UNIT 1
BASIC CONCEPTS OF ALGEBRA

Contents:

Unit Introduction

Mathematics has played the most important role in the development and understanding of the various fields of technology, science and their applications. In learning such fields, their problems have to be converted into mathematical models. To apply mathematical concepts to Technology Science and other fields a thorough understanding of Algebra is very essential. It is very important for you to learn and apply the basic concepts and operations in algebra which are described this unit.

In this unit we will teach you the basic principles and concepts involved in

I. Algebraic Relationships
II. Factorization and Simplification of Fractions
III. Solving Simple Equation and System of Linear Equations
IV. Quadratic Expressions and Equations
V. Indices and Logarithm

Session 1

Algebraic Relationships

Content:

Introduction

Knowledge of Mathematics is important for learning some Agriculture fields. In your learning process in such fields, you have to solve several types of mathematical problems. In this session, you can learn basic concepts, definitions in Algebra, which treats of the relations and properties of quantity by means of letter and another symbol.

1.1 Types of the Algebraic Expressions.

Consider the following algebraic expressions

(a)
I. $(4x + 3)$
II. $(2x + 1)^2$
III. $\left(2/_x - 1\right)^2$

(b)
I. $2x - 1 > 0$
II. $x - 3 < 5$
III. $3x - 5 \geq 4$

(c)
I. $(2x - 1)^2 = 4x^2 - 4x + 1$
II. $(3x - 2)^3 = 27x^3 - 54x^2 + 36x - 8$

(d)
I. $(2x - 1)^2 = 4x + 13$
II. $(x + 3)^2 = 8x^2 - 7$

The above four types (a), (b), (c) and (d) are all different in nature.

Now we are going to identify these differences.

(a)
I. $(4x + 3)$
II. $(2x + 1)^2$
III. $\left(2/_x - 1\right)^2$

We can see that the above expressions can take different values depending on the value of 'x'.

Such expressions are called **functions** (of x as in the above case).

x is called a variable. In general,x is called an independent variable.

The function of variable$x$ denoted by $f(x), h(x), k(x)$ and $g(x)$ etc.

Hence, we can denote above functions as follows,

$$f(x) = (4x + 3);\ g(x) = (2x + 1)^2;\ h(x) = \left(2/_x - 1\right)^2$$

According to the above notifications we can find

$f(1), g(1), h(1), f(0), g(0)$ and $h(0)$ as follows.

I. $f(x) = (4x + 3)$

$f(1) = (4 \times 1 + 3) = 7$

II. $g(x) = (2x + 1)^2$

$g(1) = (2 \times 1 + 1)^2 = 9$

III. $h(x) = \left(2/_x - 1\right)^2$

$h(1) = \left(2/_1 - 1\right)^2 = 1$

Also, we have

I. $f(0) = (4 \times 0 + 3) = 3$

II. $g(0) = (2 \times 0 + 1)^2 = 1$

III. Note that in the case $h(x) = \left({}^{2}/_{x} - 1\right)^2$ function does not exist when$x = 0$.

$h(0)$ is not defined.

> Now you could understand, that the values of both 'x' and $f(x)$ or $g(x)$ or $h(x)$ are variables. But when x could be given any value, then the values of $f(x)$ or $g(x)$ or $h(x)$ depended on the value of x. So $'x'$ is reffered to as the independent variable and $f(x), g(x)$ or $h(x)$ are dependent variables.

(b) $2x - 1 > 0$; $x - 3 < 5$; $(3x - 5) \geq 4$

Reading from left to right the notation ">" means "greater than" and the notation "<" means "less than" and the notation "≥" means "greater than or equal".

These types of relationships are called an inequality.

I. $2x - 1 > 0$

If $(2x - 1)$ have value greater than 0

Then
$$2x - 1 > 0$$
$$2x - 1 + 1 > 0 + 1$$
$$2x > 1$$
$$2x \times \tfrac{1}{2} > 1 \times \tfrac{1}{2}$$
$$x > \tfrac{1}{2}$$

Therefore $2x - 1 > 0 \quad \Rightarrow \quad x > \frac{1}{2}$

Consider the number line as being made up of adjacent points. We can represent all the real value of x (variable) on this number line.

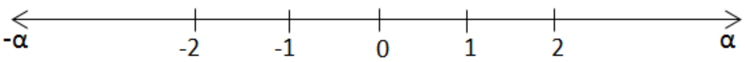


Figure 1.1.1: A number line as made up adjacent

The values of x given in the statement $2x - 1 > 0$ as $> \frac{1}{2}$, could be represented by the colored section of this line below. (See figure 1.1.2)

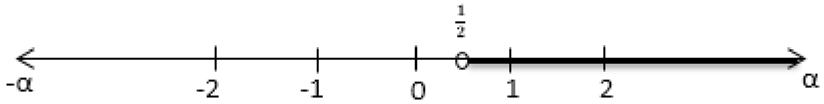


Figure 1.1.2: The values of x in the given statement, $2x - 1 > 0$ or $x > \frac{1}{2}$

II. $x - 3 < 5$

If $(x - 3)$ have a value less than 5

Then $(x - 3) \quad < 5$

$$x - 3 + 3 < 5 + 3$$

$$x \quad < 8$$

Therefore $(x - 3) < 5 \quad \Rightarrow \quad x < 8$

Then x is less than 8

Figure 1.1.3 : The values of x in the given statement, $x - 3 < 5$ or $x < 8$

III. $(3x - 5) \geq 4$

Now, we are going to consider the relationship

$$(3x - 5) \geq 4$$

$$3x - 5 + 5 \geq 4 + 5$$

$$3x \geq 9$$

$$3x \times \frac{1}{3} \geq 9 \times \frac{1}{3}$$

$$x \geq 3$$

Therefore $(3x - 5) \geq 4 \qquad x \geq 3$ which means 'x' is greater than or equal to 3. See figure 1.1.4

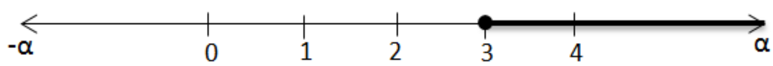


Figure 1.1.4 :The relationship of $3x - 5 \geq 4$ is equivalent to $x \geq 3$

(c)

I. $(2x-1)^2 = 4x^2 - 4x + 1$

Now, let us substitute some numerical values for x in both sides of the expression $(2x-1)^2 = 4x^2 - 4x + 1$.

Let $x = 2$.

We have the left hand side of the expression,

$LHS = (2 \times 2 - 1)^2 = (4-1)^2 = 3^2 = 9$

The right hand side of the expression

$RHS = 4(2)^2 - 4(2) + 1 = 16 - 8 + 1 = 9$

If we substitute 5 for x as before

$LHS = (2 \times 5 - 1)^2 = (10-1)^2 = 9^2 = 81$

$RHS = 4(5)^2 - 4(5) + 1 = 4 \times 25 - 20 + 1 = 81$

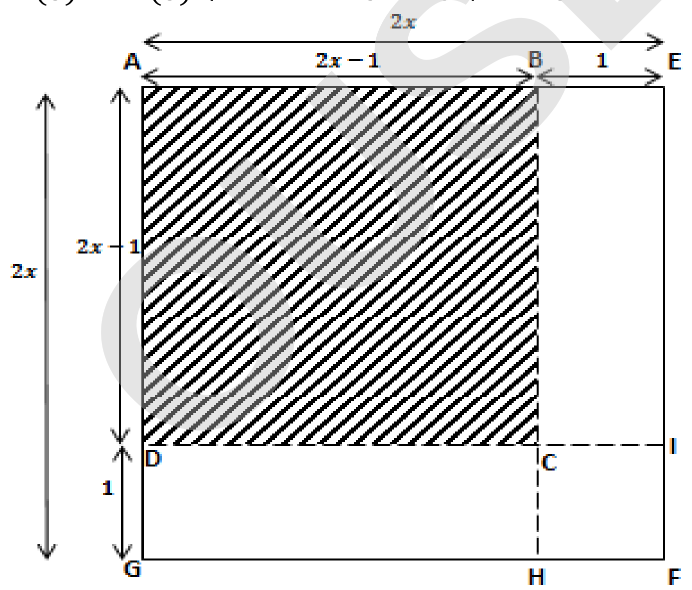


Figure 1.1.5 :Illustration to confirm that $LHS = RHS$

The area of figure 1.1.5 ABCD square =

(The area of AEFG square) – (The area of BEIC rectangular)

− (The area of DCHG rectangular) – (The area of CIFH square)

$$\text{So } (2x-1).(2x-1) = 2x.2x - 2x.1 - 1.2x + 1.1$$

$$(2x-1)^2 = 4x^2 - 4x + 1$$

$\therefore$ If we substitute whatever the numerical value for x then we can show that $LHS = RHS$.

So, it appears (but is not proved) that $LHS = RHS$ for all values of x.

The above geometric illustration confirms that this is correct.

$$\therefore (2x-1)^2 = 4x^2 - 4x + 1$$

Since these two quantities are identical

II. $(3x-2)^3 = 27x^3 - 54x^2 + 36x - 8$

When $x = 2$, $LHS = (3 \times 2 - 2)^3 = (6-2)^3 = 4^3 = 64$

$$\begin{aligned} RHS &= 27(2)^3 - 54(2)^2 + 36(2) - 8 \\ &= 27 \times 8 - 54 \times 4 + 36 \times 2 - 8 \\ &= 216 - 216 + 72 - 8 \\ &= 64 \end{aligned}$$

When $x = 5$, $LHS = (3 \times 5 - 2)^3 = (15-2)^3 = 13^3 = 2197$

$$\begin{aligned} RHS &= 27(5)^3 - 54(5)^2 + 36(5) - 8 \\ &= 27 \times 125 - 54 \times 25 + 36 \times 5 - 8 \\ &= 3375 - 1350 + 180 - 8 \\ &= 2197 \end{aligned}$$

Therefore, we can show that for all values of x,

$$(3x-2)^3 = 27x^3 - 54x^2 + 36x - 8$$

> These types of relationships are called Identities, and both sides are equal for any value of unknown quantity(x).
>
> The symbol '≡' is used to denote the equality of identities.

$$\therefore \quad (2x-1)^2 \equiv 4x^2 - 4x + 1$$

$$(3x-2)^3 \equiv 27x^3 - 54x^2 + 36x - 8$$

(d)

I. $(2x-1)^2 = 4x + 13$

Substituting $x = 2$ in both sides of the above expression, we have

$LHS = (2 \times 2 - 1)^2 = 9$

$RHS = 4 \times 2 + 13 = 21$

$LHS \neq RHS$

However, if we substitute 3 for x, then we get

$$LHS = (2 \times 3 - 1)^2 = 5^2 = 25$$

$$RHS = 4 \times 3 + 13 = 25$$

We have $LHS = RHS$ for $x = 3$

When $x = 4$

$$LHS = (2 \times 4 - 1)^2 = 7^2 = 49$$

$$RHS = 4 \times 4 + 13 = 29$$

$$LHS \neq RHS$$

Now, we can rearrange the original expression as

$$(2x - 1)^2 = 4x + 13$$

$$4x^2 - 4x + 1 = 4x + 13$$

$$4x^2 - 8x - 12 = 0$$

$$x^2 - 2x - 3 = 0$$

$$(x - 3)(x + 1) = 0$$

$$(x - 3) = 0 \text{ or } (x + 1) = 0$$

$$x = 3 \text{ or } x = -1$$

∴ From which we can see that $LHS = RHS$ is true if and only if either $x = 3$ or $x = -1$

II. $(x + 3)^2 = 8x^2 - 7$

$$x^2 + 6x + 9 = 8x^2 - 7$$

$$7x^2 - 6x - 16 = 0$$

$$(x - 2)(7x + 8) = 0$$

$$(x - 2) = 0 \text{ or } (7x + 8) = 0$$

$$x = 2 \text{ or } x = -\frac{8}{7}$$

Thus $(x + 3)^2 = 8x^2 - 7$ is true only if $x = 2$ or if $x = -\frac{8}{7}$, and the equality is not true for any other value of$x$.

> These types of expressions are called **equations,** and the equality is true only for finite number of values of a unknown quantity (or quantities).

Activity 1

Write down your own examples of different mathematical expressions.

1.2 Relations between terms in an algebraic expression

We have to recognize the basic characteristic of algebra that letters are used to represent numerical values. Since we have to use literal symbols to represent numbers, even if in a general sense, we may conclude that all basic mathematical operations valid for the numbers are valid for these literal symbols.

Any combination of numbers and literal symbols which results from algebraic expressions is known as algebraic expression. When an algebraic expression consists of several parts connected by plus $(+)$ and minus $(-)$ signs, each part (along with its sign) is known as term of the expression. If a given expression consists of the product of number of quantities, then each of these quantities or any product of them are called factors of the expression or coefficients of unknowns.

Example 1

(a) Consider the expression $6x + 9y - 3z$

$6x, 9y$ and $-3z$ are the terms of the expression (a). 6 is the factor of the unknown x or coefficient of the unknown x. 9 is the factor or the coefficient of the unknown y. -3 is the factor or coefficient of the unknown z.

(b) Consider the expression $12x - y + z$

$12x, -y$ and z are the terms of the expression (b). 12 is the factor or coefficient of the unknown x. -1 is the factor or coefficient of the unknown y and 1 is the factor or coefficient of the unknown z.

(c) Consider the expression $ax + by + c$ where a, b, c are given known constants and x and y are unknowns.

In this case, a and b are the coefficients of the unknowns x and y respectively. c is the constant term of the given expression $ax + by + c$.

(d) Consider the expression $6abxy$

In this case $6, a, b, x$ and y are the factors of the given expression $6abxy$

(e) Consider the expression $(ax + by)(cx - dy)$

Given that a, b, c and d are constants and x and y are unknowns.

In this case $(ax + by)$ and $(cx - dy)$ are the two factors of the given expression $(ax + by)(cx - dy)$.

1.3 Multinomial Expressions

Let us consider the expression $ax + by + cz + dw + et$,where a, b, c, d and e are unknown constants and x, y, c, w and $t$are unknown symbols. In this expression, we have 5 terms. The number of terms of the expressions more than 2 is called multinomial expressions. An expression containing only one term is called a monomial. For example, $7x$.. An expression containing two terms is called binomial.

Example 2

(a) $6x + 3y$

(b) $3xy + 8x^2$

(c) $7x^3y + 3xy^3$

In any given term, the numbers and literal symbols multiplying any given factor constitute the coefficient of that factor. The product of all the numbers in explicit form is known as the numerical coefficient of the term. All the terms that differ only in their numerical coefficient are known as similar terms.

Example 3

(a) $7x^4y^5$

(b) $8x - 3y + 9xy$

(c) $6x - 9x + 16x + 5y - 3y + 2$

Answers

(a) $7x^4y^5$is a monomial. It has numerical coefficient of 7. Also, we can say that the coefficient of y^5 is $7x^4$ and the coefficient of $x^4$is $7y^5$.

(b) $8x - 3y + 9xy$is a multinomial with three terms. The first term $(8x)$ has a numerical coefficient 8 and the second term $(-3y)$ has a numerical coefficient -3 and the third term $9xy$ has a numerical coefficient of 9. We can assume that y also known number. Then first term $8x$ and third term $9yx$ are similar. In this case third term has coefficient $9y$ or factor y.

Also, we can assume that x is a known number. Then second and third terms $3y, 9xy$ are similar.

1.4 The basic mathematical operations on algebraic expressions.

We can see that there are four basic mathematical operations on algebraic expressions, namely, Addition, Subtraction, Multiplication and Division.

1.4.1 Adding and subtracting on algebraic expression

We can add and subtract similar terms. All the similar terms may be combined into a single term, and the final simplified expression will be made up entirely of terms that are not similar.

Example 4

Simplify the following expressions.

(a) $3x + 4y - 3z + 6x + 7y - 2z + 9$
(b) $4x - 3y - 9z + 5y - 7z + 6x + 11y - 5z$
(c) $3y + 9z - 2x + 5y - 3z + 7$

Answers

(a) $3x + 4y - 3z + 6x + 7y - 2z + 9$
$= (3x + 6x) + (4y + 7y) - (3z + 2z) + 9$

$= x(3+6) + y(4+7) - z(3+2) + 9$

$= 9x + 11y - 5z + 9$

(b) $4x - 3y - 9z + 5y - 7z + 6x + 11y - 5z$

$= (4+6)x + (-3+5+11)y + (-9-7-5)z$

$= 10x + 13y - 21z$

(c) $3y + 9z - 2x + 5y - 3z + 7$

$= (-2)x + (3+5)y + (9-3)z + 7$

$= -2x + 8y + 6z + 7$

In writing algebraic expressions, it is often necessary to group certain terms together by using brackets. When adding and subtracting algebraic expressions, it is necessary to remove the brackets. To do so, we must change the sign of every term within the symbol if the grouping is preceded by a minus sign. If the symbols of grouping are preceded by a plus sign, each term within the symbols retains its original sign.

Example 5

Simplify the following expressions.

(a) $(3x + 9y) - 3(6x - 9y)$

(b) $-2(6x - 5y) + 3(2x - y)$

(c) $(4 - 3x^2) - (2 + 5x) + x^2$

(d) $12(3t^2 - 3t + 5) - 12t(t + 2)$

(e) $(9 + 3t)(2 - 3t) + (5 + 2t)t$

(f) $(4x^2 - 7) - 3(2x + 3)$

(g) $(3 + 2x)(4 - 3x) + (6 - x)(x + 3)$

Answers

(a) $(3x + 9y) - 3(6x - 9y)$

$= 3x + 9y - 18x + 27y$

$= (3 - 18)x + (9 + 27)y$

$= -15x + 36y$

(b) $-2(6x - 5y) + 3(2x - y)$

$= -12x + 10y + 6x - 3y$

$= (-12 + 6)x + (10 - 3)y$

$= -6x + 7y$

(c) $(4 - 3x^2) - (2 + 5x) + x^2$

$= 4 - 3x^2 - 2 - 5x + x^2$

$= (-3 + 1)x^2 - 5x + 4 - 2$

$= -2x^2 - 5x + 2$

(d) $12(3t^2 - 3t + 5) - 12t(t + 2)$

$= 36t^2 - 36t + 60 - 12t^2 - 24t$

$= (36 - 12)t^2 - (36 + 24)t + 60$

$= 24t^2 - 60t + 60$

(e) $(9 + 3t)(2 - 3t) + (5 + 2t)t$

$= 9(2 - 3t) + 3t(2 - 3t) + 5t + 2t^2$

$= 18 - 27t + 6t - 9t^2 + 5t + 2t^2$

$= (2 - 9)t^2 + (-27 + 6 + 5)t + 18$

$= -7t^2 + 16t + 18$

(f) $(4x^2 - 7) - 3(2x + 3)$

$= 4x^2 - 7 - 6x - 9$

$= 4x^2 - 6x - 16$

(g) $(3 + 2x)(4 - 3x) + (6 - x)(x + 3)$

$= 3(4 - 3x) + 2x(4 - 3x) + 6(x + 3) - x(x + 3)$

$= 12 - 9x + 8x - 6x^2 + 6x + 18 - x^2 - 3x$

$= (-6 - 1)x^2 + (-9 + 8 + 6 - 3)x + 12 + 18$

$= -7x^2 + 7x + 30$Note that (e) and (g) are based on multiplication process

Activity 2

Simplify the following algebraic expressions

I. $-3(6+2x)+2(3x-9)$

II. $(a^2-9t^2)-4(3a^2-2t^2)$

III. $2(5x+3y)-8(3y-x)$

IV. $9(2x-3y)-3(6y-3x)$

V. $(3x+2y)-3(4x+3y)$

VI. $(9x+3y)-2(5x-y)$

1.3.1 Multiplication of Algebraic Expressions

To find the product of two or monomials, we first multiply the numerical coefficients to determine the numerical coefficient of the product. Then we multiply the literal factors, remembering that the exponents may be combined only if the above is the same.

Example 6

Simplify the following expressions.

(a) $3pq^2 \times 12tp^2q \times 3t^2p$

(b) $(-2x^3y) \times (3x^2y^3) \times (5xy^2)$

(c) $(2pq)(-12ap^2)(3pqb^2)$

(d) $3x^2y(4bxy^2+2xyb+6xy)+4xy^2(2bx^2y+2bxy)$

(e) $10a^2x^3(-9ax+ab)+3ax^2(10a^2x^2+6a^2bx)$

(f) $(a^2-b^2)(a^4+a^2b^2+b^4)$

(g) $(3x^2+5y^2)(9x^4-15x^2y^2+25y^4)$

(h) $(ax+by)(bx+ay)(x+y)$

(i) $(x+2y)(2x+y)(ax+by)$

(j) $(3a+2b+c)^2$ and $(3a+2b+c)^3$

Answers

(a) $3pq^2 \times 12tp^2q \times 3t^2p$

$= 3 \times 12 \times 3 \times p^{(1+2+1)} \times q^{(2+1)} \times t^{(1+2)}$

$= 108p^4q^3t^3$

(b) $(-2x^3y) \times (3x^2y^3) \times (5xy^2)$

$= -2 \times 3 \times 5 \times x^{(3+2+1)} \times y^{(1+3+2)}$

$= -30x^6y^6$

(c) $(2pq)(-12ap^2)(3pqb^2)$

$= 2 \times -12 \times 3 \times p^{(1+2+1)}q^{(1+1)}ab^2$

$= -72p^4q^2ab^2$

(d) $3x^2y(4bxy^2 + 2xyb + 6xy) + 4xy^2(2bx^2y + 2bxy)$

$= 3x^2y.4bxy^2 + 3x^2y.2xyb + 3x^2y.6xy + 4xy^2.2bx^2y + 4xy^2.2bxy$

$= 12bx^3y^3 + 6bx^3y^2 + 18x^3y^2 + 8bx^3y^3 + 8bx^2y^3$

$= (12 + 8b)x^3y^3 + (6b + 18)x^3y^2 + 8bx^2y^3$

(e) $10a^2x^3(-9ax + ab) + 3ax^2(10a^2x^2 + 6a^2bx)$

$= 10a^2x^3.(-9ax) + 10a^2x^3.ab + 3ax^2.10a^2x^2 + 3ax^2.6a^2bx$

$= -90a^3x^4 + 10a^3bx^3 + 30a^3x^4 + 18a^3bx^3$

$= (-90 + 30)a^3x^4 + (10 + 18)a^3bx^3$

$= -60a^3x^4 + 28a^3bx^3$

(f) $(a^2 - b^2)(a^4 + a^2b^2 + b^4)$

$= a^2.a^4 + a^2.a^2b^2 + a^2.b^4 - b^2.a^4 - b^2.a^2b^2 - b^2.b^4$

$= a^6 + a^4b^2 + a^2b^4 - a^4b^2 - a^2b^4 - b^6$

$= a^6 - b^6$

(g) $(3x^2 + 5y^2)(9x^4 - 15x^2y^2 + 25y^4)$

$= 3x^2.9x^4 - 3x^2.15x^2y^2 + 3x^2.25y^4 + 5y^2.9x^4 - 5y^2.15x^2y^2 + 5y^2.25y^4$

$= 27x^6 - 45x^4y^2 + 75x^2y^4 + 45x^4y^2 - 75x^2y^4 + 125y^6$

$= 27x^6 + 125y^6$

(h) $(ax + by)(bx + ay)(x + y)$

$= (ax + by)[(bx + ay)(x + y)]$

$= (ax + by)[bx.x + bx.y + ay.x + ay.y]$

$= (ax + by)[bx^2 + (a + b)xy + ay^2]$

$= ax.bx^2 + ax.(a + b)xy + ax.ay^2 + by.bx^2 + by.(a + b)xy$
$+ by.ay^2$

$= abx^3 + ax.axy + ax.bxy + a^2xy^2 + b^2x^2y + by.axy$
$+ by.bxy + aby^3$

$= abx^3 + a^2x^2y + abx^2y + a^2xy^2 + b^2x^2y + abxy^2 + b^2xy^2$
$+ aby^3$

$= abx^3 + aby^3 + abx^2y + abxy^2 + a^2xy^2 + b^2xy^2$
$+a^2x^2y + b^2x^2y$

(i) $(x + 2y)(2x + y)(ax + by)$

$= [x(2x + y) + 2y(2x + y)](ax + by)$

$= (2x^2 + xy + 4xy + 2y^2)(ax + by)$

$= (2x^2 + 5xy + 2y^2)(ax + by)$

$= 2x^2.ax + 2x^2.by + 5xy.ax + 5xy.by + 2y^2.ax + 2y^2by$

$= 2ax^3 + 2bx^2y + 5ax^2y + 5bxy^2 + 2axy^2 + 2by^3$

(j) $(3a + 2b + c)^2$ and $(3a + 2b + c)^3$

$(3a + 2b + c)^2$

$= (3a + 2b + c)(3a + 2b + c)$

$= 3a.3a + 3a.2b + 3a.c + 2b.3a + 2b.2b + 2b.c + c.3a$
$+ c.2b + c.c$

$= 9a^2 + 6ab + 3ac + 6ab + 4b^2 + 2bc + 3ac + 2bc + c^2$

$= 9a^2 + 4b^2 + c^2 + 12ab + 6ac + 4bc$

($\therefore$ Note that

$[ap + bq + cr]^2$

$= a^2p^2 + b^2q^2 + c^2r^2 + 2apbq + 2apcr$
$+ 2bcqr$)

$$\therefore (3a + 2b + c)^3$$
$$= (3a + 2b + c)^2(3a + 2b + c)$$
$$= (9a^2 + 4b^2 + c^2 + 12ab + 6ac + 4bc)(3a + 2b + c)$$
$$= 27a^3 + 12ab^2 + 3ac^2 + 36a^2b + 18a^2c + 12abc + 18a^2b$$
$$+ 8b^3 + 2bc^2 + 24ab^2 + 12abc + 8b^2c + 9a^2c$$
$$+ 4b^2c + c^3 + 12abc + 6ac^2 + 4bc^2$$
$$= 27a^3 + 8b^3 + c^3 + 36ab^2 + 9ac^2 + 54a^2b + 27a^2c + 12b^2c$$
$$+ 6bc^2 + 36abc$$

$$\therefore (3a + 2b + c)^3$$
$$= 27a^3 + 8b^3 + c^3 + 24ab^2 + 9ac^2 + 54a^2b$$
$$+ 27a^2c + 12b^2c + 6bc^2 + 36abc$$

Activity 3

Multiply the following expressions

I. $(a + b + c)(a^2 + b^2 + c^2)$
II. $(3x + y)(3x - y)$
III. $(x^2 + 1)(x^4 - x^2 + 1)$
IV. $(x + 2)(x^2 - 3x + 2)$
V. $(a + x^2)^3$
VI. $(ax + by)^2$
VII. $3x(x^2 - 1)(x + 1)$
VIII. $(x + 3)^2(x - 3)^2$
IX. $(x + a)(x + b)(x + c)$
X. $(x + y + 1)(x + y - 1)$

1.3.2 Division of Algebraic Expressions

To find the quotient of one monomial divided by another, we have to use laws of indices.

$$\frac{a^m}{a^n} = a^{m-n}$$

$$\frac{3^4}{3^5} = 3^{(4-5)} = 3^{-1} = \frac{1}{3}$$

$$\frac{10^7}{10^2} = 10^{(7-2)} = 10^5$$

Example 7

Find the quotient of the following

(a) $32x^3y^2 \div 4x^2y$

(b) $\frac{-121x^3y^2z^3}{22xy^3}$

(c) $\frac{ab+bc+ca}{abc}$

(d) $\frac{64a^5b^2-16a^4b^3}{4a^2b^2}$

(e) $\frac{16x^4y^2-32x^3y^3+24x^2y^4}{8x^3y^2}$

(f) $\frac{a^4b^2c^5-12a^2b^2c^2+18a^3b^4c^2-6}{6a^2b^3c^4}$

Answers

(a) $\frac{32x^3y^2}{4x^2y} = \frac{32}{4}\frac{x^3}{x^2}\frac{y^2}{y} = 8x^{(3-2)}y^{(2-1)} = 8xy$

(b) $\frac{-121x^3y^2z^3}{22xy^3} = -\frac{121}{22}\frac{x^3}{x}\frac{y^2}{y^3}z^3$

$= -\frac{11}{2}x^{(3-1)}y^{(2-3)}z^3$

$= -\frac{11}{2}x^2y^{-1}z^3$

$= \frac{-11x^2z^3}{y}$

(c) $\frac{ab+bc+ca}{abc} = \frac{ab}{abc} + \frac{bc}{abc} + \frac{ca}{abc}$

$$= \frac{1}{c} + \frac{1}{a} + \frac{1}{b}$$

(d) $\frac{64a^5b^2 - 16a^4b^3}{4a^2b^2}$ $= \frac{64a^5b^2}{4a^2b^2} - \frac{16a^4b^3}{4a^2b^2}$

$$= \frac{64}{4}\frac{a^5}{a^2}\frac{b^2}{b^2} - \frac{16}{4}\frac{a^4}{a^2}\frac{b^3}{b^2}$$

$$= 16a^3 - 4a^2b$$

(e) $\frac{16x^4y^2 - 32x^3y^3 + 24x^2y^4}{8x^3y^2}$ $= \frac{16}{8}\frac{x^4}{x^3}\frac{y^2}{y^2} - \frac{32}{8}\frac{x^3}{x^3}\frac{y^3}{y} + \frac{24}{8}\frac{x^2}{x^3}\frac{y^4}{y^2}$

$$= 2x - 4y^2 + 3\frac{y^2}{x}$$

(f) $\frac{a^4b^2c^2 - 12a^2b^2c^2 + 8a^3b^4c^2 - 6}{6a^2b^3c^4}$

$$= \frac{1}{6}\frac{a^4}{a^2}\frac{b^2}{b^3}\frac{c^2}{c^4} - \frac{12}{6}\frac{a^2}{a^2}\frac{b^2}{b^3}\frac{c^2}{c^4} + \frac{8}{6}\frac{a^3}{a^2}\frac{b^4}{b^3}\frac{c^2}{c^4} - \frac{6}{6a^2b^3c^4}$$

$$= \frac{1}{6}\frac{a^2}{bc^2} - 2\frac{1}{bc^2} + \frac{4}{3}\frac{ab}{c^2} - \frac{1}{a^2b^3c^4}$$

$$= \frac{a^2}{6bc^2} - \frac{2}{bc^2} + \frac{4ab}{3c^2} - \frac{1}{a^2b^3c^4}$$

1.3.2.1 Definition of a polynomial function

Let$f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$, where $a_0, a_1, a_2, \ldots, a_n$ are all real constants and n is a positive integer.

Then $f(x)$ is a polynomial function of x, and its degree is n.

Example 7

(a) $5 + 3x^6 + x^8 + x^{12} - x^{15}$ is a polynomial of 15^{th}degree

(b) $x^3 + x + \frac{1}{x}$ is not a polynomial

(c) $\sqrt{x} + 3$ is not a polynomial

(d) $-2x + 1$ is a polynomial of 1^{st} degree

(e) $ax^2 + bx + c$ is a polynomial of 2^{nd} degree

(f) $ax^3 + b$ is a polynomial of 3^{rd} degree

(g) $8x^{10}$ is a polynomial of 10^{th} degree

Expressions (a), (b), (c), (d), (e) and (f) are multinomial. Since each contains more than one term. But expression (g) is a monomial.

1.3.2.2 Dividing one polynomial by an another

Now we are going to discuss about the steps of dividing one polynomial by a polynomial.

Step 1

Arrange the polynomial (dividend) in descending of the power of the symbol.

↓

Step 2

Divide the first term of the dividend (the polynomial to be divided) by the first term of the divisor.

↓

Step 3

Multiply the entire divisor by the first term of the quotient and subtract the product from the dividend then bring down the next term.

↓

Step 4

Divide the first term of this difference by the first term of the divisor. (This gives the second term of the quotient)

↓

Step 5

Multiply this term by each of the term of the divisor and subtract this result from the part of the dividend.

↓

Step 6

Continue the process until the remainder is either zero or term which is lower degree than the divisor.

Example 8 Simplify the following

(a) $\dfrac{16x^4+12x^3+8x^2+4x+4}{4x+1}$

(b) $\dfrac{12x^4+12x^3+20x^2+22x+12}{3x^2+6x+2}$

(c) $\dfrac{16x^6+48x^4+24x^3+12x^2+12x+6}{2x^3+2x+1}$

(d) $\dfrac{24x^7+12x^5+4x^3+4x^2+4}{12x^2+6x+1}$

Answers

(a) $\dfrac{16x^4+12x^3+8x^2+4x+4}{4x+1}$

Dividend $16x^4 + 12x^3 + 8x^2 + 4x + 4$

Divisor $4x + 1$

Step 1	Step 3	Step 5	Step 7
$\dfrac{16x^4}{4x}$	$\dfrac{8x^3}{4x}$	$\dfrac{6x^2}{4x}$	$\dfrac{5x}{2.4x}$

$$4x^3 + 2x^2 + \frac{3}{2}x + \frac{5}{8}$$

$$4x+1 \;\big)\; 16x^4 + 12x^3 + 8x^2 + 4x + 4$$

$\underline{16x^4 + 4x^3}$ ⟵ Step 2 $(4x+1) \times 4x^3$

$8x^3 + 8x^2 + 4x + 4$

$\underline{8x^3 + 2x^2}$ ⟵ Step 4 $(4x+1) \times 2x^2$

$6x^2 + 4x + 4$

$\underline{6x^2 + \frac{3}{2}x}$ ⟵ Step 6 $(4x+1) \times \frac{3x}{2}$

$\frac{5x}{2} + 4$

$\underline{\frac{5x}{2} + \frac{5}{8}}$ ⟵ Step 7 $(4x+1) \times \frac{5}{8}$

$\frac{27}{8}$

$\therefore$ Quotient $4x^3 + 2x^2 + \frac{3}{2}x + \frac{5}{8}$

Remainder $\dfrac{27}{8}$

(b) $\frac{12x^4+12x^3+20x^2+22x+12}{3x^2+6x+2}$

Dividend $\quad 12x^4 + 12x^3 + 20x^2 + 22x + 12$

Divisor $\quad 3x^2 + 6x + 2$

Step 1 $\frac{12x^4}{3x^2}$ Step 3 $\frac{-12x^3}{3x^2}$ Step 5 $\frac{36x^2}{3x^2}$

$$
\begin{array}{r|l}
 & 4x^2 - 4x + 12 \\ \hline
3x^2 + 6x + 2 & 12x^4 + 12x^3 + 20x^2 + 22x + 12 \\
 & \underline{12x^4 + 24x^3 + 8x^2} \\
 & \quad -12x^3 + 12x^2 + 22x + 12 \\
 & \quad \underline{-12x^3 - 24x^2 - 8x} \\
 & \qquad 36x^2 + 30x + 12 \\
 & \qquad \underline{36x^2 + 72x + 24} \\
 & \qquad\quad -42x - 12
\end{array}
$$

Step 2 $(3x^2 + 6x + 2) \times 4x^2$

Step 4 $(3x^2 + 6x + 2) \times -4x$

Step 6 $(3x^2 + 6x + 2) \times 12$

$\therefore$ Quotient $\quad 4x^2 - 4x + 12$

Remainder $\quad -42x - 12$

(c) $\frac{16x^6+48x^4+24x^3+12x^2+12x+6}{2x^3+2x+1}$

Dividend $\quad 16x^6 + 48x^4 + 24x^3 + 12x^2 + 12x + 6$

Divisor $\quad 2x^3 + 2x + 1$

Step 1 $\frac{16x^6}{2x^3}$ Step 3 $\frac{32x^4}{2x^3}$ Step 5 $\frac{16x^3}{2x^3}$

$$
\begin{array}{r|l}
 & 8x^3 + 16x + 8 \\ \hline
2x^3 + 2x + 1 & 16x^6 + 48x^4 + 24x^3 + 12x^2 + 12x + 6 \\
 & \underline{16x^6 + 16x^4 + 8x^3} \\
 & \quad 32x^4 + 16x^3 + 12x^2 + 12x + 6 \\
 & \quad \underline{32x^4 + 32x^2 + 16x} \\
 & \qquad 16x^3 - 20x^2 - 4x + 6 \\
 & \qquad \underline{16x^3 + 16x + 8} \\
 & \qquad\quad -20x^2 - 20x - 2
\end{array}
$$

Step 2 $(2x^3 + 2x + 1) \times 8x^3$

Step 4 $(2x^3 + 2x + 1) \times 16x$

Step 6 $(2x^3 + 2x + 1) \times 8$

$\therefore$ the quotient $8x^3 + \quad 16x \quad + 8$

Remainder $\quad -20x^2 - 20x - 2$

(d) $\frac{24x^7+12x^5+4x^3+4x^2+4}{12x^2+6x+1}$

Dividend $\quad 24x^7 + 12x^5 + 4x^3 + 4x^2 + 4$

Divisor $\quad 12x^2 + 6x + 1$

$$2x^5 - x^4 + \frac{4}{3}x^3 - \frac{7}{12}x^2 + \frac{37}{72}x + \frac{1}{8}$$

$$12x^2 + 6x + 1 \,\big)\, 24x^7 + 12x^5 + 4x^3 + 4x^2 + 4$$

$$\underline{24x^7 + 12x^6 + 2x^5}$$

$$-12x^6 + 10x^5 + 4x^3 + 4x^2 + 4$$

$$\underline{-12x^6 - 6x^5 - x^4}$$

$$16x^5 + x^4 + 4x^3 + 4x^2 + 4$$

$$\underline{16x^5 + 8x^4 + \frac{4}{3}x^3}$$

$$-7x^4 + \frac{8}{3}x^3 + 4x^2 + 4$$

$$\underline{-7x^4 - \frac{7}{2}x^3 - \frac{7}{12}x^2}$$

$$\frac{37}{6}x^3 + \frac{55}{12}x^2 + 4$$

$$\underline{\frac{37}{6}x^3 + \frac{37}{12}x^2 + \frac{37}{72}x}$$

$$\frac{3}{2}x^2 - \frac{37}{72}x + 4$$

$$\underline{\frac{3}{2}x^2 + \frac{3}{4}x + \frac{1}{8}}$$

$$-\frac{91}{72}x + \frac{31}{8}$$

$\therefore$ the quotient $2x^5 - x^4 + \frac{4}{3}x^3 - \frac{7}{12}x^2 + \frac{37}{72}x + \frac{1}{8}$

Remainder $\quad -\frac{91}{72}x + \frac{31}{8}$

Activity 4

Find the divisions of the following

(a) $\frac{16p^2q^6+32p^4q^4-64p^2q^{10}}{16p^2q^4}$

(b) $\frac{3ab^2(x+y)(x-y)+(x-y)a^2b}{a(x-y)}$

(c) $\frac{9x^2+6x+1}{3x+1}$

(d) $\frac{3x^3+17x^2+15x-20}{(3x-1)}$

(e) $\frac{x^6-1}{x-1}$

(f) $\frac{5a^2-7ab+2b^2}{(a+3b)}$

(g) $\frac{3x^2-2xy+y^2}{(x+y)}$

(h) $\frac{10x^3+8x^2+6x+4}{(x+1)}$

(i) $\frac{20x^4+10x^3+5x^2-5x+5}{x^2+5x+1}$

(j) $\frac{24x^6+12x^5+6x^4-3x^3+6x^2+6x+3}{x^3+3x^2+x+1}$

Solutions of Activities

Activity 1

(1) -36 (2) $-11a^2 - t^2$ (6) $-x + 5y$

(4) $27x - 9y$ (5) $-9x - 7y$

(3) $18x - 18y$

Activity 2

I. $a^3 + b^3 + c^3 + a^2b + b^2c + c^2a + ab^2 + bc^2 + ca^2$

II. $9x^2 - y^2$

III. $x^6 + 1$

IV. $x^3 - x^2 - 4x + 4$

V. $a^3 + 3a^2x^2 + 3ax^4 + x^6$

VI. $a^2x^2 + 2abxy + b^2y^2$

VII. $3x^4 + 3x^3 - 3x^2 - 3x$

VIII. $x^4 - 18x^2 + 81$

IX. $x^3 + (a + b + c)x^2 + (ab + bc + ca)x + abc$

X. $x^2 + 2xy + y^2 - 1$

Activity

I. $q^2 + 2p^2 - 4q^6$

II. $3b^2(x + y) + ab$

III. $(3x + 1)$

IV. Quotient $x^2 + 6x + 7$ Remainder -13

V. $x^5 + x^4 + x^3 + x^2 + 1$

VI. Quotient $5a - 22b$ Remainder $68b^2$

VII. Quotient $3x - 5y$ Remainder $6y^2$

VIII. Quotient $10x^2 - 2x + 8$ Remainder -4

IX. Quotient $20x^2 - 90x + 435$ Remainder $-2090x - 430$

X. Quotient $24x^3 - 60x^2 + 162x - 453$ Remainder $1251x^2 + 297x + 3$

Summary

- $f(x) = (ax + b)^n$ (where n is a number) is called a function of x.
  The notation $f(x)$ represents a function of x.
- $ax + by > c$ or $ax + by < c$ are called inequality,
  $ax + by > c$ denotes that $ax + by$ is greater than c.
- $(ax + b)^3 = a^3x^3 + 3a^2x^2b + 3axb^2 + b^3$
  This type of relationship is called an identity, and both sides are equal for any value of unknown quantity x.
- $2x + 3 = 4 + x$
  $ax^2 + bx + c = 0$
  $x^3 + 1 = 0$
  These types of expressions are called equations and the equality is true only for a number of values of the unknown quantity.

Learning Outcomes

On completion of this study session you should be able to

- Identify the algebraic expressions
- Apply addition, subtraction, multiplications and division of algebraic

Session 2

Factorization and Simplifications of Fractions

Introduction

In the previous session, we have discussed about the algebraic relationships. To simplify the algebraic expressions, we have to improve about the factorization and the methods of the simplifications of the fractions. Therefore, in this session, we are going to learn some additional algebraic techniques. A factor of an algebraic expression is a quantity or a term that will divide the expression.

An expression is said to be prime, when it cannot be divided by any terms (factors), other than itself and one. Simply a prime expression cannot be divided by any quantity except itself or one. An expression is said to be factorized when it has been simplified to its prime factors. Factorizing is very closely related to multiplication and division. It is very useful tool for solving equations and simplifying expressions, when it is necessary to make expressions simple. When we deal with algebraic expression, we would be able to work effectively with fractions. The basic operations on fractions, can be improved by the operations of arithmetic fractions. In this session, we demonstrate very important properties of fractions. Also, we establish the basic algebraic operations with fractions.

2.1 Factors of a Monomial and a Multinomial

Factors of Monomial

We know that a monomial consists only one term.

$12a^3, 4a^2b^3, 2x^2y^2$ are examples for the monomials.

$$12a^3 \equiv 2^2.3^1.a^3$$

$$\downarrow \quad \downarrow \quad \downarrow$$

$$3 \quad 2 \quad 4$$

$$(2+1) \qquad (1+1) \qquad (1+3)$$

$12a^3$ have $3 \times 2 \times 4 = 24$ no. of factors that are

$\{1, 2, 3, 4, 6, a, a^2, a^3, 3a, 3a^2, 6a, 6a^2, 2a, 2a^2, 4a, 4a^2, 12, 12a, 12a^2, 2a^3, 3a^3, 4a^3, 6a^3, 12a^3\}$

$4a^2b^3 = 2^2.a^2.b^3 \Rightarrow 3 \times 3 \times 4$

$4a^2b^3$ have $3 \times 3 \times 4 = 36$ no. of factors that are

$\{1, 2, 4, a, 2a, 4a, a^2, 2a^2, 4a^2b, 2b, 4b, b^2, 2b^2, b^3, 2b^3, ab, 2ab, 4ab, ab^2, 2ab^2, 4ab^2, 2a^2b, 4a^2b, a^2b^2, 2a^2b^2, 4a^2b^2, b^3, 2b^3, 4b^3, ab^3, 2ab^3, 4ab^3, a^2b^3, 2a^2b^3, 4a^2b^3\}$

$2x^2y^2 = 2^1.x^2.y^2 \Rightarrow 2 \times 3 \times 3$

$2x^2y^2$ have $2 \times 3 \times 3 = 18$ no. of factors that are

$\{1, 2, x, 2x, x^2, 2x^2, xy, y, 2y, x^2y, 2x^2y, 2xy, y^2, 2y^2, xy^2, 2xy^2, x^2y^2, 2x^2y^2\}$

Factors of multinomial

$24a^2x^2 + 12xyab + 36by^2a$ is an example for multinomial.

$24a^2x^2 + 12xyab + 36by^2a$

$24a^2x^2 \equiv 2^3.3^1.a^2x^2$

$12xyab = 2^2.3\ xyab$

$36by^2a = 2^2.3^2\ by^2a$

$\therefore$ we can see that $2^2.3.a$ is common for three terms.

$2^2.3a\{2ax^2 + \ bxy + \ 3by^2\}$

$\therefore$ The factors of the expression $24a^2x^2 + 12xyab + 36b^2y^2$ are $12a$ and $(2ax^2 + bxy + 3by^2)$

Therefore, the expression can be divided each term of the expression without a remainder by $12a$.

Then the quotient is $(2ax^2 + bxy + 3by^2)$.

Thus,$12a$ and $(2ax^2 + bxy + 3by^2)$ are the factors of $24a^2x^2 + 12xyab + 36aby^2$.

For a multinomial, if one factor is a monomial, then use the following steps to find the factors.

Step 1: Inspect the terms of the multinomial and determine a monomial that will divide into all these terms. This is one factor.

Step 2: Divide the multinomial by this monomial and find the quotient. This result is the other factor.

Example 1

Factorize the following expressions.

a) $16a^2x - 32ax^2 + 64a^2x^2$

b) $5^2a^2b^3 - 10^2a^3b^2$

c) $52a^2b^2 - 26a^3b$

d) $50a^2b^3 + 100a^3b^2$

Solutions

a) $16a^2x - 32ax^2 + 64a^2x^2$

Step 01:

$2^4a^2x - 2^5ax + 2^6a^2x^2$

Monomial factor2^4ax

Step 02:

Divide the multinomial $16a^2x - 32ax^2 + 64a^2x^2$ by 2^4ax

The quotient obtained is $(a - 2x + 4ax)$

This is the other factor of the expression.

$$\therefore 16a^2x - 32ax^2 + 64a^2x^2 \equiv (16ax)(a - 2x + 4ax)$$

b) $5^2a^2b^3 - 10^2a^3b^2$

Step 01:

Monomial factor $5^2a^2b^2$

Step 02:

Divide the multinomial$5^2a^2b^3 - 10^2a^3b^2$ by $5^2a^2b^2$

The quotient obtained is $(b - 4a)$

This is the other factor of the expression.

$$\therefore 5^2a^2b^3 - 10^2a^3b^2 \equiv 5^2a^2b^2(b - 4a)$$

c) $52a^2b^2 - 26a^3b$

Step 01:

Monomial factor $26a^2b$

Step 02:

Divide the multinomial $52a^2b^2 - 26a^3b$ by $26a^2b$

The quotient obtained is $(2b - a)$

This is the other factor of the expression.

$$\therefore 52a^2b^2 - 26a^3b \equiv 26a^2b(2b - a)$$

d) $50a^2b^3 + 100a^3b^2$

Step 01:

Monomial factor $50a^2b^2$

Step 02:

Divide the multinomial $50a^2b^3 + 100a^3b^2$ by $50a^2b^2$

The quotient obtained is $(b + 2a)$

This is the other factor of the expression.

$$\therefore 50a^2b^3 + 100a^3b^2 \equiv 50a^2b^2(b + 2a)$$

Activity 1

Factorize the following multinomial by inspection and check the expressions by multiplication

I. $20x^2y + 10xy^2$
II. $72x^2yz + 108xy^2z + 36xyz^2$
III. $32x^2y + 64y^2z + 16z^2x$
IV. $64xy^3 + 128z^2x$
V. $100a^2(x + y) - 10a(x + y)^2$
VI. $(a + b)^3x^2 - (a + b)^2x^3$
VII. $25a^2b^3 - 100a^3b^2$
VIII. $75a^2p^3 - 25a^3p^2 + 15a^5$

2.2 Factors of a Trinomial Square

A trinomial square is one which is the square of a binomial.

Thus $(x^2 + 4xy + 4y^2)$ is a trinomial square. Because it is the square of the binomial$(x + 2y)^2$, its factors$(x + 2y)(x + 2y)$. You should be able to find out whether a trinomial is a perfect square or not.

Procedure to determine whether a trinomial is a perfect square.

A perfect trinomial square must have two positive terms, each of which is the square of a monomial.

They could be identified as

Example 2

I. $x^2 + 2ax + a^2 \equiv (x + a)^2$

II. $x^2 - 2ax + a^2 \equiv (x - a)^2$

III. $x^2 + 4ax + 4a^2 \equiv (x + 2a)^2$

IV. $x^2 - 4ax + 4a^2 \equiv (x - 2a)^2$

V. $x^2 - 6ax + 9a^2 \equiv (x - 3a)^2$

VI. $x^2 + 6ax + 9a^2 \equiv (x + 3a)^2$

VII. $9x^2 + 24ax + 16a^2 \equiv (3x + 4a)^2$

VIII. $9x^2 - 24ax + 16a^2 \equiv (3x - 4a)^2$

You can see that a perfect trinomial also must have one term either positive or negative which is the twice the product of the square roots of other two terms.

Thus $\{64a^4 - 48a^2b^2 + 9b^4\}$ is a perfect trinomial square because

$$64a^4 = (8a^2)^2$$

$$9b^4 = (3b^2)^2$$

$$48a^2b^2 = 2 \times 8a^2 \times 3b^2$$

$$\therefore 64a^4 - 48a^2b^2 + 9b^4 \equiv \{8a^2 - 3b^2\}^2$$

$$\equiv (8a^2 - 3b^2)(8a^2 - 3b^2)$$

Activity 2

(a) Determine which of the following are perfect trinomial squares. If it is a perfect square, then factorize and find the positive square root.

I. $y^2 + 4y + 4$

II. $y^2 + 4y + 4$

III. $16x^2 + 16x + 1$

IV. $16x^2y^2 - 8xy + 1$

V. $x^2 + 2\sqrt{2}x + 2$

VI. $49k^2 - 42k + 9$

(b) Find the missing term needed to make the following expressions of the form of perfect trinomial squares.($a^2 + 2ab + b^2$ or $a^2 - 2ab + b^2$)

Insert only parentheses.

I. $16q^2 - 24q + (\quad)$

II. $49y^2 + 70y + (\quad)$

III. $9x^2 + (\quad)xy + 25y^2$

IV. $16 - (\quad) + 121x^2y^2$

V. $64y^2 + (\quad) + 9x^2$

VI. $64y^2 - (\quad) + 9x^2$

VII. $(a+b)^2x^2 - (\quad) + (a-b)^2y^2$

VIII. $2a^2 - 2\sqrt{2}a + (\quad)$

IX. $64b^2 - (\quad) + 625k^2$

X. $169t^2 - (\quad) + 25q^2$

2.3 Factoring Trinomials

In the above sections, we have introduced the concepts of factorization. Trinomials of the types formed from $(px + qy)(kx + ly)$ products are important expressions to be factorized, and this section is devoted to them. Let us consider the product of $(x + p)(x + q)$

$$(x + p)(x + q) \equiv x^2 + (p + q)x + pq$$

The numbers p and q are formed by analyzing the coefficient of x and the constant (pq) in the expression to be factorized.

Example 3

Factorize

I. $x^2 + 6x + 5$

II. $x^2 - 9x - 22$

III. $x^2 + 6x - 7$

IV. $x^2 - 11x + 10$

V. $x^2 - 13x - 30$

VI. $x^2 - 2x - 24$

VII. $x^2 + 3x - 18$

VIII. $x^2 - x - 20$

Answers

I. $x^2 + 6x + 5$

Constant term 5

Middle term 6

$5 = 5 \times 1$ and $6 = 5 + 1$

$\therefore\ x^2 + 6x + 5 \equiv (x + 5)(x + 1)$

II. $x^2 - 9x - 22$

Constant term -22

Middle term -9

$-22 \quad = -2 \times 11 \quad$ and $\quad -2 + 11 = 9$

$\quad\quad = -11 \times 2 \quad\quad -11 + 2 = -9$

$\therefore\ x^2 - 9x - 22 \equiv (x - 11)(x + 2)$

III. $\quad x^2 + 6x - 7$

Constant term -7

Middle term 6

$-7 \quad = 7 \times -1 \quad$ and $\quad 7 + (-1) = 6$

$\quad\quad = -7 \times 1 \quad\quad -7 + 1 = -6$

$\therefore\ x^2 + 6x - 7 \equiv (x + 7)(x - 1)$

IV. $\quad x^2 - 11x + 10$

Constant term 10

Middle term -11

$10 \quad = 2 \times 5 \quad$ and $\quad 2 + 5 = 7$

$\quad\quad = -2 \times -5 \quad\quad -2 - 5 = -7$

$\quad\quad = 1 \times 10 \quad\quad 1 + 10 = 11$

$\quad\quad = -1 \times -10 \quad\quad -1 - 10 = -11$

$\therefore\ x^2 - 11x + 10 \equiv (x - 10)(x - 1)$

V. $\quad x^2 - 13x - 30$

Constant term -30

Middle term -13

$-30 \quad = -6 \times 5 \quad$ and $\quad -6 + 5 = -1$

$\quad\quad = 6 \times -5 \quad\quad 6 - 5 = 1$

$\quad\quad = 10 \times -3 \quad\quad 10 - 3 = 7$

$\quad\quad = -10 \times 3 \quad\quad -10 + 3 = -7$

$\quad\quad = 15 \times -2 \quad\quad 15 - 2 = 13$

$\quad\quad = -15 \times 2 \quad\quad -15 + 2 = -13$

$\therefore\ x^2 - 13x - 30 \equiv (x - 15)(x + 2)$

According to above method, we can factorize

VI. $\quad x^2 - 2x - 24 = (x - 6)(x + 4)$

VII. $\quad x^2 + 3x - 18 = (x + 6)(x - 3)$

VIII. $x^2 - x - 20 = (x-5)(x+4)$

Activity 3

Factorize each of the following expressions

I. $y^2 - 8y - 20$

II. $m^2 - 19m + 48$

III. $x^4 + x^2 - 90$

IV. $y^6 + 3y^3 - 108$

V. $x^2 - 3xy + 2y^2$

VI. $x^2 + 7x - 8$

Factorizing in the form $(ax^2 + bxy + cy^2)$

When Factorizing the expression $(ax^2 + bxy + cy^2)$, the coefficient a, we have to find the factors of a and the coefficient c, we have to find the factors of c. Among the factors of a and c, we have to select suitable values to set the middle term.

Example 4

Factorize each of the following expressions.

I. $6x^2 - 5xy - 6y^2$

II. $12x^2 + xy - 6y^2$

III. $20x^2 + 13xy - 15y^2$

IV. $27x^2 - 3xy - 2y^2$

V. $15x^2 - 11xy + 2y^2$

Answers

I. $6x^2 - 5xy - 6y^2$

Possible Combinations	Middle term
$(3x-2y)(2x+3y)$	$5xy$
$\mathbf{(3x+2y)(2x-3y)}$	$\mathbf{-5xy}$
$(6x-y)(x+6y)$	$35xy$
$(6x+y)(x-6y)$	$-35xy$

$\therefore 6x^2 - 5xy - 6y^2 \equiv (3x+2y)(2x-3y)$

Also $6x^2 + 5xy - 6y^2 \equiv (3x - 2y)(2x + 3y)$

$6x^2 + 35xy - 6y^2 \equiv (6x - y)(x + 6y)$

$6x^2 - 35xy - 6y^2 \equiv (6x + y)(x - 6y)$

II. $12x^2 + xy - 6y^2$

Possible Combinations	Middle term
$\boldsymbol{(3x - 2y)(4x + 3y)}$	$\boldsymbol{xy}$
$(3x + 2y)(4x - 3y)$	$-xy$

$\therefore 12x^2 + xy - 6y^2 \equiv (3x - 2y)(4x + 3y)$

III. $20x^2 + 13xy - 15y^2$

Possible Combinations	Middle term
$(2x + 5y)(10x - 3y)$	$44xy$
$(2x - 5y)(10x + 3y)$	$-44xy$
$(5x - y)(4x + 15y)$	$71xy$
$(5x + y)(4x - 15y)$	$-71xy$
$\boldsymbol{(5x - 3y)(4x + 5y)}$	$\boldsymbol{13xy}$
$(5x + 3y)(4x - 5y)$	$-13xy$

$\therefore 20x^2 + 13xy - 15y^2 \equiv (5x - 3y)(4x + 5y)$

IV. $27x^2 - 3xy - 2y^2$

Possible Combinations	Middle term
$(27x - 2y)(x + y)$	$25xy$
$(27x + 2y)(x - y)$	$-25xy$
$(3x - 2y)(9x + y)$	$-15xy$
$(3x + 2y)(9x - y)$	$15xy$
$(9x - 2y)(3x + y)$	$3xy$
$\boldsymbol{(9x + 2y)(3x - y)}$	$\boldsymbol{-3xy}$

$\therefore 27x^2 - 3xy - 2y^2 \equiv (9x + 2y)(3x - y)$

V. $15x^2 - 11xy + 2y^2$

Possible Combinations	Middle term
$(3x - 2y)(5x - y)$	$-13xy$

$(15x - y)(x - 2y)$	$-31xy$
$(15x - 2y)(x - y)$	$-17xy$
$\mathbf{(5x - 2y)(3x - y)}$	$\mathbf{-11xy}$

$$\therefore\ 15x^2 - 11xy + 2y^2 \equiv (5x - 2y)(3x - y)$$

Activity 4

Factorize each of the following expressions

I. $18x^2 + 9xy - 2y^2$

II. $12x^2 - 7xy - 10y^2$

III. $21x^2 + 23xy + 6y^2$

IV. $15x^2 - 31xy + 14y^2$

V. $8x^2 - 10xy - 33y^2$

VI. $15x^2 + 13xy - 20y^2$

VII. $6x^2 + 29xy + 28y^2$

VIII. $18x^2 - 27xy - 5y^2$

2.4 Factors of the difference of two squares

The expressions of the form $(A^2 - B^2)$ are called the difference of two squares.

$$(A + B)(A - B) = A^2 + BA - AB - B^2$$
$$= A^2 - B^2$$
$$\therefore\ (A^2 - B^2) \equiv (A + B)(A - B)$$

Using the above product, we can factorize the difference between two squares into two binomials in one step. One factor is the sum and the other is the difference of the square roots of these squares.

Example 5

Factorize the following expressions.

I. $16a^4 - 81b^4$

II. $(x + 2)^2 - x^2$

III. $(a + b - c)^2 - (a - b + c)^2$

IV. $a^2 - 2ab + b^2 - c^2$

V. $(x + y)^2 - (x - y)^2$

VI. $625x^8 - 81$

Answers

I.
$$\begin{aligned} 16a^4 - 81b^4 &= (4a^2)^2 - (9b^2)^2 \\ &= (4a^2 + 9b^2)(4a^2 - 9b^2) \\ &= (4a^2 + 9b^2)[(2a)^2 - (3b)^2] \\ &= (4a^2 + 9b^2)(2a + 3b)(2a - 3b) \end{aligned}$$

II.
$$\begin{aligned} (x+2)^2 - x^2 &= [(x+2) + x][(x+2) - x] \\ &= (2x + 2)(2) \\ &= 4(x + 1) \end{aligned}$$

III.
$$\begin{aligned} (a+b-c)^2 - (a-b+c)^2 &= [(a+b-c) + (a-b+c)] \\ &\quad\ [(a+b-c) - (a-b+c)] \\ &= (2a)(a+b-c-a+b-c) \\ &= 2a.2(b-c) \\ &= 4a(b-c) \end{aligned}$$

IV.
$$\begin{aligned} a^2 - 2ab + b^2 - c^2 &= (a-b)^2 - c^2 \\ &= (a-b+c)(a-b-c) \end{aligned}$$

V.
$$\begin{aligned} (x+y)^2 - (x-y)^2 &= [(x+y) + (x-y)][(x+y) - (x-y)] \\ &= 2x.2y \\ &= 4xy \end{aligned}$$

VI.
$$\begin{aligned} 625x^8 - 81 &= (25x^4)^2 - 9^2 \\ &= (25x^4 + 9)(25x^4 - 9) \\ &= (25x^4 + 9)[(5x)^2 - 3^2] \\ &= (25x^4 + 9)(5x + 3)(5x - 3) \end{aligned}$$

Activity 5

1) Factorize the following expressions
   a) $256x^4 - 81$
   b) $a^2 + 2ab + b^2 - c^2$
   c) $(x+4)^2 - (2x+1)^2$
   d) $\pi[(r+3)^2 - r^2]$
   e) $9(u+v)^2 - 4(u-v)^2$

f) $(2n+1)^2-(2n-1)^2$
g) $8(x+y)^2-18(x-y)^2$
h) 400^2-4^2
i) $12x^3-3xy^2$
j) $100x^2-9(x-y)^2$

2) Show that ifa given number " a"can be expressed as the difference between two squares, then find the factors.

3) If $n$is any integer greater than one, then show that $(n+1)^3-(n+1)$ is always divisible by6.

2.5 Factors of the sum and the difference of two cubes

Consider the products of $(a+b)(a^2-ab+b^2)$and $(a-b)(a^2+ab+b^2)$

$$(a+b)(a^2-ab+b^2) \equiv a^3-a^2b+ab^2+a^2b-ab^2+b^3$$
$$\equiv a^3+b^3$$
$$\therefore \boxed{a^3+b^3 \equiv (a+b)(a^2-ab+b^2)} \leftarrow \boxed{1}$$

$$(a-b)(a^2+ab+b^2) \equiv a^3+a^2b+ab^2-a^2b-ab^2-b^3$$
$$\equiv a^3-b^3$$
$$\therefore \boxed{a^3-b^3 \equiv (a-b)(a^2+ab+b^2)} \leftarrow \boxed{2}$$

$\boxed{1}$ gives us the factors of sum of two cubes, and $\boxed{2}$ gives us the factors of difference of two cubes.

Example 6

Factorize the following expressions.

I. $256a^6-4b^6$
II. $\frac{2}{3}\pi(r+dr)^3-\frac{2}{3}\pi r^3$
III. $(3x+2y)^3-64(x+y)^3$
IV. 100^3-1

Answers

I. $256a^6 - 4b^6$

$= 4[(4a^2)^3 - (b^2)^3]$

$= 4(4a^2 - b^2)(16a^4 + 4a^2b^2 + b^4)$

$= 4(2a + b)(2a - b)(16a^4 + 4a^2b^2 + b^4)$

II. $\frac{2}{3}\pi(r + dr)^3 - \frac{2}{3}\pi r^3$

$= \frac{2}{3}\pi[(r + dr)^3 - r^3]$

$= \frac{2}{3}\pi[r + dr - r][(r + dr)^2 + r(r + dr) + r^2]$

$= \frac{2}{3}\pi dr(3r^2 + 3rdr + dr^2)$

III. $(3x + 2y)^3 - 64(x + y)^3$

$= [3x + 2y]^3 - [4(x + y)]^3$

$= [3x + 2y - 4(x + y)][(3x + 2y)^2 + (3x + 2y).4(x + y)$
$+ 16(x + y)^2]$

$= -(x + 2y)(9x^2 + 12xy + 4y^2 + 12x^2 + 20xy + 8y^2 + 16x^2$
$+ 32xy + 16y^2)$

$= -(x + 2y)(37x^2 + 64xy + 28y^2)$

IV. $100^3 - 1^3 = (100 - 1)(100^2 + 100 + 1)$

$= 99(10000 + 101)$

$= 99 \times 10101$

Activity 6

Factorize the following expressions.

a) $(p + 3q)^3 - (p - 3q)^3$

b) $(x + y)^3 - x^3$

c) $\sin^3\theta + 1$

d) $\sin^6\theta - \cos^6\theta$

e) $(4x)^6 - 1$

2.6 Simplification of Algebraic Fractions

A fraction is an indicated division. Therefore $\frac{3}{7}, \frac{12}{13}, \frac{2x+y}{z}, \frac{x^2+2x+1}{(x-1)}$ are examples for the fractions. In general, $\frac{P}{Q}$ is a fraction and P (The part of the fraction above the line) is called the **numerator** and Q (The part of the fraction under the line) is called the **denominator.**

$$\frac{P}{Q} \quad \begin{matrix} \leftarrow \text{Numerator} \\ \leftarrow \text{Denominator} \end{matrix}$$

The **degree** of an algebraic expression (such as fraction) is determined by the largest total number of literal factors in any term of the expression.

$$4x^2y^2z + 3x^3y^2 + 2x^2y^3 + xyz$$

$xxyyz \qquad xxxyy \qquad xxyyyy \qquad xyz$

∴The largest totalnumber of literal factors is 5. Thus, the given expression is 5th degree.

A fraction whose numerator is of a smaller degree than its denominator is called a proper fraction. If the degree of the numerator is equal or greater than the degree of the denominator, then the fraction is called an improper fraction.

Proper fractions $\quad \frac{x+1}{x^2+1}, \frac{x^2+2x+3}{x^3+1}, \frac{1}{x-1}$

Improper fractions $\quad \frac{x^3+1}{x^2+1}, \frac{x^3+1}{x^3-1}, \frac{x^2+1}{x}$

The sum of an integral expression and a fraction is a mixed expression.

2.7 Basic Properties of Fractions

The basic properties of the algebraic fractions resemble those of arithmetic fractions.

a) Multiplying or dividing both the numerator and the denominator of a fraction by the same quantity (other than zero), does not change the fraction.

Thus $\frac{x^3+1}{y^2-1} = \frac{(x^3+1)k}{(y^2-1)k} = \frac{(x^3+1)/l}{(y^2-1)/l}$ where $l, m \neq 0$

b) Multiplying the numerator or diving the denominator of a fraction by a given quantity multiply the fraction by that quantity.

Thus $\frac{a^3b}{p^2q} \rightarrow \frac{a^3b\times c}{p^2q} = \frac{a^3b}{p^2q/c} = \left[\frac{a^3b}{p^2q}\right].c$

c) Dividing the numerator or multiplying the denominator of a factor divides the value of the fraction.

Thus $\frac{a^2b^3}{c^2d^3} \rightarrow \frac{a^2b^3/x}{c^2d^2} = \frac{a^2b^3}{c^2d^2\times x} = \left(\frac{a^2b^3}{c^2d^2}\right) \div x$

Two fractions are to be equivalent if one can be obtained from the other by use of the above properties of fractions. Thus

I. $\frac{ax}{2}$ and $\frac{15a^3x^3}{30a^2x^2}$

II. $\frac{32ab^2c}{7pq^2r} = \frac{64}{14}\frac{apb^2q^2rc}{p^2q^4r^2}$

III. $\frac{3a}{2c^2} = \frac{6a^2bc}{4c^3a^2b}$ are equivalent fractions.

2.8 Addition and Subtraction of Algebraic Fraction

In arithmetic, fractions are transformed to common denominator before to add or subtract. Similarly, literal fractions must be transformed into fractions having a common denominator before they are added or subtracted.

In the case of computation in arithmetic, the common denominator should be the least common denominator. In algebraic fraction the common denominator is called the least common multiple.

Least Common Multiple (LCM)

The least common multiple (LCM) of several algebraic expressions is the product of all factors of the various expression, each factor being taking the greatest number of times that it occurs in any one of the given expressions. In other words, the LCM is the smallest quantity which is divisible by all the several algebraic expressions.

Example 7

a) Find the LCM of

I. $12ax-9x^2, 6a^3-18a^2x^2$ and $18a^3+36a^2x+24ax^2$

II. $18a^2x^3-6a^3x^2, 6x^3a^2-54a^3x^2$

III. $x^3+ax^2-a^2x-a^3, x^3-a^2x, x^3-a^3$

b) Simplify

I. $\frac{4x^2-1}{12x^2}-\frac{5+2x}{10x}+\frac{1}{15}$

II. $\frac{5x}{2x+4}+\frac{3x-1}{x^2-4}+\frac{1}{x-2}$

III. $1+\frac{4x}{4x-1}+\frac{32x^2-4}{1-16x^2}-\frac{2}{16x^2-1}$

IV. $\frac{2(x-7)}{4x^2-8x+3}+\frac{2x+1}{2x^2+5x-3}$

V. $\frac{1}{x^2-5x+6}+\frac{1}{x^2-3x+2}$

VI. $\frac{2p}{p^2-q^2}+\frac{1}{q-p}+\frac{3}{2p^2+pq-3q^2}$

Answers

a)

I. $12ax-9x^2, 6a^3-18a^2x^2$ and $18a^3+36a^2x+24ax^2$

Factors $\quad 12ax-9x^2 = 3x(4a-3x)$

$6a^3-18a^2x^2 = 6a^2(a-3x^2)$

$18a^3+36a^2x+24ax^2 = 6a(3a^2+6ax+4x^2)$

$\therefore LCM = 6a^2x(4a-3x)(a-3x^2)(3a^2+6ax+4x^2)$

II. $18a^2x^3-6a^3x^2, 6x^3a^2-54a^3x^2$

Factors $\quad 18a^2x^3-6a^3x^2 = 6a^2x^2(3x-a)$

$6x^3a^2-54a^3x^2 = 6a^2x^2(x-9a)$

$\therefore LCM = 6a^2x^2(3x-a)(x-9a)$

III. $x^3 + ax^2 - a^2x - a^3, x^3 - a^2x, x^3 - a^3$

Factors $x^3 + ax^2 - a^2x - a^3$

$$= (x^3 - a^3) + ax^2 - a^2x$$
$$= (x-a)(x^2 + ax + a^2) + ax(x-a)$$
$$= (x-a)(x^2 + 2ax + a^2)$$
$$= (x-a)(x+a)^2$$

$x^3 - a^2x \quad = x(x^2 - a^2) = x(x-a)(x+a)$

$(x^3 - a^3) \quad = (x-a)(x^2 + ax + a^2)$

$\therefore LCM = x(x-a)(x+a)^2(x^2 + ax + a^2)$

b) Simplification

Step 1
Find the LCD (Least Common Denominator) which is the LCM of all the denominator

↓

Step 2
For each fraction divide the LCD by its own denominator and multiply both numerator and denominator by this

↓

Step 3
Write the numerators obtained in step 2 over a single denominator LCD

↓

Step 4

Remove the groupings sign collect terms and reduce fractions to lowest terms

Figure 2.8.1: Procedure of addition and subtraction

I. $\frac{4x^2-1}{12x^2} - \frac{5+2x}{10x} + \frac{1}{15}$

Step 1

$12x^2$	$10x$	15
$2^2.3x^2$	$2.5x$	3.5

$LCD = 2^2.3.5x^2 = 60x^2$

Step 2

$\frac{60x^2}{12x^2} \quad \frac{60x^2}{10x} \quad \frac{60x^2}{15}$

5 $\quad 6x \quad 4x^2$

$$\therefore \frac{4x^2-1}{12x^2}-\frac{5+2x}{10x}+\frac{1}{15}$$

$$\frac{5(4x^2-1)}{12x^2\times 5}-\frac{6x(5+2x)}{10x.6x}+\frac{4x^2}{60x^2}$$

$$\frac{20x^2-5}{60x^2}-\frac{(30x+12x^2)}{60x^2}+\frac{4x^2}{60x^2}$$

Step 3

$$\frac{20x^2-5-(30x+12x^2)+4x^2}{60x^2}$$

Step 4

$$\frac{20x^2-5-30x-12x^2+4x^2}{60x^2}$$

$$\frac{12x^2-30x-5}{60x^2}$$

$$\therefore \frac{4x^2-1}{12x^2}-\frac{5+2x}{10x}+\frac{1}{15}=\frac{12x^2-30x-5}{60x^2}$$

II. $\frac{5x}{2x+4}+\frac{3x-1}{x^2-4}+\frac{1}{x-2}$

$$=\frac{5x}{2(x+2)}+\frac{3x-1}{(x+2)(x-2)}+\frac{1}{x-2}$$

$$=\frac{5x(x-2)+2(3x-1)+2(x+2)}{2(x+2)(x-2)}$$

$$=\frac{5x^2-10x+6x-2+2x+4}{2(x+2)(x-2)}$$

$$=\frac{5x^2-2x+2}{2(x^2-4)}$$

III. $1+\frac{4x}{4x-1}+\frac{32x^2-4}{1-16x^2}-\frac{2}{16x^2-1}$

$$= 1 + \frac{4x}{4x-1}+\frac{32x^2-4}{(1-4x)(1+4x)}-\frac{2}{(4x-1)(4x+1)}$$

$$=\frac{(4x-1)(4x+1)+4x(4x+1)-(32x^2-4)-2}{(4x-1)(4x+1)}$$

$$=\frac{16x^2-1+16x^2+4x-32x^2+4-2}{(4x-1)(4x+1)}$$

$$= \frac{4x+1}{(4x-1)(4x+1)}$$

$$= \frac{1}{4x-1}$$

IV. $\frac{2(x-7)}{4x^2-8x+3} + \frac{2x+1}{2x^2+5x-3}$

$$= \frac{2(x-7)}{(2x-3)(2x-1)} + \frac{2x+1}{(2x-1)(x+3)}$$

$$= \frac{2(x-7)(x+3)+(2x+1)(2x-3)}{(2x-1)(2x-3)(x+3)}$$

$$= \frac{2(x^2-7x+3x-21)+(4x^2+2x-6x-3)}{(2x-1)(2x-3)(x+3)}$$

$$= \frac{6x^2-12x-45}{(2x-1)(2x-3)(x+3)}$$

V. $\frac{1}{x^2-5x+6}+\frac{1}{x^2-3x+2}$

$$= \frac{1}{(x-2)(x-3)}+\frac{1}{(x-2)(x-1)}$$

$$= \frac{x-1+x-3}{(x-1)(x-2)(x-3)}$$

$$= \frac{2x-4}{(x-1)(x-2)(x-3)}$$

$$= \frac{2}{(x-1)(x-3)}$$

VI. $\frac{2p}{p^2-q^2} + \frac{1}{q-p} + \frac{3}{2p^2+pq-3q^2}$

$$= \frac{2p}{(p-q)(p+q)} + \frac{1}{q-p} + \frac{3}{(2p+3q)(p-q)}$$

$$= \frac{2p(2p+3q)-(p+q)(2p+3q)+3(p+q)}{(p-q)(p+q)(2p+3q)}$$

$$= \frac{2p^2+6pq-2p^2-2pq-3pq-3q^2+3p+3q}{(p-q)(p+q)(2p+3q)}$$

$$= \frac{pq+3p+3q-3q^2}{(p-q)(p+q)(2p+3q)}$$

Activity 7

Simplify the following expression.

(a) $\frac{3x+2}{x^2-16}+\frac{x+5}{(x+4)^2}$

(b) $\frac{3y+6}{y^2-y-6}-\frac{12}{y^2-2y-3}$

(c) $\frac{a^2-2a}{a^2-a-2}-\frac{3a}{6a-4}+\frac{5a}{6a^2+2a-4}$

(d) $\frac{3}{8(a-x)}-\frac{1}{8(a+x)}-\frac{a-2x}{4(a^2+x^2)}$

(e) $\frac{3x-5}{3x^2-2x-5}-\frac{(3x+5)}{3x^2+2x-5}+\frac{2x^2}{x^2-1}$

(f) $\frac{a^2-2ab}{a^2+ab-6b^2}-\frac{ab-7b^2}{a^2-ab-42b^2}$

(g) $\frac{1}{2x^2+3x-2}-\frac{1}{3x^2+7x+2}-\frac{1}{6x^2-x-1}$

(h) $\frac{5y}{2(y-1)(y-3)}-\frac{15(y-1)}{16(y-3)(y-2)}-\frac{9(y+3)}{16(y+1)(y-2)}$

(i) $\frac{5(2b-3)}{11(6b^2+b-1)}+\frac{7b}{6b^2+7b-3}-\frac{12(3b+1)}{11(4b^2+8b+3)}$

(j) $\frac{3}{8(a+b)}-\frac{1}{8(a-b)}+\frac{a+2b}{4(a^2+b^2)}$

(k) $\frac{2x-1}{x^2+x}+\frac{2x+1}{x^2-x}+\frac{4x+2}{x-x^2}$

(l) $\frac{3-2m}{3+2m}+\frac{2m+3}{2m-3}+\frac{12}{4m^2-9}$

(m) $\frac{a}{(a-x)^2}+\frac{3a}{x^2+ax-2a^2}+\frac{1}{2a+x}$

(n) $\frac{a+b-c}{(a-b)(a-c)}+\frac{b+c-a}{(b-c)(b-a)}+\frac{c+a-b}{(c-a)(c-b)}$

(o) $\frac{m^2nr}{(m-n)(m-r)}+\frac{n^2rm}{(n-r)(n-m)}+\frac{r^2mn}{(r-m)(r-n)}$

Solutions of Activities

Activity 1

I. $20x^2y+10xy^2\equiv 10xy(2x+y)$

II. $72x^2yz+108xy^2z+36xyz^2\equiv 36xyz(2x+3y+z)$

III. $32x^2y+64y^2z+16z^2x\equiv 16(2x^2y+4y^2z+z^2x)$

IV. $64xy^3 + 128z^2x \equiv 64x(y^3 + z^2)$

V. $100a^2(x + y) - 10a(x + y)^2 \equiv 10a(x + y)[10a - (x + y)]$

VI. $(a + b)^3x^2 - (a + b)^2x^3 \equiv (a + b)^2x^2(a + b - x)$

VII. $25a^2b^3 - 100a^3b^2 \equiv 25a^2b^2(b - 4a)$

VIII. $75a^2p^3 - 25a^3p^2 + 15a^5 \equiv 5a^2(15p^3 - 5ap^2 + 3a^3)$

Activity 2

I. $(y + 2)^2$

II. Is not a perfect square

III. Is not a perfect square

IV. $(4xy - 1)^2$

V. $\left(x + \sqrt{2}\right)^2$

VI. $(7k - 3)^2$

(a)

I. $9;\ (4q - 3)^2$

II. $25;\ (7y + 5)^2$

III. $30;\ (3x + 5y)^2$

IV. $88xy;\ (4 - 11xy)^2$

V. $48xy;\ (8y + 3x)^2$

VI. $48xy;\ (8y - 3x)^2$

VII. $2(a + b)(a - b)xy;\ \{(a + b)x - (a - b)y\}^2$

VIII. $1;\ \left(\sqrt{2}a + 1\right)^2$

IX. $400bk;\ (8b - 25k)^2$

X. $130tq;\ (13t - 5q)^2$

Activity 3

I. $y^2 - 8y - 20 = (y - 10)(y + 2)$

II. $m^2 - 19m + 48 = (m - 16)(m - 3)$

III. $x^4 + x^2 - 90 = (x^2 - 9)(x^2 + 10)$

IV. $y^6 + 3y^3 - 108 = (y^3 + 12)(y^3 - 9)$

V. $x^2 - 3xy + 2y^2 = (x - 2y)(x - y)$

VI. $x^2 + 7x - 8 = (x + 8)(x - 1)$

Activity 4

I. $18x^2 + 9xy - 2y^2 \equiv (6x - y)(3x + 2y)$

II. $12x^2 - 7xy - 10y^2 \equiv (4x - 5y)(3x + 2y)$

III. $21x^2 + 23xy + 6y^2 \equiv (7x + 3y)(3x + 2y)$

IV. $15x^2 - 31xy + 14y^2 \equiv (5x - 7y)(3x - 2y)$

V. $8x^2 - 10xy - 33y^2 \equiv (4x - 11y)(2x + 3y)$

VI. $15x^2 + 13xy - 20y^2 \equiv (5x - 4y)(3x + 5y)$

VII. $6x^2 + 29xy + 28y^2 \equiv (2x + 7y)(3x + 4y)$

VIII. $18x^2 - 27xy - 5y^2 \equiv (3x - 5y)(6x + y)$

Activity 5

1)

a) $256x^4 - 81 \quad = (16x^2)^2 - 9^2$

$= (16x^2 + 9)(16x^2 - 9)$

$= (16x^2 + 9)[(4x)^2 - 3^2]$

$= (16x^2 + 9)(4x + 3)(4x - 3)$

b) $a^2 + 2ab + b^2 - c^2 = (a + b)^2 - c^2 = (a + b + c)(a + b - c)$

c) $(x + 4)^2 - (2x + 1)^2 \quad = [x + 4 + 2x + 1][x + 4 - (2x + 1)]$

$= (3x + 5)(3 - x)$

d) $\pi[(r + 3)^2 - r^2] \quad = \pi(r + 3 + r)(r + 3 - r)$

$= 3\pi(2r + 3)$

e) $9(u + v)^2 - 4(u - v)^2$

$= [3(u + v)]^2 - [2(u - v)]^2$

$= [3(u + v) + 2(u - v)][3(u + v) - 2(u - v)]$

$= (5u + v)(u + 5v)$

f) $(2n + 1)^2 - (2n - 1)^2$

$= [(2n + 1) + (2n - 1)][(2n + 1) - (2n - 1)]$

$= 4n.2$

$= 8n$

g) $8(x + y)^2 - 18(x - y)^2$

$= 2\{[2(x + y)]^2 - [3(x - y)]^2\}$

$= 2[2(x + y) - 3(x - y)][2(x + y) + 3(x - y)]$

$= 2(5y - x)(5x - y)$

h) $400^2 - 4^2 \quad = 4^2(100^2 - 1^2)$

$= 16(100 - 1)(100 + 1)$

$= 16 \times 99 \times 101$

i) $12x^3 - 3xy^2 \quad = 3x[(2x)^2 - y^2]$

$= 3x(2x + y)(2x - y)$

j) $100x^2 - 9(x-y)^2 \quad = [10x]^2 - [3(x-y)]^2$

$= [10x - 3(x-y)][10x + 3(x-y)]$

$= (7x+3y)(13x-3y)$

2) $91 \quad \equiv 100 - 9$

$= 10^2 - 3^2$

$= (10-3)(10+3)$

$= 7 \times 13$

3) $(n+1)^3 - (n+1) \quad \equiv (n+1)[(n+1)^2 - 1]$

$= (n+1)(n+1+1)(n+1-1)$

$= n(n+1)(n+2)$

This expression represents the product of three consecutive integers.

Activity 6

a) $(p+3q)^3 - (p-3q)^3$

$= [p+3q-(p-3q)][(p+3q)^2 + (p+3q)(p-3q) + (p-3q)^2]$

$= 6q(3p^2 + 9q^2)$

b) $(x+y)^3 - x^3$

$= [x+y-x][(x+y)^2 + x(x+y) + x^2]$

$= y(3x^2 + 3xy + y^2)$

c) $\sin^3\theta + 1 = (\sin\theta + 1)(\sin^2\theta - \sin\theta + 1)$

d) $\sin^6\theta - \cos^6\theta$

$= (\sin^2\theta)^3 - (\cos^2\theta)^3$

$= (\sin^2\theta - \cos^2\theta)(\sin^4\theta + \sin^2\theta\cos^2\theta + \cos^4\theta)$

$= (\sin\theta - \cos\theta)(\sin\theta + \cos\theta)(\sin^4\theta + \sin^2\theta\cos^2\theta + \cos^4\theta)$

e) $(4x)^6 - 1 \quad = [(4x)^2]^3 - 1^3$

$= [(4x)^2 - 1][(4x)^4 + (4x)^2 + 1]$

$= (4x-1)(4x+1)(256x^4 + 16x^2 + 1)$

Activity 7

(a) $\frac{4x^2+5x+28}{(x-4)(x+4)^2}$

(b) $\frac{3}{y+1}$

(c) $\frac{a}{2(a+1)}$

(d) $\frac{ax(2a+x)}{2(a^4-x^4)}$

(e) 2

(f) $\frac{a^2+5ab-3b^2}{(a+3b)(a+6b)}$

(g) 0

(h) $\frac{1}{y+1}$

(i) $\frac{1}{2b+1}$

(j) $\frac{a^2-2b^2}{2(a^4-b^4)}$

(k) $\frac{4}{x+1}$

(l) $\frac{12(2m+1)}{4m^2-9}$

(m) $\frac{x}{(a-x)^2}$

(n) $\frac{2(bc+ca+ab-a^2-b^2-c^2)}{(a-b)(b-c)(c-a)}$

(o) 0

Summary

The factors of the multinomial can be written as

I. $abx + acy \equiv a(bx + cy)$

II. $x^2 + 2xy + y^2 \equiv (x + y)^2$

III. $x^2 - 2x + y^2 \equiv (x - y)^2$

IV. $x^2 - y^2 \equiv (x - y)(x + y)$

V. $x^2 + (m + l)x + ml \equiv (x + m)(x + l)$

VI. $x^2 - (m + l)x + ml \equiv (x - m)(x - l)$

VII. $x^2 - (m - l)x - ml \equiv (x - m)(x + l)$

VIII. $x^3 + y^3 \equiv (x + y)(x^2 - xy + y^2)$

IX. $x^3 - y^3 \equiv (x - y)(x^2 + xy + y^2)$

X. $abx^2 + (ad + bc)xy + cdy^2 \equiv (ax + cy)(bx + dy)$

- A fraction is an indicated division.
  Thus $\frac{P}{Q}$ where P is the numerator and Q is the denominator.
- A fraction whose numerator is of a smaller degree than its denominator is a proper fraction. If the degree of the numerator is equal or greater than the degree of the denominator, then the fraction is improper fraction.
- The Least Common Multiple (LCM) is the smallest quantity, which is divisible by all the several algebraic expressions.

Learning Outcomes

On completion of this study session you should be able to

- Find factors of monomial, multinomial, Trinomial Square, difference of two squares and sum and difference of two cubes.
- Simplify the algebraic fractions.

Session 3
Solving Simple Equations and Systems of Linear Equations.

Introduction

Most of the problems in the engineering field, which can be expressed using Algebra, involve equations in various forms. Solving of questions is one of the important methods in mathematics. Some equations are easy to solve, but some are very complex. Solving of equations efficiently requires a lot of time and practice. But after acquiring this skill, you will be able to solve different types of equations. Also, it is important to understand that analysis of any physical applications or studying of other topics in mathematics require knowledge of solving various types of equations.

3.1 Axioms for Solving Simple Equations

To solve an equation, we find the values of the unknown that satisfy it. There is a basic rule to follow when solving an equation. **Perform the same mathematical operation on both sides of the equation.** When we do this, isolate the unknown and thus to find the values. The following axioms are frequently used in solving equations. You should study and understand each one.

(a) If equal quantities are added to equal numbers, then the sums are equal.

Eg: $x - 3 = 5 \Longrightarrow (x - 3) + 3 = 5 + 3.$

$\therefore x = 8.$

(b) If equal quantities are subtracted from equal quantities, then the remainders are equal.

Eg: $x + 13 = 23 \Longrightarrow (x + 13) - 13 = 23 - 13$

$\therefore x = 10.$

(c) If equal quantities are multiplied by equal quantities, then the products are equal.

Eg: $\frac{x}{9} = 8 \Longrightarrow \frac{x}{9} \times 9 = 8 \times 9$

$\therefore x = 72$

(d) If equal quantities are divided by equal quantities (other than zero) then the results are equal.

Eg: $4x = 16 \Longrightarrow 4x \div 4 = 16 \div 4$

$\therefore x = 4$

(e) Quantities that are equal to the same quantities or equal quantities are equal to each other. Receive

$x = z$, and $y = z$ then $x = y$.

Example 1

Solve the following equations.

(i) $\frac{3}{2}x + 8 = 3x - 3$

(ii) $\frac{3}{2}x - 1 = 2x - 5$

(iii) $3x - 6 = 7x - 16$

(iv) $2x - 9 = 7x - 19$

(v) $4x + 3 = 2x + 17$

(vi) $6(x - 3) - 13(x - 2) = 1$

(vii) $\frac{3x-13}{7}+\frac{11-4x}{3}=0$

Solutions

(i)
$$
\begin{aligned}
\frac{3}{2}x+8 &= 3x-3\\
\frac{3}{2}x+8-3x-8 &= 3x-3-3x-8\\
\frac{3}{2}x-3x &= -11\\
\left(\frac{3}{2}-3\right)x &= -11\\
\left(\frac{3-6}{2}\right)x &= -11\\
-\frac{3}{2}x &= -11\\
-\frac{3}{2}x\times\frac{-2}{3} &= -11\times\frac{-2}{3}\\
x &= \frac{22}{3}
\end{aligned}
$$

(ii)
$$
\begin{aligned}
\frac{3}{2}x-1 &= 2x-5\\
\frac{3}{2}x-1+1-2x &= 2x-5-2x+1\\
\frac{3}{2}x-2x &= -4\\
\frac{3x-2\times 2x}{2} &= -4\\
\frac{-1}{2}x &= -4\\
\frac{-1}{2}\times(-2)x &= -4\times -2\\
x &= 8
\end{aligned}
$$

(iii)
$$
\begin{aligned}
3x-6 &= 7x-16\\
3x-6+6-7x &= 7x-16+6-7x\\
-4x &= -10\\
-4x\times\frac{-1}{4} &= -10\times\frac{-1}{4}\\
x &= \frac{10}{4} = \frac{5}{2}
\end{aligned}
$$

(iv)
$$
\begin{aligned}
2x-9 &= 7x-19\\
2x-7x &= -19+9\\
-5x &= -10\\
x &= 2
\end{aligned}
$$

(v)
$$
\begin{aligned}
4x+3 &= 2x+17\\
4x-2x &= 17-3
\end{aligned}
$$

$$2x = 14$$

$$x = \frac{14}{2} = 7$$

(vi) $$6(x-3) - 13(x-2) = 1$$

$$6x - 18 - 13x + 26 = 1$$

$$6x - 13 = 1 + 18 - 26$$

$$-7x = -7$$

$$x = 1$$

(vii) $$\frac{3x-13}{7} + \frac{11-4x}{3} = 0$$

$$3(3x-13) + 7(11-4x) = 0$$

$$9x - 39 + 77 - 28x = 0$$

$$9x - 28x = 39 - 77$$

$$-19x = -38$$

$$x = \frac{-38}{-19} = 2$$

Activity 1

Solve the following equations

(i) $20(7x+4) - 18(3x+4) - 5 = 25(x+5)$

(ii) $\frac{7x+2}{5} = \frac{4x-1}{2}$

(iii) $18 - 5(x+1) = 3(x-1)$

(iv) $\frac{1}{3}(x+1) + \frac{1}{4}(x+3) = \frac{1}{5}(x+4) + 16$

(v) $6(x-1) - (3x+11) + 7 = 0$

(vi) $\frac{7x-4}{15} + \frac{x-1}{3} = \frac{3x-1}{5} - \frac{x+7}{10}$

(vii) $\frac{3}{2}(x-1) - \frac{2}{3}(x+2) + \frac{1}{4}(x-3) = 4$

(viii) $\frac{x-2}{4} - \frac{2x-5}{4} - 1 + \frac{3}{20}x = 0$

Example 2

Solve the following equations.

(i) $(x+1)(x+2) = x(x+7) - 6$

(ii) $(x+1)^2 + (x-2)^2 = 2x^2 - 5$

(iii) $(x-5)^2 - 4(3-x) = 8x + (x+2)^2$

(iv) $\frac{(3x-4)(3x+1)}{3} - \frac{(8x-11)(x+1)}{4} = \frac{(6x-1)(2x-3)}{12}$

(v) $(x-3)(x-4) - 2x(x-3) = x(11-x)$

Solution

(i)
$$\begin{aligned}(x+1)(x+2) &= x(x+7) - 6\\ x^2 + x + 2x + 2 &= x^2 + 7x - 6\\ (7x - 3x) &= 2 + 6\\ 4x &= 8\\ x &= 2\end{aligned}$$

(ii)
$$\begin{aligned}(x+1)^2 + (x-2)^2 &= 2x^2 - 5\\ x^2 + 2x + 1 + x^2 - 4x + 4 &= 2x^2 - 5\\ 2x &= 10\\ x &= 5\end{aligned}$$

(iii)
$$\begin{aligned}(x-5)^2 - 4(3-x) &= 8x + (x+2)^2\\ x^2 - 10x + 25 - 12 + 4x &= 8x + x^2 + 4x + 4\\ -18x &= -25 + 12 + 4\\ 18x &= 9\\ x &= \frac{1}{2}\end{aligned}$$

(iv)
$$\begin{aligned}\frac{(3x-4)(3x+1)}{3} - \frac{(8x-11)(x+1)}{4} &= \frac{(6x-1)(2x-3)}{12}\\ 4(3x-4)(3x+1) - 3(8x-11)(x+1) &= (6x-1)(2x-3)\\ 4[9x^2 - 9x - 4] - 3[8x^2 - 3x - 11] &= 12x^2 - 20x + 3\\ 36x^2 - 36x - 16 - 24x^2 + 9x + 33 &= 12x^2 - 20x + 3\\ (-36 + 9 + 20)x &= 16 - 33 + 3\\ -7x &= -14\\ x &= 2\end{aligned}$$

(v)
$$\begin{aligned}(x-3)(x-4) - 2x(x-3) &= x(11-x)\\ x^2 - 3x - 4x + 12 - 2x^2 + 6x &= 11x - x^2\\ (6 - 3 - 4 - 11)x &= -12\\ -12x &= -12\\ x &= 1\end{aligned}$$

Activity 2

Solve the following equations

(i) $3x(2x+1) - 11x = 6(x+7)(x-8) + 320$

(ii) $(3x+4)(4x-1) - (7x-2)(x+1) = (5x-3)(x-2) - 1$

(iii) $(3x-2)(2x-3) - (2x-1)(x-2) = (2x-3)^2 - 6x$

(iv) $\frac{1}{6}(2x+9) - \frac{1}{10}(x^2-1) = \frac{3}{20}x - \frac{1}{10}(x-5)(x+3)$

(v) $3 + \frac{(2x-1)(3x-2)}{9} - \frac{x^2}{3} = \frac{x^2-2}{3}$

3.2 Systems of Linear Equations

The equations, so far, we have encountered were examples of equations called linear equations. In general, an equation is called linear in a given set of variables (unknowns), if each term contains only one variable to the first power or a constant.

For example, $ax + by + cz = d$ is a linear equation.

But $ax^2 + bx + c = 0$ and $axy + bz + c = 0$, where x, y and z varables are not linear equations.

An equation that can be written in the form of $ax + by + cz = d$ is known as a linear equation with three unknowns.

Many problems in engineering involve the solutions of system of linear equations.

Systems of linear equations are also known as simultaneous linear equations.

3.3 Solving Systems of Two Linear Equations with Two Unknowns

Let us consider the following two linear equations each containing the same two unknowns x and y.

$$a_1x + b_1y = c_1$$

$$a_2x + b_2y = c_2.$$

A solution of this system is any pair of values (x, y) that satisfies both equations. There are different methods to find this solution.

We discuss two methods of solving systems of linear equations, namely, graphical method and algebraic method.

Method 1: Graphical Method

When we solve two simultaneous linear equations in two unknowns graphically, we have to plot each line and determine whether they intersect or not. If they intersect, then the coordinate of the point of intersection is the unique solution of corresponding two simultaneous linear equations. If they do not intersect in other words they are parallel, then the two simultaneous linear equations have no unique solution have infinite number of solutions. This method may of course lead to approximate the values if the lines cross at values between those chosen to determine the graph.

Example 3

Solve the equations $3x - 2y = 6$, $x + 2y = 6$

$3x - 2y = 6$	$x + 2y = 6$
$x = 0, y = -3$	$x = 0, y = 3$
$y = 0, x = 2$	$y = 0, x = 6$
$A \equiv (0,-3), B \equiv (2,0)$	$A \equiv (0,3), B \equiv (6,0)$

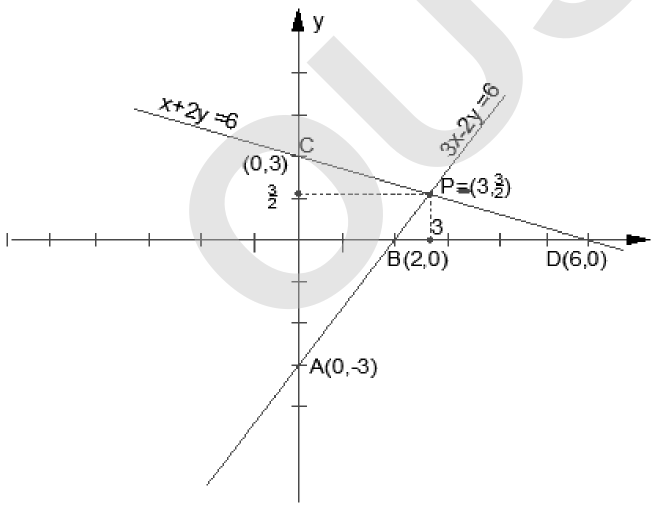


Figure 3.3.1 Graph of straight lines

The line AB represents $3x - 2y = 6$ and the line CD represents

$x + 2y = 6$. We have the intersecting point of the above lines

$P \equiv \left(3, \frac{3}{2}\right)$.

Thus, the solution of the system of equations $x + 2y = 6$ and $3x - 2y = 6$ is $x = 3$ and $y = \frac{3}{2}$

Example 4: Solve the system of equations $4x + y = 4$, $4y - x = 4$.

$4x + y = 4$

$x = 0, y = 4, \ A \equiv (0,4)$

$y = 0, x = 1, \ B \equiv (1,0)$

$4y - x = 4$

$y = 0, x = -4, C \equiv (-4,0)$

$x = 0, y = 1, D \equiv (0,1)$

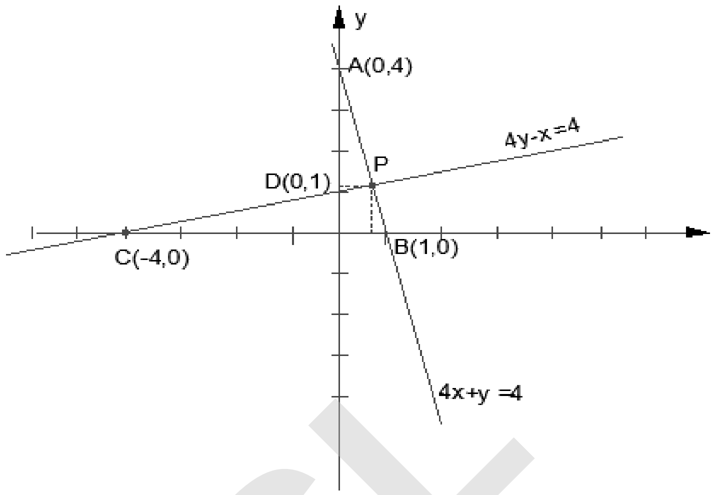


Figure 3.3.2 Graph of the straight Lines

The equation of the line AB represents by $4x + y = 4$ and the equation of the line CD represents by $4y - x = 4$ $i.e -x + 4y - 4 = 0$.. We have the intersecting point $P \equiv (0.88, \ 1.17)$. $\therefore$ The solution of the above two equations is approximately $x = 0.88$ and $y = 1.17$.

Method 2: Algebraic Method

Now, we know that, finding a solution for a system of equations by the graphical method, the results are usually approximate. If exact solutions are required, we must go to the other methods. Now we are going to discuss the Algebraic method of finding the solutions of a system of equations.

(a) Substitution Method

Example 5

Solve the system equation

(I) $4x + y = 16$

$2x - y = 2$

(II) $5x - 3y = 12$

$x + 3y = 12$

Solutions

(I) $4x + y = 16 \longrightarrow (1)$

$2x - y = 2 \quad \longrightarrow (2)$

$\therefore$ from (2), $\; y = 2x - 2$

Substitute to equation (1)

$4x + (2x - 2) = 16$

$6x \;\; = 18$

$x \;\; = 3$

$\therefore y = 2(3) - 2 = 4$

$\therefore$ the solution is $x = 3, y = 4$.

(II) $5x - 3y = 12 \longrightarrow (1)$

$x + 3y = 12 \;\longrightarrow (2)$

From (2), $x = 12 - 3y$.

Substitute to equation (1)

$5(12 - 3y) - 3y \;= 12$

$60 - 15y - 3y = 12$

$18y = 48$

$y = \frac{48}{18} = \frac{8}{3}, \quad x = 12 - 8 = 4$

$\therefore x = 4, y = \frac{8}{3}$

(b) Equating numerical coefficient method

By this method, we multiply each equation by a chosen suitable number, so that the coefficient of one of the unknowns will be numerically the same in both equations.

If these numerically equal coefficients have the same sign, we subtract one equation from the other. If the numerical equal coefficients have opposite signs, we add the two equations. After adding or subtracting, we have a simple linear equation with one known, which we then solve for the unknown. We substitute this value into the one of the equations to obtain the value of the other unknown.

Example 6

Solve the following system of equations.

(a) $4x + 3y = 10$

$2x + y = 4$

(b) $4x - 8y = 32$

$6x + 3y = -3$

Answers

(a) $4x + 3y = 10 \longrightarrow (1)$

$2x + \ \ y = \ \ 4 \ \longrightarrow (2)$

$(2) \times 3 \quad 6x + 3y = 12 \ \longrightarrow (3)$

$(3) - (1) \ \ 2x = 2$

$x = 1$

$\therefore$ From (2)

$2 \cdot 1 + y = 4$

$y = 4 - 2$

$y = 2.$

(b) $4x - 8y = 32 \ \longrightarrow (1)$

$6x + 3y = -3 \ \longrightarrow (2)$

$(1) \times 3 \quad 12x - 24y = 96 \ \longrightarrow (3)$

$(2) \times 8 \quad 48x + 24y = -24 \longrightarrow (4)$

$(3) + (4) \ \ 60x = 72$

$x = \frac{72}{60} = \frac{6}{5}$

From (2) $6 \times \frac{6}{5} + 3y = -3$

$3y \quad = -3 - \frac{36}{5}$

$3y \quad = \frac{-15-36}{5} = -\frac{51}{5}$

$y = -\frac{51}{15} \ = -\frac{17}{5}$

$\therefore$ The solution is $x = \frac{6}{5}, y = -\frac{17}{5}$

We can solve the general system of linear equations with two unknowns in the following method.

The linear system of equations

$$a_1x + b_1y = c_1 \longrightarrow (1)$$

$$a_2x + b_2y = c_2 \longrightarrow (2)$$

Substitution method

From equation $a_1x = c_1 - b_1y$, $x = \left(\frac{c_1 - b_1y}{a_1}\right)$

$\therefore$ the value of x, substitute in to the equation (2)

$$a_2\left(\frac{c_1 - b_1y}{a_1}\right) + b_2y = c_2$$

$$(a_1b_2 - a_2b_1)y = (a_1c_2 - a_2c_1)$$

$$y = \frac{(a_1c_2 - a_2c_1)}{(a_1b_2 - a_2b_1)}$$

$$x = \frac{1}{a_1}\left\{c_1 - b_1\frac{(a_1c_2 - a_2c_1)}{(a_1b_2 - a_2b_1)}\right\}$$

$$= \frac{1}{a_1}\left\{\frac{a_1b_2c_1 - a_2b_1c_1 - a_1b_1c_2 + a_2b_1c_1}{(a_1b_2 - a_2b_1)}\right\}$$

$$x = \frac{(b_2c_1 - b_1c_2)}{(a_1b_2 - a_2b_1)}.$$

$\therefore$ the solutions, $x = \frac{(b_2c_1 - b_1c_2)}{(a_1b_2 - a_2b_1)}$, $y = \frac{(a_1c_2 - a_2c_1)}{(a_1b_2 - a_2b_1)}$

where $a_1b_2 - a_2b_1 \neq 0$.

If $a_1b_2 - a_2b_1 = 0$, then there is no unique solution for the given system of equations.

By using method of equating numerical coefficients;

$$a_1x + b_1y = c_1 \longrightarrow (1)$$

$$a_2x + b_2y = c_2 \longrightarrow (2)$$

$$(1) \times b_2 \quad a_1b_2x + b_1b_2y = b_2c_1 \longrightarrow (3)$$

$$(1) \times b_1 \quad a_2b_1x + b_1b_2y = b_1c_2 \longrightarrow (4)$$

$$(3) - (4) \quad (a_1b_2 - a_2b_1)x = b_2c_1 - b_1c_2$$

$$x = \frac{(b_2c_1 - b_1c_2)}{(a_1b_2 - a_2b_1)}.$$

$\therefore$ now substitute the value of x into the equation (1)

$$a_1\left(\frac{b_2c_1-b_1c_2}{a_1b_2-a_2b_1}\right)+b_1y \qquad = c_1$$

$$b_1y \; = \frac{c_1(a_1b_2-a_2b_1)-a_1(b_2c_1-b_1c_2)}{a_1b_2-a_2b_1}$$

$$y = \frac{(a_1c_2-a_2c_1)}{(a_1b_2-a_2b_1)}$$

$$\therefore \text{ solution } x = \frac{(b_2c_1-b_1c_2)}{(a_1b_2-a_2b_1)};\; y = \frac{(a_1c_2-a_2c_1)}{(a_1b_2-a_2b_1)};\; a_1b_2 - a_2b_1 \neq 0.$$

Activity 3

Solve the following system of equations by using

(a) The method of substitution and

(b) Elimination by equating numerical coefficients;

I. $x + 2y = 4$
$2x + 3y = 7$

II. $2x + y = 23$
$4x - y = 19$

III. $5y - 3x = 85$
$12y + 5x = 21$

IV. $3x + 2y = 118$
$x + 5y = 191$

V. $12x + 9 = y$
$18y - 5x = 56\frac{1}{2}$

VI. $23x - 11y = 1$
$15x + 7y = 29$

VII. $14x + 9y = 19$
$9x + 14y = 4$

VIII. $13x + 11y = 70$
$11x + 13y = 74$

IX. $9x - 11y = 15$
$7x - 13y = 25$

3.4 Solving Systems of Three Linear Equations with Three Unknowns

There are systems of linear equations that consist of three, four and even many unknowns. Now, we are going to discuss the algebraic method of solving a system of three linear equations with three unknowns.

Any system of three linear equations can be written as follows.

$$a_1x + b_1y + c_1z = d_1$$

$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

The solution of this system is the values of x, y and z which satisfy all the three equations simultaneously. The method of solving involves multiplying two of the equations by the suitable numbers, to eliminate one of the unknowns in these equations. We then repeat this process, using a different pair of the original equations, being sure that we eliminate the same unknown as we did between the first pair of equations. Then we have two linear equations with two unknowns and it can be solved by any one of the methods previously discussed. The unknowns originally eliminated, may then be found by substituting into one of the original equations.

Example 7

(a) $4x - 5y + 6z = 3$

$8x - 7y - 3z = 9$ $7x -$

$8y + 9z = 6$

(b) $5z - 3x = 4(1 + y)$

$2(x + 2z) = 8 + 3y$

$2y + 3z = 14 - x$

Solutions

(a) $4x - 5y + 6z = 3 \longrightarrow (1)$

$8x - 7y - 3z = 9 \longrightarrow (2)$

$7x - 8y + 9z = 6 \longrightarrow (3)$

$(2) \times 2;\ 16x - 14y - 6z = 18 \longrightarrow (4)$

$(2) \times 3;\ 24x - 21y - 9z = 27 \longrightarrow (5)$

$(1) + (4);\ 20x - 19y = 21 \longrightarrow (6)$

$(3) + (5);\ 31x - 29y = 33 \longrightarrow (7)$

$(6) \times 29;\ 580x - 551y = 609 \longrightarrow (8)$

$(7) \times 19;\ 589x - 551y = 627 \longrightarrow (9)$

$(9) - (8);\ 9x = 18$

$\therefore x = 2.$

From (6)

$$20 \times 2 - 19y = 21$$
$$19y = 40 - 21$$
$$y = 1$$

$\therefore$ from (1)

$$4 \times 2 - 5 \times 1 + 6z = 3$$

$$6z = 3 + 5 - 8$$

$$z = 0$$

$\therefore$ the solution of the system is $x = 2$, $y = 1$, $z = 0$.

(b) $5z - 3x = 4(1 + y)$

$5z - 3x - 4y = 4 \longrightarrow (1)$

$2(x + 2z) = 8 + 3y$

$2x - 3y + 4z = 8 \longrightarrow (2)$

$2y + 3z = 14 - x$

$2y + 3z + x = 14 \longrightarrow (3)$

$(3) \times 2;\ 4y + 6z + 2x = 28 \longrightarrow (4)$

$(4) - (2);\ 7y + 2z = 20 \longrightarrow (5)$

$(3) \times 3;\ 6y + 9z + 3x = 42 \longrightarrow (6)$

$(1) + (6);\ 2y + 14z = 46 \longrightarrow (7)$

$(5) \times 7;\ 49y + 14z = 140 \longrightarrow (8)$

$(8) - (7);\ 47y = 94$

$y = 2$

$\therefore$ from (6) $7 \times 2 + 2z = 20$

$2z = 6$

$z = 3$

$\therefore$ from (3) $x = 14 - 2y - 3z$

$= 14 - 2 \times 2 - 3 \times 3$

$x = 1$

Solution $x = 1$, $y = 2$, $z = 3$.

Activity 4

Solve the following systems of equations

I. $2x + y + z = 8$
$5x - 3y + 2z = 3$
$7x + y + 3z = 20$

II. $5x - 4y + z = 3$
$3x + y - 2z = 31$
$x + 4y + z = 15$

III.
$$\begin{aligned} 4x - 5y + 6z &= 3 \\ 8x - 7y - 3z &= 9 \\ 7x - 8y + 9z &= 6 \end{aligned}$$

IV.
$$\begin{aligned} x + y &= 6 \\ y + z &= 10 \\ z + x &= 20 \end{aligned}$$

V.
$$\begin{aligned} 7x + 5y - 7z &= -8 \\ 4x + 2y - 3z &= 0 \\ 5x - 4y + 4z &= 35 \end{aligned}$$

VI.
$$\begin{aligned} 3x + y - z &= 3 \\ 2x - y + 3z &= 20 \\ 7x + y + z &= 23 \end{aligned}$$

Solutions of Activities

Activity 1

(I)
$$\begin{aligned} 20(7x + 4) - 18(3x + 4) - 5 &= 25(x + 5) \\ 140x + 80 - 54x - 72 - 5 &= 25x + 125 \\ (140 - 54 - 25)x &= 125 + 72 + 5 - 80 \\ 61x &= 122 \\ x &= 2 \end{aligned}$$

(II)
$$\begin{aligned} \frac{7x+2}{5} &= \frac{4x-1}{2} \\ 2(7x + 2) &= 5(4x - 1) \\ (20 - 14)x &= 4 + 5 \\ 6x &= 9 \\ x &= \frac{3}{2} \end{aligned}$$

(III)
$$\begin{aligned} 18 - 5(x + 1) &= 3(x - 1) \\ 5(x + 1) - 3(x - 1) &= 18 \\ (5 - 3)x &= 18 - 3 - 5 \\ 2x &= 10 \\ x &= 5 \end{aligned}$$

(IV)
$$\begin{aligned} \frac{1}{3}(x + 1) + \frac{1}{4}(x + 3) &= \frac{1}{5}(x + 4) + 16 \\ 20(x + 1) + 15(x + 3) &= 12(x + 4) + 60 \times 16 \\ (20 + 15 - 12)x &= 960 + 48 - 20 - 45 \\ 23x &= 943 \\ x &= 41 \end{aligned}$$

(V)
$$\begin{aligned} 6(x - 1) - (3x + 11) + 7 &= 0 \\ (6 - 3)x &= 6 + 11 - 7 \end{aligned}$$

$$3x = 10$$

$$x = \frac{10}{3}$$

(VI) $$\frac{7x-4}{15} + \frac{x-1}{3} = \frac{3x-1}{5} - \frac{x+7}{10}$$

$$2(7x-4) + 10(x-1) = 6(3x-1) - 3(x+7)$$

$$14x + 10x - 18x + 3x = 8 + 10 - 6 - 21$$

$$9x = -9$$

$$x = -1$$

(VII) $$\frac{3}{2}(x-1) - \frac{2}{3}(x+2) + \frac{1}{4}(x-3) = 4$$

$$18(x-1) - 8(x+2) + 3(x-3) = 48$$

$$(18-8+3)x = 18+16+9+48$$

$$13x = 91$$

$$x = 7$$

(VIII) $$\frac{x-2}{4} - \frac{2x-5}{4} - 1 + \frac{3}{20}x = 0$$

$$5(x-2) - 5(2x-5) - 20 + 3x = 0$$

$$(5-10+3)x = 10-25+20$$

$$-2x = 5$$

$$x = -\frac{5}{2}$$

Activity 2

(i) $$3x(2x+1) - 11x = 6(x+7)(x-8) + 320$$

$$6x^2 + 3x - 11x = 6(x^2 + 7x - 8x - 56) + 320$$

$$(3 - 42 + 48 - 11)x = 320 - 336$$

$$-2x = -16$$

$$x = 8$$

(ii) $$(3x+4)(4x-1) - (7x-2)(x+1) = (5x-3)(x-2) - 1$$

$$12x^2 + 16x - 3x - 4 - (7x^2 - 2x + 7x - 2) = 5x^2 - 3x - 10x + 6 - 1$$

$$(16 - 3 + 2 - 7 + 3 + 10)x = 4 - 2 + 6 - 1$$

$$21x = 7$$

$$x = \frac{1}{3}$$

(iii) $$(3x-2)(2x-3) - (2x-1)(x-2) = (2x-3)^2 - 6x$$

$$6x^2 - 4x - 9x + 6 - [2x^2 - x - 4x + 2] = 4x^2 - 12x +$$

$9 - 6x$

$$
\begin{aligned}
-(4+9-1-4+12+6)x &= -6+2+9 \\
-26x &= 5 \\
x &= -\frac{5}{26}
\end{aligned}
$$

(iv)

$$
\begin{aligned}
\frac{1}{6}(2x+9)-\frac{1}{10}(x^2-1) &= \frac{3}{20}x-\frac{1}{10}(x-5)(x+3) \\
10(2x+9)-6(x^2-1) &= 9x-6(x-5)(x+3) \\
20x+90-6x^2+6 &= 9x-6(x^2-5x+3x-15) \\
(20-30+18-9)x &= 90-90-6 \\
-x &= -6 \\
x &= 6
\end{aligned}
$$

(v)

$$
\begin{aligned}
3+\frac{(2x-1)(3x-2)}{9}-\frac{x^2}{3} &= \frac{x^2-2}{3} \\
27+(2x-1)(3x-2)-3x^2 &= 3x^2-6 \\
27+6x^2-3x-4x+2-3x^2 &= 3x^2-6 \\
(-3-4)x &= -6-29 \\
7x &= 35 \\
x &= 5
\end{aligned}
$$

Activity 3

(I)	$x=2$, $y=1$	(II)	$x=7$, $y=8$	(III)	$x=-15$, $y=8$
(IV)	$x=16$, $y=35$	(V)	$x=-\frac{1}{2}$, $y=3$	(VI)	$x=1$, $y=2$
(VII)	$x=2$, $y=-1$	(VIII)	$x=2$, $y=4$	(IX)	$x=-2$, $y=-3$

Activity 4

(i)	$x=2, y=3, z=1$	(ii)	$x=5, y=4, z=-6$
(iii)	$x=2, y=1, z=0$	(iv)	$x=8, y=-2, z=12$
(v)	$x=3, y=-3, z=2$	(vi)	$x=3, y=-2, z=4$

Summary

By performing the same operation (addition, subtraction, multiplication or division) on the both sides of an equation, the two sides remain equal. Thus, we may add the same number to both sides, subtract the same number from both sides, multiply both sides by the same number or divide both sides by the same number (other than zero), then both sides are equal.

Since a solution of a system of linear equations with two unknowns in a pair of values (x, y) which satisfies both equations. Graphically, the solution would be the coordinates of the point of intersection of the two lines.

The first method involves in elimination by substitution.

The second method involves in elimination by equating numerical coefficient.

Learning Outcomes

On completion of this study session, you should be able to

- Solve simple and linear equations.
- Solve simultaneous linear equations with two or three unknowns.

Session 4

Quadratic Expressions and Equation

Contents:

Introduction

Quadratic Expression could be seen in the time of Menachemus (350 BC). He discovered conic sections and quadratic functions which are used in applications in Technology and Science. For instance, in our day to day life, we have probably noticed a light source which is reflected by parabolic mirror, such as a flash light or an automobile head lamp. Another example is the parabolic radar reflection which is important in air traffic and defense systems. These examples can be described by quadratic functions. In this session, we discuss how to solve quadratic equations. These equations could be found in applied problems in many areas of Technology.

4.1 Quadratic Expression

A quadratic expression is defined by a polynomial of second degree in the form $y = f(x) = ax^2 + bx + c,$ where a, b and c are constant and $a \neq 0$.

We can give some examples for Quadratic expressions as follows.

$$y = f(x) = 3x^2 - 2x + 1$$

$y = f(x) = x^2 + 1$

$y = f(x) = 3x^2 - x$

$y = g(t) = t^2 - t$

$y = f(t) = -t^2 + 2t + 4$

4.2 Graphs of Quadratic Expressions

We can plot the graph of a quadratic expression by calculating several ordered pairs that satisfies the expression, $y = f(x) = ax^2 + bx + c$

Example 1

1. Sketch the Graphs of the following quadratic expressions.

(i) $y = f(x) = 2x^2 + 3x - 2 \quad -2.5 \le x \le 1$

(ii) $y = g(x) = 4x^2 - 12x + 9 \quad -0.5 \le x \le 3.0$

(iii) $y = h(x) = 2x^2 - 8x + 9 \quad -1 \le x \le 5$

(iv) $y = k(x) = 8x^2 - 2 \quad -2 \le x \le 2$

(v) $y = l(x) = -4x^2 + 16x - 7 \quad -0 \le x \le 4.0$

(vi) $y = t(x) = -9x^2 + 6x - 1 \quad -{}^{2}/_{3} \le x \le {}^{4}/_{3}$

(vii) $y = j(x) = -4x^2 + 4x - 2 \quad -1.0 \le x \le 2.0$

(viii) $y = n(x) = -4x^2 - 4x \quad -2.5 \le x \le 1.5$

Solution

In each example given above, we have to examine the values of a and $b^2 - 4ac$. Later, we discuss the importance of these two values of a and $b^2 - 4ac$.

(i) $y = f(x) = 2x^2 + 3x - 2,\ \ a = 2,\ \ a > 0$

$b^2 - 4ac = 3^2 - (4 \times 2 \times -2) = 25 > 0$

$i.e\ b^2 - 4ac > 0$

Now we can find the relevant ordered pairs of x and y;

$y = f(x) = 2x^2 + 3x - 2$

x	-2.5	-2	-1	-0.75	-0.5	0	0.5	1.0
y	3	0	-3	-3.125	-3.00	-2	0	3

Table 4.2.1

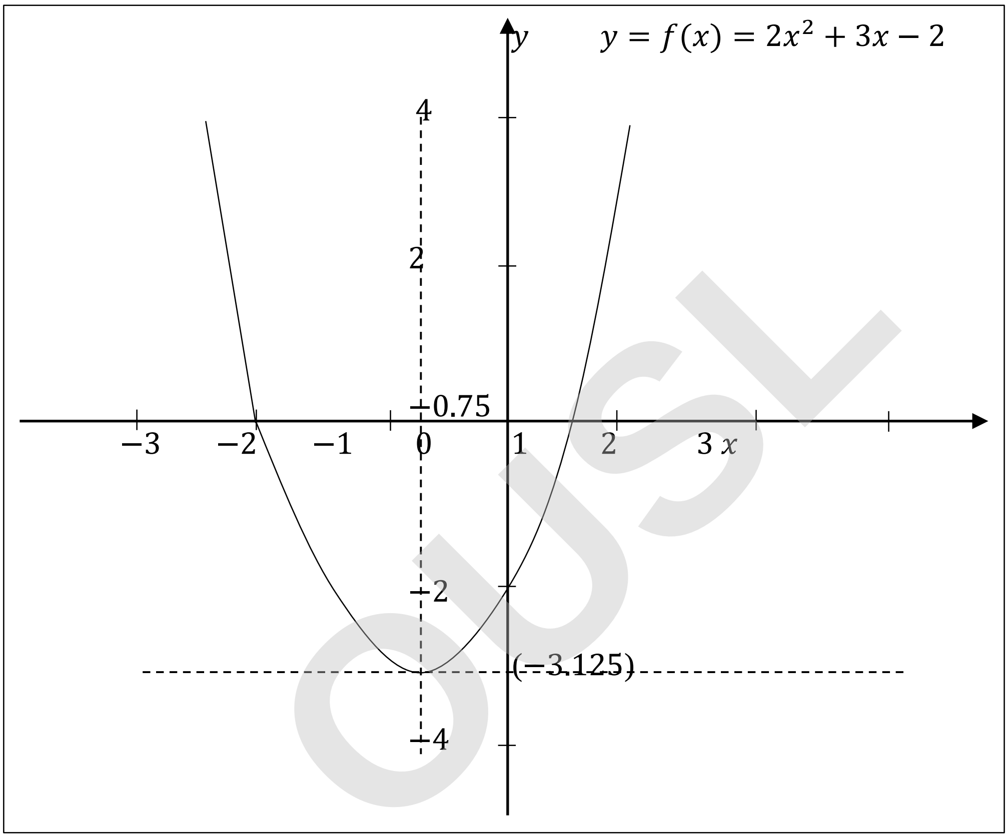


Figure 4.2.1

By observing the above graph, you can see the following features

The minimum point of the graph is $\left(-\frac{3}{4}, -\frac{25}{8}\right)$ or $(-0.75, -3.125)$

The symmetrical axis of the graph $x = -\frac{3}{4} = -0.75$

When$x = -2$ and $x = 0.5$, $f(x) = 0$

There are two intersecting points between the $x -$ axis and the graph.

$f(x) < 0, \quad -2 < x < 0.5$

$f(x) > 0, \quad -2 < x$ or$< x > 0.5$

$f(x) \geq -\frac{25}{8}$

(ii) $y = g(x) = 4x^2 - 12x + 9 \quad \therefore a = 4 \quad (a > 0)$

$b = -12, c = 9$

$\therefore\ b^2 - 4ac = (-12)^2 - 4.4.9 = 0$

$\therefore b^2 - 4ac = 0$

Now we can find the relevant ordered pairs of x and y

$$y = g(x) = 4x^2 - 12x + 9$$

x	-0.5	0	0.5	1.0	1.5	2.0	2.5	3.0
$y = g(x)$	16	9	4	1	0	1	4	9

Table 4.2.2

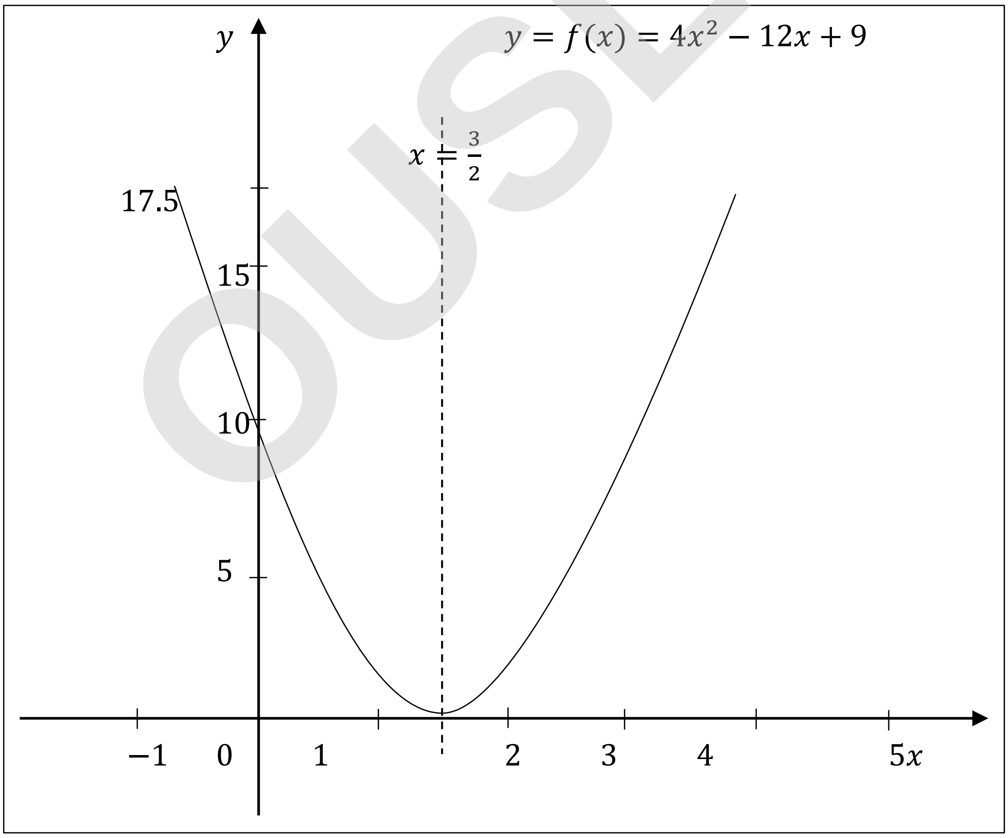


Figure 4.2.2

By observing the above graph, you can see the following features. The minimum point of the graph $\left(\frac{3}{2}, 0\right)$ or $(1.5, 0)$ the symmetrical axis of the graph $x = \frac{3}{2}$ When $x = \frac{3}{2},\ y = 0$

There is only one interesting point between the $x -$ axis and the graph.

When $x \neq \frac{3}{2}$; $g(x) > 0$ and when $x \neq \frac{3}{2}$; $g(x) = 0$.

(iii) $y = h(x) = 2x^2 - 8x + 9$; $a = 2$

$\therefore a > 0\ b = -8, c = 9$

$b^2 - 4ac = (-8)^2 - 4.2.9 < -8$

$b^2 - 4ac < 0$

Now we can find the relevant ordered pairs of x and y

$y = h(x) = 2x^2 - 8x + 9$

x	-1	0	1	2	3	4	5
$y = h(x)$	19	9	3	1	3	9	19

Table 4.2.3

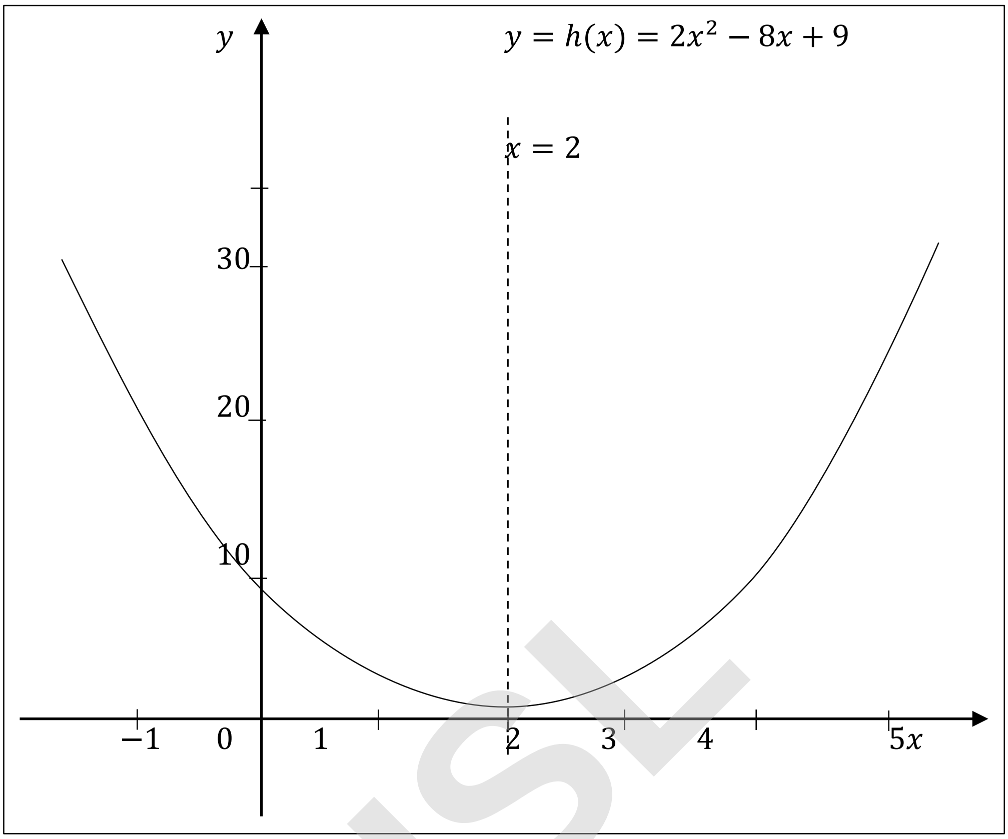


Figure 4.2.3

By observing the above graph, you can see the following features.

The minimum point of the graph is $(2, 1)$.

The symmetrical axis of the graph is $x = 2$.

There are no intersecting points between the $x-$ axis and the graph.

For all values of $x, h(x) > 0,\ h(x) \geq 1$

(iv) $y = k(x) = 8x^2 - 2$ $\quad a = 8$

$\therefore a > 0, b = 0, c = -2$

$b^2 - 4ac = -4.8.(-2) = 64$

$\therefore b^2 - 4ac > 0$

Now we can find the relevant ordered pairs of x and y

$$y = k(x) = 8x^2 - 2$$

x	-2	-1	-0.5	0	0.5	1	2
$y = k(x)$	30	6	0	-2	0	6	30

Table 4.2.4

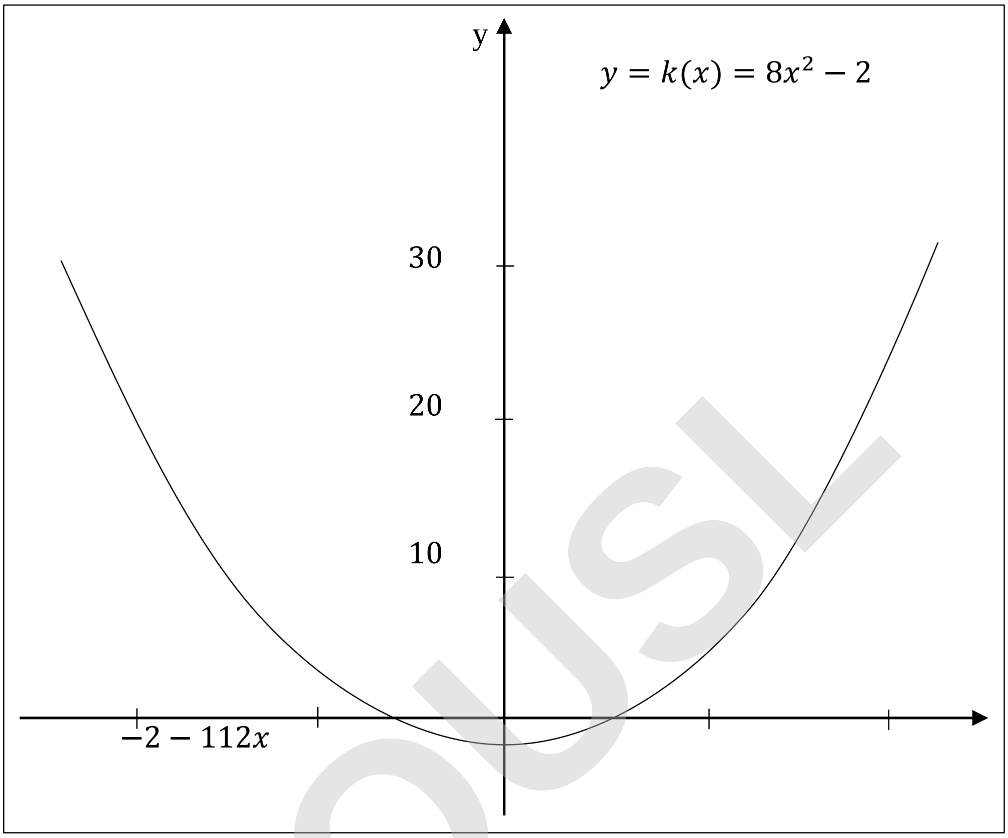


Figure 4.2.4

By observing the above graph, you can see the following features.

The minimum point of the graph is $(0, -2)$.

The symmetrical axis of the curve is $x = 0$. When $x = -0.5$ or $x = 0.5$ $k(x) = 0$.

There are two intersecting points with the $x -$ axis and the graph.

$$k(x) \geq -2$$

$$-0.5 > x \text{ and } x > 0.5\,,\;\; k(x) > 0$$

$$-0.5 < x < 0.5 \quad k(x) < 0$$

(v) $\quad y = l(x) = -4x^2 + 16x - 7 \quad a = -4 \;\; \therefore a < 0$

$\quad b = 16,\;\; c = -7 \;\Rightarrow\; b^2 - 4ac = (16)^2 - 4(-4)(-7) = 144$

$$\therefore b^2 - 4ac > 0$$

Now we can find the relevant ordered pairs of x and y;

$$y = l(x) = -4x^2 + 16x - 7$$

x	0	0.5	1	1.5	2	2.5	3	3.5	4
y	-7	0	5	8	9	8	5	0	-7

Table 4.2.5

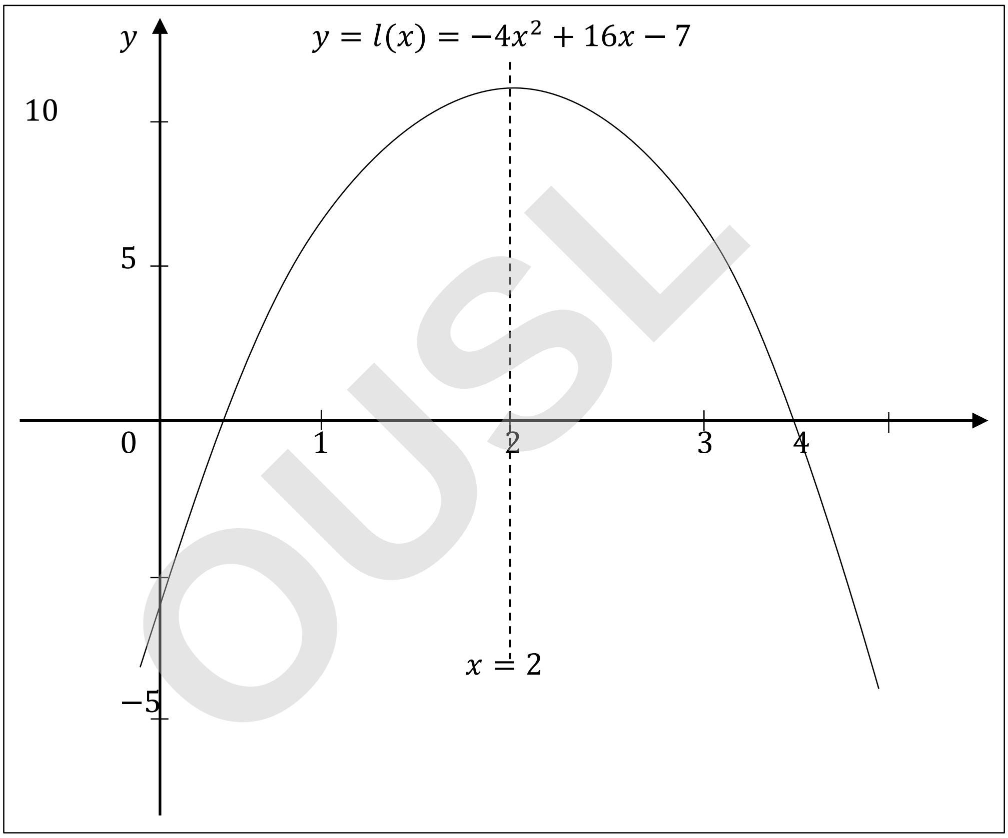


Figure 4.2.5

By observing the above graph, you can see the following features.

The maximum point of the graph is (2,9).

The symmetrical axis of the curve is *x*=2.

When $x = 0.5$ or $x = 3.5$ $l(x) = 0$

There are two intersecting point between the $x-$ axis and the curve.

$$l(x) \leq 9$$

When $0.5 > x$ and $x > 3.5$, $l(x) < 0$

$0.5 < x < 3.5$ $\quad l(x) > 0$

(vi) $y = t(x) = -9x^2 + 6x - 1 \quad a = -9 \ \therefore a < 0$

$b = 6, \qquad c = -1 \quad b^2 - 4ac = 6^2 - 4(-9)(-1) = 0$

$\therefore b^2 - 4ac = 0$

Now we can find the relevant ordered pairs of $x$and y;

$$y = t(x) = -9x\text{^}2 + 6x - 1$$

x	$-\frac{2}{3}$	$-\frac{1}{3}$	0	$\frac{1}{3}$	$\frac{2}{3}$	1	$\frac{4}{3}$
$y = t(x)$	-9	-4	-1	0	-1	-4	-9

Table 4.2.6

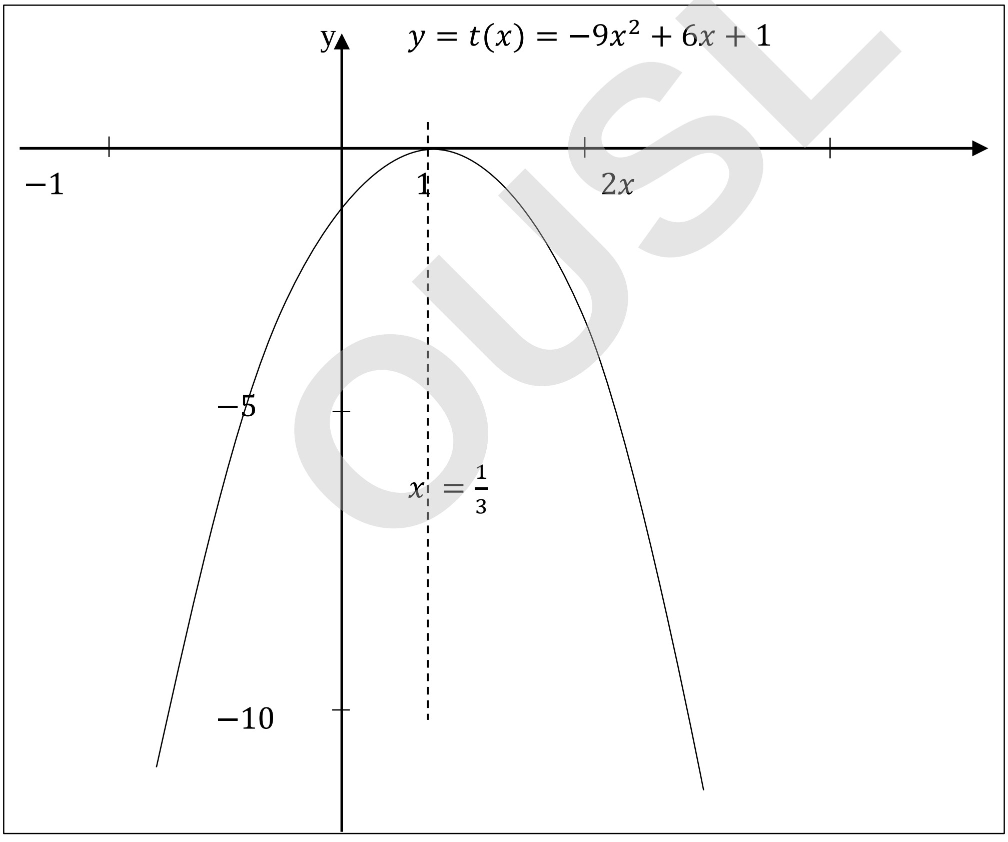


Figure 4.2.6

By observing the above graph, you can see the following features.

The maximum point of the graph is $(\frac{1}{3}, 0)$.

The symmetrical axis of the curve is $x = \frac{1}{3}$. When $= \frac{1}{3}$, $t(x) = 0$

There is only one intersecting point between the $x -$ axis and the curve.

$t(x) \leq 0$

$x = \frac{1}{3}$, $t(x) = 0$ Otherwise $f(x) < 0$

(vii) $y = j(x) = -4x^2 + 4x - 2$ $a = -4$ $\therefore a < 0$

$\therefore a < 0, b = 4, c = -2$

$b^2 - 4ac = 4^2 - 4(-4)(-2) = -16$

$\therefore b^2 - 4ac < 0$

Now we can find the relevant ordered pairs of x and y;

$$y = j(x) = -4x^2 + 4x - 2$$

x	-1	-0.5	0	0.5	1	1.5	2
$y = j(x)$	-10	-5	-2	-1	-2	-5	-10

Table 4.2.7

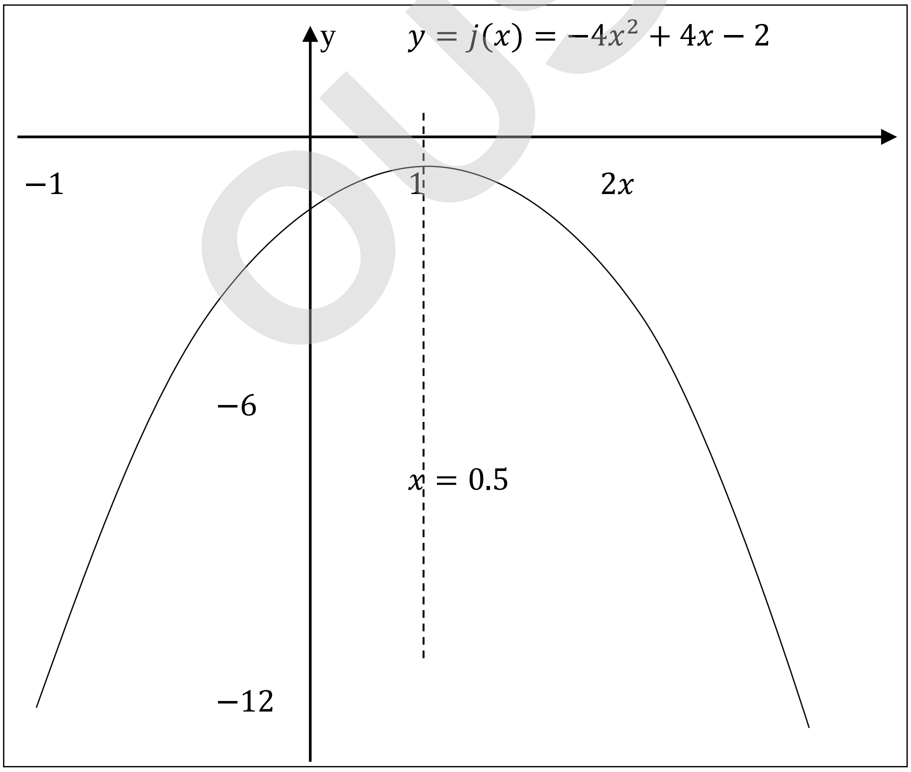


Figure 4.2.7

By observing the above graph, you can see the following features.

The maximum point of the graph is $(0.5, -2)$.

The symmetrical axis of the curve is $x \ = \ 0.5$.

There is only one intersecting point between the curve and $x -$ axis

$$j(x) \leq -2$$

$\therefore j(x) \leq 0$ for all value of x

(viii) $y = n(x) = -4x^2 - 4x$

$a = -4,\ \ a < 0\ \ b = -4\ ,\ \ c = 0$

$b^2 - 4ac = (-4)^2 = 16$

$\therefore b^2 - 4ac > 0$

Now we can find the relevant ordered pairs of $x$and y;

$y = n(x) = -4x^2 - 4x$

x	-2.5	-2	-1.5	-1	-0.5	0	0.5	1	1.5
$y = n(x)$	-15	-8	-3	0	1	0	-3	-8	-15

Table 4.2.8

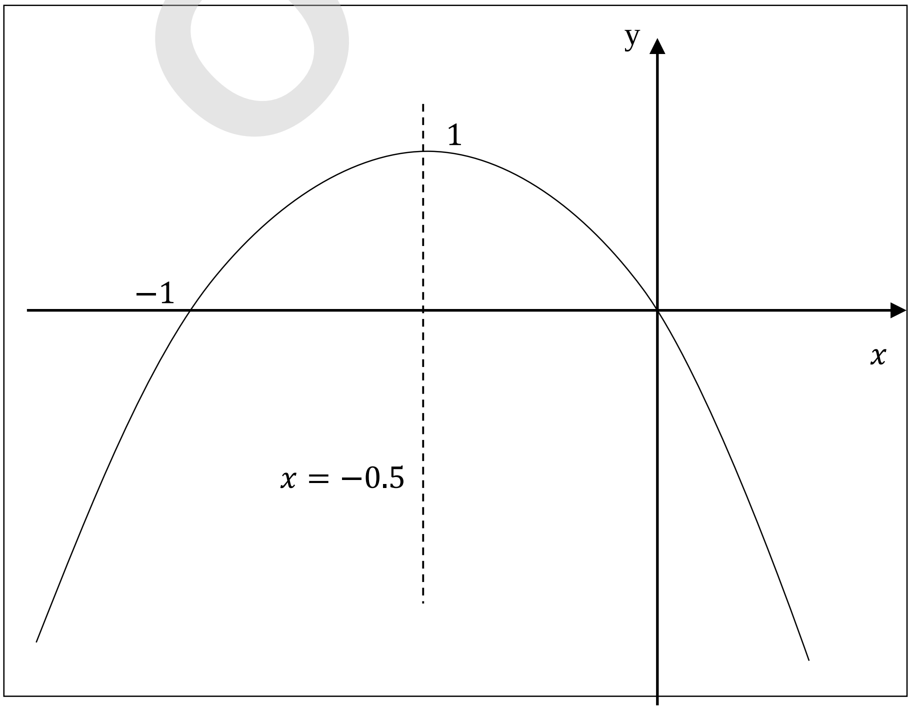


Figure 4.2.8the graph of $y = n(x) = -4x^2 - 4x$

By observing the above graph, you can see the following features.

The maximum point of the graph $(-0.5, 1)$.

The symmetrical axis of the curve $x = -0.5$.

When $x = -1$ or $x = 0$, $n(x) = 0$

There are two intersecting point between the curve and $x -$ axis

$$n(x) \leq 1$$

When $-1 > x$ and $x > 0$, $n(x) < 0$

$-1 < x < 0$ $\quad\quad$ $n(x) > 0$

Activity 1

Sketch the graphs of the curves of the following quadratic expressions.

(i) $y = f(x) = -2x^2 - 12x - 27 , -1 \leq x \leq 6$

(ii) $y = g(x) = -3x^2 + 10x - 3 , -2 \leq x \leq 5$

(iii) $y = k(x) = 2x^2 + 5x + 2 , -4 \leq x \leq 2$

(iv) $y = l(x) = 3x^2 - 12x + 17, -2 \leq x \leq 6$

(v) $y = f(x) = -x^2 - 4x - 8, \ -4 \leq x \leq 2$

(vi) $y = g(x) = 2 + x - x^2, \ -3 \leq x \leq 3$

(vii) $y = k(x) = x^2 + 4x - 8 , \ -5 \leq x \leq 3$

(viii) $y = l(x) = x^2 - 2x + 6 , \ -3 \leq x \leq 5$

For each case find the maximal or minimal value and the symmetrical axis of the curves.

Also find the range of x when $f(x) > 0$, $f(x) < 0$. Explain these results considering the values of a and $b^2 - 4ac$.

4.3 Analysis of the Quadratic Expression

In the previous section, we have determined the maximum or minimum points, the symmetric axis and the values $x$such that$f(x) = 0$ by the sketching of the relevant curves.

Now we are going to discuss the above properties of the quadratic expressions using the general quadratic expression.

The general quadratic expression is given by

$$f(x) = ax^2 + bx + c\ ; a \neq 0$$

$f(x) = a\left[x^2 + \frac{b}{a}x + \frac{c}{a}\right]$Since $a \neq 0$

$$f(x) = a\left[x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 + \frac{c}{a} - \frac{b^2}{4a^2}\right]$$

$$f(x) = a\left[\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a^2}\right]$$

$$f(x) = a\left(x + \frac{b}{2a}\right)^2 + \left(\frac{4ac - b^2}{4a}\right)$$

$(b^2 - 4ac)$ is an important quantity for a quadratic expression. It is called discriminate and is denoted by Δ.

You can see that any given quadratic expression Δdoes not depend on the value of x. Therefore,Δ is a constant for a given quadratic expression.

$$y = f(x) = a\left\{x + \frac{b}{2a}\right\}^2 + \frac{4ac - b^2}{4a}$$

$$y = f(x) = a\left\{x + \frac{b}{2a}\right\}^2 + \frac{\Delta}{4a}$$

$$y = f(x) = a\left[\left\{x + \frac{b}{2a}\right\}^2 + \frac{\Delta}{4a^2}\right]$$

You should keep in mind that nature of the graph

$y = f(x) = ax^2 + bx + c$ depends on the values of a and Δ.

Case (I)

$a > 0,$ and $b^2 - 4ac > 0$

From the - Activity 4. (i) and (iv)

(i) $y = f(x) = 2x^2 + 3x - 2$ $\quad a > 0\ ,\ \Delta > 0$

(ii) $y = k(x) = 8x^2 - 2$ $\quad a > 0\ ,\ \Delta > 0$

(i) $y = f(x) = 2x^2 + 3x - 2$

$$y = f(x) = 2\left[x^2 + \frac{3}{2}xx\right] - 2$$

$$y = f(x) = 2\left[x^2 + \frac{3}{2}xx + \left(\frac{3}{4}x\right)^2 - \frac{9}{16}\right] - 2$$

$$y = f(x) = 2\left[\left(x + \frac{3}{4}\right)^2\right] - \left(\frac{9}{8} + 2\right)$$

$$y = f(x) = 2\left(x + \frac{3}{4}\right)^2 - \frac{25}{8}$$

Compare with $f(x) = ax^2 + bx + c,\ \ a > 0\ ,\ b^2 - 4ac\ \geq 0$

$$y = f(x) = a\left\{x + \frac{b}{2a}\right\}^2 + \frac{4ac - b^2}{4a}$$

$$y = f(x) = a\left[\left\{x + \frac{b}{2a}\right\}^2 + \frac{\Delta}{4a^2}\right]$$

When $a > 0,$ the minimal value of $a\left(x + \frac{b}{2a}\right)^2$ is zero.

When $x = \frac{-b}{2a},\ f(x) = \frac{4ac-b^2}{4a}$

This is the minimal value of $f(x)$ and it is given at $x = \frac{-b}{2a}$

∴ Minimum point $\left[\frac{-b}{2a},\ \frac{-(b^2-4ac)}{4a}\right]$

$f(x) = a\left\{x + \frac{b}{2a}\right\}^2 + \frac{\Delta}{4a}$

When $x = -\frac{b}{2a} \pm \lambda$

$$f\left(\left[\frac{-b}{2a} \pm \lambda\right]\ \right) = a\lambda^2 + \frac{\Delta}{4a}$$

∴ The distance from $x = \frac{-b}{2a}$ towards both sides the value of $f(x)$ are equal.

You can see this property from Table 4.2.1.

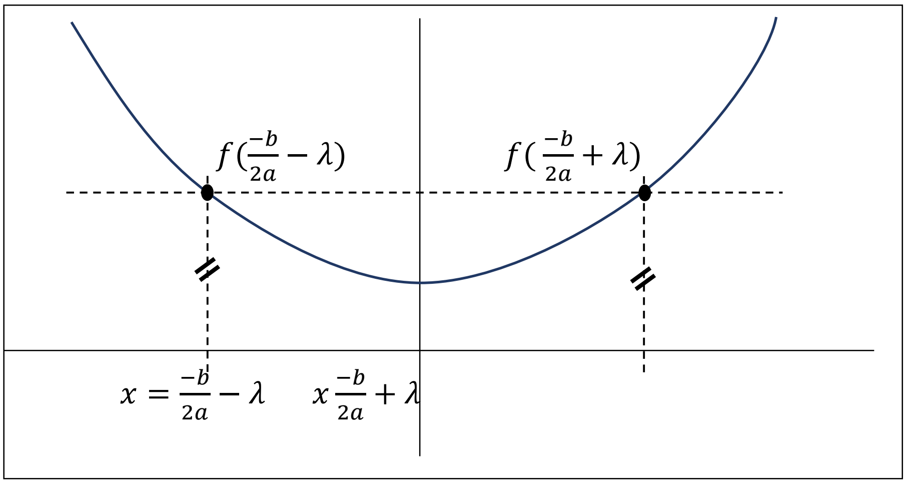


Figure 4.3.1

$\therefore$ The symmetrical axis of the curve is $x = -\frac{b}{2a}$

Also, you can see when $\Delta > 0$ Since $(b^2 - 4ac) > 0]$the graph of $f(x)$, meets the x $-$axis at two distinct points.

In the Activity 4. (i) $f(x) = -2x^2 + 3x - 2$

Minimal value the curve is $\left(-\frac{3}{4}, -\frac{25}{8}\right)$

The symmetrical axis is $x = -\frac{3}{4}$

$f(x) = 0$ when $x = -2$ and $x = \frac{1}{2}$

Activity 1(iv)

$y = k(x) = 8x^2 - 2$

$$y = 8\left[x^2 - \frac{1}{4}\right] \qquad \Delta = 4 \times 2 \times 8 = 64$$

$\therefore$the minimal value of $k(x) = -\frac{1}{4}$

Minimum point $(0, -\frac{1}{4})$

Symmetrical axis $x = 0$

$f(x) = 0$ when $x = \pm\frac{1}{2}$

Case (II)

$a > 0, \quad b^2 - 4ac = 0$

From the Activity 1 (ii)

$$y = g(x) = 4x^2 - 12x + 9 \quad a = 4, a > 0$$

$$\Delta = b^2 - 4ac = (-12)^2 - 4 \times 4 \times 9 = 0$$
$$y = g(x) = 4x^2 - 12x + 9 \equiv (2x-3)^2$$

In general, $y = f(x) = a\left(x + \frac{b}{2a}\right)^2$

$a > 0$ ∴the minimum value of $a\left(x + \frac{b}{2a}\right)^2 = 0$

∴When $x = -\frac{b}{2a}$ $y = 0$ This is the minimum value of y and it is given at $x = -\frac{b}{2a}$

∴The minimum point $\left(-\frac{b}{2a}, 0\right)$

The symmetrical axis is $x = -\frac{b}{2a}$

the Activity 1 (ii)

$g(x) = 4x^2 - 12x + 9 = (2x+3)^2$

The minimal value of the function is at the point$\left(-\frac{3}{2}, 0\right)$. The symmetric axis $x = -\frac{3}{2}$

$f(x) = 0$ when $x = -\frac{3}{2}$.

Case (III)

$a > 0, \quad b^2 - 4ac < 0$

$f(x) = ax^2 + bx + c$

$$f(x) = a\left\{x + \frac{b}{2a}\right\}^2 + \frac{4ac - b^2}{4a}$$

Since a>0 , The minimal value of $a\left\{x + \frac{b}{2a}\right\}^2 = 0$

∴ The minimum value of $f(x) = \frac{4ac-b^2}{4a}$, when $x = -\frac{b}{2a}$

Since , $b^2 - 4ac < 0$ and $a > 0$

The minimum value of $f(x)$, $\frac{4ac-b^2}{4a}$ is a positive value

Symmetrical axis $x = -\frac{b}{2a}$

Minimum point $\left(-\frac{b}{2a}, \frac{4ac-b^2}{4a}\right)$

Let consider the Activity 1 (iii)

$$y = h(x) = 2x^2 - 8x + 9; \ a = 2, \ b = -8, \ c = 9$$
$$a > 0, \ b^2 - 4ac = (-8)^2 - 4 \times 2 \times 9 = -8 < 0$$
$$y = h(x) = 2[(x-2)^2] + 9 - 8$$

$$y = h(x) = 2(x-2)^2 + 1$$

$\therefore$ The minimal value of $h(x) = 1$

$\therefore$ The minimum point (2, 1)

Symmetrical axis x=2

There is no real values of x such that $h(x) = 0$

Now we can illustrate the cases $a > 0$ and $b^2 - 4ac \gtreqless 0$ by the graphically

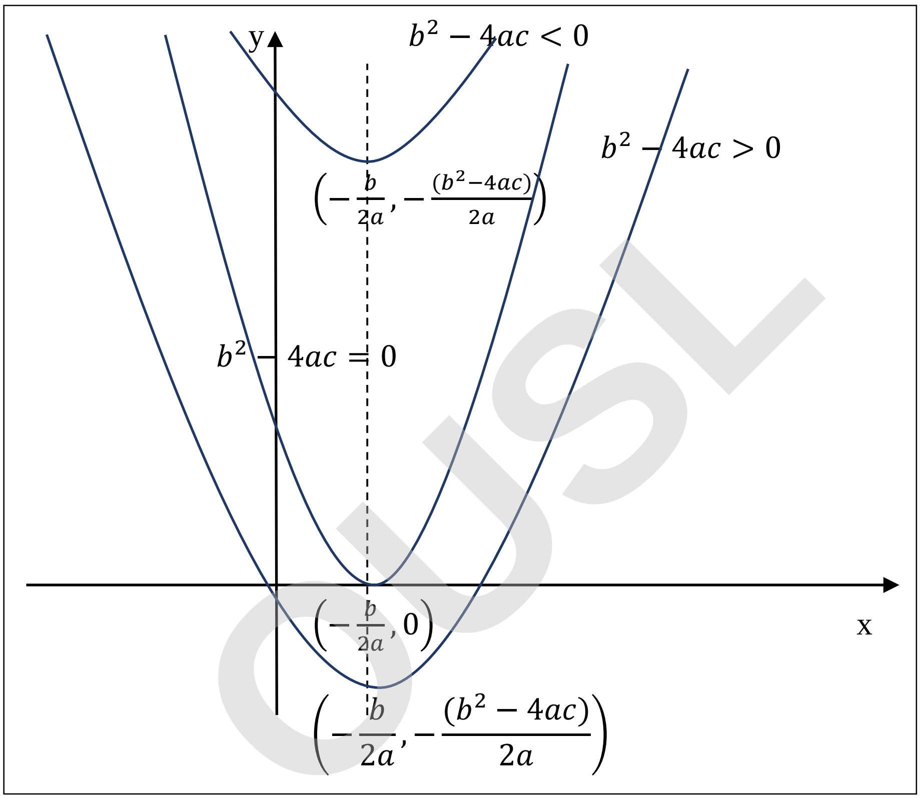


Figure 4.3.2

$$y = f(x) = ax^2 + ba + c$$

$$f(x) = a\left\{x + \frac{b}{2a}\right\}^2 + \frac{4ac - b^2}{4a}$$

Case (iv)

$$a < 0, \qquad b^2 - 4ac > 0$$

Since $a < 0, a\left\{x + \frac{b}{2a}\right\}^2 \leq 0$

The maximum value of $a\left\{x + \frac{b}{2a}\right\}^2 = 0$

$\therefore\ f(x) \leq -\dfrac{b^2-4ac}{4a}$

$\therefore$ The minimum value of $f(x) = -\frac{b^2-4ac}{4a}$

$\therefore$ The minimum point of curve $=\left(-\frac{b}{2a},\ -\frac{b^2-4ac}{4a}\right)$

The symmetric axis of $f(x)$ is $x = -\frac{b}{2a}$

Let consider the Activity 4 (v)

$y = l(x) = -4x^2 + 16x - 7; a = -4, b = 16, c = -7$

$\Delta = b^2 - 4ac = (16)^2 - 4 \times (-4) \times (-7) > 0$

$a < 0$

$y = l(x) = -4(x^2 - 4x) + 1$

$y = l(x) = -4(x-2)^2 + 9$

$\therefore$ The minimum value of $l(x) = 9$

$\therefore$ The minimum point (2, 9) when $l(x) = 0$

$$-4(x-2)^2 + 9 = 0$$

$$(x-2)^2 = \frac{9}{4} = \pm\frac{3}{2}$$

$x = \left(2 \pm \dfrac{3}{2}\right)$

$\therefore\ x = 2 + \dfrac{3}{2} = \dfrac{7}{2}\ ,\ \ x = 2 - \dfrac{3}{2} = \dfrac{1}{2}$

$\therefore f(x) = 0$ when $x = \dfrac{1}{2}$ and $\dfrac{7}{2}$

Case (v)

$a < 0, \qquad b^2 - 4ac = 0$

$F(x) = ax^2 + ba + c$

$F(x) = a\left\{x + \dfrac{b}{2a}\right\}^2$

Since $a > 0$, $\therefore a\left\{x + \frac{b}{2a}\right\}^2 \leq 0$

The maximal value of $a\left\{x + \frac{b}{2a}\right\}^2 = 0$

$\therefore F(x) \leq 0$

$\therefore$ The minimal value of $F(x) \leq 0$

Minimum point of the curve $(-\frac{b}{2a}, 0)$

Symmetric axis $x = -\frac{b}{2a}$

$F(x) = 0$ at $x = -\dfrac{b}{2a}$

Now we can consider Activity 4.2 (vi)

$y = t(x) = -9x^2 + 6x - 1 \quad a = -9 < 0$

$\Delta = b^2 - 4ac = 6^2 - 4(-9)(-1) = 0$

$y = t(x) = -(3x-1)^2$

$\therefore$The maximal value of $t(x) = 0$ when $x = \frac{1}{3}$ the maximum point is $\left(\frac{1}{3}, 0\right)$

Symmetrical axis is $x = \frac{1}{3}$

$t(x) = 0$ at $x = \frac{1}{3}$

Case (vi)

$a < 0\ , b^2 - 4ac < 0$

$F(x) = ax^2 + ba + c$

$$F(x) = a\left\{x + \frac{b}{2a}\right\}^2 - \left[\frac{b^2 - 4ac}{4a}\right]$$

$$a\left\{x + \frac{b}{2a}\right\}^2 \leq 0$$

$$\therefore F(x) \leq -\left(\frac{b^2 - 4ac}{4a}\right)$$

Since $b^2 - 4ac < 0,\ a < 0$

$$-\left(\frac{b^2 - 4ac}{4a}\right) < 0$$

$\therefore$ The maximal value of $F(x)$ is negative value

The maximum point is$\left[-\frac{b}{2a}, -\left(\frac{b^2-4ac}{4a}\right)\right]$

Symmetric axis is $x = -\frac{b}{2a}$

There is no real solution for $F(x) = 0$

Let consider Activity 1 (vii)

$y = j(x) = -4x^2 + 4x - 2$

$a = -4, b = 4, c = -2$

$\Delta = b^2 - 4ac = 4^2 - 4(-4)(-2) < 0$

$y = j(x) = -4(x^2 - x) - 2$

$$y = j(x) \ = -4\left\{\left(x - \frac{1}{2}\right)^2 + \frac{1}{4}\right\} - 2$$

$$y = j(x) \ = -4\left(x - \frac{1}{2}\right)^2 - 1$$

$$-4\left(x - \frac{1}{2}\right)^2 \leq 0$$

$\therefore j(x) \leq -1$

$\therefore$The maximal value of $j(x) = -1$

Maximum point $(\frac{1}{2}, 1)$

Symmetric axis $x = \frac{1}{2}$

You can see there is no solution for $j(x) = 0$

Now we can illustrate the case $a < 0$ and $b^2 - 4ac \lesseqgtr 0$ by the graphically.

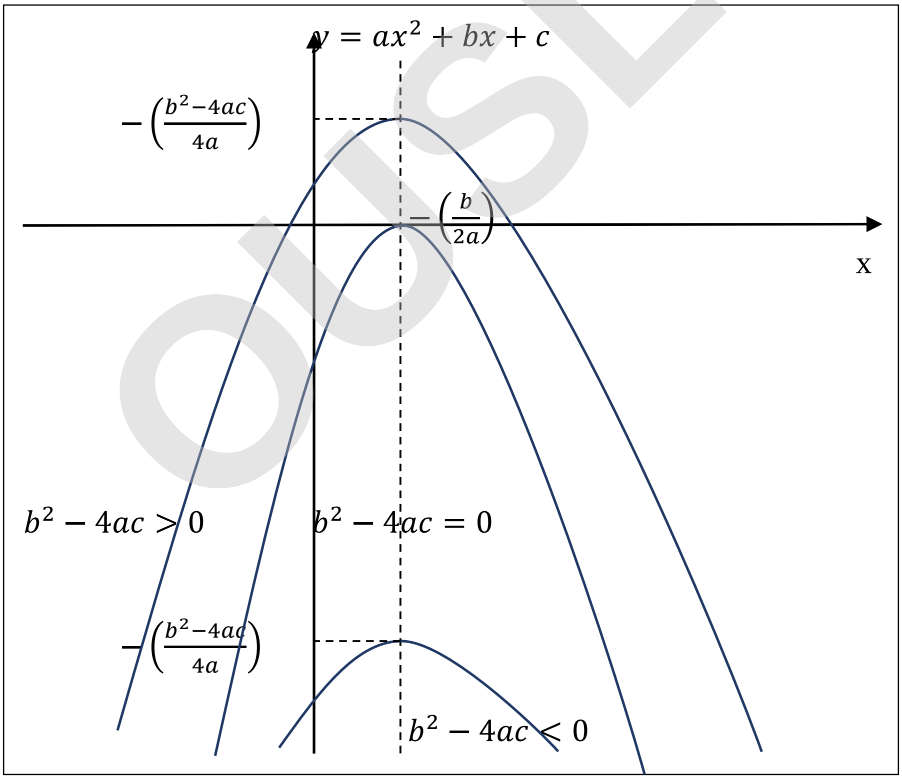


Figure 4.3.3.

Example 3

2. Without sketching the curves

(i) Find the greatest value of the function $f(x) = 4 - 8x - 4x^2$

(a) Hence find the least value of $g(x) = \frac{1}{f(x)} = \frac{1}{4-8x-4x^2}$

(b) Also find the greatest value of

$$k(x) = f(x) + 5 = 9 - 8x - 4x^2$$

$h(x) = -3f(x) + 5$

(ii) (a) Show that $6x^2 + 6x + 3$ is always positive for all real value of x.

(b) For what value of $k$are the following functions positive for all real values?

(i) $f(x) = kx^2 + 4x + 3$

(ii) $g(x) = 2kx^2 + 8x + 6$

(iii) $h(x) = x^2 + 2(k-1)x + k^2$

(iv) $i(x) = 3x^2 + 9x + 2k$

(v) $j(x) = (k-2)x^2 + 8kx + 8k$

(iii) In each of the following functions complete the square and hence obtain the range of the function

(i) $f(x) = 3x^2 - 4x + 8$

(ii) $g(x) = 2x^2 + 3x + 5$

(iii) $k(x) = -4x^2 - 12x + 7$

(iv) $h(x) = -4x^2 - 4x - 1$

(i) $f(x) = 4 - 8x - 4x^2$

$= -4\{x^2 + 2x\} + 4$

$= -4\{x^2 + 2x + 1\} + 4 + 4$

$= -4(x+1)^2 + 8$

$\therefore$ $4(x+1)^2 \leq 0$ when $x = -1$, $4(x+1)^2 = 0$

$\therefore f(x) \leq 8$ $\therefore$ The greatest value of $f(x)$ is 8

(a) $g(x) = \frac{1}{f(x)} = \frac{1}{4-8x-4x^2} = \frac{1}{-4(x+1)^2+8}$

$-4(x+1)^2 \leq 0$ $\therefore$ $g(x) \geq \frac{1}{8}$

$\therefore$ The minimal (least) value of $g(x) \geq \frac{1}{8}$ when $x = -1$.

(b) $k(x) = f(x) + 5 = -4(x+1)^2 + 5 + 8$

$$= -4(x+1)^2 + 13$$

$$\therefore \quad k(x) \leq 13$$

The maximal (greatest) value of $k(x) = 13$ when $x = -1$

$$h(x) = -3f(x) + 5$$

$$h(x) = -3[-4(x+1)^2 + 8] + 5$$

$$h(x) = -12(x+1)^2 - 24 + 5$$

$$h(x) = -12(x+1)^2 - 19$$

$\therefore$ The minimal (least) value of $h(x)$ is -19.

(ii) (a) $f(x) = 6x^2 + 6x + 3$

$$= 6[x^2 + x] + 3$$

$$= 6[x^2 + x + \left(\frac{1}{2}\right)^2] + 3 - \frac{6}{4}$$

$$= 6\left(x + \frac{1}{2}\right)^2 + \frac{3}{2}$$

$$\left(x + \frac{1}{2}\right)^2 \geq 0 \quad x \in \mathbb{R}$$

$$\left(x + \frac{1}{2}\right)^2 = 0 \quad x = -\frac{1}{2}$$

$\therefore \quad x \in \mathbb{R} \quad f(x) \geq \frac{3}{2}$ (This is least value of (x))

$\therefore \quad x \in \mathbb{R} \quad f(x) > 0$

(b) If $f(x) = ax^2 + bx + c$

If $a > 0$ and $b^2 - 4ac < 0, f(x) > 0$ for all real values of x.

(i)

$$f(x) = kx^2 + 4x + 3$$

In this case $a = k$

$\therefore k > 0 \quad b^2 - 4ac = 4^2 - 4 \times k \times 3$

$$= 16 - 12k$$

$$b^2 - 4ac < 0$$

$$16 - 12k < 0$$

$$16 < 12k$$

$$k > \frac{16}{12} \qquad k > \frac{4}{3}$$

$k > 0,\ k > \frac{4}{3}$ both satisfied with $k > \frac{4}{3}$

(ii)

$g(x) = 2kx^2 + 8x + 6$

$a = 2k \quad \therefore 2k > 0 \implies k > 0$

$b^2 - 4ac = 8^2 - 4 \times 2k \times 6$

$= 64 - 48k$

$64 - 48k < 0$

$48k > 64$

$$k > \frac{4}{3}$$

$k > 0$ and $k > \frac{4}{3}$ both are satisfying with $k > \frac{4}{3}$

(iii)

$h(x) = x^2 + 2(k-1)x + k^2$

$a = 1 \quad b^2 - 4ac = (k-1)^2 - 4k^2$

$= 4k^2 - 4k + 1 - 4k^2$

$= 1 - 4k$

$b^2 - 4ac < 0$

$1 - 4k < 0$

$$k > \frac{1}{4}$$

(iv)

$i(x) = 3x^2 + 9x + 2k$

$a = 3 \quad b^2 - 4ac = 9^2 - 4 \times 3 \times 2k$

$= 81 - 24k$

$b^2 - 4ac < 0$

$81 - 24k < 0$

$$k > \frac{81}{24}$$

$$k > \frac{27}{8}$$

(v) $l(x) = (k-2)x^2 + 8kx + 8k$

$a = (k-2) \quad \therefore a > 0 \implies k - 2 > 0$

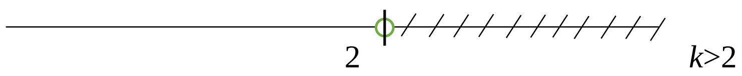
$k > 2$

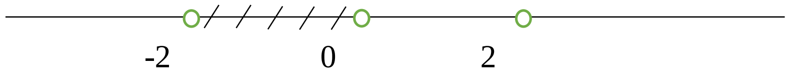
$b^2 - 4ac = (8k)^2 - 4(k-2)8k$

$$= 64k^2 - 32(k-2)k$$

$$= 32k[2k - k + 2]$$

$$= 32k(k+2)$$





$$k(k+2) < 0$$

$$-2 < k < 0$$

You can see that there is not values of $k$satisfying both the candidates.

$\therefore$ $l(x)$ Never positive for all real values of x.

(iii)

(i) $f(x) = 3x^2 - 4x + 8$

$$= 3\left[x^2 - \frac{4}{3}x + \frac{8}{3}\right]$$

$$= 3\left[x^2 - \frac{4}{3}x + \left(\frac{2}{3}\right)^2 + \frac{8}{3} - \frac{4}{9}\right]$$

$$= 3\left[x - \frac{2}{3}\right]^2 + 3\left[\frac{2}{3} - \frac{4}{9}\right]$$

$$= 3\left[x - \frac{2}{3}\right]^2 + 3\left[\frac{24-4}{9}\right]$$

$$= 3\left[x - \frac{2}{3}\right]^2 + \frac{20}{3}$$

$f(x) \geq \frac{20}{3}$

(ii) $g(x) = 2x^2 + 3x + 5$

$$= 2\left[x^2 - \frac{3}{2}x + \frac{5}{2}\right]$$

$$= 2\left[x^2 - \frac{3}{2}x + \left(\frac{3}{4}\right)^2 + \frac{5}{2} - \frac{9}{16}\right]$$

$$= 2\left[x - \frac{3}{4}\right]^2 + 2\left[\frac{40-9}{16}\right]$$

$$= 2\left[x - \frac{2}{3}\right]^2 + \frac{31}{8}$$

$g(x) \geq \frac{31}{8}$

(iii) $k(x) = -4x^2 - 12x + 7$

$= -4[x^2 + 3x] + 7$

$$= -4\left[x^2 + 3x + \left(\frac{3}{2}\right)^2\right] + 7 - 9$$

$= -4\left[x - \frac{3}{2}\right]^2 - 2$

$k(x) \leq -2$

(iv) $h(x) = -4x^2 - 4x - 1$

$= -[2x + 1]^2$

$h(x) \leq 0$

Example 4

(i) Find the greatest or least value of the following functions.

(a) $3x^2 - 4x - 3$

(b) $3x^2 - 4$

(c) $1 - 2x + x^2$

(d) $7 - 2x - x^2$

(e) $3 + 4x - x^2$

(ii) Sketch the graphs of the following quadratic functions, stating the axis of symmetry, the greatest of or least value of $f(x)$ and the value of x at which it occurs where $f(x)$ is,

(a) $x^2 - 4x + 10$

(b) $x^2 - 6x - 8$

(c) $2x^2 - 6x + 13$

(d) $4x - 5 - x^2$

(e) $2 - 5x - 3x^2$

Solution

3. Least value $-\frac{13}{3}$, when $x = \frac{2}{3}$
4. Least value -4 , when $x = 0$
5. Least value 0 , when $x = 1$
6. Least value 8 , when $x = -1$
7. Least value 7 , when $x = 2$
8.

(i)

(a) Axis of symmetry $x = 2$, minimum point $(2, 6)$ $f(x)_{min} = 6$

(b) Axis of symmetry $x = 3$, minimum point $(3, -17)$$f(x)_{min} = -17$

(c) Axis of symmetry $x = \frac{9}{4}$, minimum point $(\frac{9}{4}, \frac{17}{2})$

$$f(x)_{min} = \frac{17}{2}$$

(d) Axis of symmetry $x = 2$, minimum point $(2, -1)$ $f(x) \leq -1$

(e) Axis of symmetry $x = \frac{5}{6}$, minimum point $(\frac{5}{6}, \frac{49}{12})$ $f(x)_{max} = \frac{49}{12}$

4.4 Solution of Quadratic Equation

In the previous section we discussed about the function $f(x) = ax^2 + bx + c$, when $f(x) = 0$

i.e., $ax^2 + bx + c = 0 \quad a \neq 0$ is called quadratic equation.

The solutions of the quadratic equation we can find in two methods.

(1) Method – Factorization method

(2) Method – Complete Square method

Factorizing Method

Example 5

(a) $x^2 - 6x + 5 = 0$

(b) $x^2(x - 2) - x(x^2 - 3) = 0$

(c) $x^2 + \left(\sqrt{3} + \frac{1}{\sqrt{3}}\right)x - 1 = 0$

(d) $x^2 + 9x = 36$

(e) $4x^2 - 36x + 81 = 0$

(f) $3x^2 - 4x + 1 = 0$

Solution:

(a) $x^2 - 6x + 5 = 0$

$(x-5)(x-1) = 0$

$x - 5 = 0$ or $x - 1 = 0$

$x = 5$ or $x = 1$

$\therefore$ The roots of the equations are $x = 5$ and $x = 1$

(b) $x^2(x-2) - x(x^2-3) = 0$

$x[x(x-2) - (x^2-3)] = 0$

$x[x^2 - 2x - x^2 + 3] = 0$

$x(3-2x) = 0$

$x = 0$ or $3 - 2x = 0$

$x = 0$ or $x = \dfrac{3}{2}$

$\therefore$ The roots of the equations are x=0 and $x = \frac{3}{2}$

(c) $x^2 + \left(\sqrt{3} + \frac{1}{\sqrt{3}}\right)x - 1 = 0$

$x^2 + \left(\sqrt{3} + \dfrac{1}{\sqrt{3}}\right)x - \sqrt{3}.\dfrac{1}{\sqrt{3}} = 0$

$\left(x + \sqrt{3}\right)\left(x - \dfrac{1}{\sqrt{3}}\right) = 0$

$x + \sqrt{3} = 0$ or $x - \dfrac{1}{\sqrt{3}} = 0$

$x = -\sqrt{3}$or $x = \dfrac{1}{\sqrt{3}}$

$\therefore$ The roots of the equations are $x = -\sqrt{3}$ and $x = \frac{1}{\sqrt{3}}$

(d) $x^2 + 9x = 36$

$x^2 + 9x - 36 = 0$

$(x + 12)(x - 3) = 0$

$x + 12 = 0$ or $x - 3 = 0$

$x = -12\ or\ x = 3$

∴ The roots of the equations are $x = -12$ and $x = 3$

(e) $4x^2 - 36x + 81 = 0$

$(2x - 9)^2 = 0$

$x = \frac{9}{2}$

∴ The roots of the equations are $\frac{9}{2}$ and $\frac{9}{2}$

(f) $3x^2 - 4x + 1 = 0$

$(3x - 1)(x - 1) = 0$

$3x - 1 = 0$ or $x - 1 = 0$

$x = \frac{1}{3}$ or $x = 1$

∴ The roots of the equations are $x = \frac{1}{3}$ or$x = 1$

Solution by completing the square

Many quadratic equations cannot be solving by the rational factorizing. This is observed in applied situations.

In this section, we develop a method that could be used to solve any quadratic equation. The method is called the completing of the square.

Example 5

9. By using the method of completing the square, find the roots of the following equations.

(i) $3x^2 + 4x - 8 = 0$

(ii) $2x^2 + 3x - 7 = 0$

(iii) $6x^2 + 4x = 9$

(iv) $\frac{x-1}{3} = \frac{5}{x} + 1$

Solution

(i) $3x^2 + 4x - 8 = 0$

$3x^2 + 4x = 8$

$x^2 + \frac{4}{3}x = \frac{8}{3}$

$x^2 - \frac{4}{3}x + \left(\frac{2}{3}\right)^2 = \frac{8}{3} + \frac{4}{3}$

$\left[x - \frac{2}{3}\right]^2 = \frac{28}{9}$

$x - \frac{2}{3} = \pm\frac{\sqrt{28}}{3}$

$x = \frac{2}{3} + \frac{\sqrt{28}}{3}$ or $x = \frac{2}{3} - \frac{\sqrt{28}}{3}$

$x = \frac{-2 + 5.29}{3} \;\; or \;\; x = \frac{-2 - 5.29}{3}$

$x = 1.10$ or $x = -2.43$

Solutions

$x = 1.10$ or $x = -2.43$

(ii) $2x^2 + 3x - 7 = 0$

$2x^2 + 3x = 7$

$x^2 + \frac{3}{2}x = \frac{7}{2}$

$x^2 - \frac{3}{2}x + \left(\frac{3}{4}\right)^2 = \frac{9}{16} + \frac{7}{2} = \frac{65}{16}$

$\left[x - \frac{3}{4}\right]^2 = \frac{65}{16}$

$x - \frac{2}{3} = \pm\sqrt{\frac{65}{16}} = \pm\frac{\sqrt{65}}{4}$

$x - \frac{2}{3} \pm \frac{8.06}{4}$

$x = \frac{8.06 - 3}{4}$ or $x = \frac{-(8.06 + 3)}{4}$

$x = 1.127$ or $x = -2.77$

(iii) $6x^2 + 4x = 9$

$$x^2 + \frac{2}{3}x = \frac{3}{2}$$

$$x^2 - \frac{2}{3}x + \left(\frac{1}{3}\right)^2 = \frac{3}{2} + \frac{1}{9}$$

$$\left[x - \frac{1}{3}\right]^2 = \frac{27+2}{18} = \frac{29}{18} = \frac{58}{36}$$

$$x - \frac{1}{3} = \pm\frac{\sqrt{58}}{6} = \pm\frac{7.62}{6}$$

$$x = \frac{7.62}{6} - \frac{1}{3} \text{ or } x = -\left(\frac{7.62}{6} + \frac{1}{3}\right)$$

$x = 0.94$ or $x = -1.60$

(iv) $\frac{x-1}{3} = \frac{5}{x} + 1$

$$x(x-1) = 15 + 3x$$

$$x^2 - 4x = 15$$

$$x^2 - 4x + 4 = 15 + 4 = 19$$

$$(x-2)^2 = 19$$

$$x - 2 = \pm\sqrt{19} = \pm 4.36$$

$$x = 2 \pm 4.36$$

$x = 6.36$ or $x = -2.36$

** How do we determine the number, which must be added to the complete the square.

We know that $\{x + a\}^2 = x^2 + 2ax + a^2$

$$\{x - a\}^2 = x^2 - 2ax + a^2$$

1st Step; Divide each term by the coefficient of x^2, (Then the coefficient of x^2 becomes 1)

Example 6

$$5x^2 - 6x = 9$$

$$x^2 + \frac{6}{5}x = \frac{9}{5}$$

2nd Step; Divide the coefficient of the x term by 2 and square that value and add to the both sides of the equation

$$x^2 - \frac{6}{5}x + \left(\frac{3}{5}\right)^2 = \frac{9}{5} + \left(\frac{3}{5}\right)^2$$

Then the left hand side becomes a complete square

$$\left[x-\frac{3}{5}\right]^2=\frac{9}{5}+\frac{9}{25}=\frac{54}{25}$$

$$x-\frac{3}{5}=\pm\sqrt{\frac{54}{25}}=\ \pm\frac{\sqrt{54}}{5}=\ \pm\frac{7.35}{5}$$

$$x=\left(\frac{3\pm 7.35}{5}\right)$$

$x=2.07$ or $x=-0.87$

Example 7

1) Solve the following quadratic equation by using the factorization.
   (a) $40x-16x^2=0$
   (b) $(x+5)^3=x^3+125$
   (c) $2x^2+7x+6=0$
   (d) $3x^2+13x+4=0$
   (e) $2x^2+7x+6=3$
2) Solve the following quadratic equation by the method of completing the square.

(a) $x^2+6x-8=0$

(b) $3x^2-4x-2=0$

(c) $2x^2-2x=9$

Solution

(1)

(a) $x=0\ or\ x=\frac{5}{2}$

(b) $x=0\ or\ x=-5$

(c) $x=-3\ or\ x=-\frac{1}{2}$

(d) $x=-4\ or\ x=-\frac{1}{3}$

(e) $x=-\frac{1}{2}$ or $x=-3$

(2)

(a) $x=1.12\ or\ x=-7.12$

(b) $x = 1.72 \ or \ x = -0.39$

(c) $x = 2.68 \ or \ x = -1.68$

4.5 Solutions of an Equation

$$ax^2 + bx + c = 0$$

$a \neq 0, \qquad a, b, c \in \mathbb{R}$

$$ax^2 + bx + c = 0$$

By the process of the completing the square we have the following equation

$ax^2 + bx + c = 0$

$ax^2 + bx = -c$

$x^2 + \frac{b}{a}x = -\frac{c}{a}$

$x^2 - \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = \left(\frac{b}{2a}\right)^2 - \frac{c}{a}$

$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a} = \frac{b^2 - 4ac}{4a^2}$

$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$

$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}$

$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$ or $x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$

If the roots of the equation $ax^2 + bx + c = 0$ are α, β

$\alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \ \beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$

From the above we make the following observation regarding the nature of the quadratic equation.

The nature of the roots of the equation of $ax^2 + bx + c = 0$ depends on the value $b^2 - 4ac$ (Discriminate) Δ

(i) If $b^2 - 4ac > 0$; $\sqrt{b^2 - 4ac}$ can be evaluated, the roots of the equation $ax^2 + bx + c = 0$ α and β are real and distinct.

(ii) If $b^2 - 4ac = 0 \quad \therefore \sqrt{b^2 - 4ac} = 0$

$\therefore\ \alpha = \beta = -\dfrac{b}{2a}$

The roots of the equation are real and equal

(iii) If $b^2 - 4ac < 0$ Then $\sqrt{b^2 - 4ac}$ has no real value

$\therefore$ Then α and β are not real.

The equation $ax^2 + bx + c = 0$ has real root if and only if $b^2 - 4ac \geq 0$.

Example 8

(1) Find the range of values of p which, the quadratic equation $px^2 + (p+3)x + p = 0 \quad (p \neq 0)$ has real roots.

Find the value of p if $x = -2$ is one of roots of the given equation and find the other root in this case.

(2) Find the range of k for which the equation $x^2 - 2x + k - 4 = 0$ has real roots.

(3) Prove that the equations.

(i) $\lambda x^2 + (\lambda - 1)x + 1 - 2\lambda = 0$

(ii) $x^2 + 2(\lambda + 1)x - 2\lambda - 3 = 0$

Have real roots for the real value of λ

Solution:

(1) $px^2 + (p+3)x + p = 0$

In the case $b = p + 3$ $a = c = p$

$\therefore\ \Delta = b^2 - 4ac = (p+3)^2 - 4p.p$

For real roots $\Delta \geq 0$

$(p+3)^2 - 4p^2 \geq 0$

$(p + 3 + 2p)(p + 3 - 2p) \geq 0$

$(3p + 3)(3 - p) \geq 0$

$(p+1)(p-3) \leq 0$

$3 \geq p \geq -1$ The range of p

Such that real roots of the given equation.

If $x = -2$ $\quad px^2 + (p+3)x + p = 0$

$p(-2)^2 + (p+3)(-2) + p = 0$

$4p - 2p - 6 + p = 0$

$$3p - 6 = 0$$

$$p = 2$$

$\therefore$ The given equation $2x^2 + 5x + 2 = 0$

$$(2x + 1)(x + 2) = 0$$

$$2x + 1 = 0 \ \ or \ x + 2 = 0$$

$$x = -\frac{1}{2} \ or \ x = \ -2$$

(2) $x^2 - 2x + k - 4 = 0$

$a = 1, b = -2, c = k - 4$

$$\Delta = b^2 - 4ac$$

$$= (-2)^2 - 4(1)(k - 4)$$

$$= 4 - 4k + 16$$

$$= 20 - 4k$$

For real roots $\Delta \geq 0$

$$20 - 4k \geq 0 \qquad\qquad 4k \leq 20$$

$$k \leq 5$$

(3) (i) $\lambda x^2 + (\lambda - 1)x + 1 - 2\lambda = 0$

$\Delta = (\lambda - 1)^2 - 4\lambda(1 - 2\lambda)$

$= \lambda^2 - 2\lambda + 1 - 4\lambda + 8\lambda^2$

$= 9\lambda^2 - 6\lambda + 1$

$= (3\lambda - 1)^2$

$\therefore (3\lambda - 1)^2 \geq 0$

If $\lambda = \frac{1}{3}$ the given equation has two equal real roots otherwise the equation has real distinct roots for any value of λ

(ii) $x^2 + 2(\lambda + 1)x - 2\lambda - 3 = 0$

$\Delta = b^2 - 4ac = 2^2(\lambda + 1)^2 - 4(1)(-1)(2\lambda + 3)$

$= 4[(\lambda + 1)^2 + (2\lambda + 3)]$

$= 4[\lambda^2 + 4\lambda + 4]$

$= 4(\lambda + 2)^2$

If $\lambda = -2$ the given equation has two equal real roots. Otherwise $[\lambda \neq -2]$ the given equation has two real distinct roots.

Activity 2

(a) Find the possible values of λ (or range of value of λ)

(i) $x^2 + (\lambda + 9)x + 9 = 0$ has two real equal roots.

(ii) $x^2 + 2(\lambda - 4)x + \lambda + 1 = 0$ has two real distinct roots

(iii) $x^2 + (\lambda - 3)x + \lambda = 0$ has no real roots

(b) Let $f(x) = x^2 + 2ax + 2a^2 + a + 1 = 0$. $Find$the discriminate Δ for the above equation. Hence show that for all real values of a, the equation $x^2 + 2ax + 2a^2 + a + 1 = 0$ has no real roots.

Summary

$(x) = ax^2 + bx + c = 0 \quad (a \neq 0)$

$$\equiv a\left[x + \frac{b}{2a}\right]^2 - \left[\frac{b^2 - 4ac}{4a}\right]$$

(i) $x = -\frac{b}{2a}$ is the symmetric axis of the curve $y = f(x)$

(ii) If $a > 0$ $f(x)$has a minimum value $-\left[\frac{b^2-4ac}{4a}\right]$ and the minimum point of the curve is $\left\{-\frac{b}{2a}, -\left[\frac{b^2-4ac}{4a}\right]\right\}$

If $a < 0$ $f(x)$ has maximum value $-\left[\frac{b^2-4ac}{4a}\right]$ and the maximum point $\left\{-\frac{b}{2a}, -\left[\frac{b^2-4ac}{4a}\right]\right\}$

(iii) If $a < 0$ $f(x)$ and $b^2 - 4ac < 0$ then $f(x) > 0$ for all real value of x.

If $a < 0$ $f(x)$ and $b^2 - 4ac < 0$ then $f(x) < 0$ for all real value of x.

(iv) $ax^2 + bx + c = 0 \quad (a \neq 0)$ Then the roots of the Equation is

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The roots real and distinct if $b^2 - 4ac > 0$ When $b^2 - 4ac = 0$
Then real and equal roots.
If $b^2 - 4ac < 0$equation has no real roots.

Learning Outcomes

On completion of this study session should be able to

- Plot the graph of quadratic expression with marking the coordinates
- Sketch the graph of quadratic expression without marking the coordinates.
- Find the maximum or minimum value of a quadratic expression find the roots of quadratic equation by plotting the graph.
- Find the solution of the quadratic equation by factorizing and completing the square.
- Find the roots when the quadratic equation is given and distinguish the nature of the roots of the quadratic equation.

Session 5

Indices and Logarithms

Contents:

Introduction

In this session, we are going to discuss about the indices and logarithms. In the beginning of this session the basic concepts and elementary operations of indices and logarithms are explained.

Indices or exponents are very important in computational work.

5.1 Review of Indices

The lows of indices (Exponent) we can have expressed as follows.

(i) $a^m \times a^n = a^{m+n}; \quad m, n \in \mathbb{R},$

(ii) $a^m \div a^n = a^{m-n}$

(iii) $(a^m)^n = a^{mn}$

(iv) $(ab)^n = a^n b^n$

(v) $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$

(vi) $a^0 = 1$

(vii) $a^n = \frac{1}{a^n}$

Activity 1

Prove the above rules for positive integers m, n. Note that these rules were originally defined for positive integers of m and n. But we can prove that these rules are also true for negative integers and rational numbers.

Example 1

By assuming that the above rules are true for positive integers of m and n. Deduce that the following results are valid for any negative integer values of m and n.

(i) $a^m \times a^n$

(ii) $\frac{a^m}{a^n} = \frac{a^{-p}}{a^{-q}} = \frac{1}{a^p} \cdot \frac{1}{\frac{1}{a^q}}$

(iii) $(a^m)^n = (a^{-p})^{-q}$

(iv) $(ab)^n = (ab)^{-q} = \frac{1}{(ab)^q}$

(v) $\left(\frac{a}{b}\right)^n = \left(\frac{a}{b}\right)^{-q}$ $where\ b \neq 0$

Answer:

Let $m = -p$ and $n = -q$

Then $p, q \in \mathbb{R}$

$$a^m \times a^n$$

$$L.H.S. = a^{-p}.a^{-q} = \frac{1}{a^p}.\frac{1}{a^q} \text{ (rule } (vii))$$

$$= \frac{1}{a^{p+q}} \text{ rule}(i)$$

$$= a^{-(p+q)} \text{ rule}(vii)$$

$$= a^{(-p)+(-q)}$$

$$= a^{m+n}$$

(i) $\frac{a^m}{a^n} = \frac{a^{-p}}{a^{-q}} = \frac{1}{a^p} \cdot \frac{1}{1/a^q}$ rule (vii)

$= \dfrac{1}{a^{(p-q)}} = a^{-(p-q)}$rule (vii)

$= a^{-p+q}$

$= a^{m-n}$

(ii) $(a^m)^n = (a^{-p})^{-q}$

$= \dfrac{1}{(a^{-p})^q}$ rule (vii)

$= \dfrac{1}{\left(\frac{1}{a^{-p}}\right)^q}$ rule (vii)

$= \dfrac{1}{\frac{1}{a^{pq}}}$ rule (iii)

$= a^{pq} = a^{(-p)\times(-q)}$

$= a^{m\times n}$

(iii) $(ab)^n = (ab)^{-q} = \frac{1}{(ab)^q}$ rule(vii)

$= \dfrac{1}{a^q b^q}$ rule (iv)

$= \dfrac{1}{a^q} \cdot \dfrac{1}{b^q}$

$= a^{-q}.a^{-q}$ rule (vii)

$= a^n.a^n$

(iv) $\left(\frac{a}{b}\right)^n = \left(\frac{a}{b}\right)^{-q} \quad b \neq 0$

$= \dfrac{1}{\left(\frac{a}{b}\right)^q}$ *rule* (vii)

$= \dfrac{1}{\frac{a^q}{b^q}}$ *rule* (v)

$= \dfrac{b^q}{a^q} \quad (a \neq 0)$

$= \dfrac{a^{-q}}{b^{-q}}$ rule (vii)

$$= \frac{a^n}{b^n}$$

Example 2

Deduce that the above results are valid for rational numbers.

Let $m = \frac{c}{d}$ $n = \frac{e}{f}$ where $c, d, e, f \in \mathbb{Z}$ (c, d, e, f are integers)

(i) $a^m \times a^n$

(ii) $\frac{a^m}{a^n}$

(iii) $(a^m)^n$

Answer:

(i) $a^m \times a^n = a^{\frac{c}{d}} \times a^{\frac{e}{f}}$

$$= a^{\frac{cf}{df}} \times a^{\frac{fe}{df}}$$

$$= \left[a^{\frac{1}{df}}\right]^{cf} \times \left[a^{\frac{1}{df}}\right]^{ef}$$

Assume that $a^{\frac{1}{df}} = x$

$$\therefore a^m \times a^n = (x)^{cf} \times (x)^{ef}$$

$$= (x)^{(cf+de)} \text{rule } (i)$$

$$= \left[\frac{1}{a^{df}}\right]^{(cf+de)} \quad \text{rule } (i)$$

$$= a^{\frac{(cf+de)}{df}}$$

$$= a^{(\frac{c}{d}+\frac{e}{f})}$$

$$= a^{(m+n)}$$

(ii) $\frac{a^m}{a^n} = \frac{a^{c/d}}{a^{e/f}} = \frac{a^{cf/df}}{a^{de/df}} = \frac{\left(a^{1/df}\right)^{cf}}{\left(a^{1/df}\right)^{de}}$

$$= \frac{x^{cf}}{x^{de}}$$

$$= x^{(cf-de)}$$

$$= \left(a^{1/df}\right)^{cf-de}$$

$$= a^{\frac{cf-de}{df}}$$

$$= a^{\frac{c}{d}-\frac{e}{f}}$$

$$= a^{m-n}$$

(iv) $(a^m)^n = (a^{c/d})^{e/f} a^{1/d} = y$

$$= (y^c)^{e/f} a^{c/d} = y^c$$

$$= \{(y^c)^e\}^{1/f}$$

$$= y^{ce/f} y^{ce/f} = t$$

$$t = \sqrt[f]{y^{ce}} = \sqrt[f]{a^{ce/d}}$$

$$= a^{ce/df}$$

$$= a^{c/d \cdot e/f}$$

$$= a^{mn}$$

Note that you can apply all these laws (without proof) for irrational numbers of *m* and *n*.

Example 3

Note that if $n \in Z^+ a^{1/n} = \sqrt[n]{(a)}$ n^{th} root of a

(i) $a^{1/6} . a^{3/4}$

(ii) $\frac{3k^{3/4}}{k^{-2}}$

(iii) $\frac{a^{-4/5} . a^3}{a^{-6/5}}$

(iv) $(16a^4b^8)^{1/4}$

(v) $\frac{9x^{-3/2} . y^{3/4} . 3y^{-1/2}}{18x^{-2}x^{2/3}}$

(vi) $\left\{\frac{64a^{3/2}b^{-1/2}}{a^{1/3}b^2}\right\}^{-1/2}$

(vii) $(a^4)^{3/4}a^3$

(viii) $\frac{a^{3/10}}{a^{-2/3} . a^2}$

Solution:

(i) $a^{1/6}.a^{3/4} = a^{1/6+3/4}$

$= a^{\frac{2+9}{12}}$

$= a^{11/12}$

(ii) $\frac{3k^{3/4}}{k^{-2}} = 3k^{3/4-(-2)}$

$= 3k^{3/4+2}$

$= 3k^{11/4}$

(iii) $\frac{a^{-4/5}.a^3}{a^{-6/5}} = a^{-4/5+3-(-6/5)}$

$= a^{-4/5+3+6/5}$

$= a^{2/5+3}$

(iv) $(16a^4b^8)^{1/4} = (2^4)^{1/4}(a^4)^{1/4}[(b^2)^4]^{1/4}$

$= 2.ab^2$

(v) $\frac{9x^{-3/2}.y^{3/4}.3y^{-1/2}}{18x^{-2}x^{2/3}} = \frac{9}{18} \times 3x^{(-3/2-(-2)-2/3)}y^{(3/4-1/2)}$

$= \frac{3}{2}x^{(-3/2+2-2/3)}y^{(\frac{3-2}{4})}$

$= \frac{3}{2}x^{(\frac{-9+12-4}{6})}y^{(\frac{1}{4})}$

$= \frac{3}{2}x^{(\frac{-1}{6})}y^{(\frac{1}{4})}$

$= \frac{3y^{1/4}}{2x^{1/6}}$

(vi) $\left\{\frac{64a^{3/2}b^{-1/2}}{a^{1/3}b^2}\right\}^{-1/2} = \frac{1}{\left\{\frac{64a^{3/2}b^{-1/2}}{a^{1/3}b^2}\right\}^{1/2}}$

$$= \frac{\left(a^{1/3}b^{2}\right)^{\frac{1}{2}}}{\left(8^{2}a^{3/2}b^{-1/2}\right)^{\frac{1}{2}}}$$

$$= \frac{a^{1/6}b^{1}}{8a^{3/4}b^{-1/4}}$$

$$= \frac{1}{8}a^{1/6-3/4}.\,b^{1-(-1/4)}$$

$$= \frac{1}{8}a^{\left(\frac{2-9}{12}\right)}b^{\frac{5}{4}} \qquad\qquad = \frac{1}{8}a^{-7/12}b^{5/4}$$

$$= \frac{b^{5/4}}{8a^{7/12}}$$

(vii) $\quad (a^{4})^{3/4} \times a^{3} = a^{3} \times a^{3}$

$$= a^{3+3}$$

$$= a^{6}$$

(viii) $\quad \frac{a^{3/10}}{a^{-2/3}.a^{2}} = a^{3/10-(-2/3)-2}$

$$= a^{3/10+2/3-2}$$

$$= a^{\frac{9+20-60}{30}}$$

$$= a^{-31/30}$$

$$= \frac{1}{a^{31/30}}$$

Example 4

Simplify the following, and express as a positive index.

(i) $\quad \dfrac{\left(\frac{a^{2}}{b^{2}}\right)^{1/4}.\left(\frac{c^{2}}{b^{-1}}\right)^{1/3}}{\left(\frac{a^{-2/3}}{bc^{-2}}\right)^{1/4}}$

(ii) $\quad \dfrac{\left(\frac{a^{-2}b}{a^{3}b^{-4}}\right)^{-3}}{\left(\frac{ab^{-1}}{a^{-3}b^{2}}\right)^{5}}$

(iii) $\left(\frac{27x^3}{8a^{-3}}\right)^{-2/3}$

(iv) $(x^a y^{-b})^3 \times (x^3 y^2)^a$

(v) $(c^3)^{2/3}.\left(c^{-1/2}\right)^{4/5}.\left(c^{4/5}\right)^{1/2}$

(vi) $\dfrac{\left(\frac{x^3}{y^4}\right)^3}{\left(\frac{y^{-2}}{x^2}\right)^{-4}}$

(vii) $\left[\dfrac{\left\{\frac{a^{1/3}}{(b^{-1})^{1/4}}.\left(\frac{b^{1/4}}{a^{1/2}}\right)^2\right\}}{\left(\frac{a^{-1/2}}{b^{-1/2}}\right)}\right]^6$

Solution:

(i) $$\frac{\left(\frac{a^2}{b^2}\right)^{1/4}.\left(\frac{c^2}{b^{-1}}\right)^{1/3}}{\left(\frac{a^{-2/3}}{bc^{-2}}\right)^{1/4}} = \left(\frac{a^2}{b^2}\right)^{1/4}.\left(\frac{c^2}{b^{-1}}\right)^{1/3} \times \left(\frac{bc^{-2}}{a^{-2/3}}\right)^{1/4}$$

$$= (a^2b^{-2})^{1/4}.(c^2b)^{1/3}.\left(a^{2/3}bc^{-2}\right)^{1/4}$$

$$= a^{1/2}b^{-1/2}c^{2/3}b^{1/3}a^{1/6}b^{1/4}c^{-1/2}$$

$$= a^{1/2+1/6}.b^{-1/2+1/3+1/4}.c^{2/3-1/2}$$

$$= a^{2/3}.b^{\frac{-6+4+3}{12}}.c^{\frac{4-3}{6}}$$

$$= a^{2/3}.b^{1/12}.c^{1/6}$$

(ii) $$\frac{\left(\frac{a^{-2}b}{a^3b^{-4}}\right)^{-3}}{\left(\frac{ab^{-1}}{a^{-3}b^2}\right)^5} = \frac{1}{\left(\frac{a^{-2}b}{a^3b^{-4}}\right)^3} \times \frac{1}{\left(\frac{ab^{-1}}{a^{-3}b^2}\right)^5}$$

$$= \frac{1}{\left(\frac{a^{-2}b}{a^3b^{-4}}\right)^3 \times \left(\frac{ab^{-1}}{a^{-3}b^2}\right)^5}$$

$$= \frac{1}{\frac{a^{-6}b^3}{a^9b^{-12}} \times \frac{a^5b^{-5}}{a^{-15}b^{10}}}$$

$$= \frac{a^9b^{-12} \times a^{-15}b^{10}}{a^{-6}b^3 \times a^5b^{-5}}$$

$$= \frac{a^{-6}b^{-2}}{a^{-1}b^{-2}}$$

$$= a^{-5} = \frac{1}{a^5}$$

(iii) $\left(\frac{27x^3}{8a^{-3}}\right)^{-2/3} = \frac{1}{\left(\frac{27x^3}{8a^{-3}}\right)^{2/3}}$

$$= \frac{(2^3a^{-3})^{2/3}}{(3^3x^3)^{2/3}}$$

$$= \frac{2^2a^{-2}}{3^2x^2}$$

$$= \frac{4}{9x^2a^2}$$

(iv) $(x^a y^{-b})^3 \times (x^3y^2)^a = x^{3a}y^{-3b} \times x^{3a}y^{2a}$

$$= x^{6a}y^{(2a-3b)}$$

(v) $(c^3)^{2/3}.\left(c^{-1/2}\right)^{4/5}.\left(c^{4/5}\right)^{1/2} = c^2.c^{-2/5}.c^{2/5}$

$$= c^2$$

(vi) $\frac{\left(\frac{x^3}{y^4}\right)^3}{\left(\frac{y^{-2}}{x^2}\right)^{-4}} = \left(\frac{x^3}{y^4}\right)^3 \times \frac{1}{\left(y^{-2}/x^2\right)^{-4}}$

$$= \frac{x^9}{y^{12}} \times \frac{1}{x^8/y^{-8}}$$

$$= \frac{x^9 \times x^{-8}}{y^{12}y^8}$$

$$= \frac{x^1}{y^{20}}$$

(vii) $\left[\frac{\left\{\frac{a^{1/3}}{(b^{-1})^{1/4}}.\left(\frac{b^{1/4}}{a^{1/2}}\right)^2\right\}}{\left(\frac{a^{-1/2}}{b^{-1/2}}\right)}\right]^6 = \left[\frac{\left(\frac{a^{1/3}}{b^{-1/4}}\right).\left(\frac{a^{1/2}}{b^{1/2}}\right)}{\left(\frac{a^{-1/2}}{b^{-1/2}}\right)}\right]^6$

$$= \left[\frac{\left(\frac{a^{(1/3)+(1/2)}}{b^{(-1/4)+(1/2)}}\right)}{\left(\frac{b^{1/2}}{a^{1/2}}\right)}\right]^6$$

$$= \left[b^{-1/4} \times a^{4/3}\right]^6$$

$$= b^{-6/4} \times a^8$$

$$= b^{-3/2} a^8$$

Example 5

Simplify the following expressions

(i) $\{a^{1/3}(a^{-1/2}b^{-1/3}[a^2b^2]^{2/3})^{-1/2}\}^6$

(ii) $(a^{-1})^{-2} \times \left(a^{-1/2}\right)^2$

(iii) $\frac{(2^{p+1})^q}{(2^{q+1})^p} \times \frac{2^{2p}}{2^{2q}} \times 2^q$

(iv) $\{4a^{-1}b^2c^{1/2}\}^{1/3} \times \{12a^3b^{-2/3}c^2\}^{1/4} \div \{108a^{-3}b^2c^{-4}\}^{1/12}$

(v) $\left(4^{1/2} \times 4^{2/3} \times 4^{3/5}\right) \div 4^{4/15}$

Solution

(i) $\frac{1}{a^{1/12}b^3}$

(ii) a

(iii) 2^p

(iv) $2ca^{2/3}b^{1/3}$

(v) 8

5.2 Logarithms

Let us consider the equation $y = a^x$ where a is some constant greater than zero. If x is a variable, then the exponent (a^x) is also a variable.

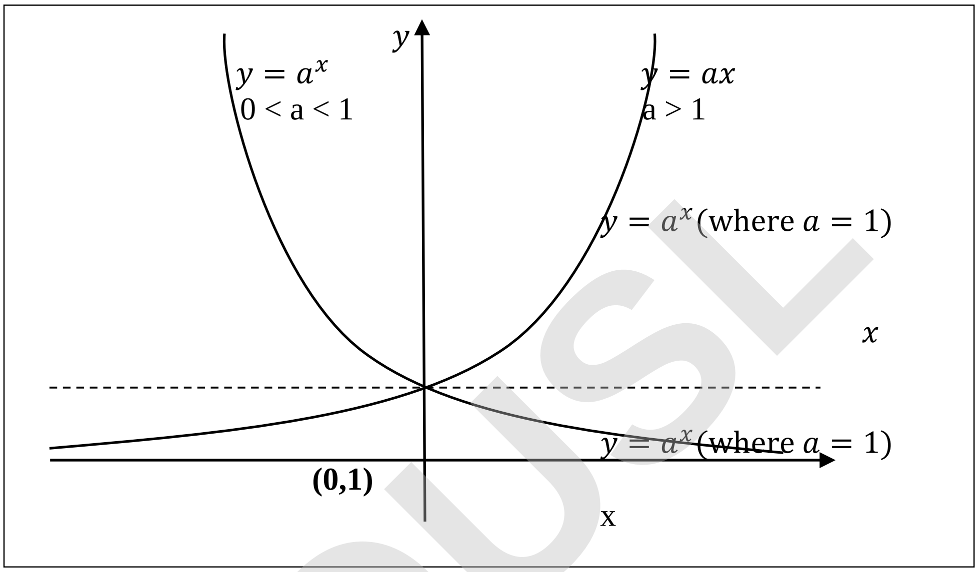


Figure 5.2.1 : the graph of $y = a^x$If $y = a^x$, then x is the logarithm of *y* to the base *a*. It is denoted by $\log_a y$. i.e. $y = a^x \iff x = \log_a y$

Example 6

$y = 3^x$ is an exponential function,

$$y = 3^x \iff x = \log_3 y$$

$\therefore$ x is the logarithm of y to the base 3.

$3^5 = 243 \iff 5 = \log_3 243$

$\therefore$ The logarithm of 243 to the base 3 is 5.

$3^{-3} = \frac{1}{27} \iff \log_3 {}^1/_{27} = -3$

Therefore, the exponent may be negative or positive, but the base must be always positive.

Example 7

(i) $128^{4/7} = 16 \iff \log_{128} 16 = 4/7$

(ii) $343^{2/3} = 49 \iff \log_{343} 49 = 2/3$

(iii) $\log_9 729 = 3 \iff 9^3 = 729$

(iv) $\log_6 1296 = 4 \iff 6^4 = 1296$

Note

$128^{4/7} = 16$ in logarithmic form $4/7 = \log_{128} 16$

$343^{2/3} = 49$ in logarithmic form $2/3 = \log_{343} 49$

$\log_9 729 = 3$ in logarithmic form $729 = 9^3$

$\log_6 1296 = 4$ in logarithmic form $1296 = 6^4$

Example 8

(i) Express the given expressions in a logarithmic form

(a) $32^{4/5} = 16$

(b) $64^{2/3} = 46$

(c) $5^3 = 125$

(d) $2^{-9} = \frac{1}{512}$

(e) $e^{-x} = t$

(f) $a^3 = b$

(ii) Express the given equations in an exponential form

(a) $\log_4 256 = 4$ (b) $\log_6\left(1/216\right) = -3$

(c) $\log_{1/2} 16 = -4$ (d) $\log_4\left(1/512\right) = -9$

(e) $\log_a b = 3$ (f) $\log_5 625 = 4$

Solution

(i) (a) $4/5 = \log_{32} 16$ (b) $2/3 = \log_{64} 46$

(c) $3 = \log_5 125$ (d) $-9 = \log_2\left(1/512\right)$

(e) $-x = \log_e t$ (f) $3 = \log_a b$

(ii) (a) $256 = 4^4$ (b) $\frac{1}{216} = 6^{-3}$

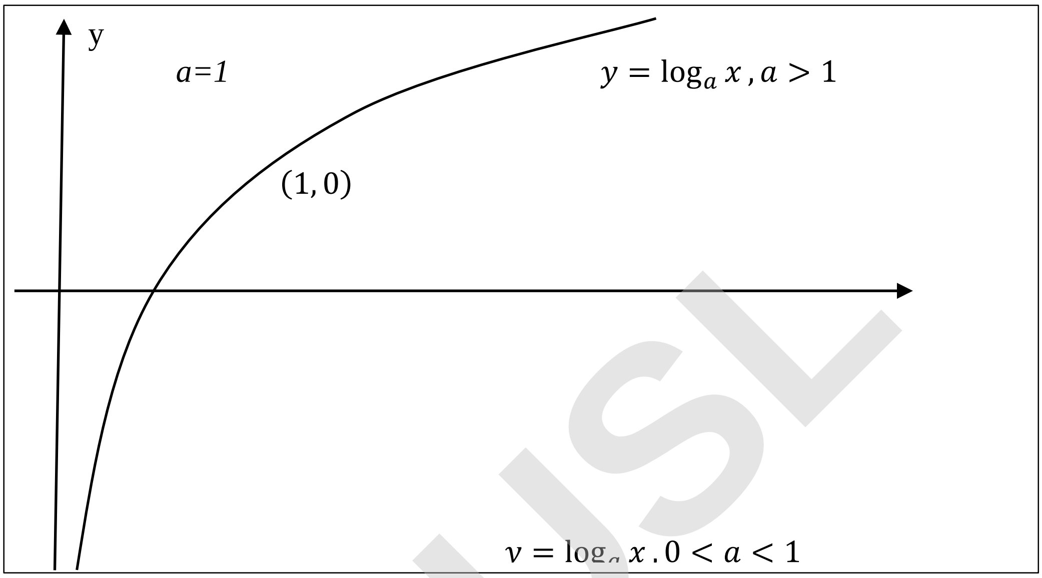
(c) $16 = \left({}^{1}/_{2}\right)^{-4}$ (d) $\frac{1}{512} = 2^{-9}$

(e) $b = a^3$ (f) $625 = 5^4$

5.3 The graph of $y = \log_a x$

In 5.2.1 figure we see that the graph of $y = a^x$

$y = \log_a x \iff x = a^y$



∴ Now we can change the x axis and y axis in the figure 5.2.1. It becomes the curve $y = \log_a x$

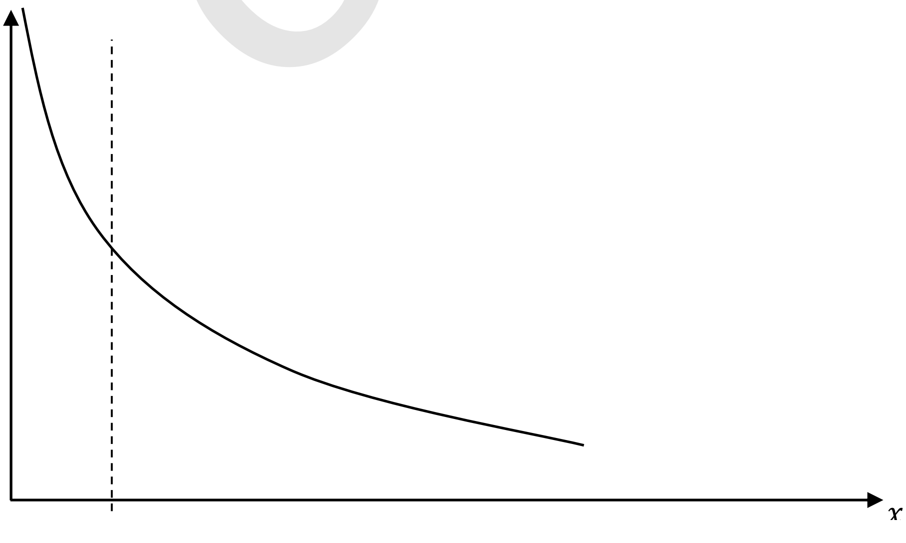


Figure 5.3.1 : the graph of $y = log_a x$

5.4 Rules of Logarithms

	Logarithms	Indices
(i)	$\log_a xy = \log_a x + \log_a y$	$a^x \times a^y = a^{x+y}$
(ii)	$\log_a\left({}^{x}/_{y}\right) = \log_a x - \log_a y$	$\frac{a^x}{a^y} = a^{x-y}$
(iii)	$\log_a x^n = n \log_a x$	$(a^x)^n = a^{nx}$

Example 9

(a) Without using the logarithm tables find the values of the following

(i) $\frac{\log\sqrt{27}+\log 8-\log\sqrt{1000}}{\log 1.2}$

(ii) $\frac{1}{6}\sqrt{\frac{3\log 1728}{1+{}^{1}/_{2}\log 0.36+{}^{1}/_{3}\log 8}}$

(b) If $x^2 + y^2 = 7xy$, Prove that

$$\log(x+y) = \log 3 + \frac{1}{2}\log x + \frac{1}{2}\log y$$

(c) Find the value of

$$\log_2 3 \log_3 4 \log_4 5 \log_5 6 \log_6 7 \log_7 8$$

Answer:

(a) (i)

$$\frac{\log\sqrt{27}+\log 8-\log\sqrt{1000}}{\log 1.2}$$

$$= \frac{\log 3^{3/2} + \log 2^3 - \log 10^{3/2}}{\log 1.2}$$

$$= \frac{\log 3^{3/2} + \log 4^{3/2} - \log 10^{3/2}}{\log 1.2}$$

$$= {}^{3}/_{2}\left\{\frac{\log 3 + \log 4 - \log 10}{\log 1.2}\right\}$$

$$= {}^{3}/_{2}\left\{\frac{\log\frac{3\times 4}{10}}{\log 1.2}\right\}$$

$$= {}^{3}/_{2}\left\{\frac{\log 1.2}{\log 1.2}\right\}$$

$$= {}^{3}/_{2}$$

(iii)

$$\frac{1}{6}\sqrt{\frac{3\log 1728}{1 + {}^{1}/_{2}\log 0.36 + {}^{1}/_{3}\log 8}}$$

$$= \frac{1}{6}\sqrt{\frac{3\log 1728}{1 + {}^{1}/_{2}\log(0.6)^2 + {}^{1}/_{3}\log 2^3}}$$

$$= \frac{1}{6}\sqrt{\frac{3\log 1728}{1 + \log 0.6 + \log 2}}$$

$$= \frac{1}{6}\sqrt{\frac{3\log 1728}{1 + \log 1.2}}$$

$$= \frac{1}{6}\sqrt{\frac{3\log 12^3}{1 + \log 1.2}}$$

$$= \frac{1}{6}\sqrt{\frac{9\log 12}{\log 10 + \log 1.2}}$$

$$= \frac{1}{6}\sqrt{\frac{9\log 12}{\log 12}}$$

$$= \frac{1}{6}\sqrt{9}$$

$$= \frac{3}{6}$$

$$= \frac{1}{2}$$

(b) $x^2 + y^2 = 7xy$

$x^2 + 2xy + y^2 = 9xy$

$(x + y)^2 = 9xy$

By taking logarithm both side

$\log(x + y)^2 = \log 9xy$

$$2\log(x+y) = \log 9 + \log x + \log y$$

$$2\log(x+y) = \log 3^2 + \log x + \log y$$

$$2\log(x+y) = 2\log 3 + \log x + \log y$$

$$\log(x+y) = \log 3 + \frac{1}{2}\log x + \frac{1}{2}\log y$$

(c) $\log_2 3 . \log_3 4 . \log_4 5 . \log_5 6 . \log_6 7 . \log_7 8$

$$\log_2 3 = a \iff 3 = 2^a \quad -(1)$$

$$\log_3 4 = a \iff 4 = 3^b \quad -(2)$$

$$\log_4 5 = a \iff 5 = 4^c \quad -(3)$$

$$\log_5 6 = a \iff 6 = 5^d \quad -(4)$$

$$\log_6 7 = a \iff 7 = 6^e \quad -(5)$$

$$\log_7 8 = a \iff 8 = 7^f \quad -(6)$$

$$From\ (1)\ and\ (2) \quad 4 = (2^a)^b = 2^{ab} \quad -(7)$$

$$From\ (7)\ and\ (3) \quad 5 = (2^{ab})^c = 2^{abc} \quad -(8)$$

$$From\ (8)\ and\ (4) \quad 6 = (2^{abc})^d = 2^{abcd} \quad -(9)$$

$$From\ (9)\ and\ (5) \quad 7 = (2^{abcd})^e = 2^{abcde} \quad -(10)$$

$$From\ (10)\ and\ (6) \quad 8 = (2^{abcde})^f$$

$$8 = 2^{abcdef}$$

$$2^3 = 2^{abcdef}$$

$$abcdef = 3$$

$$\log_2 3 . \log_3 4 . \log_4 5 . \log_5 6 . \log_6 7 . \log_7 8 = 3$$

Example 10

(i) Simplify $\log\frac{133}{65} + 2\log\frac{13}{7} - \log\frac{143}{90} + \log\frac{77}{171}$

(ii) Without using logarithm tables show that

$$2\left[\log\sqrt{125} + \log 27 - \log\sqrt{1000}\right] = 3[\log 9 - \log 2]$$

(iii) Simplify $\log_{10}\left({}^{5}/_{4}\right) + \log_{10} 14 - \log_{10}\left({}^{79}/_{3}\right)$

(iv) Express each of the following as a logarithm of a single quantity

(a) $\log_8 5 - \log_8 25$

(b) $\log_4 3^4 + \log_4 3$

(c) $\log_b y^2 + \log_b y^2 - \log_b \sqrt{y}$

(v) Find the exact value of each of the given logarithm

(a) $\log_2 \left({}^{1}/_{64}\right)$

(b) $\log_5 5^{0.2}$

(c) $\log_4 \sqrt[3]{64}$

(d) $\log_3 243$

(vi) Show that the following by using the definition of the logarithm

$(\alpha) \log_a b \,.\log_b a = 1$

$(\beta) \log_a b \,.\log_b c.\log_c a = 1$

Hence show that $\log_x y = \frac{\log_a y}{\log_a x}$, where $a; x, y$ are positive

Solution

(i) $\log 2$ (iii) ${}^{15}/_{9}$

(iv) (a) $\log_8 {}^{1}/_{5}$ (b) $5 \log_4 3 = \log_4 243$ (c) 10

(v) (a) -6 (b) 0.2 (c) 1 (d) 5

Summary

(I) The rule of indices

(i) $a^m \times a^n = a^{m+n}$

(ii) $a^m \div a^n = a^{m-n}$

(iii) $(a^m)^n = a^{mn}$

(iv) $a^0 = 1$

(v) $a^{-n} = {}^{1}/_{a^n}$

(II) If $y = a^x$ then $x = \log_a y \quad (a > 0)$ this is the definition of logarithm

(III) The laws of logarithm, where a > 0,

$\log_a xy = \log_a x + \log_a y$

$\log_a {}^{x}/_{y} = \log_a x - \log_a y$

$\log_a x^n = n \log_a x$

Learning Outcomes

On completion of this study session, you should have

- Define indices and logarithms.
- Apply the rules of indices and rules of logarithms to simplify algorithmic expressions.
- Sketch the graph of $y = a^x$ and $y = \log_a x$ for any a > 0