PAIR CORRELATION CONJECTURE

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Contents

1. Introduction

Montgomery introduces the function $F(\alpha)$ defined as

$$F(\alpha) = F(\alpha, T) = \left(\frac{T}{2\pi}\log T\right)^{-1} \sum_{0<\gamma,\gamma'\leqslant T} T^{i\alpha(\gamma-\gamma')} w(\gamma-\gamma'),$$

where α and $T \geqslant 2$ are both real and $w(u) = 4/(4+u^2)$. It can be readily observed that F is real-valued as $F(\alpha) = \overline{F(\alpha)}$ and F is even; $F(\alpha) = F(-\alpha)$. It is however not immediately obvious that $F(\alpha) \geqslant 0$. In the statement of main theorem in his paper, Montgomery states that if $T > T_0(\epsilon)$, then $F(\alpha) \geqslant -\epsilon$ for all α, which is a weaker statement.

The idea behind the proof of nonnegativity of $F(\alpha)$ is that we can decouple the term $w(\gamma-\gamma')$ not as a product $g(\gamma)g(\gamma')$ but as an improper integral $\int_{-\infty}^{\infty} g(\gamma, x)g(\gamma', x)\,dx$.

We consider the improper integral

$$\int_{-\infty}^{\infty} \frac{dx}{(1+(x-a)^2)(1+(x-b)^2)}.$$

Let

$$f(z) = \frac{1}{(1+(z-a)^2)(1+(z-b)^2)}.$$

Let $R > 0$. We take γ_R be the line segment from $-R$ to R and Γ_R to be the semicircle of radius R in positive orientation, i.e., $\Gamma_R(t) = Re^{it}$ with $t \in [0, \pi]$. Then by Cauchy's residue theorem we have

$$\int_{\gamma_R+\Gamma_R} f(z)\,dz = 2\pi i(\operatorname{Res}(f, a+i) + \operatorname{Res}(f, b+i)).$$

It can be easily seen that the integral of f over Γ_R goes to 0 as $R \to \infty$ due to the following estimate

$$\left|\int_{\Gamma_R} f(z)\,dz\right| \leqslant \frac{\pi R}{((R-a)^2-1)((R-b)^2-1)}.$$

Hence we have

$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{R\to\infty} \int_{\gamma_R+\Gamma_R} f(z)\,dz = 2\pi i(\operatorname{Res}(f, a+i) + \operatorname{Res}(f, b+i)).$$

Now observe that

$$\begin{aligned}\operatorname{Res}(f, a+i) &= \lim_{z\to a+i}(z-a-i)f(z)\\ &= \lim_{z\to a+i}\frac{1}{(z-a+i)(z-b-i)(z-b+i)}\\ &= \frac{1}{2i(a-b)(a-b+2i)}.\end{aligned}$$

Similarly, we have

$$\operatorname{Res}(f, b+i) = \frac{1}{2i(b-a)(b-a+2i)} = \frac{1}{2i(a-b)(a-b-2i)}.$$

Thus we have

$$\operatorname{Res}(f, a+i) + \operatorname{Res}(f, b+i) = \frac{1}{i(4+(a-b)^2)}$$

and the integral evaluates to

$$\int_{-\infty}^{\infty}\frac{dx}{(1+(x-a)^2)(1+(x-b)^2)} = \frac{2\pi}{4+(a-b)^2} = \frac{\pi}{2}w(a-b).$$

We can write

$$\left(\frac{T}{2\pi}\log T\right)F(\alpha) = \frac{2}{\pi}\sum_{0<\gamma,\gamma'\leqslant T}T^{i\alpha(\gamma-\gamma')}\int_{-\infty}^{\infty}\frac{dx}{(1+(x-\gamma)^2)(1+(x-\gamma')^2)}.$$

Since we have a finite sum we can interchange the integral and sum to obtain

$$\begin{aligned}\left(\frac{T}{2\pi}\log T\right)F(\alpha) &= \frac{2}{\pi}\int_{-\infty}^{\infty}\sum_{0<\gamma,\gamma'\leqslant T}\frac{T^{i\alpha(\gamma-\gamma')}}{(1+(x-\gamma)^2)(1+(x-\gamma')^2)}\\ &= \frac{2}{\pi}\int_{-\infty}^{\infty}\left|\sum_{0<\gamma\leqslant T}\frac{T^{i\alpha\gamma}}{1+(x-\gamma)^2}\right|^2 dx.\end{aligned}$$

The Montgomery's function $F(\alpha)$ should be thought of as a family of functions parametrized by T. We now state the main result of Montgomery concerning the behavior of $F(\alpha)$ in the unit interval.

THEOREM 1.1. *Assume RH to be true. Then for every fixed* $0 \leqslant \alpha < 1$ *we have*

$$F(\alpha) = (1+o(1))T^{-2\alpha}\log T + \alpha + o(1) \tag{1.1}$$

as $T \to \infty$. *This estimate holds uniformly for* $0 \leqslant \alpha \leqslant 1-\epsilon$.

Later Montgomery along with Goldston showed that the estimate 1.1 holds uniformly for $0 \leqslant \alpha \leqslant 1$.

Note that the function $T^{-2|\alpha|}\log T$ behaves in the limit as a Dirac δ-function as $T^{-2|\alpha|}\log T \to 0$ as $T \to \infty$ for every $\alpha \neq 0$ and

$$\int_{-A}^{A}T^{-2|\alpha|}\log T\,d\alpha = 2\int_0^A T^{-2\alpha}\log T\,d\alpha = \big[-T^{-2\alpha}\big]_0^A = 1 - T^{-2A}$$

tends to 1 as $A \to \infty$.