Stochastic volatility's orderly smiles

Lorenzo Bergomi *and* ***Julien Guyon*** *derive an expansion of the volatility surface of general stochastic volatility models at second order in volatility of volatility that is accurate for a wide range of strikes. They characterise the shape of stochastic volatility smiles in terms of three effective quantities that compactly summarise the joint dynamics of spot and volatilities in the model*

Stochastic volatility models generate an implied volatility surface as well as its associated dynamics. While Monte Carlo simulation is always an option, a fast and accurate approximation of the volatility surface is a useful implement for assessing any given model. In this article, we obtain such an approximation at second order in the volatility of volatility for a general class of stochastic volatility models.

Using this approximation, we derive some structural properties of stochastic volatility smiles: we highlight the dependence of the level of short-term at-the-money (ATM) skew and curvature on the short-term ATM volatility, and we also link the decay of ATM skew and curvature for long maturities to the decay of the spot/volatility and volatility/volatility covariance functions. We finally provide explicit expressions for a family of Heston-like models and the two-factor Bergomi model and assess the expansion's accuracy for the two-factor model using realistic levels of volatility of volatility.

Our approach uses the operator formulation of the time-dependent perturbation technique commonly used in quantum mechanics (Schulman, 2005) to generate order-by-order corrections to the unperturbed price, rather than the traditional method of perturbing the partial differential equation (PDE) itself.

We consider the following dynamics for a multi-factor diffusive stochastic volatility model:

$$
\begin{aligned}
dx_t &= -\frac{1}{2}\xi_t^t dt + \sqrt{\xi_t^t}\, dZ_t^1, \qquad x_0 = x \\
d\xi_t^u &= \lambda(t,u,\xi_t)\cdot dZ_t, \qquad \xi_0^u = \xi^u
\end{aligned} \tag{1}
$$

where $x_t = \ln(S_t)$ and $\xi_t \equiv (\xi_t^u, u \geq t)$ stands for the instantaneous forward variance curve at time t. ξ_t^u is the instantaneous forward variance for date u observed at time t. The volatility $\lambda = (\lambda_1, \dots, \lambda_d)$ of forward instantaneous variances takes values in $\mathbb{R}^d$, and $Z = (Z^1, \dots, Z^d)$ is a d-dimensional Brownian motion. The first component of the Brownian motion, Z^1, drives the spot dynamics. Spot/variance covariance is thus modelled through λ_1. We assume that the asset pays no dividends and for the sake of simplicity take zero rates and repos.

Forward variances ξ^u are driftless. Their initial value can be calibrated on market prices of variance swap (VS) contracts: $\xi_0^u = d/du(\hat{\sigma}_u^2 u)$, where $\hat{\sigma}_u$ is the implied VS volatility for maturity u. Alternatively, the ξ_0^u can be chosen so as to recover market prices of other instruments, such as ATM vanilla options for all maturities. Second-generation stochastic volatility models, such as the Bergomi model (2005), in which the dynamics of the ξ^u is directly modelled, are naturally expressed in the form of equation (1). By contrast, first-generation stochastic volatility models, such as the Heston (1993) or the double-lognormal model (Gatheral, 2008), are built on an autonomous dynamics for the instantaneous variance $V_t = \xi_t^t$. This imposes structural constraints on the shape of the initial variance curve ξ_0^u. Still, first-generation models can be cast as forward variance models and our analysis applies to them as well (see the example of the Heston model below).

For a general model (1), the pricing equation for European-style payouts is not analytically solvable. While many different approximations have been proposed for specific first-generation models, the general case of forward variance models has not been considered in the literature. In this article, we derive an approximation of the smile produced by the generic model (1) at second order in the volatility of volatility. To this end, we introduce a scaling factor ε for the volatilities of instantaneous forward variances: $\lambda \to \varepsilon\lambda$. Expanding at order two in ε is thus exactly equivalent to expanding at order two in λ. Our derivation relies on the property that in model (1) the volatility of S_t incorporates no local volatility component, and that λ does not depend on S_t. Mixed local/stochastic volatility models lie outside our scope, and the stochastic volatility must be an autonomous process.

While it is obvious that the smile produced by stochastic volatility models is generated by the covariance of forward variances with themselves and S_t, our aim in this article is to pinpoint which functionals of these covariances determine the vanilla smile. We also demonstrate that the accuracy of a second-order expansion is practically adequate whenever far out-of-the-money strikes are not considered.

In this respect, it is important to ensure that implied volatilities of some specific payouts are unchanged, so that the overall volatility level is not altered as ε is varied. In our framework, VS volatilities stay unchanged by construction. This stands in contrast with other types of approximations whose accuracy is compromised by the fact that the overall volatility level in the model changes as ε increases.

Expanding the price

Consider a vanilla option whose payout at maturity T is $g(x_T)$. Its value at time t is $P(t, x_t, \xi_t)$. P is a function of x and a functional of the whole variance curve ξ. P solves the equation:

$$
(\partial_t + H_t)P = 0 \tag{2}
$$

with terminal condition $P(t = T, x, \xi) = g(x)$, where:

$$
H_t = H_t^0 + W_t^1 + W_t^2, \qquad H_t^0 = \frac{\xi^t}{2}\left(\partial_x^2 - \partial_x\right)
$$

$$
W_t^1 = \int_t^T du\, \mu(t,u,\xi)\,\partial_{x\xi^u}^2
$$

$$
W_t^2 = \frac{1}{2}\int_t^T du \int_t^T du'\, \nu(t,u,u',\xi)\,\partial_{\xi^u\xi^{u'}}^2
$$

with: