Problem. Consider the function f defined on $\mathbb{R}$ by

$$f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Prove that f is infinitely differentiable on $\mathbb{R}$, and that $f^{(n)}(0) = 0$ for all $n \geqslant 1$. Conclude that f does not have a power series expansion at 0.

Solution. We show by induction that for every $n \geqslant 1$ there is a polynomial P_n such that $f^{(n)}(x) = e^{-1/x^2} P_n(1/x)$ if $x \neq 0$ and $f^{(n)}(0) = 0$. This can be shown via induction. For $n = 1$ we have $f'(x) = (2/x^3)e^{-1/x^2}$ and so $P_1(x) = 2x^3$. Moreover, f is differentiable at 0 as $e^{-1/x^2} = O(x^k)$ (near 0) for every $k \geqslant 1$.

Now suppose that $f^{(n)}(x) = e^{-1/x^2} P_n(1/x)$ for some $n \geqslant 1$ and $f^{(n)}(0) = 0$. Then note that

$$f^{(n+1)}(x) = (2/x^3)e^{-1/x^2} P_n(1/x) + e^{-1/x^2} P_n'(1/x)(-1/x^2).$$

Thus $P_{n+1}(x) = 2x^3 P_n(x) - x^2 P_n'(x)$. Since $(1/x^k)e^{-1/x^2} \to 0$ for every $k \geqslant 1$ we obtain that $f^{(n+1)}(0) = 0$.

Now suppose for the sake of contradiction that f has a power series expansion $f = \sum_{n=0}^{\infty} a_n x^n$ at 0. Then $a_n = (1/n!) f^{(n)}(0) = 0$ for every $n \geqslant 0$. This implies that f is identically 0 in a neighborhood of 0 which is a contradiction.