ESTIMATES OF ARITHMETIC FUNCTIONS

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1. Partial summation formula and its applications

The Abel's summation by parts formula is an extremely important and ubiquitous tool in analytic number theory which is frequently employed to estimate the partial sums of an arithmetic function $a : \mathbb{N} \to \mathbb{C}$ weighted by some smooth function f.

Theorem 1.1. *Let $a : \mathbb{N} \to \mathbb{C}$ be an arithmetic function, let $0 < x < y$ be real numbers and $f : [x, y] \to \mathbb{C}$ be a continuously differentiable function. Then we have*

$$\sum_{x<n\leqslant y} a(n)f(n) = A(y)f(y) - A(x)f(x) - \int_x^y A(t)f'(t)\, dt,$$

where $A(t) = \sum_{n\leqslant t} a(n)$.

Proof. Let $m = \lfloor x \rfloor$ and $M = \lfloor y \rfloor$. We can rewrite the weighted sum as

$$\sum_{x<n\leqslant y} a(n)f(n) = \sum_{n=m+1}^{M} a(n)f(n).$$

By definition $a(n) = A(n) - A(n-1)$ so we can replace $a(n)$ to get

$$\begin{aligned}\sum_{n=m+1}^{M} a(n)f(n) &= \sum_{n=m+1}^{M} (A(n) - A(n-1))f(n) \\ &= \sum_{n=m+1}^{M} A(n)f(n) - \sum_{n=m}^{M-1} A(n)f(n+1) \\ &= A(M)f(M) - A(m)f(m+1) - \sum_{n=m+1}^{M-1} A(n)(f(n+1) - f(n)) \qquad (1.1)\end{aligned}$$

Since $f(n+1)-f(n)=\int_n^{n+1} f'(t)\,dt$ and $A(t)=A(n)$ for all $t\in[n,n+1)$, we get

$$\begin{aligned}\sum_{m+1}^{M-1} A(n)(f(n+1)-f(n)) &= \sum_{m+1}^{M-1} A(n)\int_n^{n+1} f'(t)\,dt\\ &= \sum_{m+1}^{M-1}\int_n^{n+1} A(t)f'(t)\,dt\\ &= \int_{m+1}^{M} A(t)f'(t)\,dt. \qquad (1.2)\end{aligned}$$

Substituting (1.1) into (1.2), we get

$$\sum_{n=m+1}^{M} a(n)f(n) = A(M)f(M)-A(m)f(m+1)-\int_{m+1}^{M} A(t)f'(t)\,dt. \qquad (1.3)$$

Because we are missing the terms $\int_x^{m+1} A(t)f'(t)\,dt$ and $\int_M^y A(t)f'(t)\,dt$ parts, we try to evaluate what they look like. Using Fundamental Theorem of Calculus and the fact that $A(t)=A(x)$ for $t\in[x,m+1)$, we get

$$\begin{aligned}\int_x^{m+1} A(t)f'(t)\,dt &= A(x)f(m+1)-A(x)f(x)\\ &= A(m)f(m+1)-A(x)f(x). \qquad (1.4)\end{aligned}$$

Doing a similar calculation for $\int_M^y A(t)f'(t)\,dt$ yields

$$\int_M^y A(t)f'(t)\,dt = A(y)f(y)-A(M)f(M). \qquad (1.5)$$

Using (1.4) and (1.5), one can easily turn (1.3) into the required form. $\square$

Corollary 1.2. *Let $a:\mathbb{N}\to\mathbb{C}$ be an arithmetic function and let $f:[1,x]\to\mathbb{C}$ be a continuously differentiable function where $x\geqslant 1$. Then we have*

$$\sum_{n\leqslant x} a(n)f(n) = A(x)f(x)-\int_1^x A(t)f'(t)\,dt.$$

Proposition 1.3. *If $x\geqslant 1$, then we have*

$$\sum_{n\leqslant x}\frac{1}{n} = \log x+\gamma+O(1/x), \qquad (1.6)$$

where γ is the Euler-Mascheroni constant.

Proof. Taking $a(n)=1$ and $f(x)=1/x$ in the summation by parts formula, we get

$$\sum_{n\leqslant x}\frac{1}{n} = \frac{\lfloor x\rfloor}{x}+\int_1^x \frac{\lfloor t\rfloor}{t^2}\,dt. \qquad (1.7)$$

Substituting $\lfloor x \rfloor = x - \{x\}$ in (1.7), we get

$$\begin{aligned}\sum_{n \leqslant x} \frac{1}{n} &= 1 - \frac{\{x\}}{x} + \int_1^x \frac{1}{t}\,dt - \int_1^x \frac{\{t\}}{t^2}\,dt \\ &= 1 + O(1/x) + \log x - \int_1^x \frac{\{t\}}{t^2}\,dt \\ &= 1 + O(1/x) + \log x - \int_1^\infty \frac{\{t\}}{t^2}\,dt + \int_x^\infty \frac{\{t\}}{t^2}\,dt.\end{aligned}$$

Letting $C = 1 - \int_1^\infty \{t\}/t^2\,dt$, we obtain

$$\sum_{n \leqslant x} \frac{1}{n} = \log x + C + O(1/x) + \int_x^\infty \frac{\{t\}}{t^2}\,dt.$$

We can bound the improper integral as

$$\int_x^\infty \frac{\{t\}}{t^2}\,dt \leqslant \int_x^\infty \frac{1}{t^2}\,dt = \frac{1}{x}$$

and so

$$\int_x^\infty \frac{\{t\}}{t^2}\,dt = O(1/x).$$

It thus follows that

$$\sum_{n \leqslant x} \frac{1}{n} = \log x + C + O(1/x).$$

It can be easily seen by taking limit as x approaches ∞ that $C = \gamma$. □

Note that in the above proof we obtained an expression for γ in terms of integral;

$$\gamma = 1 - \int_1^\infty \frac{\{t\}}{t^2}\,dt.$$

The *Riemann zeta function*, denoted ζ, is defined as the convergent series

$$\zeta(s) = \sum_{n=1}^\infty \frac{1}{n^s}$$

for $s > 1$ and as

$$\zeta(s) = \lim_{x \to \infty} \left(\sum_{n \leqslant x} \frac{1}{n^s} - \frac{x^{1-s}}{1-s} \right) \tag{1.8}$$

for $0 < s < 1$. In order for the above definition to make sense we need to prove that the limit in (1.8) indeed exists. This will be done in the course of proving the following result.

PROPOSITION 1.4. *If $x \geqslant 1$, then for $s > 0$ and $s \neq 1$ we have*

$$\sum_{n \leqslant x} \frac{1}{n^s} = \frac{x^{1-s}}{1-s} + \zeta(s) + O(x^{-s}). \tag{1.9}$$

PROOF. We apply the Abel's summation by parts formula with $a(n) = 1$ and $f(x) = x^{-s}$. For $x \geqslant 1$ we then get

$$\sum_{n \leqslant x} \frac{1}{n^s} = \frac{\lfloor x \rfloor}{x^s} + s \int_1^x \frac{\lfloor t \rfloor}{t^{s+1}}\,dt$$

Substituting $\lfloor x \rfloor = x - \{x\}$, we obtain

$$\begin{aligned}\sum_{n \leqslant x} \frac{1}{n^s} &= x^{1-s} - \frac{\{x\}}{x^s} + s\int_1^x \frac{1}{t^s}\,dt - s\int_1^x \frac{\{t\}}{t^{s+1}}\,dt \\ &= x^{1-s} + s\left(\frac{x^{1-s}}{1-s} - \frac{1}{1-s}\right) - s\int_1^x \frac{\{t\}}{t^{s+1}}\,dt + O(x^{-s}) \\ &= \frac{x^{1-s}}{1-s} + \frac{s}{s-1} - s\int_1^\infty \frac{\{t\}}{t^{s+1}}\,dt + s\int_x^\infty \frac{\{t\}}{t^{s+1}}\,dt + O(x^{-s}). \end{aligned} \tag{1.10}$$

By the comparison test for improper integrals, we have

$$\int_x^\infty \frac{\{t\}}{t^{s+1}}\,dt \leqslant \int_x^\infty \frac{1}{t^{s+1}}\,dt = \frac{x^{-s}}{s}.$$

Thus, the equation (1.10) turns into

$$\sum_{n \leqslant x} \frac{1}{n^s} = \frac{x^{1-s}}{1-s} + \frac{s}{s-1} - s\int_1^\infty \frac{\{t\}}{t^{s+1}}\,dt + O(x^{-s}). \tag{1.11}$$

Taking the limit as x approaches ∞, we get

$$\begin{aligned}\lim_{x\to\infty}\left(\sum_{n \leqslant x} \frac{1}{n^s} - \frac{x^{1-s}}{1-s}\right) &= \lim_{x\to\infty}\left(\frac{s}{s-1} - s\int_1^\infty \frac{\{t\}}{t^{s+1}}\,dt + O(x^{-s})\right) \\ &= \frac{s}{s-1} - s\int_1^\infty \frac{\{t\}}{t^{s+1}}\,dt\end{aligned}$$

Note that if $s > 1$ then

$$\lim_{x\to\infty}\left(\sum_{n \leqslant x} \frac{1}{n^s} - \frac{x^{1-s}}{1-s}\right) = \lim_{x\to\infty}\sum_{n \leqslant x} \frac{1}{n^s} = \sum_{n=1}^\infty \frac{1}{n^s} = \zeta(s).$$

Consequently, for every $s > 0$ with $s \neq 1$ we have

$$\zeta(s) = \frac{s}{s-1} - s\int_1^\infty \frac{\{t\}}{t^{s+1}}\,dt. \tag{1.12}$$

Finally, using this identity along with (1.11) we get the desired result. □

Comparing the estimates (1.7) and (1.9) it seems plausible that

$$\lim_{s\to 1}\left(\zeta(s) + \frac{x^{1-s}}{1-s}\right) = \log x + \gamma$$

This indeed holds (see Exercise 1).

Corollary 1.5. *If $x \geqslant 1$ and $s > 1$, we have*

$$\sum_{n > x} \frac{1}{n^s} = O(x^{1-s}).$$

Proposition 1.6. *If $x \geqslant 1$, then for any $\alpha \geqslant 0$ we have*

$$\sum_{n \leqslant x} n^\alpha = \frac{x^{1+\alpha}}{1+\alpha} + O(x^\alpha). \tag{1.13}$$

PROOF. We assume that $\alpha > 0$. Applying the Abel summation by parts formula with $a(n) = 1$ and $f(x) = x^\alpha$, we get

$$\sum_{n \leqslant x} n^\alpha = \lfloor x \rfloor x^\alpha - \alpha \int_1^x \lfloor t \rfloor t^{\alpha-1}\, dt.$$

Substituting $\lfloor x \rfloor = x - \{x\}$, we obtain

$$\sum_{n \leqslant x} n^\alpha = x^{\alpha+1} - \{x\}x^\alpha - \alpha \int_1^x t^\alpha\, dt + \alpha \int_1^x \{t\} t^{\alpha-1}\, dt$$

$$= x^{1+\alpha} + O(x^\alpha) - \alpha \left(\frac{x^{1+\alpha}}{1+\alpha} - \frac{1}{1+\alpha} \right) + \alpha \int_1^x \{t\} t^{\alpha-1}\, dt. \tag{1.14}$$

Note that

$$\alpha \int_1^x \{t\} t^{\alpha-1}\, dt \leqslant \alpha \int_1^x t^{\alpha-1}\, dt = x^\alpha - 1 = O(x^\alpha).$$

Hence (1.14) simplifies to

$$\sum_{n \leqslant x} n^\alpha = \frac{x^{1+\alpha}}{1+\alpha} + \frac{\alpha}{1+\alpha} + O(x^\alpha)$$

Observe that $O(x^\alpha)$ absorbs the constant $\alpha/(1+\alpha)$ and thus we get the desired result.

If $\alpha = 0$ then we get

$$\sum_{n \leqslant x} n^\alpha = \sum_{n \leqslant x} 1 = \lfloor x \rfloor = x - \{x\} = x + O(1).$$

But this agrees with the asymptotic formula (1.13). □

2. Estimate of partial sums of the divisor function

We now turn our attention to the divisor function $d = 1 * 1$. We will first show that $d(n)$ behaves like $\log n$ on average. Then we will improve the error term for the partial sum of $d(n)$ using Dirichlet hyperbola method.

Since $d(n) = \sum_{d|n} 1$, we have

$$\sum_{n \leqslant x} d(n) = \sum_{n \leqslant x} \sum_{d|n} 1 = \sum_{n \leqslant x} \sum_{qd=n} 1 = \sum_{qd \leqslant x} 1.$$

Thus the divisor sum can now be written as

$$\sum_{n \leqslant x} d(n) = \sum_{d \leqslant x} \sum_{q \leqslant x/d} 1 = \sum_{d \leqslant x} \left(\frac{x}{d} + O(1) \right) = x \sum_{d \leqslant x} \frac{1}{d} + O(x).$$

Now we use the asymptotic formula for the Harmonic sum and obtain

$$\begin{aligned} \sum_{n \leqslant x} d(n) &= x \left(\log x + \gamma + O\left(1/x\right) \right) + O(x) \\ &= x \log x + \gamma x + O(1) + O(x) \\ &= x \log x + O(x) \end{aligned}$$

Note that the term γx does not give us any information as error is of the order x thus it gets engulfed by the big error term $O(x)$. Hence we get

$$\sum_{n \leqslant x} d(n) = x \log x + O(x).$$

Thus the average order of $d(n)$ is $\log n$ since

$$\sum_{n\leqslant x} d(n) \sim x\log x \quad \text{as } x\to\infty.$$

Dirichlet obtained sharper estimate for $\sum_{d\leqslant x} d(n)$ with the error term being $O(\sqrt{x})$. We prove the Dirichlet's result below.

THEOREM 2.1. *For all $x\geqslant 1$, we have*

$$\sum_{n\leqslant x} d(n) = x\log x + (2\gamma-1)x + O(\sqrt{x}), \tag{2.1}$$

where γ is the Euler-Mascheroni constant.

PROOF. The trick to prove this stronger estimate is to exploit the symmetry of q and d in the sum

$$\sum_{n\leqslant x} d(n) = \sum_{qd\leqslant x} 1.$$

We can split this sum into sums as

$$\begin{aligned}
\sum_{n\leqslant x} d(n) = \sum_{qd\leqslant x} 1 &= \sum_{\substack{qd\leqslant x\\ d\leqslant q}} 1 + \sum_{\substack{qd\leqslant x\\ q\leqslant d}} 1 - \sum_{\substack{qd\leqslant x\\ d=q}} 1\\
&= 2\sum_{\substack{qd\leqslant x\\ d\leqslant q}} 1 - \sum_{d\leqslant\sqrt{x}} 1\\
&= 2\Bigg(\sum_{d\leqslant\sqrt{x}}\ \sum_{d\leqslant q\leqslant x/d} 1\Bigg) - \lfloor\sqrt{x}\rfloor\\
&= 2\sum_{d\leqslant\sqrt{x}}\left(\left\lfloor\frac{x}{d}\right\rfloor - d + 1\right) - \lfloor\sqrt{x}\rfloor\\
&= 2\sum_{d\leqslant\sqrt{x}}\left(\left\lfloor\frac{x}{d}\right\rfloor - d\right) + \lfloor\sqrt{x}\rfloor\\
&= 2\sum_{d\leqslant\sqrt{x}}\left(\frac{x}{d} - d + O(1)\right) + O(\sqrt{x})\\
&= 2x\sum_{d\leqslant\sqrt{x}}\frac{1}{d} - 2\sum_{d\leqslant\sqrt{x}} d + O(\sqrt{x})
\end{aligned}$$

Finally using estimates (1.3) and (1.6) we obtain

$$\begin{aligned}
\sum_{n\leqslant x} d(n) &= 2x\Bigg(\log\sqrt{x} + \gamma + O\left(\frac{1}{\sqrt{x}}\right)\Bigg) - 2\Bigg(\frac{x}{2} + O(\sqrt{x})\Bigg) + O(\sqrt{x})\\
&= x\log x + (2\gamma-1)x + O(\sqrt{x}),
\end{aligned}$$

which is the desired result. $\square$

Exercises

EXERCISE 1. Show that

$$\lim_{s\to 1}\left(\zeta(s)+\frac{x^{1-s}}{1-s}\right)=\log x+\gamma.$$

for every $x>0$.

EXERCISE 2. Let $Q(x)$ denote the number of squarefree integers not exceeding x. Show that

$$Q(x)=\frac{6}{\pi^2}x+O(\sqrt{x}).$$

(Hint: n is squarefree if and only if $\mu(n)^2=\sum_{d^2\,|\,n}\mu(d)=1$.)

Solutions

SOLUTION 1. Note that for $s>0$ with $s\neq 1$ we have

$$\begin{aligned}\zeta(s)+\frac{x^{1-s}}{1-s}&=\frac{x^{1-s}}{1-s}+\frac{s}{s-1}-s\int_1^\infty\frac{\{t\}}{t^{s+1}}\,dt\\&=-\left(\frac{x^{1-s}-1}{s-1}\right)+1-s\int_1^\infty\frac{\{t\}}{t^{s+1}}\,dt.\end{aligned}$$

Since the derivative of x^{1-s} at $s=1$ is $-\log x$ and

$$\lim_{s\to 1}\int_1^\infty\frac{\{t\}}{t^{s+1}}\,dt=\int_1^\infty\frac{\{t\}}{t^2}\,dt=1-\gamma$$

we obtain the desired result.

SOLUTION 2. Using the identity $\mu(n)^2=\sum_{d^2\,|\,n}\mu(d)$ we can rewrite $Q(x)$ as

$$\begin{aligned}Q(x)&=\sum_{n\leqslant x}\mu^2(n)=\sum_{n\leqslant x}\sum_{d^2\,|\,n}\mu(d)=\sum_{d\leqslant\sqrt{x}}\sum_{\substack{n\leqslant x\\ d^2\,|\,n}}\mu(d)\\&=\sum_{d\leqslant\sqrt{x}}\mu(d)\sum_{\substack{n\leqslant x\\ d^2\,|\,n}}1=\sum_{d\leqslant\sqrt{x}}\mu(d)\left\lfloor\frac{x}{d^2}\right\rfloor\\&=x\sum_{d\leqslant\sqrt{x}}\frac{\mu(d)}{d^2}+O\left(\sum_{d\leqslant\sqrt{x}}|\mu(d)|\right)=x\sum_{d\leqslant\sqrt{x}}\frac{\mu(d)}{d^2}+O(\sqrt{x})\end{aligned}\tag{2.2}$$

We now push d off to ∞ in the above sum and incur an error due to the tail which fortunately is only $O(\sqrt{x})$ as seen can be seen by

$$\sum_{d\leqslant\sqrt{x}}\frac{\mu(d)}{d^2}=\sum_{d=1}^{\infty}\frac{\mu(d)}{d^2}+O\left(\sum_{d>\sqrt{x}}\frac{1}{d^2}\right)=\frac{6}{\pi^2}+O\left(\frac{1}{\sqrt{x}}\right).$$

Using this estimate in (2.2), we obtain the desired result.