FUNCTIONAL ANALYSIS

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1. Banach spaces

Throughout our notes we will write $\mathbb{K}$ to denote $\mathbb{R}$ or $\mathbb{C}$.

1.1. Normed spaces.

Definition 1.1. Let X be a vector space over $\mathbb{K}$. A *norm* on X is a map $||\cdot|| : \mathbb{K} \to \mathbb{R}$ satisfying the following properties:

(i) (positive definiteness) $||x|| \geqslant 0$ for all $x \in X$ and $||x|| = 0$ if and only if $x = 0$.
(ii) (absolute homogeneity) For every $x \in X$ and $\lambda \in \mathbb{K}$, $||\lambda x|| = |\lambda| \, ||x||$.
(iii) (triangle inequality) For every $x_1, x_2 \in X$, $||x_1 + x_2|| \leqslant ||x_1|| + ||x_2||$.

The ordered pair $(X, ||\cdot||)$ is called a *normed space*.

A norm $||\cdot||$ induces a metric on X defined as

$$d(x, y) = ||x - y|| \, .$$

This metric satisfies translation invariance and absolute homogeneity property. Due to this not every metric is induced by a norm. Think of the discrete metric on X. It does not satisfy absolute homogeneity.

Problem 1.1. Let X be a normed space and let Y be a subspace of X. Show that $\overline{Y}$ is also a subspace of X.

Claim 1.2. Let X be a normed space. The open ball $B_r(x)$ is convex.

We can replace open with closed above.

Example 1.3. Let $1 \leqslant p < \infty$ and let $n \in \mathbb{N}$. Then for $x \in \mathbb{K}^n$ we define

$$||x||_p = \left(\sum_{k=1}^{n} |x_k|^p \right)^{1/p} . \tag{1.1}$$

It is easy to see that $||\cdot||_p$ satisfies positive definiteness and absolute homogeneity. However, the triangle inequality is not obvious except for the case $p = 1$. We have to show that for any $x, y \in \mathbb{K}^n$ we have the inequality

$$||x+y||_p \leqslant ||x||_p + ||y||_p \,.$$

If $||x||_p + ||y||_p = 0$, then $||x||_p = ||y||_p = 0$ and so we have $x = y = 0$. The inequality now follows trivially. Thus suppose that $||x+y||_p \neq 0$. We can now rewrite (1.1) as

$$\left|\left|\frac{x}{||x||_p + ||y||_p} + \frac{y}{||x||_p + ||y||_p}\right|\right|_p \leqslant 1.$$

and we have

$$\left|\left|\frac{x}{||x||_p + ||y||_p}\right|\right|_p + \left|\left|\frac{y}{||x||_p + ||y||_p}\right|\right|_p = 1.$$

Due to this we can restrict ourselves to the case when $||x||_p + ||y||_p = 1$.

Now suppose that $||x||_p + ||y||_p = 1$. Let $\lambda = ||x||_p$. Then we have $||y||_p = 1 - \lambda$. If $\lambda = 0$ or $\lambda = 1$, then (1.1) follows trivially. Thus we suppose that $0 < \lambda < 1$. We can rewrite the inequality $||x+y||_p \leqslant 1$ as

$$\left|\left|\lambda\left(\frac{x}{\lambda}\right) + (1-\lambda)\left(\frac{y}{1-\lambda}\right)\right|\right|_p \leqslant 1$$

where we have

$$\left|\left|\frac{x}{\lambda}\right|\right|_p = 1 = \left|\left|\frac{y}{1-\lambda}\right|\right|_p.$$

Hence it suffices to show that if $||x||_p = ||y||_p = 1$ and $0 \leqslant \lambda \leqslant 1$, then

$$||\lambda x + (1-\lambda)y||_p \leqslant 1.$$

Because the function $t \mapsto t^p$ is convex (as $\frac{d^2t^p}{dt^2} = p(p-1)t^{p-2} \geqslant 0$) for $t \in [0, \infty)$, we have the convexity bound

$$(\lambda a + (1-\lambda)b)^p \leqslant \lambda a^p + (1-\lambda)b^p$$

for $a, b \geqslant 0$ and $0 \leqslant \lambda \leqslant 1$. Applying this we obtain

$$\begin{aligned}
||\lambda x + (1-\lambda)y||_p^P &= \sum_{k=1}^{n} |\lambda x_k + (1-\lambda)y_k|^p \\
&\leqslant \sum_{k=1}^{n} (\lambda|x_k| + (1-\lambda)|y_k|)^p \\
&\leqslant \lambda \sum_{k=1}^{n} |x_k|^p + (1-\lambda) \sum_{k=1}^{n} |y_k|^p \\
&= \lambda + (1-\lambda) = 1.
\end{aligned}$$

This completes the proof of the triangle inequality for the norm $||\cdot||_p$.

Example 1.4. Let $n \in \mathbb{N}$. Then we define the ∞-*norm* on $\mathbb{K}^n$ as

$$||x||_\infty = \max_{1 \leqslant k \leqslant \infty} |x_k|.$$

It is easy to verify that $||\cdot||_\infty$ is indeed a norm on $\mathbb{K}^n$. The triangle inequality is particularly easy to verify.

Example 1.5. Let $1 \leqslant p < \infty$. We define the ℓ^p space as

$$\ell^p = \left\{ \{a_n\}_{n\in\mathbb{N}} \subset \mathbb{K} : \sum_{n=1}^{\infty} |a_n|^p < \infty \right\},$$

i.e., ℓ^p is the collection of all p-summable sequences. It can be easily seen that ℓ^p is a subspace of $\mathbb{K}^{\mathbb{N}}$. To see why it is closed under addition note that

$$(a+b)^p \leqslant (2\max(a,b))^p = 2^p \max(a,b)^p \leqslant 2^p(a^p + b^p)$$

for any nonnegative numbers a, b. We define the p-*norm* on ℓ^p as

$$||a||_p = \left(\sum_{n=1}^{\infty} |a_n|^p \right)^{1/p},$$

where $a = \{a_n\}_{n\in\mathbb{N}}$. It is easy to see that $||\cdot||_p$ satisfies the properties of the norm besides the triangle inequality. Let $a = \{a_n\}_{n\in\mathbb{N}} \in \ell^p$, $b = \{b_n\}_{n\in\mathbb{N}} \in \ell^p$. Then for a fixed $N \in \mathbb{N}$ we have

$$\left(\sum_{n=1}^{N} |a_n + b_n|^p \right)^{1/p} \leqslant \left(\sum_{n=1}^{N} |a_n|^p \right)^{1/p} + \left(\sum_{n=1}^{N} |b_n|^p \right)^{1/p} \leqslant ||a||_p + ||b||_p$$

due to the triangle inequality for $\mathbb{K}^N$. Now taking the limit as $N \to \infty$ we get

$$||a+b||_p = \lim_{N\to\infty} \left(\sum_{n=1}^{N} |a_n + b_n|^p \right)^{1/p} \leqslant ||a||_p + ||b||_p .$$

Thus $||\cdot||_p$ is a norm on ℓ^p.

Example 1.6. Let ℓ^∞ denote the space of all bounded sequences, i.e.,

$$\ell^\infty = \left\{ \{a_n\}_{n\in\mathbb{N}} \subset \mathbb{K} : \sup_{k\in\mathbb{N}} |a_k| < \infty \right\}$$

Then we define the ∞-*norm* on ℓ^∞ as follows:

$$||a||_\infty = \sup_{k\in\mathbb{N}} |a_k|,$$

where $a = \{a_n\}_{n\in\mathbb{N}} \in \ell^\infty$. It is easy to verify that $||\cdot||_\infty$ is indeed a norm on ℓ^∞.

Example 1.7. Let X be a measure space with measure μ and let $1 \leqslant p < \infty$. The space $L^p(X)$ of all equivalence classes of p-integrable functions (f is equivalent to g if $f = g$ a.e.) is a normed space under the norm

$$||f||_p = \left(\int_X |f|^p \, d\mu \right)^{1/p}$$

Moreover, for a measurable function $f : X \to \overline{\mathbb{R}}$ (or $\mathbb{C}$) we define

$$||f||_\infty = \inf\{\alpha \in [0,\infty] : |f| \leqslant \alpha \text{ a.e.}\}$$

The space $L^\infty(X)$ is defined to be the collection of equivalence classes of essentially bounded functions, i.e., $||f||_\infty < \infty$.

Let $n \in \mathbb{N}$. If $X = \{1,\ldots,n\}$ and μ is the counting measure, then $L^p(X) = \mathbb{K}^n$ as counting measure of any nonempty set is nonzero and so $[x] = [y] \Leftrightarrow x = y$ (here we are identifying the equivalence class containing a single element with element itself). Moreover, we have

$$||x||_p = \left(\int_X |x|^p \, d\mu\right)^{1/p}$$

(here $|x|$ denotes the function $k \mapsto |x_k|$). The triangle inequality for $\mathbb{K}^n$ is now just the Minkowski inequality for space $L^p(X)$. Similarly, we have $(\mathbb{K}^n, ||\cdot||_\infty) = (L^\infty(X), ||\cdot||_\infty)$.

Example 1.8. Let X be a metric space and let $\mathcal{B}_{\mathbb{K}}(X)$ be the space of all bounded $\mathbb{K}$-valued functions on X. Then $\mathcal{B}_{\mathbb{K}}(X)$ is a normed space over $\mathbb{K}$ under the norm

$$||f||_\infty = \sup_{x\in X} |f(x)|.$$

Moreover, the space $\mathcal{BC}_{\mathbb{K}}(X)$ of all bounded and continuous $\mathbb{K}$-valued functions on X is normed subspace of $\mathcal{B}_{\mathbb{K}}(X)$.

Two norms $||\cdot||_1$ and $||\cdot||_2$ on a vector space X are said to be *equivalent* if there are positive constants A, B such that

$$A\,||x||_2 \leqslant ||x||_1 \leqslant B\,||x||_2\,.$$

It can be easily seen that this is indeed an equivalence relation.

All p-norms $(1 \leqslant p \leqslant \infty)$ on $\mathbb{K}^n$ are equivalent as

$$||x||_\infty \leqslant ||x||_p \leqslant n^{1/p}\,||x||_\infty$$

for every $1 \leqslant p < \infty$.

1.2. **Banach spaces.**

Definition 1.9. A complete normed space is said to be a *Banach space.*

Claim 1.10. If X is a vector space and $||\cdot||_1$ are $||\cdot||_2$ are two equivalent norms on X, then $(X, ||\cdot||_1)$ is a Banach space if and only if $(X, ||\cdot||_2)$ is a Banach space.

Example 1.11. $\mathbb{K}$ is a Banach space as $\mathbb{R}$ is complete.

Example 1.12. $(\mathbb{K}^n, ||\cdot||_1)$ is a Banach space. To see this let $\{a_k\}_{k\in\mathbb{N}} \subset \mathbb{K}^n$ be a Cauchy sequence. Then it can be easily seen that each component sequence $\{a_{k,i}\}_{k\in\mathbb{N}} \subset \mathbb{K}$ is Cauchy (for $1 \leqslant i \leqslant n$). Because $\mathbb{K}$ is complete, it follows that $a_{k,i} \to a_i$ for some $a_i \in \mathbb{K}$. Taking $a = (a_1,\ldots,a_n)$ it now easily follows that $a_k \to a$.

Since all p-norms on $\mathbb{K}^n$ are equivalent, it follows that $(\mathbb{K}^n, ||\cdot||_p)$ is a Banach space for each $1 \leqslant p \leqslant \infty$.

Example 1.13. Let $1 \leqslant p < \infty$. Then ℓ^p is a Banach space. Let $\{a_k\}_{k\in\mathbb{N}} \subset \ell^p$ be Cauchy. Then it can be easily seen that each component sequence $\{a_{k,n}\}_{k\in\mathbb{N}} \subset \mathbb{K}$ is also Cauchy (for $n \in \mathbb{N}$) as $|a_{k,n} - a_{\ell,n}| \leqslant ||a_k - a_\ell||_p$. It now follows from the completeness of $\mathbb{K}$ that $a_{k,n} \to a_n$ for some $a_n \in \mathbb{K}$.

Let $\epsilon > 0$ be fixed. Since the sequence $\{a_k\}_{k\in\mathbb{N}}$ is Cauchy, there is a $K \in \mathbb{N}$ such that $||a_k - a_\ell|| \leqslant \epsilon$ for every $k, \ell \geqslant K$. For a fixed $N \in \mathbb{N}$ we then have

$$\left(\sum_{n=1}^{N} |a_{k,n} - a_{\ell,n}|^p\right)^{1/p} \leqslant ||a_k - a_\ell||_p \leqslant \epsilon.$$

Keeping k fixed and rolling ℓ off to ∞ we get

$$\left(\sum_{n=1}^{N} |a_{k,n} - a_n|^p\right)^{1/p} = \lim_{\ell\to\infty} \left(\sum_{n=1}^{N} |a_{k,n} - a_{\ell,n}|^p\right)^{1/p} \leqslant \epsilon.$$

Finally, taking the limit as $N \to \infty$ we obtain

$$||a_k - a||_p = \lim_{N\to\infty} \left(\sum_{n=1}^{N} |a_{k,n} - a_n|^p\right)^{1/p} \leqslant \epsilon$$

for $k \geqslant K$. This not only shows that $a = \{a_n\}_{n\in\mathbb{N}} \in \ell^p$ but also that $a_k \to a$ in ℓ^p.

Example 1.14. The space ℓ^∞ is a Banach space. Let $\{a_k\}_{k\in\mathbb{N}} \subset \ell^\infty$ be a Cauchy sequence. Then each component sequence $\{a_{k,n}\}_{n\in\mathbb{N}} \subset \mathbb{K}$ is also Cauchy as $|a_{k,n} - a_{\ell,n}| \leqslant ||a_k - a_\ell||_\infty$. Thus $a_{k,n} \to a_n$ as $k \to \infty$ for some $a_n \in \mathbb{K}$. Let $\epsilon > 0$ be fixed. Then there is a $K \in \mathbb{N}$ such that $||a_k - a_\ell|| \leqslant \epsilon$ for every $k, \ell \geqslant K$. Note that

$$|a_{k,n} - a_{\ell,n}| \leqslant ||a_k - a_\ell||_\infty \leqslant \epsilon$$

for $k, \ell \geqslant K$ and $n \in \mathbb{N}$. Keeping k fixed and rolling ℓ off to ∞ we get

$$|a_{k,n} - a_n| \leqslant \epsilon$$

for every $k \geqslant K$ and $n \in \mathbb{N}$ and so we have $||a_k - a||_\infty \leqslant \epsilon$, where $a = \{a_n\}_{n\in\mathbb{N}}$. This implies that the sequence $\{a_n\}_{n\in\mathbb{N}}$ lies in ℓ^∞ and that $||a_k - a||_\infty \leqslant \epsilon$ for every $k \in \mathbb{N}$. Hence, $a_k \to a$ in ℓ^∞.

Example 1.15. The space c_o of all sequences $\{a_n\}_{n\in\mathbb{N}} \subset \mathbb{K}$ that converge to 0 is a Banach space under the $||\cdot||_\infty$ norm. It can be easily seen that it is a vector subspace of ℓ^∞. Moreover, if $\{a_k\}_{k\in\mathbb{N}} \subset c_0$ is a Cauchy sequence, then we know that it converges to some $a = \{a_n\}_{n\in\mathbb{N}} \in \ell^\infty$ as ℓ^∞ is a Banach space. It just remains to show that $a_n \to 0$ as $n \to \infty$ which is quite easy to show.

Example 1.16. Let X be a measure space. The space $L^p(X)$ is a Banach space for each $1 \leqslant p \leqslant \infty$. The proof of this fact is usually proved in measure theory courses and the results that $\mathbb{K}^n$ and ℓ^p are Banach spaces also follow from this.

1.3. **Bounded linear operators.**

Definition 1.17. A linear operator $\Lambda : X \to Y$ between two normed spaces is said to be *bounded* if there is a $C > 0$ such that

$$||\Lambda x|| \leqslant C\,||x||$$

for every $x \in X$.

Definition 1.18. The norm of a linear operator $\Lambda : X \to Y$ between two normed spaces is defined as

$$||\Lambda|| = \sup_{\substack{x\in X\\ ||x||=1}} ||\Lambda x||$$

It can be easily seen that a linear operator $\Lambda : X \to Y$ is bounded if and only if $||\Lambda|| < \infty$. Moreover we have

$$||\Lambda|| = \sup_{\substack{x\in X\\ ||x||\leqslant 1}} ||\Lambda x||$$

Claim 1.19. Let $\Lambda : X \to Y$ be a linear operator between normed spaces. Then Λ is bounded if and only if Λ maps bounded sets to bounded sets.

Claim 1.20. Let X and Y be normed spaces and let $\Lambda : X \to Y$ be a linear operator. If Λ is not bounded, then there there exists a sequence $\{x_n\} \subset X$ such that $||x_n|| \to 0$ and $||\Lambda x_n|| \to \infty$ as $n \to \infty$.

Theorem 1.21. Let X and Y be normed spaces and let $\Lambda : X \to Y$ be a linear operator. Then Λ is bounded if and only if Λ is continuous.

Proof. If $\Lambda : X \to Y$ is bounded, then it easily follows by linearity that Λ is Lipschitz continuous and hence is continuous in particular.

Now suppose that Λ is continuous. Then there is a $\delta > 0$ such that $||\Lambda(x)||_Y \leqslant 1$ whenever $||x||_X \leqslant \delta$. Now if $||x||_X = 1$, then $||\delta x||_X = \delta$ and so we have

$$||\Lambda(\delta x)||_Y \leqslant 1.$$

This implies that

$$||\Lambda(x)||_Y \leqslant \frac{1}{\delta}$$

for every $x \in X$ with $||x||_X = 1$. Hence, we have $||\Lambda|| < \infty$, i.e., Λ is bounded. □

Theorem 1.22. Let X be a finite-dimensional normed space over $\mathbb{K}$ and let $\{v_1, \dots, v_n\}$ be a basis of X. Then the map $\Lambda : \mathbb{K}^n \to X$ defined as

$$\Lambda(\alpha) = \alpha_1 v_1 + \cdots + \alpha_n v_n,$$

where $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{K}^n$, is a bijection and a bounded linear operator. Moreover, the inverse map Λ^{-1} is also bounded linear operator.

Proof. It is easy to see that Λ is an isomorphism of vector spaces (due to rank-nullity theorem as $\dim X = \dim \mathbb{K}^n = n$).

We now see that Λ is a bounded linear operator as

$$||\Lambda(\alpha)||_X = \left|\left|\sum_{i=1}^{n} \alpha_i v_i\right|\right|_X \leqslant \sum_{i=1}^{n} |\alpha_i|\,||v_i||_X \leqslant ||\alpha||_2 \sum_{i=1}^{n} ||v_i||_X\,,$$

where $||\alpha||_2$ denotes the Euclidean norm on $\mathbb{K}^n$. This shows that Λ is bounded and hence continuous.

Now suppose for the sake of contradiction that Λ^{-1} is not bounded. Then there is a sequence $\{x_n\} \subset X$ such that $||x_n||_X = 1$ for every $n \in \mathbb{N}$ and

$$\lim_{n\to\infty} \left|\left|\Lambda^{-1}(x_n)\right|\right|_2 = \infty.$$

Now let

$$\tilde{x}_n = \frac{x_n}{||\Lambda^{-1}(x_n)||_2} \quad \text{and} \quad \beta_n = \frac{\Lambda^{-1}(x_n)}{||\Lambda^{-1}(x_n)||_2}$$

for $n \in \mathbb{N}$. Then we have $\Lambda(\beta_n) = \tilde{x}_n$. Moreover, we have

$$\lim_{n\to\infty} ||\tilde{x}_n||_X = \lim_{n\to\infty} \frac{1}{||\Lambda^{-1}(x_n)||_2} = 0. \tag{1.2}$$

Since $\{\beta_n\}$ is a bounded sequence ($||\beta_n||_2 = 1$), it has a convergent subsequence $\{\beta_{n_k}\}$ (as every component sequence is bounded and $\mathbb{K}$ has the Heini-Borel property). Let $\beta = \lim_{k\to\infty} \beta_{n_k}$. Then we have $||\beta||_2 = 1$ and in particular $\beta \neq 0$. As Λ is continuous, we have

$$\lim_{k\to\infty} \Lambda(\beta_{n_k}) = \Lambda(\beta).$$

But we also have

$$\lim_{k\to\infty} \Lambda(\beta_{n_k}) = \lim_{k\to\infty} \tilde{x}_{n_k} = 0$$

due to (1.2). This shows that $\Lambda(\beta) = 0$ and so $\beta = 0$, a contradiction.

□

Corollary 1.23. Every finite-dimensional normed space is a Banach space.

Corollary 1.24. In a finite-dimensional space, all norms are equivalent.

Proof. Let $||\cdot||_\triangle$ and $||\cdot||_\square$ be norms on X. Suppose $\dim X = n$ and let $\Lambda : \mathbb{K}^n \to X$ be bijective linear operator. Then we know that Λ is a homeomorphism between $(\mathbb{K}^n, ||\cdot||_2)$ and $(X, ||\cdot||_\triangle)$ and also between $(\mathbb{K}^n, ||\cdot||_2)$ and $(X, ||\cdot||_\square)$. Because of the boundedness of Λ and Λ^{-1} (both with respect to $||\cdot||_\triangle$ and $||\cdot||_\square$) there are positive constants A, B, C, D such that

$$A\left|\left|\Lambda^{-1}(x)\right|\right|_2 \leqslant ||x||_\triangle \leqslant B\left|\left|\Lambda^{-1}(x)\right|\right|_2 \quad \text{and} \quad C\left|\left|\Lambda^{-1}(x)\right|\right|_2 \leqslant ||x||_\square \leqslant D\left|\left|\Lambda^{-1}(x)\right|\right|_2.$$

From this we get

$$\frac{A}{D}\,||x||_\square \leqslant ||x||_\triangle \leqslant \frac{B}{C}\,||x||_\square\,,$$

i.e., $||\cdot||_\triangle$ and $||\cdot||_\square$ are equivalent.

□