UNIT 5

NUMERICAL PRECISION, ACCURACY & ERRORS

Session 18
Scope of numerical computation and error in calculations

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Solution of linear equations

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Interpolation

Introduction

The study of numerical approximation that opposed to symbolic manipulations are called Numerical methods. These methods naturally have applications mainly in every fields of engineering and the physical sciences. In the 21st century, the life sciences, social sciences, medicine, business and even the arts have adopted elements of scientific computations. The development in computing power in computers has revolutionized the use of mathematical models in science and engineering. In subtle numerical methods are required to implement these detailed models of the world. Presently, numerical methods could be used to solve differential equations, to solve system of linear equations and to solve non – linear equations etc.

Numerical methods can only be applied to real-world measurements by translation into digits rather than providing exact symbolic answers. Numerical method gives approximate solutions within specified error bounds.

Session 18

Scope of Numerical Computation and Error in Calculations

Contents

Introduction

This session contains an extensive overview of numerical analysis in order for students to gain a good understanding of both fundamental topics and for them to familiarize themselves with the concepts, basic methods, and current techniques. The subject introduce more basic, fundamental material, a greater emphasis on solving the sorts of equations that all engineers must understand and be able to apply: equations with derivatives in which a sufficiently close first approach to the subject matter means that students will come away with the concepts and tools they need to be able to interpret the results. The subject focuses on opening students' minds to as wide a range of methods and applications as possible, so that they end up with a solid background and users of numerical methods.

18.1 Problem solving by mathematical modeling

We like to compose a simple case study which the learner capable of solving up to a point. So that concepts of problem solving, and terminology could be introduced while going through the steps of solving the problem.

Case Study

An organization manufacturing instruments that precision wishes to manufacture a large number of components each made of a thin wire to be connected end to end, in the shape shown in Figure .18.1.

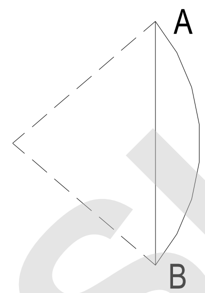

Figure 18.1

To avoid wastage, pieces of 20 cm length arc to be taken, and the radius of the arc is to be 10 cm which is due to the availability of blocks create arcs of circles.

The problem is to find the length AB of the straight portion of a wire correct to the nearest millimeter.

Such a problem is described as a design problem.

How do people tackle design problems?

A clever craftsman with long experience may imagine this construction in his mind and work out with an angle such as half a right angle that may very well use the value in his application. This common-sense type of approach for problems solving where the experience, intuition of the solver provides the solution is described as being heuristic.

Yet another approach is to build a similar model to scale but small in size and get the appropriate angle. This approach is based on the principle of physical modeling. This exercise cannot be done known to reduce by scaling.

A more scientific approach is to describe the problem in mathematical terms, as in the following example.

Example 18.1

Convert the problem mentioned to mathematical form. State any approximations you make.

Solution

The first approximation that will be made is to regard the wire as having negligible cross section.

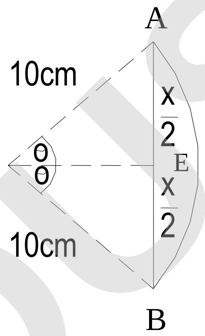

Figure 17.2

It in natural for the learner to imagine that the wire had been bent to the prescribed shape, and to take the required length AB to BE x 2.

Also, if $\angle AOB = 2\theta$ RADIS, then, Sin$\theta = x/20$

i.e. θ = Sin^{-1}(x/20)

The arc length AB = 10 $\times 2\theta$ = 20 Sin^{-1} (x/20)

Since the length of the wire is 20 cm

$$x + 20\ \mathrm{Sin}^{-1}\frac{x}{20} = 20$$

This equation is somewhat clumsy. If we take x in terms of, θ then,

i.e. $20 \text{ Sin}\theta + 20\theta = 20$

$$\text{Sin}\theta + \theta = 1$$

which looks simple enough to proceed further. Once we solve this equation for θ, we can find x.

What we just completed was an exercise where known and unknown values are expressed symbolically and mathematical relations are derived based on physical properties. This activity can be relations derived the model.

Now we have a mathematical problem in the nature of solving the nonlinear equation in θ, derived earlier.

Example 18.2

Explore the possibility of solving the equation by means of any known method in algebra, calculus, etc.

(It is advisable not to spend more than 10 minutes for this)

Solution

We cannot think of a known method to find and exact solution. However, it may be possible to find an approximates solution by graphical or other means certainly, there seem to be no method that give θ in terms of standard functions such as logarithms or trigonometric functions of some number or numbers. We describe this situation as the equation under consideration not having an analytical solution.

What then can we do?

Of course, we must satisfy that the equation could be solved. In other words, we must try to analyze the mathematical problem to examine whether there is a solution which is real (in this case), if so, how many and this what range of values.

If the mathematical model does not yield a result of the type that is required for the application, in is meaningless looking for solutions.

Here again, common sense could be a starting point, as seen in the next example.

Example 18.3

Examine whether the arrangement given in the problem, (i.e. bending the wire to the prescribed shape) is physically feasible and if so, how many angles are possible.

Based on above, make a common-sense judgement on properties of the solution of the equation θ + Sinθ = 1.

Solution

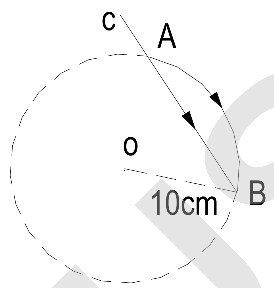

Figure 18.3

Imagine a circle of radius 10cm. Fix one end of the wire at a point A on the circumference. Spread the wire on the arc AB. For smaller angle AOB, the end point C will be outside the circle.

But for one position of the angle, C will be exactly at A. Thus, physically, there will be only one angle $\angle AOB = 2\theta$. The equation θ +Sinθ = 1 can be expected to have only one real solution for θ .

But it is not always that common sense along will help. Mathematically also we may be also to analyze the mathematical problem for properties of the solution, as seen in the following example.

Example 18.4

Show by graphical or other means that the equation θ + Sinθ = 1, has a unique real solution and find a range of values for this solution.

Solution

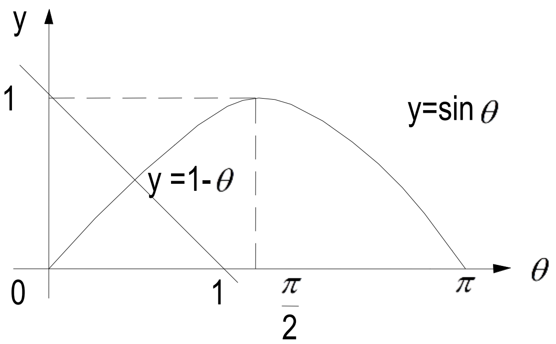

Figure 18.4

As shown in Figure 18.4, the intersection of the curve y = Sinθ, and line y = 1–θ, stands for the solution of the given equation.

Since there is only one point of intersection, the given equation has only one real solution. Also, the solution lies in the interval $(0, \pi/2)$.

This analysis gives more quantitative information than that based on the common sense approach.

We now need an approximate solution of the equation.

Example 18.5

By plotting graphs of Figure 18.4, obtain an approximate solution.

Solution

The solution $\theta \approx 0.5$ radian just obtain is an approximate one found by means of a non-analytical method. Such a solution is described as a numerical solution.

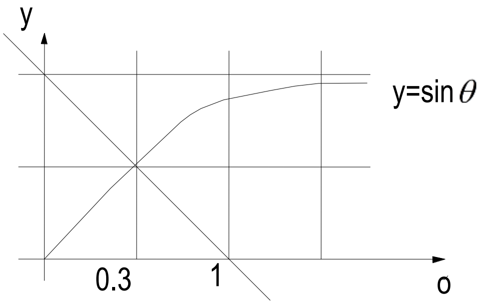

Figure 18.5

The next question we face is whether we accept this solution. This depends on what use we are going to make of the solution, which will decide what accuracy, is required in the solution.

So, we analyze the solution for its accuracy. For instance, in one example, the curve may not be drawn properly. In any case, it is only possible to read to only the nearest first decimal place, a value on this graph.

This sort of analysis on accuracy of the numerical solution is a primary function of numerical analysis.

We have to no examine whether this result is good enough taking into account that the length of the straight path of the bent wire is to be found to the nearest millimeter.

Since this length x is such that

$$x = 20 \text{ Sin}\,\theta$$

$$\Delta x = 20\, Cos\theta\, \Delta\theta$$

$$|\Delta x| = 20 |Cos\theta| \;\; |\Delta\theta|$$

i.e. $\quad |\Delta x| \leq 20|\Delta\theta|$, since $|Cos\theta| \leq 1$

Since x is to be measured in centimeters to the nearest millimeter, $|\Delta x|$ should be such that

$$|\Delta x| \leq 0.05$$

This will be so if;

$$20\ |\Delta\theta| \le 0.05$$

$$|\Delta\theta| \le \frac{0.05}{20} = 0.0025$$

Accordingly, even if the value θ found from the graph is correct, to the first decimal place, it is not acceptable.

But, it is more than sufficient if we find θ correct to the third decimal place of a radian, since if $|\Delta\theta| \le 0.0005$, then $|\Delta\theta| \le 0.0025$, certainly.

Starting from θ = 0.5, the learner could possible think of a technique along the following considerations. Let the values of Sinθ and 1–θ be tabulated and compared. If the value of Sin θ is less than the other, then θ has to be increased by a small amount such as 0.1.

With the use of a pocket calculator, the values of Sinθ and 1–θ can be "balanced" against each other and the value θ increased or decreased accordingly. The process can be continued until the value of θ cannot be changed at the 3rd decimal place.

Example 18.6

Solve the equation Sinθ = 1–θ, by the method described earlier.

Solution

The results are tabulated as follows:

θ	Sinθ	1 –θ	Remarks
0.500	0.479	0.500	θ has to be increased
0.600	0.565	0.400	too high θ
0.550	0.523	0.450	still too high θ
0.525	0.501	0.475	still too high
0.513	0.491	0.487	still to high

0.506	0.485	0.494	θ too low
0.510	0.488	0.490	still too low
0.512	0.490	0.488	too high
0.511	0.489	0.489	is the numerical solution

Length of chord = 20 Sin (0.511) cm

= 9.8 cm

Obviously, this method of "balancing" needed more resources such as time and equipment such as a calculator or a log table, compared to the need for a graph paper, in the earlier method.

The examination of resources and material required for a particular method to solve a given problem and the comparison of such requirements is also a function of numerical analysis.

18.2 Scope of numerical methods

Steps for problem solving by the use of numerical methods

1. State the design problem
2. Prepare mathematical model
3. Simplify the mathematical problem (MP) arising from 2.
4. If the MP can be solved by analytic methods, obtain such a solution, apply to design and stop; else go to next step.
5. Analyze the MP can be properties of solution.
6. Apply a numerical method and obtain a numerical solution.
7. If the accuracy of the solution is acceptable, then apply solution to the design and se go to 2.

The learner has already found one area of study where numerical methods are applicable, that is in the solution of a non – linear equation. The graphical method was one numerical method for this purpose. To have a solution of greater accuracy, we need other methods, and some such methods will be introduced in a later session.

Likewise, numerical methods could be applied to a host of other areas such as integration, curve fitting, solution of systems of equations (linear and nonlinear), ordinary differential equations etc. These will be encountered in later sections.

18.3 Numerical analysis and numerical computation

The linear has already found two aspects of numerical analysis. One is to estimate the error of numerical solution. Generally, it is not the exact error of these solutions that need be found by analysis, but upper bounds for the errors. If these bounds are within prescribed tolerance levels, then the solution could be applied in the design.

It is appropriate quote form F.B. Hildebrand from the introduction to his book "Numerical Methods" published in 1956 (Mc Graw – Hill).

"Generally, the numerical analyst does not strive for exactness. Instead, he attempts to devise a method which will yield an approximation differing from exactness by less than a specified tolerance.

Another function for numerical analysis is to compare alternate numerical methods for the same task, for their efficiency. By efficiency, we mean proper allocation of resources such as time that are required either from persons or from computers that are utilized to apply the numerical methods. Another resource is of course the computer memory capacity.

The study of numerical methods and analysis oriented towards computations by means of calculators, computers and other devices can be described as numerical computations.

Activity 18.1

1. A log of wood of specific gravity 3/4 and the shape of a right circular cylinder floats in water with the axis horizontal. It is required to find what proportion of a vertical diameter will be submerged.
   What sort of solution can be given along heuristic approach and for what reasons?

   What are the weak points in the arguments?

2. Trace the step of solving the above problem find a numerical solution to the problem.

18.4 Measurement of error

If the exact value of a quantity is x and y computed value $\bar{x}$, then the following measurement are defined.

error $= e = x - \bar{x}$

absolute error $= |e|$

relative error $= \dfrac{e}{|x|}$

percentage error $= \dfrac{e}{|x|} \times 100\%$

If there is a value E such that $|e| < E$, then the value E is described as the limiting absolute error.

Since $|e| = |x - \bar{x}| \leq E$

then $\boxed{\bar{x} - E \leq x \leq \bar{x} + E}$

Similarly, if Δ is the limiting relative error

$$|\delta| = \frac{|e|}{|x|} \le \Delta$$

i.e. $E = |x|\Delta$

Example 18.7

The density of water in a certain measurement was found to be $0.998 \pm 0.0001 \text{gm.cm}^{-3}$. Find the limiting relative error of this result.

Solution

$$\Delta \approx \frac{e}{x}$$

$$= \frac{0.0001}{0.998}$$

$$= 0.0001$$

Percentage limiting relative error = 0.01%

Example 18.8

If a value R = 29.25, has relative error of 0.12 find range of values for R.

E = 29.25 x 0.01 $\approx$ 0.03

$$29.25 - 0.03 \le R \le 29.25 + 0.03$$

$$2922 \le R \le 29.28$$

18.9 Types of error

For this study we shall deal with three types of errors in computations.

1. Fundamental Errors

These are errors in the data or in the assumptions made an in the approximations made in constructing the mathematical models. We are not concerned in this type of error in the present study.

2. Truncation Error

This is the error that is caused when a finite number of terms is taken from an infinite series, for the purpose of a computation.

The Taylor series plays important role in this type of computations where the limiting value of the truncation error can be easily found.

The expansion for a function of a single variable can be expressed as follows provided that the function satisfies the required differentiability properties for the following expression to be meaningful.

$$F\,(a + h) = f(a) + \frac{h}{1!}f'(a) + \ldots\ldots + \frac{h^n}{n!}f^{(n)}\,(a) + R_n$$

$$\text{Where } R_n = \frac{h^{n+1}}{(n+1)!}\,f^{(n+1)}\,(a + ch)$$

$$0 < c < 1.$$

Example 18.9

Expand Sin x as a power series in x up to the term in x^3, and use the approximation to calculate Sin (0.1). Find the limiting absolute error and also obtain Sin 0.1 from a scientific pocket calculated earlier.

Solution

By taking $f(x)$ = Sin x and applying Maclaren's theorem

$$\text{Sin } x = x - \frac{x^3}{6} + e; \qquad e = \frac{x^4}{4!}\ \text{Sin } c$$

$$\text{Sin } 0.1 = 0.1 - \frac{0.1^3}{6} \approx 0.09983$$

$$|e| = \left|\frac{x^4}{4!} Sin\, c\right|$$

$$= \frac{0.1^4}{4!}\left|Sin\, c\right|$$

$$\leq \frac{0.1^4}{24}, \text{ since } \left|Sin\, c\right| \leq 1$$

i.e. $|e| \leq 0.000004$

From a calculator with 8 digit display,

Sin 0.1 = 0.0998334

If we take this as the correct value, then

e' = 0.998334 – 0.09983

$|e'| = 0.00000006 < |e|$

What is seen here is that the actual error is usually much less than the limiting error.

3. Rounding error (or Round off error)

We will discuss this as a separate sub section in greater detail.

18.6 Rounding

(or Round off error)

The effect of rounding a number x to the Dth decimal digit according to that algorithm giving the number $\bar{x}$, causes an error e such that

$$\boxed{|e| \leq \frac{1}{2} \times 10^{-D}}$$

Where as usual e = $x - \bar{x}$

In order to get a feeling about the effect of rounding on arithmetic operations, let us work the following example.

Example 18.10

Square the rounded numbers obtained for $\sqrt{13}$ in the three cases given below and make observations on how to keep the effect of rounding off of errors as small as possible.

Solution

$x^2 = 13$

(a) $\overline{x_1} = 3.6$ $\qquad\qquad \overline{x}_1^{\,2} = 12.96$

(b) $\overline{x}_2 = 3.61$ $\qquad\qquad \overline{x}_2^{\,2} = 13.0321$

(c) $\overline{x}_3 = 3.605$ $\qquad\qquad \overline{x}_3^{\,2} = 12.996025$

To keep the rounding off error small as possible, we have to round decimal numbers to large numbers of digits, so that each number has small errors.

It is sometimes possible to work with a lesser number of decimal places but still obtain greater accuracy by avoiding operations such as;

(i) Subtracting almost equal numbers form one another

(ii) Dividing by small numbers

(iii) Multiplying by large numbers

You may question how example (ii) and (iii) can be accomplished. Consider the following operations.

$$1352.21 \times 408.121 = 1.35231 \times 4.08121 \times 10^5$$

$$= 5.5191 \times 10^5$$

$$\frac{4312.325}{0.2158} = \frac{4.312325}{2.158} \times 10^3$$

$$= 1.9983 \times 10^3$$

The following example will show you that higher accuracy can be achieved even with working with less decimal places, by taking precautions mentioned.

Example 18.11

$\sqrt{13.01} = 3.606938$ and $\sqrt{13} = 3.605551$

Find $\sqrt{13.01} - \sqrt{13}$ by working with;

(a) 4 decimal places (b) 2 decimal places and

(c) 2 decimal places but taking precautions mentioned

Solution

(a) $\sqrt{13.01} - \sqrt{13} = 3.6069 - 3.6056 = 0.0013$

(b) $\sqrt{13.01} - \sqrt{13} = 3.61 - 3.61 = 0.00$

(c) $\sqrt{13.01} - \sqrt{13} = \left(\sqrt{13.01} - \sqrt{13}\right)\dfrac{\left(\sqrt{13.01} + \sqrt{13}\right)}{\left(\sqrt{13.01} + \sqrt{13}\right)}$

$$= \frac{13.01 - 13}{\sqrt{13.01} + \sqrt{13}}$$

$$= \frac{0.01}{3.61 + 3.60} = \frac{0.01}{7.21}$$

$$= 1.39 \times 10^{-3}$$

18.7 Errors of computation and rounding results

In this sub-section, we shall discuss matters as indicated below:

Errors of Arithmetic Operations

1. Addition

If numbers $\bar{x}_1, \bar{x}_2, ..., \bar{x}_n$ have errors $\Delta x_1, \Delta x_2, ...\Delta x_n$ respectively, then

$x = \bar{x}_1 + \bar{x}_2 + ... + \bar{x}_n$ has error e where

$e = \Delta x_1 + \Delta x_2 + ...+ \Delta x_n$

$$e < |\Delta x_1| + |\Delta x_2| + \ldots\ldots\ldots\ldots + |\Delta x_n|$$

2. Subtraction

For numbers $\overline{x_1}, \overline{x_2}$

$\overline{x}_1 - \overline{x}_2$ has error e, where

$$e \le \Delta x_1 - \Delta x_2$$

$$|e| \le |\Delta x_1| + |\Delta x_2|$$

3. Multiplication

$x = \overline{x}_1 \overline{x}_2 \ldots\ldots\ldots \overline{x}_n$ has error e, where

$$\frac{e}{x} = \frac{\Delta x_1}{x_1} + \frac{\Delta x_2}{x_2} + \ldots\ldots\ldots\ldots\ldots\ldots + \frac{\Delta x_n}{x_n}$$

$$\boxed{\left|\frac{e}{x}\right| \le \left|\frac{\Delta x_1}{x_1}\right| + \left|\frac{\Delta x_2}{x_2}\right| + \ldots\ldots\ldots\ldots\ldots\ldots\ldots + \left|\frac{\Delta x_n}{x_n}\right|}$$

4. Product of powers

$$x = x_1^{\alpha} x_2^{\beta} \ldots\ldots\ldots\ldots\ldots\ldots\ldots x_n^{\gamma}$$

$$\boxed{\frac{e}{\overline{x}} = |\alpha| \left|\frac{\Delta x_1}{\overline{x}_1}\right| + |\beta| \left|\frac{\Delta x_2}{\overline{x}_2}\right| + \ldots\ldots\ldots\ldots\ldots + |\gamma| \left|\frac{\Delta x_n}{\overline{x}_n}\right|}$$

The proof of these results is left to the learner.

Example 18.12

$\sqrt{13.01} \approx 3.61$ with an error of – 0.0006

$\sqrt{13} \approx 3.60$ with an error of 0.0006

What is the error of $\sqrt{13.01} - \sqrt{13} = 0.01$

Solution

$$\Delta x_1 = -0.0006, \quad \Delta x_2 = 0.0006$$

i.e. $\quad |e| < 0.0006 + 0.0006$

$\quad = 0.0012$

Activity 18.2

1. Using the power series expansion for $\log_e (1 + x)$, calculate $\log_e 1.2$ accurate to 3 decimal places.
2. The criterion for a pass in a certain exam is that the average mark should not be less than 40 marks. Is a candidate who gets an average of 39.5 entitled for a pass? Argue your case. If the 40 in the criterion is changed to 40.0, what happens to the candidate?
3. In an experiment to measure g using the simple pendulum method N = 100, t = 200.60 sec, l = 100.00 cm with possible errors of ± 0.01 in each of the measurement of t and l. Find g from the formula $\frac{t}{N} = 2\pi\sqrt{\frac{l}{g}}$, together with the limiting error.
4. (i) Given that $\sqrt{4.1} = 2.0248$, work with two decimal digits to calculate $\frac{1}{\sqrt{4.1}-2}$ to highest accuracy possible.

(ii) Two resistances r_1 =10.000 Ω and r_2 = 20.000 Ω when combined in parallel produce and equivalent resistance R given by;

$1/R = 1/r_1 + 1/r_2$.

Give the value of R (correct to the appropriate decimal place).

Summary

In this session, we have introduced the basic concepts related to Scope of Numerical Computation and Error in Calculations. Initially, we have discussed some case studies involving in numerical computations. We have constructed some equations non – linear equations using case studies. We have described the errors when solving non – linear equations. We have discussed the methods of rounding errors. We have calculated errors when solving non – linear equations.

Learning outcomes

At the completion of the lesson, the learner should be able to:

1. Appreciate the approach to problem solving using scientific considerations and the need by numerical methods
2. Identify features of numerical methods in numerical analysis
3. Estimate errors of calculations
4. Appreciate the need and workout means reducing errors in calculations

Session 19

Solutions of Non – Linear Equations

Contents

Introduction

In general, some non – linear equations cannot be solved without numerical method to approximate the solution to the equation. The example of a single equation illustrates some of the problems that are considered in this session. Newton's iteration method for solution of linear equations is initially described and illustrated for the case of a linear equation. The discussion of Newton's method is then expanded to the simultaneous solution of many equations.

19.1 Choice of the initial approximation

In session 1, the learner encountered a "balancing" method to solve an equation, where an attempt was made by starting with an approximate solution (or solutions) to improve it. This type of activity is called iterations. Here we now look at different method of obtaining an initial approximation to a given equation.

1. Graphical Method: This has been discussed in session 1.
2. Taylor's Method : Here terms of an equation can be expanded as power series and only the first few terms be taken, as in the following example.

   To solve; x + Sin x = 1

   Take $x + x - x^3/3 + x^5/5 = 1$

   Neglecting higher powers of x, then $x + x = 1$

   $$x = 0.5$$

   $x_0 = 0.5$ is a suitable initial approximation.

3. Physical considerations: Some property of the physical problem which gave rise to the numerical problem could be used to get a suitable initial approximation. If for instance, the unknown in an equation is the coefficient of fiction, then it is well known that μ is between 0 and 1. Thus $\mu_0 = 0.5$ is a suitable initial approximation.

   Make a guess: In the absence of any of the methods discussed above, best thing is to make a guess, usually a convenient number such 0 or 1 or 1/2.

19.2 The method of bisection

Let the learner re-discover this method to solve an equation for a real solution, by trying to solve an equation encountered in session 1, along suggestions made in the following example. However, studied as the balancing method in session 1.

Example 19.1

In attempting to solve x + Sin x – 1 = 0, take

$$f(x) = x + \text{Sin } x - 1,$$

with $x_0 = 0$, $x_1 = \pi/2$, find the sign of $f(x_0)\, f(x_1)$.

By sketching the graph for $y = f(x)$ from $x = 0$ to $\pi/2$, give a new range for the solution of the above equation.

Solution

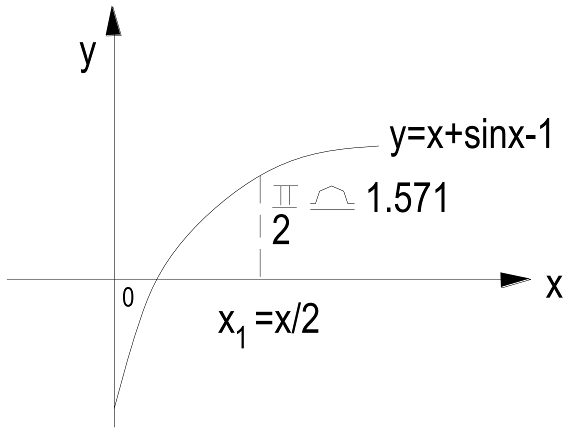

Figure 18.1

$$f(x_0) = f(0) = 0 + Sin\,0 - 1 = -1$$

$$f(x_1) = f(\pi/2) = \frac{\pi}{2} + Sin\frac{\pi}{2} - 1 = \frac{\pi}{2}$$

$$f(x_0)\,f(x_1) < 0$$

Since $f(x)$ is continuous from x_0 to x_1, and as it changes sign,

$$f(x) = 0$$

has an add number of real roots between 0 and $\pi/2$. Thus there is at least one root. But $y \geq 0$ for all x.

Therefore, y is non-decreasing for all x. Hence $f(x) = 0$ has one and only one solution in the range 0 to $\pi/2$.

Example 19.2

In the above example, bisect the line from x_0 to x_1 on x axis at

$x_2 = \frac{1}{2}(x_0 + x_1)$. Find the sign of $f(x_0)\,f(x_2)$, and hence a new range for the solution (taken as x*).

Solution

$$x_2 = \frac{1}{2}(0 + \frac{\pi}{2}) = \frac{\pi}{4}$$

$$f(x_2) = \frac{\pi}{4} + Sin\frac{\pi}{4} - 1$$

$$> 0.7857 + 0.7071 – 1 > 0$$

$$f(x_0)\ f(x_2) < 0$$

Thus, x^* is in the interval x_0 = 0 to x_2 = 0.7854

If we pause and think carefully, the method of bisection will become clear.

We can take x_2 as new x_1, (i.e. $x_1 \leftarrow x_2$), and x_0 remaining same, repeat the process.

On the other hand, if $f(x_0)\ f(x_2)$ was greater than zero, then $x_0 \leftarrow x_2$, with x_1 remaining as the same. Next repeat the process as shown in the next example.

Example 19.3

Repeat the procedure as mentioned until the distance between x_0 and x_1, that is, $|x_1 - x_0| \leq 0.005$, and take x_2 as the numerical solution, at that stage. Tabulate your results from the start.

Solution

x_0	x_1	$\|x_1 - x_0\|$	x_2	$f(x_0)\ f(x_2)$	Range for x^*
0.0	π/2 = 1.5708	1.5708	0.7854	<0	x_0 to x_2
0.0	0.7854	0.7854	0.3927	>0	x_2 to x_1
0.3927	0.7854	0.3927	0.5890	<0	x_0 to x_2

0.3927	0.5890	0.1964	0.4908	>0	x_2 to x_1
0.4908	0.5890	0.0982	0.5399	<0	x_0 to x_2
0.4908	0.5399	0.0491	0.5154	<0	x_0 to x_2
0.4908	0.5154	0.0246	0.5031	>0	x_2 to x_1
0.5031	0.5154	0.0123	0.5092	>0	x_2 to x_1
0.5092	0.5154	0.0062	0.5123	<0	x_0 to x_2
0.5092	0.5123	0.0031	0.5108	---	

In the last row of the table, that is the 10th cycle,

$$|x_1 - x_0| = 0.0031 < 0.005.$$

This is when we decided to take x_2 as the numerical solution,

$$x^* \approx 0.5123$$

We now ask the question to which decimal place is the solution correct? We leave the student to construct the algorithm as an activity.

A terminology has to be now introduced.

Iteration

We used the expression 10th cycle in referring to the 10th row in the table above. The better expression is iteration cycle. This method of using values obtained in one iteration cycle to get improved results in the next cycle is called an iterative process.

We can take x_0 $^{(10)}$ to indicate that this statics for the value of x_0 in the 10th iteration.

i.e. x_0 $^{(10)}$ = 0.5092

Question

What is $x_2^{(3)}$?

Answer

$x_2^{(3)} = 0.5890$

Convergence of the Method of Bisection

If $x_2^{(n)}$ .... > a limit as the n ———— $> \infty$, then we say that the iterative scheme is convergent.

For a function F(x) which is continuous in the interval (α, β) where $\alpha = x_0^{(0)}$, $\beta = x_1^{(0)}$, $x_2^{(0)} \in \left(x_0^{(0)}, x_1^{(0)}\right)$, from the manner in which $x_0^{(1)}$, $x_1^{(1)}$ were selected,

$$\left|x_1^{(1)} - x_0^{(1)}\right| = \frac{1}{2}|\beta - \alpha|, \quad x_2^{(1)} \in \left(x_0^{(1)}, x_1^{(1)}\right)$$

Similarly,

$$\left|x_1^{(n)} - x_0^{(n)}\right| = \frac{1}{2}\left|x_1^{(n-1)} - x_0^{(n-1)}\right|$$

$$= \frac{1}{2^n}|\beta - \alpha|,$$

$$x_2^{(n)} \in \left(x_0^{(n)}, x_1^{(n)}\right)$$

As n $\to \infty$, $\left|x_1^{(n)} - x_0^{(n)}\right| \to 0$

i.e. $x_1^{(n)}$, $x_2^{(n)}$, $x_0^{(n)} \to a$ limit = x^*

Error of Approximation

$$\in_n = \left|x^* - x_2^{(n)}\right| \leq \left|x_1^{(n)} - x_0^{(n)}\right|$$

$$\boxed{\in_n < \frac{1}{2^n}|\beta - \alpha|} \text{ ---------------- } (1)$$

Example 19.4

Round the numerical value obtained in Example 18.3, to the appropriate decimal place.

Solution

$$|\epsilon_{10}| < \frac{1}{2^{10}}\left|\frac{\pi}{2} - 0\right| \approx 0.00153$$

$$\leq 0.005 = \frac{1}{2} \times 10^{-2}$$

i.e. x^* is correct only to the 2nd decimal place.

$$\therefore \quad x^* = 0.51$$

19.3 Method of false position (Regula Falsi)

This is a method developed by Sir Isaac Newton and is described as a second order method like the previous one, in the sense two approximation values are required to get a third.

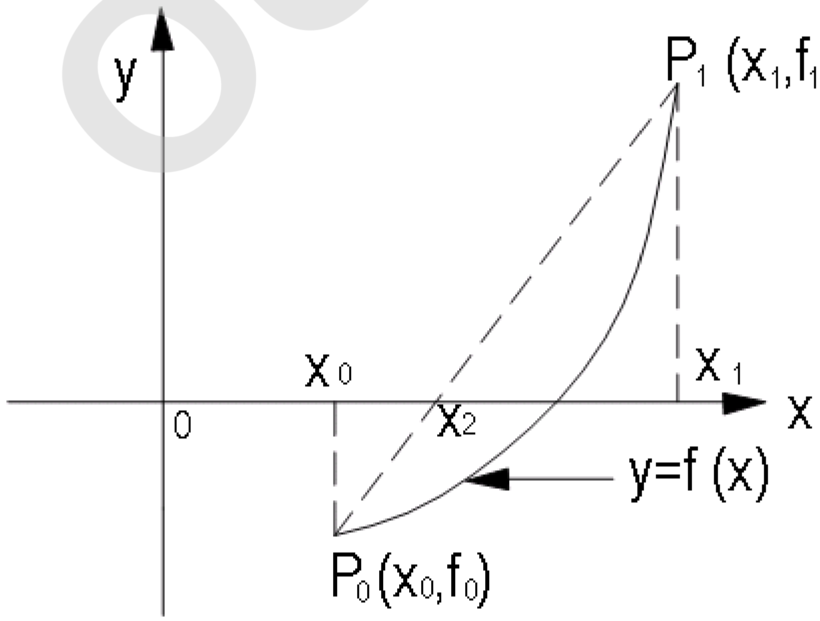

Figure 18.2

As in Figure 18.2, if x_0, x_1 are approximate solutions the point of intersection of the line $(x0, f0)$, $(x_1,\ f_1)$ with the x-axis is taken as x_2, where $f_0 = f(x_0)$, $\quad f_1 = f(x_1)$.

Example 18.5

Derive an expression for x_2 in terms of $(x_0, f_0), (x_1,\ f_1)$.

Solution

From co-ordinate geometry, since $(x_2, 0)$ is the point on p_0p_1,

$$\frac{0 - f_0}{x_2 - x_0} = \frac{f_1 - f_0}{x_1 - x_0}$$

i.e.

$$x_2 = x_0 - \frac{f_0(x_1 - x_0)}{f_1 - f_0}$$

$$x_2 = \frac{x_0 f_1 - x_1 f_0}{f_1 - f_0}$$

Thus, x_0, x_1 give x_2; x_1 and x_2 give x_3 and so on. In the general solution x_{n-1}, x_n gives, with $f_i = f\ (x_i\ )$

$$\boxed{x_{n+1} = \frac{x_{n-1} f_n - x_n f_{n-1}}{f_n - f_{n-1}}} \quad \text{----------------} \quad (2)$$

Example 19.6

Perform five iterations of the Method of False Position to solve $f(x) = d + Sin\, x - 1 = 0$, by taking $x_0 = 0$, , $x_1 = \pi/2 \approx 1.5708$ and tabulate your result (See Example 18.3). Work to 4 decimal places.

Solution

n	x_{n-1}	x_n	f_{n-1}	f_n	x_{n+1}
1	0.	0.5708	– 1	0.5708	0.3634

2	0.5708	0.3634	0.5708	– 0.2812	0.4318
3	0.3634	0.4318	– 0.2812	– 0.1496	0.5096
4	0.4318	0.5096	– 0.1496	– 0.0026	0.5110
5	0.5096	0.5110	– 0.0026	0.0002	0.5109

For a given problem iterations can be stopped at the end of the n^{th} iteration, when for a given tolerance, (i.e. a prescribed small positive numerical) there is a positive integer n, such that

$$\left|x_{n+1} - x_n\right| \leq \in$$

The value x_{n+1} could be taken as the numerical solution. Iterations can also be stopped when the function value $f(x_n)$ is such that $\left|f(x_n)\right| \leq \in$ with $\in > 0$ given.

19.4 Simple iterative method

People have been seeking to improve efficiency of different system and methods used for different purposes. The method of bisection certainly does not appear to be the most efficient way of looking for a root.

Cannot the learner attempt to stretch his or her imagination and look for faster methods, now that the concept of iteration is very much clear. Let the following problem be posed.

Example 19.7

Attempt to solve the equation $x^3 - 2 = 0$, by constructing schemes according to two rearrangements.

A) $\quad x = \dfrac{1}{5}(2 + 5x - x^3)$

B) $x = x^3 + x - 2$

Start with same initial approximation 1.2, and perform six iterations for each scheme.

Solution

It is easy to see that both schemes give the same equation $x^3 - 2 = 0$.

Scheme A	**Scheme B**
$x_{n+1} = \frac{1}{5}(2 + 5x_n - x^3{}_n)$	$x_{n+1} = x^3{}_n + x_n - 2$
$x_0 = 1.2$	$x_0 = 1.2$
$x_1 = 1.2544$	$x_1 = 0.928$
$x_2 = 1.2596$	$x_3 = -0.273$
$x_3 = 1.2599$	$x_4 = -2.274$
$x_4 = 1.2599$	$x_5 = -16.033$
$x_5 = 1.2599$	$x_6 = -4143.29$
$x_6 = 1.2599$	

It seems that the scheme A converges whereas B does not converge, as we say, it diverges. We shall now examine convergence in detail.

Convergence of the Simple Iterative Method

Let us try to find the condition for convergence,

Let $x = g(x)$ be the rearrangement of the given equation $f(x) = 0$. Then the iterative scheme with $x_0, x_1, \dots, x_k$ a closed interval R_1 is given by

$$\boxed{x_{k+1} = g(x_k)} \quad \text{-- (3)}$$

i.e. $x_{k+1} - x_k = g(x_k) - g(x_{k-1})$

Assuming that g (x) is differentiable in R_1

$$g(x_k) - g(x_{k-1}) = (x_k - x_{k-1})g'(c_k) c_k \in R_1$$

i.e. $x_{k+1} - x_k = (x_k - x_{k-1})g'(c_k)$ ------------------ (4)

$$|x_{k+1} - x_k| = |x_k - x_{k-1}| \; |g'(c_k)|$$

If an upper bound for $|g'(x)|$ in R_1 is L.

i.e. $\boxed{|x_{k+1} - x_k| \le L|x_k - x_{k-1}|}$ ----------------- (5)

Writing $k = 1, 2, \ldots \ldots \ldots \ldots \ldots \ldots n$ and multiplying left and right – hand sides.

$\boxed{|x_{n+1} - x_n| < (L)^n |x_1 - x_0|}$ --------------- (6)

A sufficient condition for $|x_{n+1} - x_n| \longrightarrow 0$, as $n \longrightarrow \infty$ is L < 1.

The property is best stated as the following theorem.

Theorem

If the iterative scheme $x_{k+1} = g(x_k)$, k = 0, 1, 2, …….., n with given x_0, produces the sequence $x_0, x_1, \ldots.$ where each x_i (i = 0, 1, …………, n) ,belongs to some closed interval R_1 in which g(x) is continuous and differentiable, then a sufficient is that;

$$L = |g'(x)| < 1$$

Where L = an upper bound for $|g'(x)|$ in R_1.

Example 19.8

Given that, $1 < \sqrt[3]{2} < 3/2$, examine whether the condition for convergence is satisfied by the schemes A and B of Example 2.2.

Solution

Scheme A – $g(x) = \frac{1}{5}\left(2 + 5x - x^3\right)$

$$g'(x) = \frac{1}{5}\left(5 - 3x^2\right) = \left(1 - \frac{3}{5}x^2\right)$$

for $1 \leq x \leq \frac{3}{2}$, sup $\left|g'(x)\right|$ is given when $x = 1$

i.e. L = $\left|1 - \frac{3}{5}\right| = \frac{2}{5} < 1$

The above sufficient condition for convergence is satisfied.

Scheme B – It is left to the learner to show that in this case L > 1.

19.5 Newton – Raphson method

The learner may remember it took 24 iterations to solve the equation x + Sin x = 1 to 3 decimal places using the simple iterative method. Newton's method (programmed for the first time by Raphson) is an attempt to make convergence with less iterations.

The learner could attempt to derive the Newton's formula by studying the Regula - Falsi method and in trying to improve it. In this method, looking at it graphically two initial values x_0, x_1 give two points P_0, P_1 respectively on the graph y = f (x), where f (x) = 0 is the equation to be solved (ref. Figure 2.2). If the chord P_0, P_1 intersects the x – axis at x_2, then x_2 is taken as the improved solution. You may be able to derive the Newton's method if you can think along above lines, as in the following example.

Example 19.5

By taking a single point x_0 as the initial approximation, extend the idea used in the Regula Falsi method to find x_1, an improved solution.

Solution

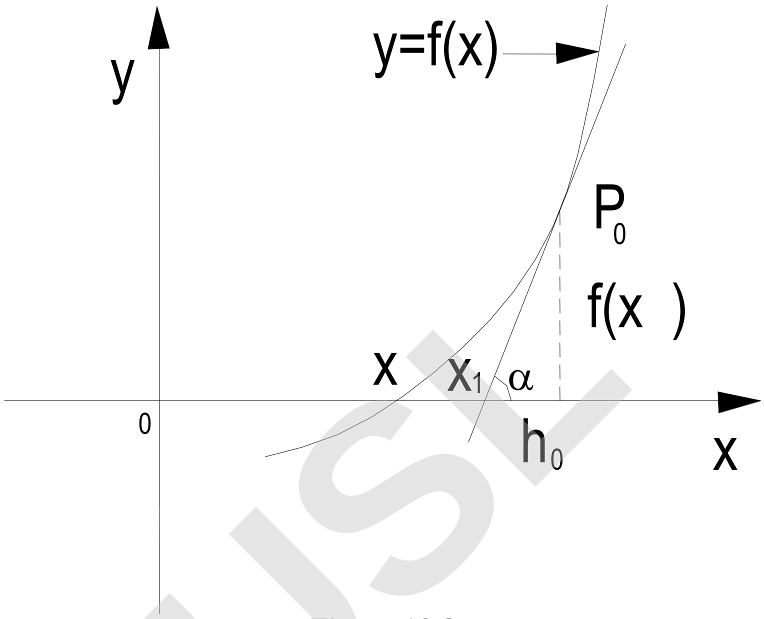

Figure 19.3

It is great if you got the idea that the tangent at P_0 to curve $y = f(x)$ corresponding to $x = x_0$ can be used in place of the chord mentioned in the earlier method.

Let $x_1 = x_0 - h_0$

For angle α shown, Tan α = $\frac{f(x_0)}{h}$, if $h_0 \neq 0$

But Tan α = $f'(x_0)$

i.e. $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$, if $f'(x_0) \neq 0$

If $x_1, x_2, \ldots, x_k$ are found starting from x_0, and if the terms $f'(x) = 0$, for x close to x^*, where these iteration values are found, then the Newton's formula is

$$\boxed{x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}} \quad \text{----------------} \ (7)$$

There is another derivation of this based on Taylor's expansion.

Example 19.10

Solve the equation x + Sin x – 1 = 0, using Newton – Raphson method with convergence to 3 decimal places. Take x_0 = 0.5

Solution

Let $\quad f(x) = x + Sin\, x - 1$

$$f'(x) = 1 + Cos\, x$$

$$x_{k+1} = x_k + \frac{f(x_k)}{f'(x_k)} \quad \text{gives,}$$

$$x_{k+1} = x_k - \frac{x_k + Sin\, x_k - 1}{1 + Cos\, x_k}$$

$$x_{k+1} = \frac{(x_k Cos\, x_k - Sin\, x_k + 1)}{(1 + Cos\, x_k)}$$

k	x_{k+1}
0	0.5
1	0.511
2	0.511

Convergence of the Newton – Raphson Method

Following properties are generally established in text books.

1. The condition for convergence is that $f\,'(x) \; = \; 0$, for x at the solution or near the solution. Modified methods can be used even for the case $f\,'(x) \; = \; 0$ at the solution.
2. If $x_{k+1} = x_k - x^*$, where x^* is the solution, then following properties have been established.

   For simple iterative method $\left|\in_{k+1}\right| \; \alpha \; \left|\in_k\right|$

   For Newton's method $\left|\in_{k+1}\right| \; \alpha \; \left|\in_k\right|^2$

The significance of these properties is that Newton's method takes about half the number of iterations that these required for the other method, starting from the same initial approximation, with other conditions being the same.

Example 19.11

Here is an interesting example for you to try. How can you perform division without actually dividing? For instance, try to find 1/5 by solving the equation $f(x) = \frac{1}{x} - 5 = 0$.

Solution

To find 1/Q, Q ≠ 0, take $f\;(x) \; = \; \frac{1}{x} - Q$

$$f\,'\,(x) \; = \; - \; \frac{1}{x^2}$$

Newton's formula gives, $x_{k+1} = \; x_k - \dfrac{\frac{1}{x_k} - Q}{-\frac{1}{x_k{}^2}}$

i.e. $x_{k+1} = \; x_k + \; x_k(1 - \mathrm{Q}.x_k)$

Take x_0 as a fraction if $|Q| > 1$, and iterate.

To find 1/5, Q = 5 $\Rightarrow x_{k+1} = x_k + x_k(1 - 5x_k)$.

Taking x_0 = 0.3

x_1 = 0.1875

x_2 = 0.1992

x_4= 0.1999

x_5 = 0.2000

x_6 = 0.2000 ==> … 1/5 = 0.2000

Activity 19.1

1. It is required to find $\sqrt[3]{2}$ to five decimal places, by solving the equation $x^3 - 2 = 0$ by means of the method of bisection. Estimate the number of iterations that will be required, and show this is so by performing the iterations.

   Use the fact $1 < \sqrt[3]{2} < 2$

2. Use the method of false position to find $\sqrt[3]{2}$ to 3 decimal places.
3. Construct an iterative scheme that satisfies the condition for convergence of the simple iterative scheme to solve the equation $x^3 - 2 = 0$, given $1 < \sqrt[3]{2} < 3.2$. Estimate the number of iterations that may be needed for convergence to 3 decimal places.

   [Hint take equation as $x = g(x)$, where $g(x) = x - \alpha(x^3 - 2)$. Obtain a convenient positive fraction for α, which is 1/2, 1/3, etc. so that the condition of convergence is satisfied at $x = 1$ and $x = 3/2$]

4. The coefficient of friction μ in a structural joint satisfies the equation,

   $$\mu \ e^{2\mu\pi} = 1$$

   Find μ to 3 decimal places.

5. If water is pumped from a depth h meters and delivered through a pipe of diameter D cm at a rate Q liters per minute, by the use of a water pump of capacity P watts, then it is known that Q satisfies the following equation (to which values of constants have been substituted).

$$\frac{5}{3}Qh + \frac{1}{27}\frac{Q^3}{D^4} = 10^{-7p}$$

Write a computer program to input P and to calculate Q to first decimal place, for h varying from 1 to 10m; in steps of 1m and D from 1 to 2cm in steps of ¼ cm.

Summary

In this session, we have discussed some methods of solving non – linear equations that cannot be solved without numerical method. To solve non – linear equations, we have introduced methods, namely, method of bisection, method of false position, simple iterative method and Newton – Raphson method. In the above methods. We have presented error calculations in each method. We have solved some non – linear equations using above methods. Newton – Raphson iteration method for solution of linear equations has been initially described and illustrated for the case of a linear equation. In Newton – Raphson method has been then expanded to the simultaneous solution of many equations.

Learning Outcomes

At the completion of the session the learner will be able to perform the following activities.

- look for suitable initial approximations.
- use iteration method such as bisection, simple iteration, Regular – Falsi and Newton – Raphson to solve equation.
- examine convergence and estimate the number of iterations that may be needed before starting the iterations.

Session 20

Interpolation

Contents

Introduction

Interpolation polynomials are of particular importance in numerical analysis and so we devote special attention to them in this session. We study both divided differences for arbitrarily spaced points and ordinary differences for equally spaced points. Not only can differences be neatly arranged in tables, for convenient hand computation, but they permit one to easily estimate the error in the approximations. Hence, methods based on differences are useful for digital computers as they suggest very efficient computing techniques and can be used for checking the accuracy of a calculation.

In this session we construct special forms of the interpolation formula, use interpolation of polynomials to determine formula for the interpolated function and examine the error in interpolation.

20.1 Forward differences

The calculus of finite differences is much used in numerical processes such as interpolation, detection of errors etc.

Let $y_0, y_1, y_2, \ldots\ldots$ be the values of a function $y = f(x)$ corresponding to equally spaced values $x_0, x_1, x_2, \ldots\ldots\ldots$ of the independent variable x. The following notation is used to designate differences.

$$Y_1 - Y_0 = \Delta Y_0$$

$$Y_2 - Y_1 = \Delta Y_1$$

$$\ldots\ldots\ldots\ldots\ldots\ldots$$

$$Y_n - Y_{n-1} = \Delta Y_{n-1}$$

Note:

The subscript attached to the Y is taken the name as the subscript of the second member of the difference.

The differences designated above are called forward differences of the first order or simply first differences. Second order differences are denoted as follows.

$$\Delta Y_1 - \Delta Y_0 = \Delta^2 Y_0$$

$$\Delta Y_2 - \Delta Y_1 = \Delta^2 Y_1$$

In general, $(n + 1)^{th}$ order differences are obtained from those of n^{th} order by the formulas;

$$\Delta^n Y_1 - \Delta^n Y_0 = \Delta^{n+1} Y_0$$

$$\Delta^n Y_2 - \Delta^n Y_1 = \Delta^{n+1} Y_1$$

By successive substitutions we easily show that

$$\Delta^2 Y_0 = Y_2 - 2Y_1 + Y_0$$

$$\Delta^3 Y_0 = Y_3 - 3Y_2 + 3Y_1 - Y_0$$

and in general

$$\Delta^{n} Y_0 = \sum_{k=0}^{n} (-1)^k \, {}^{n}c_k \, Y_{n-k} \qquad \text{---------------} \quad (1)$$

In a similar manner, by successive eliminations from equation (1), we obtain

$$Y_n = y_0 + n\,\Delta Y_0 + \frac{n(n-1)}{2!}\Delta^2 Y_0 + \ldots\ldots + \Delta^{n} Y_0$$

$$= \sum_{k=1}^{n} {}^{n}c_k \; \Delta^{k}\, Y_0 \qquad \text{---------------} \quad (2)$$

Formula (2) may be expressed in the following easily remembered symbolic form

$$y_n = (1 + \Delta)^n \, y_0$$

A difference table is set up according to the scheme below.

x_0	y_0				
		Δy_0			
x_1	y_1		$\Delta^2 y_0$		
		Δy_1		$\Delta^3 y_0$	
x_2	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$
		Δy_2		$\Delta^3 y_1$	
x_3	y_3		$\Delta^2 y_2$		—
		Δy_3		—	
x_4	y_4		—		
		—			
—	—				

Each entry in the table of differences is the difference of the adjacent entries in the column to the left.

Example 20.1

From a table of difference for y = Sin x with entries $x = 0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6.$

Solution

The table appears as follows.

x	**y**				
0	0.00000				
9983					
0.1	0.09983		– 99		
		9884		–100	
0.2	0.19867		–199		4
		9685		–96	
0.3	0.29552		–295		2
		9390		–94	
0.4	0.38942		–389		3
		9001		–91	
0.5	0.47943		–480		
		8521			
0.6	0.56464				

Note that in the column of difference it saves space to omit the decimal point and this omission need cause no confusion since we can easily restore the decimal point if needed.

If y = p(x) is any polynomial of degree n in x whose values $y_0, y_1, y_2, ...$are tabulated for equally spaced value $x_0, x_1, x_2, ...$ with interval h, we may make the change of variable $x = hs + x_0$, after which $p(x)$ becomes a polynomial $Q(s)$ of degree n in s such that;

$$Y_k = Q(k), \quad k = 0, 1, 2, \ldots..$$

The polynomial Q(s) can be expressed in terms of factorials

$$Q(s) = a_0 s^{(n)} + a_1 s^{(n-1)} + \ldots\ldots\ldots + a_n$$

Which gives $\Delta s^{(n)} = ns^{(n-1)}$

And in general $\Delta^k s^{(n)} = n^{(k)} s^{(n-k)}$

Hence $\Delta_n^k p(x) = \Delta_1^k Q(s) = a_0 n^{(k)} s^{(n-k)} + a_1 (n-1)^{(k)} s^{(n-1-k)} + \ldots\ldots\ldots\ldots;$

where the subscript on the Δ indicates the interval for which the differences are formed. It is evident that the kth difference of any polynomial of degree n is a polynomial of degree n – k. In particular the nth difference is a constant and all differences of order higher than n are zero.

20.2 Error propagation

The method of difference is frequently employed to check a table of functional values. If such a table is to be useful, the tabular interval h must be chosen with respect to the number of decimal places in such a way that the n^{th} differences are negligible to the given number of decimal places for a comparatively low value of n.

Suppose that a table has been calculated, the difference formed and suppose that differences of say the fifth order are negligible. If same entry in the table has an error of magnitude E, the effect of this error will propagate itself through the difference table in the manner shown below.

y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
__		__		__	
	__		__		__ E
__		__		__E	
	__		__E		__–5E
__		__E		__–4E	
	__E		__–3E		__10E
__E		__–2E		__6E	
	__–E		__3E		__–10E
__		__E		__–4E	

Since the fifth difference is normally negligible, the effect of the error on the 5^{th} difference becomes quite conspicuous and we may not only locate the entry that is incorrect by even estimate the amount of the error. But if many entries have errors, this becomes complicated.

In the use of finite difference there is another source of irregularity which is due to inaccuracies caused by rounding off the maximum effect would number of decimal places. The maximum effect would occur if quantities neglected in each entry alternated in sign and were equal to half a unit in the last decimal place retained. It is the number of decimals retained, we set $e = \frac{1}{2} \times 10^{-n}$ and consider the difference table.

Y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
__+e				
	__–2e			
__–e		__+4e		
	__+2e		__–8e	
__+e		__–4e		__+16e
	__–2e		__+8e	
__–e		__+4e		
	__+2e			
__+e				

It appears that the maximum possible effect on the n^{th} difference due to rounding off is $2^n e$, or $2^{n-1} \times 10^{-n}$.

20.3 Newton's forward difference interpolation formula

An interesting and useful set of interpolation formulas can be derived by supposing that the interpolating polynomial is expressed in terms of factorial polynomials. To do this we first make the change of variable $x = hs + x_0$, where h is the interval.

Note: $x^{(n)} = \dfrac{x!}{(x-1)!}$

Let y_s be the interpolating polynomial of degree *n* and let,

$$y_s = a_0 + a_1 s + a_2 s^{(2)} + \ldots\ldots\ldots + a_n s^{(n)} \text{ ---------------- (1)}$$

Where the '*a*' s are undetermined coefficients. Then if we apply the operation ^{k} to equation-(1) we have;

$$\Delta^k \ y_s = \sum_{i=0}^{n} a_i \, \Delta^k \ s^{(1)}$$

In this equation, let *s* = 0,

$$^k s^{(i)} \equiv 0 \qquad \text{if} \quad i < k$$

$$^k s^{(k)} \equiv k\,!$$

$$^k s^{(i)} \equiv 0 \quad \text{for } s = 0 \quad \text{if} \quad i > k.$$

Hence (1) becomes

$$\Delta^k y_0 = k!\, a_k$$

$$a_k = \Delta^k y_0 / k!$$

With there values for the coefficient; equation (1) becomes,

$$y_s = y_0 + \Delta y_0 \, s + \Delta^2 y_0 \binom{s}{2} + \ldots\ldots + \Delta^n y_0 \binom{s}{n} \quad \text{-----------}$$

--- (2)

Note :

$$\binom{n}{r} = \frac{n!}{(n-r)!\ r!}$$

This is Newton's Forward difference formula. It can be expressed in a neat symbolic form as the first $(n+1)$ terms in the expansions of

$$(1 + \Delta)^s \, y_0$$

Note that the values of y actually entering into the formation of (2) are $y_0, y_1, y_2, \dots, y_n$

20.4 Newton's backward difference interpolation formula

Here we assume that the interpolation polynomial is of the form

$y_s = a_0 + a_1\, s + a_2\, (s+1)^{(2)} + a_3(s + 2)^{(3)} + \dots + a_n(s + n - 1)^{(n)}$ ------------ (1)

Then $^k\, y_s = \sum_{i=0}^{n} a_i\ \Delta^k\, (s + i - 1)^{(i)}$

and again

$$\Delta^k\, (s + i - 1)^{(i)} \equiv 0 \ \text{ if } \ i < k$$

$$\Delta^k\, (s + k - 1)^{(k)} = k\,!$$

$$\Delta^k\, (s + i - 1)^{(i)} = 0 \text{ for } \ s = -k \ \text{ if } \ i > k$$

So that when $s = -k$, we have

$$\Delta^k\, y_{-k} = k\,!\, a_k\,,$$

and equation (1) becomes

$$y_s = y_0 + \Delta y_{-1}\, s + \Delta^2 y_{-2}\binom{s+1}{2} + \Delta^3 y_{-3}\binom{s+2}{3} + \dots\dots + \Delta^n y_{-n}\binom{s+n-1}{n}$$

--------------- (2)

The values of y which enter this formula are y_0, y_{-1}, y_{-n}.

Since $\binom{s+k-1}{k} = (-1)^k \binom{-s}{k}$; the substitution, $t = -s$ enable us to write equation (2) in the form ;

$$y_s = y_0 - y_{-1}\, t + (-1)^2\, y_{-2}\binom{t}{2} + \dots\dots + (-1)^n\, y_{-n}\binom{t}{n}\ ;$$

$$\text{where } s = -t = \frac{(x - x_0)}{h}$$

This is Newton's Interpolation formula with Backward differences.

Since the starting point for numbering the subscripts is immaterial, (2) may equally well be written as

$$y_s = y_n - \Delta y_{n-1}\, t + \Delta^2 y_{n-2} \binom{t}{2} - \ldots\ldots\ldots\ldots + (-1)^n\, \Delta^n y_0 \binom{t}{n},$$

where $s = -t = (x - x_n)/h$, and where the values of y involved are $y_0, y_1, \ldots\ldots, y_n$.

To make clear the relationship between these two formulas, let us each one up for the same set of values $y_0, y_1, \ldots\ldots\ldots, y_n$. First of all the table of differences is calculated as follows.

x_0	y_0						
		Δy_0					
x_1	y_1		$\Delta^2 y_0$				
		Δy_1		$\Delta^3 y_0$			
x_2	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$		
		Δy_2		$\Delta^3 y_1$		$\Delta^5 y_0$	
x_3	y_3		$\Delta^2 y_2$		$\Delta^4 y_1$		$\Delta^6 y_0$
		Δy_3		$\Delta^3 y_2$		$\Delta^5 y_1$	
x_4	y_4		$\Delta^2 y_3$		$\Delta^4 y_2$		
		Δy_4		$\Delta^3 y_3$			
x_5	y_5		$\Delta^2 y_4$				
		Δy_5					
x_6	y_6						

Then Newton's Forward difference interpolation formula is

$$y_s = y_0 + \Delta y_0\binom{s}{1} + \Delta^2 y_0\binom{s}{2} + \Delta^3 y_0\binom{s}{3} + \Delta^4 y_0\binom{s}{4} + \Delta^5 y_0\binom{s}{5} + \Delta^6 y_0\binom{s}{6}$$

in which s = $(x - x_0)/h$.

This uses the differences in the top diagonal

$$y_s = y_6 - \Delta y_5\binom{s}{1} + \Delta^2 y_4\binom{s}{2} - \Delta^3 y_3\binom{s}{3} + \Delta^4 y_2\binom{s}{4} - \Delta^5 y_1\binom{s}{5} + \Delta^6 y_0\binom{s}{6}$$

in which s = $(x_6 - x)/h$. This uses the differences in the bottom diagonal.

Example 20.2

From the values of Sin x at 10^0 intervals, calculate

(1) Sin 7^0, (2) Sin 86^0.

Solution

The table of values and the difference array appears as follows.

x	**y(Sin x)**	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
0.	.00000					
		17365				
10.	.17365		–528			
		16837		–511		
20.	.34202		–1939		31	
		15798		–480		14
30.	.50000		–1519		45	
		14279		–435		18
40.	.64279		–1954		63	
		12325		–372		2
50.	.76604		–2326		65	
		9999		–307		21
60.	.86604		–2633		86	
		7366		–221		–4
70.	.93969		–2854		82	
		4512		–139		
80.	.98481		–2999			
		1519				
90.	1.00000					

(1) Since 7^0 lies near the beginning of the table, Newton's forward formula is appropriate. We have h = 10^0, $x_0 = 0$, $x = 10s$, so that for $x = 7^0$, we have $s = 0.7$

Newton's forward formula, using the top diagonal

$$y_s = 0 + .1736s - .00528\binom{s}{2} - 0.00511\binom{s}{3} + 0.00031\binom{s}{4} + 0.00014\binom{s}{5}$$

Since $\binom{s}{k} = \dfrac{s(s-1)\ldots\ldots\ldots\ldots(s-k+1)}{k!}$,

we obtain,

y_s = (.17365)(.7) + (.00528)(.105) – (.00511)(.0455) – (.00031)(.02616)

+ (.00014)(.01727)

= 0.12187

(2) Since 86^0 occurs near the end of the table, we use backward formula. We have h = 10, $x_0 = 90^0$, $x - 90^0 = 10s$, since $x = 86^0$, $s = -0.4$.

Then using the bottom diagonal, we obtain

y = 1.0000 – (.01519)(.4) + (.02993)(.12) + (.00139)(.064) – (.00082)(.0416) + (.00004)(.02995)

= 0.99757

20.5 Lagrange's interpolation

The formula

$$I(x) = y_0 L_0^{(n)}(x) + y_1 L_1^{(n)}(x) + \ldots\ldots + y_n L_n(x) \quad \text{----------} (1)$$

Where $L_j^{(n)}(x)$ is given by

$$L_j^{(n)}(x) = \frac{(x-x_0)\ldots\ldots\ldots\ldots(x-x_{j-1})(x-x_{j+1})\ldots\ldots\ldots\ldots(x-x_n)}{(x_j-x_0)\ldots\ldots\ldots\ldots(x_j-x_{j-1})(x_j-x_{j+1})\ldots\ldots\ldots\ldots(x_j-x_n)} \quad \text{---------} \ (2)$$

gives the interpolating polynomial for the (n +1) points (x_0, y_0), (x_1, y_1) …..(x_n, y_n). Formula (1) is known as Lagrange's interpolation coefficients.

These coefficients have several properties which merit attention. First of all $L_j^{(n)}(x)$ formed for the n + 1 points (x_0, y_0), (x_1, y_1) ………(x_n, y_n) is a polynomial of degree n which vanishes at $x = x_0$,………,$x = x_{j-1}$, $x = x_j$, …….$x = x_n$ but at $x = x_j$, it assumes the values.

Secondly, they depend on the given x's and are entirely independent of their y's.

Thirdly, the Lagrange's Coefficients . possess the useful property that their form remains invariant if the variable x is replaced by a new variables though the transformation x = hs + a, where h, a; are constants & h $\neq$ 0. By substituting x = hs + a, $x_0 = hs_0 + a$, $x_1 = hs_1 + a$.

We may verify directly that

$$L_0^{(n)}(x) = \frac{(x-x_1)(x-x_2)\ldots\ldots\ldots\ldots(x-S_n)}{(x_0-x_1)(x_0-x_2)\ldots\ldots\ldots\ldots(x_0-x_n)}$$

$$= \frac{(s-s_1)(s-s_2)\ldots\ldots\ldots\ldots(s-s_n)}{(s_0-s_1)(s_0-s_2)\ldots\ldots\ldots(s_0-s_n)}$$

and similarly, for the other coefficients.

Fourth, if $\pi_{n+1}(x)$ denotes the product of the (n+1) factors $(x - x_0)$, $(x - x_1)$, …., $(x - x_n)$;

We obtain by differentiating the product & substituting $x = x_0$.

$$\pi'_{n+1}(x_0) = (x_0 - x_1)(x_0 - x_2)\ldots\ldots\ldots\ldots(x_0 - x_n)$$

Similarly

$$\pi'_{n+1}(x_1) = (x_1 - x_0)(x_1 - x_2)\ldots\ldots\ldots(x_1 - x_n) \text{ etc.}$$

Hence it is clear that

$$L_j^{(n)}(x) = \frac{\pi_{n+1}(x)}{(x - x_j)\pi'_{n+1}(x_j)} \quad \text{--------------} \quad (3)$$

Because of the invariant property, if x = hs + a, equation (3) may be expressed as

$$L_j^{(n)}(x) = \frac{\pi_{n+1}(s)}{(s - s_j)\pi'_{n+1}(s_j)}$$

where in this case

$$\pi_{n+1}(s) = (s - s_0)(s - s_1)\ldots\ldots\ldots\ldots(s - s_n)$$

Equation (3) values possible a convenient and compact form for writing Lagrange's formula as

$$I(x) = \pi_{n+1}(x)\sum_{j=0}^{n}\frac{y_j}{(x - x_j)\pi'_{n+1}(x_j)}$$

It will be noted that Lagrange's formula contains the values y_0, y_1, ………, y_n explicitly, a fact of considerately importance for subsequent applications.

Example 20.3

Construct a polynomial having the value given by the table.

x	:	1	3	4	6
y	:	–7	5	8	14

Solution

By substituting in formula (1)

$$y = (-7)\frac{(x-3)(x-4)(x-6)}{(-2)(-3)(-5)} + \frac{(5)(x-1)(x-4)(x-6)}{(2)(-1)(-3)}$$

$$+\ (8)\frac{(x-1)(x-3)(x-6)}{(3)(1)(-2)}+(14)\frac{(x-1)(x-3)(x-4)}{(5)(3)(2)}$$

$$y = \frac{1}{5}\left(x^3-13x^2+69x-92\right)$$

Example 20.4

Obtain the Lagrangian coefficient for the points

$x =$ 3.4 3.5 3.7 3.8 4.0

Solution

It is convenient first of all to introduce a new variable by the relation; $x = \frac{1}{10}$ s + 3.4. So that the values of '*s*' corresponding to the given *x*'s will be the integers 0, 1, 2, 3, 4, 6; and computer terms of the form $\frac{\pi(s)}{s-s_j}$ by synthetic division.

$$\pi(s) = s\,(s-1)\,(s-3)\,(s-4)\,(s-6)$$

$$= s^5 - 14s^4 + 67s^3 - 126^2 + 72s$$

Since $\pi'(s_j) = \frac{\pi(s)}{(s-s_j)} \mid (s \neq s_j)$;

the value of $\pi'(s_j)$ is now obtainable by synthetic division followed by substitution.

For e.g. when s = 4 we arrange the computation in the form

4	1	–14	+16	–126	+72	–0
		4	– 40	108	–72	0
	1	–10	27	–18	0	0
		4	– 24	12	–24	
	1	– 6	3	– 6	–24	

The numbers in the 3rd line are the coefficients in the polynomial $\dfrac{\pi(s)}{(s-4)}$ and the last number in the last line is $\pi'(4)$, so that

$$L_3^{(4)}(x) = \frac{1}{24}(s^4 - 10s^3 + 27s^2 - 18s).$$

In this manner we get

$$L_0^{(4)}(x) = \frac{1}{72}(s^4 - 14s^3 + 67s^2 - 126s + 72)$$

$$L_1^{(4)}(x) = \frac{1}{30}(s^4 - 13s^3 + 54s^2 - 72s)$$

$$L_2^{(4)}(x) = \frac{1}{18}(s^4 - 11s^3 + 34s^2 - 24s)$$

$$L_4^{(4)}(x) = \frac{1}{180}(s^4 - 8s^3 + 19s^2 - 12s)$$

Where x and s are related by, $x = \frac{1}{10}s + 3$.

When it is desired to compute the Lagrangian coefficient for some x, it is advantageous to arrange the computation in a compact and systematic form. For $x_0, x_1, \ldots\ldots$ are not equally spaced the following scheme is suggested and is illustrated for the case n = 3. We first set up the square arrange of differences.

$$\begin{array}{cccc}
\underline{x - x_0} & x_0 - x_1 & x_0 - x_0 & x_0 - x_3 \\
x_1 - x_0 & x - x_1 & x_1 - x_2 & x_1 - x_3 \\
x_2 - x_0 & x_2 - x_1 & x - \underline{x_2} & x_2 - x_3 \\
x_3 - x_0 & x_3 - x_1 & x_3 - x_2 & x - x_3
\end{array}$$

Let the product of the numbers in the principle diagonal be denoted by (x), product of numbers is the first row be D_1, second row be D_2 etc. Then it is evident that

$$L_0^{(3)}(x) = \pi(x)/D_0\,,\;\; L_1^{(3)}(x) = \pi(x)/D_1$$

$$L_2^{(3)}(x) = \pi(x)/D_2\,,\;\; L_3^{(3)}(x) = \pi(x)/D_4$$

and the interpolated value y is

$$y = \pi(x)\,.s$$

Where $s = \dfrac{y_0}{D_0} + y_1/D_1 + y_2/D_2 + y_3/D_3$

Example 20.5

The values of Sin x for certain angles 0^0, 45^0, etc. are given as follows.

x	Sin x
0	.0000
30	.5000
45	.707104
60	.866025
90	1.00
120	.866025
135	.707107

150 .500

180 .00

From this table obtain Sin 105^0 by interpolation.

Solution

Using change of variable x = 15 so that the given values of s are 0, 2, 3, 4, 6, 8, 9, 10, 12 and s = 7. The complete computation is arranged as follows.

	D_i	y_i	y_i / D_i
1 -2 -3 -4 -6 -8 -9 -10 -12	8709120	.00000	0
2 5 -1 -2 -4 -6 -7 -8 -10	-268800	.5000	-18601×10^{-10}
3 1 4 -1 -3 -5 -6 -7 -9	68040	.707107	103925×10^{-10}
4 2 1 3 -2 -4 -5 -6 -8	-46080	.866025	-187939×10^{-10}
6 4 3 2 1 -2 -3 -4 -6	20736	1.00	482253×10^{-10}
8 6 5 4 2 -1 -1 -2 -4	15360	.866025	563818×10^{-10}
9 7 6 5 3 1 -2 -1 -3	-34020	.707107	-267850×10^{-10}
10 8 7 6 4 2 1 -3 -2	161280	.500	31002×10^{-10}
12 10 9 8 6 4 3 2 -5	-6220800	.000	0

$$\pi(x) = 12600\, s = 766608 \times 10^{-10}$$

$$y = 0.965926$$

Activity 20.1

1. From a table of common logarithms write the values of log 2.0, log 2.1, log 2.2, log 2.3, log 2.4, log 2.5. Then find the log 2.03, log 2.15 by interpolation.
2. From a table of log sins write the values of log Sin 15^0, 20^0, 25^0, 30^0, 35^0 and find log Sin 16^0 30, 19^0 30, by interpolation
3. From a table of common logarithms obtain log 125, log 126, log 128, log 129.

   Calculate log 127 by interpolation

4. From a table of Cos x with x in radians at intervals of 0.1, calculate Cos x for x = 5.347.

Summary

In this session, we have introduced Interpolation of polynomials, which are of importance in numerical computations. We have presented both divided differences for arbitrarily spaced points and ordinary differences for equally spaced points. We have described the error propagation in each methods of interpolation, and the computing techniques that can be used for checking the accuracy of a calculation.

Learning Outcomes

At the end of this session you will be able to:

use the methods of interpolation such as Newton’s forward and backward difference formula.

- Study error propagation

- Use Lagrange interpolation and estimate error bounds of interpolation.