$$= X, x = A$$

$$\therefore \lim_{X \to A} \frac{X^n - A^n}{X - A} = nA^{n-1}$$

$$= nx^{n-1}.$$

$$\boxed{\therefore \frac{\boldsymbol{d}}{\boldsymbol{dx}} \boldsymbol{x^n} = \boldsymbol{nx^{n-1}}}$$

ii. $f(x) = \sin x$

$$f'(x) = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{\sin(x+\delta x) - \sin x}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{2\cos\left(x + \frac{\delta x}{2}\right)\sin\frac{\delta x}{2}}{\delta x}$$

$$= \lim_{\delta x \to 0} \left\{\cos\left(x + \frac{\delta x}{2}\right) \frac{\sin\frac{\delta x}{2}}{\frac{\delta x}{2}}\right\}$$

$$= \left(\lim_{\delta x \to 0} \cos\left(x + \frac{\delta x}{2}\right)\right)\left(\lim_{\delta x \to 0} \frac{\sin\frac{\delta x}{2}}{\frac{\delta x}{2}}\right)$$

$$= (\cos x) \times 1$$

$$\boxed{\therefore \frac{\boldsymbol{d}}{\boldsymbol{dx}} \boldsymbol{\sin x} = \boldsymbol{\cos x}}$$

iii. $f(x) = \tan x = \frac{\sin x}{\cos x}$

$$\frac{d}{dx}\tan x = \frac{\cos x \frac{d}{dx}\sin x - \sin x \frac{d}{dx}\cos x}{(\cos x)^2}$$

$\therefore$ (From the Quotient Rule)

$$\frac{d}{dx}(\tan x) = \frac{\cos x \cos x - \sin x(-\sin x)}{(\cos x)^2}$$

$$= \frac{(\cos x)^2 + (\sin x)^2}{(\cos x)^2}$$

$$\frac{d}{dx}(\tan x) = (\sec x)^2$$

$$\boxed{\boldsymbol{\frac{d}{dx}(\tan x) = (\sec x)^2}}$$

iv. $f(x) = \csc x$

$$\frac{d}{dx}\csc x = \frac{d}{dx}\left(\frac{1}{\sin x}\right)$$

$$= \frac{\sin x \frac{d}{dx}(1) - 1 \cdot \frac{d}{dx}\sin x}{(\sin x)^2}$$

$$= \frac{(\sin x) \times 0 - \cos x}{(\sin x)^2}$$

$$= -\frac{\cos x}{\sin x} \cdot \frac{1}{\sin x} = -\cot x \csc x$$

$$\boxed{\frac{d}{dx}\csc x = -\cot x \csc x}$$

v. $f(x) = \sec x$

$$\frac{d}{dx}\sec x = \frac{d}{dx}\left(\frac{1}{\cos x}\right)$$

$$= \frac{\cos x \frac{d}{dx}(1) - 1 \cdot \frac{d}{dx}\cos x}{(\cos x)^2}$$

$$= \frac{(\cos x) \times 0 - 1 \times (-\sin x)}{(\cos x)^2}$$

$$= \frac{\sin x}{\cos x} \cdot \frac{1}{\cos x} = \frac{\tan x}{\cos x}$$

$$= \tan x \sec x$$

vi. $f(x) = \cot x$

$$\frac{d}{dx}\cot x = \frac{d}{dx}\left(\frac{\cos x}{\sin x}\right)$$

$$= \frac{\sin x \frac{d}{dx}\cos x - \cos x \frac{d}{dx}\sin x}{(\sin x)^2}$$

$$= \frac{\sin x\,[-\sin x] - (\cos x)^2}{(\sin x)^2}$$

$$= -\frac{[(\cos x)^2 + (\sin x)^2]}{(\sin x)^2}$$

$$= -(\csc x)^2$$

$$\boxed{\frac{d}{dx}\cot x = -(\csc x)^2}$$

vii. $f(x) = e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$

$f(x) = e^x$ exponential function

$$\frac{d}{dx}f(x) = \frac{d}{dx}e^x = \frac{d}{dx}\left[1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots\right]$$

$$= 0 + 1 + \frac{2x}{2!} + \frac{3x^2}{3!} + \cdots + \frac{nx^{n-1}}{n!} + \cdots$$

$$= 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^{n-1}}{(n-1)!} + \cdots$$

$$= e^x$$

$$\boxed{\therefore \frac{d}{dx}e^x = e^x}$$

Example 5

Differentiate the following functions;

i. $\frac{x}{1+\tan x}$

ii. $\frac{\sin x+\cos x}{\sin x-\cos x}$

iii. $\frac{\sec x+\tan x}{\sec x-\tan x}$

iv. $\frac{\sec x-1}{\sec x+1}$

v. $\frac{x\sin x-\cos x}{x\cos x-\sin x}$

vi. $(1+2\tan x)(5+4\cos x)$

vii. $x^4\sin x$

viii. $(x\sin x+\cos x)(x\cos x-\sin x)$

ix. $-3\cot x\sin x+5\tan x\cos x$

x. $5\sec x+4\cos x$

Solutions

i. $y=\frac{x}{1+\tan x}$

$$\frac{dy}{dx}=\frac{(1+\tan x)\frac{d}{dx}(x)-x\frac{d}{dx}(1+\tan x)}{(1+\tan x)^2}$$

$$=\frac{(1+\tan x)\cdot 1-x(0+(\sec x)^2)}{(1+\tan x)^2}$$

$$=\frac{1+\tan x-x(\sec x)^2}{(1+\tan x)^2}.$$

ii. $y=\frac{\sin x+\cos x}{\sin x-\cos x}$

$$\frac{dy}{dx}=\frac{(\sin x-\cos x)\frac{d}{dx}(\sin x+\cos x)-(\sin x+\cos x)\frac{d}{dx}(\sin x-\cos x)}{(\sin x-\cos x)^2}$$

$$\frac{dy}{dx}=\frac{(\sin x-\cos x)(\cos x-\sin x)-(\sin x+\cos x)(\sin x+\cos x)}{(\sin x-\cos x)^2}$$

$$=-\frac{(\cos x-\sin x)^2+(\sin x+\cos x)^2}{(\sin x-\cos x)^2}$$

$$=-\frac{(\cos x)^2-2\sin x\cos x+(\sin x)^2+(\sin x)^2+2\sin x\cos x+(\cos x)^2}{(\sin x-\cos x)^2}$$

$$\frac{dy}{dx}=-\frac{2}{(\sin x-\cos x)^2}.$$

iii. $y=\frac{\sec x+\tan x}{\sec x-\tan x}$

$$\frac{dy}{dx}=\frac{(\sec x-\tan x)\frac{d}{dx}(\sec x+\tan x)-(\sec x+\tan x)\frac{d}{dx}(\sec x-\tan x)}{(\sec x-\tan x)^2}$$

$$\frac{dy}{dx} = \frac{(\sec x - \tan x)[\sec x \tan x + (\sec x)^2] - (\sec x + \tan x)[\sec x \tan x - (\sec x)^2]}{(\sec x - \tan x)^2}$$

$$\frac{dy}{dx} = \frac{\sec x\,(\sec x - \tan x)[\sec x + \tan x] - \sec x\,(\sec x + \tan x)[\tan x - \sec x]}{(\sec x - \tan x)^2}$$

$$\frac{dy}{dx} = \frac{2\sec x(\sec x - \tan x)[\sec x + \tan x]}{(\sec x - \tan x)^2}$$

$$= \frac{2\sec x}{(\sec x - \tan x)^2}\ ;\because (\sec x)^2 - (\tan x)^2 \equiv 1.$$

iv. $y = \frac{\sec x - 1}{\sec x + 1}$

$$\frac{dy}{dx} = \frac{(\sec x + 1)\frac{d}{dx}(\sec x - 1) - (\sec x - 1)\frac{d}{dx}(\sec x + 1)}{(\sec x + 1)^2}$$

$$= \frac{(\sec x + 1)\sec x \tan x - (\sec x - 1)\sec x \tan x}{(\sec x + 1)^2}$$

$$= \frac{\sec x \tan x[\sec x + 1 - \sec x + 1]}{(\sec x + 1)^2}$$

$$= \frac{2\sec x \tan x}{(\sec x + 1)^2}.$$

v. $y = \frac{x \sin x - \cos x}{x \cos x - \sin x}$

$$\frac{dy}{dx} = \frac{(x\cos x - \sin x)\frac{d}{dx}(x\sin x - \cos x) - (x\sin x - \cos x)\frac{d}{dx}(x\cos x - \sin x)}{(x\cos x - \sin x)^2}$$

$$\frac{dy}{dx} = \frac{(x\cos x - \sin x)\left[x\frac{d}{dx}(\sin x) + \sin x\frac{dx}{dx} - \frac{d}{dx}(\cos x)\right] - (x\sin x - \cos x)\left[x\frac{d}{dx}(\cos x) + \cos x\frac{dx}{dx} - \frac{d}{dx}(\mathrm{si}\right.}{(x\cos x - \sin x)^2}$$

$$\frac{dy}{dx} = \frac{(x\cos x - \sin x)[x\cos x + \sin x - (-\sin x)] - (x\sin x - \cos x)[x(-\sin x) + \cos x - \mathrm{co}}{(x\cos x - \sin x)^2}$$

$$= \frac{(x\cos x - \sin x)[x\cos x + 2\sin x] + (x\sin x - \cos x)x\sin x}{(x\cos x - \sin x)^2}$$

$$= \frac{x^2(\cos x)^2 - x\sin x\cos x + 2x\sin x\cos x - 2(\sin x)^2 + x^2(\sin x)^2 - x\sin x\cos x}{(x\cos x - \sin x)^2}$$

$$= \frac{x^2[(\cos x)^2 + (\sin x)^2] - 2(\sin x)^2}{(x\cos x - \sin x)^2}$$

$$= \frac{x^2 - 2(\sin x)^2}{(x\cos x - \sin x)^2}.$$

vi. $y = (1 + 2\tan x)(5 + 4\cos x)$

$$\frac{dy}{dx} = (1 + 2\tan x)\frac{d}{dx}(5 + 4\cos x) - (5 + 4\cos x)\frac{d}{dx}(1 + 2\tan x)$$

$$= (1 + 2\tan x)(-4\sin x) + (5 + 4\cos x)(2(\sec x)^2)$$

$$= -4\sin x - 8\sin x\tan x + 10(\sec x)^2 + 8\sec x.$$

vii. $y = x^4 \sin x$

$$\frac{dy}{dx} = x^4\frac{d}{dx}(\sin x) + \sin x\frac{d}{dx}(x^4)$$

$$= x^4\cos x + 4x^3\sin x.$$

viii. $y = (x\sin x + \cos x)(x\cos x - \sin x)$

$$\frac{dy}{dx} = (x\sin x + \cos x)\frac{d}{dx}(x\cos x - \sin x)$$

$$+ (x\cos x - \sin x)\frac{d}{dx}(x\sin x + \cos x)$$

$$= (x\sin x + \cos x)\left[x\frac{d}{dx}(\cos x) + \cos x\frac{dx}{dx} - \frac{d}{dx}(\sin x)\right]$$

$$+ (x\cos x - \sin x)\left[x\frac{d}{dx}(\sin x) + \sin x\frac{dx}{dx} + \frac{d}{dx}(\cos x)\right]$$

$$= (x\sin x + \cos x)[-x\sin x + \cos x - \cos x]$$

$$+ (x\cos x - \sin x)[x\cos x + \sin x - \sin x]$$

$$= -(x\sin x + \cos x)x\sin x + (x\cos x - \sin x)x\cos x$$

$$= -x\{\sin x\,(x\sin x + \cos x) - \cos x\,(x\cos x - \sin x)\}$$

$$= -x\{x(\sin x)^2 + \sin x\cos x - x(\cos x)^2 + \sin x\cos x\}$$

$$= -x\{2\sin x\cos x + x((\sin x)^2 - (\cos x)^2)\}$$

$$= -2x\sin x\cos x + x^2((\cos x)^2 - (\sin x)^2)$$

$$= -x\sin 2x + x^2\cos 2x.$$

ix. $y = -3\cot x\sin x + 5\tan x\cos x$

$$= -3\cos x + 5\sin x$$

$$\frac{dy}{dx} = -3\frac{d}{dx}(\cos x) + 5\frac{d}{dx}(\sin x) = -3(-\sin x) + 5\cos x$$

$$= 3\sin x + 5\cos x$$

x. $y = 5\sec x + 4\cos x$

$$\frac{dy}{dx} = 5\frac{d}{dx}(\sec x) + 4\frac{d}{dx}(\cos x) = 5\sec x\tan x - 4\sin x$$

Activity 1

1) Find the derivatives of the functions

a) $f(x) = 2x^2 + 3x + 4$ at $x = 2$

b) $g(x) = \frac{1}{x+1}$ at $x = 2$

c) $h(x) = \frac{x+2}{x+1}$ at $x = 0$

d) $k(x) = \frac{1}{3x+2}$ at $x = 1$

e) $l(x) = 5x^2 + 9x + 3$ at $x = 2$

2) Find the derivatives of the following functions with respect x, from first principle.

a) $x^3 - 27$

b) $(x-1)(x-2)$

c) $\frac{1}{x^3}$

d) $5x^2 + 3x + 6$

e) $\frac{2-x}{3+x}$

f) $\frac{1}{3x+2}$

3) Find the derivatives of the following functions w.r.t. x

a) $(x^2 - 5x + 6)(x^3 + 2)$

b) $(x^2 - 5x + 6)\sec x$

c) $\frac{3\cos x+4\sin x}{3\sin x+4\cos x}$

d) $\frac{2x^2-7}{3x^2+7}$

e) $\frac{3x^2-2}{x^2+7}$

f) $\frac{\sin x-\cos x}{\sin x+\cos x}$

g) $\frac{x+\cos x}{x-\sin x}$

h) $\frac{\sec x-\tan x}{\sec x+\tan x}$

i) $\frac{\sec x+1}{\sec x-1}$

j) $\frac{x\cos x+\sin x}{x\sin x+\cos x}$

k) $\frac{\csc x-\cot x}{\csc x+\cot x}$

l) $\frac{\sin x-x\cos x}{x\sin x+\cos x}$

m) $\frac{\cos x}{\sin x+\cos x}$

n) $\frac{1-\cos x}{1+\cos x}$

o) $\frac{x\sin x}{\cos x-\sin x}$

14.5 Derivative of a Function of a Function (Chain Rule)

If y is a differentiable function of u and u is a differentiable function of x;

Example 6

$$y=(3x+2)^3+2(3x+2)^2+4(3x+2)+5$$

$y=u^3+2u^2+4u+5$ and $u=(3x+2)$.

$$\frac{dy}{dx}=\lim_{\delta x\to 0}\frac{\delta y}{\delta x}$$

$$=\lim_{\delta x\to 0}\left(\frac{\delta y}{\delta u}\cdot\frac{\delta u}{\delta x}\right)$$

$$=\lim_{\delta x\to 0}\frac{\delta y}{\delta u}\cdot\lim_{\delta x\to 0}\frac{\delta u}{\delta x}$$

$$=\lim_{\delta u\to 0}\frac{\delta y}{\delta u}\cdot\lim_{\delta x\to 0}\frac{\delta u}{\delta x}$$

$$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}$$

$$\boxed{\therefore\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}}$$

$$y=(3x+2)^3+2(3x+2)^2+4(3x+2)+5$$

$$y = u^3 + 2u^2 + 4u + 5u = (3x + 2)$$

$$\frac{dy}{du} = 3u^2 + 4u + 4 \frac{du}{dx} = 3$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= (3u^2 + 4u + 4) \cdot 3$$

$$= 3\{3(3x + 2)^2 + 4(3x + 2) + 4\}$$

$$= 9(3x + 2)^2 + 12(3x + 2) + 12.$$

Derivative of Logarithm function

Let $y = \log_e x \Leftrightarrow x = e^y$

$\frac{d}{dy}(e^y) = e^y$ (14.4 (vii))

$$\frac{dx}{dy} = e^y$$

$$\frac{dx}{dy} = x$$

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{1}{\frac{\delta x}{\delta y}}$$

$$= \lim_{\delta y \to 0} \frac{1}{\frac{\delta x}{\delta y}}$$

$$= \frac{1}{\frac{dx}{dy}}.$$

$$\frac{d}{dx}(\log_e x) = \frac{1}{\frac{d}{dy}(e^y)} = \frac{1}{x}$$

$$\boxed{\frac{d}{dx}(\log_e x) = \frac{1}{x}, \qquad x \neq 0}$$

Example 7

Differentiate the following functions w.r.t. x.

a) $y = \sin(\log x)$

b) $y = (1 + \tan x)^{\frac{1}{2}}$

c) $y = \log(\sin 5x)$

d) $y = \frac{(\sin x)^2}{1+(\cos x)^2}$

e) $y = \log\left(\sqrt{\frac{1-x}{1+x}}\right)$

f) $y = \sqrt{\frac{1-\tan x}{1+\tan x}}$

g) $y = \log\left(\frac{\sin x}{1+\cos x}\right)$

h) $y = \log\left(\sqrt{\frac{1+\tan x}{1-\tan x}}\right)$

i) $y = \left\{x - \sqrt{1 + x^2}\right\}^n$

j) $y = (\sin x)^n$

k) $y = (\cos x)^m$

l) $y = (\sin x)^3(\cos x)^2$

m) $y = \sqrt{\frac{1+\cos 2x}{1-\cos 2x}}$

n) $y = (x^2 + 3)^2(x^3 + 1)^3$

o) $\frac{(\sec x)^2+(\tan x)^2}{(\sec x)^2-(\tan x)^2}$

Solutions

a) $y = \sin(\log x)$

Let $u = \log x$.

$$y = \sin u$$

$$\frac{dy}{du} = \cos u, \frac{du}{dx} = \frac{1}{x}$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \cos u \cdot \frac{1}{x} = \frac{1}{x}\cos(\log x).$$

b) $y = (1 + \tan x)^{\frac{1}{2}}$

$$u = 1 + \tan x.$$

$$y = u^{\frac{1}{2}}$$

$$\frac{dy}{du} = \frac{1}{2}u^{\frac{1}{2}-1} = \frac{1}{2}u^{-\frac{1}{2}} \quad , \quad \frac{du}{dx} = (\sec x)^2$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{2}u^{-\frac{1}{2}} \cdot (\sec x)^2$$

$$= \frac{1}{2}(1+\tan x)^{-\frac{1}{2}} \times (\sec x)^2 = \frac{(\sec x)^2}{2\sqrt{(1+\tan x)}}.$$

c) $y = \log(\sin 5x)$

$$u = \sin 5x$$

$$\frac{dy}{du} = \frac{1}{u}\frac{du}{dx} = 5\cos 5x$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{u} \cdot 5\cos 5x = \frac{5\cos 5x}{\sin 5x} = 5\cot 5x.$$

d) $y = \frac{(\sin x)^2}{1+(\cos x)^2}$

$$u_1 = \sin x \frac{du_1}{dx} = \cos x$$

$$y_1 = (\sin x)^2 = (u_1)^2$$

$$\frac{dy_1}{dx} = \frac{dy_1}{du_1} \cdot \frac{du_1}{dx} = 2u_1 \cdot \cos x$$

$$\frac{d}{dx}(\sin x)^2 = 2\sin x \cdot \cos x$$

$$u_2 = \cos x, \frac{du_2}{dx} = -\sin x$$

$$y_2 = (\cos x)^2 = (u_2)^2$$

$$\frac{dy_2}{dx} = \frac{dy_2}{du_2} \cdot \frac{du_2}{dx}$$

$$= 2u_2 \cdot (-\sin x)$$

$$= -2\cos x \sin x$$

$$\therefore \frac{d}{dx}[1+(\cos x)^2] = -2\cos x \sin x$$

$$y = \frac{(\sin x)^2}{1+(\cos x)^2}$$

$$\frac{dy}{dx} = \frac{[1+(\cos x)^2]\frac{d}{dx}(\sin x)^2 - (\sin x)^2\frac{d}{dx}[1+(\cos x)^2]}{[1+(\cos x)^2]^2}$$

$$= \frac{[1+(\cos x)^2](2\sin x\cos x) - (\sin x)^2(-2\cos x\sin x)}{[1+(\cos x)^2]^2}$$

$$= \frac{2\sin x\cos x[1+(\cos x)^2+(\sin x)^2]}{[1+(\cos x)^2]^2} = \frac{4\sin x\cos x}{[1+(\cos x)^2]^2}.$$

e) $y = \log\left(\sqrt{\frac{1-x}{1+x}}\right) = \log\left(\sqrt{1-x}\right) - \log\left(\sqrt{1+x}\right)$

$$= \frac{1}{2}\{\log(1-x) - \log(1+x)\} \longrightarrow (1)$$

Let $u_1 = 1 - x$ & $y_1 = \log(1-x)$ *then*, $y_1 = \log u_1$

$$\frac{du_1}{dx} = -1 \qquad \frac{dy_1}{du_1} = \frac{1}{u_1}$$

$$\frac{dy_1}{dx} = \frac{dy_1}{du_1}\cdot\frac{du_1}{dx} = \frac{1}{u_1}\times(-1) = -\frac{1}{1-x}$$

$$\frac{d}{dx}\log(1-x) = -\frac{1}{1-x}.$$

Let $u_2 = 1 + x$ *and* $y_2 = \log(1+x)$, *then* $y_2 = \log u_2$

Now $\frac{du_2}{dx} = 1$ and $\frac{dy_2}{du_2} = \frac{1}{u_2}$

By the chain rule, we have $\frac{dy_2}{dx} = \frac{dy_2}{du_2}\cdot\frac{du_2}{dx} = \frac{1}{1+x}\cdot(1) = \frac{1}{1+x}(1)$

$$\frac{d}{dx}\log(1+x) = \frac{1}{1+x}.$$

From (1), $\frac{dy}{dx} = \frac{1}{2}\left\{\frac{d}{dx}\log(1-x) - \frac{d}{dx}\log(1+x)\right\}$

$$= \frac{1}{2}\left\{-\frac{1}{1-x} - \frac{1}{1+x}\right\} = -\frac{1}{2}\frac{(1+x+1-x)}{(1-x)(1+x)}$$

$$= \frac{1}{1-x^2}.$$

$$\frac{du}{dx} = (1+\tan x)\frac{d}{dx}(1-\tan x) - (1-\tan x)\frac{d}{dx}(1+\tan x)$$

$$= (1+\tan x)[0-(\sec x)^2] - (1-\tan x)[0+(\sec x)^2]$$

$$= -(\sec x)^2[1+\tan x+1-\tan x]$$

$$= -2(\sec x)^2$$

f) $y = \sqrt{\frac{1-\tan x}{1+\tan x}}\, let\ u = \frac{1-\tan x}{1+\tan x}$

Then, $y = u^{\frac{1}{2}}$

Now, $\frac{dy}{du} = \frac{1}{2}u^{\left(\frac{1}{2}-1\right)} = \frac{1}{2}u^{-\frac{1}{2}} = \frac{1}{2}\left[\frac{1-\tan x}{1+\tan x}\right]^{-\frac{1}{2}} = \frac{1}{2}\left[\frac{1+\tan x}{1-\tan x}\right]^{\frac{1}{2}}$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= \frac{1}{2}\left[\frac{1+\tan x}{1-\tan x}\right]^{\frac{1}{2}} \times [-2(\sec x)^2] = -(\sec x)^2\left[\frac{1+\tan x}{1-\tan x}\right]^{\frac{1}{2}}.$$

g) $y = \log\left(\frac{\sin x}{1+\cos x}\right)\ \ let\ u = \frac{\sin x}{1+\cos x}$

$$\frac{du}{dx} = \frac{(1+\cos x)\frac{d}{dx}(\sin x) - \sin x\frac{d}{dx}(1+\cos x)}{(1+\cos x)^2}$$

$$= \frac{(1+\cos x)\cos x - \sin x\,(0-\sin x)}{(1+\cos x)^2}$$

$$= \frac{\cos x + (\cos x)^2 + (\sin x)^2}{(1+\cos x)^2}$$

$$\frac{du}{dx} = \frac{1+\cos x}{(1+\cos x)^2} = \frac{1}{1+\cos x}$$

Also, $y = \log u$

$$\frac{dy}{du} = \frac{1}{u} = \frac{1}{\frac{\sin x}{1+\cos x}} = \frac{1+\cos x}{\sin x}$$

$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \left(\frac{1+\cos x}{\sin x}\right) \cdot \left(\frac{1}{1+\cos x}\right) = \frac{1}{\sin x} = cosecx$

h) $y = \log\left(\sqrt{\frac{1+\tan x}{1-\tan x}}\right)$

$$= \log(1+\tan x)^{\frac{1}{2}} - \log(1-\tan x)^{\frac{1}{2}}$$

$$y = \frac{1}{2}\{\log(1+\tan x) - \log(1-\tan x)\}$$

$$u_1 = 1+\tan x\ and\ u_2 = 1-\tan x$$

$$\frac{du_1}{dx} = (\sec x)^2 \; and \; \frac{du_2}{dx} = -(\sec x)^2$$

$$y_1 = \log(1 + \tan x)$$

$y = \frac{1}{2}(y_1 - y_2)$; $\quad y_1 = \log u_1$

$$\frac{dy_1}{du_1} = \frac{1}{u_1} = \frac{1}{1 + \tan x}$$

$$\frac{dy_1}{dx} = \frac{dy_1}{du_1} \cdot \frac{du_1}{dx} = \frac{1}{u_1} \times (\sec x)^2 = \frac{(\sec x)^2}{1 + \tan x}$$

$$y_2 = \log(1 - \tan x)$$

$$y_2 = \log u_2 \; where \; u_2 = 1 - \tan x$$

Now $\frac{dy_2}{du_2} = \frac{1}{u_2}\frac{du_2}{dx} = -(\sec x)^2$

$$\frac{dy_2}{dx} = \frac{dy_2}{du_2} \cdot \frac{du_2}{dx} = \frac{1}{u_2} \times [-(\sec x)^2] = -\frac{(\sec x)^2}{1 - \tan x}$$

$$y = \frac{1}{2}\{\log(1 + \tan x) - \log(1 - \tan x)\}$$

$$= \frac{1}{2}(y_1 - y_2)$$

$$\frac{dy}{dx} = \frac{1}{2}\left[\frac{dy_1}{dx} - \frac{dy_2}{dx}\right]$$

$$= \frac{1}{2}\left[\frac{(\sec x)^2}{1+\tan x} - \left(-\frac{(\sec x)^2}{1-\tan x}\right)\right]$$

$$= \frac{(\sec x)^2}{2}\left[\frac{1}{1 + \tan x} + \frac{1}{1 - \tan x}\right]$$

$$= \frac{(\sec x)^2}{2}\frac{(1 - \tan x + 1 + \tan x)}{(1 + \tan x)(1 - \tan x)}$$

$$\frac{dy}{dx} = \frac{(\sec x)^2}{(1-(\tan x)^2)}.$$

i) $y = \left\{x - \sqrt{1 - x^2}\right\}^n$

$$let \; u = x - (1 - x^2)^{\frac{1}{2}} \; and \; v = (1 - x^2) \quad , y_2 = v^{\frac{1}{2}}$$

$$\frac{dv}{dx} = -2x \; and \; \frac{dy_2}{dv} = \frac{1}{2}v^{\frac{1}{2}-1} = \frac{1}{2}v^{-\frac{1}{2}}$$

$$\frac{d}{dx}(1-x^2)^{\frac{1}{2}} = \frac{dy_2}{dv}\cdot\frac{dv}{dx} = \frac{1}{2}v^{-\frac{1}{2}}\cdot(-2x) = \frac{1}{2}(1-x^2)^{-\frac{1}{2}}(-2x)$$

$$= -\frac{x}{(1-x^2)^{\frac{1}{2}}}$$

$y = u^n;$ $\qquad u = x - (1-x^2)^{\frac{1}{2}}$

$$\frac{du}{dx} = 1 - \left(-\frac{x}{(1-x^2)^{\frac{1}{2}}}\right) = 1 + \frac{x}{\sqrt{(1-x^2)}}$$

$$\frac{du}{dx} = \left(\frac{\sqrt{(1+x^2)}+x}{\sqrt{(1-x^2)}}\right)$$

$$y = u^n$$

$$\frac{dy}{du} = nu^{n-1}$$

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = nu^{n-1}\left(\frac{\sqrt{(1+x^2)}+x}{\sqrt{(1-x^2)}}\right)$$

$$= \frac{n\left(x-\sqrt{1-x^2}\right)^{n-1}\left(\sqrt{(1+x^2)}+x\right)}{\sqrt{1-x^2}}$$

$$= n\left(x-\sqrt{1-x^2}\right)^{n-2}\frac{\left(x-\sqrt{1-x^2}\right)\left(x+\sqrt{(1-x^2)}\right)}{\sqrt{1-x^2}}$$

$$= n\left(x-\sqrt{1-x^2}\right)^{n-2}\left[\frac{x^2-(1-x^2)}{\sqrt{1-x^2}}\right]$$

$$= \frac{n\left(x-\sqrt{1-x^2}\right)^{n-2}(2x^2-1)}{\sqrt{1-x^2}}.$$

j) $y = (\sin x)^n$ $\quad let\ u = \sin x$

$$\frac{du}{dx} = \cos x$$

Also, $y = u^n$

$$\frac{dy}{du} = nu^{n-1}$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= nu^{n-1} \cdot \cos x$$

$$\frac{d}{dx}(\sin x)^n = n(\sin x)^{n-1} \cdot \cos x.$$

k) $y = (\sin x)^n$

$$let\ u = \sin x$$

$$\frac{du}{dx} = \cos x$$

Now $y = u^m$

$$\frac{dy}{du} = mu^{m-1}$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= mu^{m-1} \cdot (-\sin x)$$

$$= -m(\cos x)^{m-1} \cdot \sin x.$$

l) $y = (\sin x)^3(\cos x)^2$

$$\frac{dy}{dx} = (\sin x)^3 \frac{d}{dx}(\cos x)^2 + (\cos x)^2 \frac{d}{dx}(\sin x)^3$$

$$\frac{d}{dx}(\cos x)^2 = -2(\cos x)^{2-1} \sin x = -2\cos x \sin x$$

$$\frac{d}{dx}(\sin x)^3 = 3(\sin x)^{3-1} \cos x = 3(\sin x)^2 \cos x$$

$$\frac{dy}{dx} = (\sin x)^3(-2\cos x \sin x) + (\cos x)^2(3(\sin x)^2 \cos x)$$

$$= -2(\sin x)^4 \cos x + 3(\sin x)^2(\cos x)^3$$

$$= (\sin x)^2 \cos x\, [-2(\sin x)^2 + 3(\cos x)^2].$$

m) $y = \sqrt{\frac{1+\cos 2x}{1-\cos 2x}} = \sqrt{\frac{1+2(\cos x)^2-1}{1-(1-2(\sin x)^2)}} = \sqrt{\frac{2(\cos x)^2}{2(\sin x)^2}}$

Let $y = \cot x$

$$\frac{dy}{dx} = -\cot x\,(\csc x)^2$$

n) $y = (x^2 + 3)^2(x^3 + 1)^3$

Let $u_1 = x^2 + 3$

$$\frac{du_1}{dx} = 2x$$

$$y_1 = (x^2 + 3)^2 = {u_1}^2$$

$$\frac{dy_1}{du_1} = 2u_1$$

$$\frac{dy_1}{dx} = \frac{dy_1}{du_1} \cdot \frac{du_1}{dx}$$

$$= 2u_1 \times 2x$$

$$\frac{d}{dx}(x^2 + 3)^2 = 2(x^2 + 3)2x$$

$$y_2 = (x^3 + 1)^3, y_2 = {u_2}^3,\ u_2 = x^3 + 1$$

$$\frac{du_2}{dx} = 3x^2$$

$$\frac{dy_2}{du_2} = 3{u_2}^2$$

$$\frac{dy_2}{dx} = \frac{dy_2}{du_2} \cdot \frac{du_2}{dx}$$

$$= 3{u_2}^2 \times 3x^2$$

$$\frac{d}{dx}(x^3 + 1)^3 = 3(x^3 + 1)^2 \cdot 3x^2$$

$$\frac{dy}{dx} = (x^2 + 3)^2 \frac{d}{dx}(x^3 + 1)^3 + (x^3 + 1)^3 \frac{d}{dx}(x^2 + 3)^2$$

$= (x^2 + 3)^2 \times 3(x^3 + 1)^2 \cdot 3x^2 + (x^3 + 1)^3 \times 2(x^2 + 3)(2x) = (x^3 + 1)^2(x^2 + 3)x \times [9(x^2 + 3)x + 4(x^3 + 1)]$

$$= x(x^3 + 1)^2(x^2 + 3)[13x^3 + 27x + 4].$$

o) $y = \frac{(\sec x)^2 + (\tan x)^2}{(\sec x)^2 - (\tan x)^2}$ we know $(\sec x)^2 - (\tan x)^2 \equiv 1$

$$\therefore y = (\sec x)^2 + (\tan x)^2 = (1 + (\tan x)^2) + (\tan x)^2$$

$$y = 1 + 2(\tan x)^2$$

Let $u = \tan x$

Then, $\frac{du}{dx} = (\sec x)^2$

$$\frac{du^2}{du} = 2u$$

$$\frac{du^2}{dx} = \frac{du^2}{du} \cdot \frac{du}{dx} = 2u \cdot (\sec x)^2$$

$$= 2 \tan x \, (\sec x)^2$$

$$\frac{dy}{dx} = 2\frac{d}{dx}(\tan x)^2 = 2 \cdot 2 \tan x \, (\sec x)^2$$

$$\frac{dy}{dx} = 4 \tan x \, (\sec x)^2.$$

Activity 2

Find the derivative of the following functions.

a) $y = \tan(\log x)$

b) $y = \log(\log x)$

c) $y = \sqrt{\frac{1+x}{1-x}}$

d) $y = \log\left(x + \sqrt{1 - x^2}\right)$

e) $y = \sqrt{\frac{1-\sin x}{1+\sin x}}$

f) $y = \sqrt{\frac{\sec x+\tan x}{\sec x-\tan x}}$

g) $y = \left(x + \sqrt{x^2 + a^2}\right)^n$

h) $y = (\sin x)^m (\cos x)^n$

i) $y = \log\left(\frac{\sin x}{1+\cos x}\right)$

j) $y = (\sin x)^5 \sin(x^5)$

Solutions to Activities

Activity 1

1)

a) 11

b) $-\frac{1}{9}$

c) -1

d) $-\frac{3}{25}$

e) 29

2)

a) $3x^2$

b) $2x-3$

c) $-\frac{3}{x^4}$

d) $10x+3$

e) $-\frac{5}{(3+x)^2}$

f) $-\frac{3}{(3x+2)^2}$

3)

a) $5x^4-20x^3+18x^2+4x-10$

b) $(2x-5)\sec x+(x^2-5x+6)\sec x\tan x$

c) $\frac{7}{(3\sin x+4\cos x)^2}$

d) $\frac{91x-6x^3}{(3x^2+7)^2}$

e) $\frac{46x}{(x^2+7)^2}$

f) $\frac{2}{(\sin x+\cos x)^2}$

g) $\frac{(x-1)(\sin x+\cos x)+1}{(x-\sin x)^2}$

h) $\frac{-2\sec x}{(\sec x+\tan x)^2}$

i) $\frac{-2\sec x\tan x}{(\sec x-1)^2}$

j) $-\frac{x(x+\sin 2x)}{(x\sin x+\cos x)^2}$

k) $2\csc x\,[\csc x-\cot x]^2$

l) $\frac{x^2}{(x\sin x+\cos x)^2}$

m) $-\frac{1}{(\sin x+\cos x)^2}$

n) $\frac{2\sin x}{(1+\cos x)^2}$

o) $\frac{x+\cos x\sin x-(\sin x)^2}{(\cos x-\sin x)^2}$

Activity 2

a) $\frac{1}{x}(\sec(\log x))^2$

b) $\frac{1}{x \log x}$

c) $\frac{1}{\sqrt{1+x}(1-x)^{\frac{3}{2}}}$

d) $\frac{1}{\sqrt{1+x^2}}$

e) $\sec x\,(\tan x - \sec x)$

f) $\sec x\,(\sec x + \tan x)$

g) $\frac{n\left(x+\sqrt{x^2+a^2}\right)^n}{\sqrt{x^2+a^2}}$

h) $m(\cos x)^{n+1}(\sin x)^{m-1} -$
$n(\sin x)^{m+1}(\cos x)^{n-1}$

i) $\cot x + \frac{\sin x}{1+\cos x}$

j) $5(\sin x)^4\{x^4 \sin x \cos(x^5) +$
$\cos x \sin(x^5)\}$

Summary

The derivative of the function $f(x)$ at $x = a$ is defined as

$$f^{'}(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

The derivative of the function $f(x)$ at the general point 'x' defined as

$$f^{'}(x) = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}.$$

Fundamental Theorem on Derivative

a) Derivative of a constant function is zero.

b) If $\frac{d}{dx}\big(f(x)\big) = f^{'}(x)$ then $\frac{d}{dx}\big(kf(x)\big) = kf^{'}(x)$, where k is a constant.

c) $\frac{d}{dx}\{f(x) \pm g(x)\} = \frac{d}{dx}\big(f(x)\big) \pm \frac{d}{dx}\big(g(x)\big)$.

d) If $\frac{d}{dx}f(x) = f^{'}(x)$then $\frac{d}{dx}f(ax) = af^{'}(ax)$.

e) $\frac{d}{dx}[f(x)g(x)] = f(x) \cdot \frac{d}{dx}g(x) + g(x) \cdot \frac{d}{dx}f(x)$

f) $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}f(x) - f(x)\frac{d}{dx}g(x)}{\big(g(x)\big)^2}$

Derivatives of some standard functions.

$$\frac{d}{dx}x^n = nx^{n-1}$$

$$\frac{d}{dx}\sin x = \cos x$$

$$\frac{d}{dx}\cos x = -\sin x$$

$$\frac{d}{dx}(\tan x) = (\sec x)^2$$

$$\frac{d}{dx}\csc x = -\csc x \cot x$$

$$\frac{d}{dx}\sec x = \sec x \tan x$$

$$\frac{d}{dx}\cot x = -(\csc x)^2$$

$$\frac{d}{dx}e^x = e^x$$

$$\frac{d}{dx}(\log_e x) = \frac{1}{x}$$

If $y = f(u),\ u = g(x)$

$$\therefore \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}, \quad \frac{dx}{dy} \neq 0$$

Learning outcomes

On completion of this session you should be able to find the differential coefficients (derivatives) of the functions by using the definition and the fundamental rules.

Session 15

Applications of Derivatives

Contents

Introduction

In the previous session, we have discussed the derivative of the function. These derivatives have a wide range of applications in science, engineering, economics and social sciences. In this session, we use the concept of derivative to determine the rate of change of quantities and the slope of the tangent to a curve at a given point. Also, we apply of the derivatives to determine of the maximal and minimal values of important functions. The concept of increasing and decreasing functions plays a significant role in finding the maxima and minima of functions. We use derivatives and its applications to sketch graphs of certain functions.

15.1 Geometrical meaning of derivative

Let $y = f(x)$ be a function that has a continuous curve. Let a be a any point in its domain.

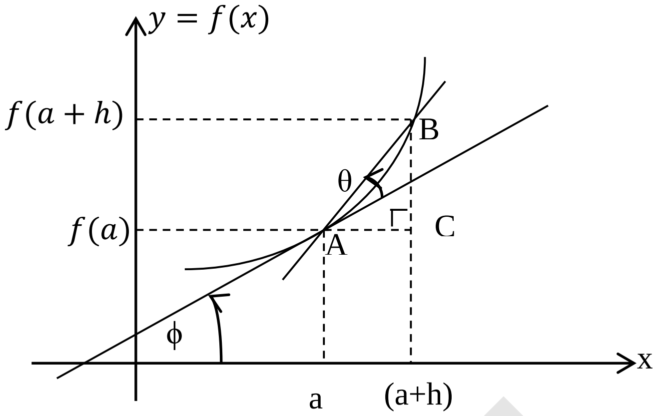


Figure 15.1.1.

Let $A \equiv \{a, f(a)\}$ be any point on the curve and $B \equiv \{a+h, f(a+h)\}$ some neighboring point on it.

Then the slope of the chord $AB = \tan\theta$

$$\tan\theta = \frac{AB}{AC} = \frac{f(a+h) - f(a)}{h}$$

As the point B comes nearer and nearer to the point A along the curve, the chord BA approaches the tangent at the point A. This happens when $\lim_{h\to 0}\frac{f(a+h)-f(a)}{h} = \lim_{B\to A}(\text{slope of the chord} AB)$.

Since $f'(a) = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$

$\therefore f'(a) =$the slope of the tangent at the point $A = \tan\varphi$

Also, we say that $f'(a)$ is the gradient of the curve at the point

$$A \equiv \{a, f(a)\}$$

15.2 Increasing and Decreasing Functions

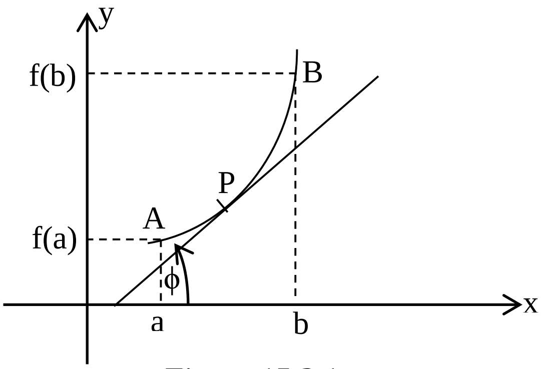


Figure 15.2.1

If $x_2 > x_1 \Longrightarrow f(x_2) > f(x_1)$, then f is called as increasing function. In the figure 15.2.1.you can see that $b > a$ and $f(b) > f(a)$.

$\therefore f$ is an increasing function.

Also, you can observe that if we draw a tangent to the curve between the points A and B the tangent makes acute angle φ with the positive direction of the $x -$ axis.

Since $0 < \varphi < \frac{\pi}{2}, \tan\varphi > 0$.

Thus, the gradient at the point P is positive.

$$x \in (a, b) \Longrightarrow f'(x) > 0$$

$x \in (a, b) f'(x) > 0$ then f is an increasing function in the interval (a, b).

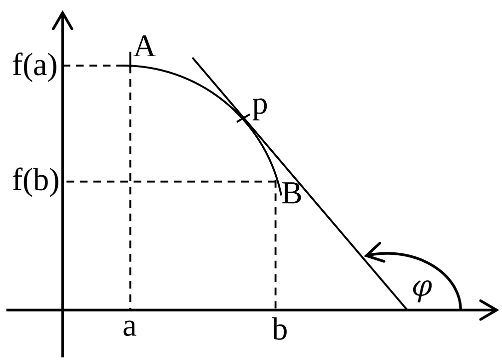


Figure 15.2.2.

If $x_2 > x_1 \Longrightarrow f(x_2) < f(x_1)$ then f is called a decreasing function. In the figure 15.2.2 you can see that $b > a$ and $f(b) < f(a)$. $\therefore f$is a decreasing function.

Also, you can observe that if we draw a tangent to the curve between the points A and B the tangent makes obtuse angle φ with the positive direction of the $x-$ axis.

$\therefore \frac{\pi}{2} < \varphi < \pi$ and hence, $\tan\varphi < 0$.

Thus, the gradient at the point P is negative.

$$\therefore x \in (a, b) \tan\varphi < 0 \qquad f'(x) < 0$$

$x \in (a, b),\ f'(x) < 0,$ then f is a decreasing function in the interval (a, b).

15.3 Rates of change of quantity

Let $y = f(x)$ be a relation between the variables x and y. Then $f'(a)$ represents the rate to change of y with respect to 'x' at $x = a$.

Let us assume that both x and y vary with time 't' (then x and y are functions of t).

We have

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} \text{ (from the chain rule)}$$

$$\frac{dy}{dt} = f'(x) \cdot \frac{dx}{dt}.$$

Example 1

A radius of a variable sphere is increasing at the rate 3cms^{-1}. How fast is the

a) Volume

b) Surface area

of the sphere increasing when the radius 10cm.

Solution

Let the volume of the sphere be V and the surface area of the sphere be A.

They are given by $V = \frac{4}{3}\pi r^3$and $A = 4\pi r^2$, where r is the radius of the sphere.

We have$\frac{dr}{dt} = 3\text{cms}^{-1}$.

$$\frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt}$$

$$= \frac{d}{dr}\left(\frac{4}{3}\pi r^3\right) \cdot \frac{dr}{dt}$$

$$= \frac{4}{3} \cdot 3 \cdot \pi r^2 \cdot \frac{dr}{dt}$$

$$= 4\pi r^2 \times 3$$

$$\frac{dV}{dt} = 12\pi r^2$$

$\therefore \left(\frac{dV}{dt}\right)_{r=10\text{cm}} = 12\pi \times 10^2 = 1200\pi\text{cm}^3/\text{s}.$

$$\frac{dA}{dt} = \frac{dA}{dr} \cdot \frac{dr}{dt}$$

$$= \frac{d}{dr}(4\pi r^2)\frac{dr}{dt}$$

$$= 8\pi r\frac{dr}{dt}$$

$$= 8\pi r \times 3$$

$$= 24\pi r$$

$\left(\frac{dA}{dt}\right)_{r=10\text{cm}} = 24\pi \times 10 = 240\pi\text{cm}^2/\text{s}.$

Example 2

A cube is expanding in such a way that its edge is change at a rate of 6mms^{-1}. Find its rate of change of its volume and the surface area when its edge is 5cm long.

Solution

Let x be the length of an edge of the cube.

It is given that $\frac{dx}{dt} = 0.6\text{cms}^{-1}$. We have the volume $V = x^3$ and the surface area $A = 6x^2$.

$$\frac{dV}{dt} = \frac{dV}{dx} \cdot \frac{dx}{dt}$$

$$= \frac{d}{dx}(x^3) \cdot \frac{dx}{dt}$$

$$= 3x^2 \cdot 0.6$$

$$= 1.8x^2$$

$\therefore \left(\frac{dV}{dt}\right)_{x=5\text{cm}} = 1.8 \times 5^{-2} = 45 \text{ cm}^3/\text{s}.$

$$\frac{dA}{dt} = \frac{dA}{dx} \cdot \frac{dx}{dt} = \frac{d}{dx}(6x^2) \cdot \frac{dx}{dt}$$

$$= 12x \cdot \frac{dx}{dt} = 12x \cdot 0.6 = 7.2x$$

$\left(\frac{dA}{dt}\right)_{x=5\text{cm}} = 7.2 \times 5 = 36 \text{ cm}^2/\text{s}.$

15.4 Tangents and Normals

We know that the derivative $f'(x)$ of a function $f(x)$ at a point $x = a$ is the slope of the tangent to the curve $y = f(x)$ at the point $[a, f(a)]$. We know that $f'(a)$ is the gradient of the tangent.

Let $y = mx + cm = f'(a)$

$$y = f'(a)x + c$$

$$\therefore [a, f(a)] \Longrightarrow f(a) = af'(a) + c$$

$$c = f(a) - af'(a)$$

$\therefore$ The equation of the tangent at the point $P(a, f(a))$ of the curve

$$y = f'(a)x + f(a) - af'(a)$$

The gradient of the normal at the point $P\big(a, f(a)\big) = -\frac{1}{f'(a)}$

$\therefore$ The equation of the normal at the point $P\big(a, f(a)\big)$

$$y = m'x + c'm'$$

$$y = -\frac{1}{f'(a)}x + c'$$

Since the point $P\big(a, f(a)\big)$ on the normal, we have

$$f(a) = -\frac{1}{f'(a)} \cdot a + c'$$

$$\therefore c' = f(a) + \frac{1}{f'(a)} \cdot a = \frac{f(a)f'(a) + a}{f'(a)}$$

$$\therefore\ y = -\frac{1}{f'(a)}x + \frac{f(a)f'(a) + a}{f'(a)}$$

$$f'(a)y + x = a + f(a)f'(a)$$

$\therefore$ The equation of the normal at the point $P\big(a, f(a)\big)$ to the curve $y = f(x)$ is

$$f'(a)y + x - [a + f(a)f'(a)] = 0$$

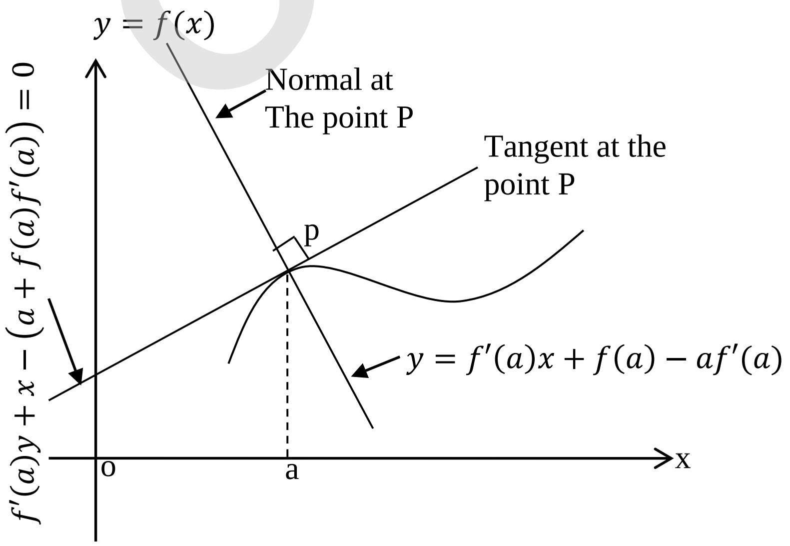


Figure 15.4.1

Example 3

a) Find the equations of the tangent and the normal to the curve $y = x^3 - x$ at $x = 2$.

b) Find the equations of tangent and normal to the curve $y = x^2 + 4x + 1$ at the point$x = 3$.

c) Find the point on the curve $y = 2x^2 - 3x + 5$ at which the tangent makes an angle of 45^0 with positive direction of x axis.

d) For the curve $y = 4x^3 - 2x^5$, find all the points at which the tangents passe through the origin.

Solution

a) $y = x^3 - x$

$$f'(x) = \frac{dy}{dx} = 3x^2 - 1$$

∴The gradient of the curve at $x = 2$

$$f'(2) = 3 \times 2^2 - 1 = 11$$

$$f(2) = 2^3 - 2 = 6$$

$$P(2,6)$$

∴ The equation of the tangent at the point $P(2,6)y = f'(2)x + c$

$$y = 11x + c$$

$P(2,6) \Longrightarrow 6 = 11 \times 2 + cc = 6 - 22 = -16.$

$y = 11x - 16.$

The gradient of the normal at $P(2,6) = -\frac{1}{f'(a)}$

∴the equation of the normal $y = -\frac{1}{11}x + c'$

Since the point $P(2,6)$ is on the normal,

$$6 = -\frac{1}{11} \times 2 + c'c' = 6 + \frac{2}{11} = \frac{66 + 2}{11} = \frac{68}{11}$$

$$y = -\frac{1}{11}x + \frac{68}{11}$$

∴ The equation of the normal at $x = 2$

$$11y + x = 68$$

b) $y = f(x) = x^2 + 4x + 1$ at $x = 3$

$$f'(x) = 2x + 4$$

$$f'(3) = 2 \times 3 + 4 = 10$$

$$f(3) = 3^2 + 4 \cdot 3 + 11 = 32$$

∴ The equation of the normal at $P(3,32)$ to the curve $y = x^2 + 4x + 1$

$$y = mx + c$$

$$m = f'(3) = 10$$

$$y = 10x + c$$

Since $P \equiv (3,32)$ is on the normal,

$$32 = 10 \times 3 + c$$

$$\therefore c = 32 - 30 = 2$$

∴ The equation of the normal at $P(3,32)$ to the curve $y = x^2 + 4x + 1$

$$y = 10x + 2.$$

The gradient of the normal to the above curve at $P(3,32) = -\frac{1}{f'(3)}$

$$= -\frac{1}{10} = m'$$

The equation of the normal at $P(3,32)$ to the above curve

$$y = m'x + c'$$

$$m' = -\frac{1}{10}$$

$$y = -\frac{1}{10}x + c'$$

Since the point $P \equiv (3,32)$ is on the normal,

$$32 = -\frac{3}{10} + c'$$

$$c' = 32 + \frac{3}{10} = \frac{323}{10}$$

Then, the equation of the normal is,

$$y = -\frac{1}{10}x + \frac{323}{10}$$

$\therefore$ The equation of the normal at the point (3,32) to the curve $y = x^2 + 4x + 1$ is $10y + x = 323$.

c) $y = f(x) = 2x^2 - 3x + 5$

$$f'(x) = 2 \times 2x - 3 = (4x - 3)$$

Given that the tangent makes an angle of 45^0 with positive direction of x axis.

$$\therefore f'(x) = \tan 45^0 = 1$$

$$4x - 3 = 1$$

$$4x = 4$$

$$x = 1, y = 2 \times 1^2 - 3 \times 1 + 5 = 4$$

$$\therefore P \equiv (1,4)$$

$\therefore$ The required point $P \equiv (1,4)$ and the equation of the tangent

$$y = mx + cm = 1$$

$$y = 1 \cdot x + c$$

Since the point $P \equiv (1,4)$ is on the tangent,

$$4 = 1 \cdot 1 + c$$

$$c = 3$$

$y = x + 3$ is the equation of the tangent

d) $y = f(x) = 4x^3 - 2x^5$

The tangent of the curve passes through the origin.

$$y = mx + c$$

$$(0,0) \Longrightarrow c = 0$$

$\therefore$ The equations of the tangents are in the form $y = mx$

$$m = f'(x) = 4(3x^2) - 2 \cdot 5x^4$$

$$= (12x^2 - 10x^4)$$

At $P_1 \equiv (x_1, y_1),\ f'(x_1) = 12{x_1}^2 - 10{x_1}^4$

$\therefore$ The equation of the tangent at $P_1 \equiv (x_1, y_1)$

$$y = mx \; where \; m = f'(x_1) = 12{x_1}^2 - 10{x_1}^4$$

$$y = (12{x_1}^2 - 10{x_1}^4)x$$

At $P_1 \equiv (x_1, y_1),\ y_1 = (12{x_1}^2 - 10{x_1}^4)x_1$

$$y_1 = 4{x_1}^3 - 2{x_1}^5$$

At the points of intersection, $(4{x_1}^3 - 2{x_1}^5) = x_1(12{x_1}^2 - 10{x_1}^4)$

$$8{x_1}^3 - 8{x_1}^5 = 0$$

$$8{x_1}^3(1 - {x_1}^2) = 0$$

$$\therefore x_1 = 0 \; or \; {x_1}^2 = 1$$

$$x_1 = 0 \; or \; x_1 = \pm 1$$

$$\Rightarrow y_1 = 0$$

$$\Rightarrow y_1 = 4(1)^3 - 2(1)^5 = 2$$

$$\Rightarrow y_1 = 4(-1)^3 - 2(-1)^5 = -2$$

The points at which the tangents pass through the origin are $(0,0)$, $(1,4)$, $(-1, -2)$.

15.5 Curve sketching

To sketch the graph of a given function, it is necessary to collect the following information about the function.

a) Intercepts on coordinate axes
b) Points and lines of symmetry
c) Asymptotes
d) Stationary points
e) Turning points & point of inflexion

a) Intercepts

The intercepts on$y$ $-$axis of a curve $y = f(x)$arethe points$(0, f(0))$.The interceptsx $-$axis are the points $(x, 0)$ such that $f(x) = 0$.

Computing the $x -$ intercepts can sometimes be extensively difficult in practice and therefore we should be content with approximate values only.

b) Symmetry

If $f(-x) = f(x)$, then f is symmetric about the y axis.

If $f(-x) = -f(x)$ then f is symmetric about the origin.

c) Asymptotes

If there is a constant 'k' such that $f(x) \longrightarrow k$as $x \longrightarrow \infty$, $x \longrightarrow -\infty$ then the line $y = k$ is called a horizontal asymptote of the curve $y = f(x)$. If there is constant 'a' such that $f(x) \longrightarrow \infty$ or $f(x) \longrightarrow -\infty$ as $x \longrightarrow a$ then the line $x = a$ is called a vertical asymptote of the curve $y = f(x)$.

Example 4

$$y = f(x) = \frac{1}{x-2} + 3$$

has a horizontal asymptote $y = 3$

$$\lim_{x \longrightarrow \infty} \left(\frac{1}{x-2} + 3 \right) = 3$$

$$\lim_{x \longrightarrow -\infty} \left(\frac{1}{x-2} + 3 \right) = 3$$

Also,

$\lim_{x\to 2^-}\left(\frac{1}{x-2}+3\right)\to -\infty$ and

$\lim_{x\to 2^+}\left(\frac{1}{x-2}+3\right)\to +\infty$

So, the curve has a vertical asymptote $x = 2$

Also, we have $f(0) = \frac{1}{0-2} + 2 = \frac{3}{2}$

$$A \equiv \left[0, \frac{3}{2}\right)$$

When $f(x) = 0;\ 0 = \frac{1}{x-2} + 2$

$$\frac{1}{2-x} = 2 \Longrightarrow 1 = 4 - 2x$$

$$2x = 3$$

$$x = \frac{3}{2}$$

$$B \equiv \left[\frac{3}{2}, 0\right]$$

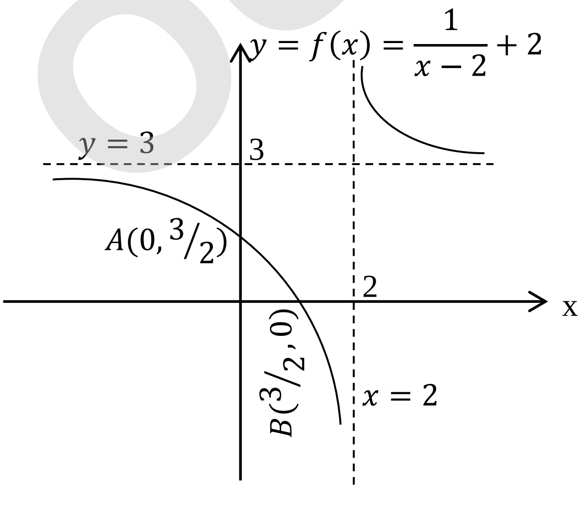


Figure 15.5.1

d) Stationary Points

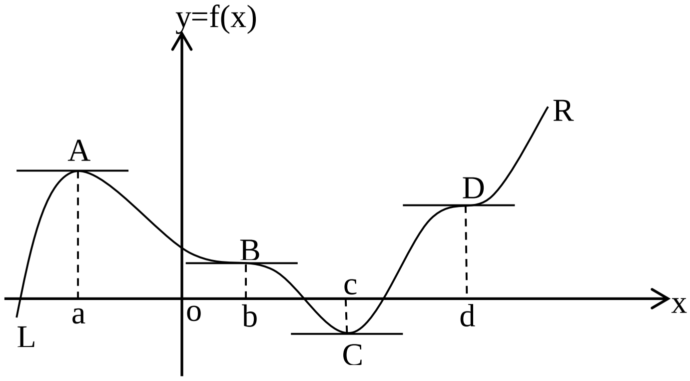


Figure 15.5.2

You can see that from the point L to the point A the curve of $y = f(x)$ is increasing. The tangent to the curve at any point between L and A makes an acute angle with the positive direction of $x -$ axis and so $f'(x) > 0$. The gradient of the curve at the point A is clearly zero. That is $f'(a) = 0$. From the point A to the point B, the curve of $y = f(x)$ is decreasing. The tangent to the curve at any point between A and B makes an obtuse angle with the positive direction of $x -$ axis and so $f'(x) < 0$.

Also, we can see that the gradient of the curve at the point B is zero. i.e. $f'(b) = 0$.

From the point B to the point C, the curve of $y = f(x)$ is decreasing.

Also, we can see that when $b < x < c$, $f'(x) < 0$.

Also, we have that the gradient of the curve at the point C is zero. i.e. $f'(c) = 0$.

Also, we have when $c < x < d$, $f'(x) > 0$.

The gradient of the point D is zero. i.e.$f'(d) = 0$

Also, we can see that when $x > d$, $f'(x) > 0$.

We have $f(a) = f(b) = f(c) = f(d) = 0$. Therefore, the points A, B, C and D are called *the* ***stationary points*.**

e) Turning Points and point of Inflexion

The points A and C (in the Figure 15.5.2.) are called the Turning Points and the point B and D are called the points of inflexion.

The identification of the turning points.

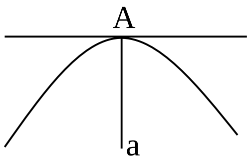


Figure 15.5.3

Again, we consider the diagram 3.5.2.

When $x < a, \quad f'(x) > 0$

When $x = a, \quad f'(x) = 0$

When $x > a, \quad f'(x) < 0$

Then the point A is identified as *a Local Maximum point* $A \equiv \{a, f(a)\}$.

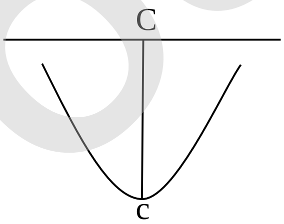


Figure 15.5.4

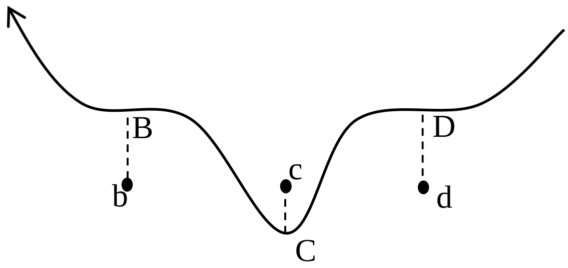
When $x < c, f'(x) < 0$

When, $x = c, f'(x) = 0$

$When\ x > c, f'(x) > 0$

Then the point $C\ is$ identified as *a e Local Minimum point* $C \equiv \{c, f(c)\}$.

Identification of Point of inflexion



Figure

When $x < b$, $f'(x) < 0$

$x = b$, $f'(x) = 0$

When $x > b$, $f'(x) < 0$

When $x < d$, $f'(x) > 0$

when $x = d$, $f'(x) = 0$

when $x > d$, $f'(x) > 0$

Then the points B and D are identified as *the points of inflexion*.

Example 5

Sketch the curve of $y = f(x) = x^4 - 8x^2$, where $x \in \mathbb{R}$.

Solution

$$y = f(x) = x^4 - 8x^2$$

When $x = 0$, $y = 0$ $\quad O \equiv (0,0)$

When $f(x) = 0$, $x^4 - 8x^2 = 0$

$$x^2(x^2 - 8) = 0$$

$$x = 0 \; or \; x^2 = 8$$

$$x = \pm 2\sqrt{2}$$

$\therefore f(x) = 0$ when $x = 0$, $x = \sqrt{2}$ and $x = -\sqrt{2}$

$$A \equiv (2\sqrt{2}, 0) \; B \equiv (-2\sqrt{2}, 0)$$

$$f(x) = x^4 - 8x^2$$

$$= x^4\left(1 - \frac{8}{x^2}\right)$$

When $x \longrightarrow \pm\infty$, $f(x) \longrightarrow +\infty$

There are no horizontal and vertical asymptotes to the curve of

$y = f(x) = x^4 - 8x^2$.

$$y = f(x) = x^4 - 8x^2$$

$$\frac{dy}{dx} = f'(x) = 4x^3 - 16x$$

$$= 4x(x^2 - 4)$$

$$= 4x(x + 2)(x - 2)$$

∴ We can see that when $x = 0$, $x = -2$ and $x = 2$, $f'(x) = 0$. There are three stationary points.

$$f'(x) = 4(x + 2)x(x - 2)$$

$when\ -2 - \delta x < x < -2\ ,\ f'(x) < 0\ \ f(x)\ is$ decreasing

when x=-2, $f'(x) = 0$

when $-2 < x < -2 + \delta x$, $f'(x) > 0$ $f(x)$ is increasing

At $x = -2$ there is a local minimum point.

$when\ 0 - \delta x < x < 0\ ,\ f'(x) > 0\ \ f(x)\ is$ increasing

when $x = 0, f(x) = 0$

when $0 < x < 0 + \delta x$, $f'(x) < 0$ $f(x)$ is decreasing

At $x = 0$ there is a local maximum point.

$when\ 2 - \delta x < x < 2\ ,\ f'(x) > 0\ \ f(x)\ is$ increasing

when $x = 2, f(x) = 0$

when $2 < x < 2 + \delta x$, $f'(x) < 0$ $f(x)$ is decreasing

At $x = 2$ there is a local minimum point

$$f(x) = x^4 - 8x^2$$

$$f(-2) = (-2)^4 - 8(-2)^2 = 16 - 32 = -16$$

$$\therefore f(0) = 0$$

$$f(2) = (2)^4 - 8(2)^2 = -16$$

$C \equiv (-2, -16)$ and $D \equiv (2, -16)$ are local minimum points.

$O \equiv (0,0)$ is the local maximum point.

Using the above details, we can sketch the curve as follows.

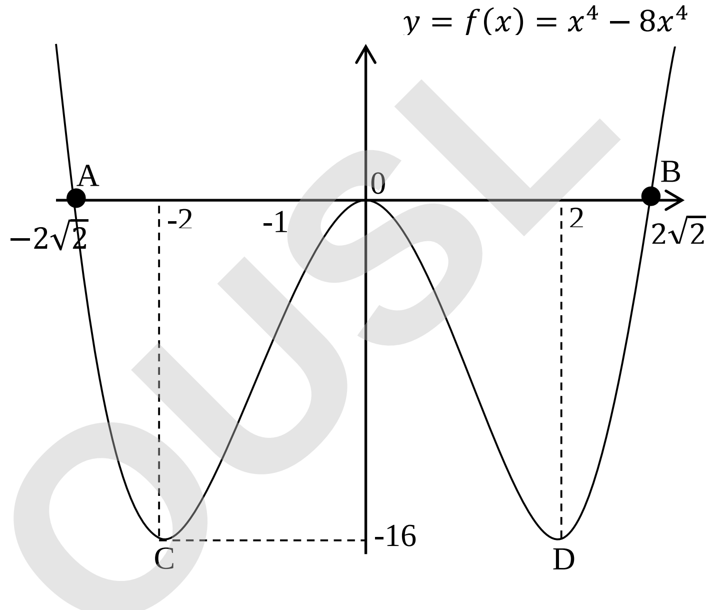


Figure 15.5.6

Example 6

Given that $f(x) = \frac{8x}{(x+1)(x^2+3)}, (x \neq -1)$

Sketch the graph of $y = f(x)$.

Solution

$$y = f(x) = \frac{8x}{(x+1)(x^2+3)}$$

$$x = 0 \Rightarrow f(0) = 0$$

$$O \equiv (0,0)$$

When $f(x) = 0,\ x = 0$

$f(-1)$ is not defined

$\therefore$ The vertical asymptote is $x + 1 = 0$

$$f(x) = \frac{8x}{(x+1)(x^2+3)}$$

$$\therefore x \to \pm\infty\ f(x) = 0$$

$\therefore$ $y = 0$ is a horizontal asymptote (x – axis)

$$f(x) = \frac{8x}{(x+1)(x^2+3)}$$

$$f^{'}(x) = \frac{(x+1)(x^2+3)\frac{d}{dx}(8x) - 8x\frac{d}{dx}\big((x+1)(x^2+3)\big)}{(x^2+3)^2(x+1)^2}$$

$f^{'}\ (x)$

$$= \frac{(x+1)(x^2+3)\cdot 8 - 8x\left[(x+1)\frac{d}{dx}(x^2+3) + (x^2+3)\frac{d}{dx}(x+1)\right]}{(x^3+3)^2(x+1)^2}$$

$$= 8\frac{\{(x+1)(x^2+3) - x[(x+1)(2x) + (x^2+3)]\}}{(x^2+3)^2(x+1)^2}$$

$$f^{'}\ (x) = 8\frac{\{x^3+x^2+3x+3-x(3x^2+2x+3)\}}{(x^2+3)^2(x+1)^2}$$

$$= 8\frac{\{-2x^3-x^2+3\}}{(x^2+3)^2(x+1)^2}$$

$$= 8\frac{(1-x)(2x^2+3x+3)}{(x^2+3)^2(x+1)^2}$$

When $f'(x) = 0$, the stationary points are given.

$$\therefore 8\frac{(1-x)(2x^2+3x+3)}{(x^2+3)^2(x+1)^2} = 0$$

$\therefore$ $x = 1$ or $2x^2 + 3x + 3 = 0$

$$f'(x) = 8\frac{(1-x)(2x^2+3x+3)}{(x^2+3)^2(x+1)^2}$$

$when\ -1 > x,\ \ f'(x) > 0 \therefore\ f(x)\ is$ increasing

$x = -1\ f'(x)\ is$ not defined, so $x = -1\ is\ a$ vertical asymptote

$1 > x > -1,\ f'(x) > 0\ f(x)\ is$ increasing

$x = 1,\ f'(x) = 0\ f(x)$ has a stationary point at $x = 1$

$x > 1, f'(x) < 0\ f(x)$ decreasing

At $x = 1, f(x)$ has the local maximum point.

$$f(1) = \frac{8 \cdot 1}{(1+1)(1+2)} = \frac{8}{6} = \frac{4}{3}$$

$\therefore$ Local maximum point $A \equiv \left[1, \frac{4}{3}\right]$

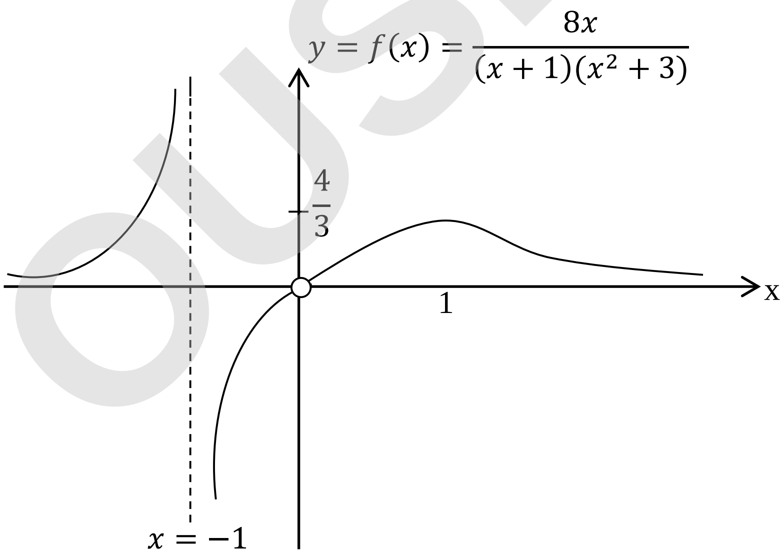


Figure 15.5.7

Example 7

Given the function $f: x \to x^3(x-2)$ where $x \in \mathbb{R}$; show that the graph of $y = f(x)$ has exactly two stationary points. Sketch the graph of $f(x)$.

Solution

$$y = f(x) = x^3(x-2)$$

$$f'(x) = x^3\frac{d}{dx}(x-2) + (x-2)\frac{d}{dx}(x^3)$$

$$= x^3 + (x-2)(3x^2)$$

$$= x^2[x + 3(x-2)]$$

$$= x^2[4x-6] \qquad\qquad f'(x) = 4x^2\left(x-\tfrac{3}{2}\right)$$

The stationary points are given by $f'(x) = 0$.

$$4x^2\left(x-\tfrac{3}{2}\right) = 0$$

$$x = 0 \text{ or } x = \tfrac{3}{2}$$

$$f(x) = x^3(x-2)$$

$$f(0) = 0$$

$$O \equiv (0,0)$$

$$f\left(\frac{3}{2}\right) = \frac{27}{8}\left(\frac{3}{2}-2\right) = -\frac{27}{16}$$

$$A \equiv (\frac{3}{2}, -\frac{27}{16})$$

∴ The coordinates of the stationary points are $O \equiv (0,0)$ and $A \equiv (\frac{3}{2}, -\frac{27}{16})$.

$$f'(x) = 4x^2\left(x-\frac{3}{2}\right)$$

$when\ 0-\delta x < x < 0$, $f'(x) < 0$ $f(x)\ is$ decreasing

when x=0 , $f'(x) = 0$

when $0 < x < 0+\delta x$, $f'(x) < 0$ $f(x)$ is decreasing

At $x = 0$ there is a point of inflexion. .

$when\ \frac{3}{2}-\delta x < x < \frac{3}{2}$, $f'(x) < 0$ $f(x)\ is$ indecreasing

when $x = 0, f(x) = 0$

At $x = \frac{3}{2}$ there is a local minimum point.

when $\frac{3}{2} < x < \frac{3}{2} + \delta x, \quad f' \ (x) < 0 \ f(x)$ is increasing

$f(x) = x^3(x - 2)$

When $x \to \pm\infty \ f(x) \to \ \pm\infty$

There are no vertical or horizontal asymptotes

$f(0) = 0 \quad 0 \equiv (0,0)$

$f(x) = 0$when$x = 0$ or $x = 2(2,0)$

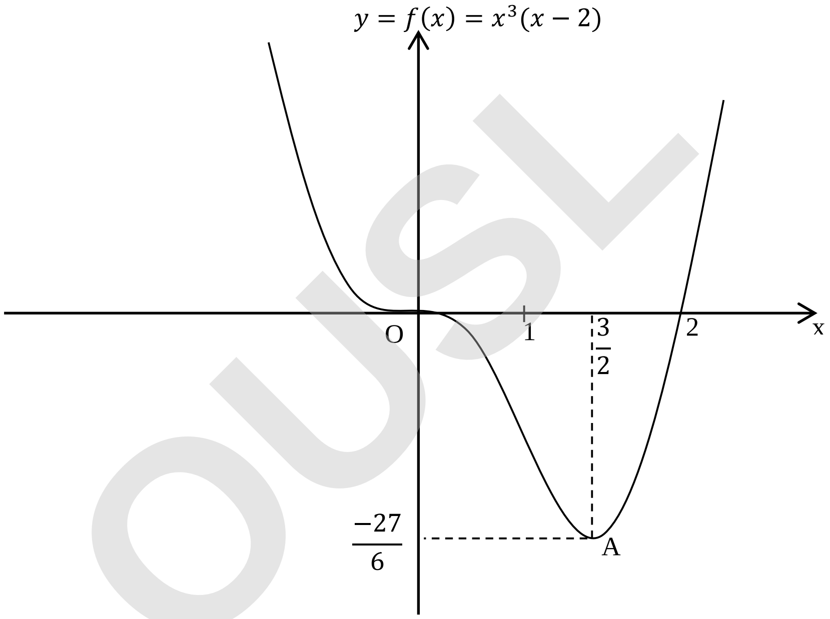


Figure 15.5.8

Example 8

1. Show that the function f defined by the function $f\colon x \to x^3 + 9(2 - x)^3$, where$x \in \mathbb{R}$ has two stationary points. Determine the nature of these stationary points.

Solution

1. $y = f(x) = x^3 + 9(2 - x)^3$

$$f'(x) = \frac{d}{dx}(x^3) + 9\frac{d}{dx}(2 - x)^3$$

$$f'(x) = 3x^2 + 9(3)(2 - x)^2(-1)$$

$$f'(x) = 3x^2 - 27(2 - x)^2$$

$$= 3x^2 - 27(4 - 4x + x^2)$$

$$= -24x^2 + 108x - 108$$

$$= -12\{2x^2 - 9x + 9\}$$

$$= -12[2x - 3][x - 3]$$

$$f'(x) = -24\left[x - \frac{3}{2}\right][x - 3]$$

$$\therefore f'(x) = 0,$$

$$\left(x - \frac{3}{2}\right)(x - 3) = 0$$

$$x = \frac{3}{2}, x = 3$$

$\therefore f(x)$ has two stationary points.

When $x = \frac{3}{2}$ and $x = 3$

$$f\left(\frac{3}{2}\right) = \left(\frac{3}{2}\right)^3 + 9\left(2 - \frac{3}{2}\right)^3 = \frac{9}{2}$$

$$f(3) = 3^3 + 9(2 - 3)^3 = 28$$

$$f'(x) = -24\left(x - \frac{3}{2}\right)(x - 3)$$

<u>At $x = -\frac{3}{2}$</u>

$x < -\frac{3}{2}, f'(x) = (-)(-)(-) < 0$ $\quad f(x)$ is decreasing

When $x = -\frac{3}{2}$ $, f'(x) = 0$

$when\ -\frac{3}{2} < x < 3,$ $\qquad f'(x) = (-)(+)(-) > 0$ $f(x)$ is increasing

$A \equiv \left(\frac{3}{2}, \frac{9}{2}\right)$ $is\ a$ Local minimum point.

<u>At $x = 3$</u>

$when\ -\frac{3}{2} < x < 3 f'(x) = (-)(+)(-) > 0 \Rightarrow f(x) is$ increasing

$when\ x = 3 \Rightarrow f'(x) = 0$

When $3 < x, f'(x) = (-)(+)(+) < 0 \quad f(x)\ is$ decreasing

$B \equiv (3,28)\ is\ a$ local maximum point.

. $A \equiv \left(\frac{3}{2}, \frac{9}{2}\right)$ is a local minimum point

$B \equiv (3,28)$ is a local maximum point.

Example 9

a) A closed right circular cylinder has volume 3456 cubic units. What should be the radius of the base so that the total surface area may be minimal value?

b) Show that a closed right circular cylinder of given surface area, and with maximum volume is such that its height is equal to the diameter of its base.

c) Show that the height of the cylinder of maximum volume that can be inscribed in a sphere of radius R is $\frac{2}{\sqrt{3}}R$. Also find the maximum volume.

d) A window of the perimeter 36m consists of a rectangle surmounted by a semicircle. Find the maximum area of the window.

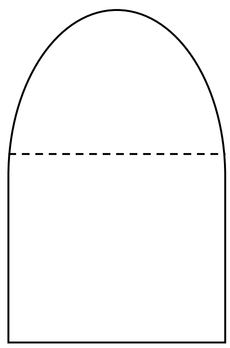


Figure 15.5.9

Solution

a) The volume of the cylinder, $V = \pi r^2 h = 3456$

the surface area of the cylinder, $A = 2\pi r^2 + 2\pi rh$

You can see that A is a function of two variables r and h.

But we have $\pi r^2 h = 3456$

$$\therefore\ h = \frac{3456}{\pi r^2}$$

$$\therefore\ A = 2\pi r^2 + 2\pi r\left(\frac{3456}{\pi r^2}\right)$$

$$= \left[2\pi r^2 + \frac{6912}{r}\right]$$

Now A is only function of an only one variable r;

$$A = \left[2\pi r^2 + \frac{6912}{r}\right]$$

$$\frac{dA}{dr} = 4\pi r - \frac{6912}{r^2}$$

$$= \frac{4\pi r^3 - 4312}{r} = \frac{4}{r}[\pi r^3 - 1728]$$

$$\frac{dA}{dr} = \frac{4}{r}[\pi r^3 - 3^3 \cdot 4^3]$$

$$\frac{dA}{dr} = \frac{4\pi}{r}\left\{r^3 - \left(\frac{1}{\pi}\right)12^3\right\}$$

$$\frac{dA}{dr} = \frac{4\pi}{r}\left\{r - \left(\frac{1}{\pi}\right)^{\frac{1}{3}}12\right\}\left\{r^2 + \left(\frac{1}{\pi}\right)^{\frac{1}{3}}12r + \left(\frac{1}{\pi}\right)^{\frac{2}{3}}12^2\right\}$$

We can see that $\frac{dA}{dr} = 0$ when only $r = 12\left(\frac{1}{\pi}\right)^{\frac{1}{3}}$

$$r < 12\left(\frac{1}{\pi}\right)^{\frac{1}{3}}, \Rightarrow \frac{dA}{dr} < 0$$

$$r > 12\left(\frac{1}{\pi}\right)^{\frac{1}{3}}, \Rightarrow \frac{dA}{dr} > 0$$

$\therefore$when $r = 12\left(\frac{1}{\pi}\right)^{\frac{1}{3}}$, A is minimum

$\therefore\therefore$when $r = 12\left(\frac{1}{\pi}\right)^{\frac{1}{3}}$, the total area may be minimum.

b) $V = \pi r^2 h \;\; A = 2\pi r^2 + 2\pi rh$

You can see that V is a function of r and h. But we have $A = 2\pi r^2 + 2\pi rh$

$$\therefore 2\pi rh = A - 2\pi r^2$$

$$h = \frac{A - 2\pi r^2}{2\pi r}$$

$$\therefore V = \pi r^2\left(\frac{A - 2\pi r^2}{2\pi r}\right) = \frac{r}{2}(A - 2\pi r^2)$$

$V = \frac{1}{2}\{Ar - 2\pi r^3\}$.

$$\frac{dV}{dr} = \frac{1}{2}[A - 6\pi r^2]$$

$$= -3\pi\left[r^2 - \frac{A}{6\pi}\right]$$

$$= -3\pi\left[r + \sqrt{\frac{A}{6\pi}}\right]\left[r - \sqrt{\frac{A}{6\pi}}\right]$$

When $r < \sqrt{\frac{A}{6\pi}} \Rightarrow \frac{dV}{dr} = (+)$

When $r > \sqrt{\frac{A}{6\pi}} \Rightarrow \frac{dV}{dr} = (-)$

$$when\ r = \sqrt{\frac{A}{6\pi}} \Rightarrow \frac{dV}{dr} = 0$$

$\therefore$ when $r = \sqrt{\frac{A}{6\pi}}$, V may be maximum

$$h = \frac{A - 2\pi r^2}{2\pi r} = \frac{A - 2\pi \frac{A}{6\pi}}{2\pi\sqrt{\frac{A}{6\pi}}}$$

$$= \frac{A - \frac{A}{3}}{\sqrt{\frac{4\pi A}{6}}} = \frac{2}{3}A\sqrt{\frac{6}{4\pi A}}$$

$$= \sqrt{\frac{6A^2}{9\pi A}} = \sqrt{\frac{2A}{3\pi}} = 2\sqrt{\frac{A}{6\pi}}$$

$$\therefore h = 2r$$

When the volume is maximum then the height of the cylinder is equal to its diameter.

c)

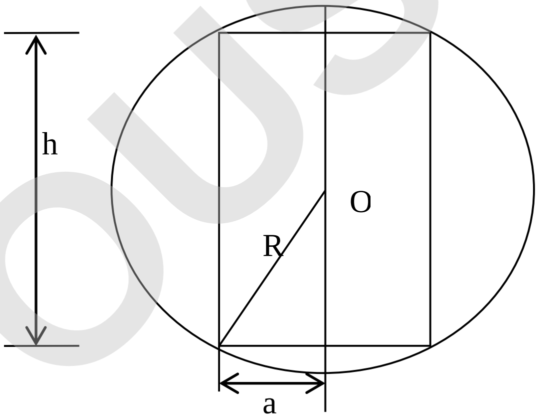


*Figure 15.5.10*Let the height of the cylinder is h and the radius a

We have $R^2 = a^2 + \left(\frac{h}{2}\right)^2$

$$a^2 = R^2 - \frac{h^2}{4}$$

The volume $V = \pi a^2 h$

$$= \pi h\left[R^2 - \frac{h^2}{4}\right]$$

$$= \pi\left[R^2 h - \frac{h^3}{4}\right]$$

$$\frac{dV}{dh} = \pi\left[R^2 - \frac{3h^2}{4}\right]$$

$$= -\frac{3\pi}{4}\left[h^2 - \frac{4R^2}{3}\right]$$

$$= -\frac{3\pi}{4}\left[h - \frac{2R}{\sqrt{3}}\right]\left[h + \frac{2R}{\sqrt{3}}\right]$$

$$when\ h < \frac{2R}{\sqrt{3}} \ \Rightarrow \frac{dV}{dh} > 0$$

$$when\ h = \frac{2R}{\sqrt{3}} \Rightarrow \frac{dV}{dh} = 0$$

$$when\ h > \frac{2R}{\sqrt{3}} \Rightarrow \frac{dV}{dh} < 0$$

∴when $h = \frac{2R}{\sqrt{3}}$, the volume of the cylinder may be maximum.

c) Perimeter of window $= P$

$$P = 36 = 2x + 2h + \pi x$$

$$(2+\pi)x + 2h = 36$$

$$2h = 36 - (2+\pi)x$$

$$h = 18 - \frac{(2+\pi)x}{2}$$

the area of the window $A = 2xh + \frac{1}{2}\pi x^2$

$$A = 2x\left[18 - \frac{(2+\pi)x}{2}\right] + \frac{1}{2}\pi x^2$$

$$= 36x - (2+\pi)x^2 + \frac{1}{2}\pi x^2$$

$$A = 36x - \left(2 + \frac{\pi}{2}\right)x^2$$

$$\frac{dA}{dx} = 36 - \left(2 + \frac{\pi}{2}\right)2x$$

$$= -\big((4+\pi)x - 36\big)$$

$$= -(4+\pi)\left[x - \frac{36}{4+\pi}\right]$$

When $x < \frac{36}{4+\pi} \Rightarrow \frac{dA}{dx} > 0$

When $x = \frac{36}{4+\pi} \Rightarrow \frac{dA}{dx} = 0$

When $x > \frac{36}{4+\pi} \Rightarrow \frac{dA}{dx} < 0$

$\therefore$when $x = \frac{36}{4+\pi}$, the area of the window is to be maximum.

$$A_{\max} = 36x - \left(2 + \frac{\pi}{2}\right)x^2$$

$$= \frac{36 \times 36}{(4+\pi)} - \frac{(4+\pi)}{4}\,\frac{36 \times 36}{(4+\pi)^2}$$

$$= \frac{36 \times 36}{(4+\pi)} - \frac{9 \times 36}{(4+\pi)}$$

$$= \frac{36 \times 27}{(4+\pi)} = \left(\frac{972}{4+\pi}\right)$$

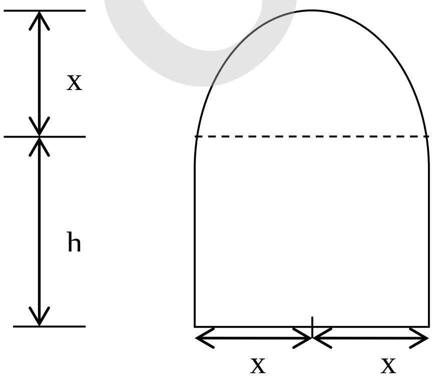


Figure 15.5.11

Activity 1

1)

a) A balloon gets inflated so that the rate of change of volume is proportional to its radius. Initially its radius is 2 units and after 1 second it is 3 units. Find its radius at time t.

b) The radius of a balloon is increasing at the rate of 10cm per second. At what rate is the surface area of the balloon is increasing when its radius15cm?

c) An edge of a variable cube is increasing at the rate of 3cm per second. Find the rate of changing the volume of the cube when the edge is 100cm long.

d) The area of an equilateral triangle increases at a rate of $2\sqrt{3}\text{cm}^2\text{s}^{-1}$. At what rate is the side increasing when it is 4cm lomg?

2)

a) Find the slope (gradient) of tangent and the normal to the curve $y = x^3 - x$at $x = 2$.

b) Find the equation of the tangent and the normal to the curve $y = -5x^2 + 6x + 7$ at the point $\left(\frac{1}{2}, \frac{35}{4}\right)$.

c) Find the equation of normal lines to the curve $y = x^3 - 3x$ which are parallel to the line $x + 9y = 14$.

d) Show that the tangent to the curve $y = 2x^3 - 3$ at the point where $x = 2$ and $x = -2$ are parallel.

e) For the curve $y = x^2 + 3x + 4$, find all the points at which tangent passes through the origin.

3)

a) Show that the graph $y = f(x) = x^3(x - 2)$ has exactly one turning point and determine its nature.

b) Show that the graph $y = f(x) = 3x^5 + 5x^3 - 30x + 10$ has exactly two stationary points. Determine the nature of these stationary points. Sketch the graph of $y = f(x)$.

c) Find the coordinates and nature of the stationary points on the curve given by $y = x^2(1 - x^2)$. Sketch the graph of $y = f(x)$.

d) Given that $f(x) = \frac{2x^2+1}{(x-1)^2}$. Find the turning points, asymptotes of the curve $y = f(x)$. Sketch the graph of $y = f(x)$.

e) Let $f(x) = (x - 1)^2(x + 1)$. Sketch the graph of $y = f(x)$ indicating the coordinates of the points it meets the coordinate axes and the turning points.

4)

a) An open tank with a square base and vertical sides is to be constructed from a metal sheet so as to hold a given quantity of water. Show that the cost of the material will be least, when depth of the tank is half of its width.

b) The section of a corner window is a rectangle surmounted by an equilateral triangle. Given that the perimeter16m. Find the width of window in order that maximum light may be admitted.

c) A wire of length 25m is to be cut into two pieces. One of the pieces is to be made into a square and the other into a circle. What should be the length of two pieces so that the total area of the square and the circle is minimal value.

d) A cylindrical thin can without a lid is to be made of a sheet metal. If S is the area used without waste, V the volume of the can and r is the radius of the cross-section, then prove that $2V = Sr - \pi r^3$. If S is given, prove that the volume of the can becomes the greatest value when the ratio of the height to the diameter is $1:2$.

e) The sum of the perimeter of two rectangles1.98m. The ratio of length to breadth is $3:2$ for one rectangle and for the other is $4:3$. Find the minimum value for the sum of their areas.

Solutions to Activities

Activity 1

1)

a) $\sqrt{5t+4}$

b) $1200\pi\text{cm}^2/\text{s}$

c) $900\ \text{cm}^3/\text{s}$

d) $1\ \text{cm}/\text{s}$

2)

a) Slope of Tangent $= 11$

Slope of Normal $= -\frac{1}{11}$

b) $4x - 4y + 33 = 4x + 4y - 37 = 0$

c) $x + 9y - 20 = 0,\ \ x + 9y + 20 = 0$

e) $(2, 14), (-2, 2)$

3)

a) $\left[\frac{3}{2}, -\frac{27}{16}\right]$ Local minimum point

b) $(1, -12)$ Local Minimum

$[-1, 32]$ Local Maximum

c) Local Minimum $(0, 0)$

Local Maximum $\left[\frac{1}{\sqrt{2}}, \frac{1}{4}\right]$

Local maximum $\left[-\frac{1}{\sqrt{2}}, \frac{1}{4}\right]$

d) Local maximum point $\left[-\frac{1}{2}, \frac{10}{9}\right]$

Horizontal asymptote $y = 1$

Vertical asymptote $x = 1$

e) $[-1,0]$, $[0,1]$, Local minimum point $[1,0]$

Local Maximum point $\left[-\frac{1}{3}, \frac{32}{21}\right]$

4)

a) $\frac{16}{6-\sqrt{3}}$

c) $\frac{25\pi}{4+\pi}, \frac{100}{4+\pi}$

e) $0.1188\ \mathrm{m}^2$

Summary

The geometrical meaning of the derivative at a point is the gradient of the curve at that point.

If $a < x < b, f'(x) > 0$ then $f(x)$ is an increasing function in the interval (a, b).

If $c < x < d, f'(x) < 0$ then $f(x)$ is a decreasing function in the interval (a, b).

$$f'(x) = \frac{dy}{dx}$$

Is the rate of change in y with respect to the variable x.

If the equation of the tangent at the point $x = a$to the curve $y = f(x)$ is

$$y - f(a) = f'(a)(x - a)$$

The equation of the normal at the point $x = a$the curve $y = f(x)$is

$y - f(a) = -\frac{1}{f'(a)}(x - a)$.

If there is a constant 'k' such that $f(x) \longrightarrow k$ as $x \longrightarrow \infty$ and $x \longrightarrow -\infty$ then $y = k$ is a horizontal asymptote of the curve $y = f(x)$;

If there is constant 'a' such that $f(x) \longrightarrow \infty$ or $f(x) \longrightarrow -\infty$ as $x \longrightarrow a$ then the line $x = a$ is called a vertical asymptote.

i. $x < a, \Rightarrow\ f'(x) > 0$

$x = a \Rightarrow f'(x) = 0$

$x > a \Rightarrow f'(x) < 0$

There is a local maximum point at $x = a$ to the curve $y = f(x)$.

ii. $x < c \Rightarrow f'(x) < 0$

$x = c \Rightarrow f'(x) = 0$

$x > c \Rightarrow f'(x) > 0$

There is a local minimum point at $x = c$ to the curve $y = f(x)$.

iii. $x < b \Rightarrow f'(x) < 0$

$x = b \Rightarrow f'(x) = 0$

$x > b \Rightarrow f'(x) > 0$

There is a point of inflexion at $x = b$ to the curve $y = f(x)$.

iv. $x < d \Rightarrow f'(x) > 0$

$x = d \Rightarrow f'(x) = 0$

$x > d \Rightarrow f'(x) > 0$

There is a point of inflexion at $x = d$ to the curve $y = f(x)$.

When $f'(x) = 0$ i.e.

$f(a) = f(b) = f(c) = f(d) = 0$

Then at $x = a$, $x = b$, $x = c$, $x = d$ the curve $y = f(x)$ has stationary points.

The Local maximum points and the Local minimum points are called the "Turning Points".

Learning outcomes

At the end of this study session you will be able to

- Use the geometrical meaning of the derivatives.
- Identify the increasing and decreasing functions
- Solve the problems involving the rate of changing.

- Identify the vertical and horizontal asymptotes of the given curve $y = f(x)$the local minimum, local maximum point and point of inflexion of a given function.
- Sketching of the simple curves
- Apply the above theory for solving the maximum minimum problems

Session 16

Introduction to Integration

Contents

Introduction

We have discussed methods for finding the derivatives of function in the previous sessions. We will now turn our attention to reversing the operation of derivative (differentiation).

In this session, for a given derivative of a function, we attempt to find the function corresponding to that derivative. This process is called “Anti differentiation”.

For an example, if the derivative of a function is $4x^3$, we know the function could be $f(x) = x^4$ because $\frac{d}{dx}(x^4) = 4x^3$. But the function $f(x) = x^4 + c$ (where c is a constant) $\frac{d}{dx}(x^4 + c) = 4x^3$. $\therefore$ We have if $f(x) = x^4 + c$ then $f^{'}(x) = 4x^3$, where c is any arbitrary constant. This process of finding antiderivative or antidifferentiation is called *Integration*. In this session, we are going to study some basic methods of integration.

16.1 Notation

If the derivative (differential of coefficient) of $\{F(x)+c\}$is $f(x)$, where c is any arbitrary constant,

$$\frac{d}{dx}\{F(x)+c\}=f(x).$$

Then we say that the anti-derivative or anti differentiation or Integral of $f(x)$ is $\{F(x)+c\}$.

We denote it as

$$\int f(x)dx=F(x)+c.$$

For an example $\int 4x^3dx=x^4+c$.

Note that $\int \cdot\, dx$ is the notation of integration.

The symbol $\int \cdot$stands for integral of, $f(x)$ is the integrand, x the variable of integration and dx denotes the integration with respect to 'x'.

16.2 Indefinite Integral

We know that if $\frac{d}{dx}F(x)=f(x)$then $\int f(x)dx=F(x)$. Also, for any c constant

$$\frac{d}{dx}[F(x)+c]=\frac{d}{dx}F(x)+\frac{d}{dx}(c)=\frac{d}{dx}F(x)=f(x)$$

$\int f(x)dx=F(x)+c$, where c is an arbitrary constant. This shows that $F(x)$ and $F(x)+c$ are both integrals of the same function $f(x)$. Thus, for different values of c, we obtain different integrals of $f(x)$. Therefore, the integral of $f(x)$ is not unique. By considering of this property, $F(x)$ called indefinite integral of $f(x)$.

16.3 Some Standard Rules on Integration

i. If k is any real valued constant and $f(x)$ is a function of x, then

$$\int kf(x)\,dx = k\int f(x)\,dx.$$

ii. The integral of the sum or difference of two functions is equal to the sum or difference of their integrals

$$\int \{f_1(x) \pm f_2(x)\}\,dx = \left\{\int f_1(x)\,dx\right\} \pm \left\{\int f_2(x)\,dx\right\}.$$

iii. If $\int f(x)\,dx = F(x) + C$ then $\int f(ax+b)\,dx = \frac{1}{a}F(ax+b) + C$.

16.4 Some standard Function's integration

The following results are a direct consequence of the definition of Indefinite integrals.

i. $\frac{d}{dx}\left\{\frac{x^{n+1}}{n+1}\right\} = \frac{1}{n+1}(n+1)x^n = x^n$

$\Rightarrow \int x^n\,dx = \frac{1}{n+1}x^{n+1} + c, \ (n \neq -1)$

ii. $\frac{d}{dx}\log|x| = \frac{1}{x}$

$$\Rightarrow \int \frac{1}{x}\,dx = \log_e|x| + c$$

iii. $\frac{d}{dx}\log|f(x)| = \frac{f'(x)}{f(x)}$

$$\Rightarrow \int \frac{f'(x)}{f(x)}\,dx = \log|f(x)| + c$$

iv. $\frac{d}{dx}\left(2\sqrt{f(x)}\right) = \frac{2}{2}f'(x)[f(x)]^{-\frac{1}{2}} = \frac{f'(x)}{\sqrt{f(x)}} \Rightarrow$

$$\int \frac{f'(x)}{\sqrt{f(x)}}\,dx = 2\sqrt{f(x)} + c$$

v. $\frac{d}{dx}(-\cos x) = \sin x \Rightarrow$

$$\int \sin x\,dx = -\cos x + c$$

vi. $\frac{d}{dx}(\sin x) = \cos x \Rightarrow$

$$\int \cos x \, dx = \sin x + c$$

vii. $\frac{d}{dx}(\tan x) = (\sec x)^2 \Longrightarrow$

$$\int (\sec x)^2 \, dx = \tan x + c$$

viii. $\frac{d}{dx}(-\csc x) = \csc x \cot x \Longrightarrow \int \csc x \cot x \, dx = -\csc x + c$

ix. $\frac{d}{dx}(\sec x) = \sec x \tan x \Longrightarrow$

$$\int \sec x \tan x \, dx = \sec x + c$$

x. $\frac{d}{dx}(-\cot x) = (\csc x)^2 \Longrightarrow$

$$\int (\csc x)^2 \, dx = -\cot x + c$$

xi. $\frac{d}{dx}(e^x) = e^x \Longrightarrow$

$$\int e^x \, dx = e^x + c$$

xii. $\frac{d}{dx}(x) = 1 \Longrightarrow$

$$\int 1 \, dx = x + c$$

xiii. $\frac{d}{dx}(c) = 0 \Longrightarrow$

$$\int 0 \, dx = c$$

Example 1

Evaluate the following integrals.

1) $\int (3x-2)^3 \, dx$

2) $\int x^2 \left(1 + \frac{1}{x^2}\right) dx$

3) $\int \frac{x^3-1}{x^2} dx$

4) $\int (3x^2 + 4x + 5) \, dx$

5) $\int \left(x^2 + x^{\frac{1}{2}} + x^{-\frac{1}{2}}\right) dx$

6) $\int \frac{(x^3-x^2+x-1)}{x-1} dx$

7) $\int \frac{1-\sin x}{(\cos x)^2} dx$

8) $\int \frac{(\sec x)^2}{(\csc x)^2} dx$

9) $\int \frac{2-3\sin x}{(\cos x)^2} dx$

10) $\int (\sec(ax+b))^2 \, dx$

11) $\int \frac{1}{\sqrt{x+a}+\sqrt{x+b}}\,dx$

12) $\int \frac{1}{\sqrt{x}+\sqrt{x+1}}\,dx$

13) $\int \frac{2}{(2x+1)^2}\,dx$

14) $\int \frac{(\cos x)^2}{1+\sin x}\,dx$

15) $\int \frac{x^4+1}{x^2-1}\,dx$

16) $\int \frac{1}{(x+1)^2}\,dx$

17) $\int \frac{1}{(x+1)(x-2)}\,dx$

18) $\int \frac{6x+8}{(3x-1)(x+1)}\,dx$

19) $\int \frac{(2x^2+3x+4)}{6x^2+x-1}\,dx$

20) $\int x\sqrt{x+3}\,dx$

21) $\int \frac{x-1}{\sqrt{x+4}}\,dx$

22) $\int (1+x)\sqrt{1-x}\,dx$

23) $\int \sqrt{1+\sin(2x)}\,dx$

24) $\int \sqrt{1+\cos(2x)}\,dx$

25) $\int \sqrt{1-\cos(2x)}\,dx$

26) $\int (\tan\theta)^2\,d\theta$

27) $\int \sec(3\theta)\tan(3\theta)\,d\theta$

28) $\int \sec\theta\,d\theta$

29) $\int \sqrt{\frac{1-\sin x}{1+\sin x}}\,dx$

30) $\int \frac{\cos x+\sin x}{\cos x-\sin x}\,dx$

31) $\int \frac{6x^2+12x+8}{2x^3+6x^2+8x+1}\,dx$

32) $\int \frac{2x+3}{\sqrt{x^2+3x+5}}\,dx$

33) $\int \tan\theta\,d\theta$

34) $\int \frac{e^x}{1+e^x}\,dx$

35) $\int \frac{2x+3}{x^2+3x+5}\,dx$

solutions

1) $\int (3x-2)^3\,dx = \frac{1}{4}(3x-2)^4 + c$

Here, we use the result
$$\int f(ax+b)\,dx = \frac{1}{a}F(ax+b)+c$$

2) $\int x^2\left(1+\frac{1}{x^2}\right)dx = \int (x^2+1)\,dx = \frac{x^3}{3}+x+c$

Here, we use the result
$$\int x^n\,dx$$

3) $\int \frac{x^3-1}{x^2} dx = \int \left(\frac{x^3}{x^2} - \frac{1}{x^2}\right) dx$

$$= \int (x - x^{-2})\, dx$$

$$= \frac{1}{2}x^2 - \frac{1}{-2+1}x^{-2+1}$$

$$= \frac{1}{2}x^2 + x^{-1} + c$$

$$= \frac{1}{2}x^2 + \frac{1}{x} + c$$

4) $\int (3x^2 + 4x + 5)\, dx = 3\frac{1}{2+1}x^{2+1} + \frac{4}{1+1}x^{1+1} + 5x + c$

$$= x^3 + 2x^2 + 5x + c$$

$$\int x^n\, dx = \frac{1}{n+1}x^{n+1}$$

5) $\int \left(x^2 + x^{\frac{1}{2}} + x^{-\frac{1}{2}}\right) dx = \frac{1}{3}x^3 + \frac{1}{\frac{1}{2}+1}x^{\frac{1}{2}+1} + \frac{1}{-\frac{1}{2}+1}x^{-\frac{1}{2}+1} + c$

$$= \frac{1}{3}x^3 + \frac{2}{3}x^{\frac{3}{2}} + 2x^{\frac{1}{2}} + c$$

6) $\int \frac{(x^3-x^2+x-1)}{x-1} dx = \int \frac{(x^3-1)-(x^2-x)}{x-1} dx$

$$= \int \frac{(x-1)[x^2+x+1] - x(x-1)}{(x-1)} dx$$

$$= \int x^2 + x + 1 - x\, dx = \int (x^2 + 1)\, dx$$

$$= \frac{x^3}{3} + x + c$$

7) $\int \frac{1-\sin x}{cos^2 x} dx = \int \left(\frac{1}{cos^2 x} - \frac{\sin x}{cos^2 x}\right) dx$

$$= \int (sec^2 x - \sec x \tan x)\, dx$$

$$= \int sec^2 x\, dx - \int \sec x \tan x\, dx$$

$$= \tan x - \sec x + c$$

Here, we apply following results to get the integration of the given function.

$$\int (\sec x)^2 \, dx = \tan x$$

$$\int \sec x \tan x \, dx = \sec x$$

8) $\int \frac{sec^2 x}{csc^2 x} dx = \int \frac{(\sin x)^2}{(\cos x)^2} dx = \int tan^2 x \, dx$

$$= \int (sec^2 x - 1) \, dx = tanx - x + c$$

$$\boxed{\int sec^2 x \, dx = tanx \quad \int (-1) \, dx = -x}$$

9) $\int \frac{2-3\sin x}{cos^2 x} dx = \int \frac{2}{cos^2 x} dx - \int 3 \frac{\sin x}{cos^2 x} dx$

$$= \int 2sec^2 x \, dx - \int 3 \tan x \sec x \, dx$$

$$= 2 \tan x - 3 \sec x + c$$

$$\boxed{\int sec^2 x \, dx = tanx \quad \int \sec x \tan x \, dx = \sec x}$$

10) $\int sec^2(ax+b) \, dx = \frac{1}{a} \tan(ax+b) + c$

11)

$$\int \frac{1}{\sqrt{x+a} + \sqrt{x+b}} dx = \int \frac{\left(\sqrt{x+a} - \sqrt{x+b}\right)}{\left(\sqrt{x+a} + \sqrt{x+b}\right)\left(\sqrt{x+a} - \sqrt{x+b}\right)} dx$$

$$\boxed{\int f(ax+b) \, dx = \frac{1}{a} F(ax+b)}$$

$$= \int \frac{\left(\sqrt{x+a} - \sqrt{x+b}\right)}{(x+a)-(x+b)}\,dx = \int \frac{\left(\sqrt{x+a} - \sqrt{x+b}\right)}{(a-b)}\,dx$$

$$= \frac{1}{a-b}\left\{\int \sqrt{x+a}\,dx - \int \sqrt{x+b}\,dx\right\}$$

$$= \frac{1}{a-b}\left[\frac{1}{\left(\frac{1}{2}+1\right)}(x+a)^{\left(\frac{1}{2}+1\right)} - \frac{1}{\left(\frac{1}{2}+1\right)}(x+b)^{\left(\frac{1}{2}+1\right)}\right]$$

$$= \frac{1}{a-b}\left[\frac{2}{3}(x+a)^{\frac{3}{2}} - \frac{2}{3}(x+b)^{\frac{3}{2}}\right] + c$$

$$= \frac{2}{3(a-b)}\left[(x+a)^{\frac{3}{2}} - (x+b)^{\frac{3}{2}}\right] + c$$

$$\int (x+a)^{\frac{1}{2}}\,dx = \frac{1}{\left(\frac{1}{2}+1\right)}(x+a)^{\left(\frac{1}{2}+1\right)} = \frac{2}{3}(x+a)^{\frac{3}{2}}$$

12) $\int \frac{1}{\sqrt{x}+\sqrt{x+1}}\,dx = \int \frac{\left(\sqrt{x+1}-\sqrt{x}\right)}{\left(\sqrt{x+1}+\sqrt{x}\right)\left(\sqrt{x+1}-\sqrt{x}\right)}\,dx$

$$= \int \frac{\left(\sqrt{x+1} - \sqrt{x}\right)}{(x+1)-x}\,dx = \int \frac{\left(\sqrt{x+1} - \sqrt{x}\right)}{1}\,dx$$

$$= \int \sqrt{x+1}\,dx - \int \sqrt{x}\,dx$$

$$= \frac{1}{\left(\frac{1}{2}+1\right)}(x+1)^{\left(\frac{1}{2}+1\right)} - \frac{1}{\left(\frac{1}{2}+1\right)}x^{\left(\frac{1}{2}+1\right)} + c$$

$$= \frac{2}{3}(x+1)^{\frac{3}{2}} - \frac{2}{3}x^{\frac{3}{2}} + c$$

$$= \frac{2}{3}\left[(x+1)^{\frac{3}{2}} - x^{\frac{3}{2}}\right] + c$$

$$\boxed{\int \sqrt{ax+b}\,dx = \frac{1}{\left(\frac{1}{2}+1\right)}(ax+b)^{\left(\frac{1}{2}+1\right)} = \frac{2}{3}(ax+b)^{\frac{3}{2}}}$$

13) $\int \frac{2}{(2x+1)^2}\,dx = 2\int (2x+1)^{-2}\,dx$

$$= \frac{2}{-2+1}(2x+1)^{-2+1} \times \frac{1}{2}$$

$$= -2(2x+1)^{-1} \times \frac{1}{2}$$

$$= -\frac{1}{(2x+1)} + c$$

$$\boxed{\int (ax+b)^n \, dx = \frac{1}{n+1}(ax+b)^{n+1} \times \frac{1}{a}}$$

14) $\int \frac{cos^2 x}{1+\sin x} dx = \int \frac{1-sin^2 x}{1+\sin x} dx = \int \frac{(1-\sin x)(1+\sin x)}{(1+\sin x)} dx$

$$= \int (1-\sin x) \, dx$$

$$= x + \cos x + c$$

$$\boxed{\int 1 \, dx = x; \ \int \sin x \, dx = -\cos x}$$

15) $\int \frac{x^4+1}{x^2-1} dx$

$$x^4 + 1 \equiv (x^2-1)(x^2+1) + 2$$

$$\frac{x^4+1}{x^2-1} = (x^2+1) + \frac{2}{(x^2-1)}$$

$$\frac{1}{x^2-1} = \frac{A}{x-1} + \frac{B}{x+1}$$

$$1 \equiv A(x+1) + B(x-1)$$

$x = 1, A = \frac{1}{2};$

$x = -1; B = -\frac{1}{2}$

$$\therefore \frac{1}{x^2-1} = \frac{\frac{1}{2}}{x-1} + \frac{-\frac{1}{2}}{x+1}$$

$$\int \frac{x^4+1}{x^2-1} dx = \int \left\{ (x^2+1) + 2\left[\frac{\frac{1}{2}}{x-1} + \frac{\frac{1}{2}}{x+1} \right] \right\} dx$$

$$= \int \left(x^2 + 1 + \frac{1}{x-1} + \frac{1}{x+1} \right) dx$$

$$= \frac{x^3}{3} + x + \log_e|x-1| + \log_e|x+1| + c$$

$$\boxed{\int x^n \, dx = \frac{1}{n+1} x^{n+1}; \ \int \frac{f'(x)}{f(x)} dx = \log_e|f(x)|}$$

16) $\int \frac{1}{(x+1)^2} dx = \int (x+1)^{-2} \, dx = \frac{1}{(-2+1)} (x+1)^{-2+1} + c$

$$= -1(x+1)^{-1} + c = -\frac{1}{(x+1)} + c$$

$$\boxed{\int (x+a)^n \, dx = \frac{1}{n+1} (x+a)^{n+1}}$$

17) $\int \frac{1}{(x+1)(x-2)} dx$

$$\frac{1}{(x+1)(x-2)} \equiv \frac{A}{x+1} + \frac{B}{x-2}$$

$$1 \equiv A(x-2) + B(x+1)$$

$x = 2; \ B = \frac{1}{3} \ and \ x = -1; \ \ A = -\frac{1}{3}$

$$\int \frac{1}{(x+1)(x-2)} dx = -\frac{1}{3} \int \frac{1}{(x+1)} dx + \frac{1}{3} \int \frac{1}{(x-2)} dx$$

$$= -\frac{1}{3} \log_e|x+1| + \frac{1}{3} \log_e|x-2| + c$$

$$\boxed{\int \frac{f'(x)}{f(x)} dx = \log_e|f(x)|}$$

18) $\int \frac{6x+8}{(3x-1)(x+1)} dx$

$$\frac{6x+8}{(3x-1)(x+1)} = \frac{A}{(3x-1)} + \frac{B}{(x+1)}$$

$$6x + 8 \equiv A(x+1) + B(3x-1)$$

When $x = -1; -6 + 8 = B(-3-1) \Rightarrow B = -\frac{1}{2}$

$$x = \frac{1}{3} \Rightarrow \frac{6}{3} + 8 = A\left(\frac{1}{3} + 1\right)$$

$$10 = A\frac{4}{3} \Rightarrow A = \frac{15}{2}$$

$$\therefore \frac{6x+8}{(3x-1)(x+1)} = \frac{\frac{15}{2}}{(3x-1)} + \frac{-\frac{1}{2}}{(x+1)}$$

$$\therefore \int \frac{6x+8}{(3x-1)(x+1)}dx = \frac{15}{2}\int \frac{1}{(3x-1)}dx - \frac{1}{2}\int \frac{1}{(x+1)}dx$$

$$= \frac{5}{2}\int \frac{3}{(3x-1)}dx - \frac{1}{2}\int \frac{1}{(x+1)}dx$$

$$= \frac{5}{2}\log_e|(3x-1)| - \frac{1}{2}\log_e|(x+1)| + c$$

$$\left\{\int \frac{f'(x)}{f(x)}dx = \log_e|f(x)|\right\}$$

19) $\int \frac{(2x^2+3x+4)}{6x^2+x-1}dx$

$$(2x^2 + 3x + 4) = \lambda(6x^2 + x - 1) + \mu x + r$$

Equating the coefficient of x^2

$2 = 6\lambda; \quad \lambda = \frac{1}{3}$

Coefficient of x; $\quad 3 = \lambda + \mu$

$\Rightarrow 3 = \frac{1}{3} + \mu$

$\Rightarrow \mu = \frac{8}{3}$

Considering coefficients of $x^0, we\ have\ 4 = -\lambda + r$

$$r = 4 + \frac{1}{3} = \frac{13}{3}$$

$$6x^2 + x - 1 \equiv (3x-1)(2x+1)$$

$$\frac{(2x^2+3x+4)}{6x^2+x-1} \equiv \frac{1}{3} + \frac{\frac{8}{3}x + \frac{13}{3}}{(6x^2+x-1)}$$

$$\frac{8x+13}{(3x-1)(2x+1)} = \frac{A}{(3x-1)} + \frac{B}{(2x+1)}$$

$$8x + 13 \equiv A(2x+1) + B(3x-1)$$

$$x = \frac{1}{3} \Rightarrow \frac{8}{3} + 13 = A\left(\frac{2}{3} + 1\right)$$

$$\frac{47}{3} = A\left(\frac{5}{3}\right) \Rightarrow A = \frac{47}{5}$$

$$x = -\frac{1}{2} \Rightarrow 8\left(-\frac{1}{2}\right) + 3 = B\left(-\frac{3}{2} - 1\right)$$

$$\Rightarrow -1 = B\left(-\frac{5}{2}\right)$$

$$\Rightarrow B = \frac{2}{5}$$

$$\int \frac{(2x^2+3x+4)}{6x^2+x-1}\,dx = \int \frac{1}{3}\,dx + \frac{1}{3}\int \frac{\frac{47}{5}}{(3x-1)}\,dx + \frac{1}{3}\int \frac{\frac{2}{5}}{(2x+1)}\,dx$$

$$= \frac{1}{3}\int 1\,dx + \frac{47}{15}\int \frac{3}{(3x-1)}\,dx + \frac{1}{15}\int \frac{2}{(2x+1)}\,dx$$

$$= \frac{1}{3}x + \frac{47}{15}\log_e|(3x-1)| + \frac{1}{15}\log_e|(2x+1)| + c$$

$$\boxed{\int \frac{f'(x)}{f(x)}\,dx = \log_e|f(x)|}$$

20) $\int x\sqrt{x+3}\,dx = \int (x+3-3)\sqrt{x+3}\,dx$

$$\int \left[(x+3)\sqrt{x+3} - 3\sqrt{x+3}\right]dx$$

$$= \int (x+3)^{\frac{3}{2}}\,dx - 3\int (x+3)^{\frac{1}{2}}\,dx$$

$$= \frac{1}{\frac{3}{2}+1}(x+3)^{\frac{3}{2}+1} - \frac{3}{\frac{1}{2}+1}(x+3)^{\frac{1}{2}+1} + c$$

$$= \frac{2}{5}(x+3)^{\frac{3}{5}} - 2(x+3)^{\frac{3}{2}} + c$$

$$\boxed{\int (x+a)^n\,dx = \frac{1}{n+1}(x+a)^{n+1}}$$

21) $\int \frac{x-1}{\sqrt{x+4}}\,dx = \int \frac{x+4-5}{\sqrt{x+4}}\,dx$

$$= \int \frac{x+4}{\sqrt{x+4}}\,dx - 5\int \frac{1}{\sqrt{x+4}}\,dx$$

$$= \int \sqrt{x+4}\,dx - 5\int (x+4)^{-\frac{1}{2}}\,dx$$

$$= \frac{1}{\frac{1}{2}+1}(x+4)^{\frac{1}{2}+1} - \frac{5}{-\frac{1}{2}+1}(x+4)^{-\frac{1}{2}+1} + c$$

$$= \frac{2}{3}(x+4)^{\frac{3}{2}} - 10(x+4)^{\frac{1}{2}} + c$$

$$\boxed{\int (x+a)^n\,dx = \frac{1}{n+1}(x+a)^{n+1}}$$

22) $\int (1+x)\sqrt{1-x}\,dx$

$$(1+x) \equiv -(1-x) + 2$$

$$\int [2-(1-x)]\sqrt{1-x}\,dx$$

$$= 2\int \sqrt{1-x}\,dx - \int (1-x)\sqrt{1-x}\,dx$$

$$= 2\int \sqrt{1-x}\,dx - \int (1-x)^{\frac{3}{2}}\,dx$$

$$= 2\cdot\frac{1}{\left(\frac{1}{2}+1\right)}(1-x)^{\frac{1}{2}+1}(-1) - \frac{1}{\left(\frac{3}{2}+1\right)}(1-x)^{\left(\frac{3}{2}+1\right)}\frac{1}{(-1)} + c$$

$$= -\frac{4}{3}(1-x)^{\frac{3}{2}} + \frac{2}{5}(1-x)^{\frac{5}{2}} + c$$

$$\boxed{\int (ax+b)^n\,dx = \frac{1}{n+1}(ax+b)^{n+1}\left(\frac{1}{a}\right)}$$

23) $\int \sqrt{1+\sin(2x)}\,dx$

$$= \int \sqrt{(\sin x)^2 + (\cos x)^2 + 2\sin x\cos x}\,dx$$

$$= \int \sqrt{(\sin x + \cos x)^2}\,dx$$

$$= \int (\sin x + \cos x)\,dx$$

$$= -\cos x + \sin x + c$$

$$\int \sin x \, dx = -\cos x$$
$$\int \cos x \, dx = \sin x$$

24) $\int \sqrt{1+\cos(2x)}\, dx$

$$= \int \sqrt{1 + 2cos^2x - 1}\, dx$$
$$= \int \sqrt{2} \cos x \, dx = \sqrt{2} \sin x + c$$

$$\int \cos x \, dx = \sin x$$

25) $\int \sqrt{1-\cos(2x)}\, dx = \int \sqrt{1-(1-2sin^2x)}\, dx$

$$= \int \sqrt{2(\sin x)^2}\, dx = \sqrt{2} \int \sin x \, dx$$
$$= \sqrt{2}(-\cos x) + c$$

$$\int \sin x \, dx = -\cos x$$

26) $\int tan^2\theta \, d\theta = \int (sec^2\theta - 1)\, d\theta$

$$= \int sec^2\theta \, d\theta - \int 1 \, d\theta$$
$$= \tan\theta - \theta + c$$

$$\int sec^2\theta \, d\theta = \tan\theta$$
$$\int 1 \, d\theta = \theta$$

27) $\int \sec(3\theta)\tan(3\theta)\, d\theta = \frac{1}{3}\sec(3\theta) + c$

$$\boxed{\begin{gathered}\int \sec\theta \tan\theta \, d\theta = \sec\theta \\ \int f(ax)\, dx = \frac{1}{a}F(ax)\end{gathered}}$$

28) $\int \sec\theta \, d\theta = \int \frac{\sec\theta(\sec\theta + \tan\theta)}{\sec\theta + \tan\theta} d\theta$

$$= \int \frac{(\sec\theta)^2 + \sec\theta\tan\theta}{\sec\theta + \tan\theta} d\theta$$

$$\therefore \int \sec\theta \, d\theta = \int \frac{(\sec\theta)^2 + \sec\theta\tan\theta}{\sec\theta + \tan\theta} d\theta$$

$$\int \sec\theta \, d\theta = \log|\sec\theta + \tan\theta| + c$$

$$\boxed{\int \frac{f'(x)}{f(x)} dx = \log_e|f(x)|}$$

29) $\int \sqrt{\frac{1-\sin x}{1+\sin x}}\, dx = \int \frac{\sqrt{1-\sin x}\sqrt{1-\sin x}}{\sqrt{(1+\sin x)(1-\sin x)}} dx$

$$= \int \frac{(1-\sin x)}{\sqrt{1-(\sin x)^2}} dx$$

$$= \int \frac{1-\sin x}{\cos x} dx$$

$$= \int \frac{1}{\cos x} dx + \int \frac{-\sin x}{\cos x} dx$$

$$= \int \sec x \frac{(secx + tanx)}{(secx + tanx)} dx + \int \frac{-\sin x}{\cos x} dx$$

$$= \log_e|\sec x + \tan x| + \log_e|\cos x| + c$$

$$\boxed{\int \frac{f'(x)}{f(x)} dx = \log_e|f(x)|}$$

30) $\int \frac{\cos x+\sin x}{\cos x-\sin x}dx = \int \frac{1+\frac{\sin x}{\cos x}}{1-\frac{\sin x}{\cos x}}dx$

$$= \int \frac{1+\tan x}{1-\tan x}dx$$

$$= \int \frac{\tan\frac{\pi}{4}+\tan x}{1-\tan\frac{\pi}{4}\tan x}dx$$

$$= \int \tan\left(\frac{\pi}{4}+x\right)dx$$

$$= -\int -\frac{\sin\left(\frac{\pi}{4}+x\right)}{\cos\left(\frac{\pi}{4}+x\right)}dx$$

$$= -\log_e\left|\cos\left(\frac{\pi}{4}+x\right)\right| + c$$

$$\boxed{\int \frac{f'(x)}{f(x)}dx = \log_e|f(x)|}$$

31) $\int \frac{6x^2+12x+8}{2x^3+6x^2+8x+1}dx$

$$\frac{d}{dx}(2x^3+6x^2+8x+1) = 6x^2+12x+8$$

$$\int \frac{6x^2+12x+8}{2x^3+6x^2+8x+1}dx = \int \frac{\frac{d}{dx}(2x^3+6x^2+8x+1)}{2x^3+6x^2+8x+1}dx$$

$$= \log_e|2x^3+6x^2+8x+1| + c$$

$$\boxed{\int \frac{f'(x)}{f(x)}dx = \log_e|f(x)|}$$

32) $\int \frac{2x+3}{\sqrt{x^2+3x+5}}dx = 2\sqrt{x^2+3x+5} + c$

$$\boxed{\int \frac{f'(x)}{\sqrt{f(x)}}dx = 2\sqrt{f(x)}}$$

33) $\int \tan\theta \, d\theta = \int \frac{\sin\theta}{\cos\theta} d\theta = -\int \frac{-\sin\theta}{\cos\theta} d\theta$

$= -\log_e|\cos\theta| + c$

$$\boxed{\int \frac{f'(x)}{f(x)} dx = \log_e|f(x)| + c}$$

34) $\int \frac{e^x}{1+e^x} dx = \log_e|1 + e^x| + c$

$$\boxed{\int \frac{f'(x)}{f(x)} dx = \log_e|f(x)| + c}$$

35) $\int \frac{2x+3}{x^2+3x+5} dx = \log_e|x^2 + 3x + 5| + c$

$$\boxed{\int \frac{f'(x)}{f(x)} dx = \log_e|f(x)| + c}$$

Activity 1

Evaluate the following Integrals

1) $\int x^4 \, dx$

2) $\int \frac{1}{\sqrt{x}} dx$

3) $\int \frac{1}{3x^2} dx$

4) $\int \sqrt[5]{x^3} \, dx$

5) $\int [\sin x + \cos x - (\sec x)^2] \, dx$

6) $\int \sqrt{ax + b} \, dx$

7) $\int (5x - 3)^3 \, dx$

8) $\int (\sec(7 - 4x))^2 \, dx$

9) $\int \frac{1}{(3x+4)^2} dx$

10) $\int \frac{1}{3-5x} dx$

11) $\int \sin 2x + \cos 2x \, dx$

12) $\int (\csc x)^2 + 2 \sin 3x \, dx$

13) $\int \left(x + \frac{1}{x}\right)^3 dx$

14) $\int \left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)^3 dx$

15) $\int \left(x + \frac{1}{x}\right)\left(x^2 + \frac{1}{x^2}\right) dx$

16) $\int \frac{x^3+3x^2+4}{\sqrt{x}} dx$

17) $\int \frac{(2x-3)^3}{x^2} dx$

18) $\int \frac{x^2-4}{x+1} dx$

19) $\int x\sqrt{x+2}\, dx$

20) $\int \frac{x-1}{\sqrt{x-4}} dx$

21) $\int \frac{1}{\sqrt{x+2}-\sqrt{x+3}} dx$

22) $\int \frac{1}{\sqrt{2x+1}+\sqrt{2x+2}} dx$

23) $\int (e^x - e^{-x})^2\, dx$

24) $\int \frac{(\sin x)^2}{1+\cos x} dx$

25) $\int \frac{4-5\sin x}{(\cos x)^2} dx$

26) $\int \frac{\sec\theta}{(\tan\theta)^2} d\theta$

27) $\int \frac{\sin x}{(\cos x)^2} dx$

28) $\int \sec x\,(\sec x + \tan x)\, dx$

29) $\int \frac{2x^3-3x^2+4x-7}{x^2} dx$

30) $\int \frac{a+bx+cx^2}{x^2} dx$

16.5 Integration of Trigonometric Functions

When the integrand is a trigonometric function, we transform given function into the standard integrals or their algebraic sum by using trigonometric formulae.

We use the following trigonometric identities to find integrals of trigonometric functions.

i. $(\sin \alpha x)^2 \equiv \frac{1}{2}(1 - \cos 2\alpha x)$

ii. $(\cos \alpha x)^2 \equiv \frac{1}{2}(1 + \cos 2\alpha x)$

iii. $\sin \alpha x \equiv 2 \sin\frac{\alpha x}{2} \cos\frac{\alpha x}{2}$

iv. $(\tan \alpha x)^2 \equiv (\sec \alpha x)^2 - 1$

v. $(\cot \alpha x)^2 \equiv (\csc \alpha x)^2 - 1$

vi. $2\cos A \cos B \equiv \cos(A+B) + \cos(A-B)$

vii. $2\cos A\sin B \equiv \sin(A+B)-\sin(A-B)$

viii. $2\sin A\cos B \equiv \sin(A+B)+\sin(A-B)$

ix. $2\sin A\sin B \equiv \cos(A-B)-\cos(A+B)$

Example 2

Evaluate of the following integral.

1) $\int cos^2 nx\,dx$

2) $\int sin^2(6x+5)\,dx$

3) $\int(\cos x-3\sec x)^2\,dx$

4) $\int(3\sin x+4\csc x)^2\,dx$

5) $\int \sin 8x\sin 12x\,dx$

6) $\int \sin^2 x.\cos^2 x\,dx$

7) $\int \sin 4x\cos 2x\,dx$

8) $\int \cos 4x\sin 2x\,dx$

9) $\int \cos x\cos 2x\cos 3x\,dx$

10) $\int \frac{\sec x+\tan x}{\sec x-\tan x}dx$

11) $\int \frac{1-\cos x}{1+\cos x}dx$

12) $\int \frac{\cos x}{1+\cos x}dx$

13) $\int \frac{\tan x}{\sec x+\tan x}dx$

14) $\int \frac{sin^2 x}{1+\cos x}dx$

15) $\int \frac{sin^3 x+cos^3 x}{sin^2 x.\ cos^2 x}dx$

16) $\int(\tan x-\cot x)^2\,dx$

17) $\int(\sin x-\cos x)^2\,dx$

18) $\int \sin x\sqrt{1-\cos 2x}\,dx$

19) $\int \cos x\sqrt{1-\cos 2x}\,dx$

20) $\int \frac{\sin x+\csc x}{\tan x}dx$

21) $\int \frac{\cos 2x}{(cos^2 x)(sin^2 x)}dx$

22) $\int(2\tan x-3\cot x)^2\,dx$

23) $\int \frac{\cos x-\cos 2x}{1-\cos x}dx$

24) $\int \frac{sin^8 x-cos^8 x}{1-2sin^2 x.cos^2 x}dx$

25) $\int \frac{\cos 8x+1}{\tan 2x-\cot 2x}dx$

Solution

1) $\int cos^2nx\,dx = \int \frac{1}{2}(1+\cos 2nx)\,dx$

$$= \frac{1}{2}\left(x + \frac{1}{2n}\sin 2nx\right) + c$$

2) $\int sin^2(6x+5)\,dx$

$$= \int \frac{1-\cos 2(6x+5)}{2}dx$$

$$= \frac{1}{2}\left[x - \frac{1}{2\times 6}\sin 2(6x+5)\right] + c$$

$$\boxed{\int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b)}$$

3)

$\int(\cos x - 3\sec x)^2\,dx = \quad = \int cos^2x - 6\cos x\sec x + 9sec^2x\,dx$

$$= \int cos^2x - 6 + 9\,sec^2x\,dx$$

$$= \int \frac{(1+\cos 2x)}{2} - 6 + 9sec^2x\,dx$$

$$= \int \left(\frac{1}{2}\cos 2x + 9sec^2x - \frac{11}{2}\right)dx$$

$$= \left[\frac{1}{2}\times\frac{1}{2}\sin 2x + 9\tan x - \frac{11}{2}x\right] + c$$

$$= \frac{1}{4}\sin 2x + 9\tan x - \frac{11}{2}x + c$$

4) $\int(3\sin x + 4\csc x)^2\,dx =$

$$= \int(9sin^2x - 24\sin x\csc x + 16csc^2x)\,dx$$

$$= \int\left(\frac{9}{2}(1-\cos 2x) - 24 + 16csc^2x\right)dx$$

$$= \int\left(16csc^2x - \frac{9}{2}\cos 2x - \frac{37}{2}\right)dx$$

$$= -16\cot x - \frac{9}{2\times 2}\sin 2x - \frac{37}{2}x + c$$

$$= -16\cot x - \frac{9}{4}\sin 2x - \frac{37}{2}x + c$$

5) $\int \sin 8x \sin 12x \, dx$

$$= \frac{1}{2}\int [\cos(12x - 8x) - \cos(12x + 8x)] \, dx$$

$$= \frac{1}{2}\int [\cos 4x - \cos 20x] \, dx = \frac{1}{2}\left[\frac{1}{4}\sin 4x - \frac{1}{20}\sin 20x\right] + c$$

$$= \frac{1}{8}\sin 4x - \frac{1}{40}\sin 20x + c$$

6) $\int \sin^2 x . \cos^2 x \, dx$

$$= \int \frac{1}{4}(2\sin x \cos x)^2 \, dx$$

$$= \frac{1}{4}\int sin^2 2x \, dx = \frac{1}{8}\int (1 - \cos 4x) \, dx$$

$$= \frac{1}{8}\left[x - \frac{1}{4}\sin 4x\right] + c$$

$$= \frac{x}{8} - \frac{1}{32}\sin 4x + c$$

7) $\int \sin 4x \cos 2x \, dx = \frac{1}{2}\int 2\sin 4x \cos 2x \, dx$

$$= \frac{1}{2}\int (\sin(4x + 2x) + \sin(4x - 2x)) \, dx$$

$$= \frac{1}{2}\int (\sin 6x + \sin 2x) \, dx$$

$$= \frac{1}{2}\left\{-\frac{1}{6}\cos 6x - \frac{1}{2}\cos 2x\right\} + c$$

$$= -\left\{\frac{1}{12}\cos 6x + \frac{1}{4}\cos 2x\right\} + c$$

8) $\int \cos 4x \sin 2x \, dx = \frac{1}{2}\int 2\cos 4x \sin 2x \, dx$

$$= \frac{1}{2}\int (\sin(4x + 2x) - \sin(4x - 2x)) \, dx$$

$$= \frac{1}{2}\int (\sin 6x - \sin 2x) \, dx$$

$$= \frac{1}{2}\left[-\frac{1}{6}\cos 6x - \frac{1}{2}(-\cos 2x)\right] + c$$

$$= \frac{1}{4}\cos 2x - \frac{1}{12}\cos 6x + c$$

9) $\int \cos x \cos 2x \cos 3x \, dx$

$= \frac{1}{2}\int \cos 2x \, [2\cos 3x \cos x] \, dx = \frac{1}{2}\int \cos 2x \, [\cos(3x + x) +$
$\cos(3x - x)] \, dx = \frac{1}{2}\int \cos 2x \, [\cos 4x + \cos 2x] \, dx =$
$\frac{1}{2}\int \cos 2x \cos 4x \, dx + \frac{1}{2}\int cos^2 2x \, dx$

$$= \frac{1}{4}\int 2\cos 2x \cos 4x \, dx + \frac{1}{4}\int (\cos 4x + 1) \, dx$$

$$= \frac{1}{4}\int [\cos(4x + 2x) + \cos(4x - 2x)] \, dx + \frac{1}{4}\int \cos 4x \, dx + \frac{1}{4}\int 1 \, dx$$

$$= \frac{1}{4}\int \cos 6x \, dx + \frac{1}{2}\int \cos 2x \, dx + \frac{1}{4}\int \cos 4x \, dx + \frac{1}{4}\int 1 \, dx$$

$$= \frac{1}{4}\left[\frac{1}{6}\sin 6x\right] + \frac{1}{2}\left[\frac{1}{2}\sin 2x\right] + \frac{1}{4}\left[\frac{1}{4}\sin 4x\right] + \frac{x}{4} + c$$

$$= \frac{1}{24}\sin 6x + \frac{1}{4}\sin 2x + \frac{1}{16}\sin 4x + \frac{x}{4} + c$$

10) $\int \frac{\sec x + \tan x}{\sec x - \tan x} dx = \int \frac{(\sec x + \tan x)^2}{(\sec x - \tan x)(\sec x + \tan x)} dx$

$$= \int sec^2 x + 2\sec x \tan x + tan^2 x \, dx$$

$$= \int sec^2 x + 2\sec x \tan x + sec^2 x - 1 \, dx$$

$$= \int 2sec^2 x + 2\sec x \tan x - 1 \, dx$$

$$= [2\tan x + 2\sec x - x] + c$$

11) $\int \frac{1 - \cos x}{1 + \cos x} dx = \int \frac{(1 - \cos x)^2}{(1 + \cos x)(1 - \cos x)} dx$

$$= \int \frac{(1 - \cos x)^2}{1 - cos^2 x} dx = \int \frac{(1 - 2\cos x + cos^2 x)}{sin^2 x} dx$$

$$= \int [csc^2 x - 2\cot x \csc x + cot^2 x] \, dx$$

$$= \int [csc^2 x - 2\cot x \csc x + csc^2 x - 1] \, dx$$

$$= \int [2csc^2x - 2\cot x \csc x - 1]\, dx$$

$$= [-2\cot x + 2\csc x - x] + c$$

12) $\int \frac{\cos x}{1+\cos x} dx = \int \frac{\cos x(1-\cos x)}{(1+\cos x)(1-\cos x)} dx$

$$= \int \frac{\cos x - cos^2x}{1-cos^2x} dx = \int \frac{\cos x - cos^2x}{sin^2x} dx$$

$$= \int \left(\frac{\cos x}{sin^2x} - \frac{cos^2x}{sin^2x}\right) dx$$

$$= \int [\cot x \csc x - csc^2x]\, dx$$

$$= \int [\cot x \csc x - (1 + csc^2x)]\, dx$$

$$= \int [\cot x \csc x - csc^2x - 1]\, dx$$

$$= -\csc x + \cot x - x + c$$

13) $\int \frac{\tan x}{\sec x + \tan x} dx = \int \frac{\tan x(\sec x - \tan x)}{(\sec x + \tan x)(\sec x - \tan x)} dx$

$$= \int \left(\frac{\tan x \sec x - tan^2x}{sec^2x - tan^2x}\right) dx = \int (\tan x \sec x - tan^2x)\, dx$$

$$= \int \tan x \sec x\, dx - \int tan^2x\, dx$$

$$= \int \tan x \sec x\, dx - \int (sec^2x - 1)\, dx$$

$$= \sec x - \tan x + x + c$$

14)

$$\int \frac{sin^2x}{1+\cos x} dx = \int \frac{1 - cos^2x}{1 + \cos x} dx = \int (1 - \cos x)\, dx$$

$$= x - \sin x + c$$

15) $\int \frac{sin^3x + cos^3x}{sin^2x.\ cos^2x} dx$

$$= \int \frac{(\sin x)^3}{(\sin x)^2(\cos x)^2} + \frac{(\cos x)^3}{(\sin x)^2(\cos x)^2} dx$$

$$= \int \frac{\sin x}{cos^2 x} + \frac{\cos x}{sin^2 x} dx = \int \tan x \sec x + \cot x \csc x \, dx$$

$$= \sec x - \csc x + c$$

16) $\int (\tan x - \cot x)^2 \, dx = \int [tan^2 x - 2 \tan x \cot x + (\cot x)^2] \, dx$

$$= \int [tan^2 x - 2 + cot^2 x] \, dx$$

$$= \int [sec^2 x - 1 - 2 + csc^2 x - 1] \, dx$$

$$= \int [sec^2 x + csc^2 x - 4] \, dx$$

$$= \tan x - \cot x - 4x + c$$

17)

$$\int (\sin x - \cos x)^2 \, dx = \int [sin^2 x - 2 \sin x \cos x + cos^2 x] \, dx$$

$$= \int [1 - \sin 2x] \, dx = x + \frac{1}{2} \cos 2x + c$$

18) $\int \sin x \sqrt{1 - \cos 2x} \, dx = \int \sin x \sqrt{1 - (1 - 2cos^2 x)} \, dx$

$$= \int \sin x \sqrt{2(\cos x)^2} \, dx = \int \sqrt{2} \sin x \cos x \, dx$$

$$= \frac{\sqrt{2}}{2} \int 2 \sin x \cos x \, dx = \frac{1}{\sqrt{2}} \int \sin 2x \, dx$$

$$= -\frac{1}{2\sqrt{2}} \cos 2x + c$$

19) $\int \cos x \sqrt{1 - \cos 2x} \, dx = \int \cos x \sqrt{1 - (1 - 2cos^2 x)} \, dx$

$$= \int \cos x \sqrt{2} \cos x \, dx = \sqrt{2} \int cos^2 x \, dx = \frac{\sqrt{2}}{2} \int (\cos 2x + 1) \, dx$$

$$= \frac{1}{\sqrt{2}} \left[\frac{1}{2} \sin 2x + x \right] + c = \frac{1}{2\sqrt{2}} \sin 2x + \frac{x}{\sqrt{2}} + c$$

20) $\int \frac{\sin x+\csc x}{\tan x} dx$

$$= \int \left(\frac{\sin x}{\tan x} + \frac{\csc x}{\tan x}\right) dx$$

$$= \int \left(\cos x + \frac{\cos x}{(\sin x)^2}\right) dx = \int (\cos x + \cot x \csc x)\, dx$$

$$= \sin x - \csc x + c$$

21) $\int \frac{\cos 2x}{(\cos x)^2(\sin x)^2} dx = \int \frac{(\cos x)^2-(\sin x)^2}{(\cos x)^2(\sin x)^2} dx$

$$= \int (csc^2x - sec^2x)\, dx = -\cot x - \tan x + c$$

22) $\int (2\tan x - 3\cot x)^2\, dx$

$$= \int [4(\tan x)^2 - 12\tan x \cot x + 9(\cot x)^2]\, dx$$

$$= \int [4(\tan x)^2 + 9(\cot x)^2 - 12]\, dx$$

$$= \int [4((\sec x)^2 - 1) + 9((\csc x)^2 - 1) - 12]\, dx$$

$$= \int [4(\sec x)^2 + 9(\csc x)^2 - 1]\, dx$$

$$= 4\tan x - 9\cot x - x + c$$

23) $\int \frac{\cos x-\cos 2x}{1-\cos x} dx = \int \frac{\cos x-\left(2(\cos x)^2-1\right)}{1-\cos x} dx$

$$= -\int \frac{2(\cos x)^2 - \cos x - 1}{1-\cos x} dx$$

$$= -\int \frac{(2\cos x + 1)(\cos x - 1)}{(1-\cos x)} dx$$

$$= \int (2\cos x + 1)\, dx = 2\sin x + x + c$$

24) $\int \frac{(\sin x)^8-(\cos x)^8}{1-2(\sin x)^2(\cos x)^2} dx$

$$= \int \frac{((\sin x)^4 + (\cos x)^4)((\sin x)^2 + (\cos x)^2)((\sin x)^2 - (\cos x)^2)}{1 - 2(\sin x)^2(\cos x)^2} dx$$

$$= \int \frac{[((\sin x)^2 + (\cos x)^2)^2 - 2(\sin x)^2(\cos x)^2]((\sin x)^2 - (\cos x)^2)}{1 - 2(\sin x)^2(\cos x)^2}\,dx$$

$$= \int \left(\frac{1 - 2(\sin x)^2(\cos x)^2}{1 - 2(\sin x)^2(\cos x)^2}\right)(-\cos 2x)\,dx$$

$$= \int -\cos 2x\,dx = -\frac{1}{2}\sin 2x + c$$

25) $\int \frac{\cos 8x+1}{\tan 2x-\cot 2x}\,dx$

$$= \int \frac{2(\cos 4x)^2}{\frac{\sin 2x}{\cos 2x} - \frac{\cos 2x}{\sin 2x}}\,dx = \int \frac{2(\cos 4x)^2 \cos 2x \sin 2x}{(\sin 2x)^2 - (\cos 2x)^2}\,dx$$

$$= -\int \frac{(\cos 4x)^2 \sin 4x}{\cos 4x}\,dx$$

$$= -\int \cos 4x \sin 4x\,dx = -\frac{1}{2}\int \sin 8x\,dx$$

$$= -\frac{1}{2}\left[\frac{1}{8}(-\cos 8x)\right] + c = \frac{1}{16}\cos 8x + c$$

Activity 2

1) $\int (\sin(bx))^2\,dx$

2) $\int 2(\tan x - 3\cot x)^2\,dx$

3) $\int (\cos(3x-5))^2\,dx$

4) $\int \frac{1}{(\sin x)^2(\cos x)^2}\,dx$

5) $\int \frac{\cos 2x + 2(\sin x)^2}{(\cos x)^2}\,dx$

6) $\int \frac{1-\cos 2x}{1+\cos 2x}\,dx$

7) $\int \frac{\sin 4x}{\cos 2x}\,dx$

8) $\int \frac{\sin x + \cos x}{\sqrt{1+\sin 2x}}\,dx$

9) $\int \sqrt{1-\cos 2x}\,dx$

10) $\int \sqrt{1+\cos x}\,dx$

11) $\int \frac{1}{1+\cos x}\,dx$

12) $\int \frac{1}{1-\cos 2x}\,dx$

13) $\int \frac{\csc x}{\csc x - \cot x}\,dx$

14) $\int \frac{(\sin x)^2}{(1+\cos x)^2}\,dx$

15) $\int \sin x\sqrt{1+\cos 2x}\,dx$

16) $\int \sqrt{1-\sin 2x}\,dx$

17) $\int \cos 4x \cos 3x\,dx$

18) $\int \cos 3x \sin 2x\,dx$

19) $\int \sin 4x \sin 8x\,dx$

20) $\int \sin x \sin 2x \sin 3x\,dx$

21) $\int \cos 2x \cos 4x \cos 6x \, dx$

22) $\int \frac{1}{1-\sin x} dx$

23) $\int \frac{\sin x}{1+\sin x} dx$

24) $\int \frac{1}{1+\sec x} dx$

25) $\int (\sin 2x + \cos 2x) \, dx$

Solutions to Activities

Activity 1

1) $\frac{1}{5}x^5 + c$

2) $2\sqrt{x} + c$

3) $-\frac{1}{3x} + c$

4) $\frac{5}{8}x^{\frac{8}{5}} + c$

5) $-\cos x + \sin x - \tan x + c$

6) $\frac{2}{3a}(ax+b)^{\frac{3}{2}} + c$

7) $\frac{1}{20}(5x-3)^4 + c$

8) $-\frac{1}{4}\tan(7-4x) + c$

9) $-\frac{1}{3(3x+4)} + c$

10) $-\frac{1}{5}\log_e|3-5x| + c$

11) $-\frac{1}{2}\cos 2x + \frac{1}{2}\sin 2x + c$

12) $-\cot x - \frac{2}{3}\cos 3x + c$

13) $\frac{x^4}{4} + \frac{3}{2}x^2 + 3\log_e|x| - \frac{1}{2x^2} + c$

14) $\frac{2}{5}x^{\frac{5}{2}} - 2x^{\frac{3}{2}} + 6x^{\frac{1}{2}} + 2x^{-\frac{1}{2}} + c$

15) $\frac{1}{4}x^4 + \log_e|x| + \frac{1}{2}x^2 - \frac{1}{2x^3} + c$

16) $\frac{2}{7}x^{\frac{7}{2}} + \frac{6}{5}x^{\frac{5}{2}} + 8x^{\frac{1}{2}} + c$

17) $4 - 36x + 54\log_e|x| + \frac{27}{x}c$

18) $\frac{x^2}{2} - x - 3\log_e|x+1| + c$

19) $\frac{2}{5}(x+2)^{\frac{5}{2}} - \frac{4}{3}(x+2)^{\frac{3}{2}} + c$

20) $\frac{2}{3}(x+4)^{\frac{3}{2}} - 10\sqrt{x+4} + c$

21) $-\frac{2}{3}(x+2)^{\frac{3}{2}} - \frac{2}{3}(x+3)^{\frac{3}{2}} + c$

22) $-\frac{1}{3}\left\{(2x+1)^{\frac{3}{2}} + (2x+2)^{\frac{3}{2}}\right\} + c$

23) $\frac{1}{2}(e^{2x} - e^{-2x}) - 2x + c$

24) $x - \sin x + c$

25) $4 + \tan x - 5\sec x + c$

26) $-\csc\theta + c$

27) $\sec x + c$

28) $\sec x + \tan x + c$

29) $x^2 - 3x + 4\log_e|x| + \frac{7}{x} + c$

30) $-\frac{a}{x} + b\log_e|x| + cx + d$

Activity 2

1) $\frac{1}{2}\left[x - \frac{1}{2b}\sin 2bx\right] + c$

2) $4\tan x - 9\cot x - 25x + c$

3) $\frac{1}{2}\left[x + \frac{1}{6}\sin(6x-5)\right] + c$

4) $-2\cot 2x + c$

5) $\tan x + c$

6) $\tan x - x + c$

7) $-\cos 2x + c$

8) $x + c$

9) $-\sqrt{2}\cos x + c$

10) $2\sqrt{2}\sin\left(\frac{x}{\sqrt{2}}\right) + c$

11) $\tan\frac{x}{2} + c$

12) $-\frac{1}{2}\cot x + c$

13) $-\cot\frac{x}{2} + c$

14) $2\tan\frac{x}{2} - x + c$

15) $-\frac{1}{2\sqrt{2}}\cos 2x + c$

16) $-\cos x - \sin x + c$

17) $\frac{1}{2}\left[\frac{1}{7}\sin 7x + \sin x\right] + c$

18) $\frac{1}{2}\left[-\frac{1}{5}\cos 5x + \cos x\right] + c$

19) $\frac{1}{2}\left[\frac{1}{4}\sin 4x - \frac{1}{12}\sin 12x\right] + c$

20) $\frac{1}{4}\left[\frac{1}{6}\cos 6x - \frac{1}{4}\cos 4x - \frac{1}{2}\cos 2x\right] + c$

21) $\frac{1}{4}\left[\frac{1}{12}\sin 12x + \frac{1}{8}\sin 8x + \frac{1}{4}\sin 4x\right] + c$

22) $\tan x + \sec x + c$

23) $\sec x - \tan x + x + c$

24) $-\csc x + \cot x + c$

25) $-\frac{1}{2}\cos 2x + \frac{1}{2}\sin 2x + c$

Summary

If $\frac{d}{dx}[F(x) + c] = f(x)$

Then integration of $F(x)$ with respect to x, $\int f(x)dx = F(x) + c$.

Standard Rules on Integration.

$\int kf(x)\,dx = k\int f(x)\,dx$, where k is a constant.

$\int[f(x) \pm g(x)]\,dx = \{\int f(x)\,dx\} \pm \{\int g(x)\,dx\}$.

If $\int f(x)\,dx = F(x) + C$then $\int f(ax + b)\,dx = \frac{1}{a}F(ax + b) + C$.

Some Standard Integrals

$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + c,\ n \neq -1;\ \int \frac{1}{x}dx = \log_e|x| + c$

$$\int \frac{f'(x)}{f(x)}dx = \log_e|f(x)| + c$$

$$\int \frac{f'(x)}{\sqrt{f(x)}}dx = 2\sqrt{f(x)} + c$$

$$\int \sin x\,dx = -\cos x + c$$

$$\int \cos x\,dx = \sin x + c$$

$$\int (\sec x)^2\,dx = \tan x + c$$

$$\int (\csc x)^2\,dx = -\cot x + c$$

$$\int \csc x \cot x \, dx = -\csc x + c$$

$$\int \sec x \tan x \, dx = \sec x + c$$

$$\int \tan x \, dx = -\log_e|\cos x| + c$$

$$\int \cot x \, dx = \log_e|\sin x| + c$$

$$\int e^x \, dx = e^x + c$$

$$\int 1 \, dx = x + c$$

$$\int 0 \, dx = c$$

Learning outcomes

On completion of this study session you should be able to

- Describe the indefinite integral.
- Determine the integrals of simple polynomial functions and determine the definite integrals of simple functions.
- Apply rule of integration to find indefinite integrals.

Session 17

Integration by parts and Definite Integrals

Contents

Introduction

In this session we discuss the method integration by parts. By using the integration by parts method, we can solve several difficult integrals. We introduce the concepts of definite integrals.

The definite integrals are very important for solving engineering problems. Thus, in this session we discuss the basic properties of definite integrals.

17.1 Integration by parts

In the differential calculus, we have the differentiation of product of two functions $y = uv$

Thus $\frac{d}{dx}uv = u\frac{dv}{dx} + v\frac{du}{dx} \longrightarrow (1)$

If we integrate with respect to x both sides of the expression, the we get,

$$uv = \int u\frac{dv}{dx}dx + \int v\frac{du}{dx}dx$$

$$\therefore \int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx$$

Always $\int u \frac{dv}{dx} dx$ is represent the given integration; therefore we have to find suitable u and $\frac{dv}{dx}$ corresponds to the given function. The aim of this method $\int v \frac{du}{dx} dx$ integration is easy to find compare with $\int u \frac{dv}{dx} dx$. Therefore, we have to very carefully to find the functions $u$and $\frac{dv}{dx}$.

Example 1

Find the following indefinite integrals.

a) $\int \log_e x \, dx$

b) $\int x(\sin x)^2 \, dx$

c) $\int x^3 \cos x \, dx$

d) $\int x^3 e^x \, dx$

e) $\int e^{ax} \cos bx \, dx$

f) $\int (\sin x)^6 \, dx$

g) $\int (\tan x)^4 \, dx$

h) $\int (\sec x)^3 \, dx$

i) $\int x \sin 5x \cos 3x \, dx$

j) $\int x^3 \log_e x \, dx$

k) $\int x^2 (\log x)^2 \, dx$

l) $\int \sin(\log x) \, dx$

m) $\int \log x^x \, dx$

n) $\int (1-x)^3 \cos 2x \, dx$

o) $\int (\log_e x)^3 \, dx$

Solutions

a) $I = \int \log_e x \, dx$

This type of integrations cannot be found by using previous forms.

Also we know that $\frac{d}{dx} \log_e x = \frac{1}{x}$. Then we have to choose such that

$u = \log_e x$ and $\frac{dv}{dx} = x^0 = 1 \therefore v = \int x^0 \, dx = x \; \frac{du}{dx} = \frac{1}{x}$

$$I = \int \log_e x \, dx = \int \underset{\substack{\downarrow \\ u}}{\log_e x} \cdot \underset{\substack{\downarrow \\ \frac{dv}{dx}}}{x^0} \, dx$$

$$= uv - \int v \frac{du}{dx} dx$$

$$= \log_e x \cdot x - \int x . \frac{1}{x} dx$$

$$= x \log_e x - \int 1\, dx = x \log_e x - x + c$$

$$I = \int \log_e x\, dx = x \log_e x - x + c$$

b) $I = \int x(\sin x)^2\, dx \cos 2x = 1 - 2(\sin x)^2$

$$= \int \frac{x}{2}(1 - \cos 2x)\, dx$$

$$= \frac{1}{2}\left\{\int x\, dx - \int x \cos 2x\, dx\right\}$$

In this case $\frac{d}{dx}(x) = 1 \therefore u = x$

$$\frac{dv}{dx} = \cos 2x \; v = \frac{1}{2}\sin 2x \therefore v\frac{du}{dx} = \frac{1}{2}\sin 2x \times 1$$

$$I = \frac{1}{2}\left\{\int x\, dx - \int \underset{\substack{\downarrow\\ u}}{x} \cdot \underset{\substack{\downarrow\\ \frac{dv}{dx}}}{\cos 2x}\, dx\right\}$$

$$= \frac{1}{2}\left[\frac{x^2}{2} - \left(\frac{x}{2}\sin 2x - \int \frac{1}{2}\sin 2x\, dx\right)\right]$$

$$= \frac{1}{2}\left[\frac{x^2}{2} - \frac{x}{2}\sin 2x - \frac{1}{4}\cos 2x\right] + c$$

c) $I = \int x^3 \cos x\, dx$

We know that the consecutive derivative of x^n, gradually decreases of the index of x; $\therefore u = x^3 \; \frac{dv}{dx} = \cos x$

Therefore, we have to use the integration by parts several times.

$$I = \int x^3 \cos x\, dx u = x^3 \frac{dv}{dx} = \cos x$$

$$\frac{du}{dx} = 3x^2 v = \sin x$$

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$$

$$I = \int x^3 \cos x\, dx = x^3 \sin x - \int \sin x\,(3x^2)\, dx$$

$$= x^3 \sin x - 3\int x^2 \sin x \, dx$$

Again we have to take $u = x^2$ and $\frac{dv}{dx} = \sin x$

$\frac{du}{dx} = 2x$ and $v = -\cos x$

$$I = \int x^3 \cos x \, dx = x^3 \sin x - 3\int \underset{u}{\underset{\downarrow}{x^2}} \cdot \underset{\frac{dv}{dx}}{\underset{\downarrow}{\sin x}} \, dx$$

$$= x^3 \sin x - 3\left[-x^2 \cos x - \int 2x(-\cos x)\, dx\right]$$

$$= x^3 \sin x + 3x^2 \cos x - 6\int \underset{u}{\underset{\downarrow}{x}} \cdot \underset{\frac{dv}{dx}}{\underset{\downarrow}{\cos x}} \, dx$$

$$\frac{du}{dx} = 1v = \sin x$$

$$= x^3 \sin x + 3x^2 \cos x - 6\left[x \sin x - \int \sin x \times 1 \, dx\right]$$

$$= x^3 \sin x + 3x^2 \cos x - 6x \sin x + 6\int \sin x \, dx$$

$$= x^3 \sin x + 3x^2 \cos x - 6x \sin x - 6\cos x + c$$

d) $I = \int x^3 e^x \, dx$

Let $x^3 = u$ and $\frac{dv}{dx} = e^x$

$$\frac{du}{dx} = 3x^2 v = e^x$$

$$\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx$$

$$I = \int \underset{u}{\underset{\downarrow}{x^3}} \cdot \underset{\frac{dv}{dx}}{\underset{\downarrow}{e^x}} \, dx = x^3 e^x - \int e^x (3x^2)\, dx$$

$$I = x^3 e^x - 3\int \underset{u}{\underset{\downarrow}{x^2}} \cdot \underset{\frac{dv}{dx}}{\underset{\downarrow}{e^x}} \, dx$$

$$\frac{du}{dx} = 2x$$

$$I = x^3 e^x - 3\left[x^2 e^x - \int 2x e^x \, dx\right]$$

$$= x^3e^x - 3x^2e^x + 6\int \underset{\underset{u}{\downarrow}}{x} \cdot \underset{\underset{\frac{dv}{dx}}{\downarrow}}{e^x}\, dx$$

$$= x^3e^x - 3x^2e^x + 6\left[xe^x - \int e^x\, dx\right]$$

$$= x^3e^x - 3x^2e^x + 6xe^x - 6\int e^x\, dx$$

$$I = x^3e^x - 3x^2e^x + 6xe^x - 6e^x + c$$

e) $I = \int e^{ax} \cos bx\, dx$

In this case, you can form$u = e^{ax}$, $\frac{dv}{dx} = \cos bx$ or $u = \cos bx$, $\frac{dv}{dx} = e^{ax}$

At first we can choose $u = e^{ax}\, \frac{dv}{dx} = \cos bx$

$$\frac{du}{dx} = ae^{ax} v = \frac{1}{b}\sin bx$$

$$I = \int \underset{\underset{u}{\downarrow}}{e^{ax}} \cdot \underset{\underset{\frac{dv}{dx}}{\downarrow}}{\cos bx}\, dx$$

$$= \frac{1}{b}e^{ax}\sin bx - \int \frac{1}{b}\sin(bx)\, ae^{ax}\, dx$$

$$= \frac{1}{b}e^{ax}\sin bx - \frac{a}{b}\int \underset{\underset{u}{\downarrow}}{e^{ax}} \cdot \underset{\underset{\frac{dv}{dx}}{\downarrow}}{\sin bx}\, dx$$

$$\frac{du}{dx} = ae^{ax} v = -\frac{1}{b}\cos bx$$

$$= \frac{1}{b}e^{ax}\sin bx - \frac{a}{b}\left[-\frac{1}{b}e^{ax}\cos bx - \int ae^{ax}\left(-\frac{1}{b}\right)\cos bx\, dx\right]$$

$$I = \frac{1}{b}e^{ax}\sin bx + \frac{a}{b^2}e^{ax}\cos bx - \frac{a^2}{b^2}\int e^{ax}\cos bx\, dx$$

$$I = \frac{1}{b}e^{ax}\sin bx + \frac{a}{b^2}e^{ax}\cos bx - \frac{a^2}{b^2}I\left\{I = \int e^{ax}\cos bx\, dx\right\}$$

$$\left(1 + \frac{a^2}{b^2}\right)I = \frac{1}{b}e^{ax}\sin bx + \frac{a}{b^2}e^{ax}\cos bx$$

$$(b^2 + a^2)I = be^{ax}\sin bx + ae^{ax}\cos bx$$

$$I = \frac{e^{ax}}{a^2 + b^2}[b\sin bx + a\cos bx] + c$$

$$\therefore \int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2 + b^2}[b \sin bx + a \cos bx] + c$$

Now we can try in the other way.

$$u = \cos bx \quad \frac{dv}{dx} = e^{ax}$$

$$\frac{du}{dx} = -b \sin bx \quad v = \frac{1}{a} e^{ax}$$

$$I = \int \underset{\frac{dv}{dx}}{\underset{\downarrow}{e^{ax}}} \cdot \underset{u}{\underset{\downarrow}{\cos}} bx \, dx = \left[\cos bx \cdot \frac{1}{a} e^{ax}\right] - \int \frac{1}{a} e^{ax} (-b \sin bx) \, dx$$

$$= \frac{1}{b} e^{ax} \cos bx + \frac{b}{a} \int \underset{\frac{dv}{dx}}{\underset{\downarrow}{e^{ax}}} \cdot \underset{u}{\underset{\downarrow}{\sin}} bx \, dx$$

$$v = \frac{1}{a} e^{ax} \quad \frac{du}{dx} = b \cos bx$$

$$= \frac{1}{b} e^{ax} \cos bx + \frac{b}{a}\left[\frac{1}{a} e^{ax} \sin bx - \int \frac{1}{a} e^{ax} \cdot b \cos bx \, dx\right]$$

$$I = \frac{1}{b} e^{ax} \cos bx + \frac{b}{a^2}\left[e^{ax} \sin bx - \frac{b^2}{a^2} \int e^{ax} \cos bx \, dx\right]$$

$$I = \frac{1}{b} e^{ax} \cos bx + \frac{b}{a^2} e^{ax} \sin bx - \frac{b^2}{a^2} I$$

$$I\left(1 + \frac{b^2}{a^2}\right) = \frac{1}{b} e^{ax} \cos bx + \frac{b}{a^2} e^{ax} \sin bx$$

$$I(a^2 + b^2) = e^{ax}(a \cos(bx) + b \sin(bx))$$

$$\therefore \int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2 + b^2}[a \cos bx + b \sin bx] + c$$

f) $I = \int (\sin x)^6 \, dx = \int (\underset{u}{\underset{\downarrow}{\sin}} x)^5 \cdot \underset{\frac{dv}{dx}}{\underset{\downarrow}{\sin}} x \, dx$

$$\frac{du}{dx} = 5(\sin x)^4 \cos x \quad v = -\cos x$$

$$I = -\cos x \, (\sin x)^5 - \int (-\cos x)[5(\sin x)^4 \cos x] \, dx$$

$$= -\cos x \, (\sin x)^5 + 5 \int (\sin x)^4 (\cos x)^2 \, dx$$

$$= -\cos x \, (\sin x)^5 + 5 \int (\sin x)^4 \, dx - 5 \int (\sin x)^6 \, dx$$

$$\therefore I + 5I = 6I = -\cos x\,(\sin x)^5 + 5\int (\sin x)^4\,dx$$

$$6I = -\cos x\,(\sin x)^5 + 5I_1 \longrightarrow (1)$$

$$I_1 = \int (\sin x)^4\,dx = \int \underset{u}{\underset{\downarrow}{(\sin x)^3}} \cdot \underset{\frac{dv}{dx}}{\underset{\downarrow}{\sin x}}\,dx$$

$$\frac{du}{dx} = 3(\sin x)^2\cos x \quad v = -\cos x$$

$$= -\cos x\,(\sin x)^3 - \int (-\cos x)3(\sin x)^2\cos x\,dx$$

$$= -\cos x\,(\sin x)^3 + 3\int (\sin x)^2(\cos x)^2\,dx$$

$$I_1 = -\cos x\,(\sin x)^3 + 3\int (\sin x)^2(1-(\sin x)^2)\,dx$$

$$I_1 = -\cos x\,(\sin x)^3 + 3\int (\sin x)^2\,dx - 3\int (\sin x)^4\,dx$$

$$(1+3)I_1 = -\cos x\,(\sin x)^3 + 3\int (\sin x)^2\,dx$$

$$4I_1 = -\cos x\,(\sin x)^3 + \frac{3}{2}\int (1-\cos 2x)\,dx$$

$$= -\cos x\,(\sin x)^3 + \frac{3}{2}x - \frac{3}{4}\sin 2x$$

$$I_1 = -\frac{1}{4}\cos x\,(\sin x)^3 + \frac{3}{8}x - \frac{3}{8}\sin 2x$$

$$6I = -\cos x\,(\sin x)^5 + 5\left[-\frac{1}{4}\cos x\,(\sin x)^3 + \frac{3}{8}x - \frac{3}{8}\sin 2x\right]$$

$\int (\sin x)^6\,dx = -\frac{1}{6}\cos x\,(\sin x)^5 - \frac{5}{24}\cos x\,(\sin x)^3 - \frac{5}{16}\sin 2x + \frac{5}{16}x$

g) $I = \int (\tan x)^4\,dx = \int (\tan x)^2(\tan x)^2\,dx$

$$= \int (\tan x)^2[(\sec x)^2 - 1]\,dx$$

$$= \int \underset{u}{\underset{\downarrow}{(\tan x)^2}} \cdot \underset{\frac{dv}{dx}}{\underset{\downarrow}{(\sec x)^2}}\,dx - \int (\tan x)^2\,dx$$

$$\frac{du}{dx} = 2\tan x\,(\sec x)^2 \quad v = \tan x$$

$$= (\tan x)^2 \tan x - \int (\tan x) 2 \tan x \,(\sec x)^2\, dx - \int (\tan x)^2\, dx$$

$$= (\tan x)^3 - 2\int (\tan x)^2 (\sec x)^2\, dx - \int (\tan x)^2\, dx$$

$$= (\tan x)^3 - 2\int (\tan x)^2 (1 + (\tan x)^2)\, dx - \int (\tan x)^2\, dx$$

$$\int (\tan x)^4\, dx = (\tan x)^3 - 2\int (\tan x)^2\, dx - 2\int (\tan x)^4\, dx$$

$$- \int (\tan x)^2\, dx$$

$$(1+2)\int (\tan x)^4\, dx = (\tan x)^3 - 3\int (\tan x)^2\, dx$$

$$3\int (\tan x)^4\, dx = (\tan x)^3 - 3\int [(\sec x)^2 - 1]\, dx$$

$$= (\tan x)^3 - 3\int (\sec x)^2\, dx + 3\int 1\, dx$$

$$= (\tan x)^3 - 3\tan x + 3x$$

$$\int (\tan x)^4\, dx = \frac{1}{3}(\tan x)^3 - \tan x + x + c$$

h) $I = \int (\sec x)^3\, dx$

$$I = \int (\sec x)^2 \sec x\, dx$$

$$= \int (1 + (\tan x)^2) \sec x\, dx$$

$$= \int \sec x\, dx + \int \sec x\, (\tan x)^2\, dx$$

$$= \int \sec x\, dx + \int \underset{\substack{\downarrow\\ u}}{\tan x} \cdot \underset{\substack{\downarrow\\ \frac{dv}{dx}}}{\sec x \tan x}\, dx$$

$$\frac{du}{dx} = (\sec x)^2 v = \sec x$$

$$= \int \sec x\, dx + \left[\tan x \sec x - \int \sec x\, (\sec x)^2\, dx\right]$$

$$= \int \sec x\, dx + \tan x \sec x - \int (\sec x)^3\, dx$$

$$\therefore 2\int (\sec x)^3\, dx = \int \sec x\, dx + \tan x \sec x$$

$$2\int (\sec x)^3\,dx = \ln|\sec x + \tan x| + \tan x \sec x$$

$$\int \sec x\,dx \ln|\sec x + \tan x|$$

$$\int (\sec x)^3\,dx = \tfrac{1}{2}[\ln|\sec x + \tan x| + \tan x \sec x] + c$$

i) $I = \int x \sin 5x \cos 3x\,dx$

$$\sin 5x \cos 3x = \frac{1}{2}[\sin(5x+3x) + \sin(5x-3x)]$$

$$= \frac{1}{2}[\sin 8x + \sin 2x]$$

$$\therefore I = \frac{1}{2}\int x(\sin 8x + \sin 2x)\,dx$$

$$= \frac{1}{2}\left\{\int \underset{\underset{u}{\downarrow}}{x}\cdot \underset{\underset{\frac{dv}{dx}}{\downarrow}}{\sin 8x}\,dx + \int \underset{\underset{u}{\downarrow}}{x}\cdot \underset{\underset{\frac{dv}{dx}}{\downarrow}}{\sin 2x}\,dx\right\}$$

$$v = -\frac{1}{8}\cos 8x\ v' = -\frac{1}{2}\cos 2x$$

$$= \frac{1}{2}\left\{-\frac{x}{8}\cos 8x - \int 1\cdot\left(-\frac{1}{8}\right)\cos 8x\,dx + x\times\left(-\frac{1}{2}\cos 2x\right)\right.$$

$$\left. - \int \left(-\frac{1}{2}\right)\cos 2x \times 1\,dx\right\}$$

$$= \frac{1}{2}\left[-\frac{x}{8}\cos 8x - \frac{x}{2}\cos 2x + \frac{1}{8}\int \cos 8x\,dx + \frac{1}{2}\int \cos 2x\,dx\right]$$

$$= \frac{1}{2}\left[-\frac{x}{8}\cos 8x - \frac{x}{2}\cos 2x + \frac{1}{64}\sin 8x + \frac{1}{4}\sin 2x\right] + c$$

$$= \frac{1}{128}\{(\sin 8x + 16\sin 2x) - x(8\cos 8x + 32\cos 2x)\} + c$$

j) $I = \int \underset{\underset{\frac{dv}{dx}}{\downarrow}}{x^3} \cdot \underset{\underset{u}{\downarrow}}{\log_e} x\,dx$

$$\frac{du}{dx} = \frac{1}{x} v = \frac{x^4}{4}$$

$$I = \frac{x^4}{4}\log_e x - \int \frac{x^4}{4}\times\frac{1}{x}\,dx$$

$$= \frac{x^4}{4}\log_e x - \frac{1}{4}\int x^3\,dx$$

$$= \frac{x^4}{4}\log_e x - \frac{1}{4}\times\frac{x^4}{4} + c$$

$$= \frac{x^4}{16}[4\log_e x - 1] + c$$

k) $I = \int \underset{\frac{dv}{dx}}{\underset{\downarrow}{x^2}} \cdot (\underset{u}{\underset{\downarrow}{\log}}\, x)^2\,dx$

$$v = \frac{x^3}{3}\quad \frac{du}{dx} = 2(\log x)\times\frac{1}{x}$$

$$= \frac{x^3}{3}(\log_e x)^2 - \int \frac{x^3}{3}\times\frac{2}{x}(\log x)dx$$

$$= \frac{x^3}{3}(\log_e x)^2 - \frac{2}{3}\int \underset{\frac{dv}{dx}}{\underset{\downarrow}{x^2}} \cdot (\underset{u}{\underset{\downarrow}{\log}}\, x)\,dx$$

$$v = \frac{x^3}{3}\quad \frac{du}{dx} = \frac{1}{x}$$

$$= \frac{x^3}{3}(\log_e x)^2 - \frac{2}{3}\left\{\frac{x^3}{3}\log_e x - \frac{1}{3}\int x^3\times\frac{1}{x}\,dx\right\}$$

$$= \frac{x^3}{3}(\log_e x)^2 - \frac{2}{9}x^3\log_e x + \frac{2}{9}\int x^2\,dx$$

$$\int x^2(\log x)^2\,dx = \frac{x^3}{3}(\log_e x)^2 - \frac{2}{9}x^3\log_e x + \frac{2}{9}\cdot\frac{x^3}{3} + c$$

$$= \frac{x^3}{3}(\log_e x)^2 - \frac{2}{9}x^3\log_e x + \frac{2}{27}x^3 + c$$

l) $I = \int \sin(\log_e x)\,dx$

$$\int \underset{\frac{dv}{dx}}{\underset{\downarrow}{x^0}} \cdot \sin(\underset{u}{\underset{\downarrow}{\log_e}}\, x)\,dx$$

$$\frac{dv}{dx} = x^0 v = x$$

$$\frac{du}{dx} = \cos(\log_e x)\cdot\frac{1}{x} = \frac{1}{x}\cos(\log_e x)$$

$$= x\sin(\log_e x) - \int x\cdot\frac{1}{x}\cos(\log_e x)\,dx$$

$$= x\sin(\log_e x) - \int \cos(\log_e x)\,dx$$

$$= x\sin(\log_e x) - \int \underset{\frac{dv}{dx}}{\underset{\downarrow}{x^0}} \cdot \cos(\underset{u}{\underset{\downarrow}{\log_e}}\, x)\,dx$$

$$v = x\,\frac{du}{dx} = -\sin(\log_e x)\cdot\frac{1}{x}$$

$$= x\sin(\log_e x) - \left\{x\cos(\log_e x) - \int x \times \left(-\frac{1}{x}\right)\sin(\log_e x)\,dx\right\}$$

$$I = \int \sin(\log_e x)\,dx = x(\sin(\log_e x) - \cos(\log_e x)) - \int \sin(\log_e x)\,dx$$

$$2\int \sin(\log_e x)\,dx = x(\sin(\log_e x) - \cos(\log_e x))$$

$$\int \sin(\log_e x)\,dx = \frac{x}{2}[\sin(\log_e x) - \cos(\log_e x)] + c$$

m) $I = \int (1 \underset{u}{\underset{\downarrow}{-}} x)^3 \cdot \underset{\frac{dv}{dx}}{\underset{\downarrow}{\cos}}\, 2x\,dx$

$$\frac{du}{dx} = 3(1-x)^2(-1) = -3(1-x)^2 v = \frac{1}{2}\sin 2x$$

$$I = (1-x)^3 \cdot \frac{1}{2}\sin 2x - \int \frac{1}{2}\sin 2x \times (-3(1-x)^2)\,dx$$

$$= \frac{1}{2}(1-x)^3 \sin 2x + \frac{3}{2}\int \underset{\frac{dv}{dx}}{\underset{\downarrow}{\sin}}\, 2x \cdot (1 \underset{u}{\underset{\downarrow}{-}} x)^2\,dx$$

$$v = -\frac{1}{2}\cos 2x\,\frac{du}{dx} = 2(1-x)(-1)$$

$$= \frac{1}{2}(1-x)^3 \sin 2x$$

$$+\frac{3}{2}\left\{-\frac{1}{2}(1-x)^2\cos 2x\right.$$

$$\left. - \int -\frac{1}{2}\cos 2x\,(-2)(1-x)\,dx\right\}$$

$$= \frac{(1-x)^3}{2}\sin 2x - \frac{3}{4}(1-x)^2\cos 2x - \int (1 \underset{u}{\underset{\downarrow}{-}} x)\cdot \underset{\frac{dv}{dx}}{\underset{\downarrow}{\cos}}\, 2x\,dx$$

$$v = \frac{1}{2}\sin 2x\,\frac{du}{dx} = -1$$

$$= \frac{(1-x)^3}{2}\sin 2x - \frac{3}{4}(1-x)^2\cos 2x - \frac{1}{2}(1-x)\sin 2x$$

$$+\frac{1}{2}\int \sin 2x\, dx$$

$$= \frac{1}{2}(1-x)^3\sin 2x - \frac{3}{4}(1-x)^2\cos 2x - \frac{1}{2}(1-x)\sin 2x$$

$$-\frac{1}{4}\cos 2x + c$$

n) $I = \int (\log_e x)^3\, dx = \int \underset{\frac{dv}{dx}}{\underset{\downarrow}{x^0}} \cdot \underset{u}{\underset{\downarrow}{(\log_e x)^3}}\, dx$

$$v = x \quad \frac{du}{dx} = 3(\log_e x)^2 \times \frac{1}{x}$$

$$= x(\log_e x)^3 - \int x \cdot \frac{1}{x} 3(\log_e x)^2\, dx$$

$$= x(\log_e x)^3 - 3\int (\log_e x)^2\, dx$$

$$= x(\log_e x)^3 - 3\left\{\int \underset{\frac{dv}{dx}}{\underset{\downarrow}{x^0}} \cdot \underset{u}{\underset{\downarrow}{(\log_e x)^2}}\, dx\right\}$$

$$\frac{du}{dx} = 2(\log_e x)\frac{1}{x}$$

$$= x(\log_e x)^3 - 3x(\log_e x)^2 + 6\int \log_e x\, dx$$

$$= x(\log_e x)^3 - 3x(\log_e x)^2 + 6\int \underset{\frac{dv}{dx}}{\underset{\downarrow}{x^0}} \cdot \underset{u}{\underset{\downarrow}{\log_e x}}\, dx$$

$$= x(\log_e x)^3 - 3x(\log_e x)^2 + 6\left[x\log_e x - \int x \cdot \frac{1}{x}\, dx\right]$$

$$= x(\log_e x)^3 - 3x(\log_e x)^2 + 6x\log_e x - 6\int 1\, dx$$

$$= x(\log_e x)^3 - 3x(\log_e x)^2 + 6x\log_e x - 6x + c$$

17.2 Definite Integral

Let $F(x)$ be any antiderivative of the function $f(x)$, then any two values of the independent variable x, say a and b, then the difference $F(b) - F(a)$ is called the definite integral of $f(x)$ from a to b, $(a < b)$ is denoted by

$$\int_{a}^{b} f(x)\,dx = F(b) - F(a),$$

Where$F(x)$ is any derivative of the function$f(x)$. The numbers a and b are called the limits of integration,

a −is the lower limit

b −is the upper limit

Usually $F(b) - F(a)$ is abbreviated by writing $[F(x)]_a^b$or $F(x)|_a^b$.

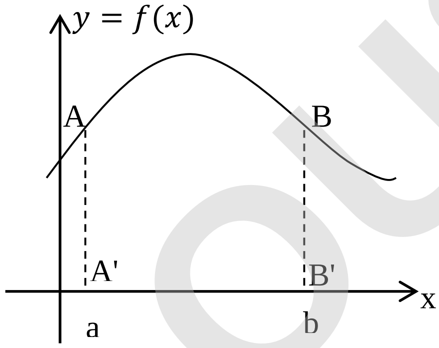


Figure 17.2.1

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Also the above definite integral $\int_a^b f(x)\,dx$ we can give very important concepts.

$$A = \text{The area}ABB'A' = \int_a^b f(x)\,dx (b > a)$$

The area bounded by the curve $y = f(x)$, x axis and the ordinates $x = a$ and $x = b$

$$A = \int_a^b f(x)\,dx$$

17.3 Properties of Definite Integrals

Now we are going to establish some properties of definite integrals, which are very useful evaluating definite integrals.

i. $\int_a^b f(x)\,dx = \int_a^b f(y)\,dy$

ii. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

iii. $\int_a^a f(x)\,dx = 0$

iv. $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

v. $\int_0^b f(x)\,dx = \int_0^b f(b-x)\,dx$

vi. $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$

Rule to Evaluate $\int_a^b f(x)\,dx$

Step 1: Evaluate the indefinite integral $\int_a^b f(x)\,dx$ and omitting the constant of the integration find $F(x)$.

Step 2: Evaluate $F(a)$ and $F(b)$.

Step 3: Evaluate $F(b) - F(a)$

Example 2

Evaluate the following definite integrals

1) $\int_0^2 \frac{x}{x^2-1}\,dx$

2) $\int_0^{\frac{\pi}{3}} \frac{\cos x}{3+4\sin x}\,dx$

3) $\int_0^{\frac{\pi}{4}} 2(\tan x)^3\,dx$

4) $\int_0^{\frac{\pi}{4}} (\sec x)^4\,dx$

5) $\int_2^3 \frac{(2-x)}{5x-6x^2}\,dx$

6) $\int_0^{\frac{\pi}{6}} \frac{1}{1+\sin x}\,dx$

7) $\int_2^4 x^2 \log_e 2x\,dx$

8) $\int_2^5 \frac{1}{(x+1)(x+2)(x+3)}\,dx$

9) $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1-\cot x}{1+\cot x}\,dx$

10) $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sin 3x \cos 5x\,dx$

11) $\int_0^2 \frac{x^2}{x^2+3x-4}\,dx$

12) $\int_0^6 \frac{1}{32-2x^2}\,dx$

13) $\int_0^3 \frac{x-1}{(x+1)(x+2)} dx$

14) $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} (\sec x)^3\, dx$

15) $\int_1^3 \frac{3x+1}{(x-2)^2(x+2)} dx$

16) $\int_0^{\frac{\pi}{2}} \frac{1}{(\sin x - \cos x)} dx$

Solutions

1) $I_1 = \int_0^2 \frac{x}{x^2-1} dx$

$\frac{x}{x^2-1} \equiv \frac{A}{x-1} + \frac{B}{x+1}$ (The process of Partial Fraction)

$$x \equiv A(x+1) + B(x-1)$$

When $x = 1, A = \frac{1}{2}$ when $x = -1 B = -\frac{1}{2}$

$$\therefore \frac{x}{x^2-1} \equiv \frac{1}{2}\left\{\frac{1}{x-1} - \frac{1}{x+1}\right\}$$

$$\therefore I_1 = \frac{1}{2}\left\{\int_0^2 \frac{1}{x-1} dx - \int_0^2 \frac{1}{x+1} dx\right\}$$

$$I_1 = \frac{1}{2}[\log_e|x-1||_0^2 - \log_e|x+1||_0^2]$$

$$\int \frac{1}{x+a} dx = \log_e|x+a|$$

$$= \frac{1}{2}[\log_e(2-1) - \log_e|0-1| - (\log_e|2+1| - \log_e(0+1))]$$

$$= \frac{1}{2}[\log_e 1 - \log_e 1 - \log_e 3 + \log_e 1]$$

$= -\frac{1}{2}\log_e 3$ since $\log_e 1 = 0$

$$= \log_e 3^{-\frac{1}{2}} = \log_e \frac{1}{\sqrt{3}}$$

2) $I = \int_0^{\frac{\pi}{3}} \frac{\cos x}{3+4\sin x} dx = \frac{1}{4}\int_0^{\frac{\pi}{3}} \frac{4\cos x}{3+4\sin x} dx$

$= \left[\frac{1}{4}\log_e|3+4\sin x|\right]_0^{\frac{\pi}{3}}$

$$\boxed{\int \frac{f'(x)}{f(x)} dx = \log_e|f(x)|}$$

$$= \frac{1}{4}\left[\log_e\left|3 + 4\sin\frac{\pi}{3}\right| - \log_e|3 + 4\sin 0|\right]$$

$$= \frac{1}{4}\left[\log_e\left|3 + 4\cdot\frac{\sqrt{3}}{2}\right| - \log_e|3+0|\right]$$

$$= \frac{1}{4}\left[\log_e\left(3 + 2\sqrt{3}\right) - \log_e 3\right]$$

$= \frac{1}{4}\log_e\left(\frac{3+2\sqrt{3}}{3}\right)$.

3) $I = \int_0^{\frac{\pi}{4}} 2(\tan x)^3\, dx$

$$\int 2(\tan x)^3\, dx = \int 2\tan x\,(\tan x)^2\, dx$$

$$= \int 2(\tan x)[(\sec x)^2 - 1]\, dx$$

$$2\left[\int (\tan x)(\sec x)^2\, dx - \int \tan x\, dx\right]$$

$$= 2\left\{\int \underset{\substack{\downarrow \\ u}}{\tan x} \cdot \underset{\substack{\downarrow \\ \frac{dv}{dx}}}{(\sec x)^2}\, dx - \int \tan x\, dx\right\}$$

$\therefore v = \tan x, \frac{du}{dx} = (\sec x)^2$

$$= 2\left\{(\tan x)^2 - \int \tan x\,(\sec x)^2\, dx - \int \tan x\, dx\right\}$$

$$= 2\left\{(\tan x)^2 - \int \tan x\,[(\tan x)^2 + 1]\, dx - \int \tan x\, dx\right\}$$

$$= 2\left\{(\tan x)^2 - \int (\tan x)^3\, dx - \int \tan x\, dx - \int \tan x\, dx\right\}$$

$$\therefore \int 2(\tan x)^3\, dx = 2(\tan x)^2 - 2\int (\tan x)^3\, dx - 4\int \tan x\, dx$$

$$\int 4(\tan x)^3\, dx = 2(\tan x)^2 - 4\int \tan x\, dx$$

$$\int 2(\tan x)^3\, dx = (\tan x)^2 - 2\int \tan x\, dx$$

$$= (\tan x)^2 - 2\int \frac{\sin x}{\cos x}\, dx$$

$$= (\tan x)^2 + 2\int \frac{-\sin x}{\cos x}\, dx$$

$$\therefore \int 2(\tan x)^3\, dx = (\tan x)^2 + 2\log_e|\cos x|$$

$$\therefore I = \int_0^{\frac{\pi}{4}} 2(\tan x)^3\, dx = [(\tan x)^2 + 2\log_e|\cos x|]_0^{\frac{\pi}{4}}$$

$$= \left[\left(\tan\frac{\pi}{4}\right)^2 + 2\log_e\left|\cos\frac{\pi}{4}\right|\right] - [(\tan 0)^2 + 2\log_e|\cos 0|]$$

$$= \left(1 + 2\log_e \frac{1}{\sqrt{2}}\right) \log_e 1 = 0$$

$$= 1 + \log_e \frac{1}{2}$$

4) $\quad I = \int_0^{\frac{\pi}{4}} (\sec x)^4 \, dx = \int_0^{\frac{\pi}{4}} (\sec x)^2 (\sec x)^2 \, dx$

$$= \int_0^{\frac{\pi}{4}} (\sec x)^2 [1 + (\tan x)^2] \, dx$$

$$= \int_0^{\frac{\pi}{4}} (\sec x)^2 \, dx + \int \underset{\frac{dv}{dx}}{\underset{\downarrow}{(\sec x)^2}} \cdot \underset{u}{\underset{\downarrow}{(\tan x)^2}} \, dx$$

$$v = \tan x \quad \frac{du}{dx} = 2\tan x \, (\sec x)^2$$

$$= [\tan x]_0^{\frac{\pi}{4}} + \left[[(\tan x)^2 \tan x]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} 2\tan x \, (\sec x)^2 \cdot \tan x \, dx \right]$$

$$= \left(\tan \frac{\pi}{4} - \tan 0\right) + \left(\left(\tan \frac{\pi}{4}\right)^3 - (\tan 0)^3\right)$$

$$- 2 \int_0^{\frac{\pi}{4}} (\tan x)^2 (\sec x)^2 \, dx$$

$$\therefore \int_0^{\frac{\pi}{4}} (\sec x)^4 \, dx = 1 + 1 - 2 \int_0^{\frac{\pi}{4}} (\tan x)^2 (\sec x)^2 \, dx$$

$$= 1 + 1 - 2 \int_0^{\frac{\pi}{4}} (\sec x)^4 \, dx + 2 \int_0^{\frac{\pi}{4}} (\sec x)^2 \, dx$$

$$3 \int_0^{\frac{\pi}{4}} (\sec x)^4 \, dx = 2 + 2[\tan x]_0^{\frac{\pi}{4}}$$

$$= 2 + 2\left(\tan \frac{\pi}{4} - \tan 0\right)$$

$$= 2 + 2$$

$$\int_0^{\frac{\pi}{4}} (\sec x)^4 \, dx = \frac{4}{3}$$

5) $I = \int_2^3 \frac{(2-x)}{5x-6x^2}\, dx$

$$\frac{(2-x)}{x(5-6x)} \equiv \frac{A}{x} + \frac{B}{(5-6x)}$$

$$(2-x) \equiv A(5-6x) + Bx$$

When $x = 0 A = \frac{2}{5}$

$$B - 6A = -1$$

$$B = -1 + \frac{12}{5} = \frac{7}{5}$$

$$\therefore I = \frac{2}{5}\int_2^3 \frac{1}{x}\, dx - \frac{7}{5}\int_2^3 \frac{1}{(5-6x)}\, dx$$

$$= \frac{2}{5}\int_2^3 \frac{1}{x}\, dx + \frac{7}{30}\int_2^3 \frac{-6}{(5-6x)}\, dx$$

$$= \frac{2}{5}[\log_e|x|]_2^3 + \frac{7}{30}[\log_e|5-6x|]_2^3$$

$$= \frac{2}{5}(\log_e 3 - \log_e 2) + \frac{7}{30}(\log_e|5-18| - \log_e|5-12|)$$

$$= \frac{2}{5}\log_e\frac{3}{2} + \frac{7}{30}(\log_e 13 - \log_e 7)$$

$$= \frac{2}{5}\log_e\frac{3}{2} + \frac{7}{30}\log_e\frac{13}{7}$$

6) $I = \int_0^{\frac{\pi}{6}} \frac{1}{1+\sin x}\, dx = \int_0^{\frac{\pi}{6}} \frac{(1-\sin x)}{(1+\sin x)(1-\sin x)}\, dx = \int_0^{\frac{\pi}{6}} \frac{(1-\sin x)}{(\cos x)^2}\, dx$

$$= \int_0^{\frac{\pi}{6}} \frac{1}{(\cos x)^2}\, dx - \int_0^{\frac{\pi}{6}} \frac{\sin x}{(\cos x)^2}\, dx$$

$$= \int_0^{\frac{\pi}{6}} (\sec x)^2\, dx - \int_0^{\frac{\pi}{6}} \tan x \sec x\, dx$$

$$= [\tan x]_0^{\frac{\pi}{6}} - [\sec x]_0^{\frac{\pi}{6}}$$

$$= \left(\tan\frac{\pi}{6} - \tan 0\right) - \left(\sec\frac{\pi}{6} - \sec 0\right)$$

$$= \frac{1}{\sqrt{3}} - 1 - \left(\frac{2}{\sqrt{3}} - 1\right)$$

$$= \frac{1}{\sqrt{3}} - \frac{2}{\sqrt{3}} = -\frac{1}{\sqrt{3}}$$

7) $I = \int_2^4 \underset{\frac{dv}{dx}}{\underset{\downarrow}{x^2}} \cdot \underset{u}{\underset{\downarrow}{\log_e}} 2x\, dx$

$$\therefore v = \frac{x^3}{3} \frac{du}{dx} = \frac{1}{2}\left(\frac{1}{2x}\right) = \frac{1}{4x}$$

$$I = \left[\frac{x^3}{3}\log_e 2x\right]_2^4 - \int_2^4 \frac{x^3}{3} \times \frac{1}{4x}\, dx$$

$$= \left(\frac{4^3}{3}\log_e 8 - \frac{2^3}{3}\log_e 4\right) - \int_2^4 \frac{x^2}{12}\, dx$$

$$= \frac{64}{3}\log_e 8 - \frac{8}{3}\log_e 4 - \left[\frac{x^3}{12 \times 3}\right]_2^4$$

$$= \frac{8}{3}[\log_e 8^8 - \log_e 4] - \left(\frac{4^3}{36} - \frac{2^3}{36}\right)$$

$$\frac{8}{3}\left[\log_e \frac{8^8}{4}\right] - \left(\frac{64 - 8}{36}\right)$$

$$\frac{8}{3}[\log_e 2 \cdot 8^7] - \frac{14}{9}$$

8) $I = \int_2^5 \frac{1}{(x+1)(x+2)(x+3)}\, dx$

Process of partial Fraction

$$\frac{1}{(x+1)(x+2)(x+3)} \equiv \frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{x+3}$$

$$1 \equiv A(x+2)(x+3) + B(x+1)(x+3) + C(x+1)(x+2)$$

When $x = -1 A = \frac{1}{2}$

$$x = -2 B = -1$$

$$x = -3 C = \frac{1}{2}$$

$$\therefore I = \frac{1}{2}\int_2^5 \frac{1}{x+1}\, dx - \int_2^5 \frac{1}{x+2}\, dx + \frac{1}{2}\int_2^5 \frac{1}{x+3}\, dx$$

$$= \frac{1}{2}[\log_e|x+1|]_2^5 - [\log_e|x+2|]_2^5 + \frac{1}{2}[\log_e|x+3|]_2^5$$

$$= \frac{1}{2}(\log_e(5+1) - \log_e(2+1)) - (\log_e(5+2) - \log_e(2+2))$$

$$+ \frac{1}{2}(\log_e(5+3) - \log_e(2+3))$$

$$= \frac{1}{2}(\log_e 6 - \log_e 3) - (\log_e 7 - \log_e 4) + \frac{1}{2}(\log_e 8 - \log_e 5)$$

$$= \frac{1}{2}\log_e \frac{6}{3} - \log_e \frac{7}{4} + \frac{1}{2}\log_e \frac{8}{5}$$

$$= \frac{1}{2}\log_e \left(\frac{6}{3} \times \frac{8}{5}\right) - \log_e \frac{7}{4}$$

$$= \log_e \left(\frac{48}{15}\right)^{\frac{1}{2}} - \log_e \frac{7}{4}$$

$$= \log_e \frac{\sqrt{48} \times 4}{\sqrt{15} \times 7}$$

9) $I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1-\cot x}{1+\cot x}\, dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1-\frac{\cos x}{\sin x}}{1+\frac{\cos x}{\sin x}}\, dx$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin x - \cos x}{\sin x + \cos x}\, dx = -\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\cos x - \sin x}{\sin x + \cos x}\, dx$$

$$= -[\log_e|\sin x + \cos x|]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$$

$$\int \frac{f'(x)}{f(x)}\, dx = \log_e|f(x)|$$

$$= -\log_e \left|\sin\frac{\pi}{3} + \cos\frac{\pi}{3}\right| + \log_e \left|\sin\frac{\pi}{6} + \cos\frac{\pi}{6}\right|$$

$$= -\log_e \left(\frac{\sqrt{3}+1}{2}\right) + \log_e \left(\frac{1+\sqrt{3}}{2}\right)$$

$$= -\log_e(1) = 0$$

10) $I = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sin 3x \cos 5x\, dx = \frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \left[\sin\left(\frac{3x+5x}{2}\right) - \sin\left(\frac{5x-3x}{2}\right)\right] dx$

$$= \frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} [\sin 4x - \sin x]\, dx$$

$$= \frac{1}{2}\left[-\frac{1}{4}\cos 4x + \cos x\right]_{\frac{\pi}{4}}^{\frac{\pi}{3}}$$

$$= \frac{1}{2}\left\{\left(-\frac{1}{4}\cos\frac{4\pi}{3} + \cos\frac{\pi}{3}\right) - \left(-\frac{1}{4}\cos\frac{4\pi}{4} + \cos\frac{\pi}{4}\right)\right\}$$

$$= \frac{1}{2}\left\{\left(-\frac{1}{4}\right) \times \left(-\frac{1}{2}\right) + \frac{1}{2} + \frac{1}{4}(-1) - \frac{1}{\sqrt{2}}\right\}$$

$$= \frac{1}{2}\left\{\frac{1}{8} + \frac{1}{2} - \frac{1}{4} - \frac{1}{\sqrt{2}}\right\}$$

$$= \frac{1}{2}\left\{\frac{3}{8} - \frac{1}{\sqrt{2}}\right\}$$

11) $I = \int_0^2 \frac{x^2}{x^2+3x-4} dx$

$$\int_0^2 \frac{x^2}{x^2 + 3x - 4} dx$$

$$\frac{x^2}{x^2 + 3x - 4} \equiv 1 + \frac{A}{(x + 4)} + \frac{B}{(x - 1)}$$

$$x^2 = (x + 4)(x - 1) + A(x - 1) + B(x + 4)$$

When $x = 11 = B(5)B = \frac{1}{5}$

$$x = -416 = A(-5)A = -\frac{16}{5}$$

$$I = \int_0^2 1\,dx - \frac{16}{5}\int_0^2 \frac{1}{x + 4} dx + \frac{1}{5}\int_0^2 \frac{1}{x - 1} dx$$

$$= [x]_0^2 - \frac{16}{5}[\log_e|x + 4|]_0^2 + \frac{1}{5}[\log_e|x - 1|]_0^2$$

$$= 2 - 0 - \frac{16}{5}\{\log_e 6 - \log_e 4\} + \frac{1}{5}(\log_e(1) - \log_e|-1|)$$

$$= 2 - \frac{16}{5}\log_e\frac{6}{4} + \frac{1}{5}(\log_e(1) - \log_e(1))$$

$$= 2 - \frac{16}{5}\log_e\frac{3}{2}$$

12) $I = \int_0^6 \frac{1}{32-2x^2} dx = \frac{1}{2}\int_0^6 \frac{1}{16-x^2} dx = \frac{1}{2}\int_0^6 \frac{1}{(4-x)(4+x)} dx$

$$\frac{1}{16 - x^2} \equiv \frac{A}{4 - x} + \frac{B}{4 + x}$$

$$1 \equiv A(4 + x) + B(4 - x)$$

When $x = 4A = \frac{1}{8}$ when $x = -4B = -\frac{1}{8}$

$$I = \frac{1}{16}\left\{\int_0^6 \frac{1}{(4-x)}\,dx - \int_0^6 \frac{1}{(4+x)}\,dx\right\}$$

$$= \frac{1}{16}\{-[\log_e|4-x|]_0^6 - [\log_e|4+x|]_0^6\}$$

$$= \frac{1}{16}\{-\log_e|4-6| + \log_e|4-0| - [\log_e|4+6| - \log_e|4+0|]\}$$

$$= \frac{1}{16}\{-\log_e 2 + \log_e 4 - \log_e 10 + \log_e 4\}$$

$$= \frac{1}{16}\log_e\left[\frac{4\times 4}{2\times 10}\right] = \frac{1}{16}\log_e\left[\frac{4}{5}\right]$$

13) $I = \int_0^3 \frac{(x-1)}{(x+1)(x+2)}\,dx$

$$\frac{x-1}{(x+1)(x+2)} \equiv \frac{A}{(x+1)} + \frac{B}{(x+2)}$$

$$x - 1 = A(x+2) + B(x+1)$$

When $x = -1 A = -2$

$$x = -2 B = 3$$

$$\therefore I = \int_0^3 \frac{-2}{(x+1)}\,dx + 3\int_0^3 \frac{1}{(x+2)}\,dx$$

$$= -2[\log_e|x+1|]_0^3 + 3[\log_e|x+2|]_0^3$$

$$= -2[\log_e(4) - \log_e(1)] + 3[\log_e(5) - \log_e(2)]$$

$$= -2\log_e 4 + 3\log_e\left(\frac{5}{2}\right)$$

$$= -\log_e 4^2 + \log_e\left(\frac{5}{2}\right)^3$$

$$= \log_e\left(\frac{25}{4}\times\frac{1}{16}\right) = \log_e\left(\frac{25}{64}\right)$$

14) $I = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} (\sec x)^3\,dx$

$$\int (\sec x)^3\,dx = \underset{\substack{\downarrow\\ \frac{dv}{dx}}}{(\sec x)^2} \cdot \underset{\substack{\downarrow\\ u}}{\sec x}$$

$$v = \tan x \quad \frac{du}{dx} = \sec x \tan x$$

$$= \sec x \tan x - \int \tan x \sec x \tan x\,dx$$

$$= \sec x \tan x - \int (\tan x)^2 \sec x \, dx$$

$$= \sec x \tan x - \int (\sec x)^3 \, dx + \int \sec x \, dx$$

$$\therefore 2\int (\sec x)^3 \, dx = \sec x \tan x + \int \sec x \, dx$$

$$= \sec x \tan x + \ln|\sec x + \tan x|$$

$$\int (\sec x)^3 \, dx = \frac{1}{2}\{\sec x \tan x + \ln|\sec x + \tan x|\}$$

$$\therefore I = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} (\sec x)^3 \, dx = \frac{1}{2}[\sec x \tan x + \ln|\sec x + \tan x|]_{\frac{\pi}{4}}^{\frac{\pi}{3}}$$

$$= \frac{1}{2}\Big[\sec\frac{\pi}{3}\tan\frac{\pi}{3} + \ln\left|\sec\frac{\pi}{3} + \tan\frac{\pi}{3}\right| - \sec\frac{\pi}{4}\tan\frac{\pi}{4}$$

$$- \ln\left|\sec\frac{\pi}{4} + \tan\frac{\pi}{4}\right|\Big]$$

$$= \frac{1}{2}\left[2\sqrt{3} + \ln|2 + \sqrt{3}| - \sqrt{2} - \ln|1 + \sqrt{2}|\right]$$

$$= \frac{1}{2}\left[2\sqrt{3} - \sqrt{2} + \ln\left(\frac{2 + \sqrt{3}}{\sqrt{2} + 1}\right)\right]$$

15) $I = \int_1^3 \frac{3x+1}{(x-2)^2(x+2)} dx$

$$\frac{3x+1}{(x-2)^2(x+2)} \equiv \frac{A}{(x-2)} + \frac{B}{(x-2)^2} + \frac{C}{(x+2)}$$

$$3x + 1 \equiv A(x-2)(x+2) + B(x+2) + C(x-2)^2$$

When $x = 2, B = \frac{7}{4}$; when $x = -2C = -\frac{5}{16}$

$$[\langle x^2 \rangle] A + C = 0 A = \frac{5}{16}$$

$$I = \frac{5}{16}\int_1^3 \frac{1}{(x-2)} dx + \frac{7}{4}\int_1^3 \frac{1}{(x-2)^2} dx - \frac{5}{16}\int_1^3 \frac{1}{(x+2)} dx$$

$$= \frac{5}{16}[\ln|x-2|]_1^3 + \frac{7}{4} \times \left[\frac{1}{(x-2)}\right]_1^3 - \frac{5}{16}[\ln|x+2|]_1^3$$

$$= \frac{5}{16}[\ln(-1) - \ln(-3)] + \frac{7}{4}\left[\frac{1}{1} - \left(\frac{-1}{1}\right)\right] - \frac{5}{16}\ln\frac{5}{3}$$

$$= -\frac{5}{16}\ln 3 + \frac{7}{4} \times 2 - \frac{5}{16}\ln\frac{5}{3}$$

$$= \frac{7}{2} - \frac{5}{16}\ln\left(3 \times \frac{5}{3}\right) = \frac{7}{2} - \frac{5}{16}\ln 5$$

16) $I = \int_0^{\frac{\pi}{2}} \frac{1}{(\sin x - \cos x)}\, dx = \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{2}\left(\frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x\right)}\, dx$

$$= \frac{1}{\sqrt{2}}\int_0^{\frac{\pi}{2}} \frac{1}{\left(\sin x \cos\frac{\pi}{4} + \cos x \sin\frac{\pi}{4}\right)}\, dx = \frac{1}{\sqrt{2}}\int_0^{\frac{\pi}{2}} \frac{1}{\sin\left(x + \frac{\pi}{4}\right)}\, dx$$

$$= \frac{1}{\sqrt{2}}\int_0^{\frac{\pi}{2}} \csc\left(x + \frac{\pi}{4}\right) dx$$

$$= \frac{1}{\sqrt{2}}\left[-\ln\left|\csc\left(x + \frac{\pi}{4}\right) + \cot\left(x + \frac{\pi}{4}\right)\right|\right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{\sqrt{2}}\left[-\ln\left|\csc\frac{3\pi}{4} + \cot\frac{3\pi}{4}\right| + \ln\left|\csc\frac{\pi}{4} + \cot\frac{\pi}{4}\right|\right]$$

$$= \frac{1}{\sqrt{2}}\left[-\ln|-1| + \ln\left|\sqrt{2} + 1\right|\right]$$

$$= \frac{1}{\sqrt{2}}\ln\left|\frac{\sqrt{2} + 1}{\sqrt{2} - 1}\right| = \frac{1}{\sqrt{2}}\ln\left(\sqrt{2} + 1\right)^2 = \frac{1}{\sqrt{2}}\ln\left(3 + 2\sqrt{2}\right)$$

Activity 1

1. Evaluate the following definite Integrals.

1) $\int_0^4 \left(\sqrt{x} - 2x + x^2\right) dx$ 2) $\int_0^4 (t^2 + 1)\, dt$

3) $\int_1^2 (4x^2 - 5x^2 + 6x + 9)\, dx$ 4) $\int_{-1}^1 (x + 1)\, dx$

5) $\int_0^1 \frac{1}{2x-3}\, dx$ 6) $\int_4^5 e^x\, dx$

7) $\int_0^{\frac{\pi}{2}} (\sin x + \cos x)\, dx$ 8) $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{\sin 2x}\, dx$

9) $\int_0^{\frac{\pi}{4}} (2(\sec x)^2 + x^3 + 2)\, dx$ 10) $\int_{\frac{\pi}{2}}^{\pi} \left(\left(\sin\frac{x}{2}\right)^2 - \left(\cos\frac{x}{2}\right)^2\right) dx$

11) $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin x}{(\cos x)^2}\, dx$ 12) $\int_0^1 \frac{x}{x+1}\, dx$

13) $\int_1^4 \frac{x^3 - 5x^2 + 2}{\sqrt{x}}\, dx$ 14) $\int_0^1 (2x + 3)^5\, dx$

15) $\int_0^{\frac{\pi}{4}} (\cos 3x)^2 \, dx$

16) $\int_0^{\frac{\pi}{4}} (\tan x)^2 \, dx$

17) $\int_{\frac{\pi}{3}}^{\frac{\pi}{4}} (\tan x + \cot x)^2 \, dx$

18) $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (\csc \theta)^2 \cos \theta \, d\theta$

19) $\int_0^{\pi} \frac{1}{1+\sin x} dx$

20) $\int_{\frac{\pi}{2}}^{\pi} \left(\frac{1-\sin x}{1-\cos x}\right) dx$

21) $\int_0^{\frac{\pi}{2}} \left(\frac{1-\cos x}{1+\cos x}\right) dx$

22) $\int_0^{\frac{\pi}{2}} \sqrt{1+\cos x} \, dx$

23) $\int_0^{\frac{\pi}{4}} \sin 3x \sin 2x \, dx$

24) $\int_0^{\frac{\pi}{4}} \sin 2x \cos 3x \, dx$

25) $\int_0^{\frac{\pi}{2}} \cos 3x \cos 2x \, dx$

26) $\int_0^{\frac{\pi}{4}} \sec x \sqrt{\frac{1-\sin x}{1+\sin x}} \, dx$

27) $\int_3^5 \frac{x^2}{x^2-4} dx$

28) $\int_1^2 \frac{1}{(x+1)(x-2)} dx$

29) $\int x\cos^2 x \, dx$

30) $\int_0^{\frac{\pi}{2}} (\sin x)^2 \, dx$

Solutions to Activity

Activity 1

1)	$\frac{32}{3}$	2)	$25\frac{1}{3}$	3)	$\frac{64}{3}$
4)	2	5)	$-\frac{1}{2}\log_e 3$	6)	$e^4(e-1)$
7)	2	8)	$\log\sqrt{3}$	9)	$2+\frac{\pi^2}{1024}+\frac{\pi}{2}$
10)	1	11)	$2-\frac{2}{\sqrt{3}}$	12)	$1-\log_e 2$
13)	$-\frac{152}{7}$	14)	$\frac{3724}{3}$	15)	$\frac{\pi}{8}-\frac{1}{12}$
16)	$1-\frac{\pi}{4}$	17)	$-\frac{2}{\sqrt{3}}$	18)	$\sqrt{2}-1$
19)	2	20)	$1-\log_e 2$	21)	$2-\frac{\pi}{2}$
22)	2	23)	$\frac{3}{5\sqrt{2}}$	24)	$\frac{1}{10}(3\sqrt{2}-4)$

25)	$\frac{3}{5}$	26)	$2-\sqrt{2}$	27)	$2+\log_e \frac{5}{11}$
28)	$\log_e \frac{9}{8}$	29)	$\frac{\pi^2}{4}$	30)	$\frac{\pi}{4}$

Summary

The formulae for integration by parts

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$$

If the Indefinite integral of the function $f(x)$is $F(x)$, then definite integral of $\int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a)$.

The properties of definite integral

i. $\int_a^b f(x)\,dx = \int_a^b f(y)\,dy$

ii. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

iii. $\int_a^a f(x)\,dx = 0$

iv. $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

v. $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

vi. $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$

Learning outcomes

On completion of this study session you should be able to

- Apply the integration by parts to find indefinite integrals.
- Evaluate the definite integrals.
- Apply the rules of definite integrals to evaluate definite integrals.