CSCI933 Machine Learning: Assignment #1

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Problem 1

$\because S^t = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} = S^{-1}$, $\therefore S$ is an orthogonal matrix.

As $(SPS^t)^t = SP^tS^t$, $PP^t = SPS^tSP^tS^t = SPP^tS^t = SP(SP)^t$ is a diagonal matrix.

$SPS^t = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} =$

Problem 2

(a) $SS^t = \begin{bmatrix} cos\alpha & sin\alpha \\ -sin\alpha & cos\alpha \end{bmatrix} \begin{bmatrix} cos\alpha & -sin\alpha \\ sin\alpha & cos\alpha \end{bmatrix} = I_2$, S is orthogonal.

(b) $SA = \begin{bmatrix} a_{11}cos\alpha + a_{21}sin\alpha & a_{12}cos\alpha + a_{22}sin\alpha \\ -a_{11}sin\alpha + a_{21}cos\alpha & -a_{12}sin\alpha + a_{22}cos\alpha \end{bmatrix}$,

$B = SAS^t = \begin{bmatrix} a_{11}cos^2\alpha + 2a_{12}sin\alpha cos\alpha + a_{22}sin^2\alpha & (a_{22}-a_{11})sin\alpha cos\alpha + a_{12}(cos^2\alpha - sin^2\alpha) \\ (a_{22}-a_{11})sin\alpha cos\alpha + a_{12}(cos^2\alpha - sin^2\alpha) & a_{11}sin^2\alpha - 2a_{12}sin\alpha cos\alpha + a_{22}cos^2\alpha \end{bmatrix}$

$tan2\alpha = \frac{2tan\alpha}{1-tan^2\alpha} = \frac{2sin\alpha cos\alpha}{cos^2\alpha - sin^2\alpha} = \frac{2a_{12}}{a_{11}-a_{22}}$,

Set $sin\alpha cos\alpha = pa_{12}, cos^2\alpha - sin^2\alpha = p(a_{11}-a_{22})$

We get $(a_{22}-a_{11})sin\alpha cos\alpha + a_{12}(cos^2\alpha - sin^2\alpha) = pa_{12}(a_{22}-a_{11}) + pa_{12}(a_{11}-a_{22}) = 0$

Therefore, $B = SAS^t$ is diagonal.

(c) $Tr[B] = a_{11}cos^2\alpha + 2a_{12}sin\alpha cos\alpha + a_{22}sin^2\alpha + a_{11}sin^2\alpha - 2a_{12}sin\alpha cos\alpha + a_{22}cos^2\alpha = a_{11} + a_{22} = Tr[A]$

Problem 3

Set A="coin is fair", B="coin is two-headed", C="heads shows both times" If the coin is fair, the probability of two heads is $P(C|A) = \frac{P(C,A)}{P(A)} = \frac{1}{4}$. If the coin is two-headed, the probability is $P(C|B) = \frac{P(C,B)}{P(B)} = 1$. Because $P(A) = P(B) = \frac{1}{2}$, therefore $P(C,B) = \frac{1}{2}, P(C,A) = \frac{1}{8}$, The conditional probability of $P(A|C) = \frac{P(A,C)}{P(C)} = \frac{P(A,C)}{P(B,C)+P(A,C)} = \frac{1}{5}$

Problem 4

(a) $P(d) = P(d|A)P(A) + P(d|B)P(B) = (\frac{C_{100}^2}{C_{1000}^2} + \frac{C_1 00^2}{C_{2000}^2})/\frac{1}{2} \approx 2.5\%$

(b) $P(A|d) = \frac{P(A,d)}{P(d)} = \frac{P(A)P(d|A)}{P(d)} = \frac{1}{5}$