WEEK 1 PROBLEMS

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Problems

1. Show that
$$\sum_{d^k\,|\,n} \mu(d) = \begin{cases} 1 & \text{if } n \text{ is } k\text{th power-free,} \\ 0 & \text{otherwise.} \end{cases}$$

2. Show that
$$\sum_{d\,|\,n} \lambda(d) = \begin{cases} 1 & \text{if } n \text{ is a square,} \\ 0 & \text{otherwise.} \end{cases}$$

3. Show that $\varphi(n) \to \infty$ as $n \to \infty$.

4. Let $e(\alpha) = e^{2\pi i \alpha}$. Show that
$$\frac{1}{q} \sum_{a=1}^{q} e\left(\frac{an}{q}\right) = \begin{cases} 1 & \text{if } q|n, \\ 0 & \text{otherwise.} \end{cases}$$

5. The Ramanujan's sum $c_q(n)$ is defined as
$$c_q(n) = \sum_{\substack{a=1 \\ (a,q)=1}}^{q} e\left(\frac{an}{q}\right).$$

Show that
$$c_q(n) = \sum_{d\,|\,(n,q)} d\mu\left(\frac{q}{d}\right).$$

Deduce that
$$\mu(q) = \sum_{\substack{a=1 \\ (a,q)=1}}^{q} e\left(\frac{a}{q}\right).$$

6. Let $d = (n, q)$. Show that
$$c_q(n) = \mu\left(\frac{q}{d}\right) \varphi(q) \varphi\left(\frac{q}{d}\right)^{-1}.$$

7. Show that $d(n) \ll_\epsilon n^\epsilon$ for every $\epsilon > 0$.

8. Using $d(n) \ll_\epsilon n^\epsilon$ or otherwise show that $\varphi(n) \gg_\epsilon n^{1-\epsilon}$ for every $\epsilon > 0$.

9. Show that $\sigma(n) \le n(1 + \log n)$ for every $n \in \mathbb{N}$.

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10. Show that
$$\frac{6n^2}{\pi^2} \le \varphi(n)\sigma(n) \le n^2$$
for every $n \in \mathbb{N}$. Also conclude that $\varphi(n) \gg n/\log n$.

11. Assuming primes are finite show that $\varphi(n) \asymp n$ and obtain a contradiction.

12. Let $\gamma(n) = \prod_{p|n} p$. Show that
$$\sum_{n=1}^{\infty} \frac{1}{n\gamma(n)} < \infty.$$

13. Show that
$$\sum_{n\le x} \frac{n}{\varphi(n)} \ll x.$$

14. Show that
$$\sum_{n\le x} \frac{1}{\varphi(n)} \ll \log x.$$

15. Prove that
$$\sum_{n\le x} \varphi(n) = \frac{3x^2}{\pi^2} + O(x\log x).$$

16. Prove that
$$\sum_{n\le x} \frac{\varphi(n)}{n} = \frac{6x}{\pi^2} + O(\log^2 x).$$

17. (i) Show that
$$\frac{n}{\varphi(n)} = \sum_{d|n} \frac{\mu(d)^2}{\varphi(d)}.$$

(ii) Using the identity in part (i) prove that
$$\sum_{n\le x} \frac{1}{\varphi(n)} = A\log x + B + O\left(\frac{\log^2 x}{x}\right)$$
for some constants A and B.

18. Let $Q(x)$ denote the number of squarefree integers $\le x$.

(i) Prove that
$$Q(x) = \frac{6}{\pi^2}x + O(\sqrt{x}).$$

(ii) Prove that for any $n \in \mathbb{N}$
$$Q(n) \ge n - \sum_{p} \left\lfloor \frac{n}{p^2} \right\rfloor.$$

(iii) Prove that
$$\sum_{p} \frac{1}{p^2} < \frac{1}{2}$$
and conclude that $Q(n) > n/2$ for all $n \in \mathbb{N}$.

(iv) Prove that every integer $n > 1$ can be written as a sum of two squarefree numbers.

19. For a prime p let $\mathcal{A}_p$ denote the set of all positive integers n such that either $n+1 \equiv 0 \pmod{p^2}$ or $n-1 \equiv 0 \pmod{p^2}$.

(i) If $[N] := \{1, \ldots, N\}$, then show that

$$\left| [N] \cap \bigcup_p \mathcal{A}_p \right| \leq 2N \sum_{p \leq N} \frac{1}{p^2} + 2\pi(N).$$

(ii) Prove that

$$\sum_p \frac{1}{p^2} < \frac{1}{4}$$

and conclude that the set of twin squarefree integers have positive density.

20. Let $\vartheta(x) = \sum_{p \leq x} \log p$.

(i) Let m, n be integers with $m \geq 2n > 0$. Show that

$$\prod_{m-n<p\leq m} p \,\Big|\, \binom{m}{n}.$$

(ii) Show that $\vartheta(2n) - \vartheta(n) \leq 2n \log 2$.
(iii) Prove that $\vartheta(x) \leq 4x \log 2$ for every $x \geq 1$.

Solutions

1. Let $n = m^k q$, where q is kth power-free. Then observe that $d^k | n$ if and only if $d|m$. Thus
$$\sum_{d^k|n} \mu(d) = \sum_{d|m} \mu(d) = e(m).$$
The assertion now follows as n is kth power-free if and only if $m = 1$.

2. Let $f(n) = \sum_{d|n} \lambda(d)$. Then $f = \lambda * 1$ is multiplicative and so it suffices to evaluate f at prime powers. Note that
$$f(p^a) = \lambda(1) + \lambda(p) + \cdots + \lambda(p^a) = 1 - 1 + \cdots + (-1)^a = \begin{cases} 1 & \text{if } a \text{ is even,} \\ 0 & \text{otherwise.} \end{cases}$$
The assertion now immediately follows from this.

3. Let $M > 0$ and let $\varphi(n) \le M$. Take $n = \prod_{i=1}^k p_i^{a_i}$. Then we have $\varphi(n) = \prod_{i=1}^k p_i^{a_i - 1}(p_i - 1) \le M$. This shows that $p_i - 1 \le M$ and $2^{a_i - 1} \le M$ for every i. Hence we have $p_i \le M + 1$ and $2^{a_i} \le 2M$ for every i. Thus the exponents a_i are bounded by $\log_2(2M)$. This shows that there are only finitely many positive integers n with $\varphi(n) \le M$ and so we conclude that $\varphi(n) \to \infty$ as $n \to \infty$.

4. Note that if $q|n$, then $e(an/q) = 1$ for every $1 \le a \le q$ and so we have
$$\frac{1}{q}\sum_{a=1}^{q} e(an/q) = \frac{1}{q}\sum_{a=1}^{q} 1 = 1.$$
Now suppose that $q \nmid n$. Then we have $e(n/q) \ne 1$ and so
$$\frac{1}{q}\sum_{a=1}^{q} e(an/q) = \frac{1}{q}\sum_{a=1}^{q} e(n/q)^a = \frac{1}{q}\left(\frac{e(n/q)^{q+1} - 1}{e(n/q) - 1} - 1\right) = 0$$
as $e(n/q)^{q+1} = e(n/q)$.

5. Observe that
$$c_q(n) = \sum_{\substack{a=1 \\ (a,q)=1}}^{q} e\left(\frac{an}{q}\right) = \sum_{a=1}^{q} e\left(\frac{an}{q}\right) \sum_{d|(a,q)} \mu(d) = \sum_{a=1}^{q} e\left(\frac{an}{q}\right) \sum_{\substack{d|a \\ d|q}} \mu(d).$$
Changing the order of summation we get
$$c_q(n) = \sum_{d|q} \mu(d) \sum_{\substack{a=1 \\ d|a}}^{q} e\left(\frac{an}{q}\right) = \sum_{d|q} \mu(d) \sum_{r=1}^{q/d} e\left(\frac{rdn}{q}\right) = \sum_{d|q} \mu(d) \sum_{r=1}^{q/d} e\left(\frac{rn}{q/d}\right).$$
Thus we can rewrite $c_q(n)$ as
$$c_q(n) = \sum_{d|q} \mu\left(\frac{q}{d}\right) \sum_{r=1}^{d} e\left(\frac{rn}{d}\right).$$
Using the identity $q^{-1}\sum_{a=1}^{q} e(an/q) = \mathbf{1}_{q|n}$ we end up with
$$c_q(n) = \sum_{\substack{d|q \\ d|n}} d\mu\left(\frac{q}{d}\right) = \sum_{d|(q,n)} d\mu\left(\frac{q}{d}\right).$$

Finally, note that

$$\mu(q) = c_q(1) = \sum_{\substack{a=1 \\ (a,q)=1}}^{q} e\left(\frac{a}{q}\right).$$

6. For a fixed n, $c_q(n)$ is a multiplicative function of q. The right side is also multiplicative as a function of q and so it suffices to verify the equality for prime powers.

7. Let $n = p_1^{a_1} \cdots p_k^{a_k}$. Then we have

$$\frac{d(n)}{n^\epsilon} = \prod_{p^a || n} \frac{a+1}{p^{a\epsilon}} \leq \prod_{\substack{p^a || n \\ p < 2^{1/\epsilon}}} \frac{a+1}{p^{a\epsilon}}$$

for if $p \geq 2^{1/\epsilon}$, then $p^\epsilon \geq 2$ and so $p^{a\epsilon} \geq 2^a \geq a+1$ which gives $(a+1)/p^{a\epsilon} \leq 1$. Now observe that

$$\frac{d(n)}{n^\epsilon} \leq \prod_{\substack{p^a || n \\ p < 2^{1/\epsilon}}} \frac{a+1}{2^{a\epsilon}} \leq \prod_{\substack{p^a || n \\ p < 2^{1/\epsilon}}} \frac{a+1}{a\epsilon \log 2}$$

as $2^{a\epsilon} = e^{a\epsilon \log 2} \geq a\epsilon \log 2$. Finally we have

$$\frac{d(n)}{n^\epsilon} \leq \prod_{p < 2^{1/\epsilon}} \frac{2}{\epsilon \log 2} \leq \left(\frac{2}{\epsilon \log 2}\right)^{\pi(2^{1/\epsilon})}.$$

This shows that $d(n) \ll_\epsilon n^\epsilon$.

8. Note that

$$\varphi(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right) \geq \frac{n}{2^{\omega(n)}} \geq \frac{n}{d(n)} \gg_\epsilon n^{1-\epsilon}$$

as $1 - \frac{1}{p} \geq \frac{1}{2}$ for every prime p and $d(n) \geq 2^{\omega(n)}$ for every $n \in \mathbb{N}$.

We present another solution which involves bounding the product $\prod_{p|n}\left(1 - \frac{1}{p}\right)$ from below. Let n be a positive integer. Then we have

$$\log \prod_{p|n} \left(1 - \frac{1}{p}\right) = \sum_{p|n} \log\left(1 - \frac{1}{p}\right) = -\sum_{p|n} \sum_{m=1}^{\infty} \frac{1}{mp^m} = -\sum_{m=1}^{\infty} \frac{1}{m} \sum_{p|n} \frac{1}{p^m}.$$

Let $p_1 < p_2 < \cdots$ be the sequence of primes. Then for $K > 1$ we have

$$\sum_{p|n} \frac{1}{p^m} \leq \sum_{k<K} \frac{1}{p_k^m} + \frac{\omega(n)}{p_K^m}.$$

This then leads to

$$\begin{aligned}\sum_{m=1}^{\infty} \frac{1}{m} \sum_{p|n} \frac{1}{p^m} &\leq \sum_{m=1}^{\infty} \frac{1}{m} \sum_{k<K} \frac{1}{p_k^m} + \omega(n) \sum_{m=1}^{\infty} \frac{1}{mp_K^m} \\ &= -\sum_{k<K} \log\left(1 - \frac{1}{p_k}\right) - \omega(n) \log\left(1 - \frac{1}{p_K}\right).\end{aligned}$$

Using the inequality $\omega(n) \le \log_2 n$ we obtain

$$\log \prod_{p|n} \left(1 - \frac{1}{p}\right) \ge \sum_{k<K} \log \left(1 - \frac{1}{p_k}\right) + \frac{\log n}{\log 2} \log \left(1 - \frac{1}{p_K}\right).$$

If we denote

$$c_K = \sum_{k<K} \log \left(1 - \frac{1}{p_k}\right) \quad \text{and} \quad \epsilon_K = -\frac{1}{\log 2} \log \left(1 - \frac{1}{p_K}\right),$$

then we have

$$\log \varphi(n) = \log n + \log \prod_{p|n} \left(1 - \frac{1}{p}\right) \ge c_K + (1 - \epsilon_K) \log n.$$

Hence we get

$$\varphi(n) \ge e^{c_K} n^{1-\epsilon_K}.$$

Since $\epsilon_K \to 0$ as $K \to \infty$ we conclude that $\varphi(n) \gg_\epsilon n^{1-\epsilon}$ for every $\epsilon > 0$.

9. Note that

$$\sigma(n) = n \sum_{d|n} \frac{1}{d} \le n \sum_{k=1}^{n} \frac{1}{k} = n(\log n + 1).$$

10. Observe that

$$\begin{aligned} \varphi(n)\sigma(n) &= \prod_{p^a||n} \varphi(p^a)\sigma(p^a) = \prod_{p^a||n} p^{a-1}(p-1) \left(\frac{p^{a+1}-1}{p-1}\right) \\ &= \prod_{p^a||n} p^{a-1}(p^{a+1}-1) = n^2 \prod_{p^a||n} \left(1 - \frac{1}{p^{a+1}}\right). \end{aligned}$$

Because the product is ≤ 1 we get the inequality $\varphi(n)\sigma(n) \le 1$. As for the other inequality note that

$$\prod_{p^a||n} \left(1 - \frac{1}{p^{a+1}}\right) \ge \prod_{p} \left(1 - \frac{1}{p^2}\right) = \frac{1}{\zeta(2)} = \frac{6}{\pi^2}.$$

11. If primes are finite then

$$\varphi(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right) \ge n \prod_{p} \left(1 - \frac{1}{p}\right).$$

Let $c = \prod_p \left(1 - \frac{1}{p}\right)$. Then we have $cn^2 \sum_{d|n} \frac{1}{d} \le \varphi(n)\sigma(n) \le n^2$ and so $\sum_{d|n} \frac{1}{d} = O(1)$, a contradiction as $\sum_{d|m!} \frac{1}{d} \ge \sum_{k=1}^{m} \frac{1}{k}$ and harmonic sums are unbounded.

12. First note that the map $n \mapsto n\gamma(n)$ is injective due to unique factorization. Now observe that

$$\begin{aligned} \sum_{n \le x} \frac{1}{n\gamma(n)} &\le \prod_{p \le x} \left(1 + \frac{1}{p^2} + \frac{1}{p^3} + \cdots\right) \\ &= \prod_{p \le x} \left(1 + \frac{1}{p^2} \frac{1}{1 - 1/p}\right) \\ &= \prod_{p \le x} \left(1 + \frac{1}{p(p-1)}\right), \end{aligned}$$

where the first inequality follows since every prime occurs in $n\gamma(n)$ occurs with power at least 2. Because the infinite product $\prod_p \left(1+\frac{1}{p(p-1)}\right)$ converges, the conclusion follows.

13. Note that

$$\frac{n}{\varphi(n)} = \prod_{p\mid n}\left(1-\frac{1}{p}\right)^{-1} = \prod_{p\mid n}\left(1+\frac{1}{p}+\frac{1}{p^2}+\cdots\right) = \sum_{\substack{m\\ p\mid m \Rightarrow p\mid n}} \frac{1}{m} = \sum_{\substack{m\\ \gamma(m)\mid\gamma(n)}} \frac{1}{m}.$$

Using this we get

$$\sum_{n\le x}\frac{n}{\varphi(n)} = \sum_{n\le x}\sum_{\substack{m\\ \gamma(m)\mid\gamma(n)}}\frac{1}{m} = \sum_{m}\frac{1}{m}\sum_{\substack{n\le x\\ \gamma(m)\mid\gamma(n)}} 1 \le \sum_{m}\frac{1}{m}\sum_{\substack{n\le x\\ \gamma(m)\mid n}} 1 \le x\sum_{m}\frac{1}{m\gamma(m)}.$$

14. Let $A(x)=\sum_{n\le x} n/\varphi(n)$. Using Abel's summation we obtain

$$\sum_{n\le x}\frac{1}{\varphi(n)} = \frac{A(x)}{x} + \int_1^x \frac{A(u)}{u^2}\,du.$$

Using the estimate $A(x) \ll x$ we have $A(x)/x = O(1)$ and

$$\int_1^x \frac{A(u)}{u^2}\,du \ll \int_1^x \frac{du}{u} = \log x.$$

This readily implies the desired result.

15. Note that

$$\sum_{n\le x}\varphi(n) = \sum_{n\le x}\sum_{d\mid n}\mu(d)\frac{n}{d} = \sum_{d\le x}\frac{\mu(d)}{d}\sum_{\substack{n\le x\\ d\mid n}} n = \sum_{d\le x}\mu(d)\sum_{q\le x/d} q.$$

Using the estimate $\sum_{n\le x} n^\alpha = \frac{x^{1+\alpha}}{1+\alpha} + O(x^\alpha)$ we get

$$\begin{aligned}\sum_{n\le x}\varphi(n) &= \frac{x^2}{2}\sum_{d\le x}\frac{\mu(d)}{d^2} + O\left(x\sum_{d\le x}\frac{|\mu(d)|}{d}\right)\\ &= \frac{x^2}{2}\left(\frac{6}{\pi^2}+O\left(\frac{1}{x}\right)\right) + O\left(x\log x\right)\\ &= \frac{3x^2}{\pi^2} + O(x\log x),\end{aligned}$$

where we use $\sum_{n=1}^\infty \mu(n)n^{-2} = 1/\zeta(2) = 6/\pi^2$ and the bounds

$$\sum_{n>x}\frac{1}{n^2} = O\left(\frac{1}{x}\right), \qquad \sum_{n\le x}\frac{1}{n} = \log x + \gamma + O\left(\frac{1}{x}\right).$$

in the penultimate step.

16. Let $A(x)=\sum_{n\leq x}\varphi(n)$. Using Abel's summation by parts formula we have

$$\begin{aligned}\sum_{n\leq x}\frac{\varphi(n)}{n} &= \frac{A(x)}{x}+\int_1^x \frac{A(u)}{u^2}\,du \\ &= \frac{3x}{\pi^2}+O(\log x)+\int_1^x\left(\frac{3}{\pi^2}+O\left(\frac{\log u}{u}\right)\right)du \\ &= \frac{6x}{\pi^2}+O\left(\log x\right)+O\left(\int_1^x \frac{\log u}{u}\,du\right) \\ &= \frac{6x}{\pi^2}+O\left(\log^2 x\right).\end{aligned}$$

A slightly better estimate can be obtained by noting that

$$\begin{aligned}\sum_{n\leq x}\frac{\varphi(n)}{n} &= \sum_{n\leq x}\sum_{d\mid n}\frac{\mu(d)}{d}=\sum_{d\leq x}\frac{\mu(d)}{d}\sum_{\substack{n\leq x\\ d\mid n}}1 \\ &= x\sum_{d\leq x}\frac{\mu(d)}{d^2}+O\left(\sum_{d\leq x}\frac{|\mu(d)|}{d}\right) \\ &= \frac{6x}{\pi^2}+O(\log x).\end{aligned}$$

17. For part (i) note that

$$\begin{aligned}\frac{n}{\varphi(n)} &= \prod_{p\mid n}\left(1-\frac{1}{p}\right)^{-1}=\prod_{p\mid n}\frac{p}{p-1}=\prod_{p\mid n}\left(1+\frac{1}{p-1}\right) \\ &= \sum_{I\subset\{p\mid n\}}\frac{1}{\prod_{p\in I}(p-1)}=\sum_{I\subset\{p\mid n\}}\frac{1}{\varphi(\prod_{p\in I}p)} \\ &= \sum_{\substack{d\mid n\\ d\,\text{sq. free}}}\frac{1}{\varphi(d)}=\sum_{d\mid n}\frac{\mu(d)^2}{\varphi(d)}.\end{aligned}$$

For part (ii) observe that

$$\sum_{n\leq x}\frac{1}{\varphi(n)}=\sum_{n\leq x}\frac{1}{n}\sum_{d\mid n}\frac{\mu(d)^2}{\varphi(d)}.$$

Changing the order of summation the above sum becomes

$$\sum_{d\leq x}\frac{\mu(d)^2}{\varphi(d)}\sum_{\substack{n\leq x\\ d\mid n}}\frac{1}{n}=\sum_{d\leq x}\frac{\mu(d)^2}{d\varphi(d)}\sum_{q\leq x/d}\frac{1}{q}.$$

Incorporating the estimate for harmonic sums we get

$$\begin{aligned}\sum_{n\leq x}\frac{1}{\varphi(n)} &= \sum_{d\leq x}\frac{\mu(d)^2}{d\varphi(d)}\left(\log\frac{x}{d}+\gamma+O\left(\frac{d}{x}\right)\right) \\ &= \log x\sum_{d\leq x}\frac{\mu(d)^2}{d\varphi(d)}-\sum_{d\leq x}\frac{\mu(d)^2\log d}{d\varphi(d)}+\gamma\sum_{d\leq x}\frac{\mu(d)^2}{d\varphi(d)}+O\left(\frac{1}{x}\sum_{d\leq x}\frac{\mu(d)^2}{\varphi(d)}\right).\end{aligned}$$

Since $\varphi(n) \gg n/\log n$ we have

$$\sum_{d\le x}\frac{\mu(d)^2}{d\varphi(d)} = \sum_{d=1}^{\infty}\frac{\mu(d)^2}{d\varphi(d)} + O\left(\sum_{d>x}\frac{\log d}{d^2}\right).$$

By comparison the error term is

$$\ll \int_x^\infty \frac{\log u}{u^2}\,du \ll \frac{\log x}{x}$$

by integral test and integration by parts. Similarly we have

$$\sum_{d\le x}\frac{\mu(d)^2\log d}{d\varphi(d)} = \sum_{d=1}^{\infty}\frac{\mu(d)^2\log d}{d\varphi(d)} + O\left(\frac{\log^2 x}{x}\right). \tag{0.1}$$

Finally,

$$\sum_{d\le x}\frac{\mu(d)^2}{\varphi(d)} \ll \sum_{d\le x}\frac{\log d}{d} \ll \log x\sum_{d\le x}\frac{1}{d} \ll \log^2 x.$$

In fact, the above sum is $\ll \log x$ as $\sum_{n\le x} 1/\varphi(n) \ll \log x$ but we can afford to lose a log power because of (0.1). Putting everything together we conclude

$$\sum_{n\le x}\frac{1}{\varphi(n)} = \log x\sum_{d=1}^{\infty}\frac{\mu(d)^2}{d\varphi(d)} + \gamma\sum_{d=1}^{\infty}\frac{\mu(d)^2}{d\varphi(d)} - \sum_{d=1}^{\infty}\frac{\mu(d)^2}{d\varphi(d)} + O\left(\frac{\log^2 x}{x}\right).$$

18. (i): Using the identity $\sum_{d^2\,|\,n}\mu(d) = \mu(n)^2$ we obtain

$$\begin{aligned} Q(x) = \sum_{n\le x}\mu(n)^2 &= \sum_{n\le x}\sum_{d^2\,|\,n}\mu(d) = \sum_{d\le\sqrt{x}}\sum_{\substack{n\le x\\ d^2\,|\,n}}\mu(d) \\ &= \sum_{d\le\sqrt{x}}\mu(d)\sum_{\substack{n\le x\\ d^2\,|\,n}}1 = \sum_{d\le\sqrt{x}}\mu(d)\left\lfloor\frac{x}{d^2}\right\rfloor \\ &= x\sum_{d\le\sqrt{x}}\frac{\mu(d)}{d^2} + O\left(\sum_{d\le\sqrt{x}}|\mu(d)|\right) = x\sum_{d\le\sqrt{x}}\frac{\mu(d)}{d^2} + O(\sqrt{x}) \end{aligned} \tag{0.2}$$

We now push d off to ∞ in the above sum and incur an error due to the tail which fortunately is only $O(\sqrt{x})$ as seen can be seen by

$$\sum_{d\le\sqrt{x}}\frac{\mu(d)}{d^2} = \sum_{d=1}^{\infty}\frac{\mu(d)}{d^2} + O\left(\sum_{d>\sqrt{x}}\frac{1}{d^2}\right) = \frac{6}{\pi^2} + O\left(\frac{1}{\sqrt{x}}\right).$$

Using this estimate in (0.2), we obtain the desired result.

(ii): Observe that for an integer n we have

$$\{1\le m\le n : m \text{ not squarefree}\} \subset \bigcup_p \{1\le m\le n : p^2|m\},$$

where p runs over all primes. It thus follows that

$$n - Q(n) \le \sum_p \left\lfloor\frac{n}{p^2}\right\rfloor,$$

which is the desired inequality.

(iii): Note that

$$\sum_p \frac{1}{p^2} < \frac{1}{4} + \sum_{k=1}^{\infty} \frac{1}{(2k+1)^2} < \frac{1}{4} + \sum_{k=1}^{\infty} \frac{1}{4k(k+1)} = \frac{1}{2}.$$

It now follows by part (ii) that

$$Q(n) \geq n - \sum_p \left\lfloor \frac{n}{p^2} \right\rfloor \geq n - n\sum_p \frac{1}{p^2} > \frac{n}{2}.$$

for every $n \in \mathbb{N}$.

(iv): Suppose for the sake of contradiction that $n > 1$ and n cannot be expressed as a sum of two squarefree integers. Then for every $a, b \in \mathbb{N}$ satisfying $a + b = n$ either a or b is not squarefree. It follows that there are at least $(n-1)/2$ (this is the number of ways n can be written as a sum of two positive integers without regard for order) integers up to $n-1$ that are not square free. Hence, we must have

$$Q(n-1) \leq (n-1) - \frac{(n-1)}{2} = \frac{n-1}{2}.$$

But this contradicts the inequality in part (iii).

19. Observe that if $n \in \mathcal{A}_p$, then $p^2 \leq n+1$ and so $p \leq n$. Consequently $[N] \cap \mathcal{A}_p = \emptyset$ for $p > N$ and so we have

$$\left| [N] \cap \bigcup_p \mathcal{A}_p \right| = \left| \bigcup_{p \leq N} [N] \cap \mathcal{A}_p \right| \leq \sum_{p \leq N} |[N] \cap \mathcal{A}_p|.$$

Since there are exactly two elements of $\mathcal{A}_p$ in any set of p^2 consecutive integers we obtain that $|[N] \cap \mathcal{A}_p| \leq 2\lceil N/p^2 \rceil$ and so

$$\left| [N] \cap \bigcup_p \mathcal{A}_p \right| \leq 2\sum_{p \leq N} \left\lceil \frac{N}{p^2} \right\rceil \leq 2N \sum_{p \leq N} \frac{1}{p^2} + 2\pi(N).$$

(ii): Note that

$$\sum_p \frac{1}{p^2} \leq \frac{1}{2^2} + \sum_{k=1}^{\infty} \frac{1}{(2k+1)^2} \leq \frac{1}{4} + \frac{1}{4}\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = \frac{1}{2}.$$

Let $C = \sum_p p^{-2}$. We have shown above that $C < \frac{1}{2}$. Thus

$$\frac{1}{N}\left| [N] \cap \bigcup_p \mathcal{A}_p \right| \leq 2C + 2\frac{\pi(N)}{N}.$$

Hence we have

$$\limsup_{N \to \infty} \frac{1}{N}\left| [N] \cap \bigcup_p \mathcal{A}_p \right| \leq 2C < 1.$$

as $\pi(N)/N \to \infty$, i.e., the set of primes has density 0. Since $[N] \cap \bigcup_p \mathcal{A}_p$ is the set of all positive integers n up to N such that either $n-1$ or $n+1$ is not squarefree, $\bigcap_p [N] \backslash \mathcal{A}_p = [N] \backslash \left([N] \cap \bigcup_p \mathcal{A}_p \right)$ is the set of all twin squarefree integers. Finally,

$$\liminf_{N \to \infty} \frac{1}{N}\left| \bigcap_p [N] \backslash \mathcal{A}_p \right| = 1 - \limsup_{N \to \infty} \frac{1}{N}\left| [N] \cap \bigcup_p \mathcal{A}_p \right| = 1 - 2C > 0.$$

This shows that the set of twin squarefree integers has positive (lower) density.

20. (i): Note that

$$\binom{m}{n} = \frac{m(m-1)\cdots(m-n+1)}{n!}.$$

If $m-n < p \leq m$, then p divides the numerator but not the denominator as $m-n+1 > n$. This implies that $\prod_{m-n<p\leq m} p$ divides $\binom{m}{n}$.

(ii): Observe that $\binom{2n}{n} \leq 2^{2n}$ for every $n \in \mathbb{N}$. Thus taking logarithm we deduce that

$$\vartheta(2n) - \vartheta(n) \leq 2n \log 2.$$

(iii): Now suppose that $2^m \leq n < 2^{m+1}$. Then we have

$$\vartheta(n) \leq \vartheta(2^{m+1}) = \sum_{k=0}^{m} (\vartheta(2^{k+1}) - \vartheta(2^k)) \leq \sum_{k=0}^{m} 2^{k+1} \log 2 = (2^{m+2} - 2) \log 2.$$

Finally note that $2^{m+2} = 4 \cdot 2^m \leq 4n$. Let $x \geq 1$ with $n = \lfloor x \rfloor$. Then we have $\vartheta(x) = \vartheta(n) \leq 4n \log 2 \leq 4x \log 2$.