The Rough Bergomi model as $H \to 0$ - skew flattening/blow up and non-Gaussian rough volatility

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Abstract

Using Jacod's stable convergence theorem, we prove the surprising result that the martingale component X_t of the log stock price for the popular Rough Bergomi stochastic volatility model (when suitably re-scaled) tends weakly to $B_{\xi_\gamma([0,t])}$ as $H \to 0$, where ξ_γ is the Gaussian multiplicative chaos measure of the Riemann-Liouville process which drives the model in the $H \to 0$ limit, and B is a Brownian motion independent of everything else. This implies that the implied volatility smile for the full rough Bergomi model with $\rho \leq 0$ is symmetric in the $H \to 0$ limit, and without re-scaling the model tends weakly to the Black-Scholes model as $H \to 0$. We also derive a closed-form expression for the conditional third moment $\mathbb{E}((X_{t+h} - X_t)^3|\mathcal{F}_t)$ (for $H > 0$) given a finite history, and $\mathbb{E}(X_T^3)$ tends to zero (or blows up) exponentially fast as $H \to 0$ depending on whether γ is less than or greater than a critical $\gamma \approx 1.61711$. We describe how to use this equation to calibrate a time-dependent correlation function $\rho(t)$ to the observed skew term structure ($\rho(t)$ satisfies an Abel integral equation for which we establish existence and uniqueness) and briefly discuss the pros and cons of a $H = 0$ model with non-zero skew for which $X_t/\sqrt{t}$ tends weakly to a non-Gaussian random variable X_1 with non-zero skewness as $t \to 0$. Finally, we also define a generalized family of non-Gaussian rough volatility models, with an associated non-Gaussian multiplicative chaos in the $H \to 0$ limit, using a similar construction to [BM03] with an independently scattered infinitely divisible random measure, and we show that the resulting GMC in the $H \to 0$ limit is locally multifractal for integer moments with a non-quadratic multifractal exponent.[1]

1 Introduction

[FGS20] consider a re-scaled Riemann-Liouville process $Z_t^H = \int_0^t (t-s)^{H-\frac{1}{2}} dW_s$ in the $H \to 0$ limit. Using's Lévy's continuity theorem for tempered distributions, they show that Z^H tends weakly to an almost log-correlated Gaussian field Z as $H \to 0$, which is a random tempered distribution, i.e. a random element of the dual of the Schwartz space $\mathcal{S}$ and can also be viewed as a Gaussian in the fractional Sobolev Hilbert space $\mathcal{H}^{-\frac{1}{2}-\epsilon}$, and one can also show convergence of $(Z - Z^H)/H$ to some $\Gamma \in \mathcal{H}^{-\frac{1}{2}-\epsilon}$, the so-called *first order correction field*. As a corollary $\xi_\gamma^H(dt) = e^{\gamma Z_t^H - \frac{1}{2}\gamma^2 \mathrm{Var}(Z_t^H)} dt$ is shown to tend to a Gaussian multiplicative chaos (GMC) random measure ξ_γ for $\gamma \in (0,1)$ as $H \to 0$. Unlike standard constructions of GMC, Z_t^H is not a backwards martingale in H so one cannot appeal to the martingale convergence theorem. We later address the more difficult "L^1-regime" where $\gamma \in [1, \sqrt{2})$ using Strassen's theorem and we show that our limiting GMC object using the RL process falls within the general setup of Shamov[Sha16] using *randomized shifts* on the Cameron-Martin space of the underlying Brownian motion. Our more involved analysis for $\gamma \in (0, \sqrt{2})$ using Strassen's theorem and [Sha16] also covers the simpler case $\gamma \in (0,1)$ but we have included the simpler L^2 analysis for the latter for pedagogical reasons. We also construct a candidate GMC for the super-critical phase $\gamma > \sqrt{2}$, using an independent stable subordinator time-changed by our Riemann-Liouville GMC (similar to section 3 in [BJRV14]) to construct an atomic GMC with the correct (locally) multifractal exponent for γ-values greater than $\sqrt{2}$, which is closely related to the non-standard branch of gravity in conformal field theory.

These results have a natural application to the popular Rough Bergomi stochastic volatility model, since ξ_γ^H is the quadratic variation of the log stock price for this model and values of H as low as .03 have been reported in empirical studies of this model (see e.g. Fukasawa et al.[FTW19]). In this article, using the aforementioned Riemann-Liouville GMC and Jacod's stable convergence theorem, we prove the surprising result that the martingale component X_t of the log stock price for the Rough Bergomi model tends weakly to $B_{\xi_\gamma([0,t])}$ as $H \to 0$ where B is a Brownian motion independent of everything else, which means the smile for the rBergomi model with $\rho \leq 0$ is symmetric in

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the $H \to 0$ limit for $\gamma \in (0,1)$. More generally for any $\gamma > 0$, we derive a closed-form expression for the conditional third moment $\mathbb{E}((X_{t+h} - X_t)^3|\mathcal{F}_t)$ (for $H > 0$, and we find that $\mathbb{E}(X_t^3)$ decays exponentially fast or blows up exponentially fast depending on whether γ is less than or greater than a critical $\gamma \approx 1.61711$. The expression for $\mathbb{E}(X_t^3)$ can then be used to calibrate a time-dependent correlation function $\rho(t)$ to the observed skew term structure, which satisfies an Abel integral equation (for which we establish existence and uniqueness) and we can also define a $H = 0$ model with non-zero skew for which $X_t/\sqrt{t}$ tends weakly to a non-Gaussian random variable X_1 with non-zero skewness as $t \to 0$. Finally, we define a new family of non-Gaussian rough volatility models, with an associated non-Gaussian multiplicative chaos in the $H \to 0$ limit, using a similar approach to [BM03] via independently scattered infinitely divisible random measures, and we show that the resulting GMC in the $H \to 0$ limit is locally multifractal for integer moments with a non-quadratic multifractal exponent.

2 The Rough Bergomi model in the $H \to 0$ limit

2.1 The skew flattening phenomenon

We consider the standard Rough Bergomi model for a stock price process X_t^H:

$$\begin{cases} dX_t^H = -\frac{1}{2}\sqrt{V_t^H} + \sqrt{V_t^H}dW_t \,, \\ V_t^H \;=\; e^{\gamma Z_t^H - \frac{1}{2}\gamma^2 \mathrm{Var}(Z_t^H)} \\ Z_t^H \;=\; \int_0^t (t-s)^{H-\frac{1}{2}}(\rho dW_s \,+\, \bar{\rho} dW_t^{\perp}) \end{cases} \tag{1}$$

where $\gamma \in (0,1)$, $|\rho| \leq 1$ and W, $W^{\perp}$ are independent Brownian motions, and (without loss of generality) we set $\tilde{X}_0^H = 0$. We let

$$\tilde{X}_t^H \;=\; \int_0^t \sqrt{V_t^H} dW_t \tag{2}$$

denote the martingale part of X^H.

Theorem 2.1 *For $\gamma \in (0,1)$, $\tilde{X}^H$ tends to $B^{\perp}_{\xi_\gamma([0,(.)])}$ stably (and hence weakly) in law on any finite interval $[0,T]$, where $B^{\perp}$ is a Brownian motion independent of everything else.*

Corollary 2.2 *From the weak convergence established in the theorem we see that*

$$\begin{aligned} \lim_{H\to 0} \mathbb{E}(e^{ikX_t^H}) &= \lim_{H\to 0} \mathbb{E}(e^{-\frac{1}{2}(ik+k^2)\int_0^t \sqrt{V_s^H} dW_s)}) \\ &= \mathbb{E}(e^{-\frac{1}{2}(ik+k^2)\xi_\gamma([0,t])}) \\ &= \mathbb{E}(e^{ik(-\frac{1}{2}\xi_\gamma([0,t]) + B_{\xi_\gamma([0,t])})}) \end{aligned}$$

which (by a well known result in Renault&Touzi[RT96]) implies that implied volatility smile for the true Rough Bergomi model in (1) is symmetric in the log-moneyness $k = \log \frac{K}{S_0}$.

Remark 2.1 We call this the *skew flattening phenomenon*, so in particular $\tilde{X}_t^H$ (for a single fixed t) tends weakly to some weakly symmetric distribution μ.

Proof. From Theorem 2.2 in [FGS20], we know that $\langle \tilde{X}^H \rangle_t$ tends to a random variable $\xi_\gamma([0,t])$ in L^2 (and hence in probability), and $\langle \tilde{X}^H, W\rangle_t = \rho \int_0^t \sqrt{V_u^H} du$. But

$$\begin{aligned} \mathbb{E}((V_t^H)^{\frac{1}{2}}) &= \mathbb{E}(e^{\frac{1}{2}(\gamma Z_t^H - \frac{1}{2}\gamma^2 \frac{1}{2H} t^{2H})}) \\ &= \mathbb{E}(e^{\frac{1}{2}\gamma Z_t^H - \frac{1}{2}\cdot\frac{1}{4}\gamma^2\cdot\frac{1}{2H} + \frac{1}{2}\cdot\frac{1}{4}\gamma^2\cdot\frac{1}{2H} - \frac{1}{2}\gamma^2 \frac{1}{4H} t^{2H}}) \;=\; e^{-\frac{1}{16H}\gamma^2 t^{2H}} \;\to\; 0 \end{aligned}$$

as $H \to 0$, so (by Markov's inequality) $\mathbb{P}(\sqrt{V_t^H} > \delta) \leq \frac{1}{\delta}\mathbb{E}(\sqrt{V_t^H}) \to 0$, so $\sqrt{V_t^H}$ tends to zero in probability, and hence

$$G_t \;:=\; \langle \tilde{X}^H, W\rangle_t \;\overset{p}{\to}\; 0\,. \tag{3}$$

Moreover, for any bounded martingale N orthogonal to W

$$\langle \tilde{X}^H, N\rangle_t \;=\; 0\,. \tag{4}$$

Thus setting $Z_t = W_t$ and applying Theorem IX.7.3 in Jacod&Shiryaev[JS03] (see also Proposition II.7.5 and Definition II.7.8 in [JS03]), we can construct an extension $(\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t), \tilde{\mathbb{P}})$ of our original filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ and a continuous Z-biased $\mathcal{F}$-progressive conditional PII martingale $\tilde{X}$ on this extension (see Definition

7.4 in chapter II in [JS03] for definition), such that $\tilde{X}^H$ converges stably (and hence weakly) to $\tilde{X}$ (see Definition 5.28 in chapter XIII in [JS03] for definition of stable convergence) for which

$$\begin{aligned}\langle \tilde{X}\rangle_t &= \xi_\gamma([0,t])\\ \langle \tilde{X},M\rangle_t &= 0\end{aligned}$$

for all continuous (bounded) martingales M with respect to the original filtration $\mathcal{F}_t$. From Proposition 7.5 and Definition 7.8 in Chapter 2 in [JS03], this means that

$$\tilde{X}_t = X'_t + \int_0^t u_s dW_s$$

where X' is an $\tilde{\mathcal{F}}_t$-local martingale and u is a predictable process on the original space $(\Omega,\mathcal{F},\mathbb{P})$. One such M is $M_t = W_{t\wedge\tau_b\wedge\tau_{-b}}$, where $\tau_b=\inf\{t: W_t=b\}$, so we have a pair of continuous local martingales (M,X) with

$$\langle \tilde{X},M\rangle_t = \langle \tilde{X},W\rangle_t = \int_0^t u_s ds = 0$$

for $t\le\tau_b\wedge\tau_{-b}$, so in fact $u_t\equiv 0$. Then applying F.Knight's Theorem 3.4.13 in [KS91] with $M^{(1)}=X$ and $M^{(2)}=W$, if $T_t=\inf\{s\ge 0:\langle X\rangle_s>t\}$, then X_{T_t} is a Brownian motion independent of W. Hence X has the same law as $B^{\perp}_{\xi_\gamma([0,t])}$ for any Brownian motion $B^\perp$ independent of W. ■

2.2 $H\to 0$ behaviour for the non re-scaled rough Bergomi model

If we replace the definition of Z^H with the usual RL process $Z_t^H=\sqrt{2H}\int_0^t(t-s)^{H-\frac{1}{2}}ds$ (as is usually done), then from Remark 2.4 in [FGS20], we know that $\xi_\gamma^H(A)$ tends Leb(A) in L^2 for any Borel set $A\subseteq[0,1]$, so adapting Theorem 2.1 for this case, we see that $\tilde{X}^H$ tends weakly to a standard Brownian motion, which means the rough Bergomi model tends weakly to the Black-Scholes model in the $H\to 0$ limit.

2.3 A closed-form expression for the skewness of $\tilde{X}_t^H$

In this subsection we compute an explicit expression for the skewness of $\tilde{X}_t^H$ (conditioned on its history), which (as a by-product) gives a more "hands-on" proof as to why the skew tends to zero as $H\to 0$, and also allows us to see how fast the skew decays and the H -value in $(0,\frac{1}{2})$ where the absolute value of the skew is maximized.

We first note that (trivially) $\tilde{X}^H$ has the same law as $\tilde{X}^H$ defined by

$$\begin{cases} d\tilde{X}_t^H = \sqrt{V_t^H}(\rho dB_t+\bar{\rho}dW_t)\,, \\ V_t^H \;= e^{\gamma Z_t^H-\frac{1}{2}\gamma^2\mathrm{Var}(Z_t^H)} \\ Z_t^H \;= \int_0^t(t-s)^{H-\frac{1}{2}}dB_t \end{cases} \tag{5}$$

where B is independent of W, and this is the version of the model we use in this subsection. We henceforth use $\mathbb{E}_t((.))$ as shorthand for the conditional expectation $\mathbb{E}((.)|\mathcal{F}_t^B)$, and we now replace the constant ρ with a time-dependent $\rho(t)$, and replace our original V_t^H process with

$$V_t^H = \xi_0(t)\,e^{\gamma Z_t^H-\frac{1}{2}\gamma^2\mathrm{Var}(Z_t^H)}$$

to incorporate a non-flat initial variance term structure.

Proposition 2.3

$$\mathbb{E}_{t_0}((\tilde{X}_T^H-\tilde{X}_{t_0}^H)^3) = 3\gamma\int_{t_0}^T\int_0^t \rho(s)\,\xi_{t_0}^{\frac{1}{2}}(s)\xi_{t_0}(t)\,e^{\frac{1}{2}\gamma^2\mathrm{Cov}_{t_0}(Z_s^HZ_t^H)-\frac{1}{8}\gamma^2\mathrm{Var}_{t_0}(Z_s^H)}(t-s)^{H-\frac{1}{2}}dsdt \tag{6}$$

where $\xi_{t_0}(t)=\xi_0(t)e^{\gamma\int_0^{t_0}(t-u)^{H-\frac{1}{2}}dB_u-\frac{\gamma^2}{4H}[t^{2H}-(t-t_0)^{2H}]}$. This simplifies to

$$\mathbb{E}((\tilde{X}_T^H)^3) = 3\rho\gamma V_0^{\frac{3}{2}}\int_0^T\int_0^t e^{\frac{1}{2}\gamma^2(R_H(s,t)-\frac{s^{2H}}{8H})}(t-s)^{H-\frac{1}{2}}dsdt < \infty \tag{7}$$

if $t_0=0$, ρ is constant and $\xi_0(t)=V_0$ for all t (i.e. flat initial variance term structure).

Proof. See Appendix B. ■

Remark 2.2 Using that $R_H(s,t) \to R^{\mathrm{fBM}}(s,t)$ as $s,t \to 0$ (for $H>0$ fixed), where $R^{\mathrm{fBM}}(s,t) = \frac{1}{2H}\frac{1}{2}(t^{2H}+s^{2H}-|t-s|^{2H})$ is the covariance function of $\frac{1}{\sqrt{2H}}W^H$ where W^H is a standard (one or two-sided) fractional Brownian motion, we find that the exponent in (7) behaves like $\frac{1}{16H}(s^{2H}+2t^{2H}-2(t-s))^{2H})$ for $s<t$ as $s,t\to 0$, and thus can effectively be ignored, so (for ρ constant)

$$\mathbb{E}((\tilde{X}^H_T)^3) \quad \sim \quad 3\rho\gamma V_0^{\frac{3}{2}} \int_0^T \int_0^t e^{\frac{1}{2}\gamma^2(R_H(s,t)-\frac{s^{2H}}{8H})}(t-s)^{H-\frac{1}{2}}dsdt \quad = \quad \frac{3\rho\gamma V_0^{\frac{3}{2}}}{(H+\frac{1}{2})(H+\frac{3}{2})}T^{H+\frac{3}{2}}$$

as $T\to 0$.

Remark 2.3 We have tested (7) against Monte Carlo estimates for $\mathbb{E}((\tilde{X}^H_T)^3)$ and the results are in very close agreement. Note that $\tilde{X}^H$ is driftless so (5) is only a toy model at the moment, but we easily adapt Proposition 2.3 and the two remarks above to incorporate the additional $-\frac{1}{2}\langle \tilde{X}^H\rangle_t$ drift term required to make $S_t = e^{\tilde{X}^H_t}$ a martingale. However, the relative contribution from this drift will disappear in the small-time limit, so we omit the tedious details, since rough stochastic volatility models are generally used (and considered more realistic) over small time horizons.

2.4 Convergence of the skew to zero

Corollary 2.4 *For $\gamma\in(0,1)$ and $0\le t\le T\le 1$, $\mathbb{E}_{t_0}((\tilde{X}^H_T-\tilde{X}^H_{t_0})^3)\to 0$ a.s. as $H\to 0$.*

Proof. See Appendix A. ■

2.5 Speed of convergence of the skew to zero

The following corollary quantifies the speed at which the skew goes to zero.

Proposition 2.5 *Let $\rho(.)$ be continuous and bounded away from zero with constant sign for t sufficiently small. Then*

$$-\lim_{H\to 0} H\log[\mathrm{sgn}(\rho)\mathbb{E}((\tilde{X}^H_T)^3)] \quad = \quad \hat{r}(\gamma) \quad = \quad \begin{cases} \frac{1}{16}\gamma^2 & 0\le\gamma\le 1\,, \\ \frac{1}{4}+\frac{1}{2}\log\gamma-\frac{3}{16} & \gamma\ge 1 \end{cases} \tag{8}$$

Proof. The proof involves many intermediate lemmas, see [Ger20] for details. ■

Remark 2.4 Note that $\hat{r}(\gamma)\le r(\gamma)\le\frac{1}{16}\gamma^2$, where $r(\gamma)=\frac{1}{16}\gamma^2 1_{\gamma^2\le 8}+\frac{1}{2}(1+\log\frac{\gamma^2}{8})1_{\gamma^2\ge 8}$ and $\hat{r}(\gamma)$ is negative for γ larger than the root at ≈ 1.61711, which makes the integral explode as $H\to 0$ for such values of γ.

2.6 Calibrating a time-dependent correlation function to the skew term structure

From (7) we see that

$$\frac{\partial}{\partial T}\mathbb{E}((\tilde{X}^H_T)^3) \quad = \quad V_0^{\frac{3}{2}}\int_0^T \rho(s)e^{\frac{1}{2}\gamma^2(R_H(s,T)-\frac{s^{2H}}{8H})}(T-s)^{H-\frac{1}{2}}ds\,. \tag{9}$$

If $g(T)=\frac{d}{dT}\mathbb{E}((\tilde{X}^H)^3_T)$ is known e.g. from the observed prices of call options in the market, this becomes a Volterra non-convolution integral equation of the first kind (also known as an Abel equation) for the unknown $\rho(.)$. Then from e.g. the first Theorem in [Atk74], we know that if $g(T)=T^\beta\tilde{g}(T)$ with $\alpha+\beta>0$ (where $\alpha=\frac{1}{2}-H$) and $\tilde{g}\in C^1[0,T]$, then (9) admits a unique solution of the form $\rho(t)=t^{\alpha+\beta-1}\tilde{f}(t)$ for some $\tilde{f}\in C^1[0,T]$. For our application, we have to then choose $\beta=H+\frac{1}{2}$ to ensure that $\rho(t)=O(1)$, and then choose V_0 to ensure that $|\rho(t)|\le 1$. If the calibrated $|\rho(t)|$ exceeds 1 for some $t>0$ for all choices of V_0 this means either the model is wrong or we need a different H-value. We could also solve for (9) numerically using an Adams scheme.

2.7 A $H=0$ model - pros and cons

Returning to Section 4.1, we can circumvent the problem of vanishing skew, by considering a toy model of the form

$$X_t \quad = \quad \sigma(\rho W_t+\bar{\rho}B^\perp_{\xi_\gamma([0,t])}) \tag{10}$$

where $\bar{\rho}=\sqrt{1-\rho^2}$, W and $\xi_\gamma([0,t])$ are defined as in Section 2.1 with $\gamma\in(0,1)$, and $B^\perp$ is a Brownian motion independent of W. Then from the tower property we see that

$$\mathbb{E}(e^{ikX_t}) \quad = \quad \mathbb{E}(\mathbb{E}(e^{ik(\alpha W_t+\beta B_{\xi_\gamma([0,t])})}|W)) \quad = \quad \mathbb{E}(e^{ik\alpha W_t-\frac{1}{2}k^2\beta^2\xi_\gamma([0,t])})$$

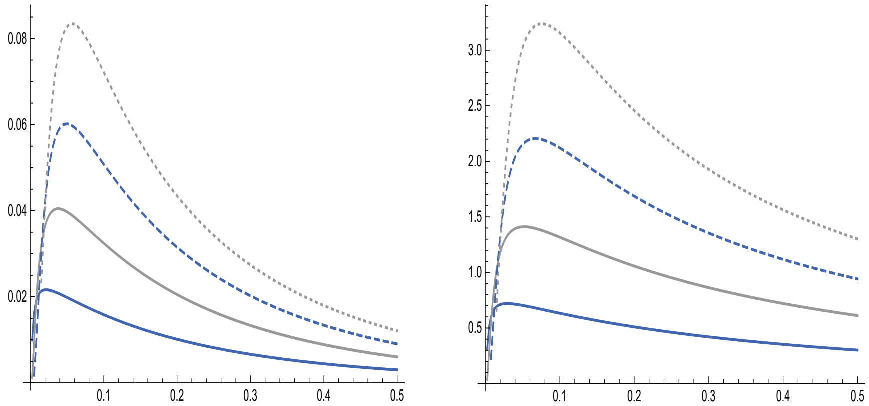

Figure 1: Here we plotted the (re-normalized) skew $\frac{1}{\rho}\mathbb{E}((\tilde{X}_T^H)^3)$ using (7) (which is independent of ρ) as a function of H for $T = .1$ (left) an $T = 1$ (right), for $\gamma = .2$ (blue), $\gamma = .4$ (grey), $\gamma = .6$ (blue dashed), and $\gamma = .8$ (grey dots). The numerical results indicate that there is a unique $H \in (0, \frac{1}{2})$ where the absolute value of the skewness is maximized.

and (from Corollary 2.3 in [FGS20]) we know that $\xi_\gamma([0,t]) \sim t\xi_\gamma([0,1])$ (i.e. self-similarity), so

$$\mathbb{E}(e^{\frac{ik}{\sqrt{t}}X_t}) \quad = \quad \mathbb{E}(e^{ik\alpha W_t/\sqrt{t}-\frac{1}{2}k^2\beta^2\xi_\gamma([0,t])/t}) \quad = \quad \mathbb{E}(e^{ik\alpha W_1-\frac{1}{2}k^2\beta^2\xi_\gamma([0,1])})$$

so X is self-similar with $X_t/\sqrt{t} \sim X_1$ for all $t > 0$, and X_1 (and hence X_t) has non-zero skewness for $\alpha \neq 0$; more specifically

$$\mathbb{E}((\frac{X_t}{\sqrt{t}})^3) \quad = \quad 4\sigma^3\rho(1-\rho^2)\gamma \tag{11}$$

and $\mathbb{E}(X_1^2) = \sigma^2$, and we can derive a similar (slightly more involved) expression for $\mathbb{E}(X_1^4)$. We note that the skewness (11) is minimized (resp. maximized) at $\rho^*_\pm = \pm\frac{1}{\sqrt{3}} \approx \pm 0.577$ (this does not imply that the density of X_t is symmetric when $\rho = \pm 1$ even though the skewness is zero in this case). The ρ component achieves the goal of a $H = 0$ model with non-zero skewness, and following a similar argument to Lemma 5 in [MT16] one can establish the following small-time behaviour for European put options in the Edgeworth Central limit theorem regime:

$$\frac{1}{\sqrt{t}}\mathbb{E}((e^{x\sqrt{t}} - e^{X_t})^+) \quad \sim \quad e^{x\sqrt{t}}\,\mathbb{E}((x - \frac{X_t}{\sqrt{t}})^+) \quad \sim \quad \mathbb{E}((x - \frac{X_t}{\sqrt{t}})^+) \quad \sim \quad \mathbb{E}((x - \bar{X}_1)^+)$$

and $\lim_{t\to 0}\hat{\sigma}_t(x\sqrt{t}, t) = C_B(x, .)^{-1}(C(x))$ for $x > 0$, where $\hat{\sigma}_t(x, t)$ denotes the implied volatility of a European call option with strike $e^{x\sqrt{t}}$ maturity t and $S_0 = 1$ ($C_B(x, \sigma)$ is the Bachelier model call price formula). Hence we see the full smile effect in the small-time FX options Edgeworth regime unlike the $H > 0$ case discussed in e.g. [Fuk17], [EFGR19], [FSV19], where the leading order term is just Black-Scholes, followed by a next order skew term, followed by an even higher order convexity term.

We can go from a toy model to a real model adding back the usual $-\frac{1}{2}\langle X\rangle_t$ drift term for the log stock price X so $S_t = e^{X_t}$ is a martingale, and in this case we lose self-similarity for X but $X_t/\sqrt{t}$ still tends weakly to a non-Gaussian random variable, and in particular $\lim_{t\to 0}\mathbb{E}((\frac{X_t}{\sqrt{t}})^3) = 4\sigma^3\rho\bar{\rho}^2\gamma$. [2]. This model overcomes two of the main drawbacks of the original Bacry et al.[BDM01] multifractal random walk (i.e. a Brownian motion evaluated at an independent MRW $M_\gamma^T([0, (.)))$, namely zero skewness and unrealistic small-time behaviour.

However, the property in (11) does not appear to be time-consistent, since if we define $\eta_t^h := \mathbb{E}((\frac{X_{t+h}-X_t}{\sqrt{h}})^3|\mathcal{F}_t)$ for $t > 0$, then $\mathbb{E}((\eta_t^h)^2) = O(h^{-\gamma^2})$ (and not $O(1)$ as we would want), so we do not pursue this model further at the present time.

Remark 2.5 It is interesting to compare (11) against the small-time behaviour for a local vol model with $\sigma(S) = \sigma_- 1_{S<S_0} + \sigma_+ 1_{S\geq S_0}$, for which Pigato[Pig19] shows that the short-time implied volatility skew behaves like

$$\frac{\sqrt{\pi}}{\sqrt{2}}\frac{\sigma_+ - \sigma_-}{\sigma_+ - \sigma_-}\frac{1}{\sqrt{t}}$$

as $t \to 0$, which attains the model-free upper bound in [Fuk10] in the limit as $\sigma_- \to 0$. Nevertheless, the extreme skew seen in [Pig19] will not hold as soon as S moves away from S_0, so this property is also not time-consistent (see also Remarks 4.3 and 4.4 in [Fuk20].

[2]We can also replace the ρW_t component of X with a second rBergomi component with a non-zero H-value, and derive similar results

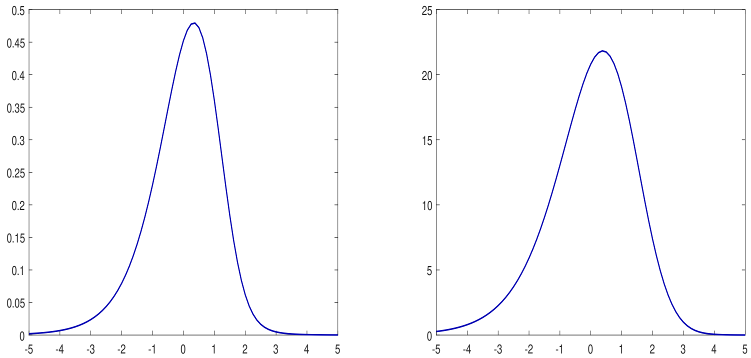

Figure 2: Here we have plotted the density of X_1 for the $H = 0$ model with skew model in (10) , for $\rho = \rho^*_- = -\frac{1}{\sqrt{3}}$ (left) and $\rho = -1$ (right) for $\gamma = .5$, using the Karhunen-Loève expansion described in [FGS20].

3 Non-Gaussian rough models, non-Gaussian multifractal exponents and the $H \to 0$ limit

We now let P be an independently scattered infinitely divisible random measure (see [BM03] for details) with

$$\mathbb{E}(e^{iqP(A)}) \;=\; e^{\varphi(q)\mu(A)}$$

for $q \in \mathbb{R}$, where $\mu(du, dw) = \frac{1}{w^2} dwdu$ denotes the Haar measure, and φ is the characteristic exponent of an infinitely divisible distribution. Let $A^H_t := \{0 \le u \le t,\, w \ge g_H(u,t)\}$ for some family of functions which satisfy the following condition:

Condition 1 *$g_H(.,t) \ge 0$ with $g_H(u,t)$ increasing in t and H.*

We define the process

$$\omega^H_t \;=\; P(A^H_t)$$

with filtration $\mathcal{F}_H = \sigma(P(A \times B) : B \subseteq [H, \infty], A, B \in \mathcal{B}(\mathbb{R}))$ (compare to a similar filtration on page 17 in [RV10]). Then

$$\begin{aligned}
\mathbb{E}(e^{iq_1\omega^H_s + iq_2\omega^H_t}) &= \mathbb{E}(e^{iq_1P(A^H_s)+iq_2P(A^H_t)}) \\
&= \mathbb{E}(e^{iq_1P(A^H_s\cap A^H_t)+iq_1P(A^H_s\cap(A^H_t)^c)+iq_2P(A^H_t\cap A^H_s)+iq_2P(A^H_t\cap(A^H_s)^c)}) \\
&= e^{\varphi(q_1)\mu(A^H_s\cap(A^H_t)^c)+\varphi(q_1+q_2)\mu(A^H_s\cap A^H_t)+\varphi(q_2)P(A^H_t\cap A^c_s)}) .
\end{aligned}$$

Similarly

$$\begin{aligned}
\mathbb{E}(e^{iq_1\omega^H_t + iq_2\omega^{H_2}_t}) &= \mathbb{E}(e^{iq_1P(A^H_t)+iq_2P(A^{H_2}_t)}) \\
&= \mathbb{E}(e^{iq_1P(A^H_t\cap A^{H_2}_t)+iq_1P(A^H_t\cap(A^{H_2}_t)^c)+iq_2P(A^{H_2}_t\cap A^H_t)+iq_2P(A^{H_2}_t\cap(A^H_t)^c)}) \\
&= e^{\varphi(q_1)\mu(A^H_t\cap(A^{H_2}_t)^c)+\varphi(q_1+q_2)\mu(A^H_t\cap A^H_t)+\varphi(q_2)P(A^H_t\cap(A^H_t)^c)}) .
\end{aligned}$$

Differentiating once in q_1 and once in q_2 and then setting $q_1 = q_2 = 0$ (and assuming $\varphi'(0) = 0$ so $\mathbb{E}(P(A)) = 0$ for all A) we find that

$$\begin{aligned}
\mathbb{E}(\omega^H_s\omega^H_t) &= \gamma^2\mu(A^H_s \cap A^H_t) \\
\mathbb{E}(\omega^H_t\omega^{H_2}_t) &= \gamma^2\mu(A^H_t \cap A^{H_2}_t)
\end{aligned}$$

where $\gamma^2 = \varphi''(0)$, and we can re-write the covariance of ω^H as

$$\mathbb{E}(\omega^H_s\omega^H_t) \;=\; \gamma^2\int_0^s\int_{g_H(u,t)}^{\infty}\frac{1}{w^2}dwdu \;=\; \gamma^2\int_0^s\frac{1}{g_H(u,t)}du$$

for $0 \le s \le t$. We now want to construct the g function so $\mathbb{E}(\omega^H_s\omega^H_t)$ matches the covariance $K_H(s,t)$ of some given family of Gaussian process $\tilde{Z}^H$ with $\tilde{Z}^H_0 = 0$ (e.g. the family of Riemann-Liouville processes indexed by H). Differentiating with respect to s, we see this is accomplished if

$$\frac{1}{g_H(s,t)} \;=\; R_s(s,t)\,.$$

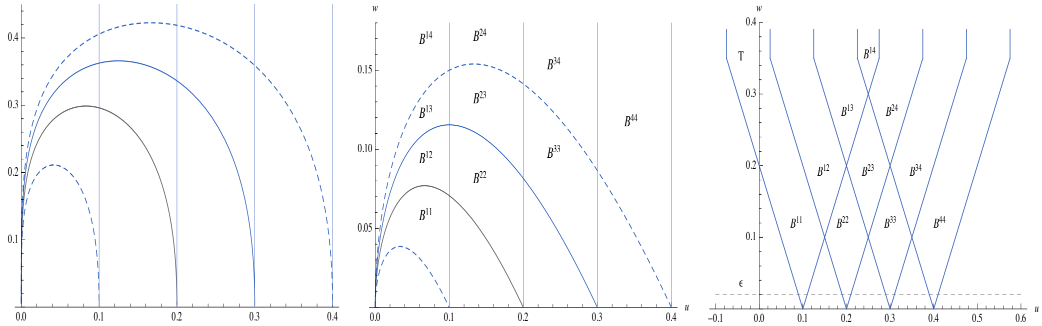

Figure 3: Here we have plotted $g_H(s,t)$ for different t values for the RL process/field with $H = .25$ (left) and $H = 0$ (middle). The middle and right plots show the corresponding $B_n^{i,j}$ sets using our method (middle) and the [BM03] construction with truncated triangles (right). Note we have dropped the n subscript when labelling the sets $B_n^{i,j}$, and $n = 4$.

We further assume that $K_H(s,t) \sim \log \frac{1}{|t-s|} + h(s,t)$ as $H \to 0$ from above, for some function h bounded away from zero, so as to ensure interesting GMC-type behaviour for the measure $\frac{e^{\omega_t^H}}{\mathbb{E}(e^{\omega_t^H})}dt$ as $H \to 0$.

We can do this explicitly for the re-scaled Riemann-Liouville process $\tilde{Z}_t^H = \gamma \int_0^t (t-s)^{H-\frac{1}{2}} dW_s$ to obtain

$$g_H(s,t) \;=\; \frac{1}{\gamma} \frac{2 s^{\frac{1}{2}-H} t^{\frac{3}{2}-H}}{\Gamma(\frac{1}{2}+H)(t(1+2H)\,{}_2F_1(1,\frac{1}{2}-H,\frac{3}{2}+H,\frac{s}{t}) + s(1-2H)\,{}_2F_1(2,\frac{3}{2}-H,\frac{5}{2}+H,\frac{s}{t}))}$$

where ${}_2F_1(a,b,c,z)$ is the regularized hypergeometric function[3] (see first two plots in Figure 3), and in Appendix C we verify that Condition 1.1 in (1) is satisfied. For $H = 0$ we get

$$g_0(s,t) \;=\; \frac{\sqrt{s}\,(t-s)}{\sqrt{t}}\,.$$

Now let $\psi(z) = \varphi(-iz)$ as in [BM03] and

$$\xi_\varphi^H(dt) \;=\; \frac{e^{\omega_t^H}}{\mathbb{E}(e^{\omega_t^H})}dt\,.$$

For $H_2 < H_1$, $\omega_t^{H_2} - \omega_t^{H_1} = P(A_t^{H_2} \setminus A_t^{H_1})$ and $\omega_t^H = P(A_t^H)$ are independent for any $H \geq H_1$, so $\omega_t^{H_2}$ is an $\mathcal{F}_H$-martingale since we imposed that $\varphi'(0) = 0$ so $\mathbb{E}(P(A)) = 0$ for all A, and from this one can easily verify that $\xi_\varphi^H(A)$ is an $\mathcal{F}_H$-martingale for any Borel set A.

Remark 3.1 Note that although

$$\mathbb{E}(\omega_s^H \omega_t^H) \;=\; \mathbb{E}(\tilde{Z}_s^H \tilde{Z}_t^H)$$

this does not imply that $\mathbb{E}(\omega_s^H \omega_t^{H_2}) = \mathbb{E}(\tilde{Z}_s^H \tilde{Z}_t^{H_2})$ for $H \neq H_2$.

3.1 Computing the $\mathbb{E}(\xi_\varphi[0,T]^q)$ for $q \in \mathbb{N}$ and local multifractality with non-Gaussian multifractal exponent

Let $\bar{\omega}_t^H = \omega_t^H - \mathbb{E}(\omega_t^H)$, and define $I_n(...)$ as follows

$$\mathbb{E}(e^{\sum_{i=1}^n \bar{\omega}_{t_i}^H}) \;=\; e^{I_n^H(t_1,...,t_n)}$$

for $0 = t_0 < t_1 < ... < t_n$. Then from e.g. the middle plot in Figure 3 we see that

$$I_n^H(t_1,...,t_n) \;=\; \sum_{i=1}^n \sum_{j=i}^n \psi(j)\mu(B_n^{i,j}) - \sum_{i=1}^n \psi(1) K_H(u_i,u_i)$$

where the final sum here comes from the $\mathbb{E}(\omega_t^H)$ term in the definition of $\bar{\omega}_t^H$. Let

$$B_n^{i,j} \;=\; \{(u,w) : t_{i-1} \leq u \leq t_i,\, g_H(u,t_j) \leq w \leq g_H(u,t_{j+1})\}\,.$$

[3] we are using Mathematica's definition here

for $i = 1..n, j = i..n$, where t_{n+1} is understood to be $+\infty$. Then

$$\begin{aligned}\mu(B_n^{i,j}) &= \int_{t_{i-1}}^{t_i}\int_{g_H(u,t_j)}^{g_H(u,t_{j+1})}\frac{1}{w^2}dwdu &= \mu(\mathcal{A}^{i,j}\setminus\mathcal{A}^{i-1,j}\setminus(\mathcal{A}^{i,j+1}\setminus\mathcal{A}^{i-1,j+1}))\\ &&= \int_{t_{i-1}}^{t_i}(\frac{1}{g_H(u,t_j)}-\frac{1}{g_H(u,t_{j+1})})du\\ &&= K_H(t_i,t_j)-K_H(t_{i-1},t_j)-(K_H(t_i,t_{j+1})-K_H(t_{i-1},t_{j+1}))\end{aligned}$$

for $i = 1..n$ and $j = i..n-1$, and for $j = n$, $\mu(B_n^{i,j}) = \int_{t_{i-1}}^{t_i}\frac{1}{g_H(u,t_j)}du = K_H(t_i,t_j) - K_H(t_{i-1},t_j)$, where $\mathcal{A}_{i,j} = A_{t_i}^H \cap A_{t_j}^H$. Thus we see that

$$I_n^H(t_1,...,t_n) = \sum_{i=1}^n\sum_{j=i}^n\psi(j)(K_H(t_i,t_j)-K_H(t_{i-1},t_j)-K_H(t_i,t_{j+1})+K_H(t_{i-1},t_{j+1})) - \sum_{i=1}^n\psi(i)K_H(u_i,u_i)$$

and note that all diagonal terms cancel.

By visual inspection of the right plot in Figure 3, we see we get the exact same formula for $\mu(B_n^{i,j})$ for the [BM03] construction as well if we now *define* $B_n^{i,j}$ as

$$B_n^{i,j} = \mathcal{A}_{i,j}\setminus\mathcal{A}_{i-1,j}\setminus(\mathcal{A}_{i,j+1}\setminus\mathcal{A}_{i-1,j+1})$$

with $K_H(.,.)$ replaced by $K_\epsilon^T(.,.)$, as defined in section 2.4 in [FGS20]. Note $K_H(t_i,t_j) = 0$ if t_i or $t_j = 0$, but this is not the case for K_ϵ^T.

For the special cases $n = 1,2,3,4$ we have

$$\begin{aligned}I^H(t_1) &= 0\\ I_2^H(t_1,t_2) &= (\psi(2)-2\psi(1))K_H(t_1,t_2)\\ I_3^H(t_1,t_2,t_3) &= (\psi(2)-2\psi(1))(K_H(t_1,t_2)+K_H(t_2,t_3)) + (\psi(3)-2\psi(2)+\psi(1))K_H(t_1,t_3)\\ I_4^H(t_1,t_2,t_3,t_4) &= (\psi(2)-2\psi(1))(K_H(t_1,t_2)+K_H(t_2,t_3)+K_H(t_3,t_4))\\ &+ (\psi(3)-2\psi(2)+\psi(1))(K_H(t_1,t_3)+K_H(t_2,t_4))\\ &+ (\psi(4)-2\psi(3)+\psi(2))K_H(t_1,t_4)\end{aligned}$$

and (noting that $\psi(0) = 0$), the general formula is given by

$$I_n^H(t_1,...,t_n) = \sum_{i=1}^{n-1}(\psi(i+1)-2\psi(i)+\psi(i-1))\sum_{j=1}^{n-i}K_H(t_j,t_{j+i}) \tag{12}$$

and note that $\psi(i+1)-2\psi(i)+\psi(i-1) \geq 0$ since ψ is convex (see e.g. Lemma 2.2.31 in [DZ98]). For the log-normal case $\psi(p) = \frac{1}{2}\gamma^2p^2$, the general formula simplifies to the well known Selberg formula

$$I_n^H(t_1,t_2,..,t_n) = \gamma^2\sum_{i=1}^n\sum_{j=1}^{i-1}K_H(t_i,t_j)$$

(see also section 2.4 in [FGS20]). Returning to the general case now but specializing to the case when $K_H(s,t) = R_H(s,t)$, we see that

$$\mathbb{E}(\xi_\varphi^H([0,T]^n)) = n!\int_0^T\int_0^{t_1}...\int_0^{t_{n-1}}e^{I_n(t_1,...,t_n)}dt_1...dt_n\,. \tag{13}$$

Theorem 3.1 *For any integer $q \in [1,q^*]$ (where q^* is the largest q-value for which $\tilde{\psi}(q) > 1$ as in [BM03]) and any interval $I \subseteq [0,1]$, $\xi_\varphi^H(I)$ tends to some non-negative random variable $\xi_\varphi(I)$ as $H \to 0$ a.s. and in L^q, and $\mathbb{E}[\xi_\varphi^H(I)^q] \to \mathbb{E}[\xi_\varphi(I)^q]$.*

Proof. From the upper bound part of the sandwich equation Eq 26 in [FGS20], we have the following inequality for $0 < s < t < 1$:

$$R_H(s,t) \leq K_{l^*(H)}^\theta(s,t)$$

where $\theta = 4\cdot\sup(I)$ and $K_l^T(s,t)$ is the covariance of the model in [BM03], and $l^*(H) \downarrow 0$ as $H \downarrow 0$. Combining this with (12) and (13) with $R_H(s,t) = K_H(s,t)$ and using that all coefficients of K_H in (12) are non-negative, we see that for integer $q \in [1,q^*)$

$$\mathbb{E}[\xi_\varphi^H(I)^q] \leq \mathbb{E}[M_{l^*(H)}^\theta(I)^q] \tag{14}$$

where M_l^T is the [BM03] regularized GMC measure as defined as in Eq 17 in [FGS20], and note we are not using Kahane's inequality here since for integer $q \in [1, q^*)$ we have a Selberg-type formula as in (13) for both cases being compared, and Kahane's inequality is only known to hold for the Gaussian case. Moreover, from Lemma 3 in [BM03] we know that

$$\sup_{l>0} \mathbb{E}[M_l^\theta(I)^q] \quad < \quad \infty$$

for $q \in [1, q^*)$, so we have the uniform bound $\sup_{H>0} \mathbb{E}[\xi_\varphi^H(I)^q] < \infty$. We know from the end of the previous subsection that $\xi_\varphi^H(I)$ is an $\mathcal{F}^H$-martingale. Then (by Doob's martingale convergence theorem for continuous martingales) $\xi_\varphi^H(I)$ tends to some random variable which we call $\xi_\varphi(I)$ as $H \to 0$ a.s. and in L^q for integer $q \in [1, q^*)$. Moreover, from the reverse triangle inequality, the aforementioned L^q-convergence implies that

$$\mathbb{E}(\xi_\varphi^H(I)^q) \quad \to \quad \mathbb{E}(\xi_\varphi(I)^q)$$

as $H \to 0$, for integer $q \in [1, q^*)$. $\blacksquare$

Corollary 3.2 *We can then mimick the remaining arguments in Theorem 2.2 in [FGS20] to show that there exists a random measure ξ_φ such that $\xi_\varphi(A) = \xi_{\varphi,A}$ a.s. for any Borel set A.*

(13) combined with the upper and lower bounds on $R_H(,,)$ in Eq 26 in [FGS20] implies that ξ_φ is locally multifractal for integer $q \in [1, q^*)$ with the same multifractal exponent as the [BM03] MRM, namely $\zeta(q) = q - \psi(q)$, i.e.

$$-\lim_{\delta \to 0} \frac{\log \mathbb{E}(\xi_\varphi([t, t+\delta])^q)}{\log \delta} \quad = \quad \zeta(q)$$

for $t > 0$.

Note that the multifractality for the [BM03] MRM is exact, whereas we only have local multifractality for ξ_φ, but ξ_φ has the advantage that it is built from the family of Gaussian processes ω_t^H which have zero boundary condition which is much more natural in a financial modelling context; the [BM03] MRM only makes sense financially if we condition on the prior history and use the prediction formula obtained in [FS20] which is rather cumbersome to apply in practice.

3.2 A simple multifractal model with zero boundary condition

If we instead let $A_t^H := A_t = \{0 \vee t - T \le u \le t, t - u \le w \le T\}$, then we find that $\mathbb{E}(\omega_s \omega_t) = \log \frac{1}{|t-s|} + \log(s \vee t)$ for $0 \le s < t \le T$ and $\mathbb{E}(\omega_s \omega_t) = \log^+ \frac{T}{|t-s|}$ for $t \ge T$, where we have dropped the dependence on H here since clearly A_t does not depend on H here. This simple model has the advantage that $\omega_0 = 0$ but mimicks the behaviour of the usual Bacry-Muzy log-correlated Gaussian field for $t \ge T$.

3.3 Non-Gaussian generalized rough Bergomi models

We can now consider a non-Gaussian extension of the rough Bergomi model

$$\begin{cases} d\tilde{X}_t^H = \sqrt{V_t^H} dW_t\,, \\ V_t^H = \frac{e^{\omega_t^H + \gamma Z_t^H}}{\mathbb{E}(e^{\omega_t^H + \gamma Z_t^H})} \\ Z_t^H = \int_0^t (t-s)^{H-\frac{1}{2}} (\rho dW_s + \bar{\rho} dW_t^\perp) \end{cases}$$

with $\tilde{X}_0^H = 0$, $|\rho| \le 1$ and W, $W^\perp$ are independent Brownian motions, and ω^H independent of both Brownian motions, and the K_H associated with ω^H not necessarily equal to the covariance R_H of the RL process (but still satisfying the zero boundary condition $K_H(0,t) = 0$) for all t. Then by trivial modifications of the argument above, we can define a GMC as the a.s. limit ξ_φ of the measure $V_t^H dt$ as $H \to 0$.

We can then also consider the following toy model

$$X_t \quad = \quad \sigma(\rho W_t + \bar{\rho} B_{\xi_\varphi^H([0,t])})$$

where B is an Brownian independent of P and W. Then this model will have skew in $H \to 0$ limit, but (unlike the model in (10)) will not be self-similar in general since $\xi_\varphi^H([0,t])$ is not self-similar for general φ.

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A Proof of Corollary 2.4

For $T \leq 1$, using that $R_H(s,t) \uparrow R(s,t)$ and $(t-s)^{H-\frac{1}{2}} \uparrow (t-s)^{-\frac{1}{2}}$ we see that

$$\begin{aligned}
|\mathbb{E}_{t_0}((\tilde{X}^H_T - \tilde{X}^H_{t_0})^3)| &\leq 3|\rho|\gamma \int_0^T \int_0^t \xi^{\frac{1}{2}}_{t_0}(s)\xi_{t_0}(t) e^{\frac{1}{2}\gamma^2(R_{t_0}(s,t) - \frac{s^{2H}}{8H})}(t-s)^{-\frac{1}{2}} ds dt \\
&\leq 3|\rho|\gamma \int_0^T \int_0^t \xi^{\frac{1}{2}}_{t_0}(s)\xi_{t_0}(t) e^{\frac{1}{2}\gamma^2(R(s,t) - \frac{s^{2H}}{8H}) - \frac{1}{2}\log(t-s)} ds dt \\
&\leq 3\bar{\xi}^{\frac{1}{2}}_{t_0}(s)\bar{\xi}_{t_0}(t)\, |\rho|\gamma \int_0^T \int_0^t e^{\frac{1}{2}(1+\gamma^2)\log\frac{1}{t-s} + \frac{1}{2}\gamma^2 \bar{g}} ds dt \\
&\leq \mathit{const.} \times \mathbb{E}(M_{\sqrt{\frac{1}{2}(1+\gamma^2)}}([0,T])^2) \quad < \quad \infty
\end{aligned}$$

for $\gamma \in (0,1)$ where $M_\gamma(dt)$ is the usual [BM03] GMC, and $R_0(s,t) = \mathbb{E}_{t_0}(Z_s Z_s) = \int_{t_0}^s (s-u)^{-\frac{1}{2}}(t-u)^{-\frac{1}{2}} du ds$, $\bar{g} = 2\log(2\sqrt{2})$, $\bar{\xi}_t = \sup_{0 \leq s \leq t} \xi_s$. The result then follows from the dominated convergence theorem.

B Proof of Proposition 2.3

We first recall that for any continuous martingale M, using Ito's lemma and integrating by parts we know that $\mathbb{E}(M_t^3) = 3\mathbb{E}(\int_0^t M_s d\langle M\rangle_s) = 3\mathbb{E}(M_t \langle M\rangle_t)$. Thus we see that

$$\begin{aligned}
&\mathbb{E}_{t_0}((X^H_T - X^H_{t_0})^3) \\
= \;& 3\mathbb{E}_{t_0}((X^H_T - X^H_{t_0})(\langle X^H_T\rangle - \langle X^H_{t_0}\rangle)) \\
= \;& 3\mathbb{E}_{t_0}(\int_{t_0}^T \rho(s)\sqrt{V^H_s}\, dB_s \cdot \int_{t_0}^T V^H_t dt) \\
= \;& 3\mathbb{E}_{t_0}(\int_{t_0}^T \rho(s)\xi^{\frac{1}{2}}_{t_0}(s)\, e^{\frac{1}{2}\gamma\int_{t_0}^s (s-u)^{H-\frac{1}{2}} dB_u - \frac{1}{2}\cdot\frac{1}{2}\gamma^2 \int_{t_0}^s (s-u)^{2H-1} du}\, dB_s \cdot \int_{t_0}^T \xi_{t_0}(t)\, e^{\gamma\int_{t_0}^t (t-u)^{H-\frac{1}{2}} dB_u - \frac{1}{2}\gamma^2 \int_{t_0}^t (t-u)^{2H-1} du} dt)\,.
\end{aligned}$$

So we (formally) need to compute

$$\begin{aligned}
\delta I &= \mathbb{E}_{t_0}(e^{\frac{1}{2}\gamma\int_{t_0}^s (s-u)^{H-\frac{1}{2}} dB_u - \frac{1}{2}\cdot\frac{1}{2}\gamma^2 \int_{t_0}^s (s-u)^{2H-1} 1_{s<t} du} dB_s \cdot e^{\gamma\int_{t_0}^t (t-u)^{H-\frac{1}{2}} dB_u - \frac{1}{2}\gamma^2 \int_{t_0}^t (t-u)^{2H-1} du}) \\
&= \mathbb{E}_{t_0}(e^{\gamma\int_{t_0}^t (t-u)^{H-\frac{1}{2}} dB_u + \frac{1}{2}\gamma\int_{t_0}^s (s-u)^{H-\frac{1}{2}} dB_u - (...)} dB_s)
\end{aligned}$$

where (...) refers to the non-random terms. To this end, let $X = \gamma\int_{t_0}^t (t-u)^{H-\frac{1}{2}} dB_u + \frac{1}{2}\gamma\int_{t_0}^s (s-u)^{H-\frac{1}{2}} dB_u$ and $Y = dB_s$. Then $\mathbb{E}(XY) = \gamma(t-s)^{H-\frac{1}{2}} ds\, 1_{s<t}$ (since formally $\mathbb{E}(\frac{1}{2}\gamma\int_{t_0}^s (s-u)^{H-\frac{1}{2}} dB_u \cdot dB_s) = 0$, see end of proof for discussion on how to make this argument rigorous) and

$$\begin{aligned}
\mathbb{E}(Ye^X) &= e^{\frac{1}{2}\mathbb{E}(X^2)}\mathbb{E}(XY) \quad = \quad e^{\frac{1}{2}V_H(s,t)}\gamma(t-s)^{H-\frac{1}{2}} ds\, 1_{s<t} \\
\Rightarrow \quad \delta I &= e^{-\frac{1}{2}\gamma^2\int_{t_0}^t (t-u)^{2H-1} du - \frac{1}{2}\cdot\frac{1}{2}\gamma^2\int_{t_0}^s (s-u)^{2H-1} 1_{s<t} du}\, e^{\frac{1}{2}V_H(s,t)}\gamma(t-s)^{H-\frac{1}{2}} ds\, 1_{s<t}
\end{aligned}$$

where $V_H(s,t) = \gamma^2\int_{t_0}^t [(t-u)^{H-\frac{1}{2}} + \frac{1}{2}(s-u)^{H-\frac{1}{2}} 1_{s<t}]^2 du$. Cancelling terms in the exponent, we see that δI simplifies to

$$\begin{aligned}
\delta I &= e^{\frac{1}{2}\gamma^2\int_{t_0}^t (s-u)^{H-\frac{1}{2}}(t-u)^{H-\frac{1}{2}} du - \frac{1}{8}\gamma^2\int_{t_0}^t (s-u)^{2H-1} 1_{s<t} du)}(t-s)^{H-\frac{1}{2}} ds\, 1_{s<t} \\
&= e^{\frac{1}{2}\gamma^2 \mathrm{Cov}_{t_0}(Z^H_s Z^H_t) - \frac{1}{8}\mathrm{Var}_{t_0}(Z^H_s))}(t-s)^{H-\frac{1}{2}} ds\, 1_{s<t}
\end{aligned}$$

Then

$$\mathbb{E}_{t_0}((X^H_T - X^H_{t_0})^3) \quad = \quad 3\mathbb{E}_{t_0}\int_{t_0}^T \int_{t_0}^T \rho(s)\xi^{\frac{1}{2}}_{t_0}(s)\, \xi_{t_0}(t)\delta I dt$$

and (6) and (7) follow. Finally we recall that a general stochastic integral $\int_0^t \phi_s dM_s$ with respect to a martingale M is defined as an L^2- limit of $\int_0^t \phi_{\frac{1}{n}[ns]} dM_s$; using this construction we can rigourize the formal argument above with δI (we omit the tedious details for the sake of brevity).

C Monotonicity properties of $g_H(s,t)$ for the Riemann-Liouville case

The covariance of the RL process for $s < t < 1$ is

$$R(s,t) \quad = \quad \int_0^s (s-u)^{H-\frac{1}{2}}(t-u)^{H-\frac{1}{2}} du \quad = \quad \int_0^s u^{H-\frac{1}{2}}(t-s+u)^{H-\frac{1}{2}} du\,. \tag{C-1}$$

Differentiating this expression using the Leibniz rule we see that

$$R_s(s,t) \quad = \quad s^{H-\frac{1}{2}}t^{H-\frac{1}{2}} + (\frac{1}{2}-H)\int_0^s u^{H-\frac{1}{2}}(t-s+u)^{H-\frac{3}{2}}du$$

and recall that $g_H(s,t) = \frac{1}{R_s(s,t)}$. Then we can infer monotonicity properties of g from R_s:

- By inspection R_s is a decreasing function of t, so g is increasing in t.
- For $0 < s < t$, $(t-s+u)^{H-\frac{1}{2}}$ is a smooth function of u on $[0,s]$ so the integral term in our expression for R_s is finite $\forall t > 0$. Thus $R_s(s,t)$ tends to $+\infty$ as $s \to 0$ so $g_H(0,t) = 0$ for $t > 0$.
- For $s = t > 0$ the first term in (3) is finite but the integral diverges, so we also have $g_H(t,t) = 0$.
- For $s,t \in (0,1]^2$, $(st)^{H-\frac{1}{2}}$, $\frac{1}{2}-H$ and $u^{H-\frac{1}{2}}(t-s+u)^{H-\frac{3}{2}}$ are non-negative and decreasing in H, so $g_H(s,t)$ is increasing in H.
- By inspection, $g_H(s,t)$ is continuous for $s \in [0,t]$, and performing a Taylor series expansion of $\frac{\partial}{\partial s}g_H(s,t)(s,t)$ we can show that $\frac{\partial}{\partial s}g_H(s,t) \to -\infty$ as $s \searrow 0$ and $s \nearrow t$.

These properties can be seen in Figure 3.