# COMMUTATIVE ALGEBRA

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## 1. Rings and ideals

### 1.1. Ideals and ring homomorphisms.

**Definition 1.1.** Let $A$ be a ring. A subset $\mathfrak{a}$ of $A$ is said to be an *ideal* of $A$ if $\mathfrak{a}$ is an additive subgroup of $A$ and $\mathfrak{a}x \subset \mathfrak{a}$ for every $x \in A$.

**Proposition 1.2.** Let $A$ be a ring and let $\mathfrak{a}$ be an additive subgroup of $A$. Then $\mathfrak{a}$ is an ideal of $A$ if and only if the multiplication operation

$$(x + \mathfrak{a})(y + \mathfrak{a}) = xy + \mathfrak{a}$$

on the quotient group $A/\mathfrak{a}$ is well-defined.

**Proposition 1.3** (characterization of ideals in a quotient ring)**.** Let $A$ be a ring and let $\mathfrak{a}$ be an ideal of $A$. Then there is an inclusion preserving bijective correspondence between the ideals $\mathfrak{b}$ of $A$ containing $\mathfrak{a}$ and the ideals of $A/\mathfrak{a}$ given by $\mathfrak{b} \mapsto \mathfrak{b}/\mathfrak{a}$.

If $\pi : A \to A/\mathfrak{a}$ is the canonical projection map, then the inverse of the map $\mathfrak{b} \mapsto \mathfrak{b}/\mathfrak{a}$ above is given by $\overline{\mathfrak{b}} \mapsto \pi^{-1}(\overline{\mathfrak{b}})$.

**Claim 1.4.** Images and preimages of subrings are subrings under a ring homomorphism.

**Claim 1.5.** Preimage of an ideal under a ring homomorphism is an ideal. The image of an ideal is an ideal of the image ring.

The image of an ideal need not be an ideal. Consider the embedding $\mathbb{Z} \to \mathbb{Q}$.

**Theorem 1.6** (Isomorphism theorems)**.**

(a) Let $f : A \to B$ be a ring homomorphism. Then $A/\ker f \cong \operatorname{im} f$.
(b) Let $\mathfrak{a}$ be an ideal and let $B$ be a subring of $A$. Then $B + \mathfrak{a}$ is a subring of $A$, $B \cap \mathfrak{a}$ is an ideal of $B$ and

$$(B + \mathfrak{a})/\mathfrak{a} \cong B/(B \cap \mathfrak{a}).$$

(c) If $\mathfrak{a} \subset \mathfrak{b}$ are ideals of a ring $A$, then

$$(A/\mathfrak{a})/(\mathfrak{b}/\mathfrak{a}) \cong A/\mathfrak{b}.$$

## 1.2. **Zero-divisors, nilpotents, and units.**

**Claim 1.7.** The set of zero-divisors and units are disjoint.

A nilpotent is always a zero-divisor in a nonzero ring but the converse is not true as $\overline{3} \in \mathbb{Z}/6\mathbb{Z}$ and $\overline{x} \in k[x,y]/(xy)$ are both zero-divisors but not nilpotents.

**Problem 1.1.** Identify nilpotent elements in the ring $\mathbb{Z}/n\mathbb{Z}$.

*Solution.* An element $\overline{a} \in \mathbb{Z}/n\mathbb{Z}$ is nilpotent if and only if $\prod_{p|n} p$ divides $a$.

**Proposition 1.8.** Let $A$ be a ring $\neq 0$ Then the following are equivalent:

(a) $A$ is a field.
(b) The only ideals of $A$ are 0 and (1).
(c) Every nonzero ring homomorphism from $A$ to a ring $B$ is injective.

## 1.3. **Prime and maximal ideals.**

**Definition 1.9.** Let $A$ be a ring. A proper ideal $\mathfrak{p}$ of $A$ is said to be *prime* if $xy \in A$ implies $x \in A$ or $y \in A$.

**Example 1.10.** (1) If $A$ is an integral domain, then 0 is a prime ideal of $A$.
(2) The prime ideals of $\mathbb{Z}$ are precisely the zero ideal and ideals of the form $(p)$, where $p$ is a prime number.
(3) The ideal $(x) \subset k[x,y]$ is prime.

**Proposition 1.11.** An ideal $\mathfrak{p}$ of a ring $A$ is prime if and only if $A/\mathfrak{p}$ is an integral domain.

**Claim 1.12.** If $f : A \to B$ is a ring homomorphism and $\mathfrak{q}$ is a prime ideal of $B$, then the inverse image $f^{-1}(\mathfrak{q})$ is also prime.

*Proof.* The proof is quite simple following directly from the definition but a more instructive proof is as follows: Consider the map $\pi \circ f : A \to B/\mathfrak{q}$, where $\pi : B \to B/\mathfrak{q}$ is the canonical projection. Then $\ker(\pi \circ f) = f^{-1}(\mathfrak{q})$. Thus we have $A/f^{-1}(\mathfrak{q}) \cong (\pi \circ f)(A)$. Since $B/\mathfrak{q}$ is an integral domain, it follows that the subring $(\pi \circ f)(A)$ and hence $A/f^{-1}(\mathfrak{q})$ is an integral domain. This implies that $f^{-1}(\mathfrak{q})$ is prime in $A$. □

**Claim 1.13.** Let $f : A \to B$ be a surjective ring homomorphism and let $\mathfrak{p}$ be a prime ideal of $A$ such that $\mathfrak{p} \supset \ker f$. Then the image $f(\mathfrak{p})$ is prime in $B$.

**Definition 1.14.** Let $A$ be a ring. A proper ideal $\mathfrak{m}$ of $A$ is said to be *maximal* if there is no proper ideal strictly containing $\mathfrak{m}$.

**Proposition 1.15.** An ideal $\mathfrak{m}$ of $A$ is maximal if and only if $A/\mathfrak{m}$ is a field.

The inverse image of a maximal ideal need not be maximal. Consider the embedding $\mathbb{Z} \to \mathbb{Q}$. However, the image of a maximal ideal under a surjective ring homomorphism containing the kernel is a maximal ideal.

**Theorem 1.16.** Every nonzero ring $A$ has a maximal ideal.

*Proof.* Follows from Zorn's lemma. □

**Corollary 1.17.** If $\mathfrak{a}$ is a proper ideal of a ring $A$, then there is a maximal ideal of $A$ containing $\mathfrak{a}$.

**Corollary 1.18.** Every nonunit element is contained in some maximal ideal.

**Problem 1.2.** Let $A$ be a ring in which every element $x$ satisfies $x^n = x$ for some $n > 1$. Show that every prime ideal is maximal.

*Solution.* Let $\mathfrak{p}$ be a prime ideal of $A$. Then we know that $\mathfrak{p}$ is contained in some maximal ideal $\mathfrak{m}$ of $A$. Suppose for the sake of contradiction that $\mathfrak{p}$ is properly contained in $\mathfrak{m}$ and let $x \in \mathfrak{m}\backslash\mathfrak{p}$. Then we have $x^n = x$ for some $n > 1$ and so $x(x^{n-1} - 1) = 0 \in \mathfrak{p}$. This implies that $x^{n-1} - 1 \in \mathfrak{p}$ as $x \notin \mathfrak{p}$. It now follows that $x^{n-1} - 1 \in \mathfrak{m}$ as $\mathfrak{p} \subset \mathfrak{m}$. Finally, we get that $1 \in \mathfrak{m}$ as $x \in \mathfrak{m}$, a contradiction. Hence we must have $\mathfrak{p} = \mathfrak{m}$.

Another solution: Let $\mathfrak{p}$ be a prime ideal of $A$. Then $A/\mathfrak{p}$ is an integral domain. Let $x \in A$. Then $x^n = x$ for some $n > 1$ and so $\overline{x}^n = \overline{x}$. If $\overline{x} \neq 0$, then $\overline{x}^{n-1} = \overline{1}$ and so $\overline{x}$ is a unit. This shows that $A/\mathfrak{p}$ is a field and so $\mathfrak{p}$ is a maximal ideal.

**Claim 1.19.** If $\mathfrak{m}$ is a proper ideal of a ring $A$ such that $A\backslash\mathfrak{m} \subset A^\times$, then $\mathfrak{m}$ is the unique maximal ideal of $A$.

*Proof.* Every proper ideal $\mathfrak{a}$ is contained in $A\backslash A^\times \subset \mathfrak{m}$. □

**Claim 1.20.** If $\mathfrak{m}$ is a maximal ideal of $A$ such that $1 + \mathfrak{m} \subset A^\times$, then $\mathfrak{m}$ is the unique maximal ideal of $A$.

*Proof.* If $x$ is a nonunit element not contained in $\mathfrak{m}$, then $\mathfrak{m} + (x) = (1)$ and so $x \in 1 + \mathfrak{m} \subset A^\times$, a contradiction. □

**Problem 1.3.** Show that the only idempotents in a local ring are 0 and 1.

*Solution.* Let $x$ be an idempotent element in a ring $A$ and $\mathfrak{m}$ be the unique maximal ideal of $A$. Then $x^2 = x$ and so $x(x-1) = 0$. Because $\mathfrak{m} = A\backslash A^\times$ we get that either $x$ or $1 - x$ is a unit for if both are nonunits, then both lie in $\mathfrak{m}$ which results in $1 \in \mathfrak{m}$, a contradiction. This implies that $x = 0$ or $x = 1$.

**Claim 1.21.** In a PID every nonzero prime ideal is maximal.

### 1.4. **Nilradical and Jacobson radical.**

**Claim 1.22.** The set $\mathfrak{N}$ of all nilpotent elements in a ring $A$ form an ideal. Moreover, the ring $A/\mathfrak{N}$ does not have any nonzero nilpotent elements.

**Theorem 1.23.** Let $A$ be a ring. Then

$$\mathfrak{N} = \bigcap_{\mathfrak{p}\text{ prime}} \mathfrak{p}.$$

*Proof.* The inclusion $\subset$ is easy. For the other inclusion let $x \in \bigcap_{\mathfrak{p} \text{ prime}} \mathfrak{p}$. Suppose for the sake of contradiction that $x$ is not nilpotent. Then the collection of all ideals $\mathfrak{a}$ of $A$ for which $x^n \notin \mathfrak{a}$ for every $n \in \mathbb{N}$ has a maximal element $\mathfrak{p}$ by the Zorn's lemma. It is then easy to see that $\mathfrak{p}$ is a prime ideal and so we obtain a contradiction. □

**Problem 1.4.** Let $A$ be a ring and let $\mathfrak{N}$ be its nilradical. Show that the following are equivalent:

(a) $A$ has exactly one prime ideal.
(b) Every element of $A$ is either a unit of a nilpotent.
(c) $A/\mathfrak{N}$ is a field.

*Solution.* (a) $\Rightarrow$ (b): Let $x \in A$ be a nonunit. Then $x$ lies in some prime ideal $\mathfrak{p}$ of $A$. But by assumption $\mathfrak{p}$ is the unique prime ideal of $A$ and so $\mathfrak{N} = \mathfrak{p}$. Thus $x$ is a nilpotent.

(b) $\Rightarrow$ (c): By assumption we have $A \backslash A^\times \subset \mathfrak{N}$. This immediately shows that $\mathfrak{N}$ is the unique maximal ideal of $A$ by Claim 1.19 and so $A/\mathfrak{N}$ is a field.

(c) $\Rightarrow$ (a): If $\mathfrak{p}$ is a prime ideal of $A$, then $\mathfrak{N} \subset \mathfrak{p}$. Since $\mathfrak{N}$ is a maximal ideal we get that $\mathfrak{p} = \mathfrak{N}$. Hence, $\mathfrak{N}$ is the unique prime ideal of $A$.

**Theorem 1.24.** Let $\mathfrak{R}$ be the Jacobson radical of a ring $A$. Then $x \in \mathfrak{R}$ if and only if $1 + xy \in A^\times$ for every $y \in A$.