Time-Frequency Based Novel Method for the Design of M-Channel Wavelet Filter Banks

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***Abstract*—A Wavelet Filter Bank is perfect reconstruction filter bank with filter iterations in a particular manner converge to a smooth function.In this paper a cost function based on the sum of time-variance and frequency-variance is proposed. It is shown that the cost function not only imposes regularity or continuity of the wavelets but also localize wavelets in time and frequency. Weights can be assigned to these important properties of wavelet using the cost function.The novel design method is independent of no. of channels and nature of wavelets, orthogonal or biorthogonal.**

I. Introduction

M-channel wavelet filter banks [PSB93] [Gop90] [CBG98] are the powerful tool for time frequency analysis of signals and studied extensively in signal processing applications.Wavelet transform overcome the limitations of short time fourier transform [Dau88] [Dau90][Dau92].M-Band wavelets helps to zoom in a narrow band in high frequency region and has log decomposition in frequency spectrum.

Multirate filter banks are well understood to signal processing community.Paraunitary filter banks is the class used to design orthogonal filter banks [ZT92] [Vai90][SVN93].Mallat's work link wavelets with filter banks [Mal89].There are several studies for the design of regular and time frequency localized wavelets.Filter bank theory for time frequency localization is given in [SB87].Many authors have given the methods for design of regular filter banks by imposing the condition of regularity in the filter bank structure [ZX09] [COA05].Minimization of time-frequency product (TFP) is the method generally used to design time and frequency localized wavelets [RK10][SKPG10].A note on frequency localization and regularity is given in [Rio93].In this paper we show trade-off between wavelet regularity and localization in time and frequency.It is found that,more the wavelet frequency localized, more regular it will be, but its time localization will become poorer.

Synthesis Filter bank of a wavelet filter bank consist of iterations of upsamplers and filters.It is shown that infinite iterations satisfy the 3 scale equation,called dilation equation.If iterations generates the continuous scaling function then corresponding low pass filter will have zeros on aliasing frequencies $e^{\pm j2\pi m/M}$ for m=1,2..,M-1 [PSB93].

We introduce a new method for imposing continuity of wavelets which is based on minimizing the sum of time-variance and frequency-variance(TFS).We report a practical problem with Time-Frequency Product(TFP) and show that minimizing the cost function, Time-Frequency Sum (TFS), is a better choice.Time-frequency localized(TFL) wavelets can be designed by minimizing TFS. At one hand TFP has the problem in practical cases, TFS can do the same job without posing any problem.

While designing three channel orthogonal filter bank with filter length nine,the surface created by the cost function TFS, depend on how the TFS of scaling and wavelet function is minimized,individually or together.It is found that the surface has very few (three to six) local minimas and can be located easily.

Since,wavelet is linear combination shifted scaling functions,time variance of wavelet is always greater than scaling function and poses the challenge while designing TFL wavelets.If TFL is not important and high regularity/continuity is the only requirement than minimizing TFS of scaling function is sufficient and at the same time,since, time-variance of wavelet is always greater than the scaling function, TFS of wavelet must not be minimized.Smooth wavelets can be generated by minimizing TFS of scaling function. However, these will not be TFL.To achieve TFL wavelets, TFS of wavelet must be minimized.

It is found that minimizing sum of TFS of scaling and wavelet function helps quickly finding out the TFL wavelet but it is yet to check whether that will be the most optimal TFL wavelet solution.It may possible that minimizing TFS of wavelet function alone,can only give the most optimal TFL wavelet.It is found that minimizing TFS of scaling function,since time-variance is not an issue, gives the most optimal continuous scaling function and the corresponding wavelet will be most optimal w.r.t number of Vanishing moments(VM).Note that number of VM's of wavelet has direct relation with continuity of scaling function or the no. of zeros at aliasing frequencies [Dau92] [PSB93].

II. Synthesis Filter Bank

A. Dilation Equation

In this section it is shown that infinite iterations of upsampler and filter satisfies the 3-Scale dilation and wavelet equation.

The iterations of upsampler and low pass filter $g_0(n)$, as in the synthesis side of a filter bank, can be represented as

$$G_0^{(i)}(z) = \mathbf{\Pi}_{k=0}^{i-1} G_0(z^{3^k})$$

Associating discrete-time sequence $g_0^{(i)}(n)$ with continuous time functions $\phi^{(i)}(t)$ as follows,

$$\phi^{(i)}(t) = 3^{i/2} g_0^{(i)}(n)$$

where,

$$\frac{n}{3^i} \leq t < \frac{n+1}{3^i}$$

Note that if the length of the filter $g_0(n)$ is L, then the length of iterated filter $g_0^{(i)}(n)$ is

$$L^{(i)} = \frac{(3^{i+1}-1)}{3-1}(L-1)+1$$

and it becomes infinite as $i- > \infty$ and scaling by $1/3^i$ is necessary. This rescaling ensure that $\phi^{(i)}(t)$ is compactly supported between 0 to L-1. The factor $3^{i/2}$ preserve L2 Norm between discrete and continuous time cases. If $||g_0(n)|| = 1$ then norm will be preserved with each iteration and $||g_0^{(i)}(n)|| = 1$ and due to the scaling factor $3^{i/2}$, $||\phi^{(i)}(t)|| = 1$.

The time domain equivalent filter,after 'i' steps is given by

$$g_0^{(i)}(n) = \sum_k g_0[k] g_0^{(i-1)}[n - 3^{i-1}k]$$

Since,

$$\frac{n}{3^{i-1}} \leq 3t \leq \frac{n+1}{3^{i-1}}$$

$$\frac{n - 3^{i-1}k}{3^{i-1}} \leq 3t - k \leq \frac{n+1-3^{i-1}k}{3^{i-1}}$$

and

$$g_0^{(i)}(n) = 3^{-\frac{i}{2}} \phi^{(i)}(t)$$

we also have,

$$g_0^{(i-1)}(n) = 3^{-\frac{i-1}{2}} \phi^{(i-1)}(3t)$$

and

$$g_0^{(i-1)}(n - 3^{i-1}k) = 3^{-\frac{i-1}{2}} \phi^{(i-1)}(3t-k)$$

Substituting these in

$$g_0^{(i)}(n) = \sum_k g_0[k] g_0^{(i-1)}[n - 3^{i-1}k]$$

we get

$$\phi^i(t) = \sqrt{3} \sum_k g_0[k] \phi^{(i-1)}(3t-k)$$

By assumption, as $i- > \infty$ $\phi^{(i)}(t)$ converges to the scaling function $\phi(t)$

we get,

$$\phi(t) = \sqrt{3} \sum_k g_0[k] \phi(3t-k)$$

This is dilation equation with scale factor 3.Simialry, there exist a wavelet equation.Independent on no. of channels and orthogonality or biorthgonality, we used this method to generate the scaling or wavelet function from the filter(s).This is called cascade algorithm.

B. Wavelet Equation

From the definition of multiresolution analysis, we know that wavelet function is linear combination of scaling function.Hence ensuring the continuity of scaling function is sufficient.

For scaling function to be normalized low pass function, $\phi(w)|_{w=0} = 1$ and therefore $G_0(1) = \sqrt{3}$.Note that there is infinite product involved in frequency domain and in any other case $\phi(w)|_{w=0}$ will converge to zero or diverge to infinity. Also the norm of the filter must be one for function to be of unit energy.The filter bank structure must satisfy the conditions of sum and energy of filter coefficients,called summation and norm constraint respectively. These constraints are applicable to both the orthogonal as well as biorthogonal linear phase case.Note that summation constraint is necessary only for LPF's but energy constraint is necessary for all the filters of filter bank.In all our simulations, the summation constraint for LPF's is inbuilt in the structure for biorthogonal as well as orthogonal case.The norm constraint is inbuilt in orthgonal case and we can carry out unconstrained minimization. However, same is not true for biorthgonal case,as it provides freedom with respect to norm also. Even then we converted the optimization procedure to unconstrained by minimizing an example cost function given below.

$$cost = TFS + |||g_0(n)|| - 1)|$$

Thus, minimizing the cost function is equivalent to making the norm of the filter equal to one. Results show that many a times the filters reached do not have unity norm and it depends on the initial point the optimization started.Note that summation and norm constraint are not necessary for convergence to smooth function but for normalized area and energy of the limit function.

III. Sufficient Condition for Continuity of Scaling Function

We know that if the first derivative of a function x(t) has finite energy, then dx(t)/dt is finite for all time. Thus dx(t)=Pdt where P is finite. If limit dt tends to zero then dx(t) tends to zero which is sufficient for continuity of x(t) and x(t) is atleast in C0. Using differentiation property of scaling function, we can show that energy of first derivative is equal to frequency variance of x(t). This implies that minimizing frequency variance is sufficient to impose continuity of x(t).

If we minimize frequency variance only, then this implies an impulse in frequency domain or a constant function in time domain with infinite variance.Thus it is important to minimize time variance also. This can be done by either minimizing sum or the product of time variance and frequency variance called time-frequency summation TFS and time frequency product TFP respectively. In the section below we show that minimizing TFS is better choice.

IV. Time-Frequency Sum is Better Choice

Time-Frequency Localization of wavelet function can be achieved by minimizing TFP or TFS.Note the variables y1 and y2 corresponding to TFS and TFP respectively.

$$y1 = \sigma_f(\phi) + \sigma_t(\phi)$$

$$y2 = \sigma_f(\phi)\sigma_t(\phi)$$

where,

$$\sigma_f(\phi) = \int_{t=-\infty}^{\infty} |\frac{d\phi(t)}{dt}|^2 dt$$

$$\sigma_t(\phi) = \int_{t=-\infty}^{\infty} (t - m_f)^2 |\phi(t)|^2 dt$$

Lets define a function $p(t)$ with unit energy but concentrated only at/around t=0. Now if $\phi(t) = p(t - m_f)$ then $\sigma_t(p) = 0$ and that makes y2=0.Thus, whenever, we minimize y2, if the initial point is not proper we end up to this solution for $\phi(t)$.To get around with this problem consider the next alternative of minimizing TFS. There is no such problem. We get the gaussian solution when we minimize TFS.The same can be proved using triangle inequality.Note that equality exist when the vectors $tx(t)$ and $\frac{dx(t)}{dt}$ are collinear.The corresponding differential equation can be solved to give gaussian solution.

Note that continuity depend on the number of derivatives in L2, which has a direct relation with even moments of squared fourier transform.In the section below we check the convergence of infinite integrals with higher even moments for a function in L2.

V. Convergence of Higher Moments of a Squared Gaussian Function

Consider the infinite integrals with higher even moments of squared function for the gaussian function.

$$\int_{-\infty}^{\infty} (u^n e^{-\alpha u^2/2})^2 du = \int_{-\infty}^{\infty} u^{2n} e^{-\alpha u^2} du$$

$$= (-1)^n \int_{-\infty}^{\infty} \frac{d^n}{d\alpha^n} e^{-\alpha u^2} du = (-1)^n \frac{d^n}{d\alpha^n} \int_{-\infty}^{\infty} e^{-\alpha u^2} du$$

$$= \sqrt{\pi}(-1)^n \frac{d^n}{d\alpha^n} \alpha^{-\frac{1}{2}} = \sqrt{\frac{\pi}{\alpha}} \frac{2n-1}{(2\alpha)^n}$$

This shows that there exist functions in L2 for which infinite integral converge and the continuity of these functions depend on max(n) for which integral converge.

VI. Optimization Procedure

First, for the given lattice parameters, filter coefficients are computed. The filter coefficients are then passed to cascade algorithm to compute the scaling and wavelet functions. TFS is then computed for all the functions. Note that summation and norm constraint are inbuilt in the given's rotation based orthogonal filter bank structure.However, in biorthogonal case,while the sum constraint is imposed by a condition on the lattice parameters, the norm constraint is imposed through the cost function.In each optimization iteration lattice parameters are varied and moved in the direction of minima.

VII. Simulation Options

Three possible cost functions are

1) Minimize the TFS for scaling function only
2) Minimize the TFS for Scaling and Wavelet Functions together.
3) Minimize the TFS for Wavelet functions only.

VIII. Results

In case of three channel,length nine, orthogonal filter bank it is found that the cost function surface has very few local minimas.There are only three designs in case of first cost function,and since,two overlapping zeros at aliasing frequencies for LPF, the designs are frequency variance(continuity) optimized. Two designs were designed using the third cost function and these are found to be more localized in time and frequency.It is important to note that this cost function spend most of the time in minimizing wavelet time variance. In these designs, the two zeros which were overlapping each other,now one of them moved little away from the aliasing frequency to generate degrees of freedom for Time-Frequency Localization.

In three channel, linear phase biorthogonal filter bank, the first two cost function surfaces are found to be much simpler than the third cost function.We could get good designs with first two cost functions. However the surface for the third cost function is very complex and none of the smooth designs could be located.

The wavelets designed for orthogonal two channel case,for filter lengths from 4 to 14,have better TFP than daubechies wavelets.The filters were computed using the second cost function.

IX. Conclusion

While designing time-frequency localized wavelets,time variance of the wavelet is a critical parameter. It can be improved only at the cost of continuity, by decreasing the number of zeros at aliasing frequencies.Thus, there is continuity and TFL trade-off for wavelet.At one hand minimizing TFS for scaling function increases the continuity of wavelet,minimizing TFS for wavelet localize it in time and frequency.Thus,during the design optimization, regularity and time-frequency localization can be assigned weights separately using the second cost function.Following are the conclusions of our study in brief.

1) Minimizing TFS is a simple and elegant way of imposing regularity/continuity and time-frequency localization together.
2) We can impose any no. of zeros on aliasing frequencies by minimizing TFS of scaling function.Higher moments are not necessary. However, it is found that, using the higher moments, quickly locate the design on the surface formed by the cost function with more and more zeros on aliasing frequencies. The limit on the maximum no. of zeros comes from the filter length.
3) If continuity is the only requirement then TFS of scaling function (only) should be minimized.Minimizing along with TFS of wavelet do not give the most optimal solution.
4) While designing TFL wavelets, it is the time variance that is critical parameter and frequency variance is not an issue.
5) This procedure not only give access to TFL but regularity or continuity also.And this is the only procedure.
6) TFS with higher moments (in frequency only) helps quickly finding out the function with more no. of continuous derivatives.
7) There is practical problem with minimizing TFP and TFS must be used as a cost function.
8) For a given length wavelet, there is trade-off in Vanishing Moments and Time-localization.Some degree of freedom for TFL can be achieved by decreasing the no. zeros at aliasing frequenices or the number of Vanishing Moments.
9) Keeping the time-variance fixed it is always possible to increase or decrease the continuity by adjusting the no. of zeros at aliasing frequencies and TFS of scaling function play an active role.In this case, first, time variance is adjusted to optimal and then freq. variance or continuity is looked for because freq. variance can be easily minimized by increasing the continuity or no. of zeros on aliasing frequencies.The limit on the no. of zeros comes from the filter length.

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