Theorem. *The power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ defines a holomorphic function in its disc of convergence and the derivative of f is also a power series obtained by differentiating term by term series for f, i.e.,*

$$f'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}.$$

Moreover, f' has the same radius of convergence as f.

Proof. It is easy to verify that the radius of convergence for the power series $\sum_{n=1}^{\infty} n a_n z^{n-1}$ is the same as that of the power series $\sum_{n=0}^{\infty} a_n z^n$ since

$$\limsup_{n\to\infty} |na_n|^{1/n} = \limsup_{n\to\infty} n^{1/n}|a_n|^{1/n} = \limsup_{n\to\infty} |a_n|^{1/n}.$$

Now let R be the radius of convergence oft he power series representing f and let $|z| < R$ be fixed. We then choose $r > 0$ to be such that $|z| + r < R$, and assume $|h| < r$. Then $|z + h| \leqslant |z| + |h| < |z| + r < R$. We can then write

$$\frac{f(z+h) - f(z)}{h} - \sum_{n=1}^{\infty} n a_n z^{n-1} = \sum_{n=2}^{\infty} a_n b_n$$

where

$$b_n = \sum_{k=2}^{n} \binom{n}{k} z^{n-k} h^{k-1}.$$

It then follows that

$$\begin{aligned} |b_n| &\leqslant |h| \sum_{k=2}^{n} \binom{n}{k} |z|^{n-k} |h|^{k-2} \leqslant |h| \sum_{k=2}^{n} \binom{n}{k} |z|^{n-k} r^{k-2} \\ &\leqslant \frac{|h|}{r^2} \sum_{k=0}^{n} \binom{n}{k} |z|^{n-k} r^k = \frac{|h|}{r^2} (|z| + r)^n \end{aligned}$$

Hence, we obtain

$$\begin{aligned} \left| \frac{f(z+h) - f(z)}{h} - \sum_{n=1}^{\infty} n a_n z^{n-1} \right| &= \left| \sum_{n=2}^{\infty} a_n b_n \right| \leqslant \sum_{n=2}^{\infty} |a_n||b_n| \\ &\leqslant \frac{|h|}{r^2} \sum_{n=2}^{\infty} |a_n|(|z| + r)^n = A|h|. \end{aligned}$$

Since A is fixed, letting $h \to 0$, we conclude that $f'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}$. $\square$

Corollary. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a power series with nonzero radius of convergence. Then f is infinitely differentiable within the disc of convergence and*

$$a_n = \frac{f^{(n)}(0)}{n!}$$

for all n.

Proof. Since f is infinitely differentiable within its domain of convergence, it is easily verified via induction that

$$f^{(k)}(z) = \sum_{n=k}^{\infty} \frac{n!}{(n-k)!} a_n z^{n-k}.$$

Evaluating $f^{(k)}(z)$ at 0, we get

$$f^{(k)}(0) = k!a_k$$

which gives the desired result. $\square$