# DIRICHLET SERIES

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Contents



## 1. Convergence of Dirichlet series

Given an arithmetic function $f$ one can associate to it the *Dirichlet series*

$$L_f(s) = \sum_{n=1} \frac{f(n)}{n^s}.$$

It is a general theme in analytic number theory to study properties of an arithmetic function by means of its Dirichlet series. We will see later that one can express partial sums of an arithmetic function $f$ in terms of an integral involving $L_f(s)$ (Perron's formula). The analytic theory of Dirichlet series is very elegant and analogous to that of power series.

One of the simplest yet most important example of Dirichlet series is the Riemann zeta function, $\zeta(s)$, which is the Dirichlet series of the unit function; $\zeta(s) = L_1(s)$.

We first observe that if $s_0 = \sigma_0 + it_0$ and the Dirichlet series $L_f(s_0)$ converges absolutely, then $L_f(s)$ converges absolutely for all $s$ with $\sigma \geqslant \sigma_0$. This can be easily seen by the comparison test;

$$\sum_{n=1}^{\infty} \left| \frac{f(n)}{n^s} \right| = \sum_{n=1}^{\infty} \frac{|f(n)|}{n^\sigma} \leqslant \sum_{n=1}^{\infty} \frac{|f(n)|}{n^{\sigma_0}} = \sum_{n=1}^{\infty} \left| \frac{f(n)}{n^{s_0}} \right|.$$

This leads to the following simply proposition.

Proposition 1.1. *Let $f$ be an arithmetic function. Then there exists a number $\sigma_a(f) \in [-\infty, \infty]$ such that $L_f(s)$ converges absolutely in the half plane $\sigma > \sigma_a(f)$ and diverges absolutely*[1] *for $\sigma < \sigma_a(f)$.*

Proof. Let $\sigma_a(f)$ be the infimum of all $\sigma \in \mathbb{R}$ such that $L_f(\sigma)$ converges. The conclusion then follows simply by the comparison test. $\square$

The unique number $\sigma_a(f)$ in the above proposition is called the *abscissa of absolute convergence* of $f$. Observe that if $f$ is a bounded arithmetic function, then $\sigma_a(f) \leqslant 1$. The next lemma gives us information about the decay of the tail of Dirichlet series.

[1] By this we mean that it does not converge absolutely.

LEMMA 1.2. *If $f$ is an arithmetic function and $N \geqslant 1$ then for any fixed $c > \sigma_a(f)$ and $\sigma \geqslant c$ we have*

$$\left|\sum_{n=N}^{\infty} \frac{f(n)}{n^s}\right| \leqslant N^{-(\sigma-c)} \sum_{n=N}^{\infty} \frac{|f(n)|}{n^c}.$$

PROOF. The proof is very easy and follows by noting that

$$\left|\sum_{n=N}^{\infty} \frac{f(n)}{n^s}\right| \leqslant \sum_{n=N}^{\infty} \frac{|f(n)|}{n^\sigma} = \sum_{n=N}^{\infty} \frac{|f(n)|}{n^{\sigma-c}n^c} \leqslant N^{-(\sigma-c)} \sum_{n=N}^{\infty} \frac{|f(n)|}{n^c}. \qquad \square$$

$\square$

If we dispense with the constants, then we can write the above result concisely as $\sum_{n\geqslant N} f(n)n^{-s} \ll N^{-\sigma}$. An interesting consequence of the above lemma is the limit

$$\lim_{\sigma\to\infty} L_f(s) = f(1).$$

which holds as $L_f(s) - f(1) = \sum_{n=2}^{\infty} f(n)n^{-s} \ll 2^{-\sigma}$. This essentially shows that the value of an arithmetic function $f$ at 1 is uniquely determined by its Dirichlet series. Using induction one can in fact show that an arithmetic function is completely determined by its Dirichlet series.

The next result is very much similar in spirit to the analytic continuation and analogous to the result for power series.

THEOREM 1.3. *Let $f$ and $g$ be arithmetic functions such that Dirichlet series $L_f(s)$ and $L_g(s)$ both converge absolutely in the half plane $\sigma > \sigma_0$. Let $\{s_k\}_{k=1}^{\infty}$ be a sequence such that $\sigma_k = \mathrm{Re}(s_k) > \sigma_0$ for all $k$ and $\sigma_k \to \infty$ as $k \to \infty$. If $L_f(s_k) = Lg(s_k)$ for all $k$ then $f = g$.*

PROOF. Let $h(n) = f(n) - g(n)$ for all $n$. Then note that $L_h(s)$ converges absolutely in the half plane $\sigma > \sigma_0$ we have $L_h(s_k) = 0$ for all $k$. Our aim is to show that $h(n) = 0$ for all $n$. Suppose for the sake of contradiction that there exists an $n \in \mathbb{N}$ such that $h(n) \neq 0$. Then let $N$ be the smallest such integer. Now note that if $L_h(s) = 0$, then

$$h(N) = -N^s \sum_{n=N+1}^{\infty} \frac{h(n)}{n^s}.$$

Due to Lemma 1.2, it follows that

$$|h(N)| = \left|N^s \sum_{n=N+1}^{\infty} \frac{h(n)}{n^s}\right| \leqslant N^\sigma (N+1)^{-(\sigma-c)} \sum_{n=N+1}^{\infty} \frac{|h(n)|}{n^c} = A(c)\left(\frac{N}{N+1}\right)^\sigma,$$

where $c > \sigma_0$ is fixed and $A(c)$ is a constant depending on $c$ only. Substituting $\sigma = \sigma_k$ and taking the limit as $k \to \infty$ we get $h(N) = 0$, which is a contradiction. $\square$

It follows by the above theorem that for a Dirichlet series $L_f(s)$ that is not identically zero there exists a half plane $\sigma > c$ in which the series never vanishes.

THEOREM 1.4. *Let $f$ and $g$ be arithmetic functions such that Dirichlet series $L_f(s)$ and $L_g(s)$ both converge absolutely. Then we have $L_f(s)L_g(s) = L_(f * g, s)$.*

PROOF. Let $s$ be such that $L_f(s)$ and $L_g(s)$ both converge absolutely. Then we have

$$L_f(s)L_g(s) = \left(\sum_{n=1}^{\infty} \frac{f(n)}{n^s}\right)\left(\sum_{m=1}^{\infty} \frac{g(m)}{m^s}\right) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{f(n)g(m)}{(nm)^s}.$$

Since the double sum converges absolutely, we can make this double sum into a single sum and rearrange the terms as we wish without changing the value (see Terence Tao *Analysis I*, pg. 188, 189). Thus it follows that

$$L_f(s)L_g(s) = \sum_{k=1}^{\infty} \sum_{nm=k} \frac{f(n)g(m)}{k^s} = \sum_{k=1}^{\infty} \frac{(f * g)(k)}{k^s} = L_{f*g}(s). \qquad \square$$

$\square$

EXAMPLE 1.5. If $f$ is an arithmetic function with $f(1) \neq 0$ and such that both $L_f(s)$ and $L_{f^{-1}}(s)$ converge absolutely in some half plane $\sigma > \sigma_0$, then $L_f(s)L_{f^{-1}}(s) = 1$ in the half plane $\sigma > \sigma_0$ and so in particular we have $L_f(s) \neq 0$ for $\sigma > \sigma_0$.

EXAMPLE 1.6. If $f$ is a completely multiplicative arithmetic function then $f^{-1} = \mu f$ and so the Dirichlet series $L_{f^{-1}}(s)$ converges in the half plane of absolute convergence of $L_f(s)$. It thus follows that

$$L_{\mu f}(s) = \sum_{n=1}^{\infty} \frac{\mu(n)f(n)}{n^s} = \frac{1}{L_f(s)}$$

for $\sigma > \sigma_a(f)$. Hence, the Dirichlet series of a completely multiplicative function does not vanish in its half plane of absolute convergence.

In particular we have $\zeta(s)L_\mu(s) = L_1(s)L_\mu(s) = 1$ for $\sigma > 1$ and so $\zeta(s)$ does not vanish in the half plane $\sigma > 1$;

$$L_\mu(s) = \frac{1}{\zeta(s)}$$

PROPOSITION 1.7. *The Dirichlet series $L_f(s)$ of an arithmetic function $f$ is analytic in the half plane $\sigma > \sigma_a(f)$ and*

$$L_f'(s) = -\sum_{n=1}^{\infty} \frac{f(n)\log n}{n^s}.$$

PROOF. Note that the Dirichlet series $L_f(s)$ converges uniformly in the shifted half plane $\sigma > \sigma_a(f) + \delta$ for any $\delta > 0$. Thus $L_f(s)$ is analytic in the half plane $\sigma > \sigma_a(f) + \delta$ for any $\delta > 0$ and satisfies

$$L_f'(s) = -\sum_{n=1}^{\infty} \frac{f(n)\log n}{n^s}$$

(see Theorem 5.2 and 5.3 in Stein and Shakarchi, *Complex Analysis*). Since $\delta$ is arbitrary, the above formula holds in the half plane $\sigma > \sigma_a(f)$. $\square$

EXAMPLE 1.8. We now record some examples of Dirichlet series of familiar arithmetic functions.

(a) We have

$$\sum_{n=1}^{\infty} \frac{e(n)}{n^s} = 1$$

for every $s \in \mathbb{C}$ and so $\sigma_a(e) = -\infty$.

(b) Since we have $\mu * 1 = e$, it follows that

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \frac{1}{\zeta(s)}$$

for $\sigma > 1$.

(c) Simply differentiating the Dirichlet series of $\zeta(s)$, we get

$$\zeta'(s) = -\sum_{n=1}^{\infty} \frac{\log n}{n^s}.$$

for $\sigma > 1$.

(d) Since we have $\Lambda = \log * \mu$, it follows that

$$\frac{\zeta'(s)}{\zeta(s)} = -\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}.$$

for $\sigma > 1$.

(e) Since $d = 1 * 1$, it follows that

$$\sum_{n=1}^{\infty} \frac{d(n)}{n^s} = \zeta(s)^2$$

for $\sigma > 1$.

(f) Because the Liouville function $\lambda$ is completely multiplicative we have $\lambda^{-1} = \mu\lambda = \mu^2$ and $1 * \lambda$ is the characteristic function of the squares, it follows that $\lambda * \mu^2 = e$ and so $(1 * \lambda) * \mu^2 = 1 * (\lambda * \mu^2) = 1$. Thus we have

$$\sum_{n=1}^{\infty} \frac{\mu(n)^2}{n^s} = \frac{\zeta(s)}{\zeta(2s)}$$

for $\sigma > 1$.

(g) Due to the identity $\varphi = \mu * N$, it follows that

$$\sum_{n=1}^{\infty} \frac{\varphi(n)}{n^s} = \frac{\zeta(s-1)}{\zeta(s)}$$

for $\sigma > 2$.

Theorem 1.9. *Let $f$ be an arithmetic function. Suppose that the the Dirichlet series $L_f(s)$ converges at the point $s = s_0 = \sigma_0 + it_0$. Then for every constant $H > 0$ the Dirichlet series $L_f(s)$ converges uniformly in the sector $S = \{s \in \mathbb{C} : \sigma \geqslant \sigma_0, |t - t_0| \leqslant H(\sigma - \sigma_0)\}$.*

Proof. Let $1 \leqslant x < y$. Applying the Abel's summation formula we obtain

$$\begin{aligned} \sum_{x<n\leqslant y} \frac{f(n)}{n^s} &= \sum_{x<n\leqslant y} \frac{f(n)}{n^{s_0}} \frac{1}{n^{s-s_0}} \\ &= \frac{1}{y^{s-s_0}} \sum_{x<n\leqslant y} \frac{f(n)}{n^{s_0}} + (s-s_0) \int_x^y \left( \sum_{x<n\leqslant t} \frac{f(n)}{n^{s_0}} \right) \frac{dt}{t^{s-s_0+1}} \end{aligned} \tag{1.1}$$

Take $\epsilon > 0$ be fixed and take $x$ to be sufficiently large so that $\left|\sum_{x<n\leqslant y} f(n)n^{-s_0}\right| < \epsilon$ for every $y > x$. Now assume that $s \in S$ and $s \neq s_0$. Then we have $\sigma > \sigma_0$. Applying the triangle inequality to (1.1), we get

$$\begin{aligned}\left|\sum_{x<n\leqslant y}\frac{f(n)}{n^s}\right| &\leqslant \frac{\epsilon}{y^{\sigma-\sigma_0}}+\epsilon|s-s_0|\int_x^\infty \frac{dt}{t^{\sigma-\sigma_0+1}}\\ &\leqslant \epsilon+\epsilon\frac{|s-s_0|}{\sigma-\sigma_0}\\ &\leqslant \epsilon+\epsilon\left(\frac{(\sigma-\sigma_0)+|t-t_0|}{\sigma-\sigma_0}\right)=(2+H)\epsilon.\end{aligned}$$

Note that the inequality also holds if $s=s_0$. Hence the Dirichlet series $L_f(s)$ converges uniformly in the sector $S$. □

Corollary 1.10. *Let $f$ be an arithmetic function. If the Dirichlet series $L_f(s)$ converges at the point $s=s_0$. Then $L_f(s)$ converges uniformly in every compact subset of the half plane $\sigma>\sigma_0$. Moreover, $L_f(s)$ defines a holomorphic function in the half plane $\sigma>\sigma_0$.*

Due to above corollary there exists a unique number $\sigma_c(f)\in[-\infty,\infty]$ such that $L_f(s)$ converges for $\sigma>\sigma_c(f)$ and diverges for $\sigma<\sigma_c(f)$. We call $\sigma_c(f)$ the abscissa of conditional convergence.

Proposition 1.11. *Let $f$ be an arithmetic function. Then $\sigma_c(f)\leqslant\sigma_a(f)\leqslant\sigma_c(f)+1$.*

Proof. The inequality $\sigma_c(f)\leqslant\sigma_a(f)$ is trivial since absolute convergence implies convergence. As for the other inequality let $\sigma>\sigma_c(f)+1$. Take $\delta=\sigma-\sigma_c(f)-1$. Then we have $\sigma=\sigma_c(f)+1+\delta$. Since the Dirichlet series $L_f(\sigma_c(f)+\delta/2)$ converges, the sequence $f(n)n^{-\sigma_c(f)-\delta/2}\ll 1$ must be bounded. It thus follows that

$$\sum_{n=1}^\infty\left|\frac{f(n)}{n^s}\right|=\sum_{n=1}^\infty\frac{|f(n)|}{n^\sigma}=\sum_{n=1}^\infty\frac{|f(n)|}{n^{\sigma_c(f)+\delta/2}}\frac{1}{n^{1+\delta/2}}\ll\sum_{n=1}^\infty\frac{1}{n^{1+\delta/2}}<\infty.$$

Hence, $L_f(s)$ converges absolutely for every $s$ with $\sigma>\sigma_c(f)+1$. Thus we conclude that $\sigma_a(f)\leqslant\sigma_c(f)+1$. □

Theorem 1.12. *Let $f$ be an arithmetic function. Suppose that $L_f(s)$ has abscissa of convergence $\sigma_c$. If $0<\epsilon<\delta<1$, then*

$$L_f(s)\ll\tau^{1-\delta+\epsilon}$$

*uniformly for $\sigma\geqslant\sigma_c+\delta$, where $\tau+|t|+4$ and the implicit constants depends on $f$, $\delta$ and $\epsilon$.*

Proof. Let $x\geqslant 1$. Let $s_0=\sigma_c+\epsilon$ and let $s\in\mathbb{C}$ be such that $\sigma\geqslant\sigma_c+\delta$. If we denote $A(u)=\sum_{n\leqslant u}f(n)n^{-s_0}$, then using Abel's summation by parts formula we obtain

$$\begin{aligned}\sum_{x<n\leqslant y}\frac{f(n)}{n^s}&=\sum_{x<n\leqslant y}\frac{f(n)}{n^{s_0}}n^{s_0-s}\\ &=A(y)y^{s_0-s}-A(x)x^{s_0-s}-(s_0-s)\int_x^y A(u)u^{s_0-s-1}\,du.\end{aligned}$$

If we denote $R(u)=\sum_{n>u}f(n)n^{-s_0}$ and using the relation $A(u)+R(u)=L_f(s_0)$ we end up with

$$\sum_{x<n\leqslant y}\frac{f(n)}{n^s}=R(x)x^{s_0-s}-R(y)y^{s_0-s}+(s_0-s)\int_x^y R(u)u^{s_0-s-1}\,du.$$

Since $R(u) = O(1)$, $|y^{s_0-s}| = y^{\sigma_c+\epsilon-\sigma} \leqslant y^{\epsilon-\delta}$ and the integral to the right converges absolutely, taking the limit as $y \to \infty$ we deduce that

$$\sum_{n>x} \frac{f(n)}{n^s} = R(x)x^{s_0-s} + (s_0 - s)\int_x^\infty R(u)u^{s_0-s-1}\,du.$$

Again using $R(u) = O(1)$ it follows that

$$\begin{aligned}\sum_{n>x} \frac{f(n)}{n^s} &\ll x^{\sigma_c+\epsilon-\sigma} + |s_0 - s|\int_x^\infty u^{\sigma_c+\epsilon-\sigma-1}\,du \\ &\ll x^{\epsilon-\delta} + \frac{|\sigma_c + \epsilon - s|}{\sigma - \sigma_c - \epsilon}x^{\epsilon-\delta}.\end{aligned}$$

Because $|\sigma_c + \epsilon - s| \leqslant (\sigma - \sigma_c - \epsilon) + |t|$ and $\sigma - \sigma_c - \epsilon \geqslant \delta - \epsilon$ we note that

$$\sum_{n>x} \frac{f(n)}{n^s} \ll \tau x^{\epsilon-\delta}.$$

Since $L_f(s_0)$ converges, we deduce that $f(n)n^{-s_0}$ are bounded. As a result we get that

$$\sum_{n\leqslant x} \frac{f(n)}{n^s} = \sum_{n\leqslant x} \frac{f(n)}{n^{s_0}}\frac{1}{n^{s-s_0}} \ll \sum_{n\leqslant x} \frac{1}{n^{\sigma-\sigma_c-\epsilon}}.$$

We see that

$$\sum_{n\leqslant x} \frac{1}{n^{\sigma-\sigma_c-\epsilon}} \leqslant \sum_{n\leqslant x} \frac{1}{n^{\delta-\epsilon}} < \int_0^x \frac{du}{u^{\delta-\epsilon}} = \frac{x^{1-\delta+\epsilon}}{1-\delta+\epsilon}$$

(this is where we use the hypothesis $\delta < 1$). Putting everything together we end up with

$$L_f(s) \ll \tau x^{\epsilon-\delta} + x^{1-\delta+\epsilon},$$

which holds uniformly for $\sigma \geqslant \sigma_c + \delta$ and the implicit constant depends only on $f$, $\delta$ and $\epsilon$. Plugging $x = \tau$ we obtain the required estimate. □

Theorem 1.13 (Landau). *Let $f$ be a nonnegative arithmetic function. Suppose that $L_f(s)$ has abscissa of convergence $\sigma_c$, which is finite. Then $L_f(s)$ cannot be continued analytically to any open set containing $\sigma_c$.*

Proof. Suppose that $L_f(s)$ has an analytic continuation $F(s)$ on some open set containing $\sigma_0$. Then there is an open disc $D_\delta(\sigma_c)$ with center $\sigma_c$ and radius $\delta$. Let $\sigma_0 = \sigma_c + \delta/3$. Then observe that $D_{2\delta/3}(\sigma_0) \subset D_\delta(\sigma_c)$. Thus $F(s)$ has a power series expansion centered at $\sigma_0$ which converges on the disc $D_{2\delta/3}(\sigma_0)$;

$$F(s) = \sum_{k=0}^\infty \frac{F^k(\sigma_0)}{k!}(s - \sigma_0)^k. \tag{1.2}$$

for $s \in D_{2\delta/3}(\sigma_0)$. Since $\sigma_0 > \sigma_c$ we have

$$F^k(\sigma_0) = L_f^k(\sigma_0) = \sum_{n=1}^\infty \frac{(-\log n)^k f(n)}{n^{\sigma_0}}.$$

Plugging this into (1.2) we obtain

$$F(s) = \sum_{k=0}^\infty \frac{1}{k!}\left(\sum_{n=1}^\infty \frac{(-\log n)^k f(n)}{n^{\sigma_0}}\right)(s - \sigma_0)^k = \sum_{k=0}^\infty \sum_{n=1}^\infty \frac{(\log n)^k f(n)}{n^{\sigma_0}k!}(\sigma_0 - s)^k.$$

Now take $\sigma_0 - 2\delta/3 < \sigma < \sigma_0$. Then observe that

$$F(\sigma) = \sum_{k=0}^{\infty}\sum_{n=1}^{\infty} \frac{(\log n)^k f(n)}{n^{\sigma_0} k!}(\sigma_0 - \sigma)^k = \sum_{n=1}^{\infty} \frac{f(n)}{n^{\sigma_0}} \sum_{k=0}^{\infty} \frac{1}{k!}(\log n)^k (\sigma_0 - \sigma)^k$$

after interchanging the summation which is justified by all the quantities in the integrand being nonnegative. Hence we get

$$F(\sigma) = \sum_{n=1}^{\infty} \frac{f(n)}{n^{\sigma_0}} e^{(\log n)(\sigma_0 - \sigma)} = \sum_{n=1}^{\infty} \frac{f(n)}{n^{\sigma_0}} n^{\sigma_0 - \sigma} = \sum_{n=1}^{\infty} \frac{f(n)}{n^{\sigma}}.$$

This shows that $L_f(\sigma)$ converges for $\sigma_0 - 2\delta/3 < \sigma < \sigma_0$ but this is a contradiction as $\sigma_c = \sigma_0 - \delta/3 > \sigma_0 - 2\delta/3$. □

## 2. Euler's product formula

Theorem 2.1. *Let $f$ be a multiplicative arithmetic function such that the series $\sum_{n=1}^{\infty} f(n)$ is absolutely convergent. Then the series can be expressed as*

$$\sum_{n=1}^{\infty} f(n) = \prod_p (1 + f(p) + f(p^2) + \cdots)$$

*and the above product converges absolutely. Moreover, the series converges absolutely if and only if*

$$\sum_{p^m} |f(p^m)| < \infty. \tag{2.1}$$

Proof. The absolute convergence of the product follows immediately as

$$\sum_p \left| \sum_{k=1}^{\infty} f(p^k) \right| \leqslant \sum_p \sum_{k=1}^{\infty} |f(p^k)| \leqslant \sum_{n=1}^{\infty} |f(n)| < \infty.$$

Now consider the finite product

$$P(x) = \prod_{p \leqslant x} (1 + f(p) + f(p^s) + \cdots)$$

extended over all primes $\leqslant x$. Since this is a product of a finite number of absolutely convergent series we can multiply the series and rearrange the terms in any fashion without altering the sum. A typical term is of the form

$$f(p_1^{a_1}) \cdots f(p_r^{a_r}) = f(p_1^{a_1} \cdots p_r^{a_r})$$

since $f$ is multiplicative. By the Fundamental Theorem of Arithmetic we can write

$$P(x) = \sum_{n \in A(x)} f(n)$$

where $A(x)$ consists of integers having their primes factors $\leqslant x$. Therefore

$$\sum_{n=1}^{\infty} f(n) - P(x) = \sum_{n \in B(x)} f(n),$$

where $B(x)$ is the set of integers having at least one prime factor $> x$. Therefore we have

$$\left| \sum_{n=1}^{\infty} f(n) - P(x) \right| \leqslant \sum_{n \in B(x)} |f(n)| \leqslant \sum_{n > x} |f(n)|.$$

Note that the sum on the right goes to 0 as $x \to \infty$ since the series $\sum_{n=1}^{\infty} |f(n)|$ is convergent. Thus the product formula follows.

Note that the convergence of series in (2.1) follows immediately from the convergence of the series $\sum_{n=1}^{\infty} |f(n)|$. To prove the converse, observe that the product

$$P = \prod_p \left(1 + \sum_{m=1}^{\infty} |f(p^m)|\right)$$

converges and we have

$$P \geqslant \prod_{p \leqslant x} \left(1 + \sum_{m=1}^{\infty} |f(p^m)|\right) = \sum_{n \in A(x)} |f(n)| \geqslant \sum_{n \leqslant x} |f(n)|.$$

Thus we have $\sum_{n=1}^{\infty} |f(n)| < \infty$ as desired. $\square$

Corollary 2.2. *Let $f$ be a completely multiplicative arithmetic function such that the series $\sum_{n=1}^{\infty} f(n)$ is absolutely convergent. Then*

$$\sum_{n=1}^{\infty} f(n) = \prod_p (1 - f(p))^{-1}.$$

Remark 2.3. Note that nowhere in the proof of above theorem did we make use of the fact that the primes are infinite. We did apply the fundamental theorem of arithmetic but it does not assume the infinitude of primes.

Example 2.4. The Euler's product for $\zeta(s) = L_1(s)$ is given as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}.$$

We will now use this formula to glean information about primes. We restrict $s$ to be a real number $> 1$. Now note that $\zeta(s)$ increases as $s \to 1^+$. Hence for any $N \in \mathbb{N}$ it follows that

$$\lim_{s \to 1^+} \zeta(s) \geqslant \lim_{s \to 1^+} \sum_{n=1}^{N} \frac{1}{n^s} = \sum_{n=1}^{N} \frac{1}{n}.$$

Thus we have

$$\lim_{s \to 1^+} \zeta(s) = \infty.$$

This shows that primes cannot be finite lest we would have

$$\lim_{s \to 1^+} \zeta(s) = \lim_{s \to 1^+} \prod_p \left(1 - \frac{1}{p^s}\right)^{-1} = \prod_p \left(1 - \frac{1}{p}\right)^{-1} < \infty.$$

Working a little harder we can glean more information than just the infinitude of primes. Observe that for $s > 1$ we have

$$\log \zeta(s) = -\sum_p \log(1 - p^{-s}).$$

Since $p^{-s} \leqslant 1/2$, we can use the series expression of log and rewrite the above formula as

$$\log \zeta(s) = \sum_p \sum_{m=1}^{\infty} \frac{1}{mp^{sm}} = \sum_p \frac{1}{p^s} + \sum_p \sum_{m=2}^{\infty} \frac{1}{mp^{sm}}.$$

Note that
$$\sum_p \sum_{m=2}^{\infty} \frac{1}{mp^{sm}} \leqslant \sum_p \sum_{m=2}^{\infty} \frac{1}{p^m} = \sum_p \frac{1}{p(p-1)} < \infty.$$
Hence we have
$$\log \zeta(s) = \sum_p \frac{1}{p^s} + O(1) \leqslant \sum_p \frac{1}{p} + O(1)$$
for $s > 1$. Since $\zeta(s) \to \infty$ as $s \to 1^+$, it follows that $\sum_p \frac{1}{p} = \infty$. This result shows that the primes in some sense are not very sparse unlike the sequence of perfect squares.

## 3. Exercises

Exercise 1. Define arithmetic function $f, g$ by
$$f(n) = (-1)^{n+1}, \qquad g(n) = \begin{cases} 1 & \text{if } n \not\equiv 0 \pmod 3, \\ -2 & \text{if } n \equiv 0 \pmod 3. \end{cases}$$

(a) Show that $L_f(s) = (1 - 2^{1-s})\zeta(s)$, $L_g(s) = (1 - 3^{1-s})\zeta(s)$ for $\sigma > 1$.
(b) Prove that $\{s \in \mathbb{C} : 2^{1-s} = 1\} \cap \{s \in \mathbb{C} : 3^{1-s} = 1\} = \{1\}$ and conclude that $\zeta(s)$ has an analytic continuation to the half plane $\sigma > 0$ with a simple pole with residue 1 at $s = 1$.

Exercise 2. Show that there is no half plane in which $1/\zeta'(s)$ can be written as a convergent Dirichlet series.

Exercise 3. Let $0 < \epsilon < \delta < 1$. Then show that
$$\zeta(s) \ll \frac{\tau^{1-\delta+\epsilon}}{1-\sigma}$$
for $\delta \leqslant \sigma < 1$, where the implicit constant depends on $\epsilon$ and $\delta$.

## 4. Solutions

Solution 1. (a): Note that
$$L_f(s) - \zeta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} - 1}{n^s} = -2 \sum_{\substack{n=1 \\ 2|n}}^{\infty} \frac{1}{n^s} = -2 \sum_{k=1}^{\infty} \frac{1}{(2k)^s} = -2^{1-s}\zeta(s)$$
from which we get $L_f(s) = (1 - 2^{1-s})\zeta(s)$. Similarly we have
$$L_g(s) - \zeta(s) = -3 \sum_{\substack{n=1 \\ 3|n}}^{\infty} \frac{1}{n^s} = -3 \sum_{k=1}^{\infty} \frac{1}{(3k)^s} = -3^{1-s}\zeta(s).$$
Thus we have $L_g(s) = (1 - 3^{1-s})\zeta(s)$.

(b): Note that $\sigma_c(f) = \sigma_c(g) = 0$ since the partial sums $\sum_{n \leqslant x} f(n)$ and $\sum_{n \leqslant x} g(n)$ are bounded. Now let us define
$$\zeta_1(s) = \frac{L_f(s)}{1 - 2^{1-s}}, \qquad \zeta_2(s) = \frac{L_g(s)}{1 - 3^{1-s}}$$
on the sets $\{s \in \mathbb{C} : \sigma > 0, 2^{1-s} \neq 1\}$ and $\{s \in \mathbb{C} : \sigma > 0, 3^{1-s} \neq 1\}$ respectively. Because $\zeta_1(s)$ and $\zeta_2(s)$ are both analytic continuations of $\zeta(s)$, it follows that $\zeta_1(s) = \zeta_2(s)$ for $\sigma > 0$ with $2^{1-s} \neq 1, 3^{1-s} \neq 1$. Let us denote this common extension by $\zeta(s)$. Observe that $2^{1-s} = 1$ if and only if $s = 2\pi i n/\log 2$, $n \in \mathbb{Z}$. Similarly $3^{1-s} = 1$ if and only if

$s = 2\pi in/\log 3$, $n \in \mathbb{Z}$. If $2^{1-s} = 3^{1-s} = 1$, then we must have $s = 2\pi in/\log 2 = 2\pi im/\log 3$ for some $m, n \in \mathbb{Z}$. It follows that $n \log 3 = m \log 2$, i.e., $3^n = 2^m$. But this is only possible if $n = m = 0$. Hence we must have $s = 1$. This shows that $\zeta(s)$ is analytic everywhere in the half plane $\sigma > 0$ with $s \neq 1$.

As for $s = 1$ we find that

$$\lim_{s\to 1}(s-1)\zeta(s) = -\lim_{s\to 1} L_f(s)\left(\frac{s-1}{2^{1-s}-1}\right) = \frac{L_f(1)}{\log 2} = 1$$

as the derivative of $2^{1-s}$ at $s = 1$ is $-\log 2$ and $L_f(1) = \sum_{n=1}^{\infty}(-1)^{n+1}n^{-1} = \log 2$. Hence, $\zeta(s)$ has a pole at $s = 1$ with residue 1.

Solution 2. First we note that there is a half plane in which $\zeta'(s) \neq 0$ as $\zeta'(s) = \sum_{n=1}^{\infty}(-\log n)n^{-s}$. Now suppose for the sake of contradiction that there is an arithmetic function $f$ such that $1/\zeta'(s) = L_f(s)$ in some half plane $\sigma > \sigma_0$. Then we have $1 = \zeta'(s)L_f(s)$. We now observe that

$$1 = \lim_{\sigma\to\infty} \zeta'(s)L_f(s) = 0 \cdot f(1) = 0$$

since $\zeta'(s) \to 0$ and $L_f(s) \to f(1)$ as $\sigma \to \infty$, a contradiction.

Solution 3. Let $f(n) = (-1)^{n+1}$. Then we know that $L_f(s) = (1 - 2^{1-s})\zeta(s)$ and $\sigma_c(f) = 0$. By Theorem 1.12 we have $L_f(s) \ll \tau^{1-\delta+\epsilon}$ for $\sigma \geqslant \delta$. Note that for $\sigma < 1$ we have

$$|1 - 2^{1-s}| \geq 2^{1-\sigma} - 1 = \sum_{k=1}^{\infty} \frac{(\log 2)^k(1-\sigma)^k}{k!} \geqslant (1-\sigma)\log 2.$$

Hence, we conclude that

$$\zeta(s) = \frac{L_f(s)}{1 - 2^{1-s}} \ll \frac{\tau^{1-\delta+\epsilon}}{1-\sigma}$$

for $\delta \leqslant \sigma < 1$.