VIX derivatives valuation and estimation based on closed-form series expansions

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Abstract

We propose a new methodology to evaluate VIX derivatives. The approach is based on a closed-form Hermite series expansion, and can be applied to general stochastic volatility models. We exemplify the proposed method using the Heston model, the mean-reverting CEV model and the 3/2 model. Numerical results show that the proposed method is accurate and efficient.

Keywords: VIX derivatives; Hermite series; Stochastic volatility; Heston model; Mean-reverting CEV model; 3/2 model

AMS subject classifications: 91G80, 93E11, 93E20

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1 Introduction

The volatility index (VIX), published by the Chicago Board of Options Exchange (CBOE), measures the market expectation of the 30-day volatility implied by at-the-money S&P 500 Index option prices. Changes in VIX are negatively correlated with changes in the stock prices (usually termed as the *leverage effect*, see Florescu and Pãsãricã (2009)). Thus, the VIX index has historically served as a risk indicator sometimes termed as the "fear/gauge index" (see detailed history and explanation in Carr (2017) and references therein). On March 26th, 2004, the CBOE Futures Exchange (CFE) launched VIX futures[1], and shortly after, on February 24th, 2006, CBOE introduced options[2] written on the VIX to satisfy the market participants' demand for hedging their positions. During 2017, the average daily traded volume of VIX call and put options is around 4.3 million contracts[3].

These VIX derivatives provide a way for investors to gain direct exposure to the market volatility in addition to the indirect exposure provided by the options on the S&P500. Furthermore, they allow volatility to be treated as an asset class (see Mencia and Sentana (2013)). Using VIX derivatives has proven effective for portfolio management as well as in constructing variable annuity in insurance. The empirical study carried by Szado (2009) shows that investing in VIX and its derivatives reduces the downside risk for a typical large investment portfolio during periods of financial distress. Linking VIX to the fee structure of a variable annuity reduces the probability of default during market downturns. This was shown empirically in a CBOE white

[1] http://www.cboe.com/products/vix-index-volatility/vix-options-and-futures/vix-futures

[2] http://www.cboe.com/products/vix-index-volatility/vix-options-and-futures/vix-options

[3] http://www.cboe.com/data/historical-options-data/volume-put-call-ratios. Please note that one contract contains 100 options.

paper[4] as well as in academic studies such as Bernard et al. (2016); Cui et al. (2017); Kouritzin and MacKay (2017).

To properly use VIX derivatives, practitioners require suitable models incorporating the observed features of the S&P 500 index underlying VIX. The valuation models developed in the current literature can be broadly categorized into two classes. The first class directly models the VIX index as a stochastic process isolated from the dynamics of the underlying S&P500. Representative references include Mencia and Sentana (2013), Goard and Mazur (2013), etc. The second class of models typically uses a two-dimensional stochastic volatility process to model the value of the S&P 500 index, and the VIX derivatives are priced within this framework.

In the current work, we follow the second class of models and consider the problem of pricing VIX derivatives (both VIX futures and VIX options) when the underlying S&P500 index is modeled using a stochastic volatility process.

Consider the following general stochastic volatility model:

$$\begin{aligned} \frac{dS_t}{S_t} &= r dt + \sqrt{V_t} dW_t^{(1)}, \\ dV_t &= \psi(V_t) dt + \sigma(V_t) dW_t^{(2)}, \end{aligned} \tag{1}$$

where $\mathbb{E}[dW_t^{(1)} dW_t^{(2)}] = \rho dt$. The underlying process S_t is modeling the S&P500 index and the model is expressed under an equivalent martingale measure $\mathbb{Q}$. If a volatility proxy (such as VIX) is traded, the market is complete and Q is unique, though in general Q may not be unique and the market is incomplete.

Essentially the model above requires having an expression for the variance process V_t. Given the stochastic volatility model (1), we describe a framework

[4] See the official release from CBOE: `http://us.spindices.com/documents/additional-material/vix-for-vas.pdf`

to price any European type VIX derivatives. Suppose that at present time t the value of a European type derivative maturing at T on VIX may be expressed as:

$$C(t) = e^{-r\Delta}\mathbb{E}[g(h(V_T)) \mid \mathcal{F}_t],$$

where $h(V_T)$ is the VIX index value at maturity time T and g is the payoff for the VIX derivative. $V = \{V_t\}_t$ is the variance process in (1), and $\Delta = T - t$ is the option's tenor. If we denote the transition density function for the V_t process as $p(\Delta, V_T|V_t)$, then the general pricing formula for a European type VIX derivative is:

$$C(t, V_t) = e^{-r\Delta}\int_0^\infty g(h(V))p(\Delta, V|V_t)dV. \tag{2}$$

The valuation formula (2) indicates very clearly the two problems needed to be solved in order to produce a European type derivative price. **First**, we need to express the value of the VIX index at time T as a function of the variance process V_T, i.e., $\text{VIX}_T = h(V_T)$. **Second**, we need to calculate or express the transition density function for the variance process: $p(\Delta, V|V_t)$. If we can do both of these tasks, the rest is to do a relatively simple integration in (2).

In the existing literature, the framework above has been replicated in the context of specific stochastic volatility models. Indeed, most studies use the Heston model and extensions of the Heston model for the process in (1). The first problem we mention is solved using a particular case of Proposition 1. The second is solved by approximating or calculating the transition density of V_t.

Specifically, Zhang and Zhu (2006) propose a formula for VIX futures, when the S&P 500 index follows the Heston model. Since, the stochastic volatility process is the Cox-Ingersoll-Ross (CIR) process (Cox et al., 1985), there exists an analytic formula for the transition density $p(\Delta, V_T|V_t)$. Subsequent work providing formulas for VIX futures include: Zhu and Zhang (2007); Zhang et al. (2010); Zhang and Huang (2010); Shu and Zhang (2012); Luo and Zhang (2012).

For VIX options, Lin and Chang (2009) and Lian and Zhu (2013) assume the S&P 500 index follows a Heston model with jumps in both asset and volatility model. The variance process is a CIR model with jumps and they are able to derive the characteristic function of the conditional random variable $V_T|V_t$. The VIX options are priced using a standard Fast Fourier transform (FFT) method to obtain the transition density $p(\Delta, V_T|V_t)$. A similar approach is provided in Sepp (2008).

As we can see, the recent literature is based on the Heston model and its variants. In this paper, we propose a general pricing methodology that may be applied to a variety of stochastic volatility models. We exemplify the methodology with the Heston model (for validation purposes), mean-reverting CEV model (Chan et al., 1992) and 3/2 model (Ahn and Gao, 1999).

The paper is organized as the following: In section 2, we propose formulas to express the VIX index as a function of the variance process V_t. In section 3, we apply a Hermite series expansion method to approximate the transition probability density of the process V_t. Section 4 presents numerical results, where we show that our formulas yield consistent pricing results with existing results in the literature and also with Monte Carlo simulation results. Section 5 summarizes this paper and provides potential research topics for the future.

2 Expressing the future VIX index as a function of the variance process V_t

In this section, we discuss a general methodology to express the VIX index at a future time in the general stochastic volatility model (1). Then we discuss a special case when the variance process has an affine drift term. In this particular setting, we obtain an explicit formula for the VIX as a function of the variance.

Assuming that the S&P500 follows the stochastic volatility process (1):

$$\begin{aligned}\frac{dS_t}{S_t} &= rdt + \sqrt{V_t}dW_t^{(1)},\\ dV_t &= \psi(V_t)dt + \sigma(V_t)dW_t^{(2)},\end{aligned}$$

and following Carr and Wu (2006), we may write the square of VIX as the conditional expectation of the realized variance under the risk-neutral measure $\mathbb{Q}$ during the next 30 calendar days. Specifically,

$$\mathrm{VIX}_T^2 = \mathbb{E}^Q[RV_{T,T+\Delta T}|\mathcal{F}_T], \tag{3}$$

with RV denoting the continuous realized variance

$$RV_{T,T+\Delta T} := \frac{1}{\Delta T}\int_T^{T+\Delta T} V_s ds, \tag{4}$$

and $\Delta T = 30/365$ denotes the 30 calendar days period. Since all expectations are taken under the risk-neutral measure $\mathbb{Q}$, we will drop the superscript from the conditional expectation notation in the following derivations.

Therefore the VIX index at time T is:

$$\mathrm{VIX}_T = \sqrt{\frac{1}{\Delta T}\int_T^{T+\Delta T}\mathbb{E}[V_s|V_T]ds} = h(V_T), \tag{5}$$

where we can write the conditional expectation as a function of V_T since V_t is a Markov process.

In some specific stochastic volatility models, $\mathbb{E}[V_s|V_T]$ may be expressed in closed-form and thus the VIX index can be calculated using (5). In section 2.1 we provide a specific form for stochastic volatility models where such calculation is possible.

2.1 Stochastic volatility with affine drift

We consider stochastic volatility models where the variance process has an affine drift term: $\psi(V_t) = a + bV_t$. The model is given by

$$\begin{aligned} \frac{dS_t}{S_t} &= r dt + \sqrt{V_t} dW_t^{(1)}, \\ dV_t &= (a + bV_t)dt + \sigma(V_t) dW_t^{(2)}, \end{aligned} \tag{6}$$

The following table shows examples of stochastic volatility models where the variance process has an affine drift.

Table 1: Examples of stochastic volatility models with an affine drift in the variance process

	a	b	$\sigma(V_t)$
Heston	$\kappa\theta$	$-\kappa$	$\sigma\sqrt{V_t}$
CEV	0	0	V_t^γ
Mean-Reverting CEV	α	β	V_t^γ
Hull-White	$\kappa\theta$	$-\kappa$	σ

The next proposition expresses the VIX index as a function of the variance process in any model that follows the form of (6).

Proposition 1. *The VIX index for the model* (6) *is given by*

$$VIX_T = \sqrt{\mathbb{E}[RV_{T,T+\Delta T} \mid \mathcal{F}_T]} = \sqrt{\frac{e^{b\Delta T} - 1}{b\Delta T} V_T + \frac{a(e^{b\Delta T} - 1 - b\Delta T)}{b^2 \Delta T}}. \tag{7}$$

Note that the expression does not depend on the functional form of $\sigma(\cdot)$.

Proof. Using the continuous realized variance formula (4) and the Fubini theorem, we have

$$\mathbb{E}[RV_{T,T+\Delta T} \mid \mathcal{F}_T] = \frac{1}{\Delta T}\int_T^{T+\Delta T} \mathbb{E}\left[V_s \mid \mathcal{F}_T\right] ds. \tag{8}$$

To find $\mathbb{E}[V_s|\mathcal{F}_T]$ when $s \in [T, T+\Delta T]$, we directly integrate both sides of the SDE of V_t:

$$\int_T^s dV_t = \int_T^s (a + bV_t)dt + \int_T^s \sigma(V_t)dW_t^{(2)}. \tag{9}$$

Then we have

$$V_s = V_T + a(s-T) + b\int_T^s V_t dt + \int_T^s \sigma(V_t)dW_t^{(2)}. \tag{10}$$

V_t is a Markov process, thus the conditional expectation depends only on the instantaneous variance V_T, i.e., $\mathbb{E}\left[V_s \mid \mathcal{F}_T\right] = \mathbb{E}\left[V_s \mid V_T\right]$.

Applying the conditional expectation to (10) and once again using the Fubini theorem,

$$\mathbb{E}[V_s \mid \mathcal{F}_T] = \mathbb{E}[V_s \mid V_T] = V_T + a(s-T) + b\int_T^s \mathbb{E}[V_t \mid V_T]dt + \mathbb{E}\left[\int_T^s \sigma(V_t)dW_t^{(2)}|V_T\right]. \tag{11}$$

Since $\int_T^s \sigma(V_t)dW_t^{(2)}$ is a martingale, using the martingale definition, we have that its conditional expectation is the value at T, which is equal to 0. This is where $\sigma(\cdot)$, the volatility of variance term, disappears.

Thus, the following equation governs the evolution of $\mathbb{E}[V_s \mid V_T]$:

$$\mathbb{E}[V_s \mid V_T] = V_T + a(s-T) + b\int_T^s \mathbb{E}[V_t \mid V_T]dt. \tag{12}$$

Since the conditional expectation only depends on V_T, we denote:

$$G(s,v) := \mathbb{E}[V_s \mid V_T = v].$$

Using this notation in equation (12), we have:

$$G(s,v) = v + a(s-T) + b\int_T^s G(t,v)dt. \tag{13}$$

This is a simple linear ordinary differential equation (ODE). Applying the integration factor e^{-bu}, we obtain

$$G(u,v) = \left(v + \frac{a}{b}\right) e^{b(u-T)} - \frac{a}{b}, \tag{14}$$

or in the original notation:

$$\mathbb{E}[V_s \mid V_T] = \left(V_T + \frac{a}{b}\right) e^{b(s-T)} - \frac{a}{b}. \tag{15}$$

Now using (8), we obtain

$$\begin{aligned} \text{VIX}_T^2 &= \frac{1}{\Delta T}\int_T^{T+\Delta T} \left(V_T + \frac{a}{b}\right) e^{b(s-T)} - \frac{a}{b} ds \\ &= \frac{e^{b\Delta T}-1}{b\Delta T} V_T + \frac{a(e^{b\Delta T} - 1 - b\Delta T)}{b^2 \Delta T}. \end{aligned} \tag{16}$$

This completes the proof. □

Proposition 1 generalizes current results in literature. Indeed, if we let $b = -\kappa$, $a = \kappa\theta$, then the model reduces to the well studied Heston model. In this case, the VIX formula in Proposition 1 becomes the well known "continuous variance swap rate" formula (Broadie and Jain, 2008).

2.2 General stochastic variance process

The previous section gives a relatively straightforward expression for VIX_T as a function of V_T when the variance process has an affine drift. When the variance is following a more general stochastic process without the affine drift property, the methodology needs to be modified. The mathematical issue lies in calculating and then integrating the conditional expectation $\mathbb{E}[V_s|V_T]$ in equation (5). In this equation, if a numerical method is found to express the conditional expectation, then the proposed method is applicable. We are exemplifying this approach for the 3/2 model, where we use a Laplace transform of the realized variance. The next proposition formalizes this approach in the case when a Laplace transform exists.

Proposition 2. *Assume that for a stochastic volatility model expressed as in* (1)*, the Laplace transform of the realized variance*

$$\mathbb{E}\left[e^{\lambda \int_T^{T+\Delta T} V_s ds} | V_T\right]$$

exists in closed-form. Then from equation (5) *we have that the VIX index at* T *is:*

$$VIX_T = \sqrt{\frac{1}{\Delta T}\frac{d}{d\lambda}\mathbb{E}\left[e^{\lambda \int_T^{T+\Delta T} V_s ds} | V_T\right]\Big|_{\lambda=0}} \times 100 \tag{17}$$

where ΔT *denotes the 30 calendar day period.*

Proof. Taking the derivative of the Laplace transform above with respect to λ and then setting $\lambda = 0$:

$$\begin{aligned}\frac{\partial}{\partial\lambda}\mathbb{E}\left[e^{\lambda \int_T^{T+\Delta T} V_s ds} \mid V_T\right]\Bigg|_{\lambda=0} &= \mathbb{E}\left[\lambda \int_T^{T+\Delta T} V_s ds \mid V_T\right] \\ &= \int_T^{T+\Delta T} \mathbb{E}\left[V_s \mid V_T\right] ds.\end{aligned}$$

Therefore, equation (17) holds for the VIX index. This completes the proof. □

The "3/2 model" has the following dynamics:

$$\begin{aligned}\tfrac{dS_t}{S_t} &= rdt + \sqrt{V_t}dW_t^{(1)}, \\ dV_t &= (pV_t + qV_t^2)dt + \sigma V_t^{\frac{3}{2}} dW_t^{(2)},\end{aligned} \tag{18}$$

and it will be used as an application of Proposition 2. The model was introduced in Ahn and Gao (1999) to model the evolution of short rates. The model and its behavior is extensively studied in literature, for example Carr and Sun (2007); Drimus (2012); Jarrow et al. (2013); Baldeaux (2012), etc. The Laplace transform of the realized variance in the 3/2 model is provided in the next proposition, due to Carr and Sun (2007):

Proposition 3 (Carr and Sun (2007))**.** *In the 3/2 model, the Laplace transform of the realized variance* $\int_t^{t+\Delta T} V_s ds$ *is given by*

$$\mathbb{E}\left[e^{-\lambda \int_t^{t+\Delta T} V_s ds} | V_T\right] = \frac{\Gamma(\gamma-\alpha)}{\Gamma(\gamma)} \left(\frac{2}{\sigma^2 y_T}\right)^{\alpha} M\left(\alpha, \gamma, \frac{-2}{\sigma^2 y_T}\right) \tag{19}$$

where $y_t = V_t \frac{e^{pT}-1}{p}$, $\alpha = -\left(\frac{1}{2} - \frac{q}{\sigma^2}\right) + \sqrt{\left(\frac{1}{2} - \frac{q}{\sigma^2}\right)^2 + \frac{2\lambda}{\sigma^2}}$, $\gamma = 2\left(\alpha + 1 - \frac{q}{\sigma^2}\right)$, Γ *is the Gamma function, and* M *is the confluent hypergeometric function*

$$M(\alpha, \gamma, z) = \lim \sum_{n=0}^{\infty} \frac{(\alpha)_n}{(\gamma)_n} \frac{z^n}{n!}, \tag{20}$$

with the notation $(x)_n = \Pi_{i=0}^{n-1}(x+i)$

This proposition combined with Proposition 2 will allow us to calculate a numerical expression for VIX_T. We stress that this method works for any diffusion process which has a Laplace transform of the realized variance. For example, the 4/2 stochastic volatility model introduced by Grasselli (2017) also has a closed form Laplace transform for the realized variance.

3 Approximating the transition density function using Hermite series expansions

The second part of pricing VIX derivatives is to calculate the transition density function $p(\Delta, V_T|V_t)$ of the variance process V_t. If we have an expression for $p(\Delta, V_T|V_t)$, then from the equation (2), there is:

$$C(t, V_t) = e^{-r\Delta} \int_0^{\infty} g(h(V)) p(\Delta, V|V_t) dV,$$

and we may price any European type derivative on VIX.

In the Heston model, the transition density of V_t is well known, therefore this equation can be directly evaluated. For example, Zhang and Zhu (2006) use this approach to evaluate VIX futures. Another common method is based on the characteristic function of V_T. For a given stochastic volatility model, if there exists a closed-form formula for $\mathbb{E}[e^{iuV_t}]$, we can use the Fast Fourier Transform (see Carr and Madan (1999)) or the Fourier Cosine series approach (see Fang and Oosterlee (2008)) to express the transition density and thus price VIX derivatives.

In this paper, we propose a method that can be applied for models which have neither a closed-form transition density nor a characteristic function of V_T. We utilize the Hermite series expansion, first introduced in Aït-Sahalia (2002), to evaluate the transition density $p(\Delta, V|V_t)$ of the variance process.

Theorem 4. *(Aït-Sahalia (2002)) Consider a general one-dimensional time-homogeneous diffusion process:*

$$dV_t = \mu(V_t)dt + \sigma(V_t)dW_t. \tag{21}$$

The transition probability density of v can be approximated as:

$$\begin{aligned}\tilde{p}_V^{(K)}(\Delta, v_t \mid v_0) = \sigma(v_t)^{-1}\Delta^{-1/2}\phi\left(\frac{\gamma(v_t)-\gamma(v_0)}{\Delta^{1/2}}\right)\exp\left(\int_{\gamma(v_0)}^{\gamma(v_v)} \mu_Y(w)dw\right) \\ \times \sum_{k=0}^{K} c_k(\gamma(v_t) \mid \gamma(v_0))\frac{\Delta^k}{k!}.\end{aligned} \tag{22}$$

$c(y_t|y_0)$ is a recursive function defined as:

$$\begin{aligned}c_j(y_t \mid y_0) = j(y_t - y_0)^{-j}\int_{y_0}^{y_t}(w-y_0)^{j-1} \\ \times\left[\lambda_Y(w)c_{j-1}(w \mid y_0) + \left(\partial^2 c_{j-1}(w \mid y_0)/\partial w^2\right)/2\right]dw,\end{aligned} \tag{23}$$

and $\gamma(v)$ *is the Lamperti transformation defined as*

$$\gamma(v) = \int_{\cdot}^{v} \frac{1}{\sigma(u)} du, \tag{24}$$

where $\Delta = t - 0$ *is the time period between origin and the time* t *when density being estimated, and we denote the standard normal density function with* $\phi(z) := e^{-z^2/2}/\sqrt{2\pi}$*.*

Proof. Please refer to Aït-Sahalia (2002) for details. □

The theorem expressed in the form above is a modification of Theorem 1 in Aït-Sahalia (2002) to deal with infinite series. Particularly, equation (22) contains the truncated series at an integer level K. In the original paper, the author pointed out that the results have sufficient accuracy when $K = 2$ for most of the diffusion processes, and this statement has been confirmed by all of our numerical implementations of the formula.

Once we have expressions for VIX_T as well as for $p(\Delta, V_T|V_t)$, we can use a numerical integration (quadrature) to compute the price of the derivative as in equation (2). The next section presents the results.

4 Numerical results

Our numerical experiments include pricing results for three different models: the Heston model, the mean-reverting CEV model (affine drift for variance) and the 3/2 model (non-affine drift). As mentioned, the existing pricing formulas for VIX derivatives are generally based on the Heston model, thus we can use existing formulas as benchmarks. There are no such formulas for the other two models and we verify our implementation using Monte Carlo simulations.

4.1 VIX derivative prices under the Heston model

The Heston stochastic volatility model is

$$\begin{aligned} \frac{dS_t}{S_t} &= rdt + \sqrt{V_t}dW_t^{(1)}, \\ dV_t &= \kappa(\theta - V_t)dt + \sigma\sqrt{V_t}dW_t^{(2)}, \end{aligned} \tag{25}$$

where $\mathbb{E}\left[dW_t^{(1)}dW_t^{(2)}\right] = \rho dt$, κ is the mean reverting rate, θ is the long term variance and σ is the volatility of variance. Using Proposition 1, the VIX index in the Heston model is:

$$\mathrm{VIX}_T = \sqrt{\frac{e^{-\kappa\Delta T} - 1}{-\kappa\Delta T}V_T + \frac{\kappa\theta(e^{-\kappa\Delta T} - 1 + \kappa\Delta T)}{\kappa^2\Delta T}}. \tag{26}$$

As a benchmark, we use VIX futures formulas in Zhang and Zhu (2006) and VIX option formulas in Lian and Zhu (2013). We use the following parameter set $\kappa = 5, \theta = 0.05, \sigma = 0.5, v_0 = 0.2$.

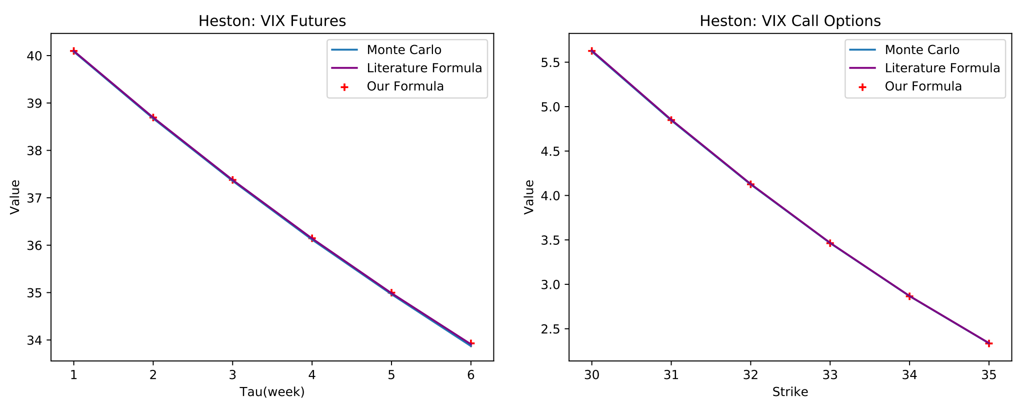

Figure 1: VIX future and option pricing: Heston model

Figure 1 shows the pricing results for VIX futures and VIX call options respectively. We also validate the results using Monte Carlo simulations (blue in the figure).

[Table 2 - 3 about here]

Numerical details may be found in tables 2 - 3 in the appendix. The results demonstrate that our numbers are very close to the analytic results and much more efficient than the Monte Carlo method.

4.2 VIX derivative prices under the mean-reverting CEV model

Chan et al. (1992) propose the following variance dynamics:

$$dV_t = (\alpha + \beta V_t)dt + \sigma V_t^{\gamma} dW_t^{(2)}, \tag{27}$$

where γ is the elasticity parameter. When β is negative, the model exhibits the mean reverting property. We apply Proposition 1, and obtain:

$$\mathrm{VIX}_T = \sqrt{\frac{e^{\beta\Delta T} - 1}{\beta\Delta T} V_T + \frac{\alpha(e^{\beta\Delta T} - 1 - \beta\Delta T)}{\beta^2 \Delta T}} \tag{28}$$

Unlike the Heston model, neither the probability transition density nor the characteristic function of V_t are available in literature. The parameter set used for this model is: $\alpha = 0.36, \beta = -6, \sigma = 1.4, v_0 = 0.2, \gamma = 1.2$.

Figure 2 shows the VIX futures and option prices under this model. For VIX futures, we use maturities between 1 week and 6 weeks. For VIX options, we show the prices for 4 week maturity contracts. All numbers and prices of options with other tenors may be found in tables 4 - 5.

[Table 4 - 5 about here]

We can draw similar conclusions as in the Heston model. Our results agree very well with the numbers obtained using Monte Carlo simulations. We stress that the formula we propose is valid for any variance process with an affine drift as in model (6).

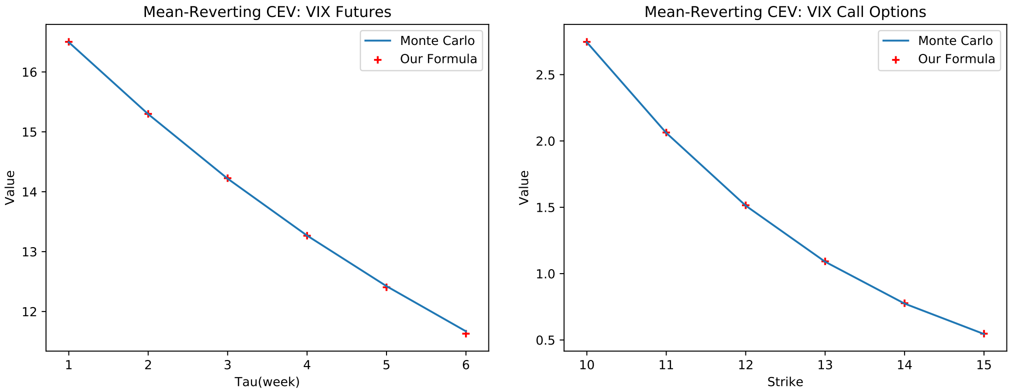

Figure 2: VIX future and option pricing: mean-reverting CEV model

4.3 3/2 Model

As discussed in section 2.2, the 3/2 model has the following dynamic for the variance process:

$$dV_t = (pV_t + qV_t^2)dt + \sigma V_t^{\frac{3}{2}} dW_t^{(2)} \tag{29}$$

The parameter set used for this model is: $p = -10.8724, q = 0.6788, \sigma = 8.1744, v_0 = 0.2029$.

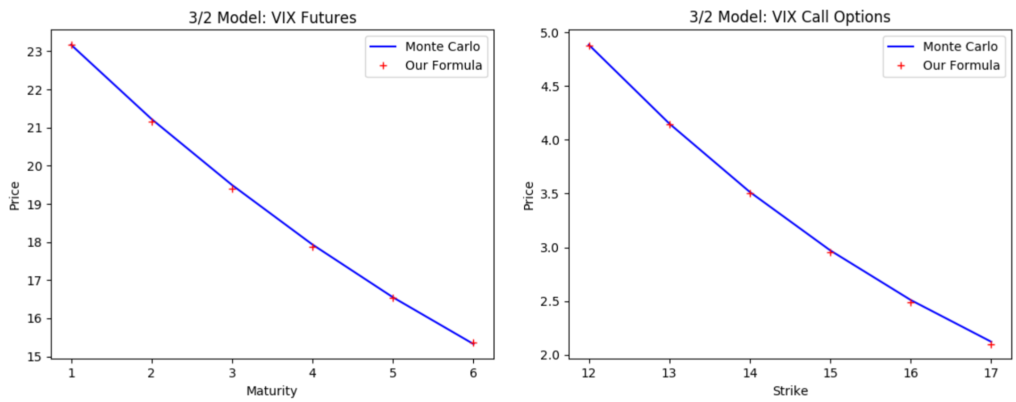

Figure 3: VIX future and option pricing: 3/2 model

[Table 6 - 7 about here]

Figure 3 shows the prices of VIX futures and options under this model. For VIX futures we use maturities between 1 week and 6 weeks.For VIX options, we show the prices for 4 week maturity contracts. We see that our results match the results from Monte Carlo simulations.

5 Conclusion and future research

In this work, we present a new methodology for pricing VIX futures and VIX options. Our method is based on closed-form Hermite series expansion, and it can be applied to general stochastic volatility models. We exemplify by Heston, mean-reverting CEV and 3/2 models. We present numerical experiments that show the accuracy and efficiency of this approach.

We consider the case when the stochastic variance process follows a time-homogeneous diffusion. We believe that it is possible to extend the method proposed to time-inhomogeneous one-dimensional diffusions, using the technique in Egorov et al. (2003). Other possible future research directions include extension to stochastic volatility models with jumps such as the models considered in Cui et al. (2017a,b), Kirkby et al. (2017). Another direction is to develop analytical methods for VIX exchange traded notes such as VXX, which is a very actively traded instrument and was recently studied in Gehrike and Zhang (2017).

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A Appendix

A.1 Pricing results with the Heston Model

Table 2: VIX future prices with the Heston model

Tau(Week)	Our Formula	Time(s)	Zhang and Zhu (2006)	Time(s)	Monte Carlo	MC StdErr	Time(s)
1	40.0979	<0.01	40.0979	<0.01	40.0834	0.0094	97.56
2	38.6937	<0.02	38.6933	<0.02	38.6739	0.0129	97.63
3	37.3774	<0.03	37.3759	<0.03	37.3562	0.0153	100.08
4	36.1464	<0.04	36.142	<0.04	36.117	0.0171	98.72
5	34.9978	<0.05	34.9882	<0.05	34.9648	0.0185	97.25
6	33.9289	<0.06	33.9109	<0.06	33.8717	0.0196	96.27

Table 3: VIX call option prices with the Heston Model

Tau(Week)	Strike	Lian&Zhu 2013	Our Formula	Monte Carlo	MC StdErr
1	30	10.103858	10.103861	10.089368	0.009407
1	31	9.104615	9.104618	9.090120	0.009402
1	32	8.106599	8.106601	8.091999	0.009387
1	33	7.111867	7.111869	7.097371	0.009345
1	34	6.124946	6.124946	6.110998	0.009251
1	35	5.154544	5.154544	5.141577	0.009066
2	30	8.726276	8.726282	8.706838	0.012727
2	31	7.747797	7.747794	7.729033	0.012586
2	32	6.785709	6.785695	6.768350	0.012362
2	33	5.849055	5.849032	5.833410	0.012030
2	34	4.949451	4.949419	4.936362	0.011564
2	35	4.100554	4.100515	4.090029	0.010954
4	30	6.491158	6.490225	6.476147	0.015356
4	31	5.638875	5.637805	5.627273	0.014754
4	32	4.834133	4.832944	4.825239	0.014041
4	33	4.085008	4.083723	4.079024	0.013218
4	34	3.398697	3.397344	3.396111	0.012295
4	35	2.780902	2.779512	2.781014	0.011294
6	30	4.906879	4.898974	4.880363	0.015470
6	31	4.200737	4.192335	4.176760	0.014583
6	32	3.552770	3.543991	3.531680	0.013620
6	33	2.966386	2.957368	2.948478	0.012596
6	34	2.443525	2.434418	2.429329	0.011532
6	35	1.984531	1.975490	1.974781	0.010446

A.2 Pricing result with mean-reverting CEV Model

Table 4: VIX future with the mean-reverting CEV Model

Tau(Week)	Our Formula	Time(s)	Monte Carlo	MC StdErr	Time(s)
1	16.5031	<0.01	16.4964	0.0072	139.27
2	15.2999	<0.01	15.296	0.0091	138.12
3	14.2262	<0.01	14.2187	0.0102	142.53
4	13.2659	<0.01	13.2682	0.0107	138.49
5	12.4046	<0.01	12.4234	0.0108	138.75
6	11.6297	<0.01	11.6693	0.0108	139.61

Table 5: VIX call option prices with the mean-reverting CEV Model

Tau(Week)	Strike	Our Formula	Monte Carlo	MC StdErr
1	10	6.505117	6.498369	0.007182
1	11	5.505691	5.498977	0.007178
1	12	4.509826	4.503204	0.007151
1	13	3.532227	3.526265	0.007025
1	14	2.610157	2.605899	0.006680
1	15	1.799769	1.798239	0.006033
2	10	5.310327	5.305145	0.009106
2	11	4.333747	4.328214	0.008988
2	12	3.404960	3.399453	0.008684
2	13	2.564744	2.560529	0.008125
2	14	1.850018	1.847112	0.007324
2	15	1.280064	1.277795	0.006359
4	10	3.433473	3.425304	0.010054
4	11	2.637672	2.630163	0.009430
4	12	1.968454	1.961493	0.008601
4	13	1.432732	1.426513	0.007646
4	14	1.021543	1.016631	0.006653
4	15	0.716573	0.712501	0.005692
6	10	2.198243	2.197031	0.009234
6	11	1.620671	1.618304	0.008335
6	12	1.174300	1.171268	0.007368
6	13	0.840449	0.837042	0.006412
6	14	0.596647	0.593328	0.005518
6	15	0.421560	0.418638	0.004712

A.3 Pricing result with 3/2 Model

Table 6: VIX future with 3/2 Model

Tau(Week)	Our Formula	Monte Carlo	MC StdErr
1	23.166911	23.170387	0.015602
2	21.14828	21.224639	0.019128
3	19.394308	19.4952	0.020415
4	17.878964	17.93356	0.020525
5	16.548306	16.550712	0.020098
6	15.360815	15.327139	0.019401

Table 7: VIX call option with 3/2 Model

Tau_Week	Strike	Hermite	Monte Carlo	MC StdErr
1	12	11.175556	11.172696	0.015602
1	13	10.176353	10.172945	0.015602
1	14	9.177840	9.173865	0.015597
1	15	8.182804	8.178454	0.015573
1	16	7.198908	7.194423	0.015496
1	17	6.240400	6.236780	0.015314
2	12	9.181210	9.232471	0.019112
2	13	8.195131	8.245110	0.019056
2	14	7.230747	7.279891	0.018917
2	15	6.306130	6.355330	0.018647
2	16	5.440180	5.489812	0.018219
2	17	4.648191	4.698327	0.017631
4	12	6.041487	6.083945	0.020043
4	13	5.199272	5.243934	0.019591
4	14	4.441915	4.488342	0.018979
4	15	3.775301	3.823351	0.018237
4	16	3.197982	3.248691	0.017398
4	17	2.703690	2.756352	0.016503
6	12	3.938066	3.915115	0.017882
6	13	3.308457	3.289830	0.017117
6	14	2.775362	2.761760	0.016268
6	15	2.328508	2.319673	0.015378
6	16	1.955928	1.953009	0.014477
6	17	1.645879	1.648538	0.013588