COMPSCI 371 Homework 5

Group Members: Mayur Sekhar, Jai Kasera, Rithvik Neti

Problem 0 (3 points)

Part 1: Hyperplane Geometry

Problem 1.1 (Exam Style)

A. Formulas for conversions between hyperplane representations

1. Normal Vector and Projection: The vector $\mathbf{w}$ is the normal vector to the hyperplane, and $\mathbf{p}$ must also be normal in order to represent the point on the hyperplane with the shortest distance to the origin. Hence, $\mathbf{p}$ is a scalar multiple of $\mathbf{w}$, i.e., $\mathbf{p} = c\mathbf{w}$. Substituting $\mathbf{x} = \mathbf{p}$ into the hyperplane equation, we get:

$$b + \mathbf{w} \cdot \mathbf{x} = 0$$

Substituting $\mathbf{p}$:

$$b + \mathbf{w} \cdot \mathbf{p} = 0$$

Since $\mathbf{p} = c\mathbf{w}$:

$$b + \mathbf{w} \cdot c\mathbf{w} = 0$$

Simplifying:

$$b + c\|\mathbf{w}\|^2 = 0$$

Solving for c:

$$c = -\frac{b}{\|\mathbf{w}\|^2}$$

Therefore, $\mathbf{p} = -\frac{b}{\|\mathbf{w}\|^2}\mathbf{w}$.

Additionally, every column of U is orthogonal to $\mathbf{w}$, but U is not unique. Hence, U is one of the possible orthonormal matrices that spans the subspace orthogonal to $\mathbf{w}$, i.e.,

$U \in \mathbf{w}^{\perp}$. If U is not already orthonormal, we can apply the Gram-Schmidt process to make it so.

Conversion from implicit representation (b, w) to canonical representation (p, U):

$$p = -\frac{b}{\|w\|^2} w$$

$$U = w^{\perp}$$

where $w^{\perp}$ denotes any orthonormal basis of the subspace orthogonal to (w).

2. Parametric to Implicit Representation:

The vector $\mathbf{w}$ is orthogonal to the hyperplane and therefore orthogonal to every column of matrix A, implying $A^T \mathbf{w} = \mathbf{0}$. This shows that $\mathbf{w}$ lies in the kernel of A^T, i.e., $\mathbf{w} \in \ker(A^T)$. Since the null space of A^T is orthogonal to the row space of A, we conclude that $\mathbf{w} \in (A)^{\perp}$.

Now, substituting $\mathbf{x} = \mathbf{a} + A\alpha$ into the equation $b + \mathbf{w} \cdot \mathbf{x} = 0$, we get:

$$b + \mathbf{w} \cdot (\mathbf{a} + A\alpha) = 0$$

Expanding:

$$b + \mathbf{w} \cdot \mathbf{a} + \mathbf{w}^T A\alpha = 0$$

Since $\mathbf{w}^T A = 0$, this simplifies to:

$$b + \mathbf{w} \cdot \mathbf{a} = 0$$

Thus:

$$b = -\mathbf{w}^T \mathbf{a}$$

Conversion from parametric representation (a, A) to implicit representation (b, w):

- $w \in A^T$.
- $b = -w \cdot a$.

B. Numerical expressions for the sample hyperplanes

1. Hyperplane H_1:

    - Implicit representation:

$$b = -3, \quad w = (0)$$

(since it's a point).

- Canonical representation:

$$p = (3), \quad U = \emptyset$$

(since it's a point in $\mathbb{R}$).

2. Hyperplane H_2:

- Implicit representation: Find w orthogonal to the vectors $u_0 = (4, 0)$ and $u_1 = (0, 3)$:

$$w = (3, 4), \quad b = -w \cdot u_0 = -12$$

- Canonical representation:

$$p = \frac{12}{25}\begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

(the point on the line closest to the origin).

$$U = \frac{1}{5}\begin{bmatrix} -4 \\ 3 \end{bmatrix}$$

3. Hyperplane H_3:

- Implicit representation:

$$b = -12, \quad w = (1, 4, 8)$$

- Canonical representation:

$$p = \frac{12}{81}\begin{bmatrix} 1 \\ 4 \\ 8 \end{bmatrix}$$

Using Gram-Schmidt to find U orthogonal to w, we get:

$$U = \begin{pmatrix} -\frac{4}{\sqrt{17}} & -\frac{8}{9\sqrt{17}} \\ \frac{1}{\sqrt{17}} & -\frac{32}{9\sqrt{17}} \\ 0 & \frac{17}{9\sqrt{17}} \end{pmatrix}$$

In [1]:
```
import numpy as np
```

```
w = np.array([1, 4, 8])
U = np.array([[-4/np.sqrt(17), -8/(9*np.sqrt(17))],
              [1/np.sqrt(17), -32/(9*np.sqrt(17))],
              [0, 17/(9*np.sqrt(17))]])
with np.printoptions(precision=6, suppress=True):
    print(np.dot(U.T, U), end='\n\n')
    print(np.dot(w, U))
```

```
[[1. 0.]
 [0. 1.]]

[0. 0.]
```

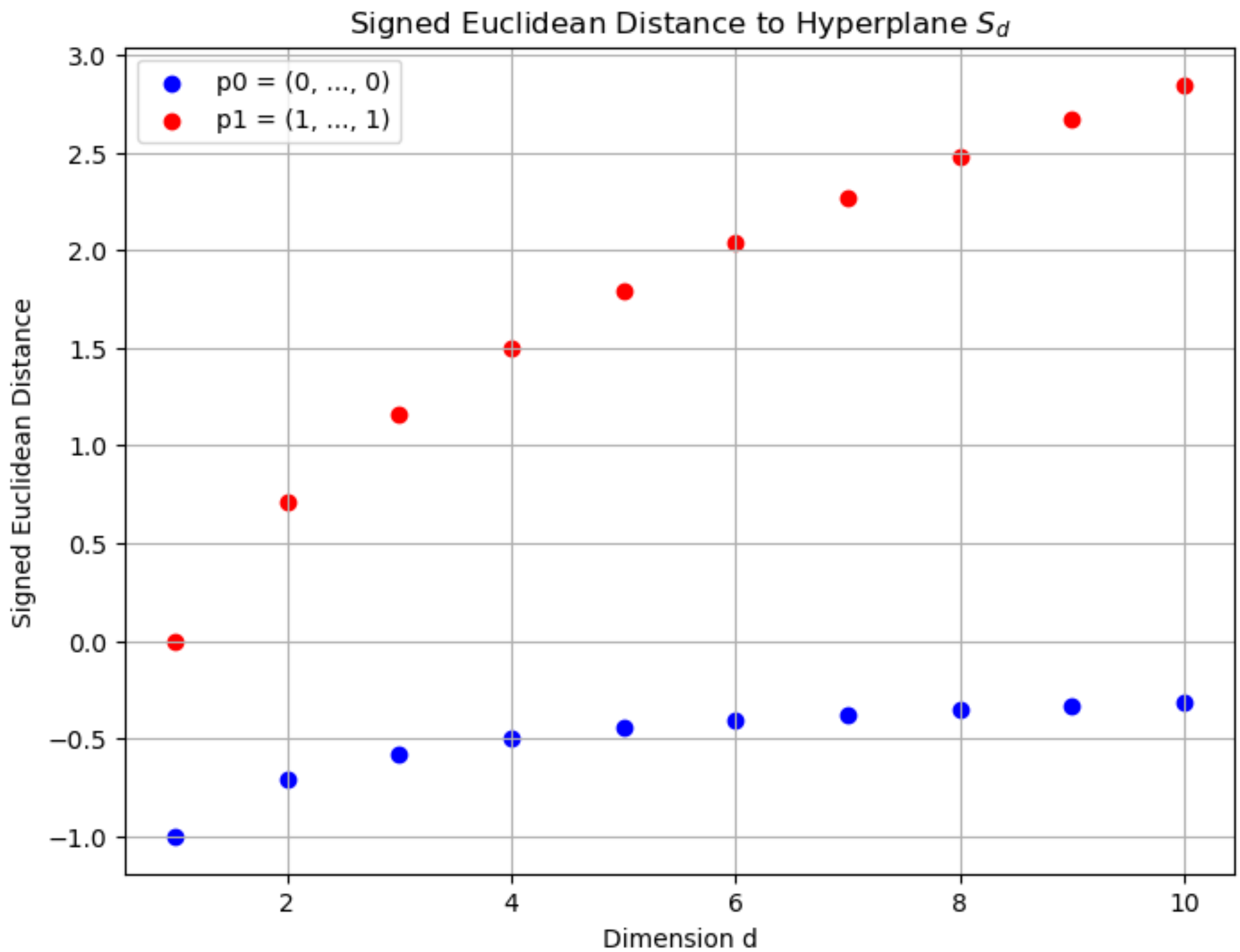
Problem 1.2 (Exam Style)

The signed Euclidean distance Δ_d from point p to the hyperplane S_d (with implicit equation $\sum_{i=1}^d x_i - 1 = 0$) can be computed as:

$$\Delta_d(p) = \frac{\sum_{i=1}^d p_i - 1}{\sqrt{d}}$$

In [2]:

```
import numpy as np
import matplotlib.pyplot as plt

def signed_distance(p, d):
    return (np.sum(p) - 1) / np.sqrt(d)

d_values = np.arange(1, 11)

p0 = np.zeros(10)
p1 = np.ones(10)

d_p0 = [signed_distance(p0[:d], d) for d in d_values]
d_p1 = [signed_distance(p1[:d], d) for d in d_values]

plt.figure(figsize=(8, 6))
plt.scatter(d_values, d_p0, label='p0 = (0, ..., 0)', color='blue')
plt.scatter(d_values, d_p1, label='p1 = (1, ..., 1)', color='red')

plt.xlabel('Dimension d')
plt.ylabel('Signed Euclidean Distance')
plt.title('Signed Euclidean Distance to Hyperplane $S_d$')
plt.legend()
plt.grid(True)

plt.show()
```



Problem 1.3 (Exam Style)

1. For $\ell_1 : x = \alpha(1,1,1)$:
   - The parametric form intersects S_3 since $\sum x_i = 1$ for $\alpha = 1/3$.
   - Point of intersection: $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.
2. For $\ell_2 : x = \alpha(1,-1,0)$:
   - Does not intersect S_3. Compute the shortest distance and closest points:
     - Distance = $\frac{1}{\sqrt{3}}$.
     - All of the points on ℓ_2, are nearest to $\mathcal{S}_3$.
     - Closest Points on line: $(t + \frac{1}{3}, -t + \frac{1}{3}, \frac{1}{3})$ ($t \in \mathbb{R}$) are on $\mathcal{S}_3$ and nearest to ℓ_2.
3. For $\ell_3 : x = (1,1,1) + \alpha(1,-1,0)$:
   - Does not intersect S_3. Compute the shortest distance and closest points:

- Distance = $\frac{2}{\sqrt{3}}$.
- All of the points on ℓ_3, are nearest to $\mathcal{S}_3$.
- Closest Points on line: $(t+\frac{1}{3}, -t+\frac{1}{3}, \frac{1}{3})$ ($t \in \mathbb{R}$) are on $\mathcal{S}_3$ and nearest to ℓ_3.

4. For ℓ_1 and ℓ_2:

   - They intersect at $(0, 0, 0)$.

5. For ℓ_2 and ℓ_3:

   - Does not intersect. Compute the shortest distance and closest points:
   - Distance = $\sqrt{3}$.
   - All of the points on ℓ_2 are nearest to ℓ_3 and all of the points on ℓ_3 are nearest to ℓ_2.

6. For ℓ_1 and ℓ_3:

   - They intersect at $(1, 1, 1)$

7. For ℓ_1 and ℓ_4: Does not intersect. Compute the shortest distance and closest points:

   - Distance = $2\sqrt{2}$.
   - Closest Points on line: $(2, 2, 2)$ on ℓ_1 is nearest to ℓ_4. and $(2, 4, 0)$ on ℓ_4 is nearest to ℓ_1.