COMPSCI 371 Homework 0

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Problem 0 (3 points)

Part 1: Linear Algebra

Problem 1.1

If A has a nontrivial null space, then there exists a nonzero vector $\vec{v}$ such that $A\vec{v} = 0$. This implies that $\vec{v}$ is an eigenvector with eigenvalue $\lambda = 0$. Since $\vec{v}$ is a nonzero vector in $\mathbb{R}^d$, it is a real vector, which means A has a real eigenvector associated with eigenvalue 0, and thus, a square real matrix A with a nontrivial null space must have real eigenvectors.

Problem 1.2

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

The characterisitc polynomial of A is found by A - λI. The characteristic polynomial is λ^2 = 1 - 0. This means that since the the possible eigenvalue is less than 0 the only possible eigenvalues and eigne vectors would be imaginary as they are the square root of negative numbers. There is a maximum of 2 eigenvalues (since we have a 2x2 matrix), but both would still be imaginary as they are the squareroot of a negative number and thus imaginary numbers (not real).

$A\vec{v} = \lambda\vec{v}$, where λ is a real eigenvalue and $\vec{v}$ is the corresponding eigenvector.

$$A\vec{v} = \lambda\vec{v} \implies \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \lambda \begin{bmatrix} a \\ b \end{bmatrix}$$

This gives the system:

$-b = \lambda a$

$a = \lambda b$

Solve for b and substitute:

$b = -\lambda a$

$a = \lambda(-\lambda a)$

$a = -a\lambda^2$

Now solve for λ:

$\lambda^2 = -1 \quad \Rightarrow \quad \lambda = \pm\sqrt{-1}$

$\lambda = i$

Since λ is imaginary(not real), the eigenvalues of A are not real, and thus, the eigenvectors will also not be real.

Problem 1.3

$A\vec{x} = \vec{b}$, $A = \begin{bmatrix} 1 & 2 & 3 \\ -2 & 1 & -1 \\ 3 & 0 & 3 \end{bmatrix}$, & $\vec{b} = \begin{bmatrix} 5 \\ 0 \\ 3 \end{bmatrix}$

Write the augmented matrix: $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 5 \\ -2 & 1 & -1 & 0 \\ 3 & 0 & 3 & 3 \end{array}\right]$

Step 1: Add 2 times Row 1 with Row 2 and Subtract 3 Row 1 from Row 3:
$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 5 \\ 0 & 5 & 5 & 10 \\ 0 & -6 & -6 & -12 \end{array}\right]$

Step 2: Divide Row 2 by 5 and Divide Row 6 by -6: $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 5 \\ 0 & 1 & 1 & 2 \\ 0 & 1 & 1 & 2 \end{array}\right]$

Step 3: Subtract Row 2 from Row 3: $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 5 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$x_2 + x_3 = 2$

$x_2 = 2 - x_3$

$x_1 + 2x_2 + 3x_3 = 5$

$x_1 + 4 - 2x_3 + 3x_3 = 5$

$x_1 = 1 - x_3$

The set of all solutions to the given linear system cna be given by

$$\vec{x} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

Problem 1.4

The value of q that we think was used in the truncated Fourier series was q = 3. This is because when q = 3 the line of best fit which is graphed is most acurate to the data and most clearly close to mapping the actual data. While q = 4 was also quite good at mapping the data, q = 3 is just a little better at mapping the actual shape and curvature of the data. Since q = 3 is left than q = 4 this using q = 3 would be an example of trying to avoid the overfitting that might ocurr from the higher q values.

In [5]:
```
import urllib.request
import ssl
from os import path as osp
import shutil
import numpy as np
import matplotlib.pyplot as plt

def retrieve(file_name, semester='fall24', homework=0):
    if osp.exists(file_name):
        print('Using previously downloaded file {}'.format(file_name))
    else:
        context = ssl._create_unverified_context()
        fmt = 'https://www2.cs.duke.edu/courses/{}/compsci371/homework/{}/{}'
        url = fmt.format(semester, homework, file_name)
        with urllib.request.urlopen(url, context=context) as response:
            with open(file_name, 'wb') as file:
                shutil.copyfileobj(response, file)
        print('Downloaded file {}'.format(file_name))
```

In [6]:
```
import pickle

# retrieve is defined in a hidden cell
file_name = 'points.pkl'
retrieve(file_name)
with open(file_name, 'rb') as file:
    t = pickle.load(file)
```

```
Using previously downloaded file points.pkl
```

In [7]:
```
t
```

Out[7]: namespace(x=array([4.52751939e-02, 4.87577107e-02, 9.99176115e-01, 6.52369112e-01,
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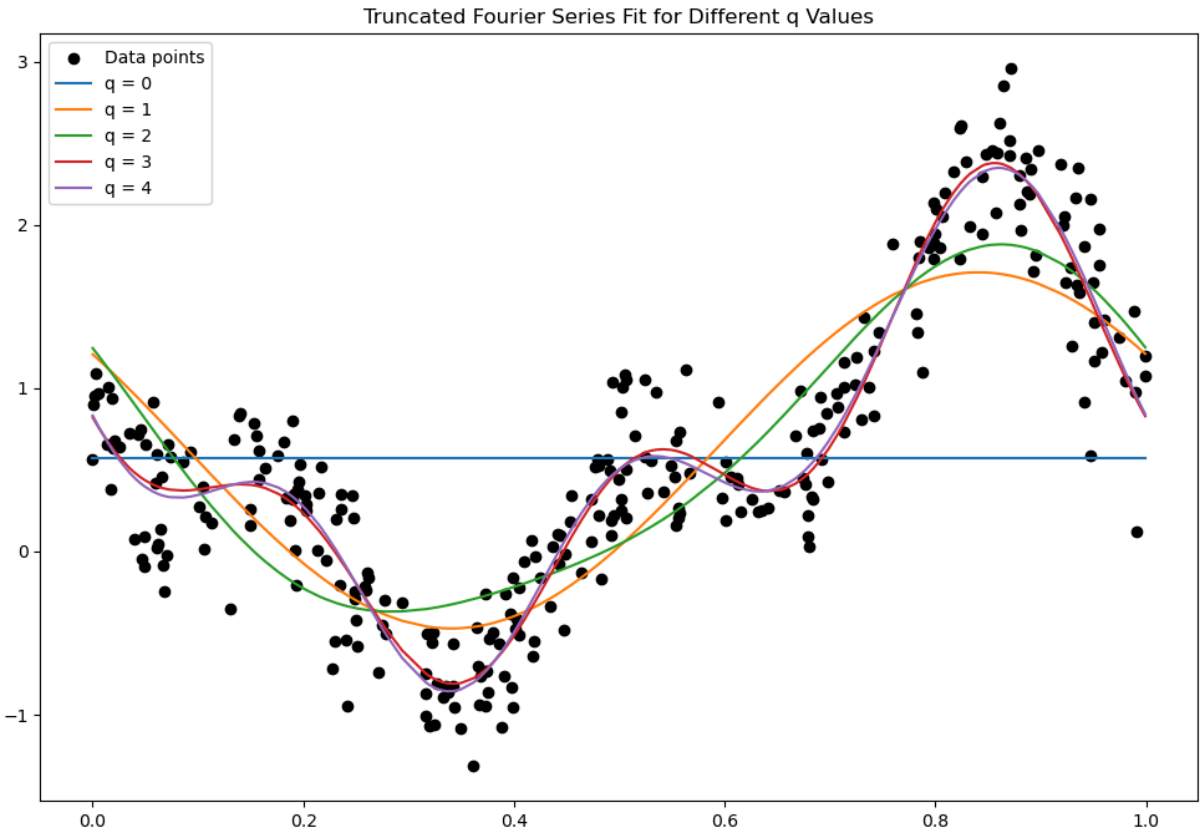
6]))
```

In [8]:

```python
def fit(x, y, q):
    n = len(x)
    A = np.ones((n, 2 * q + 1))
    for k in range(1, q + 1):
        A[:, k] = np.cos(2 * np.pi * k * x)
        A[:, q + k] = np.sin(2 * np.pi * k * x)

    c_star, _, _, _ = np.linalg.lstsq(A, y, rcond=None)
    return c_star
```

In [12]:

```python
plt.figure(figsize=(12, 8))
plt.scatter(t.x, t.y, color='black', label='Data points')

x_plot = np.sort(t.x)
for q in range(5):
    c_fit = fit(t.x, t.y, q)
    y_fit = np.ones_like(x_plot) * c_fit[0]
    for k in range(1, q + 1):
        y_fit += c_fit[k] * np.cos(2 * np.pi * k * x_plot) + c_fit[q + k] * np.sin(
    plt.plot(x_plot, y_fit, label=f'q = {q}')

plt.legend()
plt.title("Truncated Fourier Series Fit for Different q Values")
plt.show()
```



Part 2: Probability

Problem 2.1

To see if p(x, y) is a probability density function(PDF, we need to see if it meets the tw

Non-negativity means $p(x, y) \geq 0$ for all $(x, y) \in \mathbb{R}^2$.

Normalization means that the toal integral over the entire domain must $= 1$:

$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p(x, y)\, dxdy = 1$

Non-negativity:
$p(x, y) = ce^{(-1/2)(x-y)^2}$ is always non-negative because exponential functions are always

Normalization: $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p(x, y)\, dxdy = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} ce^{(-1/2)(x-y)^2} dxdy = 1$

Let $u = x - y$ and $v = x + y$

$x = \frac{u+v}{2}$

$v = (u + y) + y$

$2y = v - u$

$y = \frac{v-u}{2}$

$\frac{\partial(x,y)}{\partial(u,v)} = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$

$\det\left(\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}\right) = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) - \left(-\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{4} - \left(-\frac{1}{4}\right) = \frac{1}{2}$

$\frac{1}{2}c\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{1}{2}u^2} du dv$

$\int_{-\infty}^{\infty} e^{-\frac{1}{2}u^2} du$ is a Gaussian integral $= \sqrt{2\pi}$

$\frac{1}{2}c\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{1}{2}u^2} du dv = \frac{1}{2}c\sqrt{2\pi}\int_{-\infty}^{\infty} dv = 2c\sqrt{2\pi}\int_{-\infty}^{\infty} dv = \infty$

Since $p(x,y)$ cannot be normalized it is not a probability density function.

Problem 2.2

Given that p(x,y) is a probability density function we can normalize in order to find C.

$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(x,y) dx dy = 1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} Ce^{-\frac{1}{2}(x-y)^2} e^{-\frac{1}{2}(x+y)^2} dx dy$

$u = x - y \quad v = x + y$

$x = u + y \quad 2y = v - u$

$x = \frac{u+v}{2} \quad y = \frac{v-u}{2}$

$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \left|\begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}\right| = \left|\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}\right| = \frac{1}{2}$

$\frac{C}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{1}{2}u^2} e^{-\frac{1}{2}v^2} du dv$

$= \frac{C}{2}\int_{-\infty}^{\infty} e^{-\frac{1}{2}u^2} du \int_{-\infty}^{\infty} e^{-\frac{1}{2}v^2} dv$

$= \frac{C}{2}\left(\sqrt{2\pi}\right)\left(\sqrt{2\pi}\right)$

$= \frac{C}{2}(2\pi) = C\pi = 1$

$C = \frac{1}{\pi}$

The above was unnecessary work but done nonetheless.

The density function is symmetric and centered at 0 and therefore the mean of both x and y is 0: m $= \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

The covariance matrix $\Sigma = \begin{bmatrix} \operatorname{var}(x) & \operatorname{cov}(x,y) \\ \operatorname{cov}(x,y) & \operatorname{var}(y) \end{bmatrix}$. Because x and y are independent and identically distributed, the variance can be given by the standard normal distribution:
$\operatorname{var}(x) = \operatorname{var}(y) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} x^2 e^{-x^2} dx$

$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} x^2 e^{-x^2} dx$

$u = e^{-x^2} \quad v = \frac{-x^3}{3}$

$du = -2xe^{-x^2} dx \quad dv = x^2 dx$

$\frac{x^3}{3} e^{-x^2} + \frac{2}{3} \int \left(x^3\right) \left(-2xe^{-x^2}\right) dx$

$= \frac{1}{\sqrt{\pi}} \left(\frac{\sqrt{\pi}}{2}\right) = \frac{1}{2}$

$\operatorname{cov}(x,y) = 0$ since x and y are independent.

$\therefore \Sigma = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$

$m = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $\Sigma = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$

Problem 2.3

A has $\frac{2}{3}$ red, $\frac{1}{3}$ white.

B has $\frac{1}{3}$ red, $\frac{2}{3}$ white.

Fair coin has $\frac{1}{2}$ heads, $\frac{1}{2}$ tails.

Solve for $P(A|n\text{red})$.

$P(A) = P(\text{H}) = \frac{1}{2}, \quad P(B) = P(\text{T}) = \frac{1}{2}$

$P(R|A) = \frac{2}{3}, \quad P(W|A) = \frac{1}{3}, \quad P(R|B) = \frac{1}{3}, \quad P(W|B) = \frac{2}{3}$

Bayes' Theorem: $\quad P(A|n\text{red}) = \frac{P(n\text{red}|A) \cdot P(A)}{P(n\text{red})}$

Where $P(A|n\text{red})$ is the probability of die A being used if the first n die throws come up

Where $P(n\text{red}|A)$ is the probability of getting n reds given the die A is used.

$P(n\text{red})$ is the overall probability of getting n red outcomes.

$P(n\text{red}) = P(n\text{red}|A) \cdot P(A) + P(n\text{red}|B) \cdot P(B)$

$$P(n\text{red}|A) = \frac{2}{3}^n, \quad P(n\text{red}|B) = \frac{1}{3}^n$$

$$P(n\text{red}) = \frac{2}{3}^n \cdot \frac{1}{2} + \frac{1}{3}^n \cdot \frac{1}{2} = \frac{1}{2}((\frac{2}{3})^n + (\frac{1}{3})^n)$$

$$\therefore P(A|n\text{red}) = \frac{P(n\text{red}|A) \cdot P(A)}{P(n\text{red})} = \frac{(\frac{2}{3})^n \cdot \frac{1}{2}}{\frac{1}{2}((\frac{2}{3})^n + (\frac{1}{3})^n} = \frac{(\frac{2}{3})^n}{(\frac{2}{3})^n + (\frac{1}{3})^n}$$

$$P(A|n\text{red}) = \frac{(\frac{2}{3})^n}{(\frac{2}{3})^n + (\frac{1}{3})^n} = \frac{2^n}{3^n} \cdot \frac{3^n}{2^n + 1}$$

$$P(A|n\text{red}) = \frac{2^n}{2^n + 1}$$

Part 3: Geometry

Problem 3.1

In [15]:
```
# retrieve is defined in a hidden cell
file_name = 'lines.pkl'
retrieve(file_name)
with open(file_name, 'rb') as file:
    lines = pickle.load(file)
```

```
Using previously downloaded file lines.pkl
```

In [16]:
```
lines
```

Out[16]:
```
[namespace(A=array([[1., 0., 0.],
                    [0., 1., 0.]]),
           b=array([0., 0.])),
 namespace(A=array([[ 0.,  0., -2.],
                    [ 0.,  2.,  2.]]),
           b=array([4., 6.])),
 namespace(A=array([[2., 1., 3.],
                    [0., 5., 1.]]),
           b=array([14.,  4.]))]
```

In [33]:
```
from scipy.linalg import null_space

def parametric(A, b):
    p = np.linalg.lstsq(A, b, rcond=None)[0]
    n = null_space(A).flatten()
    n = n / np.linalg.norm(n)
    alpha_values = [0, 2]
    residuals = [np.linalg.norm(A @ (p + alpha * n) - b) for alpha in alpha_values]
    p_rounded = [float(np.round(val, 3)) for val in p]
    n_rounded = [float(np.round(val, 3)) for val in n]
    residuals_rounded = [float(np.round(res, 3)) for res in residuals]
    print(f"p = {p_rounded}, n = {n_rounded}, r = {residuals_rounded}")
    return p, n
```

```
for ns in lines:
    parametric(ns.A, ns.b)
```

```
p = [0.0, 0.0, 0.0], n = [0.0, 0.0, 1.0], r = [0.0, 0.0]
p = [0.0, 5.0, -2.0], n = [1.0, 0.0, 0.0], r = [0.0, 0.0]
p = [2.213, 0.173, 3.133], n = [-0.808, -0.115, 0.577], r = [0.0, 0.0]
```

Part 4: Calculus

Problem 4.1

$f_x = 3(x-1) - 3y^2$

Find where $f_x = 0$:

$f_x = 3(x-1)^2 - 3y^2 = 0$

$(x-1)^2 - y^2 = 0$

$x - 1 = \pm y$

$x - 1 = 0 \quad \Rightarrow \quad x = 1$

$f_y = -6xy + 6y$

Now, find where $f_y = 0$:

$f_y = -6xy + 6y = 0$

$y(1-x) = 0$

$y = 0$

Stationary point at $x = 1, y = 0$.

Find the second partial derivatives:

$f_{xx} = 6(x-1)$

$f_{yy} = -6x + 6 = 6(1-x)$

$f_{xy} = -6y$

Evaluate each at (1,0):

$f_{xx} = 6(2-1) = 0$

$f_{yy} = 6(1-1) = 0$

$f_{xy} = -6(0) = 0$

Second Derivative Test:

$f_{xx}(x,y)f_{yy}(x,y) - (f_{xy}(x,y))^2$

= $(0)(0) - (0)^2$

= 0

The second derivative test is inconclusive. Analysis of the actual structure of f(x,y) near (1,0) shows that the function behaves like a saddle point because of the cubic term of x combined with the quadratic term of y. The quadratic term of y would be at its minimum (the bottom of the parabolic function) and the cubic term of x would be a local minimum from one directions and a local maximium from another direction. Therefore the stationary point at (1,0) is a saddle point.