Theorem. *Let k be a field. Then the following statements are equivalent:*

(a) k *is algebraically closed.*

(b) *For any ideal* $\mathfrak{a}$ *in* $k[x_1, \ldots, x_n]$, $I(V(\mathfrak{a})) = \sqrt{\mathfrak{a}}$.

(c) *For every proper ideal* $\mathfrak{a}$ *of* $k[x_1, \ldots, x_n]$, $V(\mathfrak{a}) \neq \emptyset$.

(d) *The maximal ideals of* $k[x_1, \ldots, x_n]$ *are precisely of the form* $(x_1 - a_1, \ldots, x_n - a_n)$, *where* $a_i \in k$.

Proof. (a) $\Rightarrow$ (b): This is the strong form of Hilbert Nullstellensatz.

(b) $\Rightarrow$ (c): Let $\mathfrak{a}$ be a proper ideal of $k[x_1 \ldots, x_n]$. If $V(\mathfrak{a}) = \emptyset$, then $I(V(\mathfrak{a})) = I(\emptyset) = (1)$. But this is contradicts part (a) as $\sqrt{\mathfrak{a}}$ is a proper ideal as well.

(c) $\Rightarrow$ (d): Note that for any $a = (a_1, \ldots, a_n) \in \mathbb{A}^n$ the ideal $\mathfrak{m}_a = (x_1 - a_1, \ldots, x_n - a_n)$ is a maximal ideal since it is the kernel of the evaluation homomorphism $e_a : k[x_1, \ldots, x_n] \to k$ defined as $e_a(f) = f(a)$.

To see why every maximal ideal of $k[x_1, \ldots, x_n]$ is of the form $\mathfrak{m}_a$ for some $a \in \mathbb{A}^n$ let $\mathfrak{m}$ be a maximal ideal of $k[x_1, \ldots, x_n]$. By part (c) we have $V(\mathfrak{m}) \neq \emptyset$. Let $a \in V(\mathfrak{m})$. Then we have $\{a\} \subset V(\mathfrak{m})$ and so $\mathfrak{m} \subset I(V(\mathfrak{m})) \subset I(a) = \mathfrak{m}_a$. Because $\mathfrak{m}$ is maximal we deduce that $\mathfrak{m} = \mathfrak{m}_a$.

(d) $\Rightarrow$ (a): Let $f \in k[x_1]$ be a nonconstant polynomial. Then f is contained in some maximal ideal $\mathfrak{m}_a$ of $k[x_1, \ldots, x_n]$, where $a = (a_1, \ldots, a_n) \in \mathbb{A}^n$. Now note that $V(f) \supset V(\mathfrak{m}_a) = \{a\}$ and so $f(a_1) = 0$. Hence, k is algebraically closed. $\square$