COMMUTATIVE ALGEBRA

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Contents

1. Rings and Ideals

1.1. Ideals and ring homomorphisms. Let A be a ring. A additive subgroup $\mathfrak{a}$ of A is said to be an *ideal* if $x\mathfrak{a} \subset \mathfrak{a}$ for every $x \in A$.

Proposition 1.1. *Let A be a ring and let $\mathfrak{a}$ be an additive subgroup of A. Then $\mathfrak{a}$ is an ideal of A if and only if the multiplication operation*

$$(x + \mathfrak{a})(y + \mathfrak{a}) = xy + \mathfrak{a}$$

on the quotient group $A/\mathfrak{a}$ is well-defined.

Theorem 1.2 (ideals in a quotient ring). *Let A be a ring and let $\mathfrak{a}$ be an ideal of A. Then there is an inclusion preserving bijective correspondence between the ideals of A containing $\mathfrak{a}$ and the ideals of $A/\mathfrak{a}$ given by $\mathfrak{b} \mapsto \mathfrak{b}/\mathfrak{a}$.*

Proof. Consider the canonical projection $\pi : A \to A/\mathfrak{a}$. Then the map $\overline{\mathfrak{b}} \mapsto \pi^{-1}(\overline{\mathfrak{b}})$ is the inverse of the map $\mathfrak{b} \to \mathfrak{b}/\mathfrak{a}$ between the ideals of A containing $\mathfrak{a}$ and the ideals of $A/\mathfrak{a}$. $\square$

It is easy to see that the images and the preimages of subrings under a ring homomorphism are also subrings. In the case of ideals only the preimages of ideals are ideals. The image of an ideal need not be an ideal of the whole codomain. However, image of an ideal is indeed an ideal of the range. Consider the embedding of $\mathbb{Z}$ into $\mathbb{Q}$. Then $\mathbb{Z}$ is an ideal of itself but its image $\mathbb{Z}$ is not an ideal of $\mathbb{Q}$.

Theorem 1.3 (Isomorphism theorems).

(a) Let $f : A \to B$ be a ring homomorphism. Then $A/\ker f \cong f(A)$.

(b) Let $\mathfrak{a}$ be an ideal and let B be a subring of A. Then $B + \mathfrak{a}$ is a subring of A, $B \cap \mathfrak{a}$ is an ideal of B and

$$(B + \mathfrak{a})/\mathfrak{a} \cong B/(B \cap \mathfrak{a}).$$

(c) If $\mathfrak{a} \subset \mathfrak{b}$ are ideals of a ring A, then

$$(A/\mathfrak{a})/(\mathfrak{b}/\mathfrak{a}) \cong A/\mathfrak{b}.$$

1.2. **Zero-divisors, nilpotents, and units.** An element x in a ring A is said to be a *zero-divisor* if $xy = 0$ for some nonzero element y in A. Note that 0 is always a zero-divisor except when the ring is 0. A ring A is said to be an *integral domain* if $A \neq 0$ and A does not have any nonzero zero-divisors.

A ring element x is called *nilpotent* if $x^n = 0$ for some positive integer n. A nilpotent element is always a zero-divisor except when the ring is 0 in which case 0 is nilpotent but not a zero-divisor. The converse however is not true as $\overline{3} \in \mathbb{Z}/6\mathbb{Z}$ and $\overline{x} \in k[x,y]/(xy)$ are both zero-divisors but not nilpotents.

A ring element x is called *unit* if x divides 1, i.e., there is an element y such that $xy = 1$. The set of all unit elements in a ring A form a group called the *multiplicative group* of A. It is usually denoted by $A^\times$. It is easy to see that the set of zero-divisors and units are disjoint.

Note that x is a unit if and only if $(x) = (1)$. A ring A is called a *field* if $A \neq 0$ and every nonzero element is a unit.

PROPOSITION 1.4. *Let A be a ring $\neq 0$ Then the following are equivalent:*

(a) A is a field.
(b) The only ideals of A are 0 and (1).
(c) Every nonzero ring homomorphism from A to a ring B is injective.

Exercises.

EXERCISE 1.1. Identify nilpotent elements in the ring $\mathbb{Z}/n\mathbb{Z}$.

SOLUTION. An element $\overline{a} \in \mathbb{Z}/n\mathbb{Z}$ is nilpotent if and only if $\prod_{p|n} p$ divides a.