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Optimal Dynamic Hedging of Multi-Asset Options

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Abstract. Hedging and valuing multi-asset options are analyzed using the Optimal Hedge Monte-Carlo method. The average cost of hedging and the residual risks are related to the stochastic description of the underlying assets, their dependence structure, and to the option contract details. A long position in a basket of the underlying assets mixed in proportions to their hedge ratios is employed to assess a bounding rate of return on risk-capital (i.e., a *hurdle rate*) for the option trader-hedger. That hurdle rate is employed to assess bounding values of multi-asset derivative positions while accounting for hedging costs and the inevitable hedge slippage that determines the derivative trader's risk-capital. Sample calculations are provided for two-asset options where the option trader-hedger is long correlation and short correlation, such as *best-of* and *worst-of* options. The differences in hedging strategies between such options and junior and senior basket-put tranches are delineated. The dual roles of fat-tails for individual assets and uncertainty of realized correlation in controlling the irreducible hedging errors are also described.

Keywords: *Multi-Asset Option, Correlation-Trading, Hedging, Residual-Risk, Risk-Capital, Hurdle-Rate*

JEL Classification.
G13: Contingent pricing; Futures Pricing;
G11: Portfolio Choice; Investment Decisions
D81: Criteria for Decision-Making under Risk & Uncertainty

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[1,2,3] This article reflects the opinions and views of the authors and not necessarily that of their employers.

1. Introduction

The Optimal Hedge Monte-Carlo (OHMC) approach to design hedging strategies, assess residual risks, and *value* derivative contracts, developed by Bouchaud & co-workers (Bouchaud & Potters [2003]; Potters et al [2001a,b]), is applied to address multi-asset option problems. We find the hedging strategy that minimizes the P&L volatility of the trader-hedger of the multi-asset option, assess the average cost of hedging, and highlight the *residual risks*. We quantify the expected change in wealth per unit *risk-capital* of an investor holding a basket of the underlying assets mixed in proportions to their hedge ratios, and employ that to impose a bounding hurdle return rate for the multi-asset derivative trader-hedger. This hurdle rate yields bounds on the option prices that reflect the residual risks in addition to the average hedging costs associated with *attempted* replication.

The main complexities of derivative trading addressed by OHMC are how the average hedging costs and hedge slippage measures are driven by the jumpiness of the underlying and the nature of the derivative contract. By helping anticipate the range of hedging P&L, and relating it to the underlying and the derivative's terms, OHMC analysis yields insights into risk-premiums associated with option trading and *attempting* to replicate their payoffs. By juxtaposing *attempted replication* and *residual risk*, OHMC provides an integrated platform for developing trading strategy, valuing options, and risk-management.

For multi-asset option problems, the main incremental complexity - over single asset problems - arises from having to deal with the dependence between assets. In a Monte-Carlo simulation the dependence structure is specified in a model. However, even with an assumed statistical dependence model, assessing hedging costs and hedge slippage measures remains nontrivial due to the vagaries of how the multiple assets evolve in a sample path – i.e., how the *realized-covariance* or *realized-dispersion* of returns determines the efficacy of hedging. As the realized covariance can be highly variable, we address its role in controlling hedge slippage measures and provide a tutorial on *correlation trading*.

1.1 Organization

In **Section 1.2** our motivation for analyzing multi-asset options in an OHMC framework is communicated, and followed up in **Section 1.3** by the definition of model problems to be analyzed in subsequent sections. In **Section 2** the multi-asset OHMC problem is formulated, and a numerical solution methodology is provided to assess the average hedging cost, the hedge ratios, and the residual risks. We also define residual risk measures and introduce the concept of a *hurdle-rate* on the expected change in wealth per unit risk capital. In **Section 3** we define the stochastic description of assets employed later to show sample calculations of OHMC to multi-asset options. We also define *realized covariance, dispersion, & volatility* measures that are useful in interpreting the varied hedging efficacy outcomes. Sample results and sensitivities for the model problems are provided in **Section 4**. A summary of this work is provided in **Section 5**. **Appendix-A** shows details of basis functions employed in the numerical solution. **Appendix-B** shows details of the multi-dimensional OHMC numerical solution and the computational implementation. Parameter estimation of the stochastic model used to describe the assets is described in **Appendix-C**.

1.2 Motivation

Our interest in multi-asset options started with multi-name credit products that have hit headlines in the recent years. Specifically, the synthetic CDO trading activity on standardized indexes (CDX) and its bespoke cousins hit the press in late spring 2005, when US automakers were downgraded and their credit spread widening presaged widening of spread on the popular credit index and the meltdown of the implied correlation of the equity tranche (see Petrelli [2007a&b]). In the credit trading arena the understanding of the role of *realized* correlation of credit spreads of the names of the CDO pool was lacking. From a quantitative modeling perspective the situation was quite hopeless insofar as the CDO *valuation model* did not address hedging directly and did not even have spread volatility and spread correlation as an input![1] The hapless seller of credit protection on equity tranches then recognized her leveraged exposure to idiosyncrasies of the CDO reference pool and her instinct to get paid for this risk plays a role in the evolution of the equity tranche implied correlation. The seller of protection on the junior tranche of a basket derivative would like to systematically discern her P&L risk arising from the temporal variability and uncertainty of how correlation manifests itself in time-series of returns of the assets, in addition with the usual concerns one has while selling a single asset derivative. This paper provides a primer on correlation trading in general and on the risk profile of a seller and buyer of a first *first loss tranche*.

Our second encounter with the vagaries of correlation in multi-name derivatives occurred in late 2007 and through 2008. The senior and super senior tranches of standardized indexes and their bespoke cousins became rich beyond anyone's prior imagination. The concern for systematic economic downturn and the liquidity crunch showed up in an explosion in the implied correlation of the senior tranches of CDX. Once again, with spread volatility not even being an input into the CDO valuation model (among other things, including uncertain recovery!), the hapless seller of protection on senior tranches dialed up the correlation input into the CDO model. So while the junior tranche protection seller worries about low realized correlation, the senior tranche protection seller has special concern about high realized correlation. This paper provides a primer on the risk profile of a seller and buyer of a *senior tranche*.

In addition to the credit derivative trading arena in which *correlation trading* has become popular in recent years, in equity derivatives, contracts with sensitivity to the correlation of reference assets exist, and pre-date those in the credit arena. For instance, there are equity derivative contracts that guarantee an investor performance of the highest performing asset above a strike (e.g., *best of call*). Credit derivatives are harder to deal with because of the dual role of spread moves and default events and associated payoffs for the option and the hedges. This article provides a primer on correlation trading employing equity type assets. By altering the moneyness of the options in the model problems, they can be imparted a credit type behavior.

[1] The standard *risk-neutral* CDO model still does not have spread volatility and spread correlation as inputs and has not addressed residual risks of attempted replication! The *risk-neutral* model's presumption of perfect replication has created false expectations about the ability to replicate and ascribe a unique value to a derivative contract. Here we illustrate how OHMC integrates *attempted replication* with assessments of *residual risks* and provides a more rational framework for assessing value while being cognizant of risks.

1.3 Model Problems

The model problems alluded to above are stated here more formally. These problems are the basic ones in the arena of multi-asset hedging problems. Hence the importance of developing a clear understanding of hedging cost and risk-return drivers in these problems.

P1: Best of Call

Spot values: $s_1(0)$ & $s_2(0)$
Tenor: T
Strikes: K_1 & K_2
Payoff: $P = \max\{(s_1(T) - K_1)^+, (s_2(T) - K_2)^+\}$

An economic motivation of a single asset call purchaser is to have a leveraged exposure to the upside asset performance. The call purchaser gets to experience the upside asset performance with a lower initial cash outlay than what would be needed to be outright long the asset.

The call option seller can be in the business of financing the call purchaser's bullish belief about the asset. A call option seller's cheaper access to capital compared to the call-option purchaser's funding cost can be the rationale for some trades. For long dated options on seemingly less volatile assets, the skill and economics of a call-option manufacturer can have similarities with that of a loan provider.

The call option purchaser might have a bullish view on multiple assets. Then he can buy calls on both the assets separately. If he has a belief that either of two assets is likely to zoom up, say because they are in the same sector that is coming into vogue, but he can't pin down which one, then he might be inclined to buy the best of call option referencing both those assets.

P2. Best of Put

Spot values: $s_1(0)$ & $s_2(0)$
Tenor: T
Strikes: K_1 & K_2
Payoff: $P = \max\{L_1, L_2\} \quad L_1 = \{K_1 - s_1(T)\}^+ \quad L_2 = \{K_2 - s_2(T)\}^+$

The economic motivation of a single asset put purchaser is to have a positive payout if the asset under-performs, and in the event the asset performs well, to have the downside limited to the cost of the put. The put option purchaser might have potentially bearish views on multiple assets. Then he can buy puts on both the assets separately. If he has a belief that either of two assets is likely to fall sharply, say because they are in the same sector that is coming out of vogue, but he can't pin down which one, then he might be inclined to buy the best of put option referencing both those assets.

P3. Basket Put Tranche

Spot values: $s_1(0)$ & $s_2(0)$
Tenor: T
Strikes: K_1 & K_2
Attachment: N_1
Detachment: N_2
Payoff: $P = \left\{ (N_2 - N_1) - \{L - N_1\}^+ \right\}^+$; $L = L_1 + L_2$; $L_1 = \{K_1 - s_1(T)\}^+$ $L_2 = \{K_2 - s_2(T)\}^+$

In the tranche examples we address a two-name basket of European puts. The degree of moneyness of the puts can be varied by altering the put options strike relative to the spot values. The "attachment points" of the tranche, N_1 (lower attachment) and N_2 (upper attachment), determine what portion of the basket put payoff is covered by the tranche. Given the basket put strikes (K_1, K_2), the riskiness of the tranche can be altered by varying the attachment points. If the lower attachment N_1 is small the tranche is the "junior" tranche. If N_1 is large then the resultant tranche is "senior," with a greater chance of not being hit by the payoffs of the puts in the reference basket.

Junior Tranche ($N_1 = 0$)
If both the reference assets are above the put strike values at maturity then $L = 0$ and the option purchaser gets a payment of N_2 at maturity. If reference asset 1 went to zero at maturity, and the other asset was above the strike K_2, then $L = L_1 = K_1$, and the payoff to the option purchaser is $\{N_2 - K_1\}^+$. If both the puts are in the money at maturity, then the option purchaser receives N_2 minus the amount by which both the puts are in the money, floored at zero.

This contract has the flavor of the "equity tranche" of a synthetic CDO insofar as any reference put in the basket being exercised results in loss to investor. By changing the degree of moneyness of the puts, we can make the probability of loss to the first-loss risk taker become small. The dependence of the two assets plays a role too. If the assets are independent than the probability of just one put being in the money is higher, compared to the case where the assets showed perfect dependence. Since the junior-loss-tranche option purchaser loses by any of the puts being in the money, one expects her to buy this option at a lower price if she believes there is little dependence between the assets.

Senior Tranche ($N_1 > 0$)
This contract is the loss-tranche on top of the junior tranche. It incurs a principal loss (i.e., payout less than face amount $N_2 - N_1$) only if either or both the puts are exercised at maturity and the sum of the payouts on the puts exceeds N_1, the cushion. Clearly, as N_1 becomes larger, for the option purchaser, highly dependent assets increases risk. Therefore the senior tranche has a covariance dependence that is of the opposite sign compared to the junior tranche. We provide a comparison of hedge performance of the senior tranche with the junior tranche.

The packaging of puts in the zero coupon bond format above has significance beyond the distinct correlation positions. The put purchaser (i.e., tranche seller) does not have counterparty risk to the put seller (tranche buyer) – as the put seller gives the put purchaser money upfront (the *value* of the option). If the put seller ceases to exist prior to option maturity and the put options are in the money, then there is no economic loss to the tranche seller.

2. Multi-Asset Optimal Hedge Monte-Carlo

The OHMC approach puts itself in the seat of a derivatives trader and attempts to keep a trading book as flat as possible between two hedging intervals. Viewing the mark of the derivative and hedge-ratio as functions of spot, the OHMC analysis seeks to find those functions to accomplish its objectives over the ensemble of states the reference assets can take. If the derivative can be perfectly replicated then the residual risk will be zero, and the solution will correspond to the *risk-neutral* solution. This perfect replication can only be achieved if the assets are driven from a Brownian motion type process that has no excess kurtosis, and one hedges continuously. However, the premise of continuous hedging is entirely misleading for assets that exhibit jumpiness (excess kurtosis) as the residual risk achieves an irreducible value and hedging more often doesn't help – as it would only explode the transaction costs on top of the irreducible hedging error.

Even the most standard assets exhibit excess kurtosis (e.g., S&P500 daily return kurtosis is *much larger* than 3!), so perfect replication is *never* attainable. Not enough has been done in mainstream *valuation-modeling* to document the limits of replication. Instead, there are plenty of distortions of Black-Scholes [1973] results to *fit* prices. Multiple incarnations of volatility have been created to fit prices and "Q" measures (i.e., presumed *risk-neutral* measures governing *precisely* and *perfectly* replicating derivative trading!) are invoked without articulating the mechanics of replication. While perfect replication is thwarted at the whiff of jumpiness in asset returns (kurtosis), even when the traditional *valuation model* employs a fat tailed underlying process, it is used to fit prices rather than to document irreducible hedging errors.[2]

Traders have apparently not explained their risks or trades to their *valuation quant* – while trading assistants continue to book trades into officious risk-neutral valuation models on command. The assumed fiction of *perfect replication* and the associated *unique value* has also taken hold in the accounting community. Traders have had little incentive in communicating to the financial community at large that the post-Black Scholes ornate model fitting game has gone on at the cost of quantifying residual risk and developing informed views on risk-return dynamics. At issue is not a higher order correction to pricing models. A lack of quantification of leading order risks associated with option trades and a large-scale heist of risk capital has resulted from the risk-neutral models application.

The builders of *valuation models* often take no responsibility for the risk of a derivative trading book and are often not aware of the rationale for the trades. The standard vegas, deltas, and gammas can also be rendered quite ineffective in a complex book due to sensitivity hedging and the accompanying basis risks (tenor barbells, numerous cross-gammas, forward volatility sensitivity over different time-scales……) and no direct quantification of the unavoidable residual risk. When the poor state of risk management is periodically and sometimes catastrophically revealed it is often castigated as a *breakdown of the risk management model*. The *risk-neutral* derivative valuation model creates false expectations of hedge performance, which is unacceptably dangerous as the volume and complexity of derivatives have increased.

[2] The term *risk-neutral model* has degenerated to simply taking averages of the option payoff after de-trending the underlying – without even addressing the mechanics of replication. For example, one hears of *risk-neutral* "jump-diffusion" models that are fit to prices without quantifying the residual risk arising due to jump, or by assuming jumps to be a diversifiable risk factor!

We believe that the naiveté of perfect replication has to be discarded by modern derivative trading which has to embrace *attempted replication* as much as admitting and estimating *irreducible hedging errors*. The forces dictating an increase in the risk awareness of valuation models are: **(1)** broadly disseminated derivative trading risk-management mishaps; **(2)** advent of financial instruments capturing risk premiums associated with derivative trading (e.g., variance swaps); **(3)** proliferation of risk-return savvy hedge funds exploiting market agents with risk-neutral blinders about option risk premiums!; **(4)** increase in computational power and access to market data; **(5)** increasing experience and documentation of the residual-risks associated with any dynamic hedging scheme.

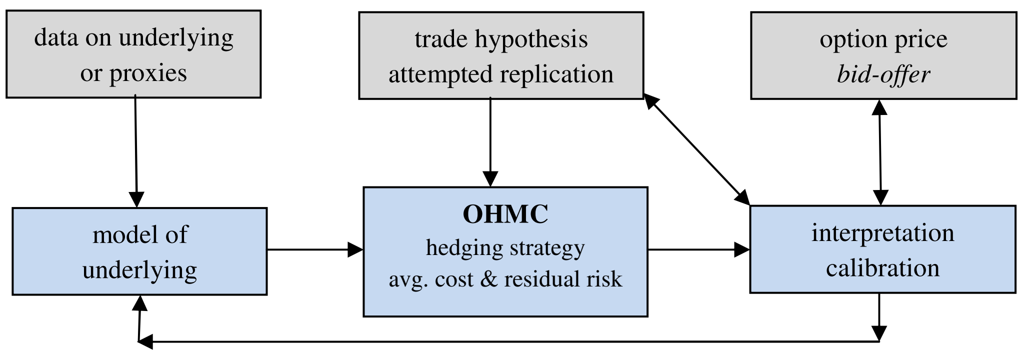

Figure 1. Information flows in OHMC based trading system

Bouchaud and co-workers (Bouchaud & Potters [2003]; Potters et al [2001a,b]) have pioneered and furthered OHMC and provided a documentation of hedging errors for single asset options, employing a variety of models of the underlying, including directly using historical return data. Their work provides a rich commentary and analysis that explains the power and limitations of Black-Scholes results in a real world hedging application. For a background into optimal hedging analysis we have provided references to the seminal works by Schweizer [1995]; Laurent & Pham [1999]; Potters et al [2001a,b]; and Bouchaud & Potters [2003]. These works are receptive to the needs of trading risk management insofar as they focus on hedge performance and residual risks. This line of investigation is to be contrasted with the *risk-neutral* models that do not address the trade risk-return and are focused only on fitting prices to vanillas, with the sole purpose of providing a means for booking day 1 P&L on exotic options without any assessment of irreducible risks in either vanilla options or exotics.

Our series of works on applying OHMC to different problems are motivated by the following: **(1)** understanding the risk-return of trading strategies; **(2)** developing buy-sell signals in option trading, and **(3)** developing risk metrics that are explicitly cognizant of attempted replication & its limitations. We also seek to expand the set of financial derivatives for which the irreducible errors in attempting replication are documented. Our prior work involved **(1)** structured products with equity underlying modeled with GARCH models (Kapoor et al [2003]); **(2)** static hedging default-risk in CDOs (Petrelli et al [2006]); **(3)** dynamic hedging of CDS options (Zhang et al [2007]; **(4)** Crash Cliquets (Petrelli et al, [2008]).

2.1 Option Seller-Hedger's Wealth Change

Consider a basket of M assets whose value is $\mathbf{s}(t_k) = \{s_1(t_k), s_2(t_k), \ldots, s_M(t_k)\}$ and an option with a payoff dependent upon the value of the basket assets

$$C(\mathbf{s}(T), T) = Payoff(\mathbf{s}(T)) \tag{1}$$

The option trader's P&L on account of the option sell position is determined by the change in *value* of the option:

$$\begin{aligned} \Delta W_{t_k}^{option}(t_k, t_{k+1}) &= C(\mathbf{s}(t_k), t_k) - G(t_k) \\ G(t_k) &= C(\mathbf{s}(t_{k+1}), t_{k+1}) df(t_k, t_{k+1}) \end{aligned} \tag{2}$$

In *attempting to hedge* the P&L driven from the option position the trader holds $\mathbf{\Phi}(\mathbf{s}(t_k), t_k)$ amount of the assets, which is the M dimensional vector of hedge ratios associated with the reference asset vector $\mathbf{s}(t_k)$: $\mathbf{\Phi}(\mathbf{s}(t_k), t_k) = \{\phi_1(\mathbf{s}(t_k), t_k), \phi_2(\mathbf{s}(t_k), t_k), \ldots, \phi_M(\mathbf{s}(t_k), t_k)\}$. The P&L generated by the hedge is determined by the change in asset values and the funding costs, which is stored in $\mathbf{H}(t_k) = \{H_1(t_k), H_2(t_k), \ldots, H_M(t_k)\}$ where

$$H_m(t_k) = \left[s_m(t_{k+1}) - \frac{s_m(t_k)}{DF_m(t_k, t_{k+1})} \right] df(t_k, t_{k+1}) \tag{3}$$

To account for different funding rates and possibly different dividends we employ asset specific funding discount factors, $DF_m(t_k, t_{k+1})$, which are possibly distinct from the discount factor $df(t_k, t_{k+1})$. The P&L from the hedge then follows

$$\Delta W_{t_k}^{hedge}(t_k, t_{k+1}) = \mathbf{\Phi}(\mathbf{s}(t_k), t_k) \bullet \mathbf{H}(t_k) \tag{4}$$

The total P&L of the option delta-hedge position follows

$$\begin{aligned} \Delta W_{t_k}(t_k, t_{k+1}) &= \Delta W_{t_k}^{option}(t_k, t_{k+1}) + \Delta W_{t_k}^{hedge}(t_k, t_{k+1}) \\ &= C(\mathbf{s}(t_k), t_k) - G(t_k) + \mathbf{\Phi}(\mathbf{s}(t_k), t_k) \bullet \mathbf{H}(t_k) \\ &= C(\mathbf{s}(t_k), t_k) - G(t_k) + \phi_n(\mathbf{s}(t_k), t_k) H_n(t_k) \end{aligned} \tag{5}$$

We are showing both a vector dot product based notation, and a sum over repeated indexes is implied in the multi-asset derivative wealth balance above.

2.2 Hedge Optimization Problem

In the OHMC approach a MC simulation of asset evolution is performed. Based on that all terms of the wealth balance (5) can be directly computed, other than the yet unknown functions of value and hedge ratio $C(\mathbf{s}(t_k),t_k)$ and $\mathbf{\Phi}(\mathbf{s}(t_k),t_k)$. These two functions are found by imposing a constraint on the average change in wealth and minimizing the wealth change variance:

$$E[\Delta W_{t_k}(t_k,t_{k+1})] = \overline{\Delta W_{t_k}}(t_k,t_{k+1}) \tag{6}$$

$$\text{minimize } \sigma^2_{\Delta W_{t_k}(t_k,t_{k+1})} = E\left[\left(\Delta W_{t_k}(t_k,t_{k+1}) - \overline{\Delta W_{t_k}}(t_k,t_{k+1})\right)^2\right] \tag{7}$$

The closest counter-part to the risk neutral approach is to set the expected wealth change to zero ($\overline{\Delta W_{t_k}}(t_k,t_{k+1})=0$) and seek the minimum variance solution. Indeed if risk can be eliminated, then any deviation from zero mean change in wealth is an "arbitrage opportunity." However, that theoretical view is inadequate for real trading because there is no derivative that can be perfectly replicated by dynamically trading the underlying, on account of excess kurtosis of the underlying. So, to grapple with the real world problem we could impose a non-zero expected change in wealth. Anecdotally, selling options on US stock indexes tends to be a positive *average* wealth change business. This is witnessed also in the equity index variance swap market, that until recently, advertised to investors double digit *expected* returns, due to the "discrepancy" between implied volatility and historical volatility.[3] The "volatility arbitrage" hedge funds track listed option prices and hedge slippage in any replication attempt and can allocate *capital* to trades based on their view of derivative trading risk-return.

2.3 Numerical Solution

We render the variational calculus problem (defined by (5), (6), & (7)) finite dimensional by representing the unknown pricing functions and hedge ratio functions in terms of sums of products of unknown time dependent coefficients and state-space dependent basis functions:

$$C(\mathbf{s}(t_k),t_k) = \hat{\mathbf{C}}(t_k)\bullet\mathbf{\Omega}(\mathbf{s}(t_k)) \;;\; \phi_m(\mathbf{s}(t_k),t_k) = \boldsymbol{\varphi}_m(t_k)\bullet\mathbf{\Omega}(\mathbf{s}(t_k)) \tag{8a}$$

The number of basis functions can be increased to obtain the desired resolution of the reference asset state-space. The notation is altered to represent dot products as sum over repeated indexes

$$C(\mathbf{s}(t_k),t_k) = \hat{C}_i(t_k)\Omega_i(\mathbf{s}(t_k)) \;;\; \phi_m(t_k) = \hat{\phi}_{m,i}(t_k)\Omega_i(\mathbf{s}(t_k)) \tag{8b}$$

The wealth change perturbation around the mean then follows

$$\Delta W'_{t_k}(t_k,t_{k+1}) = \hat{C}_j(t_k)\Omega_j(\mathbf{s}(t_k)) - G(t_k) + \hat{\phi}_{n,j}(t_k)\Omega_j(\mathbf{s}(t_k))H_n(t_k) - \overline{\Delta W_{t_k}}(t_k,t_{k+1}) \tag{9}$$

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[3] Late 2008 has witnessed levels of volatility in equity indexes that were only seen once before on *Black-Monday* in 1987. The perpetually short volatility agent was punished and the opportunistically long volatility market agent made a windfall in late 2008.

The Lagrange-multiplier technique to solve the optimization problem defined in equations (5), (6), and (7) is as follows:

$$F(t_k) = E[(\Delta W'_{t_k}(t_k, t_{k+1}))^2] + 2\gamma(t_k) E[\Delta W'_{t_k}(t_k, t_{k+1})] \tag{10}$$

$$\frac{\partial F(t_k)}{\partial \hat{C}_i(t_k)} = 0\,;\ \frac{\partial F(t_k)}{\partial \hat{\phi}_{m,i}(t_k)} = 0\,;\ \frac{\partial F(t_k)}{\partial \gamma(t_k)} = 0 \tag{11}$$

Elaborating (11)

$$\begin{gathered}
\frac{\partial F(t_k)}{\partial \hat{C}_i(t_k)} = 0 \Rightarrow E\left[\Delta W'_{t_k}(t_k, t_{k+1}) \frac{\partial \Delta W'_{t_k}(t_k, t_{k+1})}{\partial \hat{C}_i(t_k)}\right] + \gamma(t_k) E\left[\frac{\partial \Delta W'_{t_k}(t_k, t_{k+1})}{\partial \hat{C}_i(t_k)}\right] = 0 \\
\frac{\partial F(t_k)}{\partial \hat{\phi}_{m,i}(t_k)} = 0 \Rightarrow E\left[\Delta W'_{t_k}(t_k, t_{k+1}) \frac{\partial \Delta W'_{t_k}(t_k, t_{k+1})}{\partial \hat{\phi}_{m,i}(t_k)}\right] + \gamma(t_k) E\left[\frac{\partial \Delta W'_{t_k}(t_k, t_{k+1})}{\partial \hat{\phi}_{m,i}(t_k)}\right] = 0 \\
\frac{\partial F(t_k)}{\partial \gamma(t_k)} = 0 \Rightarrow E\left[\Delta W'_{t_k}(t_k, t_{k+1})\right] = 0
\end{gathered} \tag{12}$$

Based on (9) we have

$$\frac{\partial \Delta W'_{t_k}(t_k, t_{k+1})}{\partial \hat{C}_i(t_k)} = \Omega_i(\mathbf{s}(t_k));\ \frac{\partial \Delta W'_{t_k}(t_k, t_{k+1})}{\partial \hat{\phi}_{m,i}(t_k)} = \Omega_i(\mathbf{s}(t_k)) H_m(t_k) \tag{13}$$

Substituting (13) into (12) we obtain the set of linear equations to represent the constrained optimization problem:

$$\begin{gathered}
\hat{C}_j(t_k)\overline{\Omega_j(\mathbf{s}(t_k))\Omega_i(\mathbf{s}(t_k))} + \hat{\phi}_{n,j}(t_k)\overline{\Omega_j(\mathbf{s}(t_k))\Omega_i(\mathbf{s}(t_k))H_n(t_k)} \\
+ \gamma\overline{\Omega_i(\mathbf{s}(t_k))} = \overline{G(t_k)\Omega_i(\mathbf{s}(t_k))} + \overline{\Omega_i(\mathbf{s}(t_k))}\ \overline{\Delta W_{t_k}(t_k, t_{k+1})} \\[2ex]
\hat{C}_j(t_k)\overline{\Omega_j(\mathbf{s}(t_k))\Omega_i(\mathbf{s}(t_k))H_m(t_k)} + \hat{\phi}_{n,j}(t_k)\overline{\Omega_j(\mathbf{s}(t_k))\Omega_i(\mathbf{s}(t_k))H_n(t_k)H_m(t_k)} \\
+ \gamma\overline{\Omega_i(\mathbf{s}(t_k))H_m(t_k)} = \overline{G(t_k)\Omega_i(\mathbf{s}(t_k))H_m(t_k)} + \overline{\Omega_i(\mathbf{s}(t_k))H_m(t_k)}\ \overline{\Delta W_{t_k}(t_k, t_{k+1})} \\[2ex]
\hat{C}_j(t_k)\overline{\Omega_j(\mathbf{s}(t_k))} + \hat{\phi}_{n,j}(t_k)\overline{\Omega_j(\mathbf{s}(t_k))H_n(t_k)} = \overline{G(t_k)} + \overline{\Delta W_{t_k}(t_k, t_{k+1})}
\end{gathered} \tag{14}$$

Appendix A & B describe further details of the terms in (14) and its solution.

First Hedging Interval

Solving for the pricing and hedge notional is an ordinary multi-dimensional constrained minimization problem at the first time step. The wealth change for the first hedging interval follows

$$\varDelta W_{t_0}(t_0,t_1) = C(\mathbf{s}(t_0),t_0) - G(t_0) + \phi_n(\mathbf{s}(t_0),t_0) H_n(t_0) \tag{15}$$

We directly impose the expected wealth change constraint

$$E[\varDelta W_{t_0}(t_0,t_1)] = C(\mathbf{s}(t_0),t_0) - \overline{G(t_0)} + \phi_n(\mathbf{s}(t_0),t_0)\overline{H_n(t_0)} \tag{16}$$

$$C(\mathbf{s}(t_0),t_0) = \overline{\varDelta W_{t_0}(t_0,t_1)} + \overline{G(t_0)} - \phi_n(\mathbf{s}(t_0),t_0)\overline{H_n(t_0)} \tag{17}$$

The perturbation of the wealth change around its average value is related to the perturbations of *G* and *H* around their mean values by

$$\varDelta W'_{t_0}(t_0,t_1) = -G'(t_0) + \phi_n(\mathbf{s}(t_0),t_0) H'_n(t_0) \tag{18}$$

The first hedging step wealth change variance therefore follows

$$\sigma^2_{\varDelta W_{t_0}(t_0,t_1)} = E[\{-G'(t_0) + \phi_n(\mathbf{s}(t_0),t_0) H'_n(t_0)\}^2] \tag{19}$$

To find the optimal hedge ratio we find the derivative of the wealth change variance with respect to the hedge ratios:

$$\frac{d\sigma^2_{\varDelta W_{t_0}(t_0,t_1)}}{d\phi_m(\mathbf{s}(t_0),t_0)} = 2E[\{-G'(t_0) + \phi_n(\mathbf{s}(t_0),t_0) H'_n(t_0)\} H'_m(t_0)] = 0 \tag{20}$$

The set of *M* linear equations that need to be solved to find the first hedge ratios are

$$\phi_n(\mathbf{s}(t_0),t_0)\overline{H'_n(t_0) H'_m(t_0)} = \overline{G'(t_0) H'_m(t_0)} \tag{21}$$

$C(\mathbf{s}(t_0),t_0)$ is evaluated using the prior relationship with the hedge ratios in equation (17).

2.4 Residual Risks

The OHMC hedging time-grid is $\{t_0 = 0, t_1, t_2, ..., t_k, t_{k+1}, ..., t_K = T\}$. The hedging P&L between time step k and k+1 discounted to time step k, $\Delta W_{t_k}(t_k, t_{k+1})$, can be assessed for each realization in the simulated ensemble of underlying. The total change in wealth discounted to t_0 can be assessed

$$\Delta W_0(0,T) = \sum_{k=0}^{K-1} \Delta W_{t_k}(t_k, t_{k+1}) df(0, t_k)$$

The cumulative P&L from trade initiation to t_k (present valued to time t_k) follows

$$\Delta W_{t_k}(0, t_k) = \sum_{j=0}^{k-1} \Delta W_{t_j}(t_j, t_{j+1}) \left[df(t_j, t_k) \right]^{-1}$$

A trading desk is (or should be) limited in the amount of options they can write based on risk limits and a value at risk model that is typically the basis of *risk-capital.* However mainstream derivative pricing modeling has not directly addressed the linkages of derivative pricing with risk-capital or margin models. This *risk-neutral* derivative pricing theory does not admit to replication errors – hence it remains disconnected from risk capital models! As a result, big disasters in trading are driven by uncontrolled accumulation of residual risks not output in the numerous runs of the *valuation model*. This model made it easy for a derivative trading business head to see upfront P&L or carry, but it does not show residual risks[4] endemic to derivative trading. Furthermore, the premise of a perfect hedge is a key contributor to the weak risk management practices around derivatives that come to fore with surprising regularity.

If a clear methodology of understanding risk-return is not expounded at a trade level or a trading desk level, it is unlikely that risks are understood at a global portfolio level. Therefore the unchallenged invocation of *replication* and/or complete *diversification* of residual risks inside a *valuation* model is dangerous and misleading – especially for new or exotic options where historical observations of option behavior are lacking.

Expected Return on Risk Capital

The expected P&L from a derivative trade should be compared with tail losses to ensure solvency, and profitability. We define a specific solvency target and assess the risk capital associated with the derivative trade. To compute risk capital over different time intervals consistently (derivative tenor, hedging interval, etc), we employ a *target hazard rate* to assess the target survival probability over those time horizons.

survival probability/confidence-level over period τ: $p_s(\tau)$

τ interval hazard rate: $\lambda_\tau = \dfrac{-\ln(p_s(\tau))}{\tau}$

τ interval hazard rate based h interval survival probability: $p_s(h) = \exp[-\lambda_\tau h]$

[4] The risk-neutral model delivers strong results on upfront P&L by ignoring residual risks. This has built false expectations of hedge performance and has disconnected derivative valuation from understanding its risk-return.

h interval average wealth change: $\overline{\Delta W_t(t,t+h)}$

h interval $p_s(h)$ confidence level wealth change:

$$q(t;p_s(h)) \ni Probability\{\Delta W_t(t,t+h) > q(t;p_s(h))\} = p_s(h)$$

h interval $p_s(h)$ confidence level wealth change deviation from average wealth change:

$$Q(t;p_s(h)) = \overline{\Delta W_t(t,t+h)} - q(t;p_s(h))$$

h interval $p_s(h)$ confidence level expected return on risk-capital: $\Theta = \dfrac{\overline{\Delta W_t(t,t+h)}}{Q(t;p_s(h))}$

h interval $p_s(h)$ confidence level rate of expected return on risk-capital: $\theta = \dfrac{1}{h} ln(\Theta + 1)$

Note that the wealth change is defined by discounting cash-flows under the risk-free rate - that controls the discount factor $df(t, t+h)$. Therefore, the expected return on risk capital, Θ, and its rate, θ, are in reference to the risk-free rate (e.g., spread over LIBOR).

Bounding Price for Option Trader-Hedger

After recognizing the residual risks inherent to any attempted replication strategy, the derivatives trader has to develop her own criterion for attractiveness of a trade, given the anticipated average cost of attempted replication (i.e., *hedging*) and the modeled residual P&L distribution. The party providing risk-capital can have their hurdle rate below which they will not invest - that can impose a constraint on a trader.[5] Then the OHMC analysis can be used to explore the relationship between pricing and the viability of the business. The time-horizon over which the investor is willing to lock-up capital can also have an influence on hedging strategy and viability of pricing points to a trader.

We present here a bounding argument on the value of derivative contracts that utilizes the OHMC assessed *average* hedging costs and residual risk. We argue that a hedger of a derivative position should, at a minimum, compare her risk-return with that of a long position in the underlying asset on which the derivative is written in deciding the acceptable price. The *hurdle-return* set by the underlying asset can provide a bound for the derivatives trader – why should she settle for a lower expected return than a market agent that is simply long the underlying asset? To quantitatively explore this idea we employ the return on risk capital metric. For a single asset derivative trade the hurdle-return can be set by a long position in the underlying. For basket trades the composition of the long basket used to set a hurdle rate is determined by the initial hedge ratios of the derivative trade.

[5] Only if perfect replication is achievable, the unique cost of replication provided by the option buyer to seller is all the capital the seller needs to manufacture the derivative payoff. Realistically, no derivative trade is truly *self-financing* after acknowledging the need for risk-capital (or posting *margin*) prior to being able to sell derivatives.

Basket-Trade

Denoting the initial hedge ratios by w_m, the change in wealth of the basket trade is given by

$$\Delta W_t(t,t+h)\Big|_{basket} = \sum_{m=1}^{M} w_m \left[s_m(t+h) - \frac{s_m(t)}{DF_m(t,t+h)} \right] df(t,t+h)$$

Based on this basket wealth change, the metrics of return on risk-capital can be assessed, given a stochastic description of the underlying assets. The *hurdle* on return on risk capital set by the elementary basket trade is denoted by Θ_{basket}. The per-unit-time translation of the *hurdle* on return is denoted by θ_{basket}, which can be thought of as a *hurdle-rate*. These return metrics are relative to the risk-free rate embodied in the discount factors $df(t, t+h)$.

Option-Seller–Hedger Bad-Deal Bound

The option seller (payoff obligor) will try to charge an amount C_{seller} which is greater than the *average* hedging cost $C(t_0)$ so that his return on risk capital is at least equal to that resulting from holding a basket in the underlying assets:

$$\Theta_{option} = \frac{C_{seller} - C(t_0)}{-q_{seller}(0; p_s(T))} = \Theta_{basket} = e^{\theta_{basket} T} - 1 \tag{22}$$

The option seller can adopt his model of the underlying and apply the OHMC approach to assess the average cost of hedging and tail losses associated with attempted replication. The basket trade risk return can be seen as imposing a hurdle return rate on the option seller.

Option Buyer–Hedger Bad-Deal Bound

The option purchaser (payoff recipient) will try to pay an amount C_{buyer} which is less than the average wealth generated while hedging, $C(t_0)$, so that his return on risk capital is at least equal to that resulting from holding a basket in the underlying assets:

$$\Theta_{option} = \frac{C(t_0) - C_{buyer}}{-q_{buyer}(0; p_s(T))} = \Theta_{basket} = e^{\theta_{basket} T} - 1 \tag{23}$$

Like the option seller, the basket trade risk return can be seen as imposing a *hurdle* return rate on the option purchaser.

These hurdles on return rates could be exogenously applied too. For instance, an equity investor that capitalizes an option seller's enterprise (that enables him to be able to sell options) may do so on the basis of an expected rate of return over his risk-free benchmark.

Range of Option Pricing

Based on estimates of the average cost of hedging and the residual risks we have furthered bounding arguments from the points of view of a derivative seller-buyer hedger. Clearly these different agents have different pricing points at which the trades appear attractive – and if both these agents are using the same model and trade strategy, it is unlikely that they will trade with each other![6] We can have a large difference in bounding values above, depending on the confidence level of risk capital and the trade-tenor, and the asymmetry of the residual hedging P&L. This range is equal to the sum of risk-capital of option seller and buyer multiplied by the hurdle return, and can be thought of as a range of rational prices dependent on the risk-aversion of the market participants:

$$range\left(p_s(T)\right) = \left(-q_{seller} - q_{buyer}\right)\Theta_{basket} \tag{24}$$

Certainly in derivative markets one observes that the *fitted risk-neutral model parameters* show distinct regimes and a great variation in numerical values of parameters among those regimes. Model fitters tend to focus on the goodness of fit on any particular day and may not be aware of the flows of different types of market players (or residual risk for that matter!). More informed market participants are aware of the demand and supply dynamics and can sometimes associate the distinct pricing regimes with those dynamics, in addition to their views on the underlying. A prevailing pricing regime reflects a particular demand supply situation which can be disrupted by factors internal to trades (e.g., realized volatility, dispersion) or external to the trades whose pricing we debate. Even if under quiescent conditions the market trades around a certain set of implied parameters, in the event of market stress the demand-supply picture gets altered and the pricing migrates to the new regime. In our analysis the bounds on the hedger that is the payoff recipient is relevant in a market dominated by the option buyer. The bound on the hedger-option payoff obligor is relevant in a market dominated by the option seller.

We do not find the range between these prices to be troubling at all. To the contrary, we think that this range prepares one for market realities of derivative trading. Even if the pricing regime change does not happen over ones watch, the range reflects irreducible risks inherent to attempted derivative replication. Responsible trading activity requires quantifying these risks and the implied *range of "fair" prices*. The lack of focus on hedge slippage of mainstream derivative *valuation modeling* and the inability to entertain *bid-ask spread* within a theoretical framework are in stark contrast with the OHMC approach that makes it readily possible to describe a range around the average hedging costs. Model driven *mark-to-market* is almost always with respect to "mid-prices" that can mask both the buyer's and seller's risk-premium. In contrast, exchanges (e.g., CBOE) will mark its participants to liquidation values (for purposes of margin, etc). Real prices are probably driven by a mix of replications arguments, hedge slippage experiences and demand and supply considerations and the variety of views on the underlying and differing psychological predispositions towards risk-return. The *assumption* of a *perfect* replication price results in an unacceptably poor framework in dealing with real option trading and risk management.

[6] While in this work we are taking the viewpoints of derivative trader hedger, many derivative trades occur between a seller hedger and an outright purchaser of the option. The value of the option to the purchaser can then be judged by comparing with a directionally similar position in the underlying.

3. Multi-Asset Basket Description

The OHMC methodology for analyzing multi-asset options is independent of the dynamics of the underlying assets. This flexibility of the OHMC approach encourages the development of realistic models of the asset value evolution that can combine empirically observed features as well as beliefs about the asset. Such descriptions could also involve conditioning on observations of underlying and possibly other explanatory factors and should seek to incorporate empirical features of the term structures of the asset return volatility, skewness, & kurtosis.[7] For the purpose of documenting multidimensional applications of OHMC we employ a GARCH(1,1) model.

3.1 GARCH(1,1) Model

The GARCH(1,1) model is a simple and well known model of asset returns that exhibits excess kurtosis and captures some key empirical aspects of the temporal dependence of returns (Bollersev [1986]; Engle [1994]). The conditioning variable is *starting volatility*, and the jumpiness of returns associated with the return kurtosis thwarts the theoretical perfect hedge even under continuous hedging. Under GARCH(1,1) the assets and their volatility evolve as follows:

$$r_i(t_k) \equiv \frac{\Delta s_i(t_k)}{s_i(t_k)} = \left(\mu_i \Delta t + \sigma_i(t_k)\sqrt{\Delta t}\,\varepsilon_i(t_k)\right) \tag{25}$$

$$\sigma_i^2(t_k) = (1-\alpha_i-\beta_i)\sigma_i^2 + \sigma_i^2(t_{k-1})\left(\beta_i + \alpha_i \varepsilon_i^2(t_{k-1})\right) \tag{26}$$

Standard Normal random-variates generated to create the return stochastic process in (25) are denoted by $\varepsilon_i(t_k)$, and the volatility evolves per (26). The return correlation coefficient between the different assets *i* and *j* are given by

$$\rho_{r_i r_j} \equiv \frac{E[(r_i-\bar{r}_i)(r_j-\bar{r}_j)]}{\sigma_{r_i}\sigma_{r_j}} \tag{27}$$

The dependence between the returns is effected by correlating $\varepsilon_i(t_k)$ and $\varepsilon_j(t_k)$:

$$\rho_{\varepsilon_i \varepsilon_j} \equiv \overline{\varepsilon_i(t_k)\varepsilon_j(t_k)} \tag{28}$$

The correlation $\rho_{r_i r_j}$ can be different from $\rho_{\varepsilon_i \varepsilon_j}$ as the volatilities evolve per (26).

3.2 Stylized Assets

The illustration of OHMC to multi-asset option problems is best accomplished through specific examples. We have adopted 4 stylized descriptions of assets based on broad market indexes described below.

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[7] In contrast, the risk neutral approach is limited to a very specific description of the underlying (e.g., Geometric Brownian Motion) that is not supported by empirical observations. Risk neutral model fitting is focused on the "mid-market" *implied volatility surface*, and does not acknowledge the observable temporal structures of the asset return volatility, skewness, & kurtosis, as many of these features contradict the perfect replication assumption!

<u>SPX Index</u>
This is a capitalization weighted index of 500 stocks, that is designed to measure performance of the broad US economy. This index measures performance of the broad economy representing all major industries. The dividend yield as of Jun 18 2007 is 2.26%/year.
<u>SX5E</u>
The Dow Jones Euro Stoxx 50 Index is a market capitalization weighted index 50 European blue-chip stocks. Each component is capped at 10% of the index's total market capitalization. The dividend yield as of Jun 18 2007 is 4.35%/year.
<u>MXEF</u>
The MSCI EM (Emerging Markets) Index is a free-float weighted equity index. Approximately 25 emerging market country equity indexes are included in this index.
<u>SPGSCI</u>
This index is a benchmark for investment performance in commodity markets. The index is on a world-production weighted basis and encompasses non-financial commodities.

The performance of these indexes is displayed in **Figure 2**. The temporal structure of squared return perturbations are shown in **Figure 3**. The temporal correlation of the returns and squared returns are summarized in **Table 1** through the following summary statistics:

return correlation time: $$\Gamma_{r_i} \equiv \sum_{lag=1}^{252} \rho_{r_i}(lag)$$

squared return correlation time: $$\Gamma_{r_i^2} \equiv \sum_{lag=1}^{252} \rho_{r_i^2}(lag)$$

Significant long-range temporal dependencies of returns are visible in **Figure 3** and **Figure 4**. There are *cycles* of persistently high and low volatility - albeit imperfect cycles that never repeat themselves and that span a large range of time-scales.

daily return properties	SPX	SX5E	MXEF	SPGSCI
mean	0.0003674	0.0003917	0.0005066	0.0003793
stdev	0.0098440	0.0120144	0.0102889	0.0120975
skewness	-0.13	-0.09	-0.59	-0.48
kurtosis	7.25	7.98	7.10	12.66
return corr time	0.02	0.19	0.73	0.22
sqrd return corr time	14.36	23.97	7.25	4.70
corr matrix	**SPX**	**SX5E**	**MXEF**	**SPGSCI**
SPX	100.0%	41.8%	31.0%	-5.3%
SX5E	41.8%	100.0%	44.1%	-1.2%
MXEF	31.0%	44.1%	100.0%	3.6%
SPGSCI	-5.3%	-1.2%	3.6%	100.0%
GARCH(1,1) parameters	**SPX**	**SX5E**	**MXEF**	**SPGSCI**
mu	0.092577	0.098707	0.127661	0.095578
vol	0.156254	0.190705	0.163315	0.192023
alpha	0.111237	0.085978	0.162437	0.260654
beta	0.867423	0.902100	0.790740	0.645915
noise corr matrix	**SPX**	**SX5E**	**MXEF**	**SPGSCI**
SPX	100.0%	46.5%	34.5%	-6.1%
SX5E	46.5%	100.0%	49.5%	-1.4%
MXEF	34.5%	49.5%	100.0%	4.2%
SPGSCI	-6.1%	1.4%	4.2%	100.0%

Table 1. Characteristics of reference assets shown in Figures 2, 3 & 4.

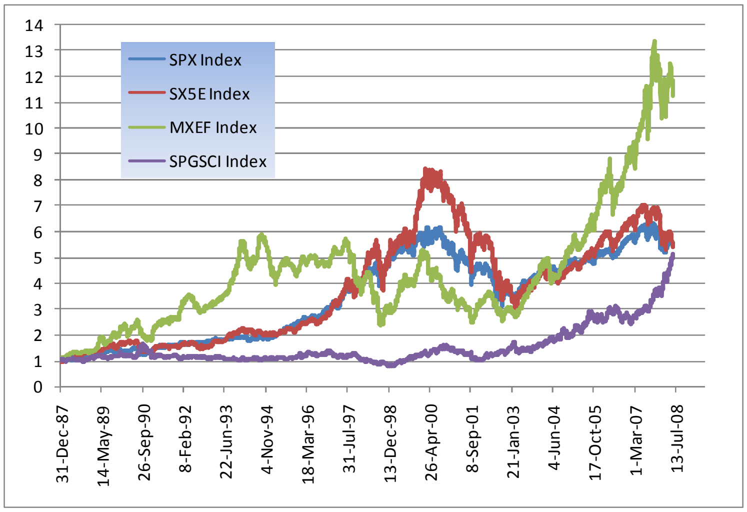


Figure 2. Performance of 4 assets employed for sample multi-asset option problems.

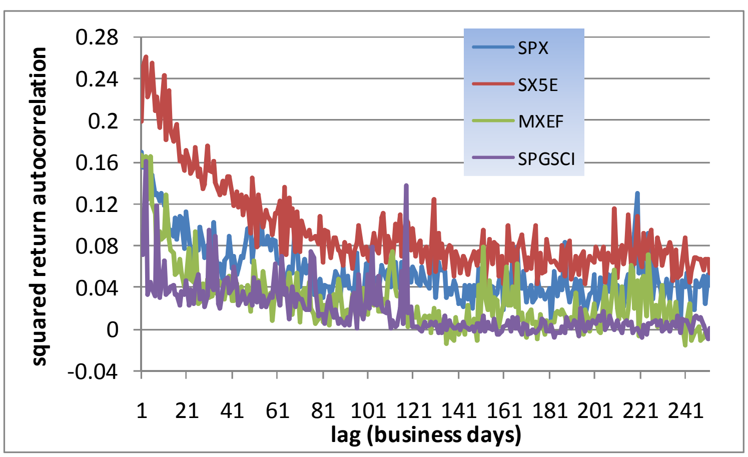


Figure 3. Temporal autocorrelation of squared returns for reference assets.

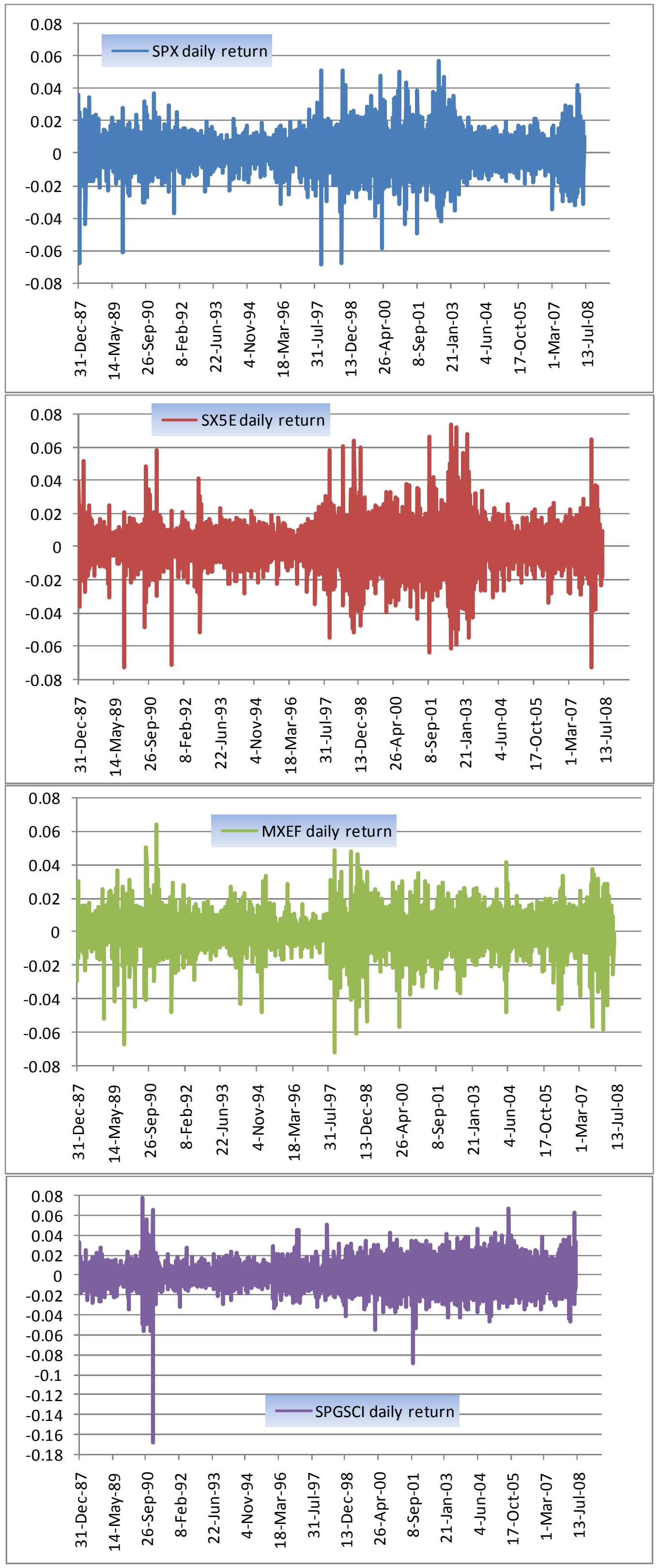


Figure 4. Daily return of four assets that are used for examples in this paper.

3.3 Realized Volatility & Dispersion

In the OHMC analysis the multi-asset option hedging strategy is assessed while accounting for the ensemble of possible outcomes over which the residual risk is sought to be minimized. While we analyze residual risk in detail from the point of view of relative value & pricing arguments, we also want to develop intuition about what kind of moves in the underlying are associated with significant hedge slips. To better understand the circumstances around hedge slippage we will monitor the realized volatility and dispersion measures as follows:

Realized Variance: $$variance[t_k, m; N] = \frac{1}{N} \sum_{i=k-N+1}^{N} r_m^2 [t_i]$$

Realized Volatility: $$volatility[t_k, m; N] = \sqrt{Variance[t_k, m; N]}$$

Realized Covariance: $$covariance[t_k, m, n; N] = \frac{1}{N} \sum_{i=k-N+1}^{N} r_m[t_i] r_n[t_i]$$

Realized Dispersion: $$dispersion[t_k, m, n; N] = \sqrt{\frac{1}{N} \sum_{i=k-N+1}^{N} \left(r_m[t_i] - r_n[t_i]\right)^2}$$

$$dispersion[t_k, m, n; N] = \sqrt{variance[t_k, m; N] + variance[t_k, n; N] - 2covariance[t_k, m, n; N]}$$

We refer to the average squared return as *realized variance*. We compare the realized variance to its ensemble average value to differentiate outcomes of low realized variance and high realized variance. The kurtosis of the return, in the GARCH model employed here, is driven by the variability of realized volatility.

We can also examine the relationship between hedging error and the product of the realized return. We refer to the average product of returns as *realized covariance*. We can compare the realized covariance to its ensemble average value to differentiate outcomes of low realized covariance and high realized covariance. Note that we have not subtracted the mean returns in our definition of realized variance and covariance.

A measure that nets realized variances with covariance is provided by the *realized dispersion*. If two assets move together and gently, the realized dispersion would be small. If either of them moved more violently and the assets do not move in tandem we would see a higher realized dispersion. It is customary to report volatility in annualized terms, assuming volatilities scale as a square-root of the time-scale. So if the returns are daily and there are 252 trading days in the year, then the volatility above is multiplied by the square-root of 252. We report realized volatilities that are annualized. Similarly, we annualize the dispersion measure in the sample results shown in the next section.

4. Hedging, Residual Risk, & Pricing

The OHMC approach enables integrating risk management, hedging strategy evaluation, and option valuation. The series of analysis envisaged in developing a derivative buying-selling hedging strategy are as follows:

(1) Develop a stochastic model of the underlying reflecting its *real-world* characteristics
The model should reflect as much real *texture* as possible and should also respond to changing environments. For instance, intervals of high and low volatility tend to show temporal persistence, so it seems essential to acknowledge the starting *volatility regime*. There is scope for using econometric models that help a trader reflect his economic views, while creating a realistic stochastic description. There is also scope for incorporating features not observed in historical data by a combination of using surrogates and factor models. There is no getting around having an *objective measure* description of the underlying.[8]

(2) Assess the hedging strategy, the average hedging cost, and the hedge slippage
OHMC provides a hedging strategy in pursuit of minimizing a hedge slippage measure. Here we illustrate P&L volatility as the hedge slippage measure and also focus on the residual risk and its tail behavior. One can attempt to minimize tail risk measures – however that will not eliminate risk and the need to develop a strategy of dealing with it. For instance, if one seeks to minimize a tail risk measure one will end up with a P&L distribution with a *fat-body*. The decision to trade has to be cognizant of the risk-return choices available to the market agent and the relative attractiveness of the derivative trade.

(3) Price by articulating a profitability criterion while allocating risk-capital
Taking risks requires risk capital. Having articulated a hedging strategy and its residual risk, the price implies a certain expected return on risk capital. The profitability criterion could be set by an investor, or by comparing with another trade. Here we employ a simple basket of the underlying assets to set a hurdle rate of return to articulate bound on prices. This is fundamentally different from the risk-neutral view that purports option prices as monoliths equal to the unique cost of *perfect replication*.

The perfect replication *mantra* seems to be adopted by control functions that are at a distance from the trade – and get to simplify their roles by conjuring absolutes out of something inherently uncertain. This is often encouraged by parties who benefit in the short-term by a continuous focus on "price verification" to a level of false numerical precision, accompanied by a complete lack of admission or recognition of risk capital associated with their businesses. If a trader is buying and selling the *identical* contracts then risks arise mainly from the credit quality of its counterparties. However when a trader sells an option and hedges by trading the underlying, the trader incurs residual risks that should be quantified as a part and parcel of *valuing* the derivative contract, as illustrated here.

[8] That an objective measure description is not needed for valuing-trading derivatives is the most obfuscating implication attributed to the *risk-neutral* model by the run of the mill *valuation quant.* That is nonsense as residual risk is the rule rather than the exception. Decisions regarding trading at specific pricing points require views on the underlying and the attendant complexity of reconciling multiple possible descriptions of the underlying.

4.1 Sample Results

For the model problems **P1**, **P2**, & **P3**, we will show summary results for 3 asset pairs: **(1)** SPX-SX5E; **(2)** SPX-MXEF; **(3)** SPX-SPGSCI. The sample problems are enumerated in **Table 2**.

	Best of Call	Best of Put	Junior Tranche	Senior Tranche
Tenor (yr)	1	1	1	1
Hedge Interval (days)	1,2,5,21	1,2,5,21	1,2,5,21	1,2,5,21
Asset1	SPX	SPX	SPX	SPX
Asset2	SX5E/MXEF/SPGSCI	SX5E/MXEF/SPGSCI	SX5E/SPGSCI	SX5E/PGSCI
Spot1 ($)	100	100	100	100
Spot2 ($)	100	100	100	100
Strike1 ($)	100	90	90	90
Strike2 ($)	100	90	90	90
Attachment ($)	-	-	0	20
Detachment ($)	-	-	20	180

Table 2. Sample problems analyzed

The risk-free discount factors are based on monthly, quarterly, semi-annual and annual discount factors shown below. The funding discount factors were taken to be identical to the risk-free discount factors.

term (yrs)	discount factor
0.08333	0.997954
0.25	0.993059
0.5	0.984696
1	0.968392

Table 3. Discount factors

The simulation results shown here are based on a GARCH(1,1) simulation[9] on a daily time grid and the OHMC solution implemented using linear basis functions. 200K Monte-Carlo paths of the asset values are created. 20 basis functions in each asset dimension were employed to represent the pricing and hedge-ratio functions of basket asset-values. The OHMC solution algorithm starts at option maturity where the simulated values of the underlying are used to assess option payoff. The hedge ratio at the hedge-instance prior to maturity is assessed by the OHMC algorithm, as is the pricing function of spot (Equation (14), (17), & (21)). The solution is thus propagated from maturity to the first time-step. After assessing the optimal hedging strategy and the corresponding breakeven-average pricing function (setting the mean change in wealth to zero) the residual risk is assessed. With information about the residual risk and average hedging costs, a rational trading decision can be made. We report bounding values of the options by imposing return on risk-capital hurdles rates that are obtained from an elementary basket holding the underlying assets, as detailed in **Section 2.4**.

[9] The initial volatility of the GARCH(1,1) model was set to long-term historical volatility for the sample calculations shown here.

P1. Best of Call

asset pair (1-2):		SPX-SX5E				SPX-MXEF				SPX-SPGSCI			
hedge interval	(days)	1	2	5	21	1	2	5	21	1	2	5	21
avg hedge cost	($)	12.64	12.63	12.62	12.55	12.26	12.25	12.22	12.12	14.34	14.32	14.30	14.21
std dev of hedge P&L	($)	2.88	3.00	3.40	5.04	2.25	2.42	2.85	4.50	2.34	2.54	3.08	4.93
	multiple of avg hedge cost	0.23	0.24	0.27	0.40	0.18	0.20	0.23	0.37	0.16	0.18	0.22	0.35
seller's hedge-asset1	%	39.1%	39.2%	39.4%	40.5%	41.8%	41.8%	42.1%	42.9%	44.3%	44.4%	44.8%	46.2%
seller's hedge-asset2	%	43.3%	43.4%	43.6%	44.8%	42.4%	42.6%	43.1%	45.1%	47.2%	47.4%	47.8%	49.2%
long-basket hurdle rate	(%/yr)	13.8%	13.8%	13.8%	13.8%	18.9%	19.0%	19.0%	19.0%	16.5%	16.5%	16.5%	16.5%
seller-hedger risk-capital (1 yr 99.9 %)	($)	20.84	20.69	21.32	24.03	15.55	15.67	16.61	20.98	15.91	16.21	17.88	23.64
	multiple of avg hedge cost	1.65	1.64	1.69	1.92	1.27	1.28	1.36	1.73	1.11	1.13	1.25	1.66
bounding risk premium for seller	($)	3.09	3.07	3.16	3.57	3.24	3.27	3.47	4.40	2.86	2.91	3.21	4.25
seller's bound	($)	15.73	15.70	15.78	16.11	15.50	15.52	15.69	16.52	17.20	17.24	17.51	18.46
buyer-hedger risk-capital (1 yr 99.9 %)	($)	12.55	12.31	12.49	17.41	9.88	10.17	10.36	15.93	6.14	6.86	8.93	16.57
	multiple of avg hedge cost	0.99	0.97	0.99	1.39	0.81	0.83	0.85	1.31	0.43	0.48	0.62	1.17
bounding risk premium for buyer	($)	1.86	1.83	1.85	2.58	2.06	2.12	2.16	3.34	1.10	1.23	1.60	2.98
buyer's bound	($)	10.78	10.81	10.76	9.96	10.20	10.13	10.06	8.77	13.24	13.09	12.69	11.23
range	$	4.96	4.90	5.02	6.15	5.30	5.39	5.63	7.74	3.96	4.15	4.82	7.23
	multiple of avg hedge cost	0.39	0.39	0.40	0.49	0.43	0.44	0.46	0.64	0.28	0.29	0.34	0.51

Table 4. Results for Best of Call

The SPX-SX5E asset pair have volatilities that are quite close to the SPX-SPGSCI assets pair. Comparison of these cases provides an insight into the impact of asset return correlation. For the best of call contract: (1) the average cost of hedging is lower when the asset correlation is higher; (2) the residual risk is higher when the correlation is higher, for both buyer and seller; (3) the increase in hedging error with hedging interval is lower when the correlation is higher.

The SPX-MXEF asset pair has a return correlation in between the SPX-SX5E and SPX-SPGSCI asset pairs. Even though the volatility of MXEF is somewhat lower than SPX and SPGSCI, the SPX-MXEF asset pair exhibits characteristics in-between the other two asset pairs analyzed here.

The vagaries of realized correlation are more acute over shorter hedge intervals, therefore hedge slippage has significant contributions from the same factors in single name derivatives - namely jumpiness (or high realized volatility) of the individual assets. Over longer hedge intervals the higher correlation in the description of the SPX-SX5E asset pair, relative to SPX-SPGSCI, manifests itself in limiting the increase in hedge slippage relative to daily hedging.

The long-basket employed to assess a bounding hurdle rate for the derivative trader is also influenced by the asset return correlation - for assets with lower correlation the expected return on risk capital can be higher (for the same drifts). So our relative value argument on the bounding price for the derivative adds/subtracts a larger fraction of the hedge slippage controlled risk capital, to the average cost of hedging for the seller/buyer of the derivative contract if the asset correlation is higher**.**

P2. Best of Put

asset pair (1-2):		SPX-SX5E				SPX-MXEF				SPX-SPGSCI			
hedge interval	(days)	1	2	5	21	1	2	5	21	1	2	5	21
avg hedge cost	($)	3.37	3.37	3.36	3.32	2.89	2.88	2.87	2.82	3.76	3.75	3.74	3.70
std dev of hedge P&L	($)	1.96	2.01	2.17	2.75	1.49	1.55	1.69	2.22	1.67	1.75	1.97	2.68
	multiple of avg hedge cost	0.58	0.60	0.65	0.83	0.51	0.54	0.59	0.79	0.44	0.47	0.53	0.72
seller's hedge-asset1	%	-13.43%	-13.32%	-13.02%	-11.98%	-15.24%	-15.16%	-14.86%	-13.76%	-16.44%	-16.34%	-15.97%	-14.47%
seller's hedge-asset2	%	-18.98%	-18.85%	-18.55%	-17.22%	-16.53%	-16.30%	-15.79%	-13.94%	-20.54%	-20.25%	-19.67%	-18.09%
long-basket hurdle rate	(%/yr)	13.66%	13.66%	13.65%	13.64%	19.07%	19.06%	19.05%	18.93%	16.36%	16.38%	16.38%	16.36%
seller-hedger risk-capital (1 yr 99.9 %)	($)	15.42	15.49	16.03	18.64	12.03	12.03	12.10	15.51	12.70	13.17	14.09	18.09
	multiple of avg hedge	4.57	4.60	4.78	5.61	4.16	4.17	4.22	5.50	3.38	3.51	3.77	4.89
bounding risk premium for seller	($)	2.26	2.27	2.34	2.72	2.53	2.53	2.54	3.23	2.26	2.34	2.51	3.22
seller's bound	($)	5.63	5.63	5.70	6.05	5.42	5.41	5.41	6.05	6.02	6.09	6.25	6.92
buyer-hedger risk-capital (1 yr 99.9 %)	($)	5.10	5.21	5.63	7.48	4.38	4.39	4.77	6.28	4.09	4.37	5.05	7.18
	multiple of avg hedge	1.51	1.55	1.68	2.25	1.51	1.52	1.66	2.23	1.09	1.16	1.35	1.94
bounding risk premium for buyer	($)	0.75	0.76	0.82	1.09	0.92	0.92	1.00	1.31	0.73	0.78	0.90	1.28
buyer's bound	($)	2.62	2.60	2.53	2.23	1.97	1.96	1.87	1.51	3.03	2.97	2.84	2.42
range	$	3.00	3.03	3.17	3.82	3.45	3.45	3.54	4.54	2.99	3.12	3.41	4.49
	multiple of avg hedge cost	0.89	0.90	0.94	1.15	1.19	1.20	1.23	1.61	0.79	0.83	0.91	1.21

Table 5. Results for Best of Put

The average hedge cost for the option with reference pairs SPX-SX5E (with a higher correlation) is lower than that for the SPX-SPGSCI pair. The option seller-hedger loss tail is wider than the option buyer loss tail, and therefore his risk-capital requirements are 2-3 times larger than the option purchaser-hedger. This is easily visible in the centered total wealth change distribution in **Figures 5, 6, & 7**. The bound on the derivative price for the seller and buyer are far apart – by approximately an amount equal to the average hedge cost amount. This range is larger than that for the best of call examples shown previously (**Table 4**), as the puts are chosen to be out of the money while the calls were chosen to be at the money. Generally speaking, the more out of the money the options are, the greater is the residual risk as a fraction of the average hedging costs, and therefore the greater the range of price between buyer and seller.[10] Like the best-of-call contract, the hedge-slippage for the asset pair with a higher correlation does not increase with hedging interval as rapidly as the asset pair with a lower asset return correlation.

The residual hedge error decreases with a decrease in hedge-interval between monthly and weekly hedging. At higher hedging frequency the hedging error is decreasing very slowly and for all practical purposes the residual risk has stagnated between hedging intervals of 1 day and 2 days. This saturation of hedging error with hedging frequency is associated with the jumpiness of the assets (excess kurtosis) as has been demonstrated for single asset problems (Petrelli et al [2008]). In all our model problems, the residual risk has practically achieved its asymptomatic levels at hedging intervals less than two days. An overlay of transaction costs on this analysis will further dispel the notion that continuous hedging helps one attain a perfect hedge and helps create *complete* markets. For markets are jumpy, and therefore hedging more often doesn't eliminate hedging error after a certain frequency, and hedging more often would simply explode the transaction costs!

[10] Our analysis hypothesizes that market agents that are option-trader-hedgers attempt to get paid for anticipated average cost of hedging, as well statistically described residual risk. This is different than the *risk-neutral* paradigm where the option price is in fact the unique cost of hedging that does not differentiate the option buyer from the seller,

P3. Basket Put Tranche

Junior Tranche

asset pair (1-2):		SPX-SX5E				SPX-SPGSCI			
hedge interval	(days)	1	2	5	21	1	2	5	21
avg hedge cost	($)	16.09	16.09	16.10	16.14	15.76	15.77	15.78	15.83
std dev of hedge P&L	($)	1.70	1.76	1.91	2.47	1.56	1.63	1.82	2.43
	multiple of avg hedge cost	0.11	0.11	0.12	0.15	0.10	0.10	0.12	0.15
seller's hedge-asset1	%	13.63%	13.55%	13.27%	12.34%	15.86%	15.77%	15.43%	14.06%
seller's hedge-asset2	%	16.34%	16.27%	16.07%	15.11%	18.52%	18.31%	17.85%	16.55%
long-basket hurdle rate	(%/yr)	13.76%	13.76%	13.77%	13.77%	16.42%	16.43%	16.44%	16.41%
seller-hedger risk-capital (1 yr 99.9 %)	($)	5.39	5.33	5.70	7.28	5.75	5.58	5.60	7.07
	multiple of avg hedge	0.33	0.33	0.35	0.45	0.36	0.35	0.35	0.45
bounding risk premium for seller	($)	0.79	0.79	0.84	1.08	1.03	1.00	1.00	1.26
seller's bound	($)	16.88	16.88	16.94	17.21	16.79	16.77	16.78	17.09
buyer-hedger risk-capital (1 yr 99.9 %)	($)	11.99	12.14	12.52	15.16	10.40	10.79	11.60	14.74
	multiple of avg hedge	0.75	0.75	0.78	0.94	0.66	0.68	0.74	0.93
bounding risk premium for buyer	($)	1.77	1.79	1.85	2.24	1.86	1.93	2.07	2.63
buyer's bound	($)	14.32	14.30	14.25	13.90	13.91	13.84	13.71	13.20
range	$	2.56	2.58	2.69	3.32	2.88	2.92	3.07	3.89
	multiple of avg hedge cost	0.16	0.16	0.17	0.21	0.18	0.19	0.19	0.25

Table 6. Results for Junior Tranche

The average hedging costs of the seller of the junior loss-tranche increases with correlation between the asset. If the purchaser of the junior tranche hedges it, she can, on the average generate more wealth if the assets have a higher correlation. This sensitivity is consistent with the usually labeled 'long correlation' exposure and 'short correlation' exposure incurred by the seller and buyer of the junior loss tranche respectively.

In a similar vein, as it takes a window of time for the average correlation to manifest itself in time series of asset returns, to some extent, hedging less often has same features as hedging under conditions of relatively higher correlation. We see that in between the hedging intervals of 1 day to 1 month, the average hedging costs of the junior tranche seller increase with hedging interval. Equivalently, the wealth created by hedging for the junior tranche buyer, on the average, increases with hedging interval.

The body risk measure, the standard deviation of the hedging P&L is increasing with hedge interval, as one expects. Albeit it is relatively stable at the daily hedging frequency, on account of the futility of hedging a jumpy asset beyond a certain frequency, and the higher uncertainty of realized correlation over small intervals.

Senior Tranche

asset pair (1-2):		SPX-SX5E				SPX-SPGSCI			
hedge interval	(days)	1	2	5	21	1	2	5	21
avg hedge cost	($)	154.31	154.32	154.32	154.33	154.58	154.59	154.59	154.59
std dev of hedge P&L	($)	1.13	1.14	1.17	1.36	0.87	0.86	0.86	0.94
	multiple of avg hedge cost	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01
seller's hedge-asset1	%	4.70%	4.62%	4.47%	3.86%	2.89%	2.86%	2.74%	2.36%
seller's hedge-asset2	%	5.77%	5.71%	5.53%	4.88%	3.69%	3.58%	3.41%	2.95%
long-basket hurdle rate	(%/yr)	13.77%	13.76%	13.75%	13.75%	16.34%	16.36%	16.37%	16.36%
seller-hedger risk-capital (1 yr 99.9 %)	($)	5.34	5.36	5.25	5.26	3.70	3.63	3.50	3.17
	multiple of avg hedge	0.03	0.03	0.03	0.03	0.02	0.02	0.02	0.02
bounding risk premium for seller	($)	0.79	0.79	0.77	0.78	0.66	0.65	0.62	0.56
seller's bound	($)	155.10	155.11	155.09	155.11	155.24	155.23	155.21	155.15
buyer-hedger risk-capital (1 yr 99.9 %)	($)	11.45	11.71	11.79	14.24	9.52	9.46	9.86	11.49
	multiple of avg hedge	0.07	0.08	0.08	0.09	0.06	0.06	0.06	0.07
bounding risk premium for buyer	($)	1.69	1.73	1.74	2.10	1.69	1.68	1.75	2.04
buyer's bound	($)	152.62	152.59	152.58	152.23	152.89	152.90	152.83	152.55
range	$	2.48	2.52	2.51	2.87	2.35	2.33	2.38	2.61
	multiple of avg hedge cost	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02

Table 7. Results for Senior Tranche

The average hedging costs of the seller of the senior tranche decreases with an increase in correlation between the assets. If the purchaser of the senior tranche hedges it, she can, on the average generate less wealth if the assets have a higher correlation. This sensitivity is consistent with the usually labeled 'short correlation' exposure and 'long correlation' exposure incurred by the buyer and seller of the senior tranche respectively.

The hedge slippage measures show some interesting features. The body risk measure, the standard deviation of the hedging P&L, is increasing with hedge interval, as one expects. Albeit, just like that for the junior tranche, it is relatively stable at the daily hedging frequency. The risk capital of the seller of the senior tranche shows an actual decrease with an increase in hedging interval. This is associated with the increase in likelihood of both assets falling below the individual strikes and by amounts enough to erode through the lower attachment point over longer time intervals. Recall that the seller of the senior tranche makes money when the assets fall enough to pierce the lower attachment point of the senior tranche. Congruent with the decrease in risk capital of senior tranche seller with hedging interval is an increase in risk capital of the senior tranche buyer. This is associated with the increase in likelihood of both assets falling below the individual strikes and by amounts enough to erode through the lower attachment point over longer time intervals.

4.2 Sensitivities

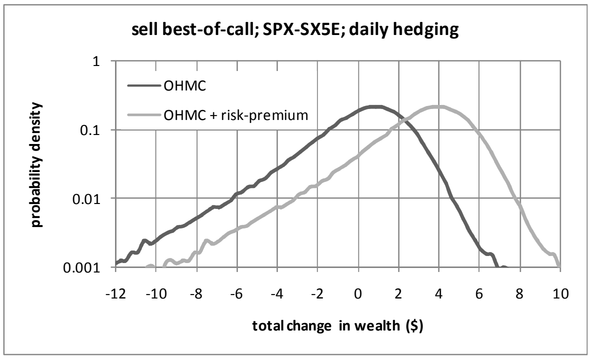
Asymmetry of Residual Risk

The asymmetry in the option traders P&L is central to understanding the dynamics of the trading strategies. Even a vanilla option seller's residual risk tends to be asymmetric, with the option seller typically having a larger loss tail compared to the gain tail, in *delta-hedging* the negative convex, negative vega position under *realistic underlying*. Our multi-asset option problems result in quite asymmetric residual P&L distributions, as shown in **Figures 5, 6, & 7**. The dark plot in these figures represent the P&L volatility minimizing OHMC solution. The lighter line represents the P&L distribution from the point of view of the option seller or buyer while accounting for the risk premium they may add to render their return on risk-capital to be no worse than simply holding a long basket of the underlying in proportions of their hedge ratios.

Our treatment of risk premiums emanate from our own trading risk-management decision-making experiences and the reality of risk-capital constraints. There are many different comparisons of risk and return that we undertake to decide on the acceptable price at which we would hold a derivative position. The *market* has agents with distinct and evolving risk-premium criterion, and not all are option seller-hedgers, as they may see value in simply holding a directional option exposure. The asymmetry of the residual P&L also guides our understanding of those agents and helps rationalize their willingness to pay for positive surprise, or have a lower P&L expectation in return for a more flattering gain to loss surprise potential, as quantified by the residual P&L distribution.

We would like to draw sharp distinctions between the type of information provided by the OHMC approach *en-route* to attempting to price and the traditional *risk-neutral* approach. The market agent trading & hedging an option is given no direct information of his residual P&L by the risk-neutral model. Instead he is given *implied* parameters that make a perfect hedging theory (assuming that the assets show no excess kurtosis) fit some vanilla market *mid* pricing. The job of risk management is not even touched by the risk-neutral model. Instead one is given a myriad of fitting parameters that have no direct bearing on risk-return of the trade at hand.

In the risk-neutral approach risk management is an afterthought to *valuation-modeling*. A regulatory requirement *value-at-risk* number is calculated with the risk-neutral model's output of *greeks* or by revaluing the *risk-neutral* model under some *stress* scenarios. For vanilla options it is possible to have long time-series of the implied parameters (less long than the underlying), to yield a *value-at-risk* estimate. However there is no reason to believe that the vanilla implied parameters adequately address the risk premiums that should guide the exotic trade. For exotics there is no common price discovery or large time series of implied parameters available. A *value-at-risk* number based on a perfect replication model with fitting parameters does not seem like a sound way to develop a trading strategy! The asymmetry inherent to option payoff structures and realistically jumpy underlyings should be more directly accessible to trading-risk management. The less than candid derivative valuation models that do not address realistic assets and do not build healthy expectations for hedge slippage deserve their fair share of blame for the poor state of risk management revealed in many financial institutions.



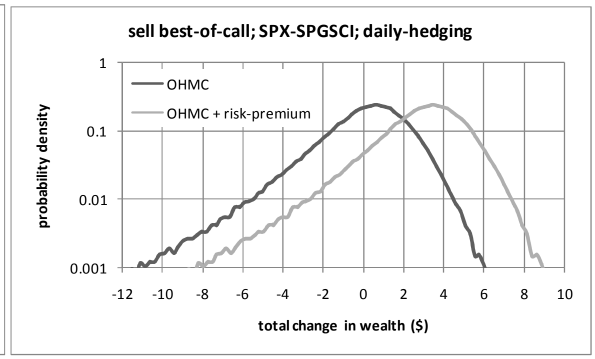


Figure 5. Total wealth change distribution for best of call option seller-hedger

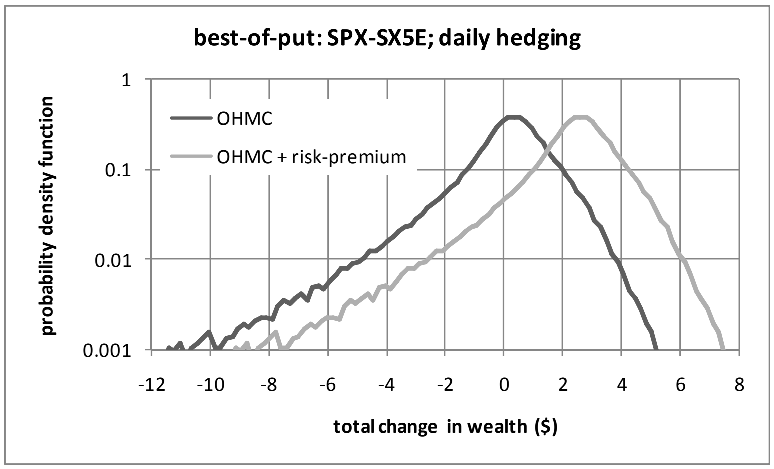


Figure 6. Total wealth change distribution for best of put option seller-hedger

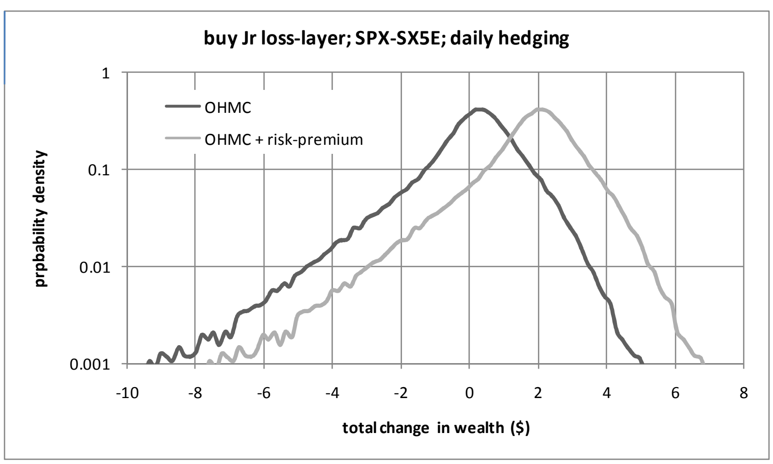


Figure 7. Total wealth change distribution of basket-put junior tranche buyer-hedger

Carry of Correlation Trader

We have shown the hedging P&L probability density to be asymmetric and the specific market agents to have a positive *most-likely outcome* and an asymmetrically larger loss tail than a gain tail. After adjusting for the plausible risk premium that we think should be a part of a bounding pricing argument, we find the resulting P&L distribution to have a positive expectation, positive most likely outcome and yet an asymmetrically larger loss tail than gain tail.

In all these configurations (**Figures 5, 6, & 7**) we showed the P&L of the *long-correlation* market agent. The long correlation trader showing positive P&L expectation is indeed rational given the asymmetry of the P&L distribution. The positive carry could well be enough to render the trade attractive based on a return on risk-capital assessment such as the one enabled by OHMC.

Positive carry could also become the sole motivator of the trade for an agent that does not calculate risk capital of the trade and does not appreciate the asymmetry of his P&L distribution. One hears anecdotally about the dangerous positive carry trade. Indeed, carry is a motivator. The series of debacles in the CDO market[11] is the most glaring recent example of carry motivations in derivatives dominating and leading the large market players into *unexpected* large losses. The annihilation of many a *short equity volatility* market agents in the latter half of 2008 also is a testimonial for continued dominance of carry motivation of trades, and the danger posed when carry is visible but the risk capital associated with a solvency target is not made available at the time of execution of the trade.

We are not trying to vilify all positive carry trades. We have met positive carry trades we liked and others we thought were not worth the risks they entailed. To differentiate a weak carry trade from a strong carry trade the expected P&L or likely P&L must be compared with the risk-capital. The risk-capital must *bake-in* the impact of hedge slippage due to excess kurtosis of assets, as well that arising from the vagaries of realized correlation. We have found the risk-neutral valuation modeling regime to be oblivious of important risk-return differentiation that are critical to trade strategy – hence the motivation for employing OHMC for trading purposes.

Realized Volatility

Table 8 shows the realized volatility associated with P&L outcomes at different confidence levels. The confidence level 0.999 means that 99.9% of the MC realizations resulted in a hedge slippage P&L to be greater than -$15.42, i.e., it is extremely unlikely that the *loss* is greater than $15.42. When the confidence level is merely 0.001 then we indeed see a gain, asymmetrically

[11] May 2005 synthetic CDO equity tranche correlation meltdown in wake of auto sector deterioration; 2007-2008 *leveraged super senior* mayhem; 2007 onwards CDO of sub-prime write downs.

smaller in magnitude than the 99.99 confidence level loss. It is extremely rare for the hedging P&L to be higher than $5.03. The realized volatility associated with the rare $15.42 loss is significantly larger than that associated with the rare gain of $5.03.

We explored the hedging of the derivative position with a position in the underlying assets. This can be extended by examining how a variance swap would have altered the hedge P&L. In fact, static hedging by variance swaps can be simply overlain on the hedging P&L simulated here and the worth of the variance swap to the OHMC hedger can be assessed. While an in-depth examination of variance swaps is a topic for another independent investigation, the story here is more complicated due to the multiple assets involved here, as discussed next.

Realized Dispersion

The sensitivity of a multi-asset option to realized correlation or dispersion of asset returns are best communicated through specific examples. In **Table 8** we present further results on the dependence of hedge performance on the realized co-movement of assets.

confidence level	total hedging P&L ($)	realized volatility (annualized %)		realized-dispersion (annualized %)
		SPX	SX5E	
0.999	-15.42	48.07	18.90	42.43
0.99	-7.18	27.61	25.45	26.29
0.85	-1.27	16.69	18.42	17.83
0.001	5.03	16.21	15.36	15.75

Table 8. Role of realized volatility and realized dispersion in controlling hedge-slippage, for Best of Put with SPX-SX5E (Daily Hedging). The total change in wealth at different confidence levels is shown in the second column. The realized volatility and realized dispersion in the MC paths corresponding to distinct total P&L confidence levels are also shown.

The realized dispersion combines the impact of realized volatility and realized correlation. It carries in it the common sense notion of co-movement of assets. The sample paths that involved assets moving sharply and in different directions display a high realized dispersion. The dispersion is driven from high realized volatility and low realized correlation or covariance. For the long-correlation option seller hedger, the loss at the distinct confidence levels is an increasing function of realized dispersion (**Table 8, Figures 8, 9, & 10**).

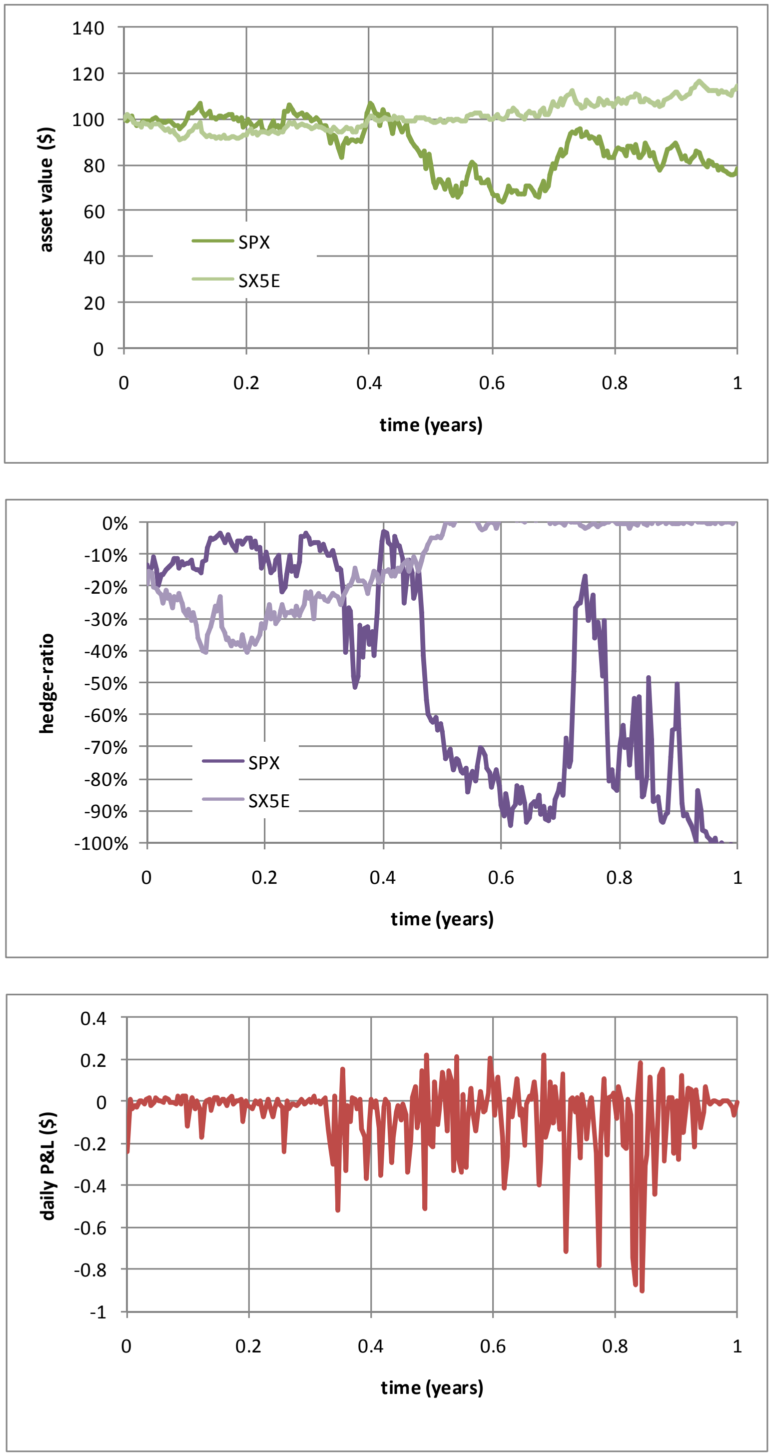


Figure 8. 99.9% confidence level hedge P&L scenario for seller of best of put (SPX-SX5E; daily hedging). The seller of the best of put is initially long-correlation (i.e., short dispersion). The total hedge slippage results in a loss of $15.42. The annualized realized volatility of SPX over the options life, 48.07%, is much larger than the historical average realized volatility. This high realized volatility and high dispersion is responsible for the large hedge slippage and loss for the option seller. This outcome of high realized dispersion results in a windfall gain for the option buyer hedger, who is short correlation and long dispersion.

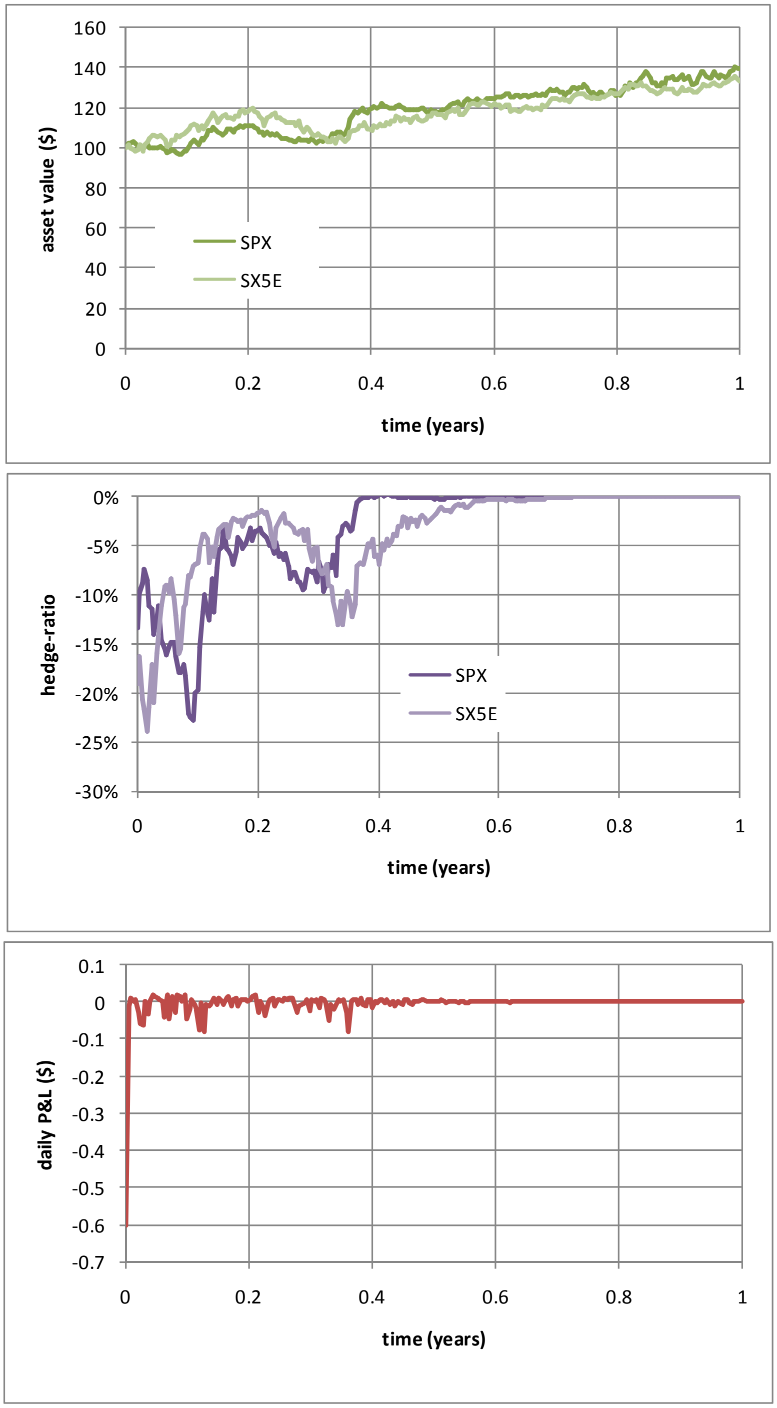


Figure 9. 85% confidence level hedge P&L scenario for seller of best-of-put (SPX-SX5E; daily hedging). The loss incurred by the option seller is one-tenth as much as that at the 99.9 confidence level, and has a large contribution from dispersion of asset values right in the beginning of the option. The asset values are not as disperse as the 99.9 confidence level case and the hedge ratios for the reference asset are not as dramatically different as that case. The realized volatility is not too different from long term historical averages, and the dispersion metric is less than half that experienced in the 99.9 confidence level wealth change realization.

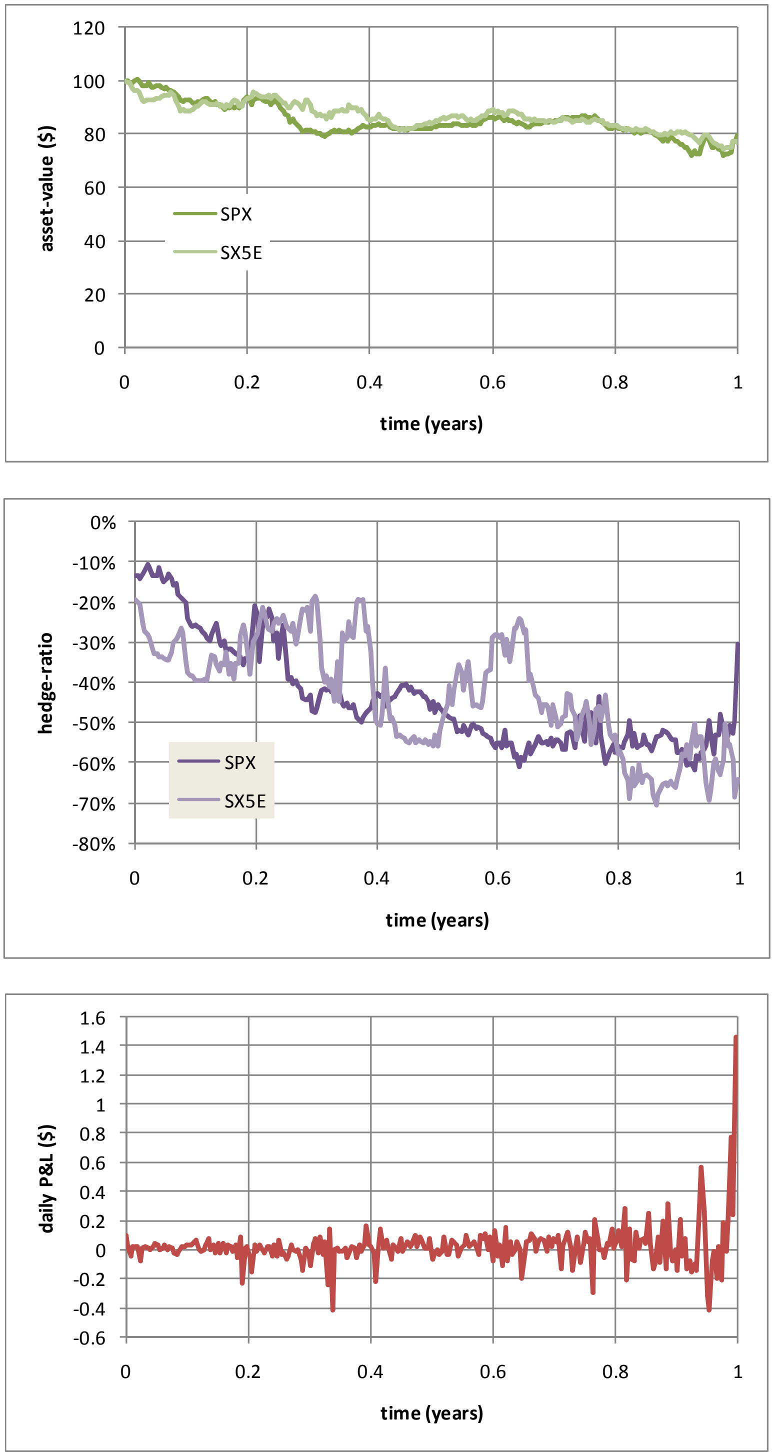


Figure 10. 0.1% confidence level hedge P&L gain scenario for seller of best of put (SPX, SX5E; daily hedging). The rare gain experienced by the option seller hedger occurs in a realization with low dispersion and low realized volatility. The option purchaser delta hedger incurs a loss as he is short correlation (long dispersion). The impact of holding variance swaps or dispersion swaps on the P&L can be overlain on such OHMC P&L realizations.

5. Summary

Correlation Trading Strategies

We examined a variety of contracts that, initially, display long and short covariance sensitivity. We found the P&L volatility minimizing hedge strategy and the P&L distribution of the trading book implementing the trading-hedging strategy. We note that the resulting distribution is asymmetric, and for the initially long-correlation trades, the most likely P&L outcome is positive, but the loss tail is longer than the gain tail. This catalogues the positive carry associated with a long-correlation trade. That carry may not be attractive compensation for tails risks relative to simply holding a long position of some combination of the underlying assets. We assessed the premium that needs to be charged over the average cost of hedging for the return over risk capital of the multi-asset option trader hedger to be no less than that of a basket of the underlying assets in proportions to the initial hedge ratios. We view such relative value based assessments overlaid on the attempted replication strategy to be more sound than the usual risk neutral approach of assuming perfect replication and fitting parameters to observed prices.

The challenges of replicating a multi-asset option are a compounded version of the challenges to replicate a single asset option. As a result of this complexity, it is even more inappropriate to ever view the P&L of a correlation trader as *arbitrage* relative to the unique cost of perfect replication. The covariance sensitive multi-asset options are particularly vexing in that the sign of the sensitivity to realized covariance can be dependent on spot values, and over small time-intervals typically associated with attempted perfect dynamic hedging, the realized covariance is particularly uncertain.

Covariance-Dependence Structures

The vagaries of realized correlation in multi-asset problems is shown to play a central role in the hedging efficacy of multi-asset derivatives. This was shown by employing a very simple model of dependence – a GARCH(1,1) model with static coefficients of correlation between the diffusive noise terms, albeit rendered more complex due to the heteroskedasticity and attendant excess kurtosis associated with GARCH(1,1). This can be extended to more ambitious renditions of the dependence structure. The OHMC algorithm is not tied to any specific rendition of the processes governing the assets underlying the derivative, and encourages developing more realistic models of the assets underlying the derivative contracts.

Risk Premiums & Expected Change in Wealth

Here we focus on assessing the trading strategy that minimizes a P&L risk measure and show the irreducible residual P&L distribution and what controls it. In doing so we also impose the condition of zero *average* change in wealth. However, after assessing the irreducible hedging error we make a simple relative value argument for pricing by adding to the *average attempted replication/hedging cost* a risk premium that would make the return on risk capital of the derivatives trader identical to that of a *delta-one* trader that is simply long the reference basket assets in a proportion equal to the initial hedge ratios. The accounting treatment of this *add-on* is debatable, especially if the accounting dictates have become increasingly tied to *models* that hide irreducible hedging errors and then purport to fit *risk-neutral* distributions to the *market*. In this back-drop, this add-on could be reasonably interpreted as a *risk-reserve* that is cognizant of the limits of replication. This reserve would also recognize that when the market demand supply dynamics shift the pricing regime can shift and that such regime shifts are commonplace in derivative trading.

This add-on of a risk premium can be made more ingrained in the OHMC method by imposing it locally in time. That would result in not setting the mean change in wealth set to zero in any hedging interval, but rather equal to the risk premium that is argued based on either of three approaches (1) relative basis (2) absolute basis, or (3) market calibration basis.

Computational Architecture

The key to speed and efficiency in effecting the OHMC solution lies in recognizing the sparseness of the matrix effecting the solution and in being able to efficiently assess the successive contribution of each simulated set of asset values to the elements of the matrix.

For the 2-asset problems analyzed here, the computational constraint arises from the storage of the MC paths of the underlying asset process. That constraint is easily overcome by simulating MC paths over shorter time intervals and storing in database/files and reading them when the pertinent hedging interval is being analyzed.

For our model problems, the full matrix involved in the optimization problem was stored in memory, despite its sparseness. This is easy to implement – however it is intensive in terms of computer memory. Therefore it is limited in the number of assets that can be practically addressed. Derivative problems involving 5 assets or less are possible within this memory inefficient implementation on an ordinary machine. For higher dimensional problems it will be necessary to more directly exploit the sparseness of the matrix and employ iterative solvers.

Trading Risk-Management Imperatives

Risk management of financial institutions is charged with ensuring solvency with a certain confidence level. To the extent risk management activities help making choices of trades that have a higher P&L expectation relative to risk, *risk management* can have a direct role in *profitability*. As the OHMC trading platform outputs integrated measures of residual risks en-route to *pricing*, it provides a means for integrating trade strategy-development, trade valuation, and risk management.

We anticipate that well publicized disasters in derivative risk management, search for profitability, a rich variety of market agents pursuing attractive return-risk tradeoffs, and maybe even central regulatory oversight needs will result in a greater acceptance of valuation methods that simultaneously highlight residual risks over methods that misleadingly invoke *perfect replication* without any direct analysis of hedging errors. We think that it is a grave mistake to view valuation modeling as a search for *risk-neutral distributions/parameters* to fit market prices with the tacit fueling of the incorrect perception of almost perfect replication-hedging. Models that highlight residual risks and enable viewing the derivative value as a means of setting a risk-return trade-off should gain acceptance.

All too often oversight of derivative trading involves enforcing *fair-valuation* as its primary goal. We think that quantification of residual risks and risk-capital needs greater attention. The false notion of a monolithic derivative price and any deviation from it as "arbitrage" and therefore day

1 P&L encourages taking outsized trading bets relative to risk-capital. For large trading counterparties to not know the risk-return tradeoffs of the bets they have entered into is a recipe for uncontrolled fear when the risks manifest themselves in the P&L in a negative way. That uncontrolled fear manifests itself as illiquidity, further widening of bid-offer, and further difficulty of *valuing* derivative books. In contrast, an honest admission of the impossibility of perfect replication with *a-priori* estimates of average hedging costs and hedging error and the associated risk-capital, as enabled by OHMC, can contribute in a more rational derivative trading dynamic.

Appendix-A

Basis Functions

In that OHMC optimization problem we are determining the value and hedge-ratio functions of spot values that minimize P&L volatility subject to an expected wealth change constraint. The optimization problem is a variational calculus problem. By replacing the unknown functions with sum-products of unknown coefficients and basis functions we turn the problem into a discrete one and we computationally determine the unknown coefficients.

The range of asset values for which a basis function has a non-zero value describes the support of the basis functions. By choosing basis functions of limited support we can render sparse the matrix defining the set of linear equations to be solved to determine the unknown coefficients (Equation (14) of the main section). Here we document two types of basis functions and their application for multi-asset OHMC problems.

A.1 Linear Basis Functions

Linear basis functions are easy to define and use. The unknown coefficients are the values of the function at the nodal locations and specification of boundary conditions is easy.

One-Dimensional Linear Basis Functions

This representation involves dividing the state space into non-overlapping *elements*, which may be defined by the *nodal* locations s_j. The basis functions maybe then associated with each node

$$\Omega_j(s) = \begin{cases} \dfrac{s - s_{j-1}}{s_j - s_{j-1}} & s_{j-1} \le s \le s_j \\ \dfrac{s_{j+1} - s}{s_{j+1} - s_j} & s_j \le s \le s_{j+1} \\ 0 & \text{otherwise} \end{cases} \tag{A1}$$

With N nodes in 1-D, the function is represented as

$$f(s) = \sum_{k=0}^{N-1} \hat{f}_k \Omega_k(s) \tag{A2}$$

Recognizing the limited support of the basis functions and identifying the adjacent nodal locations k^* and k^*+1 that s lies in between ($s_{k^*} \le s \le s_{k^*+1}$) we can represent the function as

$$f(s) = \hat{f}_{k^*}\Omega_{k^*}(s) + \hat{f}_{k^*+1}\Omega_{k^*+1}(s) \tag{A3}$$

_M_-Dimensional Linear Basis Functions

This representation involves dividing the state space into non-overlapping *elements*, which may be defined by the *nodal* locations defined by $\mathbf{s}_k \equiv \left(s_{1,j_1}, s_{2,j_2}, \ldots\ldots, s_{M,j_M}\right)$. The basis functions are then associated with each node and assessed by multiplying the uni-variate basis functions associated with each dimension

$$\Omega_k(\mathbf{s}) = \prod_{n=1}^{M} \Omega_{j_n}(s_n) \tag{A4}$$

With N_m nodes being used to discretize the state space in the mth dimension, the function is represented as

$$f(\mathbf{s}) = \sum_{k=0}^{N_1 N_2 \ldots N_M - 1} \hat{f}_k \Omega_k(\mathbf{s}) \tag{A5}$$

Recognizing the limited support of the basis functions and identifying the adjacent nodal locations in the mth dimension that s_m lies in between ($s_{j_m^*} \leq s_m \leq s_{j_m^*+1}$) the nodes of the *element* of the *state-space* that encompass a point (s_1, s_2, .., s_M) are identified as $K^* = \left\{k_1^*, k_2^*, k_3^*, \ldots\ldots k_{2^M}^*\right\}$ and the value of the function being approximated is

$$f(\mathbf{s}) = \sum_{k \in K^*} \hat{f}_k \Omega_k(\mathbf{s}) \tag{A6}$$

Two-Dimensional Linear Basis Functions

This representation involves dividing the state space into non-overlapping *elements*, which may be defined by the *nodal* locations defined by $\mathbf{s}_k \equiv \left(s_{1,j_1}, s_{2,j_2}\right)$ The basis functions maybe then associated with each node as the product of the 1-D basis functions associated with each dimension

$$\Omega_k(\mathbf{s}) = \Omega_{j_1}(s_1)\Omega_{j_2}(s_2) \tag{A7}$$

With N_m being the number of nodes in the mth dimension the function is represented as

$$f(s_1, s_2) = \sum_{k=0}^{N_1 N_2 - 1} \hat{f}_k \Omega_k(s_1, s_2) \tag{A8}$$

The adjacent nodal locations in the mth dimension that s_m lies in between ($s_{j_m^*} \leq s_m \leq s_{j_m^*+1}$) help define the nodes of the *element* of the *state-space* that encompass a point (s_1, s_2): $k_1^* = j_1^* + j_2^* N_1$; $k_2^* = k_1^* + 1$; $k_3^* = k_2^* + N_1$; $k_4^* = k_1^* + N_1$. The set of the corner nodes of this element is $K^* = \left\{k_1^*, k_2^*, k_3^*, k_4^*\right\}$ and the value of the function being approximated is

$$f(s_1, s_2) = \sum_{k \in K^*} \hat{f}_k \Omega_k (s_1, s_2) \tag{A9}$$

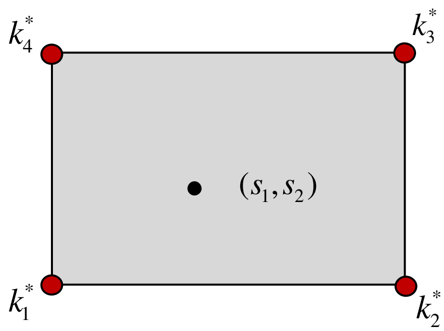


Figure A1. Local definition of nodes of element encompassing $\mathbf{s}_k = (s_1, s_2)$

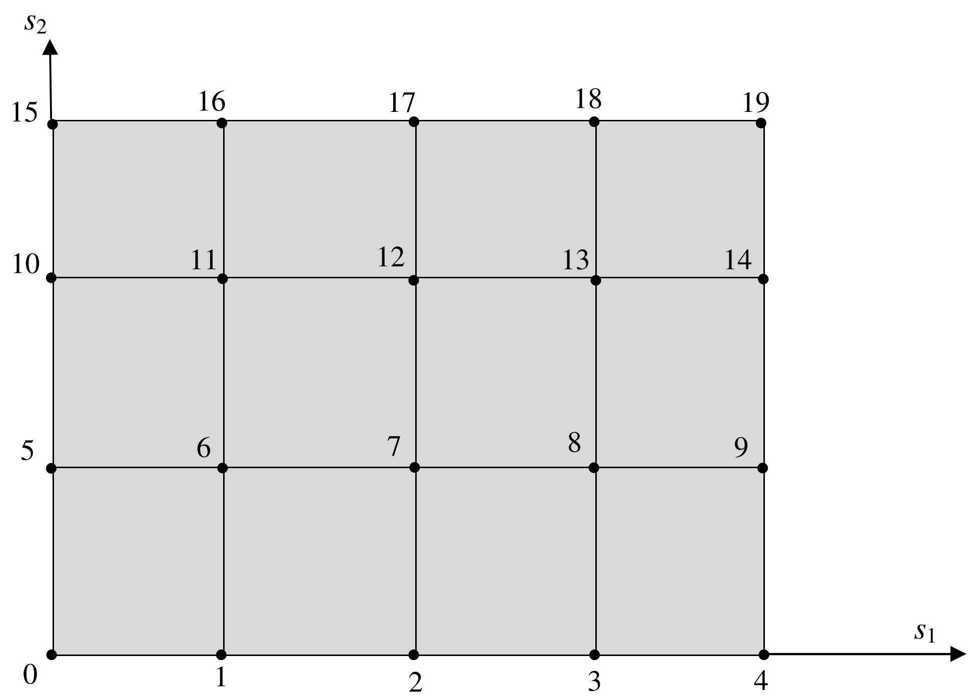


Figure A2. Sample global node numbering for 2-D problem

A.2 Hermite Cubic Basis Functions

More complex than linear basis functions, Hermite cubic polynomials enable specifying boundary conditions in terms of function values and/or its gradient directly. While linear basis functions involve a basis function and unknown coefficient per node, for Hermite cubic basis functions there are 2 unknown coefficients and basis functions associated with each node.

One-Dimensional Hermite Cubic Basis Functions

This representation involves dividing the state space into non-overlapping *elements*, which may be defined by the *nodal* locations s_k. The basis functions maybe then associated with each node

$$\omega_j(s) = \begin{cases} \dfrac{\left(s - s_{j-1}\right)^2}{\left(s_j - s_{j-1}\right)^3}\left[2\left(s_j - s\right) + \left(s_j - s_{j-1}\right)\right] & s_{j-1} \le s \le s_j \\ \dfrac{\left(s - s_{j+1}\right)^2}{\left(s_{j+1} - s_j\right)^3}\left[2\left(s - s_j\right) + \left(s_{j+1} - s_j\right)\right] & s_j \le s \le s_{j+1} \\ 0 & \text{otherwise} \end{cases} \tag{A10}$$

$$\tilde{\omega}_j(s) = \begin{cases} \dfrac{\left(s - s_{j-1}\right)^2\left(s - s_j\right)}{\left(s_j - s_{j-1}\right)^2} & s_{j-1} \le s \le s_j \\ \dfrac{\left(s - s_{j+1}\right)^2\left(s - s_j\right)}{\left(s_{j+1} - s_j\right)^2} & s_j \le s \le s_{j+1} \\ 0 & \text{otherwise} \end{cases}$$

With N nodes in 1-D, the function is represented as

$$f(s) = \sum_{k=0}^{N-1}\left(\hat{f}_k \omega_k(s) + \frac{d\hat{f}_k}{ds}\tilde{\omega}_k(s)\right) \tag{A11}$$

Recognizing the limited support of the basis functions and identifying the adjacent nodal locations k^* and k^*+1 that s lies in between ($s_{k^*} \le s \le s_{k^*+1}$) we can represent the function as

$$f(s) = \hat{f}_{k^*}\omega_{k^*}(s) + \frac{d\hat{f}_{k^*}}{ds}\tilde{\omega}_{k^*}(s) + \hat{f}_{k^*+1}\omega_{k^*+1}(s) + \frac{d\hat{f}_{k^*+1}}{ds}\tilde{\omega}_{k^*+1}(s) \tag{A12}$$

Two-Dimensional Hermite Cubic Basis Functions

This representation involves dividing the state space into non-overlapping *elements*, which may be defined by the *nodal* locations defined by $\mathbf{s}_k \equiv \left(s_{1,j1}, s_{2,j2}\right)$. The basis functions maybe then associated with each node as the product of the 1-D basis functions associated with each dimension. With N_m being the number of nodes in the mth dimension the function is represented as

$$f(s_1,s_2)=$$

$$\sum_{k=0}^{N_1N_2-1}\left(\hat{f}_k\omega_{j_1}(s_1)\omega_{j_2}(s_2)+\frac{\partial\hat{f}_k}{\partial s_1}\tilde{\omega}_{j_1}(s_1)\omega_{j_2}(s_2)+\frac{\partial\hat{f}_k}{\partial s_2}\tilde{\omega}_{j_2}(s_2)\omega_{j_1}(s_1)+\frac{\partial^2\hat{f}_k}{\partial s_1\partial s_2}\tilde{\omega}_{j_1}(s_1)\tilde{\omega}_{j_2}(s_2)\right) \quad \text{(A13)}$$

The adjacent nodal locations in the mth dimension that s_m lies in between ($s_{j_m^*}\le s_m\le s_{j_m^*+1}$) help define the nodes of the *element* of the *state-space* that encompass a point (s_1, s_2): $k_1^*=j_1^*+j_2^*N_1$; $k_2^*=k_1^*+1$; $k_3^*=k_2^*+N_1$; $k_4^*=k_1^*+N_1$. The set of the corner nodes of this element is $K^*=\{k_1^*,k_2^*,k_3^*,k_4^*\}$ and the value of the function being approximated is

$$f(s_1,s_2)=$$

$$\sum_{k\in K^*}\left(\hat{f}_k\omega_{j_1^*}(s_1)\omega_{j_2^*}(s_2)+\frac{\partial\hat{f}_k}{\partial s_1}\tilde{\omega}_{j_1^*}(s_1)\omega_{j_2^*}(s_2)+\frac{\partial\hat{f}_k}{\partial s_2}\tilde{\omega}_{j_2^*}(s_2)\omega_{j_1^*}(s_1)+\frac{\partial^2\hat{f}_k}{\partial s_1\partial s_2}\tilde{\omega}_{j_1^*}(s_1)\tilde{\omega}_{j_2^*}(s_2)\right) \quad \text{(A14)}$$

Three-Dimensional Hermite Cubic Basis Functions
This representation involves dividing the state space into non-overlapping *elements*, which may be defined by the *nodal* locations defined by $\mathbf{s}_k\equiv\left(s_{1,j_1},s_{2,j_2},s_{3,j_3}\right)$. The basis functions maybe then associated with each node as the product of the 1-D basis functions associated with each dimension. With N_m being the number of nodes in the mth dimension the function is represented as

$$f(s_1,s_2,s_3)=$$

$$\sum_{k=0}^{N_1N_2-1}\left(\begin{array}{l}\hat{f}_k\omega_{j_1}(s_1)\omega_{j_2}(s_2)\omega_{j_3}(s_3)+\frac{\partial\hat{f}_k}{\partial s_1}\tilde{\omega}_{j_1}(s_1)\omega_{j_2}(s_2)\omega_{j_3}(s_3)+\frac{\partial\hat{f}_k}{\partial s_2}\tilde{\omega}_{j_2}(s_2)\omega_{j_1}(s_1)\omega_{j_3}(s_3)+\\ \frac{\partial\hat{f}_k}{\partial s_3}\tilde{\omega}_{j_3}(s_3)\omega_{j_1}(s_1)\omega_{j_2}(s_2)+\frac{\partial^2\hat{f}_k}{\partial s_1\partial s_2}\tilde{\omega}_{j_1}(s_1)\tilde{\omega}_{j_2}(s_2)\omega_{j_3}(s_3)+\frac{\partial^2\hat{f}_k}{\partial s_1\partial s_3}\tilde{\omega}_{j_1}(s_1)\omega_{j_2}(s_2)\tilde{\omega}_{j_3}(s_3)+\\ \frac{\partial^2\hat{f}_k}{\partial s_2\partial s_3}\tilde{\omega}_{j_2}(s_2)\tilde{\omega}_{j_3}(s_3)\omega_{j_1}(s_1)+\frac{\partial^3\hat{f}_k}{\partial s_1\partial s_2\partial s_3}\tilde{\omega}_{j_1}(s_1)\tilde{\omega}_{j_2}(s_2)\tilde{\omega}_{j_3}(s_3)\end{array}\right)$$

$$\text{(A15)}$$

The adjacent nodal locations in the mth dimension that s_m lies in between ($s_{j_m^*}\le s_m\le s_{j_m^*+1}$) help define the nodes of the *element* of the *state-space* that encompass a point (s_1, s_2, s_3):

$$k_1^* = j_1^* + j_2^* N_1 + j_3^* N_1 N_2 \qquad k_2^* = k_1^* + 1 \qquad k_3^* = k_2^* + N_1 \qquad k_4^* = k_1^* + N_1$$
$$k_5^* = k_1^* + N_1 N_2 \qquad k_6^* = k_2^* + N_1 N_2 \qquad k_7^* = k_3^* + N_1 N_2 \qquad k_8^* = k_4^* + N_1 N_2$$

The set of the corner nodes of this element is $K^* = \left\{k_1^*, k_2^*, k_3^*, k_4^*, k_5^*, k_6^*, k_7^*, k_8^*\right\}$ and the value of the function being approximated is

$$f(s_1, s_2, s_3) = \sum_{k \in K^*} \left(\begin{array}{l} \hat{f}_k \omega_{j_1^*}(s_1) \omega_{j_2^*}(s_2) \omega_{j_3^*}(s_3) + \dfrac{\partial \hat{f}_k}{\partial s_1} \tilde{\omega}_{j_1^*}(s_1) \omega_{j_2^*}(s_2) \omega_{j_3^*}(s_3) + \dfrac{\partial \hat{f}_k}{\partial s_2} \tilde{\omega}_{j_2^*}(s_2) \omega_{j_1^*}(s_1) \omega_{j_3^*}(s_3) + \\ \dfrac{\partial \hat{f}_k}{\partial s_3} \tilde{\omega}_{j_3^*}(s_3) \omega_{j_1^*}(s_1) \omega_{j_2^*}(s_2) + \dfrac{\partial^2 \hat{f}_k}{\partial s_1 \partial s_2} \tilde{\omega}_{j_1^*}(s_1) \tilde{\omega}_{j_2^*}(s_2) \omega_{j_3^*}(s_3) + \dfrac{\partial^2 \hat{f}_k}{\partial s_1 \partial s_3} \tilde{\omega}_{j_1^*}(s_1) \omega_{j_2^*}(s_2) \tilde{\omega}_{j_3^*}(s_3) \\ + \dfrac{\partial^2 \hat{f}_k}{\partial s_2 \partial s_3} \tilde{\omega}_{j_2^*}(s_2) \tilde{\omega}_{j_3^*}(s_3) \omega_{j_1^*}(s_1) + \dfrac{\partial^3 \hat{f}_k}{\partial s_1 \partial s_2 \partial s_3} \tilde{\omega}_{j_1^*}(s_1) \tilde{\omega}_{j_2^*}(s_2) \tilde{\omega}_{j_3^*}(s_3) \end{array} \right) \quad \text{(A16)}$$

Appendix-B

Optimal Hedge Monte-Carlo Computational Solution

The set of linear equations effecting the constrained optimization are:

$$\hat{C}_j(t_k)\overline{\Omega_j(\mathrm{s}(t_k))\Omega_i(\mathrm{s}(t_k))}+\hat{\phi}_{n,j}(t_k)\overline{\Omega_j(\mathrm{s}(t_k))\Omega_i(\mathrm{s}(t_k))H_n(t_k)} \\ +\gamma\overline{\Omega_i(\mathrm{s}(t_k))}=\overline{G(t_k)\Omega_i(\mathrm{s}(t_k))}+\overline{\Omega_i(\mathrm{s}(t_k))}\,\overline{\Delta W_{t_k}(t_k,t_{k+1})} \tag{B1}$$

$$\hat{C}_j(t_k)\overline{\Omega_j(\mathrm{s}(t_k))\Omega_i(\mathrm{s}(t_k))H_m(t_k)}+\hat{\phi}_{n,j}(t_k)\overline{\Omega_j(\mathrm{s}(t_k))\Omega_i(\mathrm{s}(t_k))H_n(t_k)H_m(t_k)} \\ +\gamma\overline{\Omega_i(\mathrm{s}(t_k))H_m(t_k)}=\overline{G(t_k)\Omega_i(\mathrm{s}(t_k))H_m(t_k)}+\overline{\Omega_i(\mathrm{s}(t_k))H_m(t_k)}\,\overline{\Delta W_{t_k}(t_k,t_{k+1})} \tag{B2}$$

$$\hat{C}_j(t_k)\overline{\Omega_j(\mathbf{s}(t_k))}+\hat{\phi}_{n,j}(t_k)\overline{\Omega_j(\mathbf{s}(t_k))H_n(t_k)}=\overline{G(t_k)}+\overline{\Delta W_{t_k}(t_k,t_{k+1})} \tag{B3}$$

The system above is equation (14) of the main section, and is represented as

$$A_{ij}y_j=q_i \tag{B4}$$

where the rhs is stored in the vector q_i and the unknowns ($\hat{C}_i(t_k)$, $\hat{\phi}_{m,i}(t_k)$, γ) are represented by the vector y_i. We further detail the set of equations that need to be solved to find the OHMC solution.

1D Case Linear Basis Functions

number of nodes spanning asset-value space: N_1
total number of unknowns: $2N_1+1$

For linear basis functions the repeated index (denoting summation over j) also denotes the nodes used to divide the asset value space. The different terms of the set of linear equations (B4) are defined below (with $0\le i\le N_1-1$ and $0\le j\le N_1-1$):

$$y_{2j}=\hat{C}_j(t_k) \qquad q_{2i}=\overline{G(t_k)\Omega_i(s(t_k))}+\overline{\Omega_i(s(t_k))}\,\overline{\Delta W_{t_k}(t_k,t_{k+1})}$$

$$y_{2j+1}=\hat{\phi}_{1j}(t_k) \qquad q_{2i+1}=\overline{G(t_k)\Omega_i(s(t_k))H_1(t_k)}+\overline{\Omega_i(s(t_k))H_1(t_k)}\,\overline{\Delta W_{t_k}(t_k,t_{k+1})}$$

$$y_{2N_1}=\gamma \qquad q_{2N_1}=\overline{G(t_k)}+\overline{\Delta W_{t_k}(t_k,t_{k+1})}$$

$$A_{2i,2j}=\overline{\Omega_j(s(t_k))\Omega_i(s(t_k))} \qquad A_{2i,2j+1}=\overline{\Omega_j(s(t_k))\Omega_i(s(t_k))H_1(t_k)} \qquad A_{2i,2N_1}=\overline{\Omega_i(s(t_k))}$$

$$A_{2i+1,2j}=\overline{\Omega_j(s(t_k))\Omega_i(s(t_k))H_1(t_k)} \qquad A_{2i+1,2j+1}=\overline{\Omega_j(s(t_k))\Omega_i(s(t_k))H_1^2(t_k)} \qquad A_{2i+1,2N_1}=\overline{\Omega_i(s(t_k))H_1(t_k)}$$

$$A_{2N_1,2j}=\overline{\Omega_j(s(t_k))} \qquad A_{2N_1,2j+1}=\overline{\Omega_j(s(t_k))H_1(t_k)} \qquad A_{2N_1,2N_1}=0$$

2D Case Linear Basis Functions

number of nodes spanning asset-1 value space:	N_1
number of nodes spanning asset-2 value space:	N_2
total number of nodes:	N_1N_2
total number of unknowns:	$3N_1N_2+1$

The repeated index (denoting summation over j) also denotes the nodes used to divide the asset state-space. The different terms of the set of linear equations (B4) are defined below (with $0 \le i \le N_1N_2 - 1$ and $0 \le j \le N_1N_2 - 1$):

$$y_{3j} = \hat{C}_j(t_k) \qquad q_{3i} = \overline{G(t_k)\Omega_i(\mathbf{s}(t_k))} + \overline{\Omega_i(\mathbf{s}(t_k))}\,\overline{\Delta W_{t_k}(t_k, t_{k+1})}$$

$$y_{3j+1} = \hat{\phi}_{1,j}(t_k) \qquad q_{3i+1} = \overline{G(t_k)\Omega_i(\mathbf{s}(t_k))H_1(t_k)} + \overline{\Omega_i(\mathbf{s}(t_k))H_1(t_k)}\,\overline{\Delta W_{t_k}(t_k, t_{k+1})}$$

$$y_{3j+2} = \hat{\phi}_{2,j}(t_k) \qquad q_{3i+2} = \overline{G(t_k)\Omega_i(\mathbf{s}(t_k))H_2(t_k)} + \overline{\Omega_i(\mathbf{s}(t_k))H_2(t_k)}\,\overline{\Delta W_{t_k}(t_k, t_{k+1})}$$

$$y_{3N_1N_2} = \gamma \qquad q_{3N_1N_2} = \overline{G(t_k)} + \overline{\Delta W_{t_k}(t_k, t_{k+1})}$$

$$A_{3i,3j} = \overline{\Omega_j(\mathbf{s}(t_k))\Omega_i(\mathbf{s}(t_k))}$$

$$A_{3i,3j+1} = \overline{\Omega_j(\mathbf{s}(t_k))\Omega_i(\mathbf{s}(t_k))H_1(t_k)}$$

$$A_{3i,3j+2} = \overline{\Omega_j(\mathbf{s}(t_k))\Omega_i(\mathbf{s}(t_k))H_2(t_k)}$$

$$A_{3i,3N_1N_2} = \overline{\Omega_i(\mathbf{s}(t_k))}$$

$$A_{3i+1,3j} = \overline{\Omega_j(\mathbf{s}(t_k))\Omega_i(\mathbf{s}(t_k))H_1(t_k)}$$

$$A_{3i+1,3j+1} = \overline{\Omega_j(\mathbf{s}(t_k))\Omega_i(\mathbf{s}(t_k))H_1(t_k)H_1(t_k)}$$

$$A_{3i+1,3j+2} = \overline{\Omega_j(\mathbf{s}(t_k))\Omega_i(\mathbf{s}(t_k))H_2(t_k)H_1(t_k)}$$

$$A_{3i+1,3N_1N_2} = \overline{\Omega_i(\mathbf{s}(t_k))H_1(t_k)}$$

$$A_{3i+2,3j} = \overline{\Omega_j(\mathbf{s}(t_k))\Omega_i(\mathbf{s}(t_k))H_2(t_k)}$$

$$A_{3i+2,3j+1} = \overline{\Omega_j(\mathbf{s}(t_k))\Omega_i(\mathbf{s}(t_k))H_1(t_k)H_2(t_k)}$$

$$A_{3i+2,3j+2} = \overline{\Omega_j(\mathbf{s}(t_k))\Omega_i(\mathbf{s}(t_k))H_2(t_k)H_2(t_k)}$$

$$A_{3i+2,3N_1N_2} = \overline{\Omega_i(\mathbf{s}(t_k))H_2(t_k)}$$

$$A_{3N_1N_2,3j} = \overline{\Omega_j(\mathbf{s}(t_k))}$$

$$A_{3N_1N_2,3j+1} = \overline{\Omega_j(\mathbf{s}(t_k))H_1(t_k)}$$

$$A_{3N_1N_2,3j+2} = \overline{\Omega_j(\mathbf{s}(t_k))H_2(t_k)}$$

$$A_{3N_1N_2,3N_1N_2} = 0$$

Hermite-Cubic Basis Functions

In using Hermite-Cubic Basis Functions there are the multiple basis functions (and corresponding unknown coefficients) associated with each nodal local in the asset state-space. This is the main difference with linear basis functions, where each nodal location has one basis function and one unknown coefficient.

For 1D problems there are two basis functions associated with each node (A11), for 2D problems there are four, and for 3D problems there are 8 basis functions associated with each node.

The algorithm for setting up the set of linear equations can follow an approach similar to the linear basis functions, recognizing the multiplicity of basis functions per state-space nodal location.

1D Case Hermite-Cubic Basis Functions

number of nodes spanning asset state space:	N_1
total number of unknowns:	$4N_1+1$

2D Case Hermite-Cubic Basis Functions

number of nodes spanning asset 1 state space:	N_1
number of nodes spanning asset 2 state space:	N_2
total number of nodes:	N_1N_2
total number of unknowns:	$12N_1N_2+1$

3D Case Hermite-Cubic Basis Functions

number of nodes spanning asset 1 state space:	N_1
number of nodes spanning asset 2 state space:	N_2
number of nodes spanning asset 2 state space:	N_3
total number of nodes:	$N_1N_2\,N_3$
total number of unknowns:	$32N_1N_2\,N_3 + 1$

Computational Solution Structure

The OHMC methodology is certainly more complex than taking *risk-neutral* averages under a de-trended & fitted probability density function:

- The P&L evolution for every sample path of the underlying is being simulated in the search for the optimal hedge within the OHMC approach.
- The hedge slippage for every sample path of the underlying is assessed in OHMC.
- Valuation via OHMC involves understanding average hedging costs and also demands decisions regarding expected compensation for residual risks that are inevitable for assets displaying excess kurtosis.

The *risk-neutral* model is based on the premise of perfect replication that is materially false for any realistic asset due to excess kurtosis. The *risk-neutral model* is generally used to fit parameters to vanilla options and use those parameters to *value* exotics (i.e., to book day 1 P&L) without ever quantifying risk-return inherent to the attempted replication strategy for realistic assets (for vanillas or exotics). The lack of admission of irreducible hedging errors by the *risk-neutral* model is potentially dangerous, as it results in a derivatives-trader –structurer "knowing" their day 1 P&L or carry on a specific trade, without knowing the associated risk capital arising from hedging errors endemic to attempted replication! This bifurcation of P&L motivations from understanding risks is a significant contributing reason for the poor risk management that is clearly visible in turbulent market conditions.

Unlike the *risk-neutral* model, the OHMC approach provides an integrated platform to evaluate trade-hedging strategy, valuation, and risk-return. Despite this broad mandate addressed by OHMC, we find that it lends itself to robust implementation in practical settings for derivative trading desks that are keen to take up the responsibility of quantifying hedging error as a part of trading strategy.

The main classes and functions built to implement the OHMC numerical solution are catalogued here. With the few classes and functions enumerated here and a shallow object-oriented programming structure, a practical solution to the multi-asset OHMC problem can be provided.

ASSET
A class built to access reference asset values at any time step for any MC path. The marginal asset parameters and dependence structure parameters are inputs. This provides the main code access to the value for all assets, at any point in time, in any MC realization.

PAYOFF
For the European style payoffs of this paper, a stand-alone function to assess the payoff for the simulated asset paths is sufficient. This could be usefully elaborated as a separate class with several functions when dealing with more complex path dependent payoffs.

df & DF
The class df facilitates calculating discount factors over any time horizon in the options life. As the economics of holding a hedge position can be different than that implied by the time value of money embedded in the discount factor (due to spreads in borrowing costs, dividends, and subscription fees etc.) we have a funding discount factor class DF that can choose a *base* discount factor and impose a possibly term-dependent spread on top of the discount factor.

GHEVAL
A function that evaluates the “G” and “H” variables in the option trader-hedgers wealth balance.

OHMCSLEFORM
This function assembles the set of linear equations needed to solve the OHMC problem.

NODESELECT
A function that prescribes nodal values given a vector of random asset values, resulting in bins of equal probability.

BASISFNREP
An encapsulation of the concept of representing a function as a sum of coefficient multiplied by pre-specified functions of the underlying asset is effected here.

INITSLEFORM
For the first time step the problem of solving for the hedge ratio and pricing is a multi-variate constrained minimization problem that can be directly solved without any approximation beyond those employed to propagate the solution to the time-step prior to the initial time. This function outputs the hedge ratio and price.

SLESOLVE
This solves the set of linear equations formed by either OHMCSLEFORM and INITSLEFORM

PDFCDFCALC
A function that returns an estimate of the probability and cumulative density function of a random variable given an input of the MC random variates.

OHMCMAIN
The main function that assembles all the components listed above and time-steps the optimization solution from maturity to initial time. This main function handles the hedging timescale which is an explicit input into an OHMC algorithm.

Appendix-C

GARCH(1,1) Parameter Estimation

A well known model of asset returns that sidesteps the *perfect-hedge* contrivance even for vanilla options is afforded by a GARCH(1,1) description of asset returns (Bollersev [1986], Engle[1994]). The conditioning variable is *starting volatility*, and the jumpiness of returns associated with the return kurtosis thwarts the theoretical perfect hedge even under continuous hedging (see Petrelli et al [2008]). Under GARCH(1,1) the asset and its volatility evolve as follows:

$$\Delta s_k = s_k\left(\mu\Delta t + \sigma_k\sqrt{\Delta t}\,\varepsilon_k\right) \tag{C1}$$

$$\sigma_k^2 = (1-\alpha-\beta)\sigma^2 + \sigma_{k-1}^2\left(\beta + \alpha\varepsilon_{k-1}^2\right) \tag{C2}$$

Standard Normal random-variates generated to create the return stochastic process in (C1) are denoted by ε_k, and the volatility evolves per (C2).

Method of Moments Fitting to Empirical Returns

The empirical return statistics can be used to analytically specify the parameters of the GARCH(1,1) model for the evolution of the reference asset. Here are details of the unconditional moments used to infer the parameters (Carnero et al [2004]):

$$r \equiv \Delta s / s; \quad \bar{r} = \mu\Delta t \tag{C3}$$

$$r' \equiv r - \bar{r}\,,\ \sigma_r^2 \equiv E[(r-\bar{r})^2] = \Delta t\sigma^2 \tag{C4}$$

$$\kappa \equiv \frac{E[r'^4]}{\sigma_r^4} = \frac{3}{\left[1 - \frac{2\alpha^2}{1-(\alpha+\beta)^2}\right]} \tag{C5}$$

$$\rho_{r'^2}(h) = \frac{E\left[\left(r'^2(t+h\Delta t)-\sigma_r^2\right)\left(r'^2(t)-\sigma_r^2\right)\right]}{E\left[\left(r'^2(t)-\sigma_r^2\right)^2\right]} = \begin{cases} \dfrac{\alpha\left[1-(\alpha+\beta)^2+\alpha(\alpha+\beta)\right]}{1-(\alpha+\beta)^2+\alpha^2}, & h=1 \\ (\alpha+\beta)\rho_2(h-1), & h>1 \end{cases} \tag{C6}$$

$$\Gamma(n) = \sum_{j=1}^{n}\rho_{r'^2}(j) \tag{C7}$$

The empirical mean and variance of the return provide direct inferences of μ and σ. The empirical kurtosis is employed to express α as a function of β:

$$\alpha = \frac{\sqrt{(\kappa-3)\left(\kappa\left(3-2\beta^2\right)-3\right)}-\beta(\kappa-3)}{3(\kappa-1)} \tag{C8}$$

Finding parameter β that best reproduces the empirical auto-correlation of the squared returns results in a complete inference of parameters. We choose the sum of the auto-correlations to a maximum lag as the composite target (C7). As the *temporal clustering of the squared return noise* ultimately controls the temporal scale of volatility fluctuations, this sum is referred to as the *volatility clustering time*.[12] We iterate over β from 0 to 1 to find the GARCH(1,1) value of the sum of the autocorrelations that is nearest to the empirical observation.

[12] Real return data show multiple time-scales of memory. Those time-scales are visibly manifest in serial correlation of returns and its higher moments. These serial dependencies control the return skewness and kurtosis over different time-scales. Models of underlying that directly address these empirical features need to be created for developing derivative trading strategy and risk-capital models.

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