

BISECTION METHOD

In Mathematics, the bisection method is a straightforward technique to find numerical solutions of an equation with one unknown. Among all the numerical methods, the bisection method is the simplest one to solve the transcendental equation. In this article, we will discuss the bisection method with solved problems in detail.

Bisection Method Definition

The bisection method is used to find the roots of a polynomial equation. It separates the interval and subdivides the interval in which the root of the equation lies. The principle behind this method is the intermediate theorem for continuous functions. It works by narrowing the gap between the positive and negative intervals until it closes in on the correct answer. This method narrows the gap by taking the average of the positive and negative intervals. It is a simple method and it is relatively slow. The bisection method is also known as interval halving method, root-finding method, binary search method or dichotomy method.

The Bisection Method repeatedly bisects an interval and then selects a subinterval in which a root must lie. It is based on the Intermediate Value Theorem, which states that if a continuous function $f(x)$ has values of opposite signs at the endpoints of an interval, then there is at least one root within that interval.

Let us consider a continuous function “ f ” which is defined on the closed interval $[a, b]$, is given with $f(a)$ and $f(b)$ of different signs. Then by intermediate theorem, there exists a point x belong to (a, b) for which $f(x) = 0$.

Advantages of Bisection Method

- Simple and robust.
- It always converges if the initial interval contains a root.

Disadvantages of Bisection Method

- Convergence is relatively slow (linear convergence rate).
- Does not exploit the function’s values within the interval for faster convergence.

Formula

$$X_2 = (X_0 + X_1) / 2$$

Bisection Method Algorithm

Follow the below procedure to get the solution for the continuous function:

For any continuous function $f(x)$,

- Find two points, say a and b such that $a < b$ and $f(a) * f(b) < 0$

- Find the midpoint of a and b, say “t”
- t is the root of the given function if $f(t) = 0$; else follow the next step
- Divide the interval $[a, b]$ – If $f(t)*f(a) < 0$, there exist a root between t and a
– else if $f(t)*f(b) < 0$, there exist a root between t and b
- Repeat above three steps until $f(t) = 0$.

The bisection method is an approximation method to find the roots of the given equation by repeatedly dividing the interval. This method will divide the interval until the resulting interval is found, which is extremely small.

Bisection Method Example

Question: Determine the root of the given equation $x^2 - 3 = 0$ for $x \in [1, 2]$

Solution:

Given: $x^2 - 3 = 0$

Let $f(x) = x^2 - 3$

Now, find the value of $f(x)$ at $a= 1$ and $b=2$.

$$f(x=1) = 1^2 - 3 = 1 - 3 = -2 < 0$$

$$f(x=2) = 2^2 - 3 = 4 - 3 = 1 > 0$$

The given function is continuous, and the root lies in the interval $[1, 2]$.

Let “t” be the midpoint of the interval.

$$\text{I.e., } t = (1+2)/2$$

$$t = 3 / 2$$

$$t = 1.5$$

Therefore, the value of the function at “t” is

$$f(t) = f(1.5) = (1.5)^2 - 3 = 2.25 - 3 = -0.75 < 0$$

If $f(t) < 0$, assume $a = t$.

and

If $f(t) > 0$, assume $b = t$.

$f(t)$ is negative, so a is replaced with $t = 1.5$ for the next iterations.

The iterations for the given functions are:

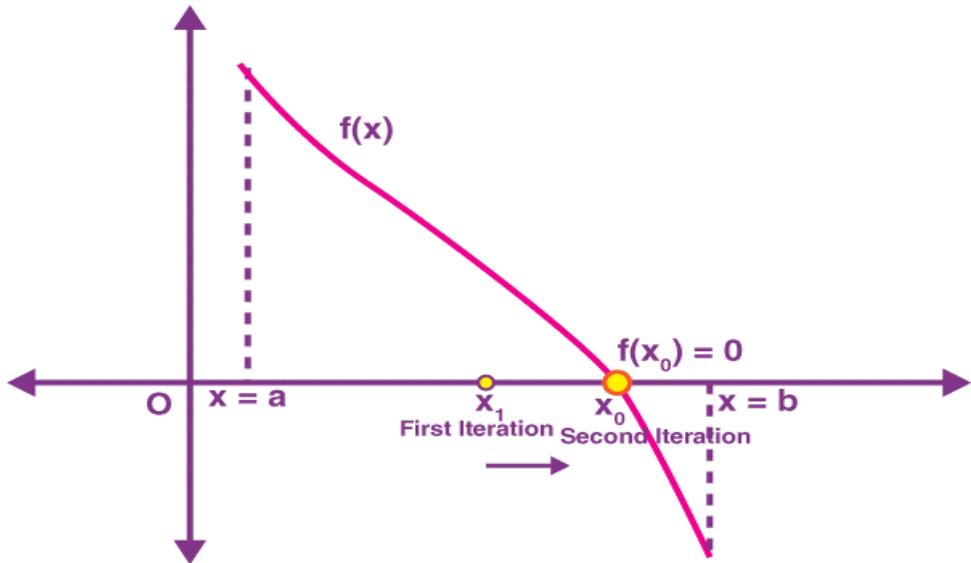
| Iterations | a | B | T | f(a) | f(b) | f(t) |
|------------|--------|--------|--------|---------|--------|---------|
| 1 | 1 | 2 | 1.5 | -2 | 1 | -0.75 |
| 2 | 1.5 | 2 | 1.75 | -0.75 | 1 | 0.062 |
| 3 | 1.5 | 1.75 | 1.625 | -0.75 | 0.0625 | -0.359 |
| 4 | 1.625 | 1.75 | 1.6875 | -0.3594 | 0.0625 | -0.1523 |
| 5 | 1.6875 | 1.75 | 1.7188 | -0.0457 | 0.0625 | -0.0457 |
| 6 | 1.7188 | 1.75 | 1.7344 | -0.0457 | 0.0625 | 0.0081 |
| 7 | 1.7188 | 1.7344 | 1.7266 | -0.0457 | 0.0081 | -0.0189 |

So, at the seventh iteration, we get the final interval $[1.7266, 1.7344]$

Hence, 1.7344 is the approximated solution.

In numerical analysis, the bisection method is an iterative method to find the roots of a given continuous function, which assumes positive and negative values at two distinct points in its domain. The main idea behind this root-finding method is to repeatedly bisect the interval, in which the function is continuous and assumes opposite sign values at the extremities of the interval. This process continues until we find a point x_0 within the interval, for which the function vanishes.

This method is based on the intermediate value theorem for continuous functions.



Algorithm for the bisection method:

- For any continuous function $f(x)$, find a closed interval $[a, b]$ such that $f(a).f(b) < 0$.
- Find the midpoint of a, b . Let $x_1 = (a + b)/2$
- If $f(x_1) = 0$, then x_1 is the root.
- If $f(x_1) \neq 0$, then
 - $f(a).f(x_1) < 0$, root of $f(x)$ lies in $[a, x_1]$, continue the above steps for interval $[a, x_1]$.
 - $f(x_1).f(b) < 0$, root of $f(x)$ lies in $[x_1, b]$, continue the above steps for interval $[x_1, b]$.

Continue the process repeatedly until we find a point x_0 in $[a, b]$ for which $f(x_0) = 0$.

Learn how to find out the roots of polynomial functions using the bisection method.

Bisection Method Questions with Solution

Follow the above algorithm of the bisection method to solve the following questions.

Question 1:

Find the root of the following polynomial function using the bisection method:

$$x^3 - 4x - 9.$$

Solution:

$$\text{Let } f(x) = x^3 - 4x - 9$$

$$f(2) = 8 - 8 - 9 = -9$$

$$f(3) = 27 - 12 - 9 = 6$$

\therefore the root lies in $[2, 3]$

First iteration:

$$x_1 = (2 + 3)/2 = 2.5$$

$$\text{Now, } f(x_1) = (2.5)^3 - 4(2.5) - 9 = -3.375$$

$$\text{Then, } f(x_1).f(3) < 0$$

Thus, the root lies in $[2.5, 3]$

Second iteration:

$$x_2 = (2.5 + 3)/2 = 2.75$$

$$\text{Now, } f(x_2) = (2.75)^3 - 4(2.75) - 9 = 0.7969$$

$$\text{Then, } f(x_1).f(x_2) < 0$$

Thus, the root lies in $[2.5, 2.75]$

Third iteration:

$$x_3 = (2.5 + 2.75)/2 = 2.625$$

$$\text{Now, } f(x_3) = (2.625)^3 - 4(2.625) - 9 = -1.4121$$

$$\text{Then, } f(x_2).f(x_3) < 0$$

Thus, the root lies in $[2.75, 2.625]$

Hence, we can make the following iteration table:

| Iterations (n) | a | B | x_n | $f(a)$ | $f(b)$ | $f(x_n)$ |
|----------------|-------|------|--------|---------|--------|----------|
| 1 | 2 | 3 | 2.5 | -8 | 6 | -3.375 |
| 2 | 2.5 | 3 | 2.75 | -3.375 | 6 | 0.7969 |
| 3 | 2.5 | 2.75 | 2.625 | -3.375 | 0.7969 | -1.4121 |
| 4 | 2.625 | 2.75 | 2.6875 | -1.4121 | 0.7969 | -0.3391 |

| | | | | | | |
|---|----------|----------|----------|---------|---------|---------|
| 5 | 2.6875 | 2.75 | 2.71875 | -0.3391 | 0.7969 | 0.2209 |
| 6 | 2.6875 | 2.71875 | 2.703125 | -0.3391 | 0.2209 | -0.0615 |
| 7 | 2.71875 | 2.703125 | 2.7109 | 0.2209 | -0.0615 | 0.0787 |
| 8 | 2.703125 | 2.7109 | 2.707 | -0.0615 | 0.0787 | 0.00849 |

∴ the root of the function is 2.707.

2) FALSE POSITION METHOD:

Regula Falsi Method

Regula Falsi is one of the oldest methods to find the real root of an equation $f(x) = 0$ and closely resembles with Bisection method. It requires less computational effort as we need to evaluate only one function per iteration.

The Regula Falsi Method improves upon the Bisection Method by using a linear interpolation to find a better approximation of the root. It calculates the root of the line segment connecting $f(a)$ and $f(b)$, which tends to be closer to the actual root than the midpoint used in the Bisection Method.

Regula Falsi Method, also known as the False Position Method, is a numerical technique used to find the roots of a non-linear equation of the form $f(x) = 0$. This method is based on the concept of bracketing, where two initial guesses, x_0 and x_1 , are chosen such that the function values at these points have opposite signs, indicating that a root lies between them.

The root finding algorithm of continuous function utilizing bracketing technique is called Regula-Falsi Method. This presents an iterative algorithm combined principle from bisection method and linear interpolation. The method works on an assumption that if a continuous function crosses 0 over an interval then there exists a root in that interval.

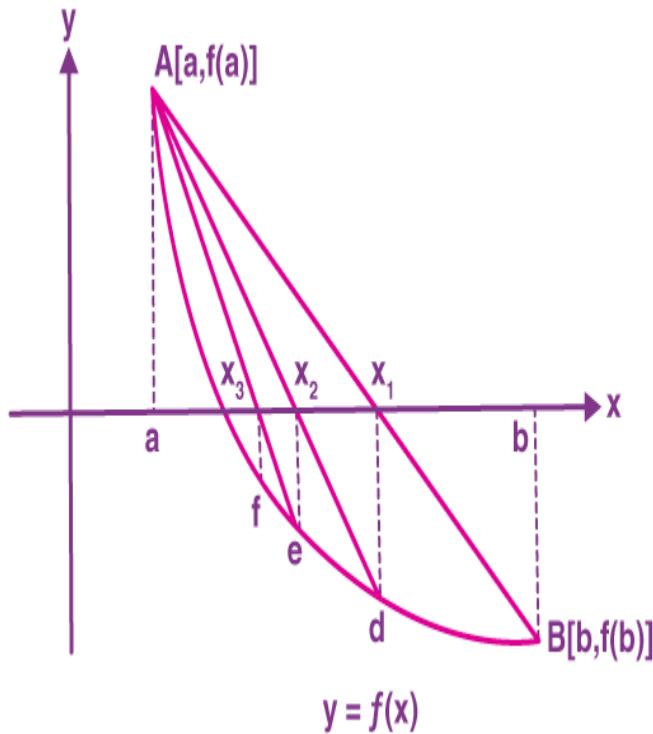
Regula Falsi Method is a numerical technique used to find the roots of a non-linear equation of the form $f(x) = 0$. This method is particularly useful when the function is continuous and the root is located between two points.

Advantages of Regula Falsi Method

- Faster convergence than the Bisection Method for functions that are approximately linear in the interval.
- Uses the function's values to better estimate the root.

Disadvantages of Regula Falsi Method

- May converge slowly if the function is not approximately linear within the interval.
- Can get stuck if the same subinterval is repeatedly chosen without significant improvement.
 - Geometrical representation of the roots of the equation $f(x) = 0$ can be shown as:



Formula

$$X_3 = \frac{X_1(f(X_2) - f(X_1))}{f(X_2) - f(X_1)}$$

Algorithm for False Position Method

1. start
2. Define function $f(x)$
3. Choose initial guesses x_0 and x_1 such that $f(x_0) f(x_1) < 0$
4. Choose pre-specified tolerable error e .
5. Calculate new approximated root as:

$$x_2 = x_0 - \frac{(x_0 - x_1) * f(x_0)}{f(x_0) - f(x_1)}$$

6. Calculate $f(x_0)$ $f(x_2)$
 - a. if $f(x_0) f(x_2) < 0$ then $x_0 = x_0$ and $x_1 = x_2$
 - b. if $f(x_0) f(x_2) > 0$ then $x_0 = x_2$ and $x_1 = x_1$

- c. if $f(x_0) f(x_2) = 0$ then go to (8)
- 7. if $|f(x_2)| > e$ then go to (5) otherwise go to (8)
- 8. Display x_2 as root.
- 9. Stop

Problem: Find a root of an equation $f(x)=x^3-x-1$

Solution:

Given equation, $x^3-x-1=0$

let $x = 0, 1, 2$

In 1st iteration :

$$f(1)=-1 < 0 \text{ and } f(2)=5 > 0$$

Root lies between these two points $x_0=1$ and $x_1=2$

$$x_2=x_0-f(x_0)$$

$$= x_1 - x_0$$

$$f(x_1)-f(x_0)$$

$$x_2=1-(-1)\cdot$$

$$= 2-1$$

$$= 5-(-1)$$

$$x_2=1.16667$$

$$f(x_2)=f(1.16667)=-0.5787 < 0$$

In 2nd iteration :

$$f(1.16667)=-0.5787 < 0 \text{ and } f(2)=5 > 0$$

Root lies between these two points $x_0=1.16667$ and $x_1=2$

$$x_3=x_0-f(x_0)$$

$$x_1 - x_0$$

$$f(x_1)-f(x_0)$$

$$x_3=1.16667-(-0.5787)$$

$$2-1.16667$$

$$5-(-0.5787)$$

$$x_3=1.25311$$

$$f(x_3)=f(1.25311)=-0.28536 < 0$$

In 3rd iteration :

$f(1.25311) = -0.28536 < 0$ and $f(2) = 5 > 0$

Root lies between these two points $x_0 = 1.25311$ and $x_1 = 2$

$$x_4 = x_0 - f(x_0) \cdot$$

$$x_1 - x_0$$

$$f(x_1) - f(x_0)$$

$$x_4 = 1.25311 - (-0.28536) \cdot$$

$$2 - 1.25311$$

$$5 - (-0.28536)$$

$$x_4 = 1.29344$$

$$f(x_4) = f(1.29344) = -0.12954 < 0$$

In 4th iteration :

$f(1.29344) = -0.12954 < 0$ and $f(2) = 5 > 0$

Root lies between these two points $x_0 = 1.29344$ and $x_1 = 2$

$$x_5 = x_0 - f(x_0) \cdot$$

$$x_1 - x_0$$

$$f(x_1) - f(x_0)$$

$$x_5 = 1.29344 - (-0.12954) \cdot$$

$$2 - 1.29344$$

$$5 - (-0.12954)$$

$$x_5 = 1.31128$$

$$f(x_5) = f(1.31128) = -0.05659 < 0$$

In 5th iteration :

$f(1.31128) = -0.05659 < 0$ and $f(2) = 5 > 0$

Root lies between these two points $x_0 = 1.31128$ and $x_1 = 2$

$$x_6 = x_0 - f(x_0) \cdot$$

$$x_1 - x_0$$

$$f(x_1) - f(x_0)$$

$$x_6 = 1.31128 - (-0.05659) \cdot$$

$$2 - 1.31128$$

$$5 - (-0.05659)$$

$$x_6 = 1.31899$$

$$f(x_6) = f(1.31899) = -0.0243 < 0$$

In 6th iteration :

$$f(1.31899) = -0.0243 < 0 \text{ and } f(2) = 5 > 0$$

Root lies between these two points $x_0 = 1.31899$ and $x_1 = 2$

$$x_7 = x_0 - f(x_0) \cdot$$

$$x_1 - x_0$$

$$f(x_1) - f(x_0)$$

$$x_7 = 1.31899 - (-0.0243) \cdot$$

$$2 - 1.31899$$

$$5 - (-0.0243)$$

$$x_7 = 1.32228$$

$$f(x_7) = f(1.32228) = -0.01036 < 0$$

In 7th iteration :

$$f(1.32228) = -0.01036 < 0 \text{ and } f(2) = 5 > 0$$

Root lies between these two points $x_0 = 1.32228$ and $x_1 = 2$

$$x_8 = x_0 - f(x_0) \cdot$$

$$x_1 - x_0$$

$$f(x_1) - f(x_0)$$

$$x_8 = 1.32228 - (-0.01036) \cdot$$

$$2 - 1.32228$$

$$5 - (-0.01036)$$

$$x_8 = 1.32368$$

The approximate root of the equation $x^3 - x - 1 = 0$ using the Regula Falsi method is 1.32368

Newton Raphson Method

Newton Raphson Method

Newton Raphson Method or Newton Method is a powerful technique for solving equations numerically. It is most commonly used for approximation of the roots of the real-valued functions. Newton Rapson Method was developed by Isaac Newton and Joseph Raphson, hence the name Newton Rapson Method. Newton Raphson Method involves iteratively refining an initial guess to converge it toward the desired root. However, the method is not efficient in calculating the roots of the polynomials or equations with higher degrees but in the case of small-degree equations, this method yields very quick result

- **What is Newton Raphson Method?**

The Newton-Raphson method which is also known as Newton's method, is an iterative numerical method used to find the roots of a real-valued function. This formula is named after Sir Isaac Newton and Joseph Raphson, as they independently contributed to its development. Newton Raphson Method or Newton's Method is an algorithm to approximate the roots of zeros of the real-valued functions, using guess for the first iteration (x_0) and then approximating the next iteration(x_1) which is close to roots, using the following formula.

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

where,

- x_0 is the initial value of x ,
- $f(x_0)$ is the value of the equation at initial value, and
- $f'(x_0)$ is the value of the first order derivative of the equation or function at the initial value x_0 .

Note: $f'(x_0)$ should not be zero else the fraction part of the formula will change to infinity which means $f(x)$ should not be a constant function.

Newton Raphson Method Formula

In the general form, the Newton-Raphson method formula is written as follows:

$$x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}$$

Where,

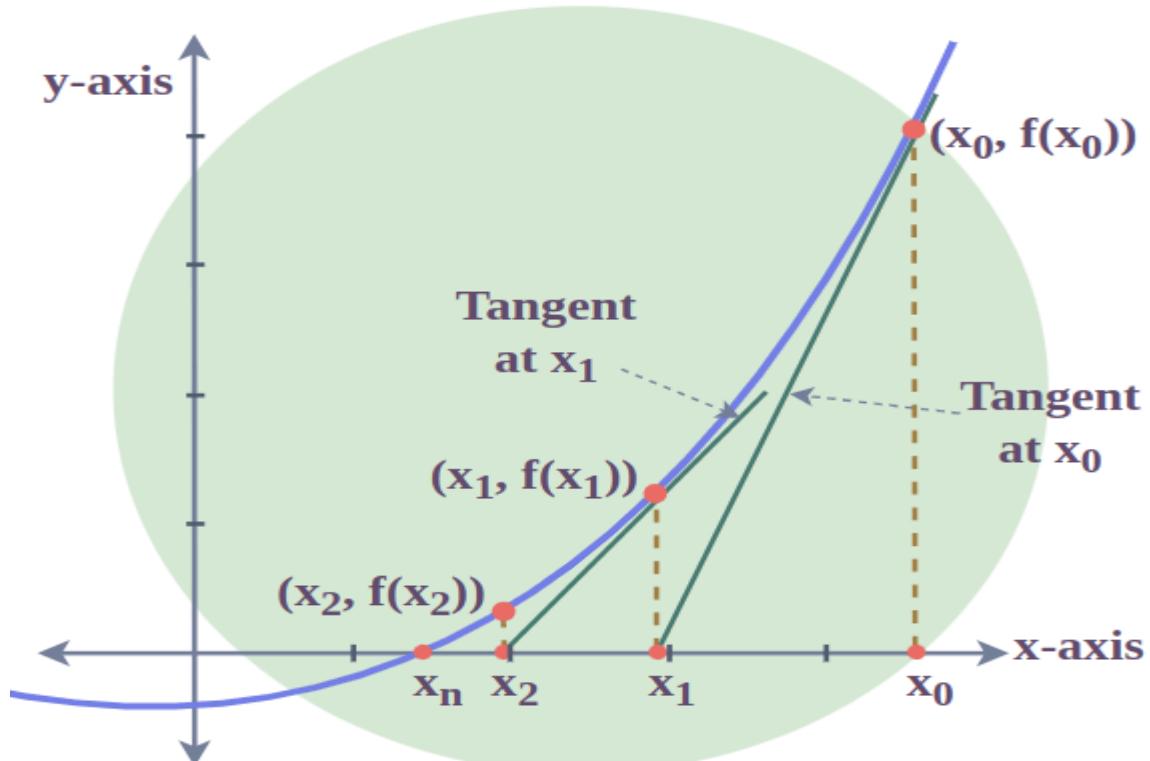
- x_{n-1} is the estimated $(n-1)$ th root of the function
- $f(x_{n-1})$ is the value of the equation at $(n-1)$ th estimated root
- $f'(x_{n-1})$ is the value of the first order derivative of the equation or function at x_{n-1}

Newton Raphson Method Calculation

Assume the equation or functions whose roots are to be calculated as $f(x) = 0$.

In order to prove the validity of Newton Raphson method following steps are followed:

Step 1: Draw a graph of $f(x)$ for different values of x as shown below:



Step 2: A tangent is drawn to $f(x)$ at x_0 . This is the initial value.

Step 3: This tangent will intersect the X-axis at some fixed point $(x_1, 0)$ if the first derivative of $f(x)$ is not zero i.e. $f'(x_0) \neq 0$.

Step 4: As this method assumes iteration of roots, this x_1 is considered to be the next approximation of the root.

Step 5: Now steps 2 to 4 are repeated until we reach the actual root x^* .

Now we know that the slope-intercept equation of any line is represented as $y = mx + c$, Where m is the slope of the line and c is the x-intercept of the line.

Using the same formula, we, get

$$y = f(x_0) + f'(x_0)(x - x_0)$$

Here $f(x_0)$ represents the c and $f'(x_0)$ represents the slope of the tangent m . As this equation holds true for every value of x , it must hold true for x_1 . Thus, substituting x with x_1 , and equating the equation to zero as we need to calculate the roots, we get:

$$0 = f(x_0) + f'(x_0)(x_1 - x_0)$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Which is the Newton Raphson method formula.

Thus, Newton Raphson's method was mathematically proved and accepted to be valid.

Convergence of Newton Raphson Method

The Newton-Raphson method tends to converge if the following condition holds true:

$$|f(x). f'(x)| < |f'(x)|^2$$

It means that the method converges when the modulus of the product of the value of the function at x and the second derivative of a function at x is lesser than the square of the modulo of the first derivative of the function at x . The Newton-Raphson Method has a convergence of order 2 which means it has a quadratic convergence.

Note:

Newton Raphson's method is not valid if the first derivative of the function is 0 which means $f'(x) = 0$. It is only possible when the given function is a constant function.

Newton Raphson Method Example

Let's consider the following example to learn more about the process of

Problem 1: Find the root of the equation $f(x) = x^3 - 5x + 3 = 0$, if the initial value is 3.

Solution:

Given $x_0 = 3$ and $f(x) = x^3 - 5x + 3 = 0$

$$f(x) = 3x^2 - 5$$

$$f(x_0 = 3) = 3 \times 9 - 5 = 22$$

$$f(x_0 = 3) = 27 - 15 + 3 = 15$$

Using Newton Raphson method:

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$\Rightarrow x_1 = 3 - 15/22$$

$$\Rightarrow x_1 = 2.3181$$

Using Newton Raphson method again:

$$x_2 = 1.9705$$

$$x_3 = 1.8504$$

$$x_4 = 1.8345$$

$$x_5 = 1.8342$$

Therefore, the root of the equation is approximately $x = 1.834$.

Regula Falsi vs Bisection Method vs Newton-Raphson Method

| Feature | Regula Falsi Method | Bisection Method | Newton-Raphson Method |
|--------------------------------|--|--|---|
| Initial Requirements | Requires two initial guesses a and b such that $f(a) \cdot f(b) < 0$ | Requires two initial guesses a and b such that $f(a) \cdot f(b) < 0$ | Requires one initial guess x_0 |
| Convergence Guarantee | Guaranteed if the function is continuous in $[a, b]$ | Guaranteed if the function is continuous in $[a, b]$ | Not guaranteed; depends on the choice of x_0 and function behaviour |
| Convergence Speed | Linear, generally faster than Bisection but slower than Newton-Raphson | Linear, generally slow | Quadratic, generally very fast |
| Formula | $x = a - \frac{f(a) \cdot (b-a)}{f(b)-f(a)}$ | $x = a + b / 2$ | $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ |
| Number of Function Evaluations | One per iteration | Two per iteration | One function and one derivative evaluation per iteration |
| Derivative Requirement | Not required | Not required | Requires first derivative |
| Application | Useful for non-differentiable functions | Useful for non-differentiable functions | Requires differentiable functions |
| Efficiency | More efficient than Bisection, less efficient than Newton-Raphson | Least efficient of the three | Most efficient if the derivative is available and well-behaved |

| Feature | Regula Falsi Method | Bisection Method | Newton-Raphson Method |
|-------------------------|--|--|--|
| Oscillatory Behaviour | Can exhibit oscillatory behaviour, slowing convergence | No oscillatory behaviour | Can exhibit oscillatory behaviour if initial guess is poor |
| Handling Multiple Roots | May converge to the nearest root | Will find a root within the interval but not necessarily the nearest | May fail to find multiple roots or diverge if poorly initialized |

What is the Gauss Elimination Method?

A system of linear equations with various unknown factors is known as a **system of linear equations**. Unknown factors can be found in various equations, as we all know. To check all of the equations that make up the system, you must find the value for the unknown factors. We can state that the given system is a **consistent independent system** if there is a single solution that indicates one value for each unknown factor. If there are several solutions, the system has an infinite number of solutions, then we call it a consistent dependent system.

What is the Gauss Elimination Method?

The method we use to perform the three types of matrix row operations on an augmented matrix obtained from a linear system of equations to find the solutions for such a system is known as the Gaussian elimination method.

Row reduction is a technique that consists of two stages: forward elimination and back substitution. The difference between these two **Gaussian elimination method** phases is the output they create, not the activities you can perform through them. The row reduction required to simplify the matrix in question into its echelon form is referred to as the forward elimination step. A goal of this stage is to demonstrate if the system of equations represented in the matrix has a single feasible solution, an unlimited number of solutions, or no solution at all.

If it is discovered that the system has no solution, there is no need to proceed to the next stage of row reduction. The Gaussian elimination with the back substitution step is carried out if solutions for the variables involved in the linear system can be found. This final step yields a reduced echelon form of the matrix, which yields the system of linear equations' general solution.

The Gaussian elimination rules are the same as the rules for the three basic row operations, in other words, you can algebraically act on a matrix's rows in the following three ways:

- Interchanging two rows, for example, $R_2 \leftrightarrow R_3$
- Multiplying a row by a constant, for example, $R_1 \rightarrow kR_1$ where k is some nonzero number
- Adding a row to another row, for example, $R_2 \rightarrow R_2 + 3R_1$

How to do Gaussian Elimination

It's all about the matrix you have in your hands and the necessary row operations to simplify it, not a series of Gaussian elimination processes to follow to solve a system of linear equations. Let's start with our first **Gauss elimination method example with solution** for a better understanding of the process and the intuition required to work through it:

Question:

Solve the following system of equations:

$$x + y + z = 2$$

$$x + 2y + 3z = 5$$

$$2x + 3y + 4z = 11$$

Solution:

Given system of equations are:

$$x + y + z = 2$$

$$x + 2y + 3z = 5$$

$$2x + 3y + 4z = 11$$

Let us write these equations in matrix form.



$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 1 & 2 & 3 & 5 \\ 2 & 3 & 4 & 11 \end{array} \right]$$

Subtracting R_1 from R_2 to get the new elements of R_2 , i.e. $R_2 \rightarrow R_2 - R_1$.

From this we get,

$$= \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 2 & 3 \\ 2 & 3 & 4 & 11 \end{array} \right]$$

Let us make another operation as $R_3 \rightarrow R_3 - 2R_1$

$$= \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 7 \end{array} \right]$$

Subtract R_2 from R_1 to get the new elements of R_1 , i.e. $R_1 \rightarrow R_1 - R_2$.

$$= \left[\begin{array}{ccc|c} 1 & 0 & -1 & -1 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 7 \end{array} \right]$$

Now, subtract R_2 from R_3 to get the new elements of R_3 , i.e. $R_3 \rightarrow R_3 - R_2$.

$$= \left[\begin{array}{ccc|c} 1 & 0 & -1 & -1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 4 \end{array} \right]$$

Here,

$$x - z = -1$$

$$y + 2z = 3$$

$$0 = 4$$

That means, there is no solution for the given system of equations.

Gauss-Seidel Method

Iterative methods Jacobi and Gauss-Seidel in numerical analysis are based on the idea of successive approximations. This iterative method begins with one or two initial approximations of the roots, with a sequence of approximations $x_1, x_2, x_3, \dots, x_k, \dots$, as $k \rightarrow \infty$, this sequence of roots converges to exact root α . For a system of equations $\mathbf{Ax} = \mathbf{B}$, we begin with an initial approximation of solution vector $\mathbf{x} = \mathbf{x}_0$, by which we get a sequence of solution vector $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k, \dots$ as $k \rightarrow \infty$, this sequence converges to the solution $\mathbf{x} = \mathbf{A}^{-1}\mathbf{B}$.

The general iterative formulas can be given as:

$$\mathbf{x}_{k+1} = \mathbf{Hx}_k ; k = 1, 2, 3, \dots$$

Where \mathbf{x}_{k+1} and \mathbf{x}_k are approximations for the exact root of $\mathbf{Ax} = \mathbf{B}$ at $(k + 1)$ th and k th iterations. \mathbf{H} is an iteration matrix that depends on \mathbf{A} and \mathbf{B} .

Convergence of Iterative Methods: The sequence of iterates $\{\mathbf{x}_k\}$ is said to be converging to the exact root \mathbf{x} , if

$$\lim_{k \rightarrow \infty} \|\mathbf{x}_k - \mathbf{x}\| \leq \epsilon$$

Where ϵ is a very small positive quantity called error tolerance or error bound.

The criterion to terminate Iteration Process: As we cannot perform the iteration process infinitely, we need some criterion to stop the iteration; we may use one or both of the criteria:

- The approximated root satisfies the given system of linear equations to a given accuracy.
- The magnitude of the difference between two successive iterates is negligible or smaller than a given error bound ε .

Jacobi Method

Jacobi method or Jacobian method is named after German mathematician Carl Gustav Jacob Jacobi (1804 – 1851). The main idea behind this method is,

For a system of linear equations:

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$

⋮

$$a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = b_n$$

To find the solution to this system of equations $\mathbf{Ax} = \mathbf{B}$, we assume that the system of equations have a unique solution and there is no zero entry among the diagonal or pivot elements of the coefficient matrix \mathbf{A} .

Now, we shall begin to solve equation 1 for x_1 , equation 2 for x_2 and so on equation n for x_n , we get

$$x_1 = 1/a_{11} [b_1 - a_{12}x_2 - a_{13}x_3 - \dots - a_{1n}x_n]$$

$$x_2 = 1/a_{22} [b_2 - a_{21}x_1 - a_{23}x_3 - \dots - a_{2n}x_n]$$

⋮

$$x_n = 1/a_{nn} [b_n - a_{n1}x_1 - a_{n2}x_2 - \dots - a_{n(n-1)}x_{n-1}]$$

By making an initial guess for the solution $\mathbf{x}^{(0)} = (x_1^{(0)}, x_2^{(0)}, \dots, x_n^{(0)})$ and substituting these values only to the right hand side of the above equations we get first approximations $\mathbf{x}^{(1)} = (x_1^{(1)}, x_2^{(1)}, \dots, x_n^{(1)})$. Continuing this process iteratively we get sequence of approximations $\{\mathbf{x}^{(k)}\}$ such that as $k \rightarrow \infty$, this sequence converges to exact solution of the system of equation up to a given error tolerance.

Convergence of Approximations

The sufficient condition for the convergence of the approximations obtained by Jacobi method is that the system of equations is **diagonally dominant**, that is, the coefficient matrix \mathbf{A} is diagonally dominant.

The matrix \mathbf{A} is said to be diagonally dominant if $|a_{ii}| \geq \sum_{j=1}^n |a_{ij}|$ for $i \neq j$. That means, the absolute value of the diagonal element is greater than or equal to the sum of all elements of the corresponding row.

How to Decide Initial Approximations?

To initiate the iterative process, we must assume some initial approximation of the possible solution for the given system of equations. If the given system of equations are diagonally dominant, then any initial approximation will converge to the exact solution. Still, if no suitable approximations are not available, then we may take $\mathbf{x}^{(0)} = \mathbf{0}$ where $x_i^{(0)} = 0$ for all i . Then, the first approximation becomes $x_i = b_i/a_{ii}$ for all i .

Gauss-Seidel Method

The Guass-Seidel method is a improvisation of the Jacobi method. This method is named after mathematicians Carl Friedrich Gauss (1777–1855) and Philipp L. Seidel (1821–1896). This modification often results in higher degree of accuracy within fewer iterations.

In Jacobi method the value of the variables is not modified until next iteration, whereas in Gauss-Seidel method the value of the variables are modified as soon as new value is evaluated. For instance, in Jacobi method the value of $x_i^{(k)}$ is not modified until the $(k + 1)$ th iteration but in Gauss-Seidel method the value of $x_i^{(k)}$ changes in in k th iteration only.

Example 1:

Solve the system of equations using the Jacobi Method

$$26x_1 + 2x_2 + 2x_3 = 12.6$$

$$3x_1 + 27x_2 + x_3 = -14.3$$

$$2x_1 + 3x_2 + 17x_3 = 6.0$$

Obtain the result correct to three decimal places.

Solution:

First, check for the convergence of approximations,

$$26 > 2 + 2$$

$$27 > 3 + 1$$

$$17 > 2 + 3$$

Hence, the given system of equations are strongly diagonally dominant, which ensures the convergence of approximations. Let us take the initial approximation, $x_1^{(0)} = 0$, $x_2^{(0)} = 0$ and $x_3^{(0)} = 0$

$$x_3^{(0)} = 0$$

First Iteration:

$$x_1^{(1)} = 1/26[12.6 - 2 \times 0 - 2 \times 0] = 0.48462$$

$$x_2^{(1)} = 1/27[-14.3 - 3 \times 0 - 1 \times 0] = -0.52963$$

$$x_3^{(1)} = 1/17[6 - 2 \times 0 - 3 \times 0] = 0.35294$$

Second Iteration:

$$x_1^{(2)} = 1/26[12.6 - 2 \times (-0.52963) + 0.35294] = 0.49821$$

$$x_2^{(2)} = 1/27[-14.3 - 3 \times 0.48462 - 1 \times 0.35294] = -0.59655$$

$$x_3^{(2)} = 1/17[6 - 2 \times 0.48462 - 3 \times (-0.52963)] = 0.38939$$

Likewise there will be modification in approximation with each iteration.

| kth iteration | 0 | 1 | 2 | 3 | 4 | 5 |
|---------------|-------|----------|----------|----------|----------|----------|
| x_1 | 0.000 | 0.48462 | 0.49821 | 0.50006 | 0.50000 | 0.50001 |
| x_2 | 0.000 | -0.52963 | -0.59655 | -0.59941 | -0.59999 | -0.60000 |
| x_3 | 0.000 | 0.35294 | 0.38939 | 0.39960 | 0.39989 | 0.40000 |

After the fifth iteration, we get $|x_1^{(5)} - x_1^{(4)}| = |0.50001 - 0.50000| = 0.00001$

$$|x_2^{(5)} - x_2^{(4)}| = |-0.6 + 0.59999| = 0.00001$$

$$|x_3^{(5)} - x_3^{(4)}| = |0.4 - 0.39989| = 0.00011$$

Since, all the errors in magnitude are less than 0.0005, the required solution is

$$x_1 = 0.5, x_2 = -0.6, x_3 = 0.4.$$

Example 2:

Solve the system of equations using the Gauss-Seidel Method

$$45x_1 + 2x_2 + 3x_3 = 58$$

$$-3x_1 + 22x_2 + 2x_3 = 47$$

$$5x_1 + x_2 + 20x_3 = 67$$

Obtain the result correct to three decimal places.

Solution:

First, check for the convergence of approximations,

$$45 > 2 + 3$$

$$22 > -3 + 2$$

$$20 > 5 + 1$$

Hence, the given system of equations are strongly diagonally dominant, which ensures the convergence of approximations. Let us take the initial approximation, $x_1^{(0)} = 0$, $x_2^{(0)} = 0$ and

$$x_3^{(0)} = 0$$

First Iteration:

$$x_1^{(1)} = 1/45[58 - 2 \times 0 - 3 \times 0] = 1.28889$$

$$x_2^{(1)} = 1/22[47 + 3 \times 1.28889 - 2 \times 0] = 2.31212$$

$$x_3^{(1)} = 1/20[67 - 5 \times 1.28889 - 1 \times 2.31212] = 2.91217.$$

Second Iteration:

$$x_1^{(2)} = 1/45[58 - 2 \times 2.31212 - 3 \times 2.91217] = 0.99198$$

$$x_2^{(2)} = 1/22[47 + 3 \times 0.99198 - 2 \times 2.91217] = 2.00689$$

$$x_3^{(2)} = 1/20[67 - 5 \times 0.99198 - 1 \times 2.00689] = 3.00166.$$

Likewise there will be modification in approximation with each iteration.

| kth iteration | 0 | 1 | 2 | 3 | 4 |
|---------------|-------|---------|---------|---------|---------|
| x_1 | 0.000 | 1.28889 | 0.99198 | 0.99958 | 1.0000 |
| x_2 | 0.000 | 2.31212 | 2.00689 | 1.99979 | 1.99999 |
| x_3 | 0.000 | 2.91217 | 3.00166 | 3.00012 | 3.00000 |

After the fourth iteration, we get $|x_1^{(4)} - x_1^{(3)}| = |1.0000 - 0.99958| = 0.00042$

$$|x_2^{(4)} - x_2^{(3)}| = |1.99999 + 1.99979| = 0.00020$$

$$|x_3^{(4)} - x_3^{(3)}| = |3.0000 - 3.00012| = 0.00012$$

Since, all the errors in magnitude are less than 0.0005, the required solution is

$$x_1 = 1.0, x_2 = 1.99999, x_3 = 3.0.$$

Rounding to three decimal places, we get $x_1 = 1.0, x_2 = 2.0, x_3 = 3.0$.