Characteristics of 3 band orthogonal wavelet system

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Outline

- Introduction
- Basic concepts
- 3 band filter bank
- Parameterization of 3 band orthogonal wavelet system
- Generating high tfp localized wavelets
- Sobolev smoothness of scaling function
- New method to calculate the smoothness of scaling functions
- Conclusion and future work
- References

- In the past 2 decades, there has been an indepth study of the 2 band wavelet system and its associated properties. We continue the research by extending the ideas to 3-band wavelet system and its properties.

- Polyphase representation permits a great level of simplification of theoretical results for both FIR and IIR filters
- There are 2 ways in which we can represent a filter in its polyphase form, they are as follows:

$$Type : 1, \quad H(z) = \sum_{l=0}^{M-1} z^{-l} E_l(z^M) \tag{1}$$

$$where, \quad E_l(z^M) = \sum_{n=-\infty}^{\infty} h(Mn+l) z^{-n} \tag{2}$$

$$Type : 2, \quad H(z) = \sum_{l=0}^{M-1} z^{(-M-1-l)} R_l(z^M)) \tag{3}$$

$$where, \quad R_l(z^M) = \sum^{\infty} h(M(n+l)-(l+1)) z^{-l} \tag{4}$$

- The filter bank with FIR filters satisfies P.R property then the following(Eqn. 5) condition on the polyphase component of analysis filter must hold true:

$$|E(z)| = z^{-n} \tag{5}$$

- A transfer matrix is said to be paraunitary if :

$$\tilde{E}(z)E(z) = cI, \quad \text{for all z} \tag{6}$$

$$\tilde{E}(z) = E_*^T(z^{-1}) \tag{7}$$

- Paraunitary property automatically satisfies Eqn. 5, hence we can generate P.R filter band using transfer functions that satisfies parauniatriness

- A general structure of how 3 channel filter bank works:

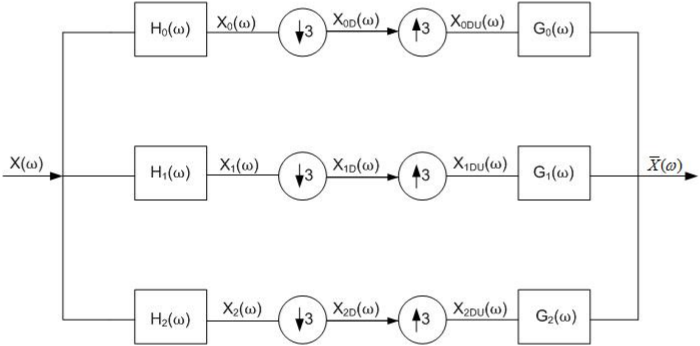

Figure : 3 channel filter bank

- $a^0(n)$, $a^1(n)$ and $a^2(n)$ are the filter coefficients of analysis filters
- Dilation equation in terms of the scaling function and filter coefficients is as follows:

$$\phi(t) = \sum_{n=0}^{N-1} a^0(n) * \phi(3t-1) \tag{8}$$

$$\psi_1(t) = \sum_{n=0}^{N-1} a^1(n) * \phi(3t-1) \tag{9}$$

$$\psi_2(t) = \sum_{n=0}^{N-1} a^2(n) * \phi(3t-1) \tag{10}$$

3 band filter bank

Generating scaling and wavelet functions via cascade algorithm

- The above equations can be equivalently explained in the following 3 ways:
- In time domain:

$$\phi(t) = \lim_{k\to\infty} a^0(3t) \bigotimes a^0(3^2 t)... \bigotimes a^0(3^k t) \tag{11}$$

- In Frequency domain:

$$\hat{\phi}(\Omega) = \lim_{k\to\infty} \hat{\phi}(0) \prod_{m=1}^{k} \left(\frac{1}{3}\right) * A^0 \left(\frac{\Omega}{3^m}\right) \tag{12}$$

- It can be also understood in the following way:

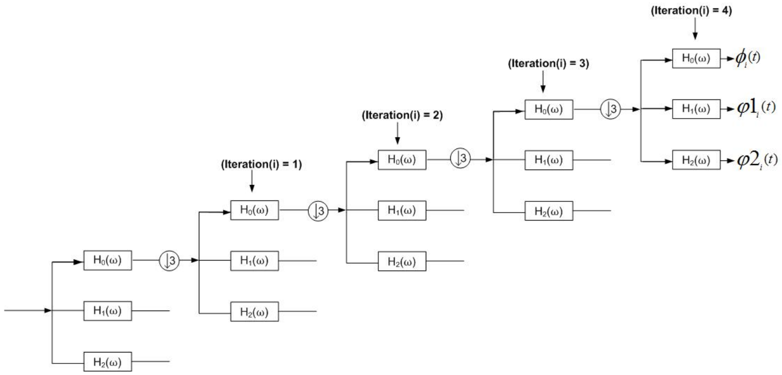

Figure : Diagrammatic view of how the cascade algorithm works

3 band filter bank
variation of scaling and wavelet function for different iterations

- Following figure shows how scaling function varies with different iterations

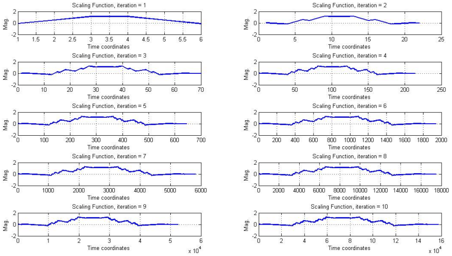

Figure : Evolution of scaling function under different levels of iteration

- As evident from the iterations, the scaling function don't change after 8 iterations

- $P[a](z)$ is called $3 - band$ orthogonal wavelet matrix if it satisfies:

$$P[a](z) * P[a](z^{-1}) = 3I \tag{13}$$

$$\sum_{k=0}^{2} p[a]_k^s(1) = \delta_{s,0} \tag{14}$$

- $1st$ (Eqn. 13) equation is the similarity between P.R and Orthogonal filter banks, $2nd$ equation (14) is the essential difference between them
- $2nd$ equation(Eqn. (14)) ensures that the generated wavelet and scaling functions will converge in L_2 norm

- Peng L. and Wang Y. in [9] gave a parameterization of subset of $3 - band$ orthogonal wavelet systems in which scaling filter has length $3 * g$ and the McMillan degree is $g - 1$. Following is the algebraic structure of the system:

$$P[a](z) = V_{g-1} V_{g-2} V_{g-3} .... V_1 H_0 \tag{15}$$

$$V_k = [I - (1 - z^{-1}) u_k u_k^T] \tag{16}$$

where u_k is any $3X1$ unit norm vector and H_0 is a real Haar Wavelet Matrix

- In theory, Time Bandwidth Product (T.B.P) is calculated both from frequency and time domain. Equations governing them are as follows:
- In time domain:

$$\sigma_t^2 = \frac{\int_{-\infty}^{\infty} t^2 |x(t)|^2 \, dt}{\int_{-\infty}^{\infty} |x(t)|^2 \, dt} \tag{17}$$

$$\sigma_\Omega^2 = \frac{\int_{-\infty}^{\infty} |dx(t)/dt|^2 \, dt}{\int_{-\infty}^{\infty} |x(t)|^2 \, dt} \tag{18}$$

$$TBP = \frac{\int_{-\infty}^{\infty} t^2 |x(t)|^2 \, dt}{\int_{-\infty}^{\infty} |x(t)|^2 \, dt} * \frac{\int_{-\infty}^{\infty} |dx(t)/dt|^2 \, dt}{\int_{-\infty}^{\infty} |x(t)|^2 \, dt} \tag{19}$$

- In Frequency Domain:

$$\sigma_t^2 = \frac{\int_{-\infty}^{\infty} |\frac{d\hat{x}(\Omega)}{d\Omega}| d\Omega}{\int_{-\infty}^{\infty} 2 * \pi * |\hat{x}(\Omega)|^2 \, d\Omega} \tag{20}$$

$$\sigma_\Omega^2 = \frac{\int_{-\infty}^{\infty} \Omega^2 |\hat{x}(\Omega)|^2 \, d\Omega}{\int_{-\infty}^{\infty} |\hat{x}(\Omega)|^2 \, d\Omega} \tag{21}$$

$$TBP = \frac{\int_{-\infty}^{\infty} |\frac{d\hat{x}(\Omega)}{d\Omega}| d\Omega}{\int_{-\infty}^{\infty} 2 * \pi * |\hat{x}(\Omega)|^2 \, d\Omega} * \frac{\int_{-\infty}^{\infty} \Omega^2 |\hat{x}(\Omega)|^2 \, d\Omega}{\int_{-\infty}^{\infty} |\hat{x}(\Omega)|^2 \, d\Omega} \tag{22}$$

- Method to calculate the Time Bandwidth Product for discrete signals:

  1. Calculate the FFT of the normalized $f(n)$
  2. The number of iterations used in cascade algorithm, provides the sampling distance in discrete time.
  3. The integrals calculated using summations
  4. In frequency domain we replace the variable Ω by an equivalent variable proportional to the FFT sequence index

- The TFP calculations for length 6 filters involve 3 variables $var1, var2, Theta$.
- Variation in $Theta$ does not change the value of TFP.
- Hence, we create a grid of the first 2 variables $var1, var2$,
- We calculate the TBP of the scaling and wavelet functions on all these points of the grid

Results: Characteristics of Filter Banks
Observations from the above algorithm

- The variation of the TBP values of the scaling and wavelet functions are similar

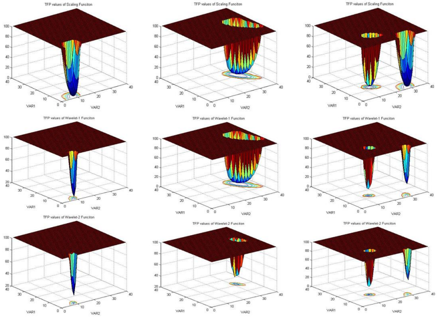

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Figure : Scaling and Wavelet Function TFP (Row-1 Corresponds to Scaling functions, Row 2-3 corresponds to wavelet function, column 1,2 and 3 corresponds to grid-1,2 and 5 respectively

- The variation of the TBP values of the scaling and wavelet functions are similar

Correlation & Count → Function ↓	Count of Low TFP points	Correlation with Scaling function	Correlation with Wavelet-1 function	Correlation with Wavelet-2 function
Scaling function	4489	4489	1293	964
Wavelet-1 function	1293	1293	1293	964
Wavelet-2 function	964	964	964	964

Figure : TBP variation for Scaling and both Wavelet Functions

- The variation of the TBP values of the scaling and wavelet functions repeat itself at regular intervals

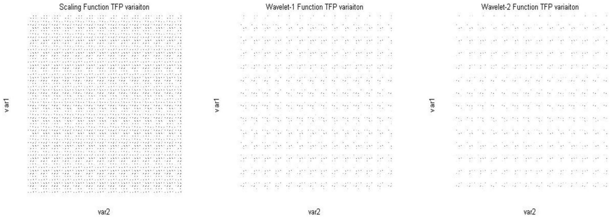


Figure : Similarity in TFP variation for both Scaling and Wavelet Functions

Results: Characteristics of Filter Banks

calculating the input parameters that give highly localized wavelets

- Create a cost function that calculates the sum of TFP values of the 2 wavelets
- Further divided the grid into 9 sub sections
- For each subsection we make an even finer grid of $var1, var2$
- At all points of this grid we calculated the TFP values and selected a set (S_1) of points that give us low TFP values
- Optimize the TBP value using the above points as input variables
- Compare them and calculate the global minima

Results: Characteristics of Filter Banks

Results for highly optimized wavelets

- Following are the results of TBP values for 5 grids out of the 9 grids that we had constructed

	var1	*var2*	*Theta*	Scaling TFP	Wavelet -1 TFP	Wavelet-2 TFP	Sum TFP Wav-1 & Wav-2
	-3.4736	-2.4712	0.5771	2.3981	12.3171	13.1788	25.4959
	-2.6908	-3.0832	1.1692	2.4050	13.2829	17.8599	31.1428
GRID-1	-2.6908	-3.0832	1.1692	2.4050	13.2829	17.8599	31.1428
	-2.6908	-3.0832	1.1692	2.4050	13.2829	17.8599	31.1428
	-3.4736	-2.4712	0.5771	2.3981	12.3171	13.1788	25.4959
	-0.4508	0.0584	1.1692	2.4050	13.2829	17.8599	31.1428
	-0.4508	0.0584	1.1692	2.4050	13.2829	17.8599	31.1428
GRID-2	-0.4508	0.0584	1.1692	2.4050	13.2829	17.8599	31.1428
	-0.4508	0.0584	1.1692	2.4050	13.2829	17.8599	31.1428
	-3.5924	0.0584	1.1692	2.4050	13.2829	17.8599	31.1428
	0.3320	0.6704	0.5771	2.3981	12.3171	13.1788	25.4959
	-0.4508	0.0584	-0.4016	2.4050	17.8599	13.1788	31.1428
GRID-5	-0.4508	0.0584	-0.4016	2.4050	17.8599	13.1788	31.1428
	-0.4508	0.0584	-0.4016	2.4050	17.8599	13.1788	31.1428
	-0.4508	0.0584	-1.9724	2.4050	13.2829	17.8599	31.1428
	-0.4508	-3.0832	1.1692	2.4050	13.2829	17.8599	31.1428
	0.4508	0.0584	1.1692	2.4050	13.2829	17.8599	31.1428
GRID-7	-0.4508	0.0584	1.1692	2.4050	13.2829	17.8599	31.1428
	0.4508	-3.0832	1.1692	2.4050	13.2829	17.8599	31.1428
	0.4508	-3.0832	1.1692	2.4050	13.2829	17.8599	31.1428
	0.3320	0.6704	0.5771	2.3981	12.3171	13.1788	25.4959
	0.3320	0.6704	0.5771	2.3981	12.3171	13.1788	25.4959
GRID-9	0.3320	0.6704	0.5771	2.3981	12.3171	13.1788	25.4959
	0.3320	0.6704	0.5771	2.3981	12.3171	13.1788	25.4959
	0.3320	0.6704	0.5771	2.3981	12.3171	13.1788	25.4959

Results: Characteristics of Filter Banks

Generating smooth wavelets with regularity $= 1$

- Wavelet functions will be a sum of the linear shifted scaling functions
- We will study only the smoothness of scaling functions and generalize the results
- Smoothness of a function is generally related with its regularity
- The parameterization imposes a regularity of order - 1 on all the scaling functions
- We will hence calculate the value of sobolev smoothness for all the filters

- Sobolev smoothness of the scaling filter tells us about the smoothness of the scaling function which is generated after cascade iterations. Calculated in the following way:
- Normalize $Q(z)$, such that $Q(1) = 1$
- Calculate the transition operator (T^Q) of $Q(z)$
- The transition operator(T^Q) of $h(n)$ captures how the cascade algorithm converges and hence the stability of scaling function under iterations
- Sobolev smoothness (S_max) is dependent on the spectral radius of T^Q
- Smaller the spectral radius, better is the Sobolev smoothness

- General unwanted characteristics of the scaling functions:
  1. A lot of small peaks
  2. Surface very rough, slope of the function is small but changes quickly
  3. Single/multiple points where function changes it values very quickly

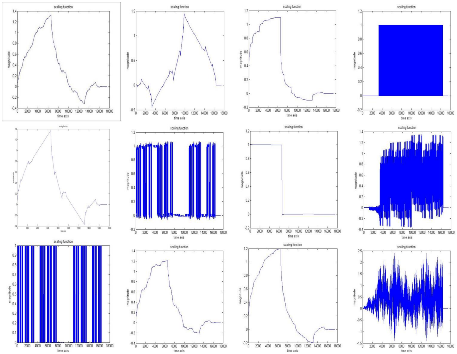


Figure : General Figures of the scaling filters

Results: Characteristics of Filter Banks

Parameterizing smoothness of scaling functions

- *cost*1 is first parameter

$$cost1 = \sum_{n=1}^{N} |f^{''}(n)|, \ where\ f^{''}(n) is\ the\ double\ derivative \qquad (23)$$

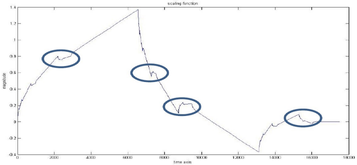


Figure : Most smooth plot when compared in the metric of cost1

- The circled regions are the point of concern, which we will refer to as small peaks. They dont get compensated by this cost function

Results: Characteristics of Filter Banks
Parameterizing smoothness of scaling functions

- *cost*2 is the second parameter

$$cost2 = \sum_{n=1}^{Threshold} |sf''(n)|, \text{ where Threshold is carefully selected} \quad (24)$$

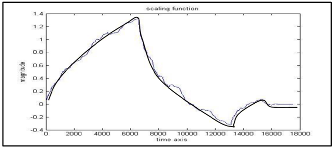


Figure : Most smooth plot when compared in the metric of cost2

- Threshold selected from (.025 to .50 of the total length), Threshold values subjectively analyzed
- Curve is very rough, slope not changing drastically but changing at almost all points hence leading to this roughness.

- *cost*3 is the second parameter

$$cost3 = count\ of\ small\ peaks \tag{25}$$

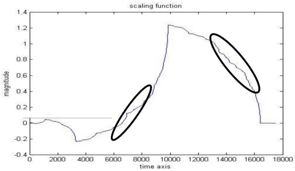


Figure : Most smooth plot when compared in the metric of cost2

- The function is not as rough as the previous one, highlighted some portion in the curve which are still rough.

Results: Characteristics of Filter Banks

Parameterizing smoothness of scaling functions

- Cost functions have their own advantages and disadvantages
- Need to have a function with appropriately weighted sum of all these 3 cost functions

$$costfunction = w1 * norm(cost1) + w2 * norm(cost2) + w3 * norm(cost3) \tag{26}$$

where, norm(cost1/2/3) are the normalized cost functions

- Aim is to decide what will be these weights w1,w2,w3.

- To calculate the weights, I have adopted the following method:
- calculate the cost functions for the grid, and normalize them respectively
- With the constraint w1+w2+w3=1, calculate the *costfuntion* for the grid
- For a particular set of $w1, w2, w3$, calculate the input which gives the least value of cost function
- Check the rank in the sorted $cost1, cost2, cost3$
- Weight corresponding to least sum of ranks will give us the smoothest functions.

Results: Characteristics of Filter Banks
Results for weights for different values of Theta

- I have applied the above method for $Theta = \pi/2, \pi/3, \pi/4, \pi/6 and \pi/9$

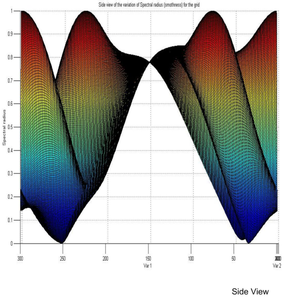
Table : Comparision of Weights for various Theta

$Theta \downarrow Weights \rightarrow$	W1	W2	W2
$\frac{\pi}{2}$	0.09	0.90	0.09
$\frac{\pi}{3}$	0.09	0.90	0.09
$\frac{\pi}{4}$	0.09	0.90	0.09
$\frac{\pi}{6}$	0.09	0.90	0.09
$\frac{\pi}{9}$	0.09	0.90	0.09

- Value of Smoothness does not change with Theta

Results: Characteristics of Filter Banks
Parameterizing smoothness of scaling functions

- Using (0.09, 0 .90, 0 .01) as the weights in my function
- Comparision of smoothness variation for my function and sobolev's



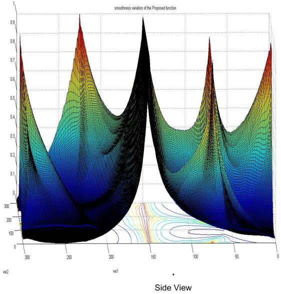


Figure : Comparision of variation of smoothness over grid for sobolev (left) and my function (right)

- The similarity of minima and maxima matches in both

Results: Characteristics of Filter Banks
Parameterizing smoothness of scaling functions

- comparison of the most smooth plots obtained from my method and sobolev's regularity

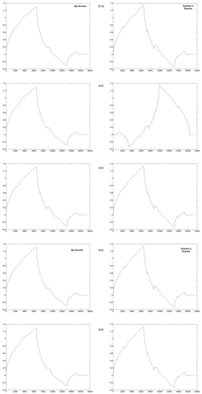


Figure : Comparision of smoothness of scaling functions for sobolev (right) and my function (left)

- comparison of the most smooth plots obtained from my method and sobolev's regularity

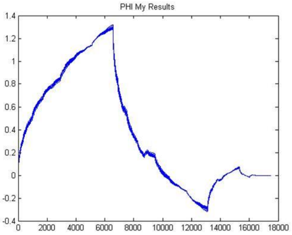


Figure : Comparision of smoothness of scaling functions for sobolev (right) and my function (left)

Conclusion

- In the first section of our investigation we have developed an algorithm to get the wavelets that are highly localized in time and frequency domain
- In the second section of our investigation we have developed a novel method to generate smooth scaling functions
- At a preliminary stage we have compared our results with the Sobolev's measure of regularity and we have found our results to be better than those obtained from his measure.

Future work

- For the algorithm giving TFP localized wavelets we need to implement a few more wavelet system designs and verify our results for all of them.
- For the Smoothness criteria we can expand our research on generating the smoothness for filters of varied lengths and compare our results.

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