Problem. A subset $S \subset \mathbb{N}$ is said to be in arithmetic progression if

$$S = \{a, a+d, a+2d, \dots\},$$

where $a, d \in \mathbb{N}$. Here d is called the step of S. Show that $\mathbb{N}$ cannot be partitioned into a finite number of subsets that are in arithmetic progression with distinct steps (except for the trivial case $a = d = 1$).

Solution. Suppose that $\mathbb{N} = S_{a_1,d_1} \sqcup \cdots \sqcup S_{a_r,d_r}$, where

$$S_{a_k,d_k} = \{a_k, a_k + d_k, a_k + 2d_k, \dots\}$$

and $a_k, d_k \in \mathbb{N}$ and d_k are all distinct. Then note that we have

$$\frac{1}{1-z} = \sum_{n\in\mathbb{N}} z^n = \sum_{k=1}^{r} \sum_{n\in S_k} z^n = \sum_{k=1}^{r} \sum_{\ell=0}^{\infty} z^{a_k+\ell d_k} = \sum_{k=1}^{r} \frac{z^{a_k}}{1-z^{d_k}} \tag{0.1}$$

for $|z| < 1$ since the series $\sum_{n\in\mathbb{N}} z^n$ converges absolutely and so we can rearrange the terms. Without loss of generality assume that $d_1 < d_2 < \cdots < d_r$ (as d_k are all distinct).

Note that if $r = 1$, then we have we have the trivial case $a_1 = d_1 = 1$. Now suppose for the sake of contradiction that $r > 1$. Let $\rho = e^{2\pi i/d_r}$ and observe that $\rho \neq 1$ as $d_r > 1$. Finally, we note that

$$\lim_{t\to 1^-} \left(\frac{1}{1-t\rho} - \sum_{k=1}^{r-1} \frac{(t\rho)^{a_k}}{1-(t\rho)^{d_k}} \right) = \frac{1}{1-\rho} - \sum_{k=1}^{r-1} \frac{\rho^{a_k}}{1-\rho^{d_k}}.$$

as $\rho^{d_k} \neq 1$ for every $1 \leqslant k < r$. However, the limit

$$\lim_{t\to 1^-} \frac{(t\rho)^{a_r}}{1-(t\rho)^{d_r}}$$

does not exist since $|\rho^{a_k}| = 1$ and $\rho^{d_r} = 1$. This is a contradiction as

$$\frac{1}{1-t\rho} - \sum_{k=1}^{r-1} \frac{(t\rho)^{a_k}}{1-(t\rho)^{d_k}} = \frac{(t\rho)^{a_r}}{1-(t\rho)^{d_r}}$$

for every $0 < t < 1$ due to (0.1).