ALGEBRAIC GEOMETRY

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1. Algebraic Sets

Let k be a field. The *affine n-space*, denoted $\mathbb{A}^n$, is defined to be the set of all n-tuples with components in k, i.e., $\mathbb{A}^n = k^n$. The reason for using this alternate notation is to ignore the vector space and ring structure on k^n and to treat $\mathbb{A}^n$ simply as a space of points. For a subset $S \subset k[x_1, \ldots, x_n]$ of polynomials we call

$$V(S) = \{a \in \mathbb{A}^n : f(a) = 0 \text{ for all } f \in S\}$$

the *zero locus* or *vanishing set* of S. Subsets of $\mathbb{A}^n$ of this form are called (*affine*) *algebraic sets*. If $S = \{f_1, \ldots, f_r\}$, then we denote $V(S)$ simply as $V(f_1, \ldots, f_r)$.

Example 1.1.

(a) $\mathbb{A}^n = V(\emptyset) = V(0)$ and $\emptyset = V(1)$ are algebraic sets.

(b) A point $a = (a_1, \ldots, a_n) \in \mathbb{A}^n$ is an algebraic set as $\{a\} = V(x_1 - a_1, \ldots, x_n - a_n)$.
Note that every finite subset of $\mathbb{A}^1$ is an algebraic set for if $a_1, \ldots, a_r \in \mathbb{A}^1$, then $V((x_1 - a_1) \cdots (x_1 - a_r)) = \{a_1, \ldots, a_r\}$. Since every nonzero polynomial in $k[x_1]$ has only finitely many roots, it follows that the algebraic subsets of $\mathbb{A}^1$ are precisely $\mathbb{A}^1$ and the finite subsets of $\mathbb{A}^1$. For instance, $\mathbb{Z}$ is not an algebraic subset of $\mathbb{C}$.

(c) Linear subspaces of $\mathbb{A}^n$ are algebraic sets. Let X be a subspace of $\mathbb{A}^n$ and let $\{v_1, \ldots, v_m\}$ be a basis of X. This basis can be extended to a basis $\{v_1, \ldots, v_n\}$ of $\mathbb{A}^n$. If $L : \mathbb{A}^n \to \mathbb{A}^n$ is the linear map defined as

$$L(a_1 v_1 + \cdots + a_n v_n) = a_{m+1} v_{m+1} + \cdots + a_n v_n,$$

then $\ker L = X$. Let $(a_{ij})_{1 \leqslant i,j \leqslant n}$ be the matrix corresponding to L. If $f_i(x) = a_{i1} x_1 + \cdots + a_{in} x_n$, then we have $X = V(f_1, \ldots, f_n)$ and so X is an algebraic set.

(d) If $X \subset \mathbb{A}^n$ and $Y \subset \mathbb{A}^m$ are algebraic sets, then so is the product $X \times Y \subset \mathbb{A}^{n+m}$.

Lemma 1.2 (properties of $V(\cdot)$).

(a) For any $S_1 \subset S_2 \subset k[x_1, \ldots, x_n]$ we have $V(S_1) \supset V(S_2)$.

(b) For any $S_1, S_2 \subset k[x_1, \ldots, x_n]$ we have $V(S_1) \cup V(S_2) = V(S_1 S_2)$.

(c) If $\{S_i\}_i$ is a collection of subsets of $k[x_1, \ldots, x_n]$, then $\bigcap_i V(S_i) = V(\bigcup_i S_i)$.

The above lemma shows that the operator $V(\cdot)$ is inclusion-reversing and that algebraic sets are closed under finite unions and arbitrary intersections. The arbitrary union of algebraic sets however need not be an algebraic set for this would imply that every subset of $\mathbb{A}^n$

is an algebraic set as a point is an algebraic set. In fact, even countable union of algebraic sets need not be an algebraic set. For instance, $\mathbb{Z}$ is not an algebraic subset of $\mathbb{C}$.

REMARK 1.3. If S is a subset of polynomials in $k[x_1, \dots, x_n]$, then it is easy to see that $V(S) = V(\langle S \rangle)$, i.e., every algebraic set is a zero locus of some ideal in $k[x_1, \dots, x_n]$. But we know by Hilbert's Basis Theorem that every ideal in $k[x_1, \dots, x_n]$ is finitely generated. Thus every algebraic set can be written as a zero locus of only finitely many polynomials.

LEMMA 1.4 (properties of $V(\cdot)$ - ideal version). *For any ideals $\mathfrak{a}, \mathfrak{b}$ in $k[x_1, \dots, x_n]$ we have*

(a) $V(\sqrt{\mathfrak{a}}) = V(\mathfrak{a})$;
(b) $V(\mathfrak{a}) \cup V(\mathfrak{b}) = V(\mathfrak{a}\mathfrak{b}) = V(\mathfrak{a} \cap \mathfrak{b})$;
(c) $V(\mathfrak{a}) \cap V(\mathfrak{b}) = V(\mathfrak{a} + \mathfrak{b})$.

Let $X \subset \mathbb{A}^n$. The *ideal* of X is defined to be

$$I(X) = \{f \in k[x_1, \dots, x_n] : f(a) = 0 \text{ for all } a \in X\}.$$

It is easily verified that $I(X)$ is indeed an ideal in $k[x_1, \dots, x_n]$. In fact, $I(X)$ is a radical ideal. Moreover, $I(\cdot)$ is also an inclusion-reversing operator.

EXAMPLE 1.5. If $a = (a_1, \dots, a_n) \in \mathbb{A}^n$, then $I(a) = (x_1 - a_1, \dots, x_n - a_n)$.

PROPOSITION 1.6.

(a) *For any ideal $S \subset k[x_1, \dots, x_n]$ we have $I(V(S)) \supset S$.*
(b) *For any algebraic set $X \subset \mathbb{A}^n$ we have $V(I(X)) = X$.*

PROOF. The part (a) and the inclusion $V(I(X)) \supset X$ in part (b) follows immediately from definition. For the reverse inclusion let $X = V(\mathfrak{a})$, where $\mathfrak{a}$ is an ideal in $k[x_1, \dots, x_n]$. Since $I(X) = I(V(\mathfrak{a})) \supset \mathfrak{a}$ by part (a), it follows that $V(I(X)) \subset V(\mathfrak{a}) = X$. $\square$

THEOREM 1.7. *Let k be a field. Then the following statements are equivalent:*

(a) k is algebraically closed.
(b) For any ideal $\mathfrak{a}$ in $k[x_1, \dots, x_n]$, $I(V(\mathfrak{a})) = \sqrt{\mathfrak{a}}$.
(c) For every proper ideal $\mathfrak{a}$ of $k[x_1, \dots, x_n]$, $V(\mathfrak{a}) \neq \emptyset$.
(d) The maximal ideals of $k[x_1, \dots, x_n]$ are precisely of the form $(x_1 - a_1, \dots, x_n - a_n)$, where $a_i \in k$.

PROOF. (a) $\Rightarrow$ (b): This is the strong form of Hilbert Nullstellensatz.

(b) $\Rightarrow$ (c): Let $\mathfrak{a}$ be a proper ideal of $k[x_1 \dots, x_n]$. If $V(\mathfrak{a}) = \emptyset$, then $I(V(\mathfrak{a})) = I(\emptyset) = (1)$. But this is contradicts part (a) as $\sqrt{\mathfrak{a}}$ is a proper ideal as well.

(c) $\Rightarrow$ (d): Note that for any $a = (a_1, \dots, a_n) \in \mathbb{A}^n$ the ideal $\mathfrak{m}_a = (x_1 - a_1, \dots, x_n - a_n)$ is a maximal ideal since it is the kernel of the evaluation homomorphism $e_a : k[x_1, \dots, x_n] \to k$ defined as $e_a(f) = f(a)$.

To see why every maximal ideal of $k[x_1, \dots, x_n]$ is of the form $\mathfrak{m}_a$ for some $a \in \mathbb{A}^n$ let $\mathfrak{m}$ be a maximal ideal of $k[x_1, \dots, x_n]$. By part (c) we have $V(\mathfrak{m}) \neq \emptyset$. Let $a \in V(\mathfrak{m})$. Then we have $\{a\} \subset V(\mathfrak{m})$ and so $\mathfrak{m} \subset I(V(\mathfrak{m})) \subset I(a) = \mathfrak{m}_a$. Because $\mathfrak{m}$ is maximal we deduce that $\mathfrak{m} = \mathfrak{m}_a$.

(d) $\Rightarrow$ (a): Let $f \in k[x_1]$ be a nonconstant polynomial. Then f is contained in some maximal ideal $\mathfrak{m}_a$ of $k[x_1, \dots, x_n]$, where $a = (a_1, \dots, a_n) \in \mathbb{A}^n$. Now note that $V(f) \supset V(\mathfrak{m}_a) = \{a\}$ and so $f(a_1) = 0$. Hence, k is algebraically closed. $\square$

EXAMPLE 1.8. Let k be an algebraically closed field. If f is a nonconstant polynomial in $k[x]$, then f splits completely into product of linear factors, i.e., we can write

$$f = c(x - a_1)^{d_1} \cdots (x - a_r)^{d_r}.$$

If $\mathfrak{a} = (f)$, then $V(\mathfrak{a}) = \{a_1, \ldots, a_r\}$ and $I(V(\mathfrak{a})) = ((x - a_1) \cdots (x - a_r))$. It is also easy to see that $\sqrt{\mathfrak{a}} = ((x-a_1) \cdots (x-a_r))$ and so our theorem above checks out, i.e., $I(V(\mathfrak{a})) = \sqrt{\mathfrak{a}}$.

REMARK 1.9. Although we have really shown in the theorem above that the condition of k being algebraically closed is necessary for the correspondence between algebraic sets and radical ideals to work but we illustrate it further with an example. Note that $\mathfrak{m} = (x^2 + 1)$ is a maximal ideal in $\mathbb{R}[x]$ since $x^2 + 1$ is an irreducible polynomial but $V(\mathfrak{m}) = \emptyset$ or that $I(V(\mathfrak{m})) = I(\emptyset) = (1) \neq \mathfrak{m} = \sqrt{\mathfrak{m}}$.

From now onwards we will assume that the underlying ground field is algebraically closed unless otherwise stated.

LEMMA 1.10 (properties of $I(\cdot)$).

(a) If $\{X_i\}_i$ is a collection of subsets of $\mathbb{A}^n$, then $I(\bigcup_i X_i) = \bigcap_i I(X_i)$.

(b) If X_1 and X_2 are algebraic sets, then $I(X_1 \cap X_2) = \sqrt{I(X_1) + I(X_2)}$.

PROOF. The part (a) follows immediately from definition. For part (b) note that

$$I(X_1 \cap X_2) = I(V(I(X_1)) \cap V(I(X_2))) = I(V(I(X_1) + I(X_2))) = \sqrt{I(X_1) + I(X_2)}. \quad \square$$

The part (b) of above lemma does not hold if X_1 and X_2 are not algebraic sets. For instance, if X_1 and X_2 are sets of positive and negative integers respectively, then $I(X_1) = 0 = I(X_2)$ and so $\sqrt{I(X_1) + I(X_2)} = 0$, whereas $I(X_1 \cap X_2) = I(\emptyset) = (1)$.

EXAMPLE 1.11. Let $X = V(y - x^2)$ and $Y = V(y)$ be algebraic subsets of $\mathbb{A}^2$. Then note that $I(X) = I(V(y - x^2)) = (y - x^2)$ and $I(Y) = I(V(y)) = (y)$ as $(y - x^2)$ and (y) are both prime ideals and thus radical ideals in $k[x, y]$. Thus we have

$$I(X) + I(Y) = (y - x^2, y) = (x^2, y).$$

Taking the radical we get $\sqrt{I(X) + I(Y)} = \sqrt{(x^2, y)} = (x, y)$. Also note that $I(X \cap Y) = I((0, 0)) = (x, y)$. Hence we indeed have the identity $I(X \cap Y) = \sqrt{I(X) + I(Y)}$.