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A CONTINUOUS-TIME GARCH PROCESS DRIVEN BY A LÉVY PROCESS: STATIONARITY AND SECOND-ORDER BEHAVIOUR

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Abstract

We use a discrete-time analysis, giving necessary and sufficient conditions for the almost-sure convergence of ARCH(1) and GARCH(1, 1) discrete-time models, to suggest an extension of the ARCH and GARCH concepts to continuous-time processes. Our 'COGARCH' (continuous-time GARCH) model, based on a single background driving Lévy process, is different from, though related to, other continuous-time stochastic volatility models that have been proposed. The model generalises the essential features of discrete-time GARCH processes, and is amenable to further analysis, possessing useful Markovian and stationarity properties.

Keywords: ARCH model; GARCH model; stability; stationarity; conditional heteroscedasticity; perpetuities; stochastic integration; Lévy process

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1. Introduction

Certain time series models known as ARCH (autoregressive conditionally heteroscedastic) and GARCH (generalised ARCH) models are popular in financial econometrics where they are designed to capture some of the distinctive features of asset-price, exchange-rate and other series. So-called stylised facts characterise financial returns data as heavy tailed, uncorrelated, but not independent, with time-varying volatility and a long-range-dependence effect evident in volatility, this last also being manifest as a 'persistence in volatility'. Various attempts have been made to capture these features in a continuous-time model, a natural extension being given by diffusion approximations to the discrete-time GARCH process as in [21] and [10] or also in [8]. These lead to stochastic volatility models of the form

$$\mathrm{d}Y_t = \sigma_t \,\mathrm{d}B_t^{(1)}, \qquad \mathrm{d}\sigma_t^2 = \theta(\gamma - \sigma_t^2)\,\mathrm{d}t + \rho\sigma_t^2\,\mathrm{d}B_t^{(2)}, \qquad t > 0, \tag{1.1}$$

where $B^{(1)}$ and $B^{(2)}$ are independent Brownian motions. For a review paper on such continuous-time GARCH models, we refer the reader to [9].

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Various related models have been suggested and investigated, many generalisations being based on Lévy processes replacing the Brownian motions and on relaxing the independence property. We refer the reader here to [2], [3] and [1] for quite sophisticated models.

The main difference between models like (1.1) and the original GARCH setup is the fact that in the GARCH modelling a single source of randomness suffices; all stylised features are then captured by the dependence structure of the model.

We adopt this idea of a single noise process and suggest a new continuous-time GARCH model, which captures all the stylised facts that the discrete-time GARCH model does. As noise process, any Lévy process is possible, its increments replacing the innovations in the discrete-time GARCH model. The volatility process is modelled by a stochastic differential equation, whose solution displays the 'feedback' and 'autoregressive' aspects of the recursion formula for the discrete-time GARCH model.

Our paper is organised as follows. We start in Section 2 with the basics, giving necessary and sufficient conditions for the existence of stable solutions to the discrete-time GARCH(1, 1) model, assuming no *a priori* conditions whatsoever; in particular, no moment or log-moment assumptions are made.

In Section 3, motivated by the structural results of the previous section, we suggest a new continuous-time GARCH(1, 1) model taking a general Lévy process as the driving process. The resulting volatility process satisfies a stochastic differential equation and is stationary under conditions analogous to those for the discrete-time GARCH model. Moreover, it is Markovian. For the continuous-time GARCH model, a bivariate state-space representation exists and is Markovian, again in analogy to the discrete-time GARCH model.

Section 4 is devoted to an investigation of the stylised facts for the volatility process as mentioned above. The second-order properties of the continuous-time GARCH model match those of the discrete-time model, as calculated moments and autocorrelation functions reveal. Moreover, the stationary volatility is heavy tailed in the sense that not all moments exist in a given parametrisation.

Finally, in Section 5 we summarise some moment properties of the GARCH process itself, showing in particular that its squared increments are positively correlated under some conditions.

2. Discrete-time ARCH(1) and GARCH(1, 1) processes

We write the discrete-time GARCH(1, 1) process in the form

$$Y_n = \varepsilon_n \sigma_n, \quad \text{where } \sigma_n^2 = \beta + \lambda Y_{n-1}^2 + \delta \sigma_{n-1}^2, \qquad n \in \mathbb{N}. \tag{2.1}$$

The random variable σ_n is the positive square root of σ_n^2 and the ε_n, $n = 1, 2, \ldots$, are independent and identically distributed (i.i.d.) nondegenerate random variables with $\mathrm{P}\{\varepsilon_1 = 0\} = 0$. The parameters β, λ and δ satisfy $\beta > 0$, $\lambda \geq 0$ and $\delta \geq 0$. When $\delta = 0$ in (2.1), the GARCH(1, 1) process reduces to the ARCH(1) process and, if $\delta = \lambda = 0$, $(Y_n)_{n\in\mathbb{N}}$ is simply a sequence of i.i.d. random variables, so we assume that $\delta + \lambda > 0$ to exclude this case. We assume some initial almost surely finite (random, in general) values for ε_0 and σ_0, independent of each other and independent of $(\varepsilon_n)_{n\geq 1}$, and let $Y_0 = \varepsilon_0 \sigma_0$. For general background on the ARCH model we refer the reader to [13], and for the GARCH model to [6]; see also [29].

There have been many empirical and theoretical investigations into properties of the models. Of major theoretical importance are conditions on the parameters in the model under which a stationary version of the process exists. Define the random variables

$$\pi_n = \pi_n(\lambda, \delta) := \prod_{i=1}^{n} (\delta + \lambda \varepsilon_i^2), \qquad n \in \mathbb{N}.$$

The next result will be used to motivate our continuous-time model. Throughout, '$\xrightarrow{\mathrm{D}}$' means 'convergence in distribution', '$\xrightarrow{\mathrm{P}}$' means 'convergence in probability' and '$\stackrel{\mathrm{D}}{=}$' means 'has the same distribution as'.

Theorem 2.1. (a) (GARCH(1, 1) process.) *Assume the above setup with $\delta > 0$ and $\lambda \geq 0$, but no further restrictions. Suppose that*

$$\mathrm{E}|\log(\delta + \lambda\varepsilon_1^2)| < \infty \quad \textit{and} \quad \mathrm{E}\log(\delta + \lambda\varepsilon_1^2) < 0. \tag{2.2}$$

Then we have stability of the mean and variance processes, that is, $Y_n \xrightarrow{\mathrm{D}} Y$ and $\sigma_n \xrightarrow{\mathrm{D}} \sigma$ as $n \to \infty$ for finite random variables Y and σ. Conversely, if (2.2) does not hold, then $\sigma_n \xrightarrow{\mathrm{P}} \infty$ and $|Y_n| \xrightarrow{\mathrm{P}} \infty$ as $n \to \infty$.

(b) (ARCH(1) process.) *Suppose that $\delta = 0$ and $\lambda > 0$. Then we have stability of $(Y_n)_{n\geq 0}$ and $(\sigma_n)_{n\geq 0}$ if (2.2) holds with $\delta = 0$, or if*

$$\mathrm{E}(\log(\lambda\varepsilon_1^2))^- = \infty \quad \textit{and} \quad \int_0^\infty x\left(\int_0^x \mathrm{P}\{\log(\lambda\varepsilon_1^2) < -y\}\,\mathrm{d}y\right)^{-1} \mathrm{d}\,\mathrm{P}\{\log(\lambda\varepsilon_1^2) \leq x\} < \infty. \tag{2.3}$$

Conversely, if (2.2) with $\delta = 0$ and (2.3) both fail, then $\sigma_n \xrightarrow{\mathrm{P}} \infty$ and $|Y_n| \xrightarrow{\mathrm{P}} \infty$ as $n \to \infty$.

Proof. Take $\delta \geq 0$ and $\lambda \geq 0$. From (2.1) we have

$$\sigma_n^2 = \beta + \lambda Y_{n-1}^2 + \delta\sigma_{n-1}^2 = \beta + (\delta + \lambda\varepsilon_{n-1}^2)\sigma_{n-1}^2, \qquad n \in \mathbb{N}, \tag{2.4}$$

where ε_{n-1} is independent of σ_{n-1}^2. Iterate this to get (cf. [16], [22, Equation (6)])

$$\sigma_n^2 = \beta \sum_{i=0}^{n-1} \prod_{j=i+1}^{n-1} (\delta + \lambda\varepsilon_j^2) + \sigma_0^2 \prod_{j=0}^{n-1} (\delta + \lambda\varepsilon_j^2), \qquad n \in \mathbb{N} \tag{2.5}$$

(take $\prod_{j=a}^{b} = 1$ when $a > b$). This relation shows that the distribution of σ_n has the form of the distribution of a discrete-time perpetuity, as in [17]. In the notation of [17], setting $M_j = M_j(\delta, \lambda) = \delta + \lambda\varepsilon_j^2$ and $Q_i = 1$, we can apply Theorem 2.1 of [17] to see that $\sigma_n^2 \xrightarrow{\mathrm{D}} \sigma^2$ for a finite random variable σ, provided that $\lim_{n\to\infty} \pi_n = 0$ almost surely. Assuming that $\lim_{n\to\infty} \pi_n = 0$ almost surely, and taking limits in (2.4), shows that σ satisfies $\sigma^2 \stackrel{\mathrm{D}}{=} \beta + (\delta + \lambda\varepsilon^2)\sigma^2$, with ε and σ independent. From (2.1) we then get $Y_n \xrightarrow{\mathrm{D}} Y$, satisfying $Y \stackrel{\mathrm{D}}{=} \sigma\varepsilon$, with ε and σ independent. If π_n does not tend to 0 almost surely, then Theorem 2.1 of [17] shows that $\sigma_n \xrightarrow{\mathrm{P}} \infty$, and then $|Y_n| \xrightarrow{\mathrm{P}} \infty$ because $\mathrm{P}\{\varepsilon_1 = 0\} = 0$. Thus, a necessary and sufficient condition for stability of the discrete ARCH(1) and GARCH(1, 1) processes is that $\pi_n \to 0$ almost surely as $n \to \infty$.

Now define

$$S_0 = 0, \qquad S_n = \sum_{i=1}^{n} X_i, \quad n \in \mathbb{N},$$

where $X_i = -\log(\delta + \lambda\varepsilon_i^2)$ for $i \in \mathbb{N}$. Since $\mathrm{P}\{\varepsilon_i \neq 0\} = 1$, the X_i and S_n are almost surely finite random variables for any $\delta \geq 0$ and $\lambda \geq 0$ such that $\delta + \lambda > 0$. Further, $\pi_n \to 0$ almost surely if and only if $S_n \to \infty$ almost surely. Let $X = X_1$, $X^+ = \max(0, X)$ and $X^- = -X + X^+$. Then, by [18] and [14], a necessary and sufficient condition for $\pi_n \to 0$ almost surely or, equivalently, $S_n \to \infty$ almost surely, is that

$$\mathrm{E}|X| < \infty \quad \text{and} \quad \mathrm{E}\,X > 0, \tag{2.6}$$

or else

$$\mathrm{E}\,X^+ = \infty \quad \text{and} \quad \int_{[0,\infty)} \frac{x}{\mathrm{E}(X^+ \wedge x)}\,\mathrm{d}\,\mathrm{P}\{X^- \leq x\} < \infty. \tag{2.7}$$

To prove (a), keep $\delta > 0$ and $\lambda \geq 0$. Now (2.6) is exactly (2.2), so we only have to check that the condition (2.7) cannot occur in this case. We do this by showing that $\mathrm{E}\,X^+ < \infty$. Note that (2.2) implies that $\delta < 1$, as does $\lim_{n\to\infty} \pi_n = 0$ almost surely. So we may keep $0 < \delta < 1$. Then, for $x > 0$,

$$\mathrm{P}\{X > x\} = \mathrm{P}\{-\log(\delta + \lambda\varepsilon_1^2) > x\} = \mathrm{P}\{\log(\delta + \lambda\varepsilon_1^2) < -x\}\,\mathbf{1}_{\{x<-\log\delta\}},$$

so

$$\mathrm{E}\,X^+ = \int_0^{-\log\delta} \mathrm{P}\{\log(\delta + \lambda\varepsilon_1^2) < -x\}\,\mathrm{d}x,$$

which is always finite, completing the proof of (a).

Next, to prove (b), keep $\delta = 0$ and $\lambda > 0$. This time (2.7) can occur, the condition being equivalent to (2.3). Alternatively, (2.6) is equivalent to (2.2) with $\delta = 0$ in this case. This proves (b).

Remark 2.1. (i) Under the *a priori* assumption that the expectations of the positive and negative parts of $\log(\delta + \lambda\varepsilon_1^2)$ are not both infinite, Nelson [22] gave a necessary and sufficient condition for stability of the ARCH(1) and GARCH(1, 1) volatility processes as $\mathrm{E}\log(\delta + \lambda\varepsilon_1^2) < 0$ (see also [26]). In the GARCH case, $\delta > 0$ and $\lambda \geq 0$, we always have $\mathrm{E}(\log(\delta + \lambda\varepsilon_1^2))^- < \infty$, and so (2.2) recovers Nelson's sufficient condition. Nelson claimed that, if (2.2) fails, then $\sigma_n \to \infty$ almost surely, but his proof is incorrect in the case $\mathrm{E}\log(\delta + \lambda\varepsilon_1^2) = 0$. Only the weak divergences, that $\sigma_n \xrightarrow{\mathrm{P}} \infty$ and $|Y_n| \xrightarrow{\mathrm{P}} \infty$ (as $n \to \infty$) as stated in our Theorem 2.1, can be claimed in general. This distinction is important in some applications.

In the ARCH case, $\delta = 0$ and $\lambda > 0$, it is easy to construct $(\varepsilon_n)_{n\in\mathbb{N}}$ such that $\mathrm{E}(\log(\lambda\varepsilon_1^2))^- = \mathrm{E}(\log(\lambda\varepsilon_1^2))^+ = \infty$ but (2.3) still holds. Thus, Theorem 2.1 extends Nelson's result for the ARCH(1) case.

(ii) The condition (2.2) obviously implies that $\delta < 1$. Conversely, if $\delta > 0$ and

$$\delta + \lambda\,\mathrm{E}(\varepsilon_1^2) < 1,$$

then (2.2) holds by an application of Jensen's inequality. Under the finite variance condition $\mathrm{E}(\varepsilon_1^2) < \infty$, Bougerol and Picard [7] gave a necessary and sufficient condition for strict stationarity of GARCH(p, q) models.

(iii) Note that $\lim_{n\to\infty} \pi_n(\lambda, \delta) = 0$ almost surely for $\lambda > 0$ and $\delta > 0$ implies that $\lim_{n\to\infty} \pi_n(\lambda, 0) = 0$ almost surely for $\lambda > 0$. Thus, the GARCH(1, 1) stability condition implies stability of the ARCH(1) process.

Remark 2.2. When Y and σ exist in Theorem 2.1 they satisfy the random equations

$$Y \overset{D}{=} \sigma\varepsilon, \quad \text{where } \sigma^2 \overset{D}{=} \beta + (\delta + \lambda\varepsilon^2)\sigma^2,$$

with $\varepsilon \overset{D}{=} \varepsilon_1$ independent of σ, as shown in the proof. Also, σ has an explicit representation as an infinite (absolutely convergent) random series:

$$\sigma^2 \overset{D}{=} \beta \sum_{i=0}^{\infty} \prod_{j=1}^{i} (\delta + \lambda\varepsilon_j^2). \tag{2.8}$$

The equation (2.8) makes it clear why $\lim_{n\to\infty} \pi_n = 0$ almost surely is necessary for the stability of the GARCH(1, 1) process, but the sufficiency comes about using deeper properties of random walks, as exploited in [17].

For conditions guaranteeing various useful properties of a stationary solution (existence of moments, tail behaviour, extremal behaviour, etc.) when it exists, Mikosch and Stărică [20] provided the most general investigation so far. Such results of course have great practical importance as well. Connections between GARCH models and the random difference equation literature have been noted by various authors, among them Goldie [16]; see [12, Section 8.4] for further references. Rather than pursue these here, we turn to a continuous-time setting.

3. A continuous-time GARCH process

Our aim now is to construct a kind of GARCH process in continuous time. We want to preserve the essential features of (2.1), that innovations feed into the volatility process, which has in addition an autoregressive aspect. We proceed from the representation (2.5). The summation in (2.5) can be written as

$$\beta \int_0^n \exp\left(\sum_{j=\lfloor s\rfloor+1}^{n-1} \log(\delta + \lambda\varepsilon_j^2) \right) \mathrm{d}s, \tag{3.1}$$

which suggests replacing the noise variables ε_j by increments of a Lévy process. Accordingly, let L be a (càdlàg) Lévy process with jumps $\Delta L_t = L_t - L_{t-}$, $t \geq 0$, defined on a probability space with appropriate filtration, satisfying the 'usual conditions'. We recall some of its properties. For each $t \geq 0$, the characteristic function of L_t can be written in the form

$$\mathrm{E}(\mathrm{e}^{\mathrm{i}\theta L_t}) = \exp\left(t\left(\mathrm{i}\gamma_L\theta - \tau_L^2\frac{\theta^2}{2} + \int_{(-\infty,\infty)} (\mathrm{e}^{\mathrm{i}\theta x} - 1 - \mathrm{i}\theta x\,\mathbf{1}_{\{|x|\leq 1\}})\Pi_L(\mathrm{d}x) \right)\right), \qquad \theta \in \mathbb{R} \tag{3.2}$$

(see [27, Theorem 8.1], [4, p. 13]). The constants $\gamma_L \in \mathbb{R}$ and $\tau_L^2 \geq 0$ and the measure Π_L on $\mathbb{R}$ form the *characteristic triplet* of L; as usual, the Lévy measure Π_L is required to satisfy $\int_{\mathbb{R}} \min(1, x^2)\Pi_L(\mathrm{d}x) < \infty$. If in addition $\int_{\mathbb{R}} \min(1, |x|)\Pi_L(\mathrm{d}x) < \infty$, then $\gamma_{L,0} := \gamma_L - \int_{[-1,1]} x\Pi_L(\mathrm{d}x)$ is called the *drift* of L. We will only be interested in the situation where Π_L is nonzero.

Keep $0 < \delta < 1$ and $\lambda \geq 0$ and, with (3.1) in mind, define a càdlàg process $(X_t)_{t\geq 0}$ by

$$X_t = -t\log\delta - \sum_{0<s\leq t} \log\left(1 + \frac{\lambda}{\delta}(\Delta L_s)^2\right), \qquad t \geq 0. \tag{3.3}$$

Then, with $\beta > 0$ and σ_0 a finite random variable, independent of $(L_t)_{t\geq 0}$, define the *left-continuous* volatility process analogously to (2.5) by

$$\sigma_t^2 = \left(\beta \int_0^t \mathrm{e}^{X_s}\,\mathrm{d}s + \sigma_0^2\right)\mathrm{e}^{-X_{t-}}, \qquad t \geq 0, \tag{3.4}$$

and define the *integrated* continuous-time GARCH (COGARCH) process $(G_t)_{t\geq 0}$ as the càdlàg process satisfying

$$\mathrm{d}G_t = \sigma_t\,\mathrm{d}L_t, \quad t \geq 0, \qquad G_0 = 0. \tag{3.5}$$

Thus, G jumps at the same times as L does, and has jumps of size $\Delta G_t = \sigma_t \Delta L_t$, $t \geq 0$. Here ΔL_t is to play the role of the innovation ε_n in the discrete-time GARCH model, and the intention is that $(G_t)_{t\geq 0}$ and $(\sigma_t^2)_{t\geq 0}$ display a kind of continuous-time GARCH-like behaviour. This indeed turns out to be the case.

We begin our analysis by first investigating the process $(X_t)_{t\geq 0}$, which has a special structure.

Proposition 3.1. *The process $(X_t)_{t\geq 0}$ is a spectrally negative Lévy process of bounded variation with drift $\gamma_{X,0} = -\log\delta$, Gaussian component $\tau_X^2 = 0$ and Lévy measure Π_X given by*

$$\Pi_X([0,\infty)) = 0$$

and

$$\Pi_X((-\infty,-x]) = \Pi_L(\{y \in \mathbb{R} : |y| \geq \sqrt{(\mathrm{e}^x - 1)\delta/\lambda}\}), \qquad x > 0.$$

Proof. That $(X_t)_{t\geq 0}$ is a Lévy process with no positive jumps is clear. The Lévy measure of $(X_t)_{t\geq 0}$ has negative component given by

$$\begin{aligned}
\Pi_X\{(-\infty,-x]\} &= \mathrm{E}\sum_{0<s\leq 1} \mathbf{1}_{\{-\log(1+(\lambda/\delta)(\Delta L_s)^2)\leq -x\}} \\
&= \mathrm{E}\sum_{0<s\leq 1} \mathbf{1}_{\{|\Delta L_s|\geq\sqrt{(\mathrm{e}^x-1)\delta/\lambda}\}} \\
&= \Pi_L\{y : |y| \geq \sqrt{(\mathrm{e}^x - 1)\delta/\lambda}\}, \qquad x > 0.
\end{aligned}$$

This means that Π_X is the image measure of Π_L under the transformation $T : \mathbb{R} \to (-\infty, 0]$; $x \mapsto -\log(1 + (\lambda/\delta)x^2)$. This shows in particular that

$$\int_{[-1,1]} |x|\Pi_X(\mathrm{d}x) = \int_{\{|y|\leq\sqrt{(\mathrm{e}-1)\delta/\lambda}\}} \log(1 + (\lambda/\delta)y^2)\Pi_L(\mathrm{d}y)$$

is finite, because $\int_{[-1,1]} y^2\Pi_L(\mathrm{d}y)$ is finite. Thus, $(X_t)_{t\geq 0}$ is a Lévy process of bounded variation (e.g. [27, Theorem 21.9]), having characteristic function

$$\mathrm{E}(\mathrm{e}^{\mathrm{i}\theta X_t}) = \exp\left(-\mathrm{i}t\theta\log\delta + t\int_{(-\infty,0)} (\mathrm{e}^{\mathrm{i}\theta x} - 1)\Pi_X(\mathrm{d}x)\right), \qquad \theta \in \mathbb{R} \tag{3.6}$$

(e.g. [27, Theorem 19.3]), showing that $\gamma_{X,0} = -\log\delta$ and $\tau_X^2 = 0$. (In fact, $(X_t)_{t\geq 0}$ is the negative of a subordinator together with a positive drift.)

We now proceed to investigate the processes $(G_t)_{t\geq 0}$ and $(\sigma_t^2)_{t\geq 0}$ given by (3.4) and (3.5).

Proposition 3.2. *The process $(\sigma_t^2)_{t\geq 0}$ satisfies the stochastic differential equation*

$$\mathrm{d}\sigma_{t+}^2 = \beta\,\mathrm{d}t + \sigma_t^2 \mathrm{e}^{X_{t-}}\,\mathrm{d}(\mathrm{e}^{-X_t}), \qquad t > 0, \tag{3.7}$$

and we have

$$\sigma_t^2 = \beta t + \log\delta \int_0^t \sigma_s^2\,\mathrm{d}s + \frac{\lambda}{\delta}\sum_{0<s<t} \sigma_s^2(\Delta L_s)^2 + \sigma_0^2, \qquad t \geq 0. \tag{3.8}$$

Proof. Set $K_t := t\log\delta$, $S_t := \prod_{0<s\leq t}(1 + (\lambda/\delta)(\Delta L_s)^2)$ and $f(k,s) := \mathrm{e}^k s$. Then use Itô's lemma in two variables (e.g. [23, Theorem 33, p. 81]) to get, from (3.3), that

$$\begin{aligned} \mathrm{e}^{-X_t} &= f(K_t, S_t) \\ &= 1 + \log\delta \int_0^t \mathrm{e}^{-X_s}\,\mathrm{d}s + \frac{\lambda}{\delta}\sum_{0<s\leq t} \mathrm{e}^{-X_{s-}}(\Delta L_s)^2, \qquad t \geq 0. \end{aligned} \tag{3.9}$$

Integration by parts gives

$$\begin{aligned} \mathrm{e}^{-X_t}\int_0^t \mathrm{e}^{X_s}\,\mathrm{d}s &= \int_{0+}^t \mathrm{e}^{-X_{s-}}\,\mathrm{d}\left(\int_0^s \mathrm{e}^{X_y}\,\mathrm{d}y\right) \\ &\quad + \int_{0+}^t \left(\int_0^s \mathrm{e}^{X_y}\,\mathrm{d}y\right)\mathrm{d}(\mathrm{e}^{-X_s}) + \left[\mathrm{e}^{-X_\cdot}, \int_0^\cdot \mathrm{e}^{X_s}\,\mathrm{d}s\right]_t, \end{aligned}$$

wherein the quadratic covariation is, in view of (3.9),

$$\left[\log\delta \int_0^\cdot \mathrm{e}^{-X_{s-}}\,\mathrm{d}s, \int_0^\cdot \mathrm{e}^{X_s}\,\mathrm{d}s\right]_t = \int_0^t \mathrm{d}[s\log\delta, s] = 0, \qquad t \geq 0.$$

Thus,

$$\mathrm{d}\left(\mathrm{e}^{-X_t}\int_0^t \mathrm{e}^{X_s}\,\mathrm{d}s\right) = \mathrm{d}t + \left(\int_0^t \mathrm{e}^{X_s}\,\mathrm{d}s\right)\mathrm{d}(\mathrm{e}^{-X_t}), \qquad t \geq 0,$$

by the associativity of the stochastic integral. So we obtain from (3.4) that (3.7) holds, from which (3.8) follows after application of (3.9).

The equation (2.4) shows that the discrete-time GARCH(1, 1) process satisfies

$$\sigma_{n+1}^2 - \sigma_n^2 = \beta - (1-\delta)\sigma_n^2 + \lambda\sigma_n^2\varepsilon_n^2, \qquad n \in \mathbb{N}_0,$$

which by summation yields that

$$\sigma_n^2 = \beta n - (1-\delta)\sum_{i=0}^{n-1}\sigma_i^2 + \lambda\sum_{i=0}^{n-1}\sigma_i^2\varepsilon_i^2 + \sigma_0^2, \tag{3.10}$$

analogously to (3.8). (Note that we use $(\sigma_n^2)_{n\in\mathbb{N}_0}$ to denote the squared discrete-time GARCH volatility process, and $(\sigma_t^2)_{t\geq 0}$ to denote the continuous-time process defined by (3.4); these are quite different processes but this should cause no confusion.) Thus, (3.8) captures the 'feedback' and 'autoregressive' aspects of the GARCH volatility process which are important features of its application.

By comparison with Theorem 2.1 we are now led to the following result.

Theorem 3.1. *Suppose that*

$$\int_{\mathbb{R}} \log\left(1 + \frac{\lambda}{\delta} y^2\right) \Pi_L(\mathrm{d}y) < -\log \delta \tag{3.11}$$

(which, since $\delta > 0$, incorporates the requirement that the integral be finite). Then $\sigma_t^2 \xrightarrow{\mathrm{D}} \sigma_\infty^2$ as $t \to \infty$ for a finite random variable σ_∞ satisfying

$$\sigma_\infty^2 \stackrel{\mathrm{D}}{=} \beta \int_0^\infty \mathrm{e}^{-X_t}\, \mathrm{d}t$$

(thus, the improper integral exists as a finite random variable, almost surely). Conversely, if (3.11) does not hold, then $\sigma_t^2 \xrightarrow{\mathrm{P}} \infty$ as $t \to \infty$.

Proof. By a continuous-time analogue of the Goldie–Maller theorem [17], due to Erickson and Maller [15], $\int_0^\infty \mathrm{e}^{-X_s}\, \mathrm{d}s$ converges almost surely to a finite random variable if $X_t \to \infty$ almost surely, and $\sigma_t^2 \xrightarrow{\mathrm{P}} \infty$ as $t \to \infty$ otherwise. By the stationarity of the increments of $(X_t)_{t\geq 0}$,

$$\mathrm{e}^{-X_t} \int_0^t \mathrm{e}^{X_s}\, \mathrm{d}s \stackrel{\mathrm{D}}{=} \int_0^t \mathrm{e}^{-X_s}\, \mathrm{d}s, \qquad t \geq 0.$$

Hence, we only need to show that (3.11) is equivalent to $X_t \to \infty$ almost surely as $t \to \infty$. Since $\Pi_X\{[0, \infty)\} = 0$, $\mathrm{E}\, X_1$ always exists (possibly, $\mathrm{E}\, X_1 = -\infty$) and $X_t/t \to \mathrm{E}\, X_1$ almost surely as $t \to \infty$ (e.g. [27, Theorem 36.3]). If $\mathrm{E}\, X_1 \leq 0$, then $X_t \to -\infty$ almost surely or $(X_t)_{t\geq 0}$ oscillates, so we need to show that $\mathrm{E}\, X_1 > 0$ if and only if (3.11) holds. From (3.6) we get that

$$\mathrm{E}\, X_1 = -\log \delta + \int_{(-\infty,0)} x \Pi_X(\mathrm{d}x) = -\log \delta - \int_{\mathbb{R}} \log\left(1 + \frac{\lambda}{\delta} y^2\right) \Pi_L(\mathrm{d}y),$$

implying the equivalence of $\mathrm{E}\, X_1 > 0$ and (3.11).

Next we show that $(\sigma_t^2)_{t\geq 0}$ is Markovian and further that, if the process is started at $\sigma_0^2 \stackrel{\mathrm{D}}{=} \sigma_\infty^2$, then it is strictly stationary.

Theorem 3.2. *The squared volatility process $(\sigma_t^2)_{t\geq 0}$, as given by (3.4), is a time-homogeneous Markov process. Moreover, if the limit variable σ_∞^2 in Theorem 3.1 exists and $\sigma_0^2 \stackrel{\mathrm{D}}{=} \sigma_\infty^2$, independent of $(L_t)_{t\geq 0}$, then $(\sigma_t^2)_{t\geq 0}$ is strictly stationary.*

Proof. Let $(\mathcal{F}_t)_{t\geq 0}$ be the filtration generated by $(\sigma_t^2)_{t\geq 0}$. Then, for $y \in [0, t)$,

$$\begin{aligned}
\sigma_t^2 &= \beta \int_0^y \mathrm{e}^{X_s}\, \mathrm{d}s \mathrm{e}^{-X_{y-}} \mathrm{e}^{-(X_{t-} - X_{y-})} + \beta \int_y^t \mathrm{e}^{X_s}\, \mathrm{d}s \mathrm{e}^{-X_{t-}} + \sigma_0^2 \mathrm{e}^{-X_{t-}} \\
&= (\sigma_y^2 - \sigma_0^2 \mathrm{e}^{-X_{y-}}) \mathrm{e}^{-(X_{t-} - X_{y-})} + \beta \int_y^t \mathrm{e}^{X_s}\, \mathrm{d}s \mathrm{e}^{-X_{t-}} + \sigma_0^2 \mathrm{e}^{-X_{t-}} \\
&= \sigma_y^2 A_{y,t} + B_{y,t}, \quad \text{say},
\end{aligned} \tag{3.12}$$

where

$$A_{y,t} := \mathrm{e}^{-(X_{t-} - X_{y-})} \quad \text{and} \quad B_{y,t} := \beta \int_y^t \mathrm{e}^{(X_s - X_{y-})}\, \mathrm{d}s \mathrm{e}^{-(X_{t-} - X_{y-})}$$

are independent of $\mathcal{F}_y$. This means that, conditional on $\mathcal{F}_y$, σ_t^2 depends only on σ_y^2, from which it follows easily that $(\sigma_t^2)_{t\geq 0}$ is a Markov process.

Next, let $\mathrm{D}[0,\infty)$ be the space of càdlàg functions on $[0,\infty)$ and define the function $g_{y,t} : \mathrm{D}[0,\infty) \to \mathbb{R}^2;\ x \mapsto (\mathrm{e}^{-(x_{t-}-x_{y-})},\ \beta\int_y^t \mathrm{e}^{-(x_{t-}-x_s)}\,\mathrm{d}s)$. Since $(X_t)_{t\geq 0}$ is a Lévy process, $(X_s)_{s\geq 0} \stackrel{\mathrm{D}}{=} (X_{s+h}-X_h)_{s\geq 0}$ for any $h>0$. Furthermore, we have that $(A_{y,t}, B_{y,t}) = g_{y,t}((X_s)_{s\geq 0})$ and $(A_{y+h,t+h}, B_{y+h,t+h}) = g_{y,t}((X_{s+h}-X_h)_{s\geq 0})$. This shows that the joint distribution of $(A_{y,t}, B_{y,t})$ depends only on $t-y$. By the independence of σ_y^2 and $(A_{y,t}, B_{y,t})$, the transition functions are thus time homogeneous.

It remains to show that $\sigma_t^2 \stackrel{\mathrm{D}}{=} \sigma_\infty^2$ for all $t>0$, provided that $\sigma_0^2 \stackrel{\mathrm{D}}{=} \sigma_\infty^2$. For calculating the distribution of

$$\sigma_{t+}^2 = \beta\int_0^t \mathrm{e}^{X_{s-}-X_t}\,\mathrm{d}s + \mathrm{e}^{-X_t}\sigma_0^2,$$

we can take any version of σ_0^2, independent of $(L_s)_{0\leq s\leq t}$, and with the distribution of σ_∞^2. A suitable choice is $\sigma_0^2 := \beta\int_0^\infty \mathrm{e}^{-(X_{s+t}-X_t)}\,\mathrm{d}s$. Then

$$\sigma_{t+}^2 = \beta\int_0^t \mathrm{e}^{(X_{(t-s)-}-X_t)}\,\mathrm{d}s + \mathrm{e}^{(X_{(t-t)-}-X_t)}\beta\int_0^\infty \mathrm{e}^{-(X_{s+t}-X_t)}\,\mathrm{d}s.$$

By the time-reversal property of Lévy processes (e.g. [4, Lemma II.2, p. 45]),

$$(X_{(t-s)-}-X_t)_{0\leq s\leq t} \stackrel{\mathrm{D}}{=} (-X_s)_{0\leq s\leq t}$$

and both processes are independent of σ_0^2 as chosen. Hence,

$$\begin{aligned}
\sigma_{t+}^2 &\stackrel{\mathrm{D}}{=} \beta\int_0^t \mathrm{e}^{-X_s}\,\mathrm{d}s + \mathrm{e}^{-X_t}\beta\int_0^\infty \mathrm{e}^{-(X_{s+t}-X_t)}\,\mathrm{d}s\\
&= \beta\int_0^t \mathrm{e}^{-X_s}\,\mathrm{d}s + \beta\int_t^\infty \mathrm{e}^{-X_s}\,\mathrm{d}s\\
&\stackrel{\mathrm{D}}{=} \sigma_0^2.
\end{aligned}$$

Since $\sigma_{t+}^2 = \sigma_t^2$ almost surely (σ_t^2 has no fixed points of discontinuity, almost surely), it follows that $\sigma_t^2 \stackrel{\mathrm{D}}{=} \sigma_0^2$ for all $t>0$.

For the process $G_t = \int_0^t \sigma_s\,\mathrm{d}L_s$, $t\geq 0$, note that, for any $y\in[0,t)$,

$$G_t = G_y + \int_{y+}^t \sigma_s\,\mathrm{d}L_s, \qquad t\geq 0.$$

Here, the integrand $(\sigma_s)_{y<s\leq t}$ depends on the past until time y only through σ_y, and the integrator L_s is independent of this past. From Theorem 3.2 we thus obtain:

Corollary 3.1. *The bivariate process $(\sigma_t, G_t)_{t\geq 0}$ is Markovian. If $(\sigma_t^2)_{t\geq 0}$ is the stationary version of the process with $\sigma_0^2 \stackrel{\mathrm{D}}{=} \sigma_\infty^2$, then $(G_t)_{t\geq 0}$ is a process with stationary increments.*

Remark 3.1. (i) The analogy between (3.8) and (3.10) is not exact, in that the parameterisation is slightly different; $1-\delta$ is replaced by $-\log\delta$ in the continuous version.

(ii) The value $\lambda = 0$ is permissible in (3.3), in which case $X_t = -t \log \delta$ for $t \geq 0$ (with $0 < \delta < 1$), and by (3.4) we have the trivial solution

$$\sigma_t^2 = \frac{\beta(1-\delta^t)}{-\log \delta} + \sigma_0^2 \delta^t, \qquad t \geq 0.$$

For the discrete-time GARCH model, from (2.5), when $\lambda = 0$,

$$\sigma_n^2 = \beta \sum_{i=0}^{n-1} \delta^{n-1-i} + \sigma_0^2 \delta^n = \frac{\beta(1-\delta^n)}{1-\delta} + \sigma_0^2 \delta^n, \qquad n \in \mathbb{N},$$

again demonstrating the correspondence between the discrete- and continuous-time versions. (The same results if we take $L \equiv 0$.)

(iii) Only $\delta > 0$ is allowed in (3.3)–(3.9). Thus, our continuous-time GARCH model does not contain a continuous-time ARCH submodel. To accommodate the case $\delta = 0$, which is the ARCH situation, we have to go back to (3.1). Then X_t should be taken as

$$X_t = -t \log \lambda - \sum_{0<s\leq t} \log(\Delta L_s)^2 \mathbf{1}_{\{\Delta L_s \neq 0\}}, \qquad t \geq 0,$$

and this is only a well-defined (Lévy) process if L is compound Poisson.

We treat this important example in the more general GARCH setup.

Example 3.1. (*The compound Poisson COGARCH*(1, 1) *model.*) Let $(L_t)_{t\geq 0}$ be a compound Poisson process, with jumps ε_n at the times T_n of an independent Poisson process $(N_t)_{t\geq 0}$. Thus, $L_t = \sum_{i=1}^{N_t} \varepsilon_i$, with $L_0 = T_0 = 0$ and $N_t = \max\{n \geq 1 : T_n \leq t\}$, $t \geq 0$. Suppose that $\mathrm{P}\{\varepsilon_1 = 0\} = 0$. Evaluated at T_n, L has the jump $\Delta L_{T_n} = L_{T_n} - L_{T_n-} = \varepsilon_n$, so $\Delta X_{T_n} = X_{T_n} - X_{T_{n-1}} = (1 - \Delta T_n) \log \delta - \log(\delta + \lambda \varepsilon_n^2)$, where the $\Delta T_n = T_n - T_{n-1}$ are i.i.d. exponential random variables. This shows that the continuous-time GARCH process evaluated at the jump times differs from a discrete-time GARCH process, due to the term $(1 - \Delta T_n) \log \delta$, though it evidently has similar characteristics. A simulation of such a process, driven by a compound Poisson process with rate 1 and standard normally distributed jump sizes, is given in Figure 1. The parameters were chosen as $\beta = 1$, $\delta = 0.95$ and $\lambda = 0.045$. For these values, a stationary distribution of $(\sigma_t^2)_{t\geq 0}$ exists and has finite second, but not third, moment (by (4.12) below). The parameters were chosen so that the simulated series is close to nonstationarity, as is often observed for financial time series.

Of course, the class of continuous-time processes given by our model is much larger than that of the compound Poisson processes. Examples currently of great interest in financial modelling are the pure jump process generated by a normal inverse Gaussian or hyperbolic process [2], [11], a variance gamma (VG) process [19], a Meixner process (e.g. [28]) or simply a stable process (e.g. [25]). These processes are not compound Poisson—they have infinitely many jumps, almost surely, in finite time intervals—and have been successfully used for financial modelling in various applications.

It is instructive to compare the process defined in (3.4) with the stochastic volatility model of Barndorff-Nielsen and Shephard [2], [3], which specifies that

$$\mathrm{d}\sigma_t^2 = -\lambda \sigma_t^2 \,\mathrm{d}t + \mathrm{d}z_{\lambda t}, \qquad t \geq 0 \tag{3.13}$$

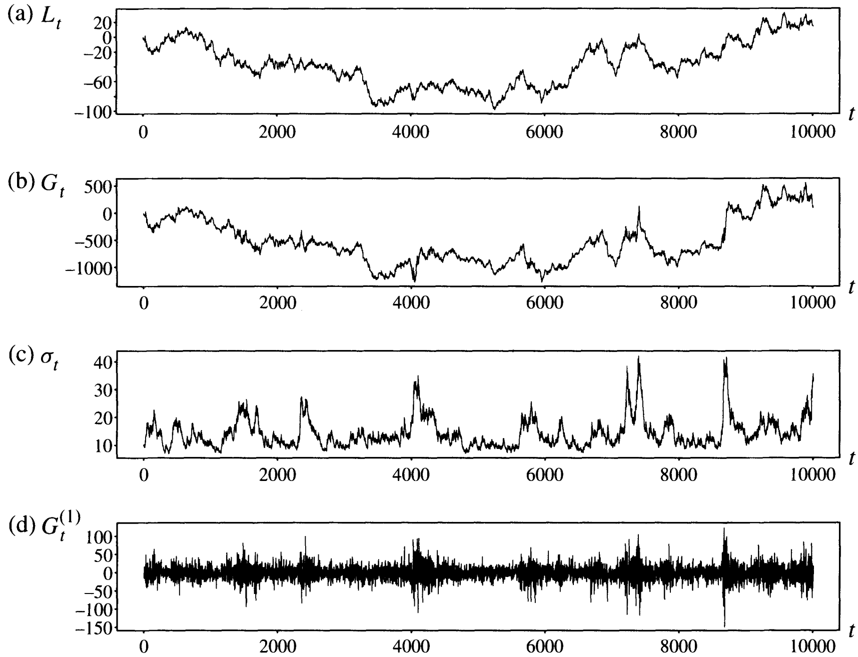

FIGURE 1: (a) A simulated compound Poisson process $(L_t)_{0\le t\le 10\,000}$ with rate 1 and standard normally distributed jump sizes with the corresponding (b) COGARCH process (G_t), (c) volatility process (σ_t) and (d) differenced COGARCH process $(G_t^{(1)})$ of order 1, where $G_t^{(1)} = G_{t+1} - G_t$. The parameters are: $\beta = 1$, $\delta = 0.95$ and $\lambda = 0.045$. The starting value was chosen as $\sigma_0 = 10$.

(with $\lambda > 0$) for a subordinator (increasing Lévy process) $(z_t)_{t\ge 0}$. The solution to (3.13) is the Ornstein–Uhlenbeck-type process

$$\sigma_t^2 = \mathrm{e}^{-\lambda t} \int_0^t \mathrm{e}^{\lambda s}\, \mathrm{d}z_{\lambda s} + \mathrm{e}^{-\lambda t}\sigma_0^2, \qquad t \ge 0. \tag{3.14}$$

By comparison with (3.4), the Lévy process $(z_t)_{t\ge 0}$ is in the integrator rather than in the integrand. A class of processes which includes both models is obtained if we let σ_t^2 have the same distribution as

$$\mathrm{e}^{-\xi_t}\sigma_0^2 + \int_0^t \mathrm{e}^{-\xi_{s-}}\, \mathrm{d}\eta_s, \qquad t \ge 0, \tag{3.15}$$

where (ξ, η) is a bivariate Lévy process. When $(\eta_t)_{t\ge 0}$ is pure drift, we get (3.4) and, when $(\xi_t)_{t\ge 0}$ is pure drift (to ∞), we get a random variable with the same distribution as the one in (3.14). Conditions for convergence of (3.15) as $t \to \infty$ are given in [15], but we do not investigate further at this stage.

An alternative stochastic volatility model was introduced in [1, Section 5], with proposed volatility the stationary process

$$\sigma(t) = \int_{-\infty}^t M(t-s)\, \mathrm{d}L(s), \qquad t \ge 0,$$

where M is a 'memory' function and $(L_t)_{t\geq 0}$ is a Lévy process such that $L(1)$ is a random variable with positive support. In [1], as well as in [2] and [3], the logarithmic price process is modelled by the stochastic differential equation

$$\mathrm{d}x^*(t) = (\mu + b\sigma^2(t))\,\mathrm{d}t + \sigma(t)\,\mathrm{d}W(t), \qquad t > 0,$$

where μ and b are constants and $(W(t))_{t\geq 0}$ is a standard Brownian motion, independent of the Lévy process $(L_t)_{t\geq 0}$. The Itô solution of this equation is given by

$$x^*(t) = \int_0^t \sigma(u)\,\mathrm{d}W(u) + \mu t + b\sigma^{2*}(t), \qquad t \geq 0,$$

where $\sigma^{2*}(t) = \int_0^t \sigma^2(u)\,\mathrm{d}u$. For $\Delta > 0$, the random variables

$$y_n = x^*(n\Delta) - x^*((n-1)\Delta), \qquad n \in \mathbb{N},$$

model the logarithmic asset returns over time periods of length Δ.

4. Second-order properties of the volatility process

In this section we derive moments and autocorrelation functions of the squared stochastic volatility process $(\sigma_t^2)_{t\geq 0}$. It is obvious from (3.4) that moments of $(\sigma_t^2)_{t\geq 0}$ correspond to certain exponential moments of $(X_t)_{t\geq 0}$. The following lemma specifies the relationships exactly.

Lemma 4.1. *Keep $c > 0$ throughout.*

(a) *Let $\lambda > 0$. Then the Laplace transform $\mathrm{E}\,\mathrm{e}^{-cX_t}$ of X_t at c if finite for some $t > 0$ or, equivalently, for all $t > 0$, if and only if $\mathrm{E}\,L_1^{2c} < \infty$.*

(b) *When $\mathrm{E}\mathrm{e}^{-cX_1} < \infty$, define $\Psi(c) = \Psi_X(c) = \log \mathrm{E}\mathrm{e}^{-cX_1}$. Then $|\Psi(c)| < \infty$, $\mathrm{E}\mathrm{e}^{-cX_t} = \mathrm{e}^{t\Psi(c)}$ and*

$$\Psi(c) = c\log\delta + \int_{\mathbb{R}} \left(\left(1 + \frac{\lambda}{\delta}y^2\right)^c - 1 \right) \Pi_L(\mathrm{d}y). \tag{4.1}$$

(c) *If $\mathrm{E}\,L_1^2 < \infty$ and $\Psi(1) < 0$, then (3.11) holds and σ_t^2 converges in distribution to a finite random variable.*

(d) *If $\Psi(c) < 0$ for some $c > 0$, then $\Psi(d) < 0$ for all $d \in (0, c)$.*

Proof. (a) By Theorem 25.17 of [27], the Laplace transform $\mathrm{E}\mathrm{e}^{-cX_t}$ is finite for some, and hence all, $t \geq 0$ if and only if

$$\int_{\{|x|>1\}} \mathrm{e}^{-cx}\,\Pi_X(\mathrm{d}x) = \int_{(-\infty,-1)} \mathrm{e}^{-cx}\,\Pi_X(\mathrm{d}x) = \int_{\{|y|>\sqrt{(\mathrm{e}-1)\delta/\lambda}\}} \left(1 + \frac{\lambda}{\delta}y^2\right)^c \Pi_L(\mathrm{d}y)$$

is finite, giving (a) (see e.g. [27, Theorem 25.3]).

(b) This follows from Theorem 25.17 of [27] and (3.6).

(c) From (4.1) we see that $\Psi(1) < 0$ is equivalent to

$$\frac{\lambda}{\delta}\int_{\mathbb{R}} y^2\,\Pi_L(\mathrm{d}y) < -\log\delta.$$

Since $\log(1 + (\lambda/\delta)y^2) < (\lambda/\delta)y^2$, this implies (3.11).

(d) Let $\Psi(c) < 0$. From (a) and (b) we conclude that $\Psi(d)$ is definable when $0 < d \leq c$. From (4.1) it then follows that $\Psi(d) < 0$ if and only if

$$\frac{1}{d}\int_{\mathbb{R}}\left(\left(1+\frac{\lambda}{\delta}y^2\right)^d - 1\right)\Pi_L(\mathrm{d}y) < -\log\delta.$$

Since the function $d \mapsto (1/d)((1+(\lambda/\delta)y^2)^d - 1)$ from $(0,\infty)$ to $\mathbb{R}$ is increasing for any fixed y, the result follows.

The next result gives the first two moments and the autocovariance function of $(\sigma_t^2)_{t\geq 0}$ in terms of the function Ψ, showing in particular that the autocovariance function decreases exponentially fast with the lag.

Proposition 4.1. *Let $\lambda > 0$, $t > 0$ and $h \geq 0$.*

(a) *The mean $\mathrm{E}\,\sigma_t^2$ is finite if and only if $\mathrm{E}\,L_1^2 < \infty$ and $\mathrm{E}\,\sigma_0^2 < \infty$. If this is so, then*

$$\mathrm{E}\,\sigma_t^2 = \frac{\beta}{-\Psi(1)} + \left(\mathrm{E}\,\sigma_0^2 + \frac{\beta}{\Psi(1)}\right)\mathrm{e}^{t\Psi(1)}, \tag{4.2}$$

where, if $\Psi(1) = 0$, the right-hand side has to be interpreted as its limit as $\Psi(1) \to 0$, that is, $\mathrm{E}\,\sigma_t^2 = \beta t + \mathrm{E}\,\sigma_0^2$.

(b) *The second moment $\mathrm{E}\,\sigma_t^4$ is finite if and only if $\mathrm{E}\,L_1^4 < \infty$ and $\mathrm{E}\,\sigma_0^4 < \infty$. In that case, the following formulae hold (with a suitable interpretation as a limit if some of the denominators are zero):*

$$\begin{aligned}\mathrm{E}\,\sigma_t^4 &= \frac{2\beta^2}{\Psi(1)\Psi(2)} + \frac{2\beta^2}{\Psi(2)-\Psi(1)}\left(\frac{\mathrm{e}^{t\Psi(2)}}{\Psi(2)} - \frac{\mathrm{e}^{t\Psi(1)}}{\Psi(1)}\right)\\ &\quad + 2\beta\,\mathrm{E}\,\sigma_0^2\left(\frac{\mathrm{e}^{t\Psi(2)} - \mathrm{e}^{t\Psi(1)}}{\Psi(2)-\Psi(1)}\right) + \mathrm{E}\,\sigma_0^4\mathrm{e}^{t\Psi(2)}, \end{aligned}\tag{4.3}$$

$$\operatorname{cov}(\sigma_t^2, \sigma_{t+h}^2) = \operatorname{var}(\sigma_t^2)\mathrm{e}^{h\Psi(1)}. \tag{4.4}$$

Proof. (a) We start with the calculation of $\mathrm{E}\,\sigma_t^2$. Using Fubini's theorem and the fact that σ_0^2 is independent of all the other quantities, we conclude from (3.4) and Lemma 4.1 that

$$\mathrm{E}\,\sigma_t^2 = \beta\,\mathrm{E}\int_0^t \mathrm{e}^{X_s - X_{t-}}\,\mathrm{d}s + \mathrm{E}\,\sigma_0^2\,\mathrm{E}\,\mathrm{e}^{-X_{t-}} = \beta\int_0^t \mathrm{E}\,\mathrm{e}^{-X_s}\,\mathrm{d}s + \mathrm{E}\,\sigma_0^2\,\mathrm{E}\,\mathrm{e}^{-X_t}$$

is finite if and only if $\mathrm{E}\,L_1^2 < \infty$ and $\mathrm{E}\,\sigma_0^2 < \infty$. Then (4.2) follows since

$$\mathrm{E}\,\sigma_t^2 = \beta\int_0^t \mathrm{e}^{s\Psi(1)}\,\mathrm{d}s + \mathrm{E}\,\sigma_0^2\mathrm{e}^{t\Psi(1)}.$$

(b) Assume that $\mathrm{E}\,L_1^4 < \infty$ and $\mathrm{E}\,\sigma_0^4 < \infty$. We calculate $\mathrm{E}\,\sigma_t^4$ as follows:

$$\begin{aligned}\mathrm{E}\,\sigma_t^4 &= \beta^2\,\mathrm{E}\left(\int_0^t \mathrm{e}^{X_s - X_t}\,\mathrm{d}s\right)^2 + 2\beta\,\mathrm{E}\,\sigma_0^2\,\mathrm{E}\int_0^t \mathrm{e}^{X_s - 2X_t}\,\mathrm{d}s + \mathrm{E}\,\sigma_0^4\,\mathrm{E}\,\mathrm{e}^{-2X_t}\\ &=: \beta^2\,\mathrm{E}\,I_1 + 2\beta\,\mathrm{E}\,\sigma_0^2\,\mathrm{E}\,I_2 + \mathrm{E}\,\sigma_0^4\mathrm{e}^{t\Psi(2)}, \quad \text{say.}\end{aligned}$$

Using the stationarity of increments, we get that

$$\begin{aligned}\left(\int_0^t \mathrm{e}^{X_s-X_t}\,\mathrm{d}s\right)^2 &\overset{\mathrm{D}}{=} \left(\int_0^t \mathrm{e}^{-X_s}\,\mathrm{d}s\right)^2\\ &= \int_0^t\int_0^t \mathrm{e}^{-X_s}\mathrm{e}^{-X_u}\,\mathrm{d}u\,\mathrm{d}s\\ &= 2\int_0^t\int_0^s \mathrm{e}^{-(X_s-X_u)}\mathrm{e}^{-2X_u}\,\mathrm{d}u\,\mathrm{d}s.\end{aligned}$$

Then, by the independence of increments,

$$\begin{aligned}\mathrm{E}\,I_1 &= 2\int_0^t\int_0^s (\mathrm{E}\mathrm{e}^{-(X_s-X_u)})(\mathrm{E}\mathrm{e}^{-2X_u})\,\mathrm{d}u\,\mathrm{d}s\\ &= 2\int_0^t\int_0^s \mathrm{e}^{(s-u)\Psi(1)}\mathrm{e}^{u\Psi(2)}\,\mathrm{d}u\,\mathrm{d}s\\ &= \frac{2}{\Psi(1)\Psi(2)} + \frac{2}{\Psi(2)-\Psi(1)}\left(\frac{\mathrm{e}^{t\Psi(2)}}{\Psi(2)} - \frac{\mathrm{e}^{t\Psi(1)}}{\Psi(1)}\right).\end{aligned}$$

By similar arguments,

$$\begin{aligned}\mathrm{E}\,I_2 &= \mathrm{E}\int_0^t \mathrm{e}^{X_s-2X_t}\,\mathrm{d}s\\ &= \mathrm{E}\int_0^t \mathrm{e}^{-2(X_t-X_s)}\mathrm{e}^{-X_s}\,\mathrm{d}s\\ &= \int_0^t \mathrm{e}^{(t-s)\Psi(2)}\mathrm{e}^{s\Psi(1)}\,\mathrm{d}s\\ &= \frac{\mathrm{e}^{t\Psi(2)} - \mathrm{e}^{t\Psi(1)}}{\Psi(2)-\Psi(1)}.\end{aligned}$$

Putting all this together, we see that $\mathrm{E}\,\sigma_t^4 < \infty$, and we obtain (4.3). The converse follows similarly.

For the proof of (4.4), let $(\mathcal{F}_t)_{t\geq 0}$ be the filtration generated by $(\sigma_t^2)_{t\geq 0}$. Then it follows from (3.12) and (4.2) that

$$\begin{aligned}\mathrm{E}(\sigma_{t+h}^2 \mid \mathcal{F}_t) &= \sigma_t^2\mathrm{e}^{h\Psi(1)} + \beta\int_0^h \mathrm{e}^{s\Psi(1)}\,\mathrm{d}s\\ &= (\sigma_t^2 - \mathrm{E}\,\sigma_0^2)\mathrm{e}^{h\Psi(1)} + \mathrm{E}\,\sigma_h^2.\end{aligned}\tag{4.5}$$

Then

$$\begin{aligned}\mathrm{E}(\sigma_{t+h}^2\sigma_t^2) &= \mathrm{E}(\sigma_t^2((\sigma_t^2 - \mathrm{E}\,\sigma_0^2)\mathrm{e}^{h\Psi(1)} + \mathrm{E}\,\sigma_h^2))\\ &= (\mathrm{E}\,\sigma_t^4 - \mathrm{E}\,\sigma_t^2\,\mathrm{E}\,\sigma_0^2)\mathrm{e}^{h\Psi(1)} + \mathrm{E}\,\sigma_t^2\,\mathrm{E}\,\sigma_h^2.\end{aligned}\tag{4.6}$$

Calculations using (4.2) show that

$$\mathrm{E}\,\sigma_t^2\,\mathrm{E}\,\sigma_h^2 - \mathrm{E}\,\sigma_t^2\,\mathrm{E}\,\sigma_{t+h}^2 = (\mathrm{E}\,\sigma_t^2\,\mathrm{E}\,\sigma_0^2 - (\mathrm{E}\,\sigma_t^2)^2)\mathrm{e}^{h\Psi(1)}.$$

Then (4.4) follows immediately from (4.6).

The following results hold for the stationary version of the volatility process. Recall from Theorem 3.2 that this is $(\sigma_t)_{t\geq 0}$ for $\sigma_0 \stackrel{\mathrm{D}}{=} \sigma_\infty$, where σ_∞ is the limit random variable from Theorem 3.1. Results related to the following proposition can be found in [5]; see also the references therein.

Proposition 4.2. *Let $\lambda > 0$. Then the kth moment of σ_∞^2 is finite if and only if $\mathrm{E}\, L_1^{2k} < \infty$ and $\Psi(k) < 0$ for $k \in \mathbb{N}$. In this case,*

$$\mathrm{E}\,\sigma_\infty^{2k} = k!\beta^k \prod_{l=1}^{k} \frac{1}{-\Psi(l)}. \tag{4.7}$$

Proof. Using Fubini's theorem and the independent and stationary increments property, it follows from Theorem 3.1 that, for $k \in \mathbb{N}$,

$$\begin{aligned}
\mathrm{E}\,\sigma_\infty^{2k} &= \beta^k\, \mathrm{E}\bigg(\int_0^\infty \mathrm{e}^{-X_t}\,\mathrm{d}t\bigg)^k \\
&= \beta^k\, \mathrm{E}\int_0^\infty \cdots \int_0^\infty \mathrm{e}^{-X_{t_1}} \cdots \mathrm{e}^{-X_{t_k}}\,\mathrm{d}t_k \ldots \mathrm{d}t_1 \\
&= k!\beta^k\, \mathrm{E}\int_0^\infty \int_0^{t_1} \cdots \int_0^{t_{k-1}} \mathrm{e}^{-(X_{t_1}-X_{t_2})}\mathrm{e}^{-2(X_{t_2}-X_{t_3})} \cdots \mathrm{e}^{-(k-1)(X_{t_{k-1}}-X_{t_k})}\mathrm{e}^{-kX_{t_k}}\,\mathrm{d}t_k \ldots \mathrm{d}t_1 \\
&= k!\beta^k \int_0^\infty \int_0^{t_1} \cdots \int_0^{t_{k-1}} \mathrm{e}^{t_1\Psi(1)}\mathrm{e}^{t_2(\Psi(2)-\Psi(1))} \cdots \mathrm{e}^{t_k(\Psi(k)-\Psi(k-1))}\,\mathrm{d}t_k \ldots \mathrm{d}t_1 \\
&= k!\beta^k \prod_{l=1}^{k} \frac{1}{-\Psi(l)},
\end{aligned}$$

provided that $\Psi(1), \ldots, \Psi(k)$ are all defined and negative. The last equality follows from analytic calculations. If $j \in \{1, \ldots, k\}$ is the first index for which $\Psi(j) \geq 0$ or $\mathrm{E}\mathrm{e}^{-jX_1} = \infty$, then the calculation shows that $\mathrm{E}\,\sigma_\infty^{2j} = \infty$. Since $\mathrm{E}\,\sigma_\infty^{2k} < \infty$ implies that $\mathrm{E}\,\sigma_\infty^{2j} < \infty$ for $j < k$, it follows from Lemma 4.1 that $\mathrm{E}\,\sigma_\infty^{2k} < \infty$ if and only if $\Psi(k)$ is defined (i.e. $\mathrm{E}\, L_1^{2k} < \infty$) and negative.

From this result we obtain the mean and second moment of σ_∞^2; we also calculate the autocovariance function of the stationary process $(\sigma_t^2)_{t\geq 0}$.

Corollary 4.1. *If $(\sigma_t^2)_{t\geq 0}$ is the stationary process with $\sigma_0^2 \stackrel{\mathrm{D}}{=} \sigma_\infty^2$, then*

$$\mathrm{E}\,\sigma_\infty^2 = \frac{\beta}{-\Psi(1)}, \tag{4.8}$$

$$\mathrm{E}\,\sigma_\infty^4 = \frac{2\beta^2}{\Psi(1)\Psi(2)}, \tag{4.9}$$

$$\operatorname{cov}(\sigma_t^2, \sigma_{t+h}^2) = \beta^2\bigg(\frac{2}{\Psi(1)\Psi(2)} - \frac{1}{\Psi^2(1)}\bigg)\mathrm{e}^{h\Psi(1)}, \qquad t, h \geq 0, \tag{4.10}$$

provided that $\mathrm{E}\, L_1^{2k} < \infty$ and $\Psi(k) < 0$ when $k = 1$ for (4.8) and when $k = 2$ for (4.9) and (4.10).

Proof. The results (4.8) and (4.9) are immediate from (4.7) for $\lambda > 0$, and (4.10) follows by inserting (4.8) and (4.9) into (4.4).

Of course, it is our goal to express the quantities Ψ_X in terms of the driving Lévy process $(L_t)_{t\geq 0}$. We obtain the following results for the existence of moments.

Theorem 4.1. *Let $k \in \mathbb{N}$, $\delta \in (0, 1)$ and $\lambda \geq 0$. Then the limit variable σ_∞^2 exists and has finite kth moment if and only if*

$$\frac{1}{k}\int_{\mathbb{R}}\left(\left(1+\frac{\lambda}{\delta}y^2\right)^k - 1\right)\Pi_L(\mathrm{d}y) < -\log\delta. \tag{4.11}$$

Proof. By Lemma 4.1, $\mathrm{E}\,L_1^{2k} < \infty$ and $\Psi(k) < 0$ imply that $\mathrm{E}\,L_1^2 < \infty$ and $\Psi(1) < 0$, which implies the stability condition (3.11). Now the condition for $\mathrm{E}\,\sigma_\infty^{2k}$ to be finite is that $\mathrm{E}\,L_1^{2k} < \infty$ and $\Psi(k) < 0$, which is (4.11).

As for the discrete-time GARCH model, the continuous-time GARCH model also turns out to be heavy tailed. This is an implication of the fact that the volatility process never has moments of all orders.

Proposition 4.3. (a) *For any Lévy process $(L_t)_{t\geq 0}$ with nonzero Lévy measure such that $\int_{\mathbb{R}}\log(1 + y^2)\Pi_L(\mathrm{d}y)$ is finite, there exist parameters $\delta, \lambda \in (0, 1)$ for which σ_∞^2 exists, but $\mathrm{E}\,\sigma_\infty^2 = \infty$.*

(b) *Let $k \in \mathbb{N}$. For any Lévy process $(L_t)_{t\geq 0}$ such that $\mathrm{E}\,L_1^{2k} < \infty$ and for any $\delta \in (0, 1)$, there exists a $\lambda_\delta > 0$ such that the limit variable σ_∞^2 exists with $\mathrm{E}\,\sigma_\infty^{2k} < \infty$ for any pair of parameters (δ, λ) such that $0 \leq \lambda \leq \lambda_\delta$.*

(c) *Suppose that $0 < \delta < 1$ and $\lambda > 0$. Then for no Lévy process $(L_t)_{t\geq 0}$ (with nonzero Lévy measure) do the moments of all orders of σ_∞^2 exist. In particular, the Laplace transform of σ_∞^2 does not exist for any negative argument.*

Proof. (a) Let $\delta_0 := \exp(-\int_{\mathbb{R}}\log(1 + y^2)\Pi_L(\mathrm{d}y))$ and $\delta_1 := \exp(-\int_{\mathbb{R}} y^2\Pi_L(\mathrm{d}y))$. Then $0 \leq \delta_1 < \delta_0 < 1$ and, for any $\lambda = \delta \in (\delta_1, \delta_0)$, (3.11) holds, but (4.11) does not.

(b) Let $\delta \in (0, 1)$ be fixed. Since $\mathrm{E}\,L_1^{2k} < \infty$, the left-hand side of (4.11) is finite for any $\lambda > 0$, and goes to zero as $\lambda \to 0$. Choosing λ sufficiently small then implies (4.11).

(c) Let $\eta > 0$ be such that $q := \Pi_L(\{y : |y| \geq \eta\}) > 0$. Then, for $k \in \mathbb{N}$,

$$\int_{\mathbb{R}}\left(\left(1+\frac{\lambda}{\delta}y^2\right)^k - 1\right)\Pi_L(\mathrm{d}y) \geq q\left(\left(1+\frac{\lambda}{\delta}\eta^2\right)^k - 1\right).$$

If all moments of σ_∞^2 existed, this would imply that

$$\left(1+\frac{\lambda}{\delta}\eta^2\right)^k - 1 < k\frac{-\log\delta}{q} \quad \text{for all } k \in \mathbb{N},$$

a contradiction.

Example 4.1. (*The compound Poisson GARCH*(1, 1) *model.*) Let $(L_t)_{t\geq 0}$ be a compound Poisson process with Poisson rate $c > 0$ and jump distribution ϑ. Then $\Pi_L = c\vartheta$. Let Y be a random variable with distribution ϑ and set $Z := \lambda Y^2/\delta$. Then, for $k \in \mathbb{N}$,

$$\int_{\mathbb{R}} \left(\left(1 + \frac{\lambda}{\delta} y^2 \right)^k - 1 \right) \Pi_L(\mathrm{d}y) = c\,\mathrm{E}((1+Z)^k - 1),$$

and $(\sigma_t^2)_{t\geq 0}$ is a stationary Markov process whose stationary distribution has finite kth moment if and only if

$$\mathrm{E}(1+Z)^k - 1 + \frac{k}{c}\log\delta < 0, \tag{4.12}$$

which is equivalent to (4.11) in this case.

5. Second-order properties of the GARCH process

In (3.5), the integrated GARCH process was defined to satisfy $\mathrm{d}G_t = \sigma_t\,\mathrm{d}L_t$ for $t > 0$, that is, G jumps at the same time as L does and has jumps of size $\Delta G_t = \sigma_t \Delta L_t$. This definition implies that, for any fixed time point t, all moments of ΔG_t are zero. It makes sense, however, to calculate moments for the increments of G in arbitrary time intervals. Consequently, for $r > 0$, set

$$G_t^{(r)} := G_{t+r} - G_t = \int_{t+}^{t+r} \sigma_s\,\mathrm{d}L_s, \qquad t \geq 0.$$

We shall restrict ourselves to the case of stationary $(\sigma_t^2)_{t\geq 0}$. Recall from Corollary 3.1 that this implies strict stationarity of $(G_t^{(r)})_{t\geq 0}$.

Proposition 5.1. *Suppose that* $(L_t)_{t\geq 0}$ *is a quadratic pure jump process (i.e.* $\tau_L^2 = 0$ *in (3.2)) with* $\mathrm{E}\,L_1^2 < \infty$ *and* $\mathrm{E}\,L_1 = 0$*, and that* $\Psi(1) < 0$*. Let* $(\sigma_t^2)_{t\geq 0}$ *be the stationary volatility process with* $\sigma_0^2 \overset{\mathrm{D}}{=} \sigma_\infty^2$*. Then, for any* $t \geq 0$ *and* $h \geq r > 0$*,*

$$\mathrm{E}\,G_t^{(r)} = 0, \tag{5.1}$$

$$\mathrm{E}(G_t^{(r)})^2 = \frac{\beta r}{-\Psi(1)}\,\mathrm{E}\,L_1^2, \tag{5.2}$$

$$\operatorname{cov}(G_t^{(r)}, G_{t+h}^{(r)}) = 0. \tag{5.3}$$

Assume further that $\mathrm{E}\,L_1^4 < \infty$ *and* $\Psi(2) < 0$*. Then*

$$\operatorname{cov}((G_t^{(r)})^2, (G_{t+h}^{(r)})^2) = \left(\frac{\mathrm{e}^{-r\Psi(1)} - 1}{-\Psi(1)} \right) \mathrm{E}\,L_1^2 \operatorname{cov}(G_r^2, \sigma_r^2)\mathrm{e}^{h\Psi(1)}. \tag{5.4}$$

Assume further that $\lambda > 0$*, that* $\mathrm{E}\,L_1^8 < \infty$ *and* $\psi(4) < 0$*, that* $\int_{[-1,1]} |x|\Pi_L(\mathrm{d}x) < \infty$*, and that* $\int_{\mathbb{R}} x^3 \Pi_L(\mathrm{d}x) = 0$*. Then the right-hand side of (5.4) is strictly positive.*

Proof. Since $(L_t)_{t\geq 0}$ is a quadratic pure jump process, its quadratic variation process is given by

$$[L]_t = \sum_{0<s\leq t} (\Delta L_s)^2, \qquad t \geq 0$$

(e.g. [23, p. 71]). Then, by the properties of the stochastic integral,

$$\mathrm{E}\, G_r^2 = \mathrm{E} \int_0^r \sigma_s^2 \,\mathrm{d}[L]_s = \mathrm{E} \sum_{0<s\le r} \sigma_s^2 (\Delta L_s)^2.$$

The last can be calculated from the compensation formula (e.g. [4, p. 7]) and (4.8) as the right-hand side of (5.2). This shows the square integrability of G_r and (5.2) then follows from stationarity of the increments of $(G_t)_{t\ge 0}$.

From the Itô isometry for square-integrable martingales as integrators (e.g. [24, Section IV.27]) it follows that

$$\mathrm{E}(G_t^{(r)} G_{t+h}^{(r)}) = \mathrm{E} \int_0^{t+h+r} \sigma_s^2 \,\mathbf{1}_{(t,t+r]}(s)\, \mathbf{1}_{(t+h,t+h+r]}(s) \,\mathrm{d}[L]_s = 0$$

for $h \ge r$. By the martingale property of $(L_t)_{t\ge 0}$, we have (5.1) and, hence, (5.3) also follows.

For the proof of (5.4), assume further that $\mathrm{E}\, L_1^4 < \infty$ and $\Psi(2) < 0$, and let E_r denote conditional expectation given $\mathcal{F}_r$, the σ-algebra generated by $(\sigma_s^2)_{0\le s\le r}$. Integration by parts, the compensation formula and use of (3.12) and (4.5) imply that

$$\begin{aligned}
\mathrm{E}_r (G_h^{(r)})^2 &= \mathrm{E}_r \biggl(2 \int_{h+}^{h+r} G_{s-} \,\mathrm{d}G_s + [G]_h^{h+r} \biggr) \\
&= \mathrm{E}_r \biggl(2 \int_{h+}^{h+r} G_{s-} \sigma_s \,\mathrm{d}L_s \biggr) + \mathrm{E}_r \int_{h+}^{h+r} \sigma_s^2 \,\mathrm{d}[L]_s \\
&= 0 + \mathrm{E}_r \sum_{h<s\le h+r} (\sigma_r^2 A_{r,s} + B_{r,s})(\Delta L_s)^2 \\
&= \mathrm{E}\, L_1^2 \int_h^{h+r} (\sigma_r^2 \,\mathrm{E}\, A_{r,s} + \mathrm{E}\, B_{r,s}) \,\mathrm{d}s \\
&= \mathrm{E}\, L_1^2 \int_h^{h+r} \mathrm{E}_r(\sigma_s^2) \,\mathrm{d}s \\
&= \mathrm{E}\, L_1^2 \int_h^{h+r} [(\sigma_r^2 - \mathrm{E}\,\sigma_0^2) \mathrm{e}^{(s-r)\Psi(1)} + \mathrm{E}\,\sigma_{s-r}^2] \,\mathrm{d}s \\
&= (\sigma_r^2 - \mathrm{E}\,\sigma_0^2)\, \mathrm{E}\, L_1^2 \int_0^r \mathrm{e}^{-s\Psi(1)} \,\mathrm{d}s \mathrm{e}^{h\Psi(1)} + \mathrm{E}\,\sigma_0^2 \,\mathrm{E}\, L_1^2 r.
\end{aligned}$$

Conditioning on $\mathcal{F}_r$ gives

$$\begin{aligned}
\mathrm{E}((G_0^{(r)})^2 (G_h^{(r)})^2) &= \mathrm{E}(G_r^2 \,\mathrm{E}_r (G_h^{(r)})^2) \\
&= \mathrm{E}\, L_1^2 \biggl(\frac{\mathrm{e}^{-r\Psi(1)} - 1}{-\Psi(1)} \biggr) \mathrm{E}(G_r^2 \sigma_r^2 - G_r^2 \,\mathrm{E}\,\sigma_0^2) \mathrm{e}^{h\Psi(1)} + \mathrm{E}\,\sigma_0^2 \,\mathrm{E}\, L_1^2 r \,\mathrm{E}\, G_r^2.
\end{aligned}$$

This shows that

$$\operatorname{cov}(G_r^2, (G_h^{(r)})^2) = \biggl(\frac{\mathrm{e}^{-r\Psi(1)} - 1}{-\Psi(1)} \biggr) \mathrm{E}\, L_1^2 \operatorname{cov}(G_r^2, \sigma_r^2) \mathrm{e}^{h\Psi(1)} + \mathrm{E}\, G_r^2 \biggl(\frac{\beta r}{-\Psi(1)} \mathrm{E}\, L_1^2 - \mathrm{E}\, G_r^2 \biggr).$$

The equation (5.4) then follows from (5.2).

Finally, assume that $\mathrm{E}\, L_1^8 < \infty$ and $\Psi(4) < 0$, that $\int_{[-1,1]} |x| \Pi_L(\mathrm{d}x) < \infty$, and that $\int_{\mathbb{R}} x^3 \Pi_L(\mathrm{d}x) = 0$; we prove that $\operatorname{cov}(G_t^2, \sigma_t^2) > 0$. First, we calculate $\mathrm{E}(G_t^2 \sigma_t^2)$. Using integration by parts,

$$G_t^2 = [G]_t + 2\int_0^t G_{s-}\, \mathrm{d}G_s = \sum_{0<s\le t} \sigma_s^2 (\Delta L_s)^2 + 2\int_0^t G_{s-}\sigma_s \, \mathrm{d}L_s.$$

Substituting from (3.8) gives

$$\frac{\lambda}{\delta} G_t^2 = \sigma_{t+}^2 - \beta t - \log\delta \int_0^t \sigma_s^2 \, \mathrm{d}s - \sigma_0^2 + 2\frac{\lambda}{\delta}\int_0^t G_{s-}\sigma_s \, \mathrm{d}L_s, \tag{5.5}$$

which we will multiply through by σ_t^2 and take expectations. Since $\int_{[-1,1]} |x| \Pi_L(\mathrm{d}x) < \infty$, $(L_t)_{t\ge 0}$ is of bounded variation and the last term in (5.5) gives rise via (3.12) to

$$\sigma_t^2 \int_0^t G_{s-}\sigma_s \, \mathrm{d}L_s = \int_{0+}^t G_{s-}\sigma_s(\sigma_s^2 A_{s,t} + B_{s,t})\, \mathrm{d}L_s, \tag{5.6}$$

wherein we substitute

$$A_{s,t} = \mathrm{e}^{X_{s-} - X_{t-}} \quad \text{and} \quad B_{s,t} = \beta \int_s^t \mathrm{e}^{X_u - X_{t-}}\, \mathrm{d}u.$$

Let $I_t := \int_{0+}^t \mathrm{e}^{X_{s-}} G_{s-}\sigma_s^3 \, \mathrm{d}L_s$. Since X_t has no fixed points of discontinuity, almost surely, to show that the A-component in (5.6) has expectation 0 it will suffice to show that $\mathrm{E}(\mathrm{e}^{-X_t} I_t) = 0$. Integration by parts gives

$$\mathrm{e}^{-X_t} I_t = \int_{0+}^t \mathrm{e}^{-X_{s-}}\, \mathrm{d}I_s + \int_{0+}^t I_{s-}\, \mathrm{d}(\mathrm{e}^{-X_s}) + C_t, \tag{5.7}$$

where C_t is the quadratic covariation. Since $\mathrm{E}\, L_1 = 0$ and $\psi(4) < 0$, I_t is a locally square-integrable zero-mean martingale and hence the first term on the right-hand side of (5.7) has expectation 0. Substituting

$$\mathrm{d}(\mathrm{e}^{-X_t}) = \mathrm{e}^{t\Psi(1)}\, \mathrm{d}(\mathrm{e}^{-X_t - t\Psi(1)} - 1) + \mathrm{e}^{-X_t}\Psi(1)\, \mathrm{d}t,$$

we can write the second term on the right-hand side of (5.7) as an integral with respect to a locally square-integrable zero-mean martingale, hence having expectation 0, plus $\Psi(1)\int_0^t \mathrm{e}^{-X_s} I_s \, \mathrm{d}s$. Since L_t is pure jump,

$$\Delta C_t = (\Delta \mathrm{e}^{-X_t})(\Delta I_t) = \frac{\lambda}{\delta} G_{t-}\sigma_t^3 (\Delta L_t)^3$$

(using (3.9)). Letting $M_t = \sum_{0<s\le t} (\Delta L_s)^3$, the quadratic covariation is

$$C_t = \frac{\lambda}{\delta}\int_{0+}^t G_{s-}\sigma_s^3 \, \mathrm{d}M_s,$$

and since M_t is a locally square-integrable martingale, with mean zero as a result of our assumption that $\int_{\mathbb{R}} x^3 \Pi_L(\mathrm{d}x) = 0$, we see that C_t has expectation 0. Taking expectations in (5.7) thus gives $\mathrm{E}(\mathrm{e}^{-X_t} I_t) = \Psi(1)\int_0^t \mathrm{E}(\mathrm{e}^{-X_s} I_s)\, \mathrm{d}s$, implying that $\mathrm{E}(\mathrm{e}^{-X_t} I_t) = 0$.

Write the B-component in (5.6) as

$$\beta\biggl(\int_0^t \mathrm{e}^{X_u-X_{t-}}\,\mathrm{d}u\biggr)\biggl(\int_{0+}^t G_{s-}\sigma_s\,\mathrm{d}L_s\biggr)-\beta\int_{0+}^t G_{s-}\sigma_s\biggl(\int_{0+}^s \mathrm{e}^{X_u-X_{t-}}\,\mathrm{d}u\biggr)\mathrm{d}L_s.$$

After integration by parts, this equals

$$\beta\int_0^t\biggl(\int_{0+}^s G_{u-}\sigma_u\,\mathrm{d}L_u\biggr)\mathrm{e}^{-(X_{t-}-X_s)}\,\mathrm{d}s+\beta\tilde{C}_t, \tag{5.8}$$

where

$$\Delta\tilde{C}_t=\biggl(\Delta\biggl(\mathrm{e}^{-X_t}\int_0^t \mathrm{e}^{X_u}\,\mathrm{d}u\biggr)\biggr)(G_{t-}\sigma_t\Delta L_t)=\frac{\lambda}{\delta}\mathrm{e}^{-X_{t-}}\biggl(\int_0^t \mathrm{e}^{X_u}\,\mathrm{d}u\biggr)G_{t-}\sigma_t(\Delta L_t)^3.$$

Here $\tilde{C}_t$ has expectation 0 again since $\int_{\mathbb{R}} x^3\Pi_L(\mathrm{d}x)=0$, so (5.8) has expectation 0. Thus, the last term in (5.5) contributes 0 to the expectation.

To deal with the other integral in (5.5), use (4.6) to write

$$\mathrm{E}(\sigma_t^2\sigma_s^2)=\operatorname{var}(\sigma_0^2)\mathrm{e}^{(t-s)\Psi(1)}+(\mathrm{E}\,\sigma_0^2)^2,$$

since we are using the stationary version. Thus, from (5.5),

$$\begin{aligned}\frac{\lambda}{\delta}\,\mathrm{E}(G_t^2\sigma_t^2)&=\mathrm{E}\,\sigma_0^4-\beta t\,\mathrm{E}\,\sigma_0^2-\log\delta\int_0^t(\operatorname{var}(\sigma_0^2)\mathrm{e}^{(t-s)\Psi(1)}+(\mathrm{E}\,\sigma_0^2)^2)\,\mathrm{d}s-(\mathrm{E}\,\sigma_0^2\sigma_t^2)+0\\&=\operatorname{var}(\sigma_0^2)(1-\mathrm{e}^{t\Psi(1)})-\beta t\,\mathrm{E}\,\sigma_0^2-\log\delta\operatorname{var}(\sigma_0^2)\frac{1-\mathrm{e}^{t\Psi(1)}}{-\Psi(1)}-t\log\delta(\mathrm{E}\,\sigma_0^2)^2.\end{aligned} \tag{5.9}$$

Note that $(\lambda/\delta)\,\mathrm{E}\,L_1^2=\Psi(1)-\log\delta$ (see (4.1)). Thus, from (5.2),

$$\begin{aligned}\frac{\lambda}{\delta}\,\mathrm{E}\,G_t^2\,\mathrm{E}\,\sigma_t^2&=\frac{\lambda\beta t\,\mathrm{E}\,L_1^2\,\mathrm{E}\,\sigma_0^2}{-\delta\Psi(1)}\\&=-\beta t\,\mathrm{E}\,\sigma_0^2-\frac{\beta t\log\delta\,\mathrm{E}\,\sigma_0^2}{-\Psi(1)}\\&=-\beta t\,\mathrm{E}\,\sigma_0^2-t\log\delta(\mathrm{E}\,\sigma_0^2)^2\end{aligned}$$

(using (4.8)). Subtracting this from (5.9) gives

$$\frac{\lambda}{\delta}\operatorname{cov}(G_t^2,\sigma_t^2)=\operatorname{var}(\sigma_0^2)\biggl(1-\mathrm{e}^{t\Psi(1)}-\log\delta\frac{1-\mathrm{e}^{t\Psi(1)}}{-\Psi(1)}\biggr),$$

which is positive.

In Figure 2 we show the simulated autocorrelation functions of σ_t and of the increment $G_t^{(1)}$, and of their squares, for the same process simulated in Figure 1. A feature of the σ and σ^2 autocorrelations is their very slow decrease with increasing lag. As expected, the sample autocorrelation functions of the increment $G_t^{(1)}$ are zero, and those of its square are positive, within sampling errors.

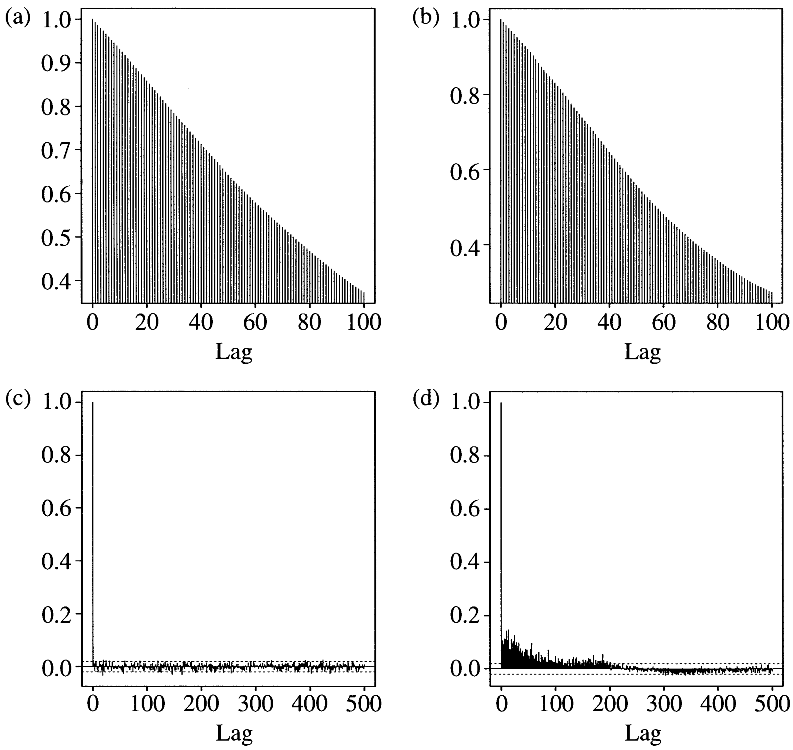

FIGURE 2: Sample autocorrelation functions of (a) σ_t, (b) σ_t^2, (c) $G_t^{(1)}$ and (d) $(G_t^{(1)})^2$ for the process simulated in Figure 1. The dashed lines in the graphs (c) and (d) show the confidence bounds $\pm 1.96/\sqrt{9999}$.

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