AFFINE VARIETIES

Algebraic sets

Let k be a field. We define the *affine n-space* to be k^n and denote it by $\mathbf{A}^n$. One reason for using this notation is to treat $\mathbf{A}^n$ as a space of points rather than a k-vector space. Given a subset S of the polynomial ring $k[x_1, \ldots, x_n]$ we define the *zero locus* or *vanishing set* of S to be

$$V(S) = \{a \in \mathbf{A}^n : f(a) = 0 \text{ for every } f \in S\}.$$

A subset X of $\mathbf{A}^n$ is said to be an (affine) *algebraic set* if $X = V(S)$ for some $S \subset k[x_1, \ldots, x_n]$; i.e., algebraic sets are solution sets of polynomial equations. If $S = \{f_1, \ldots, f_k\}$, then we write $V(S)$ simply as $V(f_1, \ldots, f_k)$. We now present some simple examples of algebraic sets.

Example 1. (a) Both $\emptyset$ and X are algebraic sets since

$$\emptyset = V(1) = V(k[x_1, \ldots, x_n]) \quad \text{and} \quad \mathbf{A}^n = V(\emptyset) = V(0).$$

(b) A point $a = (a_1, \ldots, a_n) \in \mathbf{A}^n$ is an algebraic set since

$$\{a\} = V(x_1 - a_1, \ldots, x_n - a_n).$$

(c) Every vector subspace of $\mathbf{A}^n$ is an algebraic set. Let X be a subspace of $\mathbf{A}^n$ and let $\{v_1, \ldots, v_m\}$ be its basis. Then we can extend it to a basis $\{v_1, \ldots, v_n\}$ of $\mathbf{A}^n$. Now if we define a linear map $L : \mathbf{A}^n \to \mathbf{A}^n$ defined as $L(v_k) = \delta_{k>m} v_k$. It can then be easily seen that $\ker L = X$. If $(a_{ij})_{1 \leqslant i,j \leqslant n}$ is the matrix corresponding to L and $f_i(x) = a_{i1}x_1 + \cdots + a_{in}x_n$, then $X = V(f_1, \ldots, f_n)$.

More simply one can also observe that X can be expressed as the vanishing set of the homogeneous polynomials corresponding to the rows of the matrix of the canonical projection map $\pi : \mathbf{A}^n \to \mathbf{A}^n/X$.

(d) $\mathbf{Z}$ (as a subset of $\mathbf{R}$ or $\mathbf{C}$) is not an algebraic set. Can you see why?

Observe that $V(\cdot)$ is an inclusion reversing operator; i.e., if $S_1 \subset S_2 \subset k[x_1, \ldots, x_n]$, then $V(S_1) \supset V(S_2)$. We now record some elementary properties of algebraic sets.

Lemma 2. (a) *If $S \subset k[x_1, \ldots, x_n]$ and $\mathfrak{a}$ is the ideal generated by S, then*

$$V(S) = V(\mathfrak{a}) = V(\sqrt{\mathfrak{a}}).$$

(b) *If $\{S_i\}_i \subset k[x_1, \ldots, x_n]$, then*

$$V\left(\bigcup_i S_i\right) = \bigcap_i V(S_i).$$

(c) *If $S_1, S_2 \subset k[x_1, \ldots, x_n]$, then*

$$V(S_1) \cup V(S_2) = V(S_1 S_2).$$

The proof of the above lemma follows immediately from the definition of $V(\cdot)$. Note that the part (a) above tells us that as we transition from algebraic objects (ideals) to geometric objects (algebraic sets) some information is lost; i.e., we cannot recover an ideal $\mathfrak{a}$ back from the resulting algebraic set $V(\mathfrak{a})$.

The above lemma along with Example 1(a) implies that algebraic sets form closed sets of a topology on $\mathbf{A}^n$. This topology is called the *Zariski topology* named after the influential algebraic geometer Oscar Zariski (1899 - 1986). The induced topology on an algebraic set X is called the *Zariski topology on* X. Note that the closed sets under this topology are precisely the algebraic sets contained in X.

We will see shortly that the Zariski topology is not so well-behaved or interesting in its own regard. For instance, it is not Hausdorff. Moreover, the open sets in Zariski topology are very large or equivalently the closed sets are very small. However, Zariski topology provides us a convenient large to work with. For instance, we can talk about continuous maps.

It might seem not so feasible to work with infinitely many polynomial defining an algebraic set. Fortunately, Hilbert basis theorem comes to our rescue. Recall that the Hilbert basis theorem says that if A is a Noetherian ring, then so is the polynomial ring $A[x_1, \ldots, x_n]$. Due to this it immediately follows that every algebraic set X is a zero locus of only finitely many polynomials for if $X = V(\mathfrak{a})$ for some ideal $\mathfrak{a}$ of $k[x_1, \ldots, x_n]$, then $\mathfrak{a} = (f_1, \ldots, f_k)$ for some $f_i \in k[x_1, \ldots, x_n]$ and so we have $X = V(f_1, \ldots, f_k)$.