Arithmetic Functions

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- An **arithmetic function** is a complex-valued function defined on $\mathbb{N}$, i.e., a sequence of complex numbers. While the class of arithmetic functions is broad, namely $\mathbb{C}^{\mathbb{N}}$, we will restrict our attention to only those of number-theoretic significance.

- Below are some commonly occurring arithmetic functions.

  (i) The **identity function** e is defined as $e(n) = \lfloor 1/n \rfloor$, i.e., $e(1) = 1$ and $e(n) = 0$ for $n > 1$. It is called so because, as we will see later, it acts as the identity element in a group of arithmetic functions.

  (ii) For any $\alpha \in \mathbb{C}$, the **power function** N^{α} is defined as $N^{\alpha}(n) = n^{\alpha}$. We denote N^0 by 1 and call it the **unit function**.

  (iii) For $n \in \mathbb{N}$, $\Omega(n)$ is defined to be the total number of prime factors of n counted with multiplicity. We can write it in summation notation as $\Omega(n) = \sum_{p^k | n} 1 = \sum_{p^k || n} k$. It is sometimes called **big omega function**.

(iv) For $n \in \mathbb{N}$, $\omega(n)$ is defined to be the number of prime factors of n. We can express it in summation notation as $\omega(n) = \sum_{p \mid n} 1$. It is usually called **small omega function**.

(v) The **Liouville function**, denoted λ, is defined as $\lambda(n) = (-1)^{\Omega(n)}$.

(vi) For a complex number α, the **divisor function** σ_α is defined as $\sigma_\alpha = \sum_{d \mid n} d^\alpha$. We denote σ_0 by d or τ and σ_1 by simply σ.

- We now discuss several important arithmetic functions that occur frequently throughout number theory.

- The **Möbius function**, denoted μ, is defined as

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1, \\ (-1)^k & \text{if } n = p_1 \ldots p_k, \text{ where } p_1, \ldots, p_k \\ & \text{are distinct primes,} \\ 0 & \text{otherwise.} \end{cases}$$

- The Möbius function is the signed characteristic function of squarefree positive integers. It is intimately connected to the Riemann zeta function, $\zeta(s)$, which is one of the most important function in analytic number theory.

- The estimate

$$\sum_{n \leq x} \mu(x) \ll x^{1/2 + \epsilon}$$

implies the Riemann Hypothesis, a notoriously difficult problem in analytic number theory (or arguably in all of mathematics), which states that all of the nontrivial zeros of $\zeta(s)$ lie on the critical line with $\Re s = \frac{1}{2}$.

- The convergence of the Dirichlet series $\sum_{n=1}^{\infty} \mu(n) n^{-s}$ for every s with $\Re s > \frac{1}{2}$ also implies Riemann Hypothesis.

- It is known that the Dirichlet series $\sum_{n=1}^{\infty} \mu(n) n^{-s}$ converges for every s with $\Re s = 1$ and the limit is 0 in the case $s = 1$.

- The convergence of the series $\sum_{n=1}^{\infty} \mu(n) n^{-1}$ (without regard for the value) implies prime number theorem.

- **Theorem** (divisor sum of μ) If $n \geq 1$, then we have

$$\sum_{d \mid n} \mu(d) = e(n).$$

- **Proof.** The equality follows trivially for $n = 1$. For $n > 1$ the identity follows by restricting the sum to squarefree divisors of n as shown by

$$\sum_{d \mid n} \mu(d) = \sum_{\substack{d \mid n \\ d \text{ sq. free}}} \mu(d) = \sum_{\mathcal{P} \subset \{p \mid n\}} \mu\left(\prod_{p \in \mathcal{P}} p\right)$$

$$= \sum_{\mathcal{P} \subset \{p \mid n\}} (-1)^{|\mathcal{P}|} = \sum_{k=0}^{\omega(n)} \sum_{\substack{\mathcal{P} \subset \{p \mid n\} \\ |\mathcal{P}| = k}} (-1)^{|\mathcal{P}|}$$

$$= \sum_{k=0}^{\omega(n)} \binom{\omega(n)}{k} (-1)^k = (1-1)^{\omega(n)} = 0.$$

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- As remarked earlier it is difficult to show that

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n} = 0$$

  but by simply using the above result we can bound the partial sums of this series by 1.

- **Corollary.** If $n \geq 1$, then

$$\left| \sum_{n \leq x} \frac{\mu(n)}{n} \right| \leq 1.$$

- **Proof.** Note that

$$1 = \sum_{n \leq x} e(n) = \sum_{n \leq x} \sum_{d \mid n} \mu(d) = \sum_{qd \leq x} \mu(d) = \sum_{d \leq x} \mu(d) \left\lfloor \frac{x}{d} \right\rfloor.$$

- Writing $\lfloor x/d \rfloor = x/d - \{x/d\}$ we obtain

$$1 = x \sum_{d \leq x} \frac{\mu(d)}{d} - \sum_{d \leq x} \mu(d) \left\{ \frac{x}{d} \right\}.$$

- We can bound the sum to the right by

$$\sum_{d \leq x} \left\{ \frac{x}{d} \right\} = \{x\} + \sum_{2 \leq d \leq x} \left\{ \frac{x}{d} \right\} \leq \{x\} + \lfloor x \rfloor - 1 = x - 1.$$

- Thus we have

$$\left| x \sum_{d \leq x} \frac{\mu(d)}{d} \right| \leq 1 + \sum_{d \leq x} \left\{ \frac{x}{d} \right\} \leq x.$$

This completes the proof.

- The **Euler's totient function**, denoted φ (or ϕ), is defined at n to be the number of positive integers not exceeding n that are relatively prime to n. We can rewrite $\varphi(n)$ in the summation notation as

$$\varphi(n) = \sum_{\substack{k=1 \\ (k,n)=1}}^{n} 1.$$

- The function φ is ubiquitous in number theory, particularly in prime number theory. For instance, the degree of the irreducible polynomial of $e^{2\pi i/n}$ over $\mathbb{Q}$ is $\varphi(n)$, or equivalently $[\mathbb{Q}(e^{2\pi i/n}) : \mathbb{Q}] = \varphi(n)$. The number of primitive nth roots of unity is also $\varphi(n)$ but this fact alone does not imply the aforementioned result.

- **Theorem.** If $n \geq 1$, then

$$\sum_{d \mid n} \varphi(d) = n.$$

- **Proof.** We partition the set $\{1, \ldots, n\}$ into subsets $A_d = \{1 \leq k \leq n : (k, n) = d\}$, where d is a divisor of n, and note that there is a one-to-one bijection between elements of A_d and integers $1 \leq r \leq n/d$ satisfying $(r, n/d) = 1$. This then implies that

$$n = \sum_{d \mid n} |A_d| = \sum_{d \mid n} \varphi(n/d) = \sum_{d \mid n} \varphi(d).$$

- Next we obtain a relationship between φ and μ.

- **Theorem.** If $n \geq 1$, then we have

$$\varphi(n) = \sum_{d \mid n} \mu(d) \frac{n}{d}.$$

- **Proof.** We use the formula for the divisor sum of μ to obtain

$$\varphi(n) = \sum_{k=1}^{n} e((k, n)) = \sum_{k=1}^{n} \sum_{d \mid (k,n)} \mu(d) = \sum_{k=1}^{n} \sum_{\substack{d \mid n \\ d \mid k}} \mu(d)$$

- Changing the order of summation we get

$$\varphi(n) = \sum_{d \mid n} \sum_{\substack{k=1 \\ d \mid k}}^{n} \mu(d) = \sum_{d \mid n} \mu(d) \sum_{\substack{k=1 \\ d \mid k}}^{n} 1 = \sum_{d \mid n} \mu(d) \frac{n}{d}.$$

- We have a nice product formula for $\varphi(n)$.
- **Theorem.** For $n \geq 1$ we have

$$\varphi(n) = n \prod_{p \mid n} \left(1 - \frac{1}{p} \right).$$

- **Proof.** Note that

$$\prod_{p \mid n} \left(1 - \frac{1}{p} \right) = \sum_{\mathcal{P} \subset \{p \mid n\}} \prod_{p \in \mathcal{P}} \left(\frac{-1}{p} \right) = \sum_{\mathcal{P} \subset \{p \mid n\}} \frac{(-1)^{|\mathcal{P}|}}{\prod_{p \in \mathcal{P}} p}$$

$$= \sum_{\mathcal{P} \subset \{p \mid n\}} \frac{\mu \left(\prod_{p \in \mathcal{P}} p \right)}{\prod_{p \in \mathcal{P}} p} = \sum_{\substack{d \mid n \\ d \text{ sq. free}}} \frac{\mu(d)}{d}$$

$$= \sum_{d \mid n} \frac{\mu(d)}{d} = \frac{\varphi(n)}{n}.$$

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- **Corollary** Let m and n be positive integers. Then $\varphi(mn) = \varphi(m)\varphi(n)(d/\varphi(d))$, where $d = (m, n)$. In particular, $\varphi(mn) = \varphi(m)\varphi(n)$ if m and n are coprime.

- **Proof.** Using the product formula for φ we get

$$\frac{\varphi(mn)}{mn} = \prod_{p \mid mn}\left(1 - \frac{1}{p}\right) = \prod_{p \mid m}\left(1 - \frac{1}{p}\right)\prod_{\substack{p \mid n \\ p \nmid m}}\left(1 - \frac{1}{p}\right)$$

$$= \prod_{p \mid m}\left(1 - \frac{1}{p}\right)\prod_{p \mid n}\left(1 - \frac{1}{p}\right)\prod_{\substack{p \mid n \\ p \mid m}}\left(1 - \frac{1}{p}\right)^{-1}$$

$$= \frac{\varphi(m)}{m}\frac{\varphi(n)}{n}\prod_{p \mid (n,m)}\left(1 - \frac{1}{p}\right)^{-1}$$

$$= \frac{\varphi(m)}{m}\frac{\varphi(n)}{n}\frac{d}{\varphi(d)}.$$

- Note that if m and n are prime powers and $m|n$, then it is easy to see that $\varphi(m)|\varphi(n)$. From this and multiplicativity of φ it follows generally that if $m|n$, then $\varphi(m)|\varphi(n)$.

- For a positive integer k let $A(k)$ denote the number of solutions to the equation $\varphi(n) = k$. It is easy to see that $A(k)$ is finite for every k since $\varphi(n) \to \infty$. Kevin Ford showed in a paper published in Annals of Mathematics that all of positive integers greater than 1 lie in the image of function A. It is not known if $A(k) = 1$ for some k. The answers seems to be in the negation and this is known as **Carmichael's conjecture**. Robert Carmichael originally stated this as a result instead of a conjecture in his book which he later retracted.

- The **von-Mangoldt function**, denoted Λ, is defined as

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^a \text{ for some prime } p \text{ and integer } a \geq 1 \\ 0 & \text{otherwise.} \end{cases}$$

- The von-Mangoldt function plays an important role in prime number theory. For instance, the summatory function of Λ, $\sum_{n \leq x} \Lambda(n)$, is closely related to $\zeta(s)$.

- The estimate

$$\sum_{n \leq x} \Lambda(n) = x + O(x^{1/2+\epsilon})$$

  for every positive ϵ is equivalent to Riemann Hypothesis.

- **Theorem.** If $n \geq 1$, then

$$\log n = \sum_{d \mid n} \Lambda(d).$$

- If f and g are two arithmetic functions we define their **Dirichlet product** (or *Dirichlet convolution*) to be the arithmetic function $f * g$ given by

$$(f * g)(n) = \sum_{d \mid n} f(d)g(n/d)$$

- The Dirichlet product of two arithmetic functions arises when multiplying two Dirichlet series $\sum_{n=1}^{\infty} f(n)n^{-s}$ and $\sum_{n=1}^{\infty} g(n)n^{-s}$. After multiplication and rearrangement of terms we get the new Dirichlet series $\sum_{n=1}^{\infty}(f * g)(n)n^{-s}$.

- It is easily seen that Dirichlet multiplication is both commutative and associative, i.e., for any arithmetic functions f, g, h we have

$$f * g = g * f \quad \text{and} \quad (f * g) * h = f * (g * h).$$

- We have $e * f = f$ for any arithmetic function f. Thus the set of all arithmetic functions form a commutative monoid. The next result allows us to characterize arithmetic functions that are invertible under Dirichlet multiplication.

- **Theorem.** If f is an arithmetic function with $f(1) \neq 0$, then there is a unique arithmetic function g such that $g * f = f * g = e$. The function g is given by

$$g(1) = \frac{1}{f(1)}, \qquad g(n) = -\frac{1}{f(1)} \sum_{\substack{d \mid n \\ d < n}} g(d) f(n/d) \quad \text{for } n > 1.$$

- The above result show that the set of all arithmetic functions f satisfying $f(1) \neq 0$ form an abelian group under Dirichlet multiplication.

- The Dirichlet multiplication provides a convenient notation to write relations among different arithmetic functions in a compact fashion;

$$\mu * 1 = e, \quad \varphi * 1 = N, \quad \varphi = \mu * N.$$

- **Theorem.** (Möbius inversion) Let f and g be arithmetic functions. Then
$$f(n) = \sum_{d|n} g(d)$$

  for every $n \in \mathbb{N}$ if and only if
$$g(n) = \sum_{d|n} f(d)\mu(n/d)$$

  for every $n \in \mathbb{N}$.

- **Proof.** Follow immediately by noting that $f = g * 1$ if and only if $g = f * \mu$ which is seen by multiplying by μ (or 1) and using the identity $\mu * 1 = e$.

- **Corollary.** If $n \geq 1$, then
$$\Lambda(n) = -\sum_{d|n} \mu(d) \log d.$$

- By Möbius inversion we have

$$\Lambda(n) = \sum_{d\,|\,n} \mu(d) \log \frac{n}{d} = \log n \sum_{d\,|\,n} \mu(d) - \sum_{d\,|\,n} \mu(d) \log d$$

$$= e(n) \log n - \sum_{d\,|\,n} \mu(d) \log d.$$

This completes the proof as $e(n) \log n = 0$ for every n.

Multiplicative and completely multiplicative functions

- An arithmetic function f is called **multiplicative** if f is not identically zero and

$$f(mn) = f(m)f(n) \quad \text{whenever } (m, n) = 1.$$

  A multiplicative function f is called **completely multiplicative** (or totally multiplicative) if f is not identically zero and

$$f(mn) = f(m)f(n) \quad \text{for all } m, n.$$

- The power function N^α is completely multiplicative.
- The identity function e is completely multiplicative.
- The Möbius function μ is multiplicative but not completely multiplicative as $\mu(4) = 0 \neq 1 = \mu(2)^2$.

- The Euler totient function φ is multiplicative but not completely multiplicative as $\varphi(4) = 2 \neq 1 = \varphi(2)^2$.

- **Lemma.** If f is multiplicative, then $f(1) = 1$.

- **Proof.** Let $n \in \mathbb{N}$ be such that $f(n) \neq 0$. Then $f(n) = f(1)f(n)$ as $(n, 1) = 1$ which in turn implies $f(1) = 1$.

- It immediately follows from above lemma that Λ is not multiplicative since $\Lambda(1) = 0$.

- **Theorem.** Let f be an arithmetic function with $f(1) = 1$.

  (i) f is multiplicative if and only if

  $$f(p_1^{a_1} \cdots p_k^{a_k}) = f(p_1^{a_1}) \cdots f(p_k^{a_k}),$$

  where $p_1, \ldots, p_k$ are distinct primes.

  (ii) If f is multiplicative, then f is completely multiplicative if and only if

  $$f(p^a) = f(p)^a$$

  for all primes p and all integers $a \geq 1$.

- The above result shows that a multiplicative function is uniquely determined by its values on prime powers, and a completely multiplicative function is uniquely determined by its values on primes.

- **Theorem.** If f and g are multiplicative, then so is their Dirichlet product $f * g$.

- **Proof.** Let m and n be relatively prime integers. Then observe that

$$(f * g)(mn) = \sum_{d|mn} f(d)g\left(\frac{mn}{d}\right) = \sum_{\substack{a|m \\ b|n}} f(ab)g\left(\frac{mn}{ab}\right)$$

as every divisor of mn can be uniquely written as ab, where $a|m$ and $b|n$.

- Using the multiplicativity of f and g we obtain

$$(f * g)(mn) = \sum_{\substack{a|m \\ b|n}} f(a)f(b)g\left(\frac{m}{a}\right)g\left(\frac{n}{b}\right)$$

$$= \sum_{a|m}\sum_{b|n} f(a)f(b)g\left(\frac{m}{a}\right)g\left(\frac{n}{b}\right)$$

$$= \sum_{a|m} f(a)g\left(\frac{m}{a}\right)\sum_{b|n} f(b)g\left(\frac{n}{b}\right)$$

$$= (f * g)(m)(f * g)(n).$$

- The Dirichlet product of two completely multiplicative functions need not be completely multiplicative. For instance, the divisor function $d = 1 * 1$ is not completely multiplicative as $d(4) = 3 \neq 4 = d(2)^2$ whereas 1 clearly is.

- **Theorem.** If f is multiplicative, then so is it's Dirichlet inverse f^{-1}.

- **Proof.** Suppose for the sake of contradiction that f^{-1} is not multiplicative. Then there exist positive integers m and n with $(m, n) = 1$ such that
$$f^{-1}(mn) \neq f^{-1}(m)f^{-1}(n).$$

- We choose such a pair m and n for which the product mn is the smallest. Since f is multiplicative therefore $f^{-1}(1) = 1/f(1) = 1$ and hence neither m nor n can be 1. In particular, $mn > 1$. By the construction of the product mn, $f(ab) = f(a)f(b)$ for all positive integers a and b with $(a, b) = 1$ and $ab < mn$.

- It now follows that
$$f^{-1}(mn) = - \sum_{\substack{a \mid m \\ b \mid n \\ ab < mn}} f^{-1}(ab)f\left(\frac{mn}{ab}\right)$$
$$= - \sum_{\substack{a \mid m}} f^{-1}(a)f^{-1}(b)f\left(\frac{m}{a}\right)f\left(\frac{n}{b}\right)$$

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- Splitting the sum we obtain

$$f^{-1}(mn) = -f^{-1}(n) \sum_{\substack{a \mid m \\ a < m}} f^{-1}(a) f\left(\frac{m}{a}\right)$$

$$- f^{-1}(m) \sum_{\substack{b \mid n \\ b < n}} f^{-1}(b) f\left(\frac{n}{b}\right)$$

$$- \sum_{\substack{a \mid m \\ a < m}} \sum_{\substack{b \mid n \\ b < n}} f^{-1}(a) f^{-1}(b) f\left(\frac{m}{a}\right) f\left(\frac{n}{b}\right)$$

$$= f^{-1}(n) f^{-1}(m) + f^{-1}(m) f^{-1}(n) - f^{-1}(m) f^{-1}(n)$$

$$= f^{-1}(m) f^{-1}(n).$$

This contradiction proves the result.

- There is another simpler proof of above result which goes as follows: Let g be an arithmetic function defined as

$$g(n) = \prod_{p^a \mid\mid n} f^{-1}(p^a).$$

- By definition g is a multiplicative function and so it suffices to show that $f^{-1} = g$.

- Note that

$$(g * f)(p^k) = \sum_{d \mid p^k} g(d) f(p^k/d) = \sum_{i=0}^{k} g(p^i) f(p^{k-i})$$

$$= \sum_{i=0}^{k} f^{-1}(p^i) f(p^{k-i}) = \sum_{d \mid p^k} f^{-1}(d) f(p^k/d)$$

$$= (f^{-1} * f)(p^k) = e(p^k).$$

- Because $g * f$ and e are both multiplicative functions and agree on prime powers, it follows that $g * f = e$ and so $g = f^{-1}$.

- The next result allows us to characterize completely multiplicative functions.

- **Theorem.** Let f be multiplicative. Then f is completely multiplicative if and only if $f^{-1} = \mu f$.

- **Proof.** Suppose f is completely multiplicative. Then observe that

$$(f * \mu f)(n) = \sum_{d \mid n} \mu(d) f(d) f\left(\frac{n}{d}\right) = f(n) \sum_{d \mid n} \mu(d)$$

$$= f(n) e(n) = e(n)$$

and so $f^{-1} = \mu f$.

- Conversely, assume that $f^{-1} = \mu f$. Then observe that

$$\sum_{d \mid n} \mu(d) f(d) f\left(\frac{n}{d}\right) = 0$$

for $n > 1$.

- Let $n = p^a$, where $a \geq 1$. Then, we get

$$\mu(1)f(1)f(p^a) + \mu(p)f(p)f(p^{a-1}) = 0.$$

This leads to

$$f(p^a) = f(p)f(p^{a-1}),$$

which implies that $f(p^a) = f(p)^a$. Thus f is completely multiplicative.

- The Abel's summation by parts formula is one of the most important and ubiquitous results in analytic number theory which is employed to estimate the partial sums of arithmetic functions weighted by some smooth function.

- **Theorem.** (Abel's summation by parts formula) Let $a : \mathbb{N} \to \mathbb{C}$ be an arithmetic function, let $0 < x < y$ be real numbers and $f : [x, y] \to \mathbb{C}$ be a continuously differentiable function. Then we have

$$\sum_{x < n \leq y} a(n) f(n) = A(y) f(y) - A(x) f(x) - \int_x^y A(u) f'(u) \, du,$$

where $A(u) = \sum_{n \leq u} a(n)$.

- Let $m = \lfloor x \rfloor$ and $M = \lfloor y \rfloor$. We can rewrite the weighted sum as

$$\sum_{x < n \leq y} a(n)f(n) = \sum_{n=m+1}^{M} a(n)f(n).$$

- By definition $a(n) = A(n) - A(n-1)$ so we can replace $a(n)$ to get

$$\sum_{n=m+1}^{M} a(n)f(n) = \sum_{n=m+1}^{M} (A(n) - A(n-1))f(n)$$

$$= \sum_{n=m+1}^{M} A(n)f(n) - \sum_{n=m}^{M-1} A(n)f(n+1)$$

$$= A(M)f(M) - A(m)f(m+1)$$

$$- \sum_{n=m+1}^{M-1} A(n)(f(n+1) - f(n)) \qquad (1)$$

- Since $f(n+1) - f(n) = \int_n^{n+1} f'(u)\, du$ and $A(u) = A(n)$ for all $u \in [n, n+1)$, we get

$$\sum_{m+1}^{M-1} A(n)(f(n+1) - f(n)) = \sum_{m+1}^{M-1} A(n) \int_n^{n+1} f'(u)\, du$$

$$= \sum_{m+1}^{M-1} \int_n^{n+1} A(u)f'(u)\, du$$

$$= \int_{m+1}^{M} A(u)f'(u)\, du. \qquad (2)$$

- Substituting (1) into (2), we get

$$\sum_{n=m+1}^{M} a(n)f(n) = A(M)f(M) - A(m)f(m+1)$$

$$- \int_{m+1}^{M} A(u)f'(u)\, du. \qquad (3)$$

- Using Fundamental Theorem of Calculus and observing that $A(u) = A(x)$ for $u \in [x, m+1)$, we get

$$\int_x^{m+1} A(u)f'(u)\,du = A(x)f(m+1) - A(x)f(x)$$
$$= A(m)f(m+1) - A(x)f(x). \quad (4)$$

- Doing a similar calculation for $\int_M^y A(u)f'(u)\,du$ yields

$$\int_M^y A(u)f'(u)\,du = A(y)f(y) - A(M)f(M). \quad (5)$$

Using (4) and (5), one can easily turn (3) into the required form.

- **Corollary.** Let $a : \mathbb{N} \to \mathbb{C}$ be an arithmetic function and let $f : [1, x] \to \mathbb{C}$ be a continuously differentiable function where $x \geq 1$. Then we have

$$\sum_{n \leq x} a(n)f(n) = A(x)f(x) - \int_1^x A(u)f'(u)\,du.$$

- We now employ the Abel's summation to estimate partial sums of various harmonic sums.

- **Theorem.** If $x \geq 1$, then we have

$$\sum_{n \leq x} \frac{1}{n} = \log x + \gamma + O\left(\frac{1}{x}\right), \tag{6}$$

where γ is the Euler-Mascheroni constant.

- **Proof.** Taking $a(n) = 1$ and $f(x) = 1/x$ in the summation by parts formula, we get

$$\sum_{n \leq x} \frac{1}{n} = \frac{\lfloor x \rfloor}{x} + \int_1^x \frac{\lfloor u \rfloor}{u^2} \, du. \tag{7}$$

- Substituting $\lfloor x \rfloor = x - \{x\}$ in (7), we get

$$\sum_{n \leq x} \frac{1}{n} = 1 - \frac{\{x\}}{x} + \int_1^x \frac{du}{u} - \int_1^x \frac{\{u\}}{u^2}\, du$$

$$= 1 + O\left(\frac{1}{x}\right) + \log x - \int_1^x \frac{\{u\}}{u^2}\, du$$

$$= 1 + O\left(\frac{1}{x}\right) + \log x - \int_1^\infty \frac{\{u\}}{u^2}\, du + \int_x^\infty \frac{\{u\}}{u^2}\, du.$$

- Taking $C = 1 - \int_1^\infty \{u\} u^{-2}\, du$, we obtain

$$\sum_{n \leq x} \frac{1}{n} = \log x + C + O\left(\frac{1}{x}\right) + \int_x^\infty \frac{\{u\}}{u^2}\, du.$$

- We can bound the improper integral as

$$\int_x^\infty \frac{\{u\}}{u^2}\, du \leq \int_x^\infty \frac{du}{u^2} = \frac{1}{x}$$

and so

$$\int_x^\infty \frac{\{u\}}{u^2}\, du = O\left(\frac{1}{x}\right).$$

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- It thus follows that

$$\sum_{n \leq x} \frac{1}{n} = \log x + C + O\left(\frac{1}{x}\right).$$

It can be easily seen by taking limit as x approaches ∞ that $C = \gamma$.

- Note that in the above proof we obtained the following integral expression

$$\gamma = 1 - \int_1^\infty \frac{\{u\}}{u^2} \, du.$$

- **Theorem.** If $x \geq 1$, then for any $s \in \mathbb{C}$ with $s \neq 1$ and $\sigma > 0$ we have

$$\sum_{n \leq x} \frac{1}{n^s} = \frac{x^{1-s}}{1-s} + \frac{s}{s-1} - s \int_1^\infty \frac{\{u\}}{u^{s+1}} \, du + O(x^{-\sigma}), \quad (8)$$

where the implicit constant depends on s.

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- **Proof.** We apply the Abel's summation by parts formula with $a(n) = 1$ and $f(x) = x^{-s}$. For $x \geq 1$ we then get

$$\sum_{n \leq x} \frac{1}{n^s} = \frac{\lfloor x \rfloor}{x^s} + s \int_1^x \frac{\lfloor u \rfloor}{u^{s+1}} \, du$$

Substituting $\lfloor x \rfloor = x - \{x\}$, we obtain

$$\sum_{n \leq x} \frac{1}{n^s} = x^{1-s} - \frac{\{x\}}{x^s} + s \int_1^x \frac{du}{u^s} - s \int_1^x \frac{\{u\}}{u^{s+1}} \, du$$

$$= x^{1-s} + s \left(\frac{x^{1-s}}{1-s} - \frac{1}{1-s} \right) - s \int_1^x \frac{\{u\}}{u^{s+1}} \, du$$
$$+ O(x^{-\sigma})$$

$$= \frac{x^{1-s}}{1-s} + \frac{s}{s-1} - s \int_1^\infty \frac{\{u\}}{u^{s+1}} \, du + s \int_x^\infty \frac{\{u\}}{u^{s+1}} \, du$$
$$+ O(x^{-\sigma}). \qquad (9)$$

- Finally note that

$$\left| \int_x^\infty \frac{\{u\}}{u^{s+1}} \, du \right| \leq \int_x^\infty \frac{du}{u^{\sigma+1}} = \frac{x^{-\sigma}}{\sigma}.$$

  This leads to the desired estimate.

- The Riemann zeta function, $\zeta(s)$, can be defined for $\sigma > 0$ by

$$\zeta(s) = \frac{s}{s-1} - s \int_1^\infty \frac{\{u\}}{u^{s+1}} \, du$$

- If $\sigma > 1$, then it follows from (8) that

$$\zeta(s) = \sum_{n=1}^\infty \frac{1}{n^s}.$$

- **Corollary.** If $x \geq 1$ and $\sigma > 1$, then we have

$$\sum_{n>x} \frac{1}{n^s} = O(x^{1-\sigma}),$$

  where the implicit constant depends on s.

- **Theorem.** If $x \geq 1$, then for any $s \in \mathbb{C}$ with $\sigma > 0$ we have
$$\sum_{n \leq x} n^s = \frac{x^{1+s}}{1+s} + O(x^{\sigma}), \qquad (10)$$
where the implicit constant depends on s.

- **Proof.** Applying the Abel summation by parts formula with $a(n) = 1$ and $f(x) = x^s$, we get
$$\sum_{n \leq x} n^s = \lfloor x \rfloor x^s - s \int_1^x \lfloor u \rfloor u^{s-1} \, du.$$

- Substituting $\lfloor x \rfloor = x - \{x\}$, we obtain
$$\sum_{n \leq x} n^s = x^{s+1} - \{x\}x^s - s \int_1^x u^s \, du + s \int_1^x \{u\}u^{s-1} \, du$$

$$= x^{s+1} + O(x^{\sigma}) - s\left(\frac{x^{1+s}}{1+s} - \frac{1}{1+s}\right)$$

$$+ s \int_1^x \{u\}u^{s-1} \, du. \qquad (11)$$

- Now note that

$$\left| s \int_1^x \{u\} u^{s-1} \, du \right| \leq |s| \int_1^x u^{\sigma-1} \, du = \frac{|s|}{\sigma}(x^\sigma - 1) = O_s(x^\sigma).$$

- Hence (11) simplifies to

$$\sum_{n \leq x} n^s = \frac{x^{1+s}}{1+s} + \frac{s}{1+s} + O(x^\sigma)$$

Observe that the error term $O(x^\sigma)$ absorbs the constant $s/(1+s)$ and thus we get the desired result.

- The **divisor function** $d(n)$ counts the number of divisors of n. It is a completely multiplicative function. If $n = p_1^{a_1} \cdots p_k^{a_k}$, then $d(n) = (a_1 + 1) \cdots (a_k + 1)$.
- It can be easily seen that the average order of $d(n)$ is $\log n$. Note that

$$\sum_{n \leq x} d(n) = \sum_{n \leq x} \sum_{d \mid n} 1 = \sum_{n \leq x} \sum_{qd = n} 1 = \sum_{qd \leq x} 1.$$

- Thus the divisor sum can be written as

$$\sum_{n \leq x} d(n) = \sum_{d \leq x} \sum_{q \leq x/d} 1 = \sum_{d \leq x} \left(\frac{x}{d} + O(1) \right) = x \sum_{d \leq x} \frac{1}{d} + O(x).$$

- Using the asymptotic formula for the harmonic sum we get

$$\sum_{n \leq x} d(n) = x \left(\log x + \gamma + O\left(1/x\right) \right) + O(x) = x \log x + O(x).$$

- Thus we find that

$$\frac{1}{x} \sum_{n \leq x} d(n) \sim \log x$$

  as $x \to \infty$.

- Dirichlet obtained a sharper estimate for $\sum_{d \leq x} d(n)$ with the error term being $O(\sqrt{x})$. We prove the Dirichlet's result below.

- **Theorem.** For all $x \geq 1$, we have

$$\sum_{n \leq x} d(n) = x \log x + (2\gamma - 1)x + O(\sqrt{x}), \qquad (12)$$

  where γ is the Euler-Mascheroni constant.

- **Proof.** The slick trick to prove this stronger estimate is to exploit the symmetry of q and d in the sum

$$\sum_{n \leq x} d(n) = \sum_{qd \leq x} 1.$$

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- We can split this sum as

$$\sum_{n \leq x} d(n) = \sum_{qd \leq x} 1 = \sum_{\substack{qd \leq x \\ d \leq q}} 1 + \sum_{\substack{qd \leq x \\ q \leq d}} 1 - \sum_{\substack{qd \leq x \\ d = q}} 1$$

$$= 2 \sum_{\substack{qd \leq x \\ d \leq q}} 1 - \sum_{d \leq \sqrt{x}} 1$$

- Now we first sum over d and then over q to get

$$\sum_{n \leq x} d(n) = 2 \left(\sum_{d \leq \sqrt{x}} \sum_{d \leq q \leq x/d} 1 \right) - \lfloor \sqrt{x} \rfloor$$

$$= 2 \sum_{d \leq \sqrt{x}} \left(\left\lfloor \frac{x}{d} \right\rfloor - d + 1 \right) - \lfloor \sqrt{x} \rfloor$$

$$= 2 \sum_{d \leq \sqrt{x}} \left(\left\lfloor \frac{x}{d} \right\rfloor - d \right) + \lfloor \sqrt{x} \rfloor$$

$$= 2 \sum_{d \leq \sqrt{x}} \left(\frac{x}{d} - d + O(1) \right) + O(\sqrt{x}).$$

- Thus we end up with

$$\sum_{n \le x} d(n) = 2x \sum_{d \le \sqrt{x}} \frac{1}{d} - 2 \sum_{d \le \sqrt{x}} d + O(\sqrt{x})$$

- Finally using estimates for harmonic and power sums we obtain

$$\sum_{n \le x} d(n) = 2x \left(\log \sqrt{x} + \gamma + O\left(\frac{1}{\sqrt{x}} \right) \right)$$

$$- 2 \left(\frac{x}{2} + O(\sqrt{x}) \right) + O(\sqrt{x})$$

$$= x \log x + (2\gamma - 1)x + O(\sqrt{x}),$$

which is the desired result.

- Let

$$\Delta(x) = \sum_{n \le x} d(n) - x \log x - (2\gamma - 1)x.$$

The **Dirichlet divisor problem** asks to find the smallest θ for which $\Delta(x) \ll x^{\theta + \epsilon}$ for every $\epsilon > 0$

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- In 1904, Voronoi using his Voronoi summation formula showed that $\Delta(x) \ll x^{1/3} \log x$.

- In 1916, G. H. Hardy showed that $\Delta(x) = \Omega_{\pm}(x^{1/4})$, i.e.,

$$\limsup_{x \to \infty} \frac{\Delta(x)}{x^{1/4}} > 0 \quad \text{and} \quad \liminf_{x \to \infty} \frac{\Delta(x)}{x^{1/4}} < 0.$$

- In 1922, Van der Corput showed that θ can taken to be 0.33.

- The current record (relying on Wikipedia here) is due to Huxley who showed in 2003 that θ can taken to be $131/416 = 0.3149...$ For the intermediate progress see the Wikipedia article "Divisor summatory function".

- For $x > 0$, the **Chebyshev's ψ-function** and **Chebyshev's ϑ-function** are defined as

$$\psi(x) = \sum_{p^k \leq x} \log p, \qquad \vartheta(x) = \sum_{p \leq x} \log p.$$

These are weighted sum of characteristic function of primes.

- We can rewrite $\psi(x)$ in terms of von-Mangoldt function as

$$\psi(x) = \sum_{n \leq x} \Lambda(n).$$

- The Chebyshev's functions $\psi(x)$ and $\vartheta(x)$ are related to each other due to Möbius inversion as shown by the result below.

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- **Theorem.** For $x > 0$ we have

$$\psi(x) = \sum_k \vartheta(x^{1/k}), \qquad (13)$$

$$\vartheta(x) = \sum_k \mu(k)\psi(x^{1/k}) \qquad (14)$$

- **Proof.** Note that

$$\psi(x) = \sum_{p^k \leq x} \log p = \sum_k \sum_{p \leq x^{1/k}} \log p = \sum_k \vartheta(x^{1/k}).$$

- The identity (14) follows by substituting (13) in the sum;

$$\sum_k \mu(k)\psi(x^{1/k}) = \sum_k \mu(k) \sum_\ell \vartheta(x^{1/k\ell}) = \sum_{k,\ell} \mu(k)\vartheta(x^{1/k\ell}).$$

- Collecting terms for which $k\ell$ is fixed we get

$$\sum_k \mu(k)\psi(x^{1/k}) = \sum_m \vartheta(x^{1/m}) \sum_{k|m} \mu(k) = \vartheta(x).$$

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- **Theorem.** For $x \geq 2$ we have

$$\psi(x) = \vartheta(x) + O(x^{1/2} \log x). \tag{15}$$

- **Proof.** Note that

$$\psi(x) = \sum_k \vartheta(x^{1/k}) = \sum_{k \leq \log_2 x} \vartheta(x^{1/k})$$

as $x^{1/k} < 2$ for $k > \log_2 x$ in which case $\vartheta(x^{1/k}) = 0$.

- Now using the crude estimate $\vartheta(x) \ll x \log x$ we see that

$$\psi(x) - \vartheta(x) = \sum_{2 \leq k \leq \log_2 x} \vartheta(x^{1/k}) \ll \sum_{2 \leq k \leq \log_2 x} x^{1/k} \log x^{1/k}$$

$$\ll x^{1/2} \log x,$$

where the last estimate follows as the first term is of order $x^{1/2} \log x$ and the sum of the rest of the terms is $\ll x^{1/3} \log^2 x \ll x^{1/2} \log x$.

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- The Chebyshev's functions are closely related to the prime counting function, $\pi(x)$, as indicated by the next result.

- **Theorem.** We have

$$\pi(x) \sim \frac{x}{\log x} \quad \Leftrightarrow \quad \psi(x) \sim x \quad \Leftrightarrow \quad \vartheta(x) \sim x.$$

as $x \to \infty$.

- **Proof.** The equivalence $\psi(x) \sim x \Leftrightarrow \vartheta(x) \sim x$ follows immediately from (15).

- Using Abel's summation by parts formula we have

$$\vartheta(x) = \sum_{p \leq x} \log p = \sum_{n \leq x} 1_{\mathbb{P}}(n) \log n$$
$$= \pi(x) \log x - \int_2^x \frac{\pi(u)}{u} \, du,$$

where $1_{\mathbb{P}}$ is the characteristic function of primes.

- Now suppose that $\pi(x) \sim x/\log x$. Then we have

$$\int_2^x \frac{\pi(u)}{u}\, du \ll \int_2^x \frac{du}{\log u} = \left(\int_2^{\sqrt{x}} + \int_{\sqrt{x}}^x \right) \frac{du}{\log u}$$

$$\leq \frac{\sqrt{x}}{\log 2} + \frac{x}{\log \sqrt{x}} \ll \frac{x}{\log x}$$

- Thus we have

$$\frac{\vartheta(x)}{x} = \frac{\pi(x)\log x}{x} - \frac{1}{x} \int_2^x \frac{\pi(u)}{u}\, du$$

$$= \frac{\pi(x)\log x}{x} + O\left(\frac{1}{\log x} \right)$$

It immediately follows from this that $\vartheta(x) \sim x$.

- Now suppose that $\vartheta(x) \sim x$. Again using Abel's summation by parts formula we have

$$\pi(x) = \sum_{2 \leq n \leq x} \frac{1_{\mathbb{P}}(n)\log n}{\log n} = \frac{\vartheta(x)}{\log x} + \int_2^x \frac{\vartheta(u)}{u(\log u)^2}\, du.$$

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- Since $\vartheta(x) \ll x$ we have

$$\int_2^x \frac{\vartheta(u)}{u(\log u)^2}\, du \ll \int_2^x \frac{du}{(\log u)^2} = \left(\int_2^{\sqrt{x}} + \int_{\sqrt{x}}^x\right) \frac{du}{(\log u)^2}$$

$$\leq \frac{\sqrt{x}}{(\log 2)^2} + \frac{x}{(\log \sqrt{x})^2} \ll \frac{x}{\log^2 x}.$$

- Hence we have

$$\frac{\pi(x)\log x}{x} = \frac{\vartheta(x)}{x} + \frac{\log x}{x}\int_2^x \frac{du}{u(\log u)^2}$$

$$= \frac{\vartheta(x)}{x} + O\left(\frac{1}{\log x}\right).$$

Since $\vartheta(x) \sim x$, we immediately obtain $\pi(x) \sim x/\log x$.

- It can be shown using properties of the middle binomial coefficient $\binom{n}{\lfloor n/2 \rfloor}$ that $\vartheta(x) \ll x$, which in turn implies $\psi(x) \ll x$ due to estimate (15).

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- In 1896, la Vallée Poussin showed that

$$\psi(x) = x + O(x \exp(-c\sqrt{\log x}))$$

for some positive constant c using the following zero-free region of $\zeta(s)$:

$$\zeta(s) \neq 0 \qquad \text{for} \qquad \sigma \geq 1 - \frac{c_1}{\log \tau}$$

for some positive constant c_1, where $\tau = |t| + 2$. This is usually referred as classical zero-free region.

- In 1958, Vinogradov and Korobov independently showed that

$$\zeta(s) \neq 0 \qquad \text{for} \qquad \sigma \geq 1 - \frac{c_2}{(\log \tau)^{2/3+\epsilon}}$$

for every $\epsilon > 0$, where c_2 is a positive constant depending on ϵ. It follows from this that

$$\psi(x) = x + O(x \exp(-c_3(\log x)^{3/5-\epsilon}))$$

for every $\epsilon > 0$, where c_3 is a positive constant depending on ϵ.

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- **Theorem.** For $x \geq 1$ we have
  (i)
  $$\sum_{n \leq x} \log n = x \log x - x + O(\log x),$$

  (ii)
  $$\sum_{n \leq x} \frac{\Lambda(n)}{n} = \log x + O(1),$$

  (iii)
  $$\sum_{p \leq x} \frac{\log p}{p} = \log x + O(1).$$

- **Proof.** By Abel's summation by parts formula we have

  $$\sum_{n \leq x} \log n = \lfloor x \rfloor \log x - \int_1^x \frac{\lfloor u \rfloor}{u} \, du$$

  $$= x \log x - \{x\} \log x - (x - 1) + \int_1^x \frac{\{u\}}{u} \, du$$

  $$= x \log x - x + O(\log x).$$

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- We note that

$$\sum_{n \leq x} \log n = \sum_{n \leq x} \sum_{d \mid n} \Lambda(d) = \sum_{d \leq x} \Lambda(d) \sum_{\substack{n \leq x \\ d \mid n}} 1$$

$$= \sum_{d \leq x} \Lambda(d) \left\lfloor \frac{x}{d} \right\rfloor = x \sum_{d \leq x} \frac{\Lambda(d)}{d} + O\left(\sum_{d \leq x} \Lambda(d) \right)$$

By Chebyshev estimate we have $\psi(x) = \sum_{d \leq x} \Lambda(d) \ll x$.

- It now follows that

$$\sum_{n \leq x} \log n = x \sum_{d \leq x} \frac{\Lambda(d)}{d} + O(x).$$

Now using the estimate in part (i) we immediately obtain the desired estimate.

- For part (iii) we note that

$$\sum_{n \leq x} \frac{\Lambda(n)}{n} = \sum_{p^k \leq x} \frac{\log p}{p^k} = \sum_{p \leq x} \frac{\log p}{p} + \sum_{\substack{p^k \leq x \\ k \geq 2}} \frac{\log p}{p^k}.$$

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- We now observe that the contribution of prime powers with exponent greater than 1 is very small. We find that

$$\sum_{\substack{p^k \le x \\ k \ge 2}} \frac{\log p}{p^k} \le \sum_{p \le x} \log p \sum_{k=2}^{\infty} \frac{1}{p^k} = \sum_{p \le x} \frac{\log p}{p(p-1)} \le \sum_{n=2}^{\infty} \frac{\log n}{n(n-1)}$$

- Since the series to the right converges we get

$$\sum_{\substack{p^k \le x \\ k \ge 2}} \frac{\log p}{p^k} \ll 1$$

and so we have

$$\sum_{n \le x} \frac{\Lambda(n)}{n} = \sum_{p \le x} \frac{\log p}{p} + O(1).$$

Using the estimate in part (ii) we readily get the desired estimate.

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- The part (iii) of above theorem is usually referred as Merten's first estimate.

- **Theorem.** For $x \geq 2$, we have

$$\sum_{p \leq x} \frac{1}{p} = \log \log x + c + O\left(\frac{1}{\log x}\right)$$

for some constant c.

- **Proof.** Let $1_{\mathbb{P}}$ denote the characteristic function of primes. Then we apply the Abel's summation and obtain

$$\sum_{p \leq x} \frac{1}{p} = \sum_{3/2 < n \leq x} \frac{1_{\mathbb{P}}(n) \log n}{n} \frac{1}{\log n}$$

$$= \frac{1}{\log x} \sum_{p \leq x} \frac{\log p}{p} + \int_{3/2}^{x} \left(\sum_{p \leq u} \frac{\log p}{p} \right) \frac{1}{u \log^2 u} \, du.$$

- Using Merten's first estimate, it follows that

$$\sum_{p \leq x} \frac{1}{p} = 1 + O\left(\frac{1}{\log x}\right) + \int_2^x \frac{\log u + O(1)}{u \log^2 u}\, du$$

$$= 1 + O\left(\frac{1}{\log x}\right) + \int_2^x \frac{du}{u \log u} + \int_2^x \frac{O(1)}{u \log^2 u}\, du$$

$$= \log\log x + 1 - \log\log 2 + O\left(\frac{1}{\log x}\right)$$

$$+ \int_2^x \frac{O(1)}{u \log^2 u}\, du.$$

- Observe that

$$\int_2^x \frac{O(1)}{u \log^2 u}\, du = \int_2^\infty \frac{O(1)}{u \log^2 u}\, du - \int_x^\infty \frac{O(1)}{u \log^2 u}\, du$$

and

$$\int_x^\infty \frac{O(1)}{u \log^2 u}\, du = O\left(\int_x^\infty \frac{du}{u \log^2 u}\right) = O\left(\frac{1}{\log x}\right).$$

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- Hence, we get

$$\sum_{p \leq x} \frac{1}{p} = \log \log x + 1 - \log \log 2 + \int_2^\infty \frac{O(1)}{u \log^2 u} \, du$$
$$+ O\left(\frac{1}{\log x}\right),$$

that is,

$$c = 1 - \log \log 2 + \int_2^\infty \frac{O(1)}{u \log^2 u} \, du.$$

- Let
$$M(x) = \sum_{n \leq x} \mu(x).$$

Next we show that prime number theorem is equivalent to the estimate $M(x) = o(x)$.

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- **Theorem.** We have $\psi(x) \sim x$ if and only if $M(x) = o(x)$.
- **Proof.** Assume that $\psi(x) \sim x$ and let

$$H(x) = \sum_{n \leq x} \mu(n) \log n.$$

- By Abel's summation by parts formula we have

$$H(x) = M(x) \log x - \int_1^x \frac{M(u)}{u}\, du.$$

and so using the trivial estimate $M(u) \ll u$ we obtain

$$\frac{H(x)}{x \log x} = \frac{M(x)}{x} + O\left(\frac{1}{\log x}\right)$$

Hence, it suffices to show that $H(x) = o(x \log x)$ as $x \to \infty$.

- It now follows by Möbius inversion that

$$\mu(n) \log n = - \sum_{qd=n} \mu(d) \Lambda(q).$$

- Using this we get that

$$H(x) = -\sum_{n\leq x}\sum_{qd=n}\mu(d)\Lambda(q) = -\sum_{qd\leq x}\mu(d)\Lambda(q)$$

$$= -\sum_{d\leq x}\mu(d)\psi\left(\frac{x}{d}\right).$$

- By the assumption there is a function $f(x) = o(1)$ such that $\psi(x) - x = xf(x)$. We take $1 < y < x$ to be a quantity that depends on x and the choice of which will be made later. We split the sum as

$$H(x) = -\sum_{n\leq y}\mu(n)\psi\left(\frac{x}{d}\right) - \sum_{y<n\leq x}\mu(n)\psi\left(\frac{x}{d}\right)$$

and bound each term separately.

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- First note that

$$\left| \sum_{n \leq y} \mu(n) \psi\left(\frac{x}{n}\right) \right| = \left| \sum_{n \leq y} \mu(n) \left(\frac{x}{n} + \psi\left(\frac{x}{n}\right) - \frac{x}{n}\right) \right|$$

$$\leq x \left| \sum_{n \leq y} \frac{\mu(n)}{n} \right| + \sum_{n \leq y} \left| \psi\left(\frac{x}{n}\right) - \frac{x}{n} \right|.$$

- Let $\overline{f}(x) = \sup_{u \geq x} |f(u)|$. Note that $\overline{f}(x) = o(1)$. It now follows from $\left| \sum_{n \leq x} \frac{\mu(n)}{n} \right| \leq 1$ that

$$\sum_{n \leq y} \mu(n) \psi\left(\frac{x}{n}\right) \ll x + x \sum_{n \leq y} \frac{1}{n} \left| f\left(\frac{x}{n}\right) \right|$$

$$\ll x + x \log y \, \overline{f}\left(\frac{x}{y}\right).$$

- For the other sum we observe that

$$\sum_{y < n \leq x} \mu(n) \psi\left(\frac{x}{n}\right) \ll x \psi\left(\frac{x}{y}\right).$$

- Taking $y = x / \log\log x$ we see that

$$\sum_{n \leq y} \mu(n) \psi\left(\frac{x}{n}\right) \ll x + x \log x \overline{f}(\log\log x).$$

  which implies that

$$\sum_{n \leq y} \mu(n) \psi\left(\frac{x}{n}\right) = o(x \log x)$$

  as $\overline{f}(\log\log x) = o(1)$.

- For the other sum using the estimate $\psi(x) \ll x$ which follows from assumption we find that

$$\sum_{y < n \leq x} \mu(n) \psi\left(\frac{x}{n}\right) \ll x \log\log x$$

  and so this sum is also $o(x \log x)$. Hence, we have $H(x) = o(x \log x)$.

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- Now suppose that $M(x) = o(x)$. Note that

$$\psi(x) - \lfloor x \rfloor + 2\gamma = \sum_{n \leq x} (\Lambda(n) - 1(n) + 2\gamma e(n)).$$

- Using the identities $\Lambda = \mu * \log$, $1 = \mu * d$, and $e = 1 * \mu$ we get

$$\psi(x) - \lfloor x \rfloor + 2\gamma = \sum_{n \leq x} \sum_{qd=n} \mu(d)(\log q - d(q) + 2\gamma)$$

$$= \sum_{dq \leq x} \mu(d)\left(\log q - d(q) + 2\gamma\right)$$

$$= \sum_{q \leq x} M\left(\frac{x}{q}\right)\left(\log q - d(q) + 2\gamma\right).$$

- By assumption there is a function $g(x) = o(1)$ such that $M(x) = xg(x)$. Again we take $1 < y < x$ to be some quantity depending on x which will be chosen later.

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- We now split the sum as

$$\psi(x) - \lfloor x \rfloor + 2\gamma = \sum_{q \leq y} M\left(\frac{x}{q}\right)(\log q - d(q) + 2\gamma)$$

$$+ \sum_{\substack{d \leq x/y \\ y < q \leq x/d}} \mu(d)(\log q - d(q) + 2\gamma).$$

- Note that we can bound the first sum as

$$\left| \sum_{q \leq y} M\left(\frac{x}{q}\right)(\log q - d(q) + 2\gamma) \right|$$

$$\leq x \sum_{q \leq y} \frac{1}{q}\left| g\left(\frac{x}{q}\right) \right| |\log q - d(q) + 2\gamma|.$$

- Let $\overline{g}(x) = \sup_{u \geq x}|g(u)|$. Note that $\overline{g}(x) = o(1)$. It then follows that the first sum is

$$\ll xy \log y\, \overline{g}\left(\frac{x}{y}\right) \ll xy^2\, \overline{g}\left(\frac{x}{y}\right)$$

- Let $\Delta^*(x) = \sum_{n \le x} (d(n) - \log n + 2\gamma)$. Observe that for the other sum we have

$$
\left| \sum_{\substack{d \le x/y \\ y < q \le x/d}} \mu(d)(\log q - d(q) + 2\gamma) \right|
$$

$$
\le \sum_{d \le x/y} \left| \sum_{y < q \le x/d} \log q - d(q) + 2\gamma \right|
$$

$$
= \sum_{d \le x/y} |\Delta^*(x/d) - \Delta^*(y)|
$$

$$
\ll \sum_{d \le x/y} \left(\frac{\sqrt{x}}{\sqrt{d}} + \sqrt{y} \right) \ll \frac{x}{\sqrt{y}}.
$$

- It is easy to see that $\overline{g}$ is a decreasing function and $\overline{g}(x) > 0$ for every $x \geq 1$ for if $\overline{g}(x_0) = 0$ for some $x_0 \geq 1$, then $g(x) = 0$ eventually which in turn implies that $\mu(n) = 0$ eventually, a contradiction.

- We take $y = \min\left(\sqrt{x}, \overline{g}(\sqrt{x})^{-1/3}\right)$ and note that

$$xy^2\,\overline{g}\left(\frac{x}{y}\right) \leq xy^2\,\overline{g}(\sqrt{x}) = x\,\overline{g}(\sqrt{x})^{1/3} = o(x).$$

- Since $\overline{g}(\sqrt{x}) = o(1)$ we obtain that $y \to \infty$ as $x \to \infty$ and so $x/\sqrt{y} = o(x)$. Thus we have shown that

$$\psi(x) - \lfloor x \rfloor + 2\gamma = o(x),$$

which implies $\psi(x) \sim x$.

- **Problem.** Let $p_1, \ldots, p_k$ be distinct primes. Show that the number of positive integers up to x with prime factorization of the form $p_1^{a_1} \cdots p_k^{a_k}$ is $\ll (\log x)^k$. Conclude that primes cannot be finite.

- **Solution.** If $n = p_1^{a_1} \cdots p_k^{a_k}$ and $n \leq x$, then $a_i \leq \log_2 x$ for every $1 \leq i \leq k$. Thus the number of integers up to x must be $\leq (\lfloor \log_2 x \rfloor + 1)^k \ll (\log x)^k$. But this is clearly a contradiction as $x \not\ll (\log x)^k$.

- **Problem.** Show that for every $k \in \mathbb{N}$ there are infinitely many n such that

$$\mu(n+1) = \cdots = \mu(n+k).$$

- **Solution.** Let $p_1, \ldots, p_k$ be distinct primes. Then by the Chinese Remainder Theorem there exist infinitely many positive integers n such that $n \equiv -j \mod p_j^2$ for every $1 \le j \le k$. Thus $p_j^2 | (n+j)$ for every $1 \le j \le k$ and so $n+j$ is not squarefree, i.e., $\mu(n+1) = \cdots = \mu(n+k) = 0$.

- **Problem.** Show that

$$\sum_{d^k \mid n} \mu(d) = \begin{cases} 1 & \text{if } n \text{ is } k\text{th power-free,} \\ 0 & \text{otherwise.} \end{cases}$$

- **Solution.** Let $n = m^k q$, where q is kth power-free. Then observe that $d^k \mid n$ if and only if $d \mid m$. Thus

$$\sum_{d^k \mid n} \mu(d) = \sum_{d \mid m} \mu(d) = e(m).$$

The assertion now follows as n is kth power-free if and only if $m = 1$.

- **Problem.** Show that $\omega * \mu$ is the characteristic function of primes whereas $\Omega * \mu$ is the characteristic function of prime powers.

- **Solution.** Let $1_{\mathbb{P}}$ denote the characteristic function of primes. Then note that $\omega(n) = \sum_{d \mid n} 1_{\mathbb{P}}(d)$, i.e., $\omega = 1_{\mathbb{P}} * 1$ which can be written as $\omega * \mu = 1_{\mathbb{P}}$ by Möbius inversion.

- Similarly, if $1_{\mathcal{P}}$ denotes the characteristic function of prime powers, then $\Omega(n) = \sum_{d \mid n} 1_{\mathcal{P}}(d)$, i.e., $\Omega = 1_{\mathcal{P}} * 1$ and so we have $\Omega * \mu = 1_{\mathcal{P}}$.

- **Problem.** Show that

$$\prod_{t \mid n} t = n^{d(n)/2}.$$

- **Solution.** Follows by observing that

$$\left(\prod_{t \mid n} t \right)^2 = \left(\prod_{t \mid n} \frac{n}{t} \right) \left(\prod_{t \mid n} t \right) = \prod_{t \mid n} n = n^{d(n)}.$$

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- **Problem.** Show that

$$\sum_{t \mid n} d(t)^3 = \left(\sum_{t \mid n} d(t) \right)^2.$$

- **Solution.** Denote the left-hand side by $f(n)$ and the right-hand side by $g(n)$. Note that both f and g are multiplicative and so it suffices to show that two coincide at prime powers.

- Observe that

$$
\begin{aligned}
f(p^a) = \sum_{t \mid p^a} d(t)^3 &= \sum_{k=1}^{a+1} k^3 \\
&= \left(\sum_{k=1}^{a+1} k \right)^2 = \left(\sum_{t \mid p^a} d(t) \right)^2 \\
&= g(p^a).
\end{aligned}
$$

- **Problem.** Show that $\varphi(n) \to \infty$ as $n \to \infty$.
- **Solution.** Let $M > 0$ and let $\varphi(n) \leq M$. Take $n = \prod_{i=1}^{k} p_i^{a_i}$. Then we have

$$\varphi(n) = \prod_{i=1}^{k} p_i^{a_i-1}(p_i - 1) \leq M.$$

This shows that $p_i - 1 \leq M$ and $2^{a_i-1} \leq M$ for every $1 \leq i \leq k$.

- Hence we obtain that $p_i \leq M + 1$ and $2^{a_i} \leq 2M$ for every $1 \leq i \leq k$. Thus the exponents a_i are bounded by $\log_2(2M)$. Consequently, there are only finitely many positive integers n with $\varphi(n) \leq M$. We conclude that $\varphi(n) \to \infty$ as $n \to \infty$.

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- **Problem.** Let us denote $e(\alpha) = e^{2\pi i \alpha}$. Show that

$$\frac{1}{q} \sum_{a=1}^{q} e\left(\frac{an}{q}\right) = \begin{cases} 1 & \text{if } q \mid n, \\ 0 & \text{otherwise.} \end{cases}$$

- **Solution.** Note that if $q \mid n$, then $e(an/q) = 1$ for every $1 \leq a \leq q$ and so we have

$$\frac{1}{q} \sum_{a=1}^{q} e(an/q) = \frac{1}{q} \sum_{a=1}^{q} 1 = 1.$$

- Now suppose that $q \nmid n$. Then we have $e(n/q) \neq 1$ and so

$$\frac{1}{q} \sum_{a=1}^{q} e(an/q) = \frac{1}{q} \sum_{a=1}^{q} e(n/q)^a = \frac{1}{q} \left(\frac{e(n/q)^{q+1} - 1}{e(n/q) - 1} - 1 \right)$$

$$= 0$$

as $e(n/q)^{q+1} = e(n/q)$.

- **Problem.** The *Ramanujan's sum* $c_q(n)$ is defined as

$$c_q(n) = \sum_{\substack{a=1 \\ (a,q)=1}}^{q} e\left(\frac{an}{q}\right).$$

Show that

$$c_q(n) = \sum_{d|(n,q)} d\mu\left(\frac{q}{d}\right).$$

Deduce that

$$\mu(q) = \sum_{\substack{a=1 \\ (a,q)=1}}^{q} e\left(\frac{a}{q}\right).$$

- **Solution.** Observe that

$$c_q(n) = \sum_{\substack{a=1 \\ (a,q)=1}}^{q} e\left(\frac{an}{q}\right) = \sum_{a=1}^{q} e\left(\frac{an}{q}\right) \sum_{d|(a,q)} \mu(d)$$

$$= \sum_{a=1}^{q} e\left(\frac{an}{q}\right) \sum_{\substack{d|a \\ d|q}} \mu(d).$$

- Changing the order of summation we get

$$c_q(n) = \sum_{d \mid q} \mu(d) \sum_{\substack{a=1 \\ d \mid a}}^{q} e\left(\frac{an}{q}\right) = \sum_{d \mid q} \mu(d) \sum_{r=1}^{q/d} e\left(\frac{rdn}{q}\right)$$

$$= \sum_{d \mid q} \mu(d) \sum_{r=1}^{q/d} e\left(\frac{rn}{q/d}\right).$$

- Thus we can rewrite $c_q(n)$ as

$$c_q(n) = \sum_{d \mid q} \mu\left(\frac{q}{d}\right) \sum_{r=1}^{d} e\left(\frac{rn}{d}\right).$$

- Using the identity $\frac{1}{q} \sum_{a=1}^{q} e(an/q) = \mathbf{1}_{q \mid n}$ we end up with

$$c_q(n) = \sum_{\substack{d \mid q \\ d \mid n}} d\mu\left(\frac{q}{d}\right) = \sum_{d \mid (q,n)} d\mu\left(\frac{q}{d}\right).$$

- Finally, note that

$$\mu(q) = c_q(1) = \sum_{\substack{a=1 \\ (a,q)=1}}^{q} e\left(\frac{a}{q}\right).$$

- **Problem.** Let $\ell = (n, q)$. Show that

$$c_q(n) = \mu\left(\frac{q}{\ell}\right) \varphi(q) \varphi\left(\frac{q}{\ell}\right)^{-1}.$$

- **Solution.** Note that

$$c_q(n) = \sum_{d \mid \ell} d\mu\left(\frac{q}{d}\right) = \sum_{de=\ell} d\mu\left(\frac{qe}{\ell}\right).$$

- Let $q_1 = q/\ell$. Then observe that $\mu(q_1 e) = 0$ if $(q_1, e) \neq 1$ and $\mu(q_1 e) = \mu(q_1)\mu(e)$ if $(q_1, e) = 1$. Using this we get

$$c_q(n) = \sum_{de=\ell} d\mu(q_1 e) = \sum_{\substack{de=\ell \\ (q_1, e)=1}} d\mu(q_1)\mu(e)$$

$$= \mu(q_1) \sum_{\substack{de=\ell \\ (q_1, e)=1}} d\mu(e) = \mu(q_1)\ell \sum_{\substack{de=\ell \\ (q_1, e)=1}} \frac{\mu(e)}{e}.$$

- It is easy to see that

$$\sum_{\substack{e \mid \ell \\ (e, q_1) = 1}} \frac{\mu(e)}{e} = \prod_{\substack{p \mid \ell \\ p \nmid q_1}} \left(1 - \frac{1}{p}\right) = \prod_{\substack{p \mid q \\ p \nmid q_1}} \left(1 - \frac{1}{p}\right)$$

$$= \prod_{p \mid q} \left(1 - \frac{1}{p}\right) \cdot \prod_{p \mid q_1} \left(1 - \frac{1}{p}\right)^{-1}$$

$$= \frac{\varphi(q)}{q} \cdot \frac{q_1}{\varphi(q_1)} = \frac{\varphi(q)}{\ell \varphi(q_1)}.$$

- Putting everything together we end up with

$$c_q(n) = \mu(q_1) \ell \cdot \frac{\varphi(q)}{\ell \varphi(q_1)} = \mu(q_1) \varphi(q) \varphi(q_1)^{-1}.$$

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- **Problem.** Prove that

$$\sigma(n) = \frac{\pi^2 n}{6} \sum_{q=1}^{\infty} \frac{c_q(n)}{q^2}.$$

- **Solution.** We rewrite $\sigma(n)$ as

$$\sigma(n) = n \sum_{d \mid n} \frac{1}{d} = n \sum_{d=1}^{n} \frac{1}{d} \left(\frac{1}{d} \sum_{a=1}^{d} e(an/d) \right)$$

$$= n \sum_{d=1}^{n} \frac{1}{d^2} \sum_{a=1}^{d} e(an/d)$$

since $\frac{1}{d} \sum_{a=1}^{d} e(an/d)$ is the characteristic function of the divisors of n.

- Because $\sum_{a=1}^{d} e(an/d) = 0$ for $d > n$, we can extend the above finite sum to an infinite sum as

$$\sigma(n) = n \sum_{d=1}^{\infty} \frac{1}{d^2} \sum_{a=1}^{d} e(an/d). \qquad (16)$$

- Observe that

$$\sum_{a=1}^{d} e(an/d) = \sum_{q|d} \sum_{\substack{a=1 \\ (a,d)=q}}^{d} e(an/d) = \sum_{q|d} \sum_{\substack{r=1 \\ (r,d/q)=1}}^{d/q} e(rqn/d)$$

$$= \sum_{q|d} \sum_{\substack{r=1 \\ (r,d/q)=1}}^{d/q} e(rn/(d/q)) = \sum_{q|d} c_{d/q}(n)$$

$$= \sum_{q|d} c_q(n).$$

- Substituting $\sum_{a=1}^{d} e(an/d) = \sum_{q|d} c_q(n)$ into (16) we obtain

$$\sigma(n) = n \sum_{d=1}^{\infty} \frac{1}{d^2} \sum_{q|d} c_q(n) = n \sum_{d=1}^{\infty} \sum_{q|d} \frac{c_q(n)}{d^2}.$$

- Changing the order of summation, we get

$$\sigma(n) = n \sum_{q=1}^{\infty} \sum_{\substack{d=1 \\ q|d}}^{\infty} \frac{1}{d^2} c_q(n) = n \sum_{q=1}^{\infty} \sum_{\ell=1}^{\infty} \frac{c_q(n)}{(q\ell)^2}$$

$$= n \sum_{q=1}^{\infty} \sum_{\ell=1}^{\infty} \frac{1}{\ell^2} \frac{c_q(n)}{q^2} = n \left(\sum_{\ell=1}^{\infty} \frac{1}{\ell^2} \right) \sum_{q=1}^{\infty} \frac{c_q(n)}{q^2}$$

$$= \frac{n\pi^2}{6} \sum_{q=1}^{\infty} \frac{c_q(n)}{q^2},$$

where we use $\sum_{k=1}^{\infty} 1/k^2 = \pi^2/6$ in the final equality. The change of order of summation is justified by the fact that $c_q(n) \ll_n 1$ and $d(n) \leq \sqrt{n}$.

- **Problem.** Show that $d(n) \ll_\epsilon n^\epsilon$ for every $\epsilon > 0$.
- **Solution.** Let $n = p_1^{a_1} \cdots p_k^{a_k}$. Then we have

$$\frac{d(n)}{n^\epsilon} = \prod_{p^a \| n} \frac{a+1}{p^{a\epsilon}} \leq \prod_{\substack{p^a \| n \\ p < 2^{1/\epsilon}}} \frac{a+1}{p^{a\epsilon}}$$

for if $p \geq 2^{1/\epsilon}$, then $p^\epsilon \geq 2$ and so $p^{a\epsilon} \geq 2^a \geq a+1$ which gives $(a+1)/p^{a\epsilon} \leq 1$.

- Now observe that

$$\frac{d(n)}{n^\epsilon} \leq \prod_{\substack{p^a \| n \\ p < 2^{1/\epsilon}}} \frac{a+1}{2^{a\epsilon}} \leq \prod_{\substack{p^a \| n \\ p < 2^{1/\epsilon}}} \frac{a+1}{a\epsilon \log 2}$$

as $2^{a\epsilon} = e^{a\epsilon \log 2} \geq a\epsilon \log 2$.

- Finally we have

$$\frac{d(n)}{n^\epsilon} \leq \prod_{p < 2^{1/\epsilon}} \frac{2}{\epsilon \log 2} \leq \left(\frac{2}{\epsilon \log 2} \right)^{\pi(2^{1/\epsilon})}.$$

This shows that $d(n) \ll_\epsilon n^\epsilon$.

- **Problem.** Show that $\varphi(n) \gg_\epsilon n^{1-\epsilon}$ for every $\epsilon > 0$.
- **First solution.** Note that

$$\varphi(n) = n \prod_{p \mid n} \left(1 - \frac{1}{p}\right) \geq \frac{n}{2^{\omega(n)}} \geq \frac{n}{d(n)} \gg n^{1-\epsilon}$$

  as $1 - 1/p \geq 1/2$ for every prime p and $d(n) \geq 2^{\omega(n)}$ for every $n \in \mathbb{N}$.
- **Second solution.** We present another solution which involves bounding the product $\prod_{p \mid n} (1 - 1/p)$ from below. The idea is to show that the product $\prod_{p \mid n} (1 - 1/p) \gg_\epsilon n^{-\epsilon}$ for every $\epsilon > 0$.
- Note that

$$\log \prod_{p \mid n} \left(1 - \frac{1}{p}\right) = \sum_{p \mid n} \log \left(1 - \frac{1}{p}\right) = -\sum_{p \mid n} \sum_{k=1}^{\infty} \frac{1}{kp^k}$$

$$= -\sum_{k=1}^{\infty} \frac{1}{k} \sum_{p \mid n} \frac{1}{p^k}.$$

- Now observe that for any $k \in \mathbb{N}$ and $U \geq 1$ we have

$$\sum_{p \mid n} \frac{1}{p^k} \leq \sum_{p \leq U} \frac{1}{p^k} + \frac{\omega(n)}{U^k}.$$

- Using the above inequality we get

$$\sum_{k=1}^{\infty} \frac{1}{k} \sum_{p \mid n} \frac{1}{p^k} \leq \sum_{k=1}^{\infty} \frac{1}{k} \sum_{p \leq U} \frac{1}{p^k} + \omega(n) \sum_{k=1}^{\infty} \frac{1}{kU^k}$$

$$= -\sum_{p \leq U} \log\left(1 - \frac{1}{p}\right) - \omega(n) \log\left(1 - \frac{1}{U}\right).$$

- Using the trivial inequality $\omega(n) \leq \log_2 n$ we get

$$\log \prod_{p \mid n} \left(1 - \frac{1}{p}\right) \geq \sum_{p \leq U} \log\left(1 - \frac{1}{p}\right) + \frac{\log n}{\log 2} \log\left(1 - \frac{1}{U}\right)$$

- Denoting

$$c_U = \sum_{p \leq U} \log\left(1 - \frac{1}{p}\right), \qquad \epsilon_U = -\frac{1}{\log 2}\log\left(1 - \frac{1}{U}\right)$$

we get that

$$\log \prod_{p \mid n}\left(1 - \frac{1}{p}\right) \geq c_U - \epsilon_U \log n.$$

- Finally, taking exponential we find that

$$\prod_{p \mid n}\left(1 - \frac{1}{p}\right) \geq e^{c_U} n^{-\epsilon_U}.$$

Since $\epsilon_U \to 0$ as $U \to \infty$ we obtain the desired conclusion.

- **Problem.** Show that $\sigma(n) \leq n(1 + \log n)$ for every $n \in \mathbb{N}$.
- **Solution.** Note that

$$\sigma(n) = n \sum_{d \mid n} \frac{1}{d} \leq n \sum_{k=1}^{n} \frac{1}{k} = n(\log n + 1).$$

- **Problem.** Show that

$$\frac{6n^2}{\pi^2} \leq \varphi(n)\sigma(n) \leq n^2$$

for every $n \in \mathbb{N}$. Also deduce that

$$\varphi(n) \geq \frac{6n}{\pi^2(\log n + 1)}$$

for every $n \in \mathbb{N}$.

- **Solution.** Observe that

$$\varphi(n)\sigma(n) = \prod_{p^a \| n} \varphi(p^a)\sigma(p^a) = \prod_{p^a \| n} p^{a-1}(p-1)\left(\frac{p^{a+1}-1}{p-1}\right)$$

$$= \prod_{p^a \| n} p^{a-1}(p^{a+1}-1) = n^2 \prod_{p^a \| n}\left(1 - \frac{1}{p^{a+1}}\right).$$

- Because each factor in the product is ≤ 1 we get the inequality $\varphi(n)\sigma(n) \leq 1$.

- As for the other inequality note that

$$\prod_{p^a \| n} \left(1 - \frac{1}{p^{a+1}}\right) \geq \prod_{p} \left(1 - \frac{1}{p^2}\right) = \frac{1}{\zeta(2)} = \frac{6}{\pi^2}.$$

- Since $\sigma(n) \leq n(\log n + 1)$ for every $n \in \mathbb{N}$ the last inequality follows.

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- **Problem** (Pollock, 2011) Let f and g be arithmetic functions such that $f = g * 1$.
    - (i) Show that if f and g both have finite support, then f and g must be identically zero.
    - (ii) Use part (i) to give a proof of infinitude of primes.
- **Solution.** For part (i) suppose for the sake of contradiction that f and g both have finite support. Now consider the entire function $F(z)$ defined as

$$F(z) = \sum_{n=1}^{\infty} f(n)z^n.$$

  The function $F(z)$ is a polynomial in z as f has finite support.

- Because $f = g * 1$ we find that for $|z| < 1$ we have

$$F(z) = \sum_{n=1}^{\infty} \sum_{d \mid n} g(d)z^n = \sum_{d} g(d) \sum_{\substack{n=1 \\ d \mid n}}^{\infty} z^n = \sum_{d=1}^{\infty} g(d)\frac{z^d}{1 - z^d}.$$

- The change of order of summation is justified by observing that

$$\sum_{n=1}^{\infty} \sum_{d \mid n} |g(d)||z|^n \leq A \sum_{n=1}^{\infty} n|z|^n = A \frac{|z|}{(1-|z|^2)} < \infty,$$

where $A = \max_{d \in \mathbb{N}} |g(d)|$.

- It now follows by principle of analytic continuation that

$$F(z) = \sum_{d=1}^{\infty} g(d) \frac{z^d}{1-z^d}$$

for every z with $z^d \neq 1$. Let $D = \max\{d \in \mathbb{N} : g(d) \neq 0\}$. Then the sum on the right hand side has a pole at $z = e^{2\pi i/D}$, a contradiction. Thus f and g must both be identically zero.

- To prove part (ii) note that if there are only finitely many primes, then μ must have finite support as there would then be only finitely many squarefree integers. But $\mu * 1 = e$ has finite support as well. This contradicts part (i) as μ and e are not identically zero.

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- **Problem.** Let $\gamma(n) = \prod_{p \mid n} p$. Show that

$$\sum_{n=1}^{\infty} \frac{1}{n\gamma(n)} < \infty.$$

- **Solution.** First note that the map $n \mapsto n\gamma(n)$ is injective due to unique factorization. Now observe that

$$\sum_{n \leq x} \frac{1}{n\gamma(n)} \leq \prod_{p \leq x} \left(1 + \frac{1}{p^2} + \frac{1}{p^3} + \cdots\right)$$

$$= \prod_{p \leq x} \left(1 + \frac{1}{p^2} \frac{1}{1 - 1/p}\right)$$

$$= \prod_{p \leq x} \left(1 + \frac{1}{p(p-1)}\right),$$

where the first inequality follows since every prime that divides $n\gamma(n)$ has power at least 2 in the prime factorization. Because the infinite product $\prod_p \left(1 + 1/(p(p-1))\right)$ converges, the conclusion follows.

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- **Problem.** Show that

$$\sum_{n \leq x} \frac{n}{\varphi(n)} \ll x.$$

- **Solution.** Note that

$$\frac{n}{\varphi(n)} = \prod_{p \mid n} \left(1 - \frac{1}{p}\right)^{-1} = \prod_{p \mid n} \left(1 + \frac{1}{p} + \frac{1}{p^2} + \cdots\right)$$

$$= \sum_{\substack{m \\ p \mid m \Rightarrow p \mid n}} \frac{1}{m} = \sum_{\substack{m \\ \gamma(m) \mid \gamma(n)}} \frac{1}{m}.$$

- Using this we get

$$\sum_{n \leq x} \frac{n}{\varphi(n)} = \sum_{n \leq x} \sum_{\substack{m \\ \gamma(m) \mid \gamma(n)}} \frac{1}{m} = \sum_m \frac{1}{m} \sum_{\substack{n \leq x \\ \gamma(m) \mid \gamma(n)}} 1$$

$$\leq \sum_m \frac{1}{m} \sum_{\substack{n \leq x \\ \gamma(m) \mid n}} 1 \leq x \sum_m \frac{1}{m \gamma(m)} \ll x.$$

- **Problem.** Show that

$$\sum_{n \leq x} \frac{1}{\varphi(n)} \ll \log x.$$

- **Solution.** Let $A(x) = \sum_{n \leq x} n/\varphi(n)$. Using Abel's summation we obtain

$$\sum_{n \leq x} \frac{1}{\varphi(n)} = \frac{A(x)}{x} + \int_1^x \frac{A(u)}{u^2} \, du.$$

- Using the estimate $A(x) \ll x$ we have $A(x)/x \ll 1$ and

$$\int_1^x \frac{A(u)}{u^2} \, du \ll \int_1^x \frac{du}{u} = \log x.$$

  This readily implies the desired result.

- **Problem.** Prove that

$$\sum_{n \leq x} \varphi(n) = \frac{3x^2}{\pi^2} + O(x \log x).$$

- **Solution.** Note that

$$\sum_{n \leq x} \varphi(n) = \sum_{n \leq x} \sum_{d \mid n} \mu(d) \frac{n}{d} = \sum_{d \leq x} \frac{\mu(d)}{d} \sum_{\substack{n \leq x \\ d \mid n}} n$$

$$= \sum_{d \leq x} \mu(d) \sum_{q \leq x/d} q.$$

- Using the estimate $\sum_{n \leq x} 1/n^2 = 1/x + O(1/x^2)$ we get

$$\sum_{n \leq x} \varphi(n) = \frac{x^2}{2} \sum_{d \leq x} \frac{\mu(d)}{d^2} + O\left(x \sum_{d \leq x} \frac{|\mu(d)|}{d} \right)$$

- Finally we obtain

$$\sum_{n \leq x} \varphi(n) = \frac{x^2}{2} \left(\frac{6}{\pi^2} + O\left(\frac{1}{x}\right) \right) + O\left(x \log x\right)$$

$$= \frac{3x^2}{\pi^2} + O(x \log x),$$

where we use $\sum_{n=1}^{\infty} \mu(n)/n^2 = 1/\zeta(2) = 6/\pi^2$ and the estimates

$$\sum_{n > x} \frac{1}{n^2} = O\left(\frac{1}{x}\right), \qquad \sum_{n \leq x} \frac{1}{n} = \log x + \gamma + O\left(\frac{1}{x}\right).$$

- **Problem.** Prove that

$$\sum_{n \leq x} \frac{\varphi(n)}{n} = \frac{6x}{\pi^2} + O(\log^2 x).$$

- **Solution.** Let $A(x) = \sum_{n \leq x} \varphi(n)$. Using Abel's summation by parts formula we have

$$\sum_{n \leq x} \frac{\varphi(n)}{n} = \frac{A(x)}{x} + \int_1^x \frac{A(u)}{u^2} \, du$$

$$= \frac{3x}{\pi^2} + O(\log x) + \int_1^x \left(\frac{3}{\pi^2} + O\left(\frac{\log u}{u} \right) \right) \, du$$

$$= \frac{6x}{\pi^2} + O(\log x) + O\left(\int_1^x \frac{\log u}{u} \, du \right)$$

$$= \frac{6x}{\pi^2} + O\left(\log^2 x \right).$$

- A slightly better estimate can be obtained by noting that

$$\sum_{n \le x} \frac{\varphi(n)}{n} = \sum_{n \le x} \sum_{d \mid n} \frac{\mu(d)}{d} = \sum_{d \le x} \frac{\mu(d)}{d} \sum_{\substack{n \le x \\ d \mid n}} 1$$

$$= x \sum_{d \le x} \frac{\mu(d)}{d^2} + O\left(\sum_{d \le x} \frac{|\mu(d)|}{d}\right)$$

$$= \frac{6x}{\pi^2} + O(\log x).$$

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- **Problem.**
  (i) Show that

  $$\frac{n}{\varphi(n)} = \sum_{d \mid n} \frac{\mu(d)^2}{\varphi(d)}.$$

  (ii) Using the identity in part (i) prove that

  $$\sum_{n \leq x} \frac{1}{\varphi(n)} = A \log x + B + O\left(\frac{\log^2 x}{x}\right)$$

  for some constants A and B.

- **Solution.** For part (i) note that

  $$\frac{n}{\varphi(n)} = \prod_{p \mid n} \left(1 - \frac{1}{p}\right)^{-1} = \prod_{p \mid n} \frac{p}{p-1} = \prod_{p \mid n} \left(1 + \frac{1}{p-1}\right)$$

  $$= \sum_{\mathcal{P} \subset \{p \mid n\}} \frac{1}{\prod_{p \in \mathcal{P}}(p-1)} = \sum_{\mathcal{P} \subset \{p \mid n\}} \frac{1}{\varphi(\prod_{p \in \mathcal{P}} p)}$$

  $$= \sum_{\substack{d \mid n \\ d \text{ sq. free}}} \frac{1}{\varphi(d)} = \sum_{d \mid n} \frac{\mu(d)^2}{\varphi(d)}.$$

- For part (ii) observe that

$$\sum_{n \leq x} \frac{1}{\varphi(n)} = \sum_{n \leq x} \frac{1}{n} \sum_{d \mid n} \frac{\mu(d)^2}{\varphi(d)}.$$

- Changing the order of summation the above sum becomes

$$\sum_{d \leq x} \frac{\mu(d)^2}{\varphi(d)} \sum_{\substack{n \leq x \\ d \mid n}} \frac{1}{n} = \sum_{d \leq x} \frac{\mu(d)^2}{d\varphi(d)} \sum_{q \leq x/d} \frac{1}{q}.$$

- Incorporating the estimate for harmonic sums we get

$$\sum_{n \leq x} \frac{1}{\varphi(n)} = \sum_{d \leq x} \frac{\mu(d)^2}{d\varphi(d)} \left(\log \frac{x}{d} + \gamma + O\left(\frac{d}{x}\right) \right)$$

$$= \log x \sum_{d \leq x} \frac{\mu(d)^2}{d\varphi(d)} - \sum_{d \leq x} \frac{\mu(d)^2 \log d}{d\varphi(d)}$$

$$+ \gamma \sum_{d \leq x} \frac{\mu(d)^2}{d\varphi(d)} + O\left(\frac{1}{x} \sum_{d \leq x} \frac{\mu(d)^2}{\varphi(d)} \right).$$

- Since $\varphi(n) \gg n/\log n$ we have

$$\sum_{d \le x} \frac{\mu(d)^2}{d\varphi(d)} = \sum_{d=1}^{\infty} \frac{\mu(d)^2}{d\varphi(d)} + O\left(\sum_{d>x} \frac{\log d}{d^2}\right).$$

By comparison the error term is

$$\ll \int_x^{\infty} \frac{\log u}{u^2}\, du \ll \frac{\log x}{x}$$

by integral test and integration by parts.

- Similarly we have

$$\sum_{d \le x} \frac{\mu(d)^2 \log d}{d\varphi(d)} = \sum_{d=1}^{\infty} \frac{\mu(d)^2 \log d}{d\varphi(d)} + O\left(\frac{\log^2 x}{x}\right). \quad (17)$$

- Moreover, we have

$$\sum_{d \le x} \frac{\mu(d)^2}{\varphi(d)} \ll \sum_{d \le x} \frac{\log d}{d} \ll \log x \sum_{d \le x} \frac{1}{d} \ll \log^2 x.$$

- In fact, the above sum is $\ll \log x$ as $\sum_{n \leq x} 1/\varphi(n) \ll \log x$ but we can afford to lose a log power because of (17). Putting everything together we conclude

$$\sum_{n \leq x} \frac{1}{\varphi(n)} = \log x \sum_{d=1}^{\infty} \frac{\mu(d)^2}{d\varphi(d)} + \gamma \sum_{d=1}^{\infty} \frac{\mu(d)^2}{d\varphi(d)}$$
$$- \sum_{d=1}^{\infty} \frac{\mu(d)^2}{d\varphi(d)} + O\left(\frac{\log^2 x}{x}\right).$$

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- **Problem.** Let $Q(x)$ denote the number of squarefree integers $\leq x$.

  (i) Prove that

  $$Q(x) = \frac{6}{\pi^2}x + O(\sqrt{x}).$$

  (ii) Prove that for any $n \in \mathbb{N}$

  $$Q(n) \geq n - \sum_p \left\lfloor \frac{n}{p^2} \right\rfloor.$$

  (iii) Prove that

  $$\sum_p \frac{1}{p^2} < \frac{1}{2}$$

  and conclude that $Q(n) > n/2$ for all $n \in \mathbb{N}$.

  (iv) Prove that every integer $n > 1$ can be written as a sum of two squarefree numbers.

- **Solution.** Using the identity $\sum_{d^2 \mid n} \mu(d) = \mu(n)^2$ we obtain

$$Q(x) = \sum_{n \le x} \mu(n)^2 = \sum_{n \le x} \sum_{d^2 \mid n} \mu(d) = \sum_{d \le \sqrt{x}} \sum_{\substack{n \le x \\ d^2 \mid n}} \mu(d)$$

$$= \sum_{d \le \sqrt{x}} \mu(d) \sum_{\substack{n \le x \\ d^2 \mid n}} 1 = \sum_{d \le \sqrt{x}} \mu(d) \left\lfloor \frac{x}{d^2} \right\rfloor$$

$$= x \sum_{d \le \sqrt{x}} \frac{\mu(d)}{d^2} + O\left(\sum_{d \le \sqrt{x}} |\mu(d)| \right)$$

$$= x \sum_{d \le \sqrt{x}} \frac{\mu(d)}{d^2} + O(\sqrt{x}) \tag{18}$$

- We now push d off to ∞ in the above sum and incur an error due to the tail which fortunately is only $O(\sqrt{x})$ as seen can be seen by

$$\sum_{d \leq \sqrt{x}} \frac{\mu(d)}{d^2} = \sum_{d=1}^{\infty} \frac{\mu(d)}{d^2} + O\left(\sum_{d > \sqrt{x}} \frac{1}{d^2}\right) = \frac{6}{\pi^2} + O\left(\frac{1}{\sqrt{x}}\right).$$

Using this estimate in (18), we obtain part (i).

- Observe that for an integer n we have

$$\{1 \leq m \leq n : m \text{ nonsquarefree}\} \subset \bigcup_p \{1 \leq m \leq n : p^2 | m\},$$

where p runs over all primes. It thus follows that

$$n - Q(n) \leq \sum_p \left\lfloor \frac{n}{p^2} \right\rfloor,$$

which is the desired inequality.

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- Note that

$$\sum_p \frac{1}{p^2} < \frac{1}{4} + \sum_{k=1}^{\infty} \frac{1}{(2k+1)^2} < \frac{1}{4} + \sum_{k=1}^{\infty} \frac{1}{4k(k+1)} = \frac{1}{2}.$$

It now follows by part (ii) that for every $n \in \mathbb{N}$

$$Q(n) \geq n - \sum_p \left\lfloor \frac{n}{p^2} \right\rfloor \geq n - n \sum_p \frac{1}{p^2} > \frac{n}{2}.$$

- For part (iv) suppose for the sake of contradiction that $n > 1$ and n cannot be expressed as a sum of two squarefree integers. Then for every $a, b \in \mathbb{N}$ satisfying $a + b = n$ either a or b is not squarefree. It follows that there are at least $(n-1)/2$ (this is the number of ways n can be written as a sum of two positive integers without regard for order) integers up to $n-1$ that are not square free. Hence, we must have

$$Q(n-1) \leq (n-1) - \frac{(n-1)}{2} = \frac{n-1}{2}.$$

But this contradicts the inequality in part (iii).

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- **Problem.** The Riemann zeta function, $\zeta(s)$, has Euler product representation

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}$$

for $\sigma > 1$. Using the inequality

$$|\zeta(s)| \geq \prod_p \left(1 + \frac{1}{p^\sigma}\right)^{-1}$$

for $\sigma > 1$, show that $\zeta(s) \neq 0$.

- **Solution.** It suffices to show that

$$\prod_p \left(1 + \frac{1}{p^\sigma}\right)^{-1} \neq 0.$$

Suppose for the sake of contradiction that the above product is 0. Then we have $\log \prod_{p \leq x} \left(1 + \frac{1}{p^\sigma}\right)^{-1} \to -\infty$ as $x \to \infty$.

- But note that

$$\left| \log \prod_{p \leq x} \left(1 + \frac{1}{p^\sigma} \right)^{-1} \right| \leq \sum_{p \leq x} \left(\frac{1}{p^\sigma} + \frac{1}{2p^{2\sigma}} + \frac{1}{3p^{3\sigma}} + \cdots \right)$$

$$\leq \sum_{n=1}^{\infty} \frac{1}{n^\sigma}.$$

- **Problem.** In this problem we show that $\vartheta(x) \ll x$.

  (i) Let m, n be integers with $m \geq 2n > 0$. Show that

  $$\prod_{m-n < p \leq m} p \,\Big|\, \binom{m}{n}.$$

  (ii) Show that $\vartheta(2n) - \vartheta(n) \leq 2n \log 2$.

  (iii) Prove that $\vartheta(x) \leq 4x \log 2$ for every $x \geq 1$.

- **Solution.** Note that

  $$\binom{m}{n} = \frac{m(m-1)\cdots(m-n+1)}{n!}.$$

  If $m - n < p \leq m$, then p divides the numerator but not the denominator as $m - n + 1 > n$. This implies that $\prod_{m-n < p \leq m} p$ divides $\binom{m}{n}$.

- Observe that $\binom{2n}{n} \leq 2^{2n}$ for every $n \in \mathbb{N}$. Thus taking logarithm we deduce that

  $$\vartheta(2n) - \vartheta(n) \leq 2n \log 2.$$

- Now suppose that $2^m \leq n < 2^{m+1}$. Then we have

$$\vartheta(n) \leq \vartheta(2^{m+1}) = \sum_{k=0}^{m} (\vartheta(2^{k+1}) - \vartheta(2^k))$$

$$\leq \sum_{k=0}^{m} 2^{k+1} \log 2 = (2^{m+2} - 2) \log 2.$$

Finally note that $2^{m+2} = 4 \cdot 2^m \leq 4n$. Thus $\vartheta(n) \leq 4n \log 2$ for every $n \in \mathbb{N}$.

- Let $x \geq 1$ with $n = \lfloor x \rfloor$. Then we have $\vartheta(x) = \vartheta(n) \leq 4n \log 2 \leq 4x \log 2.$

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- **Problem.** For a prime p let $\mathcal{A}_p$ denote the set of all positive integers n such that either $n + 1 \equiv 0 \pmod{p^2}$ or $n - 1 \equiv 0 \pmod{p^2}$.

  (i) If $[N] := \{1, \ldots, N\}$, then show that

  $$\left| [N] \cap \bigcup_p \mathcal{A}_p \right| \leq 2N \sum_{p \leq N} \frac{1}{p^2} + 2\pi(N).$$

  (ii) Prove that

  $$\sum_p \frac{1}{p^2} < \frac{1}{4}$$

  and conclude that the set of twin squarefree integers have positive density.

- **Solution.** Observe that if $n \in \mathcal{A}_p$, then $p^2 \leq n + 1$ and so $p \leq n$. Consequently $[N] \cap \mathcal{A}_p = \emptyset$ for $p > N$ and so we have

$$\left| [N] \cap \bigcup_p \mathcal{A}_p \right| = \left| \bigcup_{p \leq N} [N] \cap \mathcal{A}_p \right| \leq \sum_{p \leq N} |[N] \cap \mathcal{A}_p|.$$

- Since there are exactly two elements of $\mathcal{A}_p$ in any set of p^2 consecutive integers we obtain that $|[N] \cap \mathcal{A}_p| \leq 2\lceil N/p^2 \rceil$ and so

$$\left| [N] \cap \bigcup_p \mathcal{A}_p \right| \leq 2 \sum_{p \leq N} \left\lceil \frac{N}{p^2} \right\rceil \leq 2N \sum_{p \leq N} \frac{1}{p^2} + 2\pi(N).$$

- Note that

$$\sum_p \frac{1}{p^2} \leq \frac{1}{2^2} + \sum_{k=1}^{\infty} \frac{1}{(2k+1)^2} \leq \frac{1}{4} + \frac{1}{4} \sum_{k=1}^{\infty} \frac{1}{k(k+1)} = \frac{1}{2}.$$

Let $C = \sum_p p^{-2}$. We have shown above that $C < \frac{1}{2}$. Thus

$$\frac{1}{N} \left| [N] \cap \bigcup_p \mathcal{A}_p \right| \leq 2C + 2\frac{\pi(N)}{N}.$$

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- Hence we have

$$\limsup_{N \to \infty} \frac{1}{N} \left| [N] \cap \bigcup_p \mathcal{A}_p \right| \leq 2C < 1.$$

as $\pi(N)/N \to \infty$, i.e., the set of primes has density 0.

- Since $[N] \cap \bigcup_p \mathcal{A}_p$ is the set of all positive integers n up to N such that either $n-1$ or $n+1$ is not squarefree, $\bigcap_p [N] \backslash \mathcal{A}_p = [N] \backslash \left([N] \cap \bigcup_p \mathcal{A}_p \right)$ is the set of all twin squarefree integers. Finally,

$$\liminf_{N \to \infty} \frac{1}{N} \left| \bigcap_p [N] \backslash \mathcal{A}_p \right| = 1 - \limsup_{N \to \infty} \frac{1}{N} \left| [N] \cap \bigcup_p \mathcal{A}_p \right|$$

$$= 1 - 2C > 0.$$

This shows that the set of twin squarefree integers has positive (lower) density.

- **Problem.** Using Abel's summation for the partial sum $\sum_{p \leq x}(\log p)/p$ show that if the limit

$$\lim_{x \to \infty} \frac{\pi(x) \log x}{x},$$

exists then it must be 1.

Solution. Using Abel's summation by parts formula we obtain

$$\sum_{p \leq x} \frac{\log p}{p} = \frac{\pi(x) \log x}{x} - \int_2^x \pi(u) \left(\frac{1 - \log u}{u^2} \right) \, du.$$

- Employing the estimate for $\sum_{p \leq x}(\log p)/p$ we see that

$$\log x + O(1) = \frac{\pi(x) \log x}{x} + \int_2^x \frac{\pi(u) \log u}{u} \left(\frac{1}{u} - \frac{1}{u \log u} \right) \, du.$$

Hence we obtain

$$\int_2^x \frac{\pi(u) \log u}{u} \left(\frac{1}{u} - \frac{1}{u \log u} \right) \, du = \log x + O(1) \quad (19)$$

as $\pi(x)(\log x)/x \ll 1$.

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- Now let L be the limit of $\pi(x)(\log x)/x$ as $x \to \infty$ and let $\epsilon > 0$ be fixed. Let $x_0 \geq 1$ be such that $\pi(x)(\log x)/x < L + \epsilon$ for $x \geq x_0$. We break the integral

$$\int_2^x \frac{\pi(u)\log u}{u} \left(\frac{1}{u} - \frac{1}{u \log u} \right) \, du$$

at x_0. The integral over the interval $[2, x_0]$ is $O(1)$ whereas the integral over the interval $[x_0, x]$ is $\leq (L + \epsilon)\log x$. Hence, we have

$$\int_2^x \frac{\pi(u)\log u}{u} \left(\frac{1}{u} - \frac{1}{u \log u} \right) \, du \leq (L + \epsilon)\log x + O(1),$$

where the implicit constant depends on ϵ. Combining this with (19) we immediately deduce that $L \geq 1$.

- Similarly we have

$$\int_2^x \frac{\pi(u)\log u}{u} \left(\frac{1}{u} - \frac{1}{u \log u} \right) \, du \geq (L - \epsilon)\log x + O(1).$$

Again combining this with (19) it follows that $L \leq 1$. Thus we must have $L = 1$.