CHAPTER 1

The Heston Model for European Options

Abstract

In this chapter, we present a complete derivation of the European call price under the Heston model. We first present the model and obtain the various partial differential equations (PDEs) that arise in the derivation. We show that the call price in the Heston model can be expressed as the sum of two terms that each contains an in-the-money probability, but obtained under a separate measure, a result demonstrated by Bakshi and Madan (2000). We show how to obtain the characteristic function for the Heston model, and how to solve the Riccati equation from which the characteristic function is derived. We then show how to incorporate a continuous dividend yield and how to compute the price of a European put, and demonstrate that the numerical integration can be speeded up by consolidating the two numerical integrals into a single integral. Finally, we derive the Black-Scholes model as a special case of the Heston model.

MODEL DYNAMICS

The Heston model assumes that the underlying stock price, S_t, follows a Black-Scholes–type stochastic process, but with a stochastic variance v_t that follows a Cox, Ingersoll, and Ross (1985) process. Hence, the Heston model is represented by the bivariate system of stochastic differential equations (SDEs)

$$\begin{aligned} dS_t &= \mu S_t dt + \sqrt{v_t} S_t dW_{1,t} \\ dv_t &= \kappa(\theta - v_t)dt + \sigma\sqrt{v_t}dW_{2,t} \end{aligned} \tag{1.1}$$

where $E^{\mathbb{P}}[dW_{1,t}dW_{2,t}] = \rho dt$.

We will sometimes drop the time index and write $S = S_t$, $v = v_t$, $W_1 = W_{1,t}$ and $W_2 = W_{2,t}$ for notational convenience. The parameters of the model are

- μ the drift of the process for the stock;
- $\kappa > 0$ the mean reversion speed for the variance;
- $\theta > 0$ the mean reversion level for the variance;
- $\sigma > 0$ the volatility of the variance;
- $v_0 > 0$ the initial (time zero) level of the variance;

$\rho \in [-1, 1]$ the correlation between the two Brownian motions W_1 and W_2; and
λ the volatility risk parameter. We define this parameter in the next section and explain why we set this parameter to zero.

We will see in Chapter 2 that these parameters affect the distribution of the terminal stock price in a manner that is intuitive. Some authors refer to v_0 as an unobserved initial state variable, rather than a parameter. Because volatility cannot be observed, only estimated, and because v_0 represents this state variable at time zero, this characterization is sensible. For the purposes of estimation, however, many authors treat v_0 as a parameter like any other. Parameter estimation is covered in Chapter 6.

It is important to note that the volatility $\sqrt{v_t}$ is not modeled directly in the Heston model, but rather through the variance v_t. The process for the variance arises from the Ornstein-Uhlenbeck process for the volatility $h_t = \sqrt{v_t}$ given by

$$dh_t = -\beta h_t dt + \delta dW_{2,t}. \tag{1.2}$$

Applying Itō's lemma, $v_t = h_t^2$ follows the process

$$dv_t = (\delta^2 - 2\beta v_t)dt + 2\delta\sqrt{v_t}dW_{2,t}. \tag{1.3}$$

Defining $\kappa = 2\beta$, $\theta = \delta^2/(2\beta)$, and $\sigma = 2\delta$ expresses dv_t from Equation (1.1) as (1.3).

The stock price and variance follow the processes in Equation (1.1) under the historical measure $\mathbb{P}$, also called the physical measure. For pricing purposes, however, we need the processes for (S_t, v_t) under the risk-neutral measure $\mathbb{Q}$. In the Heston model, this is done by modifying each SDE in Equation (1.1) separately by an application of Girsanov's theorem. The risk-neutral process for the stock price is

$$dS_t = rS_t dt + \sqrt{v_t}S_t d\widetilde{W}_{1,t} \tag{1.4}$$

where

$$\widetilde{W}_{1,t} = \left(W_{1,t} + \frac{\mu - r}{\sqrt{v_t}}t\right).$$

It is sometimes convenient to express the price process in terms of the log price instead of the price itself. By an application of Itō's lemma, the log price process is

$$d\ln S_t = \left(\mu - \frac{1}{2}\right)dt + \sqrt{v_t}dW_{1,t}.$$

The risk-neutral process for the log price is

$$d\ln S_t = \left(r - \frac{1}{2}\right)dt + \sqrt{v_t}d\widetilde{W}_{1,t}. \tag{1.5}$$

If the stock pays a continuous dividend yield, q, then in Equations (1.4) and (1.5) we replace r by $r - q$.

The risk-neutral process for the variance is obtained by introducing a function $\lambda(S_t, v_t, t)$ into the drift of dv_t in Equation (1.1), as follows

$$dv_t = [\kappa(\theta - v_t) - \lambda(S_t, v_t, t)]dt + \sigma\sqrt{v_t}d\widetilde{W}_{2,t} \tag{1.6}$$

where

$$\widetilde{W}_{2,t} = \left(W_{2,t} + \frac{\lambda\left(S_t, v_t, t\right)}{\sigma\sqrt{v_t}}t\right). \tag{1.7}$$

The function $\lambda(S, v, t)$ is called the volatility risk premium. As explained in Heston (1993), Breeden's (1979) consumption model yields a premium proportional to the variance, so that $\lambda(S, v, t) = \lambda v_t$, where λ is a constant. Substituting for λv_t in Equation (1.6), the risk-neutral version of the variance process is

$$dv_t = \kappa^*(\theta^* - v_t)dt + \sigma\sqrt{v_t}d\widetilde{W}_{2,t} \tag{1.8}$$

where $\kappa^* = \kappa + \lambda$ and $\theta^* = \kappa\theta/(\kappa + \lambda)$ are the risk-neutral parameters of the variance process.

To summarize, the risk-neutral process is

$$\begin{aligned} dS_t &= rS_t dt + \sqrt{v_t}S_t d\widetilde{W}_{1,t} \\ dv_t &= \kappa^*(\theta^* - v_t)dt + \sigma\sqrt{v_t}d\widetilde{W}_{2,t} \end{aligned} \tag{1.9}$$

where $E^{\mathbb{Q}}[d\widetilde{W}_{1,t}d\widetilde{W}_{2,t}] = \rho dt$ and with $\mathbb{Q}$ the risk-neutral measure.

Note that, when $\lambda = 0$, we have $\kappa^* = \kappa$ and $\theta^* = \theta$ so that these parameters under the physical and risk-neutral measures are the same. Throughout this book, we set $\lambda = 0$, but this is not always needed. Indeed, λ is embedded in the risk-neutral parameters κ^* and θ^*. Hence, when we estimate the risk-neutral parameters to price options we do not need to estimate λ. Estimation of λ is the subject of its own research, such as that by Bollerslev et al. (2011). For notational simplicity, throughout this book we will drop the asterisk on the parameters and the tilde on the Brownian motion when it is obvious that we are dealing with the risk-neutral measure.

Properties of the Variance Process

The properties of v_t are described by Cox, Ingersoll, and Ross (1985) and Brigo and Mercurio (2006), among others. It is well-known that conditional on a realized value of v_s, the random variable $2c_t v_t$ (for $t > s$) follows a non-central chi-square distribution with $d = 4\kappa\theta/\sigma^2$ degrees of freedom and non-centrality parameter $2c_t v_s e^{-\kappa(t-s)}$, where

$$c_t = \frac{2\kappa}{\sigma^2(1 - e^{-\kappa(t-s)})} \tag{1.10}$$

and with $t > s$. The mean and variance of v_t, conditional on the value v_s are, respectively

$$\begin{aligned} m &= E[v_t|v_s] = \theta + (v_s - \theta)e^{-\kappa(t-s)}, \\ S^2 &= \mathrm{Var}[v_t|v_s] = \frac{v_s\sigma^2 e^{-\kappa(t-s)}}{\kappa}(1 - e^{-\kappa(t-s)}) + \frac{\theta\sigma^2}{2\kappa}(1 - e^{-\kappa(t-s)})^2. \end{aligned} \tag{1.11}$$

The effect of the mean reversion speed κ on the moments is intuitive and explained in Cox, Ingersoll, and Ross (1985). When $\kappa \to \infty$ the mean m approaches the mean reversion rate θ and the variance S^2 approaches zero. As $\kappa \to 0$ the mean approaches the current level of variance, v_s, and the variance approaches $\sigma^2 v_t(t-s)$.

If the condition $2\kappa\theta > \sigma^2$ holds, then the drift is sufficiently large for the variance process to be guaranteed positive and not reach zero. This condition is known as the Feller condition.

THE EUROPEAN CALL PRICE

In this section, we show that the call price in the Heston model can be expressed in a manner which resembles the call price in the Black-Scholes model, which we present in Equation (1.76). Authors sometimes refer to this characterization of the call price as "Black-Scholes–like" or "à la Black-Scholes." The time-t price of a European call on a non-dividend paying stock with spot price S_t, when the strike is K and the time to maturity is $\tau = T - t$, is the discounted expected value of the payoff under the risk-neutral measure $\mathbb{Q}$

$$\begin{aligned} C(K) &= e^{-r\tau} E^{\mathbb{Q}}[(S_T - K)^+] \\ &= e^{-r\tau} E^{\mathbb{Q}}[(S_T - K)\mathbf{1}_{S_T > K}] \\ &= e^{-r\tau} E^{\mathbb{Q}}[S_T \mathbf{1}_{S_T > K}] - K e^{-r\tau} E^{\mathbb{Q}}[\mathbf{1}_{S_T > K}] \\ &= S_t P_1 - K e^{-r\tau} P_2 \end{aligned} \tag{1.12}$$

where $\mathbf{1}$ is the indicator function. The last line of (1.12) is the "Black-Scholes–like" call price formula, with P_1 replacing $\Phi(d_1)$, and P_2 replacing $\Phi(d_2)$ in the Black-Scholes call price (1.76). In this section, we explain how the last line of (1.12) can be derived from the third line. The quantities P_1 and P_2 each represent the probability of the call expiring in-the-money, conditional on the value $S_t = e^{x_t}$ of the stock and on the value v_t of the volatility at time t. Hence

$$P_j = \Pr(\ln S_T > \ln K) \tag{1.13}$$

for $j = 1, 2$. These probabilities are obtained under different probability measures. In Equation (1.12), the expected value $E^{\mathbb{Q}}[\mathbf{1}_{S_T > K}]$ is the probability of the call expiring in-the-money under the measure $\mathbb{Q}$ that makes W_1 and W_2 in the risk-neutral version of Equation (1.1) Brownian motion. We can therefore write

$$E^{\mathbb{Q}}[\mathbf{1}_{S_T > K}] = \mathbb{Q}(S_T > K) = \mathbb{Q}(\ln S_T > \ln K) = P_2.$$

Evaluating $e^{-r\tau} E^{\mathbb{Q}}[S_T \mathbf{1}_{S_T > K}]$ in (1.12) requires changing the original measure $\mathbb{Q}$ to another measure $\mathbb{Q}^S$. Consider the Radon-Nikodym derivative

$$\frac{d\mathbb{Q}}{d\mathbb{Q}^S} = \frac{B_T/B_t}{S_T/S_t} = \frac{E^{\mathbb{Q}}[e^{x_T}]}{e^{x_T}} \tag{1.14}$$

where

$$B_t = \exp\left(\int_0^t r\,du\right) = e^{rt}.$$

In (1.14), we have written $S_t e^{r(T-t)} = E^{\mathbb{Q}}[e^{x_T}]$, since under $\mathbb{Q}$ assets grow at the risk-free rate, r. The first expectation in the third line of (1.12) can therefore be written as

$$\begin{aligned}
e^{-r(T-t)}E^{\mathbb{Q}}[S_T\mathbf{1}_{S_T>K}] &= S_t E^{\mathbb{Q}}\left[\frac{S_T/S_t}{B_T/B_t}\mathbf{1}_{S_T>K}\right] = S_t E^{\mathbb{Q}^S}\left[\frac{S_T/S_t}{B_T/B_t}\mathbf{1}_{S_T>K}\frac{d\mathbb{Q}}{d\mathbb{Q}^S}\right] \\
&= S_t E^{\mathbb{Q}^S}[\mathbf{1}_{S_T>K}] = S_t\mathbb{Q}^S(S_T>K) = S_t P_1.
\end{aligned} \tag{1.15}$$

This implies that the European call price of Equation (1.12) can be written in terms of both measures as

$$C(K) = S_t\mathbb{Q}^S(S_T > K) - Ke^{-r\tau}\mathbb{Q}(S_T > K). \tag{1.16}$$

The measure $\mathbb{Q}$ uses the bond B_t as the numeraire, while the measure $\mathbb{Q}^S$ uses the stock price S_t. Bakshi and Madan (2000) present a derivation of the call price expressed as (1.16), but under a general setup. As shown in their paper, the change of measure that leads to (1.16) is valid for a wide range of models, including the Black-Scholes and Heston models. We will see later in this chapter that when S_T follows the lognormal distribution specified in the Black-Scholes model, then $\mathbb{Q}^S(S_T > K) = \Phi(d_1)$ and $\mathbb{Q}(S_T > K) = \Phi(d_2)$. Hence, the characteristic function approach to pricing options, pioneered by Heston (1993), applies to the Black-Scholes model also.

THE HESTON PDE

In this section, we explain how to derive the PDE for the Heston model. This derivation is a special case of a PDE for general stochastic volatility models, described in books by Gatheral (2006), Lewis (2000), Musiela and Rutkowski (2011), Joshi (2008), and others. The argument is similar to the hedging argument that uses a single derivative to derive the Black-Scholes PDE. In the Black-Scholes model, a portfolio is formed with the underlying stock, plus a single derivative which is used to hedge the stock and render the portfolio riskless. In the Heston model, however, an additional derivative is required in the portfolio, to hedge the volatility. Hence, we form a portfolio consisting of one option $V = V(S, v, t)$, Δ units of the stock, and φ units of another option $U(S, v, t)$ for the volatility hedge. The portfolio has value

$$\Pi = V + \Delta S + \varphi U$$

where the t subscripts are omitted for convenience. Assuming the portfolio is self-financing, the change in portfolio value is

$$d\Pi = dV + \Delta dS + \varphi dU. \tag{1.17}$$

The strategy is similar to that for the Black-Scholes case. We apply Itō's lemma to obtain the processes for U and V, which allows us to find the process for Π. We then find the values of Δ and φ that makes the portfolio riskless, and we use the result to derive the Heston PDE.

Setting Up the Hedging Portfolio

To form the hedging portfolio, first apply Itō's lemma to the value of the first derivative, $V(S, v, t)$. We must differentiate V with respect to the variables t, S, and v, and form a second-order Taylor series expansion. The result is that dV follows the process

$$\begin{aligned} dV = {} & \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}dS + \frac{\partial V}{\partial v}dv + \frac{1}{2}vS^2\frac{\partial^2 V}{\partial S^2}dt \\ & + \frac{1}{2}v\sigma^2\frac{\partial^2 V}{\partial v^2}dt + \sigma\rho vS\frac{\partial^2 V}{\partial S\partial v}dt. \end{aligned} \tag{1.18}$$

We have used the fact that $(dS)^2 = vS^2(dW_1)^2 = vS^2dt$, that $(dv)^2 = \sigma^2 vdt$, and that $dSdv = \sigma vSdW_1dW_2 = \sigma\rho vSdt$. We have also used $(dt)^2 = 0$ and $dW_1dt = dW_2dt = 0$. Applying Itō's lemma to the second derivative, $U(S, v, t)$, produces an expression identical to (1.18), but in terms of U. Substituting these two expressions into (1.17), the change in portfolio value can be written

$$\begin{aligned} d\Pi &= dV + \Delta dS + \varphi dU \\ &= \left[\frac{\partial V}{\partial t} + \frac{1}{2}vS^2\frac{\partial^2 V}{\partial S^2} + \rho\sigma vS\frac{\partial^2 V}{\partial v\partial S} + \frac{1}{2}\sigma^2 v\frac{\partial^2 V}{\partial v^2}\right]dt \\ &\quad + \varphi\left[\frac{\partial U}{\partial t} + \frac{1}{2}vS^2\frac{\partial^2 U}{\partial S^2} + \rho\sigma vS\frac{\partial^2 U}{\partial v\partial S} + \frac{1}{2}\sigma^2 v\frac{\partial^2 U}{\partial v^2}\right]dt \\ &\quad + \left[\frac{\partial V}{\partial S} + \varphi\frac{\partial U}{\partial S} + \Delta\right]dS + \left[\frac{\partial V}{\partial v} + \varphi\frac{\partial U}{\partial v}\right]dv. \end{aligned} \tag{1.19}$$

In order for the portfolio to be hedged against movements in both the stock and volatility, the last two terms in Equation (1.19) must be zero. This implies that the hedge parameters must be

$$\varphi = -\frac{\partial V}{\partial v}\Big/\frac{\partial U}{\partial v}, \qquad \Delta = -\varphi\frac{\partial U}{\partial S} - \frac{\partial V}{\partial S}. \tag{1.20}$$

Substitute these values of φ and Δ into (1.19) to produce

$$\begin{aligned} d\Pi = {} & \left[\frac{\partial V}{\partial t} + \frac{1}{2}vS^2\frac{\partial^2 V}{\partial S^2} + \rho\sigma vS\frac{\partial^2 V}{\partial S\partial v} + \frac{1}{2}\sigma^2 v\frac{\partial^2 V}{\partial v^2}\right]dt \\ & + \varphi\left[\frac{\partial U}{\partial t} + \frac{1}{2}vS^2\frac{\partial^2 U}{\partial S^2} + \rho\sigma vS\frac{\partial^2 U}{\partial S\partial v} + \frac{1}{2}\sigma^2 v\frac{\partial^2 U}{\partial v^2}\right]dt. \end{aligned} \tag{1.21}$$

The condition that the portfolio earn the risk-free rate, r, implies that the change in portfolio value is $d\Pi = r\Pi dt$. Equation (1.17) thus becomes

$$d\Pi = r(V + \Delta S + \varphi U)dt. \tag{1.22}$$

Now equate Equation (1.22) with (1.21), substitute for φ and Δ, drop the dt term and re-arrange. This yields

$$\frac{\left[\frac{\partial V}{\partial t}+\frac{1}{2}vS^2\frac{\partial^2 V}{\partial S^2}+\rho\sigma vS\frac{\partial^2 V}{\partial S\partial v}+\frac{1}{2}\sigma^2 v\frac{\partial^2 V}{\partial v^2}\right]-rV+rS\frac{\partial V}{\partial S}}{\frac{\partial V}{\partial v}} = \frac{\left[\frac{\partial U}{\partial t}+\frac{1}{2}vS^2\frac{\partial^2 U}{\partial S^2}+\rho\sigma vS\frac{\partial^2 U}{\partial S\partial v}+\frac{1}{2}\sigma^2 v\frac{\partial^2 U}{\partial v^2}\right]-rU+rS\frac{\partial U}{\partial S}}{\frac{\partial U}{\partial v}} \tag{1.23}$$

which we exploit in the next section.

The PDE for the Option Price

The left-hand side of Equation (1.23) is a function of V only, and the right-hand side is a function of U only. This implies that both sides can be written as a function $f(S,v,t)$. Following Heston (1993), specify this function as

$$f(S,v,t) = -\kappa(\theta - v) + \lambda(S,v,t)$$

where $\lambda(S,v,t)$ is the price of volatility risk. An application of Breeden's (1979) consumption model yields a price of volatility risk that is a linear function of volatility, so that $\lambda(S,v,t) = \lambda v$, where λ is a constant. Substitute for $f(S,v,t)$ in the left-hand side of Equation (1.23)

$$-\kappa(\theta - v) + \lambda(S,v,t) = \frac{\left[\frac{\partial U}{\partial t}+\frac{1}{2}vS^2\frac{\partial^2 U}{\partial S^2}+\rho\sigma vS\frac{\partial^2 U}{\partial S\partial v}+\frac{1}{2}\sigma^2 v\frac{\partial^2 U}{\partial v^2}\right]-rU+rS\frac{\partial U}{\partial S}}{\frac{\partial U}{\partial v}}$$

Rearrange to produce the Heston PDE expressed in terms of the price S

$$\begin{aligned}&\frac{\partial U}{\partial t}+\frac{1}{2}vS^2\frac{\partial^2 U}{\partial S^2}+\rho\sigma vS\frac{\partial^2 U}{\partial v\partial S}+\frac{1}{2}\sigma^2 v\frac{\partial^2 U}{\partial v^2}\\&-rU+rS\frac{\partial U}{\partial S}+[\kappa(\theta-v)-\lambda(S,v,t)]\frac{\partial U}{\partial v}=0.\end{aligned} \tag{1.24}$$

This is Equation (6) of Heston (1993).

The following boundary conditions on the PDE in Equation (1.24) hold for a European call option with maturity T and strike K. At maturity, the call is worth its intrinsic value

$$U(S,v,T) = \max(0, S-K). \tag{1.25}$$

When the stock price is zero, the call is worthless. As the stock price increases, delta approaches one, and when the volatility increases, the call option becomes equal to the stock price. This implies the following three boundary conditions

$$U(0,v,t)=0,\quad \frac{\partial U}{\partial S}(\infty,v,t)=1,\quad U(S,\infty,t)=S. \tag{1.26}$$

Finally, note that the PDE (1.24) can be written

$$\frac{\partial U}{\partial t} + \mathcal{A}U - rU = 0 \tag{1.27}$$

where

$$\begin{aligned}\mathcal{A} &= rS\frac{\partial}{\partial S} + \frac{1}{2}vS^2\frac{\partial^2}{\partial S^2} \\ &+ [\kappa(\theta - v) - \lambda(S,v,t)]\frac{\partial}{\partial v} + \frac{1}{2}\sigma^2 v\frac{\partial^2}{\partial v^2} + \rho\sigma v S\frac{\partial^2}{\partial S\partial v}\end{aligned} \tag{1.28}$$

is the generator of the Heston model. As explained by Lewis (2000), the first line in Equation (1.28) is the generator of the Black-Scholes model, with $v = \sqrt{\sigma_{BS}}$, where σ_{BS} is the Black-Scholes volatility. The second line augments the PDE for stochastic volatility.

We can define the log price $x = \ln S$ and express the PDE in terms of (x, v, t) instead of (S, v, t). This leads to a simpler form of the PDE in which the spot price S does not appear. This simplification requires the following derivatives. By the chain rule

$$\frac{\partial U}{\partial S} = \frac{\partial U}{\partial x}\frac{1}{S}, \qquad \frac{\partial^2 U}{\partial v \partial S} = \frac{\partial}{\partial v}\left(\frac{1}{S}\frac{\partial U}{\partial x}\right) = \frac{1}{S}\frac{\partial^2 U}{\partial v \partial x}.$$

Using the product rule,

$$\frac{\partial^2 U}{\partial S^2} = \frac{\partial}{\partial S}\left(\frac{1}{S}\frac{\partial U}{\partial x}\right) = -\frac{1}{S^2}\frac{\partial U}{\partial x} + \frac{1}{S}\frac{\partial^2 U}{\partial S \partial x} = -\frac{1}{S^2}\frac{\partial U}{\partial x} + \frac{1}{S^2}\frac{\partial^2 U}{\partial x^2}.$$

Substitute these expressions into the Heston PDE in (1.24). All the S terms cancel, and we obtain the Heston PDE in terms of the log price $x = \ln S$

$$\begin{aligned}&\frac{\partial U}{\partial t} + \frac{1}{2}v\frac{\partial^2 U}{\partial x^2} + \left(r - \frac{1}{2}v\right)\frac{\partial U}{\partial x} + \rho\sigma v\frac{\partial^2 U}{\partial v \partial x} \\ &+\frac{1}{2}\sigma^2 v\frac{\partial^2 U}{\partial v^2} - rU + [\kappa(\theta - v) - \lambda v]\frac{\partial U}{\partial v} = 0\end{aligned} \tag{1.29}$$

where we have substituted $\lambda(S, v, t) = \lambda v$. The modern approach to obtaining the PDE in (1.29) is by an application of the Feynman-Kac theorem, which we will encounter in Chapter 12 in the context of the double Heston model of Christoffersen et al. (2009).

The PDE for P_1 and P_2

Recall Equation (1.16) for the European call price, written here using $x = x_t = \ln S_t$

$$C(K) = e^x P_1 - Ke^{-r\tau}P_2. \tag{1.30}$$

Equation (1.30) expresses $C(K)$ in terms of the in-the-money probabilities $P_1 = \mathbb{Q}^S(S_T > K)$ and $P_2 = \mathbb{Q}(S_T > K)$. Since the European call satisfies the PDE (1.29), we can find the required derivatives of Equation (1.30), substitute them into the PDE, and express the PDE in terms of P_1 and P_2. The derivative of $C(K)$ with respect to t is

$$\frac{\partial C}{\partial t} = e^x \frac{\partial P_1}{\partial t} - Ke^{-r\tau}\left[rP_2 + \frac{\partial P_2}{\partial t}\right]. \tag{1.31}$$

With respect to x

$$\frac{\partial C}{\partial x} = e^x \left[P_1 + \frac{\partial P_1}{\partial x}\right] - Ke^{-r\tau}\frac{\partial P_2}{\partial x}. \tag{1.32}$$

With respect to x^2

$$\begin{aligned}\frac{\partial^2 C}{\partial x^2} &= e^x P_1 + 2e^x \frac{\partial P_1}{\partial x} + e^x \frac{\partial^2 P_1}{\partial x^2} - Ke^{-r\tau}\frac{\partial^2 P_2}{\partial x^2} \\ &= e^x \left[P_1 + 2\frac{\partial P_1}{\partial x} + \frac{\partial^2 P_1}{\partial x^2}\right] - Ke^{-r\tau}\frac{\partial^2 P_2}{\partial x^2}.\end{aligned} \tag{1.33}$$

With respect to v, and v^2

$$\frac{\partial C}{\partial v} = e^x \frac{\partial P_1}{\partial v} - Ke^{-r\tau}\frac{\partial P_2}{\partial v}, \qquad \frac{\partial^2 C}{\partial v^2} = e^x \frac{\partial^2 P_1}{\partial v^2} - Ke^{-r\tau}\frac{\partial^2 P_2}{\partial v^2}. \tag{1.34}$$

With respect to v and x

$$\frac{\partial^2 C}{\partial x \partial v} = e^x \left[\frac{\partial P_1}{\partial v} + \frac{\partial^2 P_1}{\partial x \partial v}\right] - Ke^{-r\tau}\frac{\partial^2 P_2}{\partial x \partial v}. \tag{1.35}$$

As mentioned earlier, since the European call $C(K)$ is a financial derivative, it also satisfies the Heston PDE in (1.29), which we write here in terms of $C(K)$

$$\begin{aligned}&\frac{\partial C}{\partial t} + \frac{1}{2}v\frac{\partial^2 C}{\partial x^2} + \left(r - \frac{1}{2}v\right)\frac{\partial C}{\partial x} + \rho\sigma v\frac{\partial^2 C}{\partial v \partial x} \\ &\quad + \frac{1}{2}\sigma^2 v\frac{\partial^2 C}{\partial v^2} - rC + [\kappa(\theta - v) - \lambda v]\frac{\partial C}{\partial v} = 0.\end{aligned} \tag{1.36}$$

To obtain the Heston PDE for P_1 and P_2, Heston (1993) argues that the PDE in (1.36) holds for any contractual features of $C(K)$, in particular, for any strike price $K \geq 0$, for any value of $S \geq 0$, and for any value $r \geq 0$ of the risk-free rate. Setting $K = 0$ and $S = 1$ in the call price in Equation (1.12) produces an option whose price is simply P_1. This option will also follow the PDE in (1.36). Similarly, setting $S = 0$, $K = 1$, and $r = 0$ in (1.12) produces an option whose price is $-P_2$. Since $-P_2$ follows the PDE, so does P_2.

In Equations (1.31) through (1.35), regroup terms common to P_1, cancel e^x, and substitute the terms into the PDE in (1.36) to obtain

$$\begin{aligned}&\frac{\partial P_1}{\partial t} + \frac{1}{2}v\left[P_1 + 2\frac{\partial P_1}{\partial x} + \frac{\partial^2 P_1}{\partial x^2}\right] + \left(r - \frac{1}{2}v\right)\left[P_1 + \frac{\partial P_1}{\partial x}\right] + \rho\sigma v\left[\frac{\partial P_1}{\partial v} + \frac{\partial^2 P_1}{\partial x \partial v}\right] \\ &\quad + \frac{1}{2}\sigma^2 v\frac{\partial^2 P_1}{\partial v^2} - rP_1 + [\kappa(\theta - v) - \lambda v]\frac{\partial P_1}{\partial v} = 0.\end{aligned} \tag{1.37}$$

Simplifying, (1.37) becomes

$$\begin{aligned}&\frac{\partial P_1}{\partial t}+\left(r+\frac{1}{2}v\right)\frac{\partial P_1}{\partial x}+\frac{1}{2}v\frac{\partial^2 P_1}{\partial x^2}+\rho\sigma v\frac{\partial^2 P_1}{\partial x\partial v}\\&\quad+[\rho\sigma v+\kappa(\theta-v)-\lambda v]\frac{\partial P_1}{\partial v}+\frac{1}{2}\sigma^2 v\frac{\partial^2 P_1}{\partial v^2}=0.\end{aligned}\tag{1.38}$$

Similarly, in Equations (1.31) through (1.35) regroup terms common to P_2, cancel $-Ke^{-r\tau}$, and substitute the terms into the PDE in Equation (1.36) to obtain

$$\begin{aligned}&\frac{\partial P_2}{\partial t}+\frac{1}{2}v\frac{\partial^2 P_2}{\partial x^2}+\left(r-\frac{1}{2}v\right)\frac{\partial P_2}{\partial x}+\rho\sigma v\frac{\partial^2 P_2}{\partial v\partial x}\\&\quad+\frac{1}{2}\sigma^2 v\frac{\partial^2 P_2}{\partial v^2}+[\kappa(\theta-v)-\lambda v]\frac{\partial P_2}{\partial v}=0.\end{aligned}\tag{1.39}$$

For notational convenience, combine Equations (1.38) and (1.39) into a single expression

$$\begin{aligned}&\frac{\partial P_j}{\partial t}+\rho\sigma v\frac{\partial^2 P_j}{\partial v\partial x}+\frac{1}{2}v\frac{\partial^2 P_j}{\partial x^2}+\frac{1}{2}\sigma^2 v\frac{\partial^2 P_j}{\partial v^2}\\&\quad+(r+u_j v)\frac{\partial P_j}{\partial x}+(a-b_j v)\frac{\partial P_j}{\partial v}=0\end{aligned}\tag{1.40}$$

for $j=1,2$ and where $u_1=\frac{1}{2}$, $u_2=-\frac{1}{2}$, $a=\kappa\theta$, $b_1=\kappa+\lambda-\rho\sigma$, and $b_2=\kappa+\lambda$. This is Equation (12) of Heston (1993).

OBTAINING THE HESTON CHARACTERISTIC FUNCTIONS

When the characteristic functions $f_j(\phi;x,v)$ are known, each in-the-money probability P_j can be recovered from the characteristic function via the Gil-Pelaez (1951) inversion theorem, as

$$P_j=\Pr(\ln S_T>\ln K)=\frac{1}{2}+\frac{1}{\pi}\int_0^\infty \mathrm{Re}\left[\frac{e^{-i\phi\ln K}f_j(\phi;x,v)}{i\phi}\right]d\phi.\tag{1.41}$$

Inversion theorems can be found in many textbooks, such as that by Stuart (2010). The inversion theorem in (1.41) will be demonstrated in Chapter 3. A discussion of how the theorem relates to option pricing in stochastic volatility models appears in Jondeau et al. (2007).

At maturity, the probabilities are subject to the terminal condition

$$P_j=\mathbf{1}_{x_T>\ln K}\tag{1.42}$$

where $\mathbf{1}$ is the indicator function. Equation (1.42) simply states that, when $S_T>K$ at expiry, the probability of the call being in-the-money is unity. Heston (1993) postulates that the characteristic functions for the logarithm of the terminal stock price, $x_T=\ln S_T$, are of the log linear form

$$f_j(\phi;x_t,v_t)=\exp(C_j(\tau,\phi)+D_j(\tau,\phi)v_t+i\phi x_t)\tag{1.43}$$

where $i = \sqrt{-1}$ is the imaginary, unit, C_j and D_j are coefficients and $\tau = T - t$ is the time to maturity.

The characteristic functions f_j will follow the PDE in Equation (1.40). This is a consequence of the Feynman-Kac theorem, which stipulates that, if a function $f(\mathbf{x}_t, t)$ of the Heston bivariate system of SDEs $\mathbf{x}_t = (x_t, v_t) = (\ln S_t, v_t)$ satisfies the PDE $\partial f/\partial t - rf + \mathcal{A}f = 0$, where $\mathcal{A}$ is the Heston generator from (1.28), then the solution to $f(\mathbf{x}_t, t)$ is the conditional expectation

$$f(\mathbf{x}_t, t) = E[f(\mathbf{x}_T, T)|\mathcal{F}_t].$$

Using $f(\mathbf{x}_t, t) = \exp(i\phi \ln S_t)$ produces the solution

$$f(\mathbf{x}_t, t) = E[e^{i\phi \ln S_T}|x_t, v_t]$$

which is the characteristic function for $x_T = \ln S_T$. Hence, the PDE for the characteristic function is, from Equation (1.40)

$$\begin{aligned} -\frac{\partial f_j}{\partial \tau} + \rho\sigma v \frac{\partial^2 f_j}{\partial v \partial x} + \frac{1}{2} v \frac{\partial^2 f_j}{\partial x^2} + \frac{1}{2}\sigma^2 v \frac{\partial^2 f_j}{\partial v^2} \\ + (r + u_j v)\frac{\partial f_j}{\partial x} + (a - b_j v)\frac{\partial f_j}{\partial v} = 0. \end{aligned} \tag{1.44}$$

Note the transformation from t to τ, which explains the negative sign in front of the first term in the PDE (1.44). The following derivatives are required to evaluate (1.44)

$$\begin{aligned} &\frac{\partial f_j}{\partial \tau} = \left(\frac{\partial C_j}{\partial \tau} + \frac{\partial D_j}{\partial \tau} v\right) f_j, \quad \frac{\partial f_j}{\partial x} = i\phi f_j, \quad \frac{\partial f_j}{\partial v} = D_j f_j, \\ &\frac{\partial^2 f_j}{\partial x^2} = -\phi^2 f_j, \quad \frac{\partial^2 f_j}{\partial v^2} = D_j^2 f_j, \quad \frac{\partial^2 f_j}{\partial v \partial x} = i\phi D_j f_j. \end{aligned}$$

Substitute these derivatives into (1.44) and drop the f_j terms to obtain

$$\begin{aligned} -\left(\frac{\partial C_j}{\partial \tau} + v\frac{\partial D_j}{\partial \tau}\right) + \rho\sigma v i\phi D_j - \frac{1}{2} v\phi^2 + \frac{1}{2} v\sigma^2 D_j^2 \\ + (r + u_j v)i\phi + (a - b_j v)D_j = 0, \end{aligned} \tag{1.45}$$

or equivalently

$$v\left(-\frac{\partial D_j}{\partial \tau} + \rho\sigma i\phi D_j - \frac{1}{2}\phi^2 + \frac{1}{2}\sigma^2 D_j^2 + u_j i\phi - b_j D_j\right) - \frac{\partial C_j}{\partial \tau} + ri\phi + aD_j = 0. \tag{1.46}$$

This produces two differential equations

$$\begin{aligned} \frac{\partial D_j}{\partial \tau} &= \rho\sigma i\phi D_j - \frac{1}{2}\phi^2 + \frac{1}{2}\sigma^2 D_j^2 + u_j i\phi - b_j D_j \\ \frac{\partial C_j}{\partial \tau} &= ri\phi + aD_j. \end{aligned} \tag{1.47}$$

These are Equations (A7) in Heston (1993). The first equation in (1.47) is a Riccati equation in D_j, while the second is an ordinary derivative for C_j that can solved using straightforward integration once D_j is obtained. Solving these equations requires two initial conditions. Recall from (1.43) that the characteristic function is

$$f_j(\phi; x_t, v_t) = E[e^{i\phi x_T}] = \exp(C_j(\tau, \phi) + D_j(\tau, \phi)v_t + i\phi x_t). \tag{1.48}$$

At maturity ($\tau = 0$), the value of $x_T = \ln S_T$ is known, so the expectation in (1.48) will disappear, and consequently the right-hand side will reduce to simply $\exp(i\phi x_T)$. This implies that the initial conditions at maturity are $D_j(0, \phi) = 0$ and $C_j(0, \phi) = 0$.

Finally, when we compute the characteristic function, we use x_t as the log spot price of the underlying asset, and v_t as its unobserved initial variance. This last quantity is the parameter v_0 described earlier in this chapter, and must be estimated. We sometimes write (x_0, v_0) for (x_t, v_t), or simply (x, v).

SOLVING THE HESTON RICCATI EQUATION

In this section, we explain how the expressions in Equation (1.47) can be solved to yield the call price. First, we introduce the Riccati equation and explain how its solution is obtained. The solution can be found in many textbooks on differential equations, such as that by Zwillinger (1997).

The Riccati Equation in a General Setting

The Riccati equation for $y(t)$ with coefficients $P(t)$, $Q(t)$, and $R(t)$ is defined as

$$\frac{dy(t)}{dt} = P(t) + Q(t)y(t) + R(t)y(t)^2. \tag{1.49}$$

The equation can be solved by considering the following second-order ordinary differential equation (ODE) for $w(t)$

$$w'' - \left[\frac{P'}{P} + Q\right]w' + PRw = 0 \tag{1.50}$$

which can be written $w'' + bw' + cw = 0$. The solution to Equation (1.49) is then

$$y(t) = -\frac{w'(t)}{w(t)}\frac{1}{R(t)}.$$

The ODE in (1.50) can be solved via the auxiliary equation $r^2 + br + c = 0$, which has two solutions α and β given by

$$\alpha = \frac{-b + \sqrt{b^2 - 4c}}{2}, \quad \beta = \frac{-b - \sqrt{b^2 - 4c}}{2}.$$

The solution to the second-order ODE in (1.50) is

$$w(t) = Me^{\alpha t} + Ne^{\beta t}$$

where M and N are constants. The solution to the Riccati equation is therefore

$$y(t) = -\frac{M\alpha e^{\alpha t} + N\beta e^{\beta t}}{Me^{\alpha t} + Ne^{\beta t}} \frac{1}{R(t)}.$$

Solution of the Heston Riccati Equation

From Equation (1.47), the Heston Riccati equation can be written

$$\frac{\partial D_j}{\partial \tau} = P_j - Q_j D_j + RD_j^2 \tag{1.51}$$

where

$$P_j = u_j i\phi - \frac{1}{2}\phi^2, \quad Q_j = b_j - \rho\sigma i\phi, \quad R = \frac{1}{2}\sigma^2. \tag{1.52}$$

The corresponding second-order ODE is

$$w'' + Q_j w' + P_j R w = 0 \tag{1.53}$$

so that $D_j = -\frac{1}{R}\frac{w'}{w}$. The auxiliary equation is $r^2 + Q_j r + P_j R = 0$, which has roots

$$\alpha_j = \frac{-Q_j + \sqrt{Q_j^2 - 4P_jR}}{2} = \frac{-Q_j + d_j}{2}$$
$$\beta_j = \frac{-Q_j - \sqrt{Q_j^2 - 4P_jR}}{2} = \frac{-Q_j - d_j}{2}$$

where

$$\begin{aligned} d_j &= \alpha_j - \beta_j = \sqrt{Q_j^2 - 4P_jR} \\ &= \sqrt{(\rho\sigma i\phi - b_j)^2 - \sigma^2(2u_j i\phi - \phi^2)}. \end{aligned} \tag{1.54}$$

For notational simplicity, we sometimes omit the "j" subscript on some of the variables. The solution to the Heston Riccati equation is therefore

$$D_j = -\frac{1}{R}\frac{w'}{w} = -\frac{1}{R}\left(\frac{M\alpha e^{\alpha\tau} + N\beta e^{\beta\tau}}{Me^{\alpha\tau} + Ne^{\beta\tau}}\right) = -\frac{1}{R}\left(\frac{K\alpha e^{\alpha\tau} + \beta e^{\beta\tau}}{Ke^{\alpha\tau} + e^{\beta\tau}}\right) \tag{1.55}$$

where $K = M/N$. The initial condition $D_j(0, \phi) = 0$ implies that, when $\tau = 0$ is substituted in (1.55), the numerator becomes $K\alpha + \beta = 0$, from which $K = -\beta/\alpha$. The solution for D_j becomes

$$\begin{aligned} D_j &= -\frac{\beta}{R}\left(\frac{-e^{\alpha\tau} + e^{\beta\tau}}{-g_j e^{\alpha\tau} + e^{\beta\tau}}\right) = -\frac{\beta}{R}\left(\frac{1 - e^{d_j\tau}}{1 - g_j e^{d_j\tau}}\right) \\ &= \frac{Q_j + d_j}{2R}\left(\frac{1 - e^{d_j\tau}}{1 - g_j e^{d_j\tau}}\right) \end{aligned} \tag{1.56}$$

where

$$g_j = -K = \frac{\beta}{\alpha} = \frac{b_j - \rho\sigma i\phi + d_j}{b_j - \rho\sigma i\phi - d_j} = \frac{Q_j - d_j}{Q_j + d_j}. \tag{1.57}$$

The solution for D_j can, therefore, be written

$$D_j(\tau, \phi) = \frac{b_j - \rho\sigma i\phi + d_j}{\sigma^2}\left(\frac{1 - e^{d_j\tau}}{1 - g_j e^{d_j\tau}}\right). \tag{1.58}$$

The solution for C_j is found by integrating the second equation in (1.47)

$$C_j = \int_0^\tau ri\phi dy + a\left(\frac{Q_j + d_j}{\sigma^2}\right)\int_0^\tau\left(\frac{1 - e^{d_j y}}{1 - g_j e^{d_j y}}\right)dy + K_1 \tag{1.59}$$

where K_1 is a constant. The first integral is $ri\phi\tau$ and the second integral can be found by substitution, using $x = \exp(d_j y)$, from which $dx = d_j \exp(d_j y)dy$ and $dy = dx/(xd_j)$. Equation (1.59) becomes

$$C_j = ri\phi\tau + \frac{a}{d_j}\left(\frac{Q_j + d_j}{\sigma^2}\right)\int_1^{\exp(d_j\tau)}\left(\frac{1 - x}{1 - g_j x}\right)\frac{1}{x}dx + K_1. \tag{1.60}$$

The integral in (1.60) can be evaluated by partial fractions

$$\begin{aligned}\int_1^{\exp(d_j\tau)} \frac{1 - x}{x(1 - g_j x)}dx &= \int_1^{\exp(d_j\tau)}\left[\frac{1}{x} - \frac{1 - g_j}{1 - g_j x}\right]dx \\ &= \left[\ln x + \frac{1 - g_j}{g_j}\ln\left(1 - g_j x\right)\right]_{x=1}^{x=\exp(d_j\tau)} \\ &= \left[d_j\tau + \frac{1 - g_j}{g_j}\ln\left(\frac{1 - g_j e^{d_j\tau}}{1 - g_j}\right)\right].\end{aligned} \tag{1.61}$$

Substituting the integral back into (1.60), and substituting for d_j, Q_j, and g_j, produces the solution for C_j

$$C_j(\tau, \phi) = ri\phi\tau + \frac{a}{\sigma^2}\left[\left(b_j - \rho\sigma i\phi + d_j\right)\tau - 2\ln\left(\frac{1 - g_j e^{d_j\tau}}{1 - g_j}\right)\right] \tag{1.62}$$

where $a = \kappa\theta$. Note that we have used the initial condition $C_j(0, \phi) = 0$, which results in $K_1 = 0$. This completes the original derivation of the Heston model.

We use two functions to implement the model in Matlab, HestonProb.m and HestonPrice.m. The first function calculates the characteristic functions and returns the real part of the integrand. The function allows for the Albrecher et al. (2007) "Little Trap" formulation for the characteristic function, which is introduced in Chapter 2. The functions allow to price calls or puts, and allow for a dividend yield, as explained in the following section. To conserve space parts of the functions have been omitted.

```
function y = HestonProb(phi,...,Trap);
x = log(S);
a = kappa*theta;
d = sqrt((rho*sigma*i*phi - b)^2 - sigma^2*(2*u*i*phi - phi^2));
g = (b - rho*sigma*i*phi + d) / (b - rho*sigma*i*phi - d);
if Trap==1
    c = 1/g;
    D = (b - rho*sigma*i*phi - d)/sigma^2*((1-exp(-d*tau)) ...;
    G = (1 - c*exp(-d*tau))/(1-c);
    C = (r-q)*i*phi*tau + a/sigma^2*((b - rho*sigma*i*phi - d)...;
elseif Trap==0
    G = (1 - g*exp(d*tau))/(1-g);
    C = (r-q)*i*phi*tau + a/sigma^2*((b - rho*sigma*i*phi + d) ...;
    D = (b - rho*sigma*i*phi + d)/sigma^2*((1-exp(d*tau)) ...;
end
f = exp(C + D*v0 + i*phi*x);
y = real(exp(-i*phi*log(K))*f/i/phi);
```

The second function calculates the price of a European call $C(K)$, or European put $P(K)$, by put-call parity in Equation (1.67). The function calls the HestonProb.m function at every point of the integration grid and uses the trapezoidal rule for integration when all the integration points have been calculated, using the built-in Matlab function trapz.m. Chapter 5 presents alternate numerical integration schemes that do not rely on built-in Matlab functions.

```
function y = HestonPrice(PutCall,...,trap,Lphi,Uphi,dphi)
phi = [Lphi:dphi:Uphi];
N = length(PHI);
for k=1:N;
    int1(k) = HestonProb(phi(k),...,1);
    int2(k) = HestonProb(phi(k),...,2,);
end
I1 = trapz(int1)*dphi;
I2 = trapz(int2)*dphi;
P1 = 1/2 + 1/pi*I1;
P2 = 1/2 + 1/pi*I2;
HestonC = S*exp(-q*T)*P1 - K*exp(-r*T)*P2;
HestonP = HestonC - S*exp(-q*T) + K*exp(-r*T);
```

Pricing European calls and puts is straightforward. For example, the price a 6-month European put with strike $K = 100$ on a dividend-paying stock with spot price $S = 100$ and yield $q = 0.02$, when the risk-free rate is $r = 0.03$ and using the parameters $\kappa = 5$, $\sigma = 0.5$, $\rho = -0.8$, $\theta = v_0 = 0.05$, and $\lambda = 0$, along with the integration grid $\phi \in [0.00001, 50]$ in increments of 0.001 is 5.7590. The price of the call with identical features is 6.2528. If there is no dividend yield so that $q = 0$, then as expected, the put price decreases, to 5.3790, and the call price increases, to 6.8678.

Some applications require Matlab code for the Heston characteristic function. The HestonProb.m function can be modified to return the characteristic function itself, instead of the integrand. In certain instances, the integrand for P_j

$$\mathrm{Re}\left[\frac{e^{-i\phi \ln K} f_j(\phi; x, v)}{i\phi}\right] \tag{1.63}$$

is well-behaved in that it poses no difficulties in numerical integration. This corresponds to an integrand that does not oscillate much, that dampens quickly so that a large upper limit in the numerical integration is not required, and that does not contain portions that are excessively steep. In other instances, the integrand is not well-behaved, and numerical integration loses precision. To illustrate, we plot the second integrand ($j = 2$) in Equation (1.63), using the settings $S = 7$, $K = 10$, and $r = q = 0$, with parameter values $\kappa = 10$, $\theta = v_0 = 0.07$, $\sigma = 0.3$, and $\rho = -0.9$. The plot uses the domain $-50 < \phi < 50$ over maturities running from 1 week to 3 months. This plot appears in Figure 1.1. The integrand has a discontinuity at $\phi = 0$, but this does not show up in the figure.

The plot indicates an integrand that has a fair amount of oscillation, especially at short maturities, and that is steep near the origin. In Chapter 2, we investigate other problems that can arise with the Heston integrand.

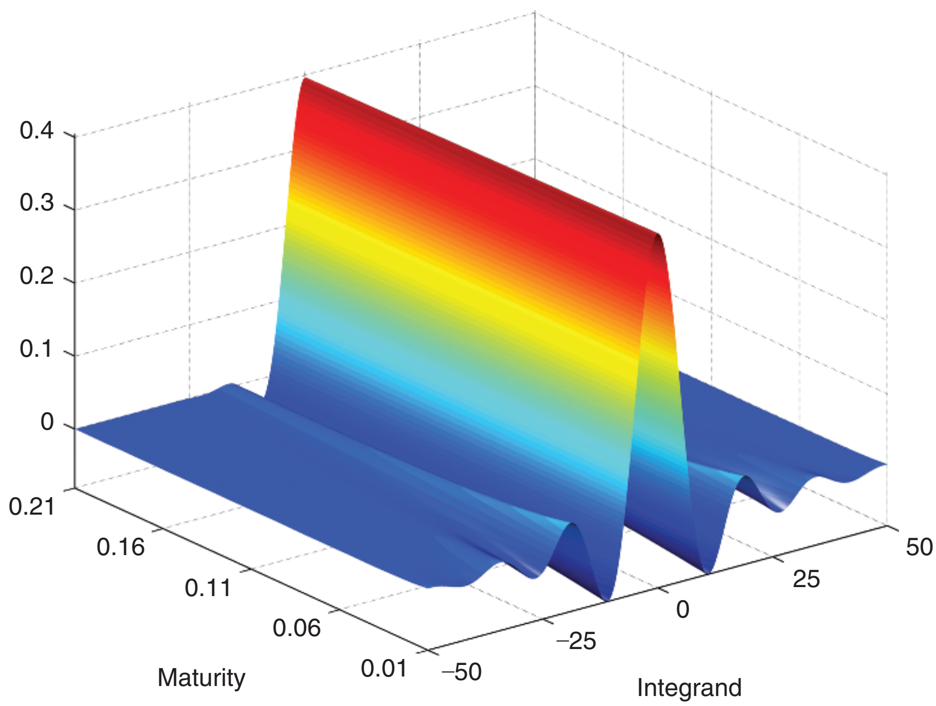

FIGURE 1.1 Heston Integrand and Maturity

DIVIDEND YIELD AND THE PUT PRICE

It is straightforward to include dividends into the model if it can be assumed that the dividend payment is a continuous yield, q. In that case, r is replaced by $r-q$ in Equation (1.4) for the stock price process

$$dS_t = (r-q)S_t dt + \sqrt{v_t} S_t d\widetilde{W}_{1,t}. \tag{1.64}$$

The solution for C_j in Equation (1.62) becomes

$$C_j = (r-q)i\phi\tau + \frac{\kappa\theta}{\sigma^2}\left[\left(b_j - \rho\sigma i\phi + d_j\right)\tau - 2\ln\left(\frac{1-g_j e^{d_j\tau}}{1-g_j}\right)\right]. \tag{1.65}$$

To obtain the price $P(K)$ of a European put, first obtain the price $C(K)$ of a European call, using a slight modification of Equation (1.12) to include the term $e^{-q\tau}$ for the dividend yield, as explained by Whaley (2006)

$$C(K) = S_t e^{-q\tau} P_1 - K e^{-r\tau} P_2. \tag{1.66}$$

The put price is found by put-call parity

$$P(K) = C(K) + K e^{-r\tau} - S_t e^{-q\tau}. \tag{1.67}$$

Alternatively, as in Zhu (2010) the put price can be obtained explicitly as

$$P(K) = K e^{-r\tau} P_2^c - S_t e^{-q\tau} P_1^c. \tag{1.68}$$

The put expires in-the-money if $x_T < \ln K$. The in-the-money probabilities in (1.68) are, therefore, the complement of those in (1.41)

$$P_j^c = \Pr(\ln S_T < \ln K) = \frac{1}{2} - \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left[\frac{e^{-i\phi\ln K} f_j(\phi; x, v)}{i\phi}\right] d\phi. \tag{1.69}$$

It is straightforward to show the equivalence of Equations (1.67) and (1.68). Finally, by an application of the Feynman-Kac theorem, which will be introduced in Chapter 12, the PDE for $x = \ln S$ is

$$\begin{aligned} &\frac{\partial U}{\partial t} + \frac{1}{2}v\frac{\partial^2 U}{\partial x^2} + \left(r - q - \frac{1}{2}v\right)\frac{\partial U}{\partial x} + \rho\sigma v\frac{\partial^2 U}{\partial v \partial x} \\ &\quad + \frac{1}{2}\sigma^2 v\frac{\partial^2 U}{\partial v^2} - rU + [\kappa(\theta - v) - \lambda v]\frac{\partial U}{\partial v} = 0. \end{aligned} \tag{1.70}$$

Equation (1.70) is simply (1.29), but with $(r - v/2)$ in the third term replaced by $(r - q - v/2)$.

CONSOLIDATING THE INTEGRALS

It is possible to regroup the integrals for the probabilities P_1 and P_2 into a single integral, which will speed up the numerical integration required in the call price calculation. Substituting the expressions for P_j from Equation (1.41) into the call price in (1.66) and re-arranging produces

$$\begin{aligned} C(K) = {} & \frac{1}{2}S_t e^{-q\tau} - \frac{1}{2}Ke^{-r\tau} \\ & + \frac{1}{\pi}\int_0^\infty \text{Re}\left[\frac{e^{-i\phi \ln K}}{i\phi}\left(S_t e^{-q\tau} f_1(\phi; x, v) - Ke^{-r\tau} f_2(\phi; x, v)\right)\right] d\phi. \end{aligned} \tag{1.71}$$

The advantage of this consolidation is that only a single numerical integration is required instead of two, so the computation time will be reduced by almost one-half. The put price can be obtained by using put-call parity, with the call price calculated using (1.71).

The integrand of the consolidated form is in the function HestonProbConsol.m.

```
function y = HestonProbConsol(phi,...,Pnum,Trap);
% First characteristic function f1
u1 = 0.5;
b1 = kappa + lambda - rho*sigma;
d1 = sqrt((rho*sigma*i*phi - b1)^2 - sigma^2*(2*u1*i*phi - phi^2));
g1 = (b1 - rho*sigma*i*phi + d1)/(b1-rho*sigma*i*phi-d1);
G1 = (1 - g1*exp(d1*tau))/(1-g1);
C1 = (r-q)*i*phi*tau + a/sigma^2 * ...;
D1 = (b1 - rho*sigma*i*phi + d1)/sigma^2 * ...;
f1 = exp(C1 + D1*v0 + i*phi*x);
% Second characteristic function f1
f2 = exp(C2 + D2*v0 + i*phi*x);
% Return the real part of the integrand
y = real(exp(-i*phi*log(K))/i/phi*(S*exp(-q*tau)*f1 - K*exp(-r*tau)*f2));
```

This function is then fed into the HestonPriceConsol.m function, which calculates the call price in accordance with Equation (1.71).

```
function y = HestonPriceConsol(PutCall,...,trap,Lphi,Uphi,dphi)
% Build the integration grid
phi = [Lphi:dphi:Uphi];
N = length(phi);
for k=1:N;
    inte(k) = HestonProbConsol(phi(k),...,1,trap);
end
I = trapz(inte)*dphi;
% The call price
HestonC = (1/2)*S*exp(-q*T) - (1/2)*K*exp(-r*T) + I/pi;
% The put price by put-call parity
HestonP = HestonC - S*exp(-q*T) + K*exp(-r*T);
```

The consolidated form produces exactly the same prices for the call and the put, but requires roughly one-half of the computation time only.

BLACK-SCHOLES AS A SPECIAL CASE

With a little manipulation, it is straightforward to show that the Black-Scholes model is nested inside the Heston model. The Black-Scholes model assumes the following dynamics for the underlying price S_t under the risk-neutral measure $\mathbb{Q}$

$$dS_t = rS_t + \sigma_{BS} S_t d\widetilde{W}_t. \tag{1.72}$$

It is shown in many textbooks, such as that by Hull (2011) or Chriss (1996) that (1.72) can be solved for the spot price S_t. This is done in two steps. First, apply Itō's lemma to obtain the process for $d \ln S_t$, which produces a stochastic process that is no longer an SDE since its drift and volatility no longer depend on S_t. Second, integrate the stochastic process to produce

$$S_t = S_0 \exp([r - \sigma_{BS}^2/2]t + \sigma_{BS}\widetilde{W}_t). \tag{1.73}$$

This implies that, at time t, the natural logarithm of the stock price at expiry $\ln S_T$ is distributed as a normal random variable with mean $\ln S_t + \left(r - \frac{1}{2}\sigma_{BS}^2\right)\tau$ and variance $\sigma_{BS}^2\tau$, where $\tau = T - t$ is the time to expiry. Consequently, the characteristic function of $\ln S_T$ in the Black-Scholes model is

$$E[e^{i\phi \ln S_T}] = \exp\left(i\phi\left[\ln S_t + \left(r - \frac{1}{2}\sigma_{BS}^2\right)\tau\right] - \frac{1}{2}\phi^2\sigma_{BS}^2\tau\right). \tag{1.74}$$

The Black-Scholes PDE is

$$\frac{\partial U}{\partial t} + \frac{1}{2}\sigma_{BS}^2 S^2 \frac{\partial^2 U}{\partial S^2} + rS\frac{\partial U}{\partial S} - rU = 0. \tag{1.75}$$

The Black-Scholes call price is given by

$$C_{BS}(K) = S_t\Phi(d_1) - Ke^{-r\tau}\Phi(d_2) \tag{1.76}$$

with

$$\begin{aligned} d_1 &= \frac{\ln(S_t/K) + (r + \sigma_{BS}^2/2)\tau}{\sigma_{BS}\sqrt{\tau}}, \\ d_2 &= \frac{\ln(S_t/K) + (r - \sigma_{BS}^2/2)\tau}{\sigma_{BS}\sqrt{\tau}} = d_1 - \sigma_{BS}\sqrt{\tau} \end{aligned} \tag{1.77}$$

where $\Phi(x)$ is the standard normal cumulative distribution function. The volatility σ_{BS} is assumed to be constant.

If we set $\sigma = 0$, the volatility of variance parameter in the Heston model, then the Brownian component of the variance process in Equation (1.1) drops out. Consequently, from (1.11) we obtain $\text{Var}[v_t|v_0] = 0$. This will produce volatility that is time-varying, but deterministic. If we further set $\theta = v_0$, then from (1.11) we get $E[v_t|v_0] = v_0$, which is time independent. This will produce volatility that is constant. Hence, setting $\sigma = 0$ and $\theta = v_0$ in the Heston model leads us to expect the same price as that produced by the Black Scholes model, with $\sigma_{BS} = \sqrt{v_0}$ as the Black Scholes implied volatility. Indeed, substituting $\sigma = 0$ and $\theta = v_0$ into the Heston PDE (1.24) along with $\lambda = 0$ produces the Black-Scholes PDE in (1.75) with $\sigma_{BS} = \sqrt{v_0}$. Consequently, the Heston price under these parameter values will be the Black-Scholes price.

To implement the Black-Scholes model as a special case of the Heston model, we cannot simply substitute $\sigma = 0$ into the pricing functions, because that will lead to division by zero in the expressions for $C_j(\tau, \phi)$ in Equation (1.62) and $D_j(\tau, \phi)$ in (1.58). Instead, we must start with the set of equations in (1.47). With $\sigma = 0$, the Riccati equation in (1.51) reduces to the ordinary first-order differential equation

$$\frac{\partial D_j}{\partial \tau} = P_j - Q_j D_j,$$

where $P_j = u_j i\phi - \frac{1}{2}\phi^2$ and $Q_j = b_j$. The solution of this ODE is

$$D_j(\tau, \phi) = \frac{\left(u_j i\phi - \frac{1}{2}\phi^2\right)\left(1 - e^{-b_j\tau}\right)}{b_j}. \tag{1.78}$$

As for the general case $\sigma > 0$, substitute (1.78) into the expression for C_j in the second equation of (1.47) and integrate to obtain

$$\begin{aligned} C_j(\tau, \phi) &= ri\phi\tau + a\int_0^\tau \frac{\left(u_j i\phi - \frac{1}{2}\phi^2\right)\left(1 - e^{-b_j y}\right)}{b_j} dy + K_1 \\ &= ri\phi\tau + \frac{a\left(u_j i\phi - \frac{1}{2}\phi^2\right)}{b_j}\left[\tau - \frac{\left(1 - e^{-b_j\tau}\right)}{b_j}\right] \end{aligned} \tag{1.79}$$

where the initial condition $C_j(0, \phi) = 0$ has been applied, which produces $K_1 = 0$ for the integration constant. Now substitute C_j and D_j from Equations (1.79) and (1.78) into the characteristic function in (1.43), and proceed exactly as in the case $\sigma > 0$. Note that the correlation coefficient, ρ, no longer appears in the expressions for C_j and D_j, which is sensible since it is no longer relevant.

Now consider the case $j = 2$. Substitute for $u_2 = -\frac{1}{2}$ and $b_2 = \kappa$ (with $\lambda = 0$) in Equations (1.78) and (1.79), set $\theta = v_0$, and substitute the resulting expressions for $D_2(\tau, \phi)$ and $C_2(\tau, \phi)$ into the characteristic function in (1.48). The second characteristic function is reduced to

$$f_2(\phi) = \exp\left(i\phi\left[x_0 + \left(r - \frac{1}{2}v_0\right)\tau\right] - \frac{1}{2}\phi^2 v_0\tau\right) \tag{1.80}$$

where $x_0 = \ln S_0$ is the log spot stock price and v_0 is the spot variance (at $t = 0$). Equation (1.80) is recognized to be (1.74), the characteristic function of $x_T = \ln S_T$ under the Black-Scholes model, with the Black-Scholes volatility as $\sigma_{BS} = \sqrt{v_0}$, as required by (1.77).

The Black-Scholes call price can also be derived using the characteristic function approach to pricing options detailed by Bakshi and Madan (2000), in accordance with Equation (1.16). If a random variable Y is distributed lognormal with mean μ and variance σ^2, its cumulative density function is

$$F_Y(y) = \Pr(Y < y) = \Phi\left(\frac{\ln y - \mu}{\sigma}\right). \tag{1.81}$$

The expectation of Y, conditional on $Y > y$ is

$$L_Y(y) = E(Y|Y > y) = \exp\left(\mu + \frac{\sigma^2}{2}\right)\Phi\left(\frac{-\ln y + \mu + \sigma^2}{\sigma}\right). \tag{1.82}$$

See, for example, Hogg and Klugman (1984) for a derivation of these formulas, which are straightforward. Under the risk-neutral measure $\mathbb{Q}$, S_T is distributed as lognormal with mean $\ln S_t + \left(r - \frac{1}{2}\sigma_{BS}^2\right)\tau$ and variance $\sigma_{BS}^2\tau$, as described earlier in this section. Substituting this mean and variance into (1.81) produces

$$\mathbb{Q}(S_T > K) = \Phi\left(\frac{\mu - \ln K}{\sigma}\right) = \Phi\left(\frac{\ln\left(S_t/K\right) + \left(r - \frac{1}{2}\sigma_{BS}^2\right)\tau}{\sigma_{BS}\sqrt{\tau}}\right) = \Phi(d_2) \tag{1.83}$$

where $1 - \Phi(x) = \Phi(-x)$ has been exploited. To obtain the other probability, apply the Radon-Nikodym derivative (1.14)

$$\begin{aligned}\mathbb{Q}^S(S_T > K) &= \int_K^\infty d\mathbb{Q}^S = \int_K^\infty \frac{d\mathbb{Q}^S}{d\mathbb{Q}}d\mathbb{Q} = \frac{e^{-r(T-t)}}{S_t}\int_K^\infty S_T q_T(x)dx \\ &= \frac{e^{-r\tau}}{S_t}E^{\mathbb{Q}}[S_T|S_T > K]\end{aligned} \tag{1.84}$$

where $q_T(x)$ is the probability density function for S_T. Substitute the mean and variance of S_T into Equation (1.82), and substitute the resulting expression in the last line of (1.84) to obtain

$$\mathbb{Q}^S(S_T > K) = \Phi\left(\frac{\ln\left(S_t/K\right) + \left(r + \frac{1}{2}\sigma_{BS}^2\right)\tau}{\sigma_{BS}\sqrt{\tau}}\right) = \Phi(d_1). \tag{1.85}$$

The Black-Scholes call price can therefore be written as the form in Equation (1.16)

$$C(K) = S_t\mathbb{Q}^S(S_T > K) - Ke^{-r\tau}\mathbb{Q}(S_T > K).$$

To obtain the result with a continuous dividend yield, replace r by $r - q$ in all the required expressions and the result follows.

The fact that $f_2(\phi)$ in Equation (1.80) is the Black-Scholes characteristic function, and not $f_1(\phi)$, is the desired result. Indeed, we will see in Chapter 2 that $f_2(\phi)$ is the "true" characteristic function in the Heston model, because it is the one obtained under the risk-neutral measure $\mathbb{Q}$. As shown by Bakshi and Madan (2000) and others, $f_1(\phi)$ can be expressed in terms of $f_2(\phi)$, so a separate expression for $f_1(\phi)$ is not required.

The function HestonProbZeroSigma.m is used to implement the Black-Scholes model as a special case of the Heston model (when $\sigma = 0$). To conserve space, only the crucial portions of the function are presented.

```
function y = HestonProbZeroSigma(phi,...,Pnum)
D = (u*i*phi - phi^2/2)*(1-exp(-b*tau))/b;
C = (r-q)*i*phi*tau + a*(u*i*phi-0.5*phi^2)/b * ...;
f = exp(C + D*theta + i*phi*x);
y = real(exp(-i*phi*log(K))*f/i/phi);
```

The function HestonPriceZeroSigma.m is used to obtain the price when $\sigma = 0$. The following Matlab code illustrates this point, using the same settings as stated earlier. Again, only the relevant parts of the code are presented.

```
d1 = (log(S/K) + (r-q+theta/2)*T)/sqrt(theta*T);
d2 = d1 - sqrt(theta*T);
BSCall = S*exp(-q*T)*normcdf(d1)  - K*exp(-r*T)*normcdf(d2);
BSPut  = K*exp(-r*T)*normcdf(-d2) - S*exp(-q*T)*normcdf(-d1);
HCall = HestonPriceZeroSigma('C',...);
HPut  = HestonPriceZeroSigma('P',...);
```

With the settings $\tau = 0.5$, $S = K = 100$, $q = 0.02$, $r = 0.03$, $\kappa = 5$, $v_0 = \theta = 0.05$, and $\lambda = 0$, the Heston model and Black-Scholes model with $\sigma_{BS} = \sqrt{v_0}$ each return 6.4730 for the price of the call and 5.9792 for the price of the put.

SUMMARY OF THE CALL PRICE

From Equation (1.66), the call price is of the form

$$C(K) = S_t e^{-q\tau} P_1 - K e^{-r\tau} P_2 \tag{1.86}$$

with in-the-money probabilities P_1 and P_2 from Equation (1.41)

$$P_j = \Pr(\ln S_T > \ln K) = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \text{Re}\left[\frac{e^{-i\phi \ln K} f_j(\phi; x, v)}{i\phi}\right] d\phi. \tag{1.87}$$

These probabilities are derived from the characteristic functions f_1 and f_2 for the logarithm of the terminal stock price, $x_T = \ln S_T$

$$f_j(\phi; x_t, v_t) = \exp(C_j(\tau, \phi) + D_j(\tau, \phi)v_t + i\phi x_t) \tag{1.88}$$

where $x_t = \ln S_t$ is the log spot price of the underlying asset, and v_t is its unobserved initial variance, which is estimated as the parameter v_0.

To obtain the price of a European call, we use the expressions for C_j and D_j in Equations (1.65) and (1.58) to obtain the two characteristic functions. To obtain the price of a European put, we use put-call parity in (1.67).

CONCLUSION

In this chapter, we have presented the original derivation of the Heston (1993) model, including the PDEs from the model, the characteristic functions, and the European call and put prices. We have also shown how the Black-Scholes model arises as a special case of the Heston model.

The Heston model has become the most popular stochastic volatility model for pricing equity options. This is in part due to the fact that the call price in the model is available in closed form. Some authors refer to the call price as being in "semi-closed" form because of the numerical integration required to obtain P_1 and P_2. But the Black-Scholes model also requires numerical integration, to obtain $\Phi(d_1)$ and $\Phi(d_2)$. In this sense, the Heston model produces call prices that are no less closed than those produced by the Black-Scholes model. The difference is that programming languages often have built-in routines for calculating the standard normal cumulative distribution function, $\Phi(\cdot)$ (usually by employing a polynomial approximation), whereas the Heston probabilities are not built-in and must be obtained using numerical integration. In the next chapter, we investigate some of the problems that can arise in numerical integration when the integrand

$$\text{Re}\left[\frac{e^{-i\phi \ln K} f_j(\phi; x, v)}{i\phi}\right]$$

is not well-behaved. We encountered an example of such an integrand in Figure 1.1.