BANACH SPACES

Throughout these notes we will write $\mathbf{K}$ to denote $\mathbf{R}$ or $\mathbf{C}$.

1. Normed spaces

Let X be a vector space over $\mathbf{K}$. A map $\|\cdot\| : X \to \mathbf{R}$ is said to be a *norm* on X if it satisfies the following properties:

(a) (nonnegativity) For every $x \in X$, $\|x\| \geqslant 0$.
(b) (positive definiteness) For every $x \in X$, $\|x\| = 0$ if and only if $x = 0$.
(c) (absolute homogeneity) For every $x \in X$ and $\lambda \in \mathbf{K}$, $\|\lambda x\| = |\lambda| \, \|x\|$.
(d) (triangle inequality) For every $x, y \in X$, $\|x + y\| \leqslant \|x\| + \|y\|$.

In this case the ordered pair $(X, \|\cdot\|)$ is called a *normed space*. It is easy to see that the norm $\|\cdot\|$ induces a metric d on X defined as

$$d(x, y) = \|x - y\| .$$

However, this metric satisfies some special properties such as translation invariance and absolute homogeneity. Due to this not every metric on X in induced by a norm. For instance, the discrete metric on X does not come from a norm as it violates absolute homogeneity.

We now present some examples of normed spaces.

Example 1. $\mathbf{K}$ is a normed space with the norm given by $\|x\| = |x|$.

Example 2 (p-norm on $\mathbf{K}^n$)**.** Let $n \in \mathbf{N}$ and $1 \leqslant p < \infty$. The *p-norm* on $\mathbf{K}^n$ is defined as

$$\|x\|_p = \left(\sum_{k=1}^{n} |x_n|^p \right)^{1/p} ,$$

where $x = (x_1, \ldots, x_n) \in \mathbf{K}^n$. It is easily verified that $\|\cdot\|_p$ satisfies the properties (a) - (c) of a norm. The triangle inequality however is not obvious unless $p = 1$. We have to show that

$$\|x + y\|_p \leqslant \|x\|_p + \|y\|_p \tag{1.1}$$

for every $x, y \in \mathbf{K}^n$. Note that if $\|x\|_p + \|y\|_p = 0$, then $\|x\|_p = \|y\|_p = 0$ and so we have $x = y = 0$ and the inequality follows trivially. Now suppose that $\|x\|_p + \|y\|_p \neq 0$. Then we can rewrite (1.1) as

$$\left\| \frac{x}{\|x\|_p + \|y\|_p} + \frac{y}{\|x\|_p + \|y\|_p} \right\|_p \leqslant 1.$$

If we take

$$u = \frac{x}{\|x\|_p + \|y\|_p} \quad \text{and} \quad v = \frac{y}{\|x\|_p + \|y\|_p},$$

then we have to show that

$$\|u+v\|_p \leqslant 1, \tag{1.2}$$

where $\|u\|_p + \|v\|_p = 1$. Let $\lambda = \|u\|_p$. Then $\|v\|_p = 1-\lambda$. If $\lambda = 0, 1$, then the inequality follows trivially as either $u = 0$ or $v = 0$ in this case. Thus assume that $0 < \lambda < 1$. Observe that (1.2) can be rewritten as

$$\left\| \lambda \left(\frac{u}{\lambda}\right) + (1-\lambda)\left(\frac{v}{1-\lambda}\right) \right\|_p \leqslant 1,$$

where we have

$$\left\|\frac{u}{\lambda}\right\|_p = \left\|\frac{v}{1-\lambda}\right\|_p = 1.$$

Thus it suffices to prove the inequality

$$\|\lambda x + (1-\lambda)y\|_p \leqslant 1$$

whenever $\|x\|_p = \|y\|_p = 1$. Since the function $t \mapsto t^p$ ($t \in [0,\infty)$) is convex for $p \geqslant 1$ (as $d^2t^p/dt^2 = p(p-1) \geqslant 0$), we have the convexity bound[1]

$$(\lambda|x_k| + (1-\lambda)|y_k|)^p \leqslant \lambda|x_k|^p + (1-\lambda)|y_k|^p.$$

for the coordinates x_k and y_k. Summing over k from 1 to n we obtain

$$\begin{aligned}
\|\lambda x + (1-\lambda)y\|_p^p &= \sum_{k=1}^n |\lambda x_k + (1-\lambda)y_k|^p \\
&\leqslant \sum_{k=1}^n (\lambda|x_k| + (1-\lambda)|y_k|)^p \\
&\leqslant \sum_{k=1}^n \left(\lambda|x_k|^p + (1-\lambda)|y_k|^p\right) \\
&= \lambda \sum_{k=1}^n |x_k|^p + (1-\lambda)\sum_{k=1}^n |y_k|^p \\
&= \lambda \|x\|_p^p + (1-\lambda)\|y\|_p^p = 1.
\end{aligned}$$

This completes the proof of the triangle inequality and thus we have established that $\|\cdot\|_p$ is a norm on $\mathbf{K}^n$ for any $1 \leqslant p < \infty$.

Example 3 (∞-norm on $\mathbf{K}^n$)**.** Let $n \in \mathbf{N}$. The ∞-*norm* on $\mathbf{K}^n$ is defined as

$$\|x\|_\infty = \sup_{1\leqslant k\leqslant n} |x_k|.$$

It is easily verified that $\|\cdot\|_\infty$ is indeed a norm on $\mathbf{K}^n$. The triangle inequality is particularly easy. Observe that we have the inequalities

$$\|x\|_\infty \leqslant \|x\|_p \leqslant n^{1/p}\|x\|_\infty$$

for any $x \in \mathbf{K}^n$ and $1 \leqslant p < \infty$. From this it follows that

$$\lim_{p\to\infty} \|x\|_p = \|x\|_\infty$$

for every $x \in \mathbf{K}^n$. This in some sense justifies the terminology.

[1] See Wikipedia article on convex functions.