ARITHMETIC FUNCTIONS

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An *arithmetic function* is a complex-valued function defined on the set of natural numbers, i.e., $f : \mathbb{N} \to \mathbb{C}$. The class of arithmetic functions is huge, namely $\mathbb{C}^{\mathbb{N}}$, but we will only be interested in arithmetic functions of number theoretic significance. Following are some common examples of arithmetic functions.

(i) The *identity function* e is defined as $e(n) = \lfloor 1/n \rfloor$, i.e., $e(1) = 1$ and $e(n) = 0$ for $n > 1$.

(ii) For any $\alpha \in \mathbb{C}$, the *power function* N^α is defined as $N^\alpha(n) = n^\alpha$. We denote N^0 by 1 and call it the *unit function*

(iii) For $n \in \mathbb{N}$, $\Omega(n)$ is defined to be the total number of prime factors of n counted with multiplicity. We can expression it in summation notation as $\Omega(n) = \sum_{p^k \mid n} 1$. It is sometimes called *big omega* function.

(iv) For $n \in \mathbb{N}$, $\omega(n)$ is defined to be the number of prime factors of n. We can write it in summation notation as $\omega(n) = \sum_{p \mid n} 1$. It is usually called *small omega* function.

(v) The *Liouville function*, denoted λ, is defined as $\lambda(n) = (-1)^{\Omega(n)}$.

We now discuss some important arithmetic functions.

The *Möbius function*, denoted μ, is defined as follows

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1, \\ (-1)^k & \text{if } n = p_1 \dots p_k, \text{ where } p_1, \dots, p_k \text{ are distinct primes,} \\ 0 & \text{otherwise.} \end{cases}$$

It is the signed characteristic function of square-free positive integers. It is intimately connected to the Riemann zeta function, $\zeta(s)$, which is one of the most important function in analytic number theory. The estimate $\sum_{n \leqslant x} \mu(x) \ll x^{1/2+\epsilon}$ implies the Riemann Hypothesis (RH), which states that all of the non-trivial zeros of $\zeta(s)$ lie on the critical line with $\Re(s) = 1/2$. In fact, the convergence of the Dirichlet series $\sum_{n=1}^{\infty} \mu(n) n^{-s}$ for every s with $\Re(s) > 1/2$ also implies RH. It is known that the Dirichlet series of μ converges for every s with $\Re(s) = 1$. This is a nontrivial result and is of difficulty similar to that of prime number theorem.

Proposition 1. *If $n \geqslant 1$, then we have*

$$\sum_{d \mid n} \mu(d) = e(n).$$

Proof. If $n = 1$, then the formula clearly holds as $\mu(1) = 1$. Now suppose that $n = \prod_{i=1}^{k} p_i^{a_i}$. Because $\mu(d)$ is nonzero if and only if d is squarefree, we can restrict the sum to divisors of

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the form $\prod_{i\in I} p_i$, where I is a subset of $[k] := \{1, \ldots, k\}$. Hence, we get

$$\sum_{d\,|\,n} \mu(d) = \sum_{I\subset[k]} \mu\left(\prod_{i\in I} p_i\right) = \sum_{I\subset[k]} (-1)^{|I|}.$$

Since for each $0 \leqslant r \leqslant k$ there are precisely $\binom{k}{r}$ subsets of $[k]$ containing exactly r elements, we obtain

$$\sum_{d\,|\,n} \mu(d) = \sum_{r=0}^{k} \binom{k}{r}(-1)^r = (-1+1)^k = 0. \qquad \square$$

Corollary 2. *If $n \geqslant 1$, then*

$$\left|\sum_{n\leqslant x} \frac{\mu(n)}{n}\right| \leqslant 1.$$

Proof. Note that

$$1 = \sum_{n\leqslant x} e(n) = \sum_{n\leqslant x}\sum_{d\,|\,n} \mu(d) = \sum_{qd\leqslant x} \mu(d) = \sum_{d\leqslant x} \mu(d) \left\lfloor \frac{x}{d} \right\rfloor.$$

Writing $\lfloor x/d \rfloor = x/d - \{x/d\}$ we obtain

$$1 = x\sum_{d\leqslant x} \frac{\mu(d)}{d} - \sum_{d\leqslant x} \mu(d) \left\{\frac{x}{d}\right\}.$$

Observe that

$$\sum_{d\leqslant x} \left\{\frac{x}{d}\right\} = \{x\} + \sum_{2\leqslant d\leqslant x} \left\{\frac{x}{d}\right\} \leqslant \{x\} + \lfloor x \rfloor - 1 = x - 1.$$

Thus we have

$$\left|x\sum_{d\leqslant x} \frac{\mu(d)}{d}\right| \leqslant 1 + \sum_{d\leqslant x} \left\{\frac{x}{d}\right\} \leqslant x.$$

This completes the proof. $\square$

The *Euler's totient function* φ is defined at n to be the number of positive integers not exceeding n that are relatively prime to n. We can rewrite $\varphi(n)$ in the summation notation as

$$\varphi(n) = \sum_{\substack{k=1 \\ (k,n)=1}}^{n} 1.$$

The function φ is ubiquitous in number theory, particularly in prime number theory. The degree of the irreducible polynomial of $e^{2\pi i/n}$ over $\mathbb{Q}$ is $\varphi(n)$ or equivalently, $[\mathbb{Q}(e^{2\pi i/n}) : \mathbb{Q}] = \varphi(n)$.

Proposition 3. *If $n \geqslant 1$, then*

$$\sum_{d|n} \varphi(d) = n.$$

Proof. We partition the set $\{1, \dots, n\}$ into subsets $A_d = \{1 \leqslant k \leqslant n : (k, n) = d\}$, where d is a divisor of n, and note that there is a one-to-one bijection between elements of A_d and integers $1 \leqslant r \leqslant n/d$ satisfying $(r, n/d) = 1$. This then implies that

$$n = \sum_{d|n} |A_d| = \sum_{d|n} \varphi(n/d) = \sum_{d|n} \varphi(d). \qquad \square$$

Proposition 4 (relationship between μ and φ)**.** *If $n \geqslant 1$, then we have*

$$\varphi(n) = \sum_{d|n} \mu(d)\frac{n}{d}.$$

Proof. We use the formula for the divisor sum of μ to obtain

$$\varphi(n) = \sum_{k=1}^{n} e((k, n)) = \sum_{k=1}^{n} \sum_{d|(k,n)} \mu(d) = \sum_{k=1}^{n} \sum_{\substack{d|n \\ d|k}} \mu(d)$$

Changing the order of summation we get

$$\varphi(n) = \sum_{d|n} \sum_{\substack{k=1 \\ d|k}}^{n} \mu(d) = \sum_{d|n} \mu(d) \sum_{\substack{k=1 \\ d|k}}^{n} 1 = \sum_{d|n} \mu(d)\frac{n}{d}. \qquad \square$$

Next we obtain a nice product formula for $\varphi(n)$.

Proposition 5. *For $n \geqslant 1$ we have*

$$\varphi(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right),$$

Proof. If $n = 1$, then the product on the right hand side is empty and so the formula trivially holds. Now let $p_1, \dots, p_k$ be the prime divisors of n let $[k] := \{1, \dots, k\}$. Then expanding the product, we get

$$\prod_{i=1}^{k} \left(1 - \frac{1}{p_i}\right) = \sum_{I \subset [k]} \prod_{i \in I} \left(-\frac{1}{p_i}\right) = \sum_{I \subset [k]} \frac{(-1)^{|I|}}{\prod_{i \in I} p_i} = \sum_{d|n} \frac{\mu(d)}{d} = \frac{\varphi(n)}{n}. \qquad \square$$

Proposition 6. *The Euler's totient function has the following properties:*

(i) $\varphi(p^a) = p^a - p^{a-1}$ *for prime p and $a \geqslant 1$.*
(ii) $\varphi(mn) = \varphi(m)\varphi(n)(d/\varphi(d))$, *where $d = (m, n)$.*
(iii) $\varphi(mn) = \varphi(m)\varphi(n)$ *if $(m, n) = 1$.*
(iv) $n|m$ *implies* $\varphi(n)|\varphi(m)$.
(v) $\varphi(n)$ *is even for $n \geqslant 3$. Moreover, if n has r distinct odd prime factors, then $2^r|\varphi(n)$.*

Proof. Part (i) follows immediately from the product formula. As for part (ii) note that

$$\begin{aligned}
\frac{\varphi(mn)}{mn} = \prod_{p|mn}\left(1-\frac{1}{p}\right) &= \prod_{p|m}\left(1-\frac{1}{p}\right)\prod_{\substack{p|n\\ p\nmid m}}\left(1-\frac{1}{p}\right) \\
&= \prod_{p|m}\left(1-\frac{1}{p}\right)\prod_{p|n}\left(1-\frac{1}{p}\right)\prod_{\substack{p|n\\ p|m}}\left(1-\frac{1}{p}\right)^{-1} \\
&= \frac{\varphi(m)}{m}\frac{\varphi(n)}{n}\prod_{p|(n,m)}\left(1-\frac{1}{p}\right)^{-1} \\
&= \frac{\varphi(m)}{m}\frac{\varphi(n)}{n}\frac{d}{\varphi(d)},
\end{aligned}$$

where $d = (m, n)$.

Part (iii) follows immediately from part (ii). For part (iv) let $n = p_1^{a_1}\cdots p_k^{a_k}$ and $m = p_1^{b_1}\cdots p_k^{b_k}$, where a_i and b_i are nonnegative. Because $a_i \leqslant b_i$, we have $\varphi(p_i^{a_i})|\varphi(p_i^{b_i})$ due to part (i). This coupled with the fact that φ is multiplicative (due to part (iii)) gives us the desired result.

Finally for part (iv) observe that if $n \geqslant 3$ and $n = 2^a$ for some positive integer a then a must be at least 2 and so $\varphi(2^a) = 2^a - 2^{a-1} = 2(2^{a-1} - 2^{a-2})$ is even. Now note that

$$\varphi(n) = n\prod_{p|n}\left(1-\frac{1}{p}\right) = \frac{n}{\prod_{p|n} p}\prod_{p|n}(p-1),$$

where the factor $n(\prod_{p|n} p)^{-1}$ is an integer. If n is not of the form 2^a, then an odd prime p divides n, and so the factor on the right must be even which implies that $\varphi(n)$ is even. Finally, if n has r distinct odd prime factors then $2^r | \prod_{p|n}(p-1)$ and hence $2^r|\varphi(n)$. □

For a positive integer k let $A(k)$ denote the number of solutions to the equation $\varphi(n) = k$. It is easy to see that $A(k)$ is finite for every k since $\varphi(n) \to \infty$. Kevin Ford showed in a paper published in Annals of Mathematics that all of positive integers greater than 1 lie in the image of function A. It is not known if $A(k) = 1$ for some k. The answers seems to be in the negation and this is known as *Carmichael's conjecture* (Robert Carmichael originally stated this as a result instead of a conjecture in his book which he later retracted).

The *von-Mangoldt function*, denoted Λ, is defined as

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^a \text{ for some prime } p \text{ and integer } a \geqslant 1, \\ 0 & \text{otherwise.} \end{cases}$$

The von-Mangoldt function Λ plays an important role in prime number theory.

Proposition 7. *If $n \geqslant 1$, then*

$$\log n = \sum_{d\,|\,n} \Lambda(d).$$

Dirichlet product

If f and g are two arithmetic functions we define their *Dirichlet product* (or *Dirichlet convolution*) to be the arithmetic function $f * g$ given by

$$(f * g)(n) = \sum_{d|n} f(d)g(n/d)$$

The Dirichlet product of two arithmetic functions arises when multiplying two Dirichlet series $\sum_{n=1}^{\infty} f(n)n^{-s}$ and $\sum_{n=1}^{\infty} g(n)n^{-s}$. After multiplication and rearrangement of terms we get the Dirichlet series $\sum_{n=1}^{\infty}(f * g)(n)n^{-s}$. It is easily seen that Dirichlet multiplication is both commutative and associative, i.e., for any arithmetic functions f, g, h we have

$$f * g = g * f \quad \text{and} \quad (f * g) * h = f * (g * h).$$

Moreover, we have $e * f = f$ for any arithmetic function f. Thus the set of all arithmetic functions form a commutative monoid. The next result allows us to characterize arithmetic functions that are invertible under Dirichlet multiplication.

Proposition 8. *If f is an arithmetic function with $f(1) \neq 0$, then there is a unique arithmetic function g such that $g * f = f * g = e$. The function g is given by*

$$g(1) = \frac{1}{f(1)}, \qquad g(n) = -\frac{1}{f(1)} \sum_{\substack{d\,|\,n \\ d<n}} g(d)f(n/d) \quad \textit{for } n > 1.$$

The above result show that the set of all arithmetic functions f satisfying $f(1) \neq 0$ form an abelian group under Dirichlet multiplication.

The Dirichlet multiplication provides a convenient notation to write relations among different arithmetic functions in a compact fashion

$$\mu * 1 = e, \quad \varphi * 1 = N, \quad \varphi = \mu * N.$$

Theorem 9 (Möbius inversion formula)**.** *Let f and g be arithmetic functions. Then*

$$f(n) = \sum_{d|n} g(d)$$

for every $n \in \mathbb{N}$ if and only if

$$g(n) = \sum_{d|n} f(d)\mu(n/d)$$

for every $n \in \mathbb{N}$.

Proof. Follow immediately by noting that $f = g * 1$ if and only if $g = f * \mu$ which is seen by multiplying by μ (or 1) and using the identity $\mu * 1 = e$. □

Corollary 10. *If $n \geqslant 1$, then*

$$\Lambda(n) = -\sum_{d\,|\,n} \mu(d) \log d.$$

Proof. By Möbius inversion we have

$$\Lambda(n) = \sum_{d\,|\,n} \mu(d) \log \frac{n}{d} = \log n \sum_{d\,|\,n} \mu(d) - \sum_{d\,|\,n} \mu(d) \log d = e(n) \log n - \sum_{d\,|\,n} \mu(d) \log d.$$

This completes the proof as $e(n)\log n = 0$ for every n. $\square$

An arithmetic function f is called *multiplicative* if f is not identically zero and

$$f(mn) = f(m)f(n) \quad \text{whenever } (m,n) = 1.$$

A multiplicative function f is called *completely multiplicative* (or *totally multiplicative*) if f is not identically zero and

$$f(mn) = f(m)f(n) \quad \text{for all } m, n.$$

We note some common examples of multiplicative functions.

(i) The power function N^α is completely multiplicative.
(ii) The identity function e is completely multiplicative.
(iii) The Möbius function μ is multiplicative but not completely multiplicative as $\mu(4) = 0 \neq 1 = \mu(2)^2$.
(iv) The Euler totient function φ is multiplicative but not completely multiplicative as $\varphi(4) = 2 \neq 1 = \varphi(2)^2$.

Note that if f is multiplicative, then $f(1) = 1$. From this it immediately follows that Λ is not multiplicative as $\Lambda(1) = 0$.

Proposition 11. *Let f be an arithmetic function with $f(1) = 1$.*

(i) f is multiplicative if and only if

$$f(p_1^{a_1} \cdots p_k^{a_k}) = f(p_1^{a_1}) \cdots f(p_k^{a_k}),$$

where $p_1, \ldots, p_k$ are distinct primes.

(ii) If f is multiplicative, then f is completely multiplicative if and only if

$$f(p^a) = f(p)^a$$

for all primes p and all integers $a \geqslant 1$.

The above result shows that a multiplicative function is uniquely determined by its values on prime powers, and a completely multiplicative function is uniquely determined by its values on primes.

Proposition 12. *If f and g are multiplicative, then so is their Dirichlet product $f * g$.*

Proof. Let m and n be relatively prime integers. Then observe that

$$(f * g)(mn) = \sum_{d|mn} f(d) g\left(\frac{mn}{d}\right) = \sum_{\substack{a|m \\ b|n}} f(ab) g\left(\frac{mn}{ab}\right)$$

as every divisor of mn can be uniquely written as ab, where $a|m$ and $b|n$. Using the multiplicativity of f and g we obtain

$$\begin{aligned}(f * g)(mn) &= \sum_{\substack{a|m \\ b|n}} f(a)f(b) g\left(\frac{m}{a}\right) g\left(\frac{n}{b}\right) = \sum_{a|m}\sum_{b|n} f(a)f(b) g\left(\frac{m}{a}\right) g\left(\frac{n}{b}\right) \\ &= \sum_{a|m} f(a) g\left(\frac{m}{a}\right) \sum_{b|n} f(b) g\left(\frac{n}{b}\right) = (f * g)(m)(f * g)(n).\end{aligned}$$

$\square$

The Dirichlet product of two completely multiplicative functions need not be completely multiplicative. For instance, the divisor function $d = 1 * 1$ is not completely multiplicative as $d(4) = 3 \neq 4 = d(2)^2$ whereas 1 clearly is.

Proposition 13. *If f is multiplicative, then so is it's Dirichlet inverse f^{-1}.*

Proof. Suppose for the sake of contradiction that f^{-1} is not multiplicative. Then there exist positive integers m and n with $(m, n) = 1$ such that

$$f^{-1}(mn) \neq f^{-1}(m)f^{-1}(n).$$

We choose such a pair m and n for which the product mn is the smallest. Since f is multiplicative therefore $f^{-1}(1) = 1/f(1) = 1$ and hence neither m nor n can be 1. In particular, $mn > 1$. By the construction of the product mn, $f(ab) = f(a)f(b)$ for all positive integers a and b with $(a, b) = 1$ and $ab < mn$. It now follows that

$$f^{-1}(mn) = -\sum_{\substack{a\,|\,m\\ b\,|\,n\\ ab<mn}} f^{-1}(ab)f\left(\frac{mn}{ab}\right) = -\sum_{\substack{a\,|\,m\\ b\,|\,n\\ ab<mn}} f^{-1}(a)f^{-1}(b)f\left(\frac{m}{a}\right)f\left(\frac{n}{b}\right)$$

Splitting the sum we obtain

$$\begin{aligned}
f^{-1}(mn) &= -f^{-1}(n)\sum_{\substack{a\,|\,m\\ a<m}} f^{-1}(a)f\left(\frac{m}{a}\right) - f^{-1}(m)\sum_{\substack{b\,|\,n\\ b<n}} f^{-1}(b)f\left(\frac{n}{b}\right)\\
&\quad -\sum_{\substack{a\,|\,m\\ a<m}}\sum_{\substack{b\,|\,n\\ b<n}} f^{-1}(a)f^{-1}(b)f\left(\frac{m}{a}\right)f\left(\frac{n}{b}\right)\\
&= f^{-1}(n)f^{-1}(m) + f^{-1}(m)f^{-1}(n) - f^{-1}(m)f^{-1}(n)\\
&= f^{-1}(m)f^{-1}(n).
\end{aligned}$$

This contradiction proves the result.

Second Proof. Let g be an arithmetic function defined as

$$g(n) = \prod_{p^a\,||\,n} f^{-1}(p^a).$$

Then g is a multiplicative function by definition and so it suffices to show that $f^{-1} = g$. Note that

$$\begin{aligned}
(g * f)(p^k) &= \sum_{d\,|\,p^k} g(d)f(p^k/d) = \sum_{i=0}^{k} g(p^i)f(p^{k-i})\\
&= \sum_{i=0}^{k} f^{-1}(p^i)f(p^{k-i}) = \sum_{d\,|\,p^k} f^{-1}(d)f(p^k/d) = (f^{-1} * f)(p^k) = e(p^k).
\end{aligned}$$

Because $g * f$ and e are both multiplicative functions and agree on prime powers, it follows that $g * f = e$ and so $g = f^{-1}$. □

Proposition 14. *Let f be multiplicative. Then f is completely multiplicative if and only if $f^{-1} = \mu f$.*

Proof. Suppose f is completely multiplicative. Then observe that

$$(f * \mu f)(n) = \sum_{d\,|\,n} \mu(d) f(d) f\left(\frac{n}{d}\right) = f(n) \sum_{d\,|\,n} \mu(d) = f(n)e(n) = e(n).$$

Conversely, assume that $f^{-1} = \mu f$. Then observe that

$$\sum_{d\,|\,n} \mu(d) f(d) f\left(\frac{n}{d}\right) = 0$$

for $n > 1$. Let $n = p^a$, where $a \geqslant 1$. Then, we get

$$\mu(1)f(1)f(p^a) + \mu(p)f(p)f(p^{a-1}) = 0.$$

It then follows that

$$f(p^a) = f(p)f(p^{a-1}).$$

This implies that $f(p^a) = f(p)^a$. Thus f is completely multiplicative. □

1. Estimates of arithmetic functions

The Abel's summation by parts formula is one of the most important and ubiquitous results in analytic number theory which is employed to estimate the partial sums of arithmetic functions weighted by some smooth function.

Theorem 15. *Let $a : \mathbb{N} \to \mathbb{C}$ be an arithmetic function, let $0 < x < y$ be real numbers and $f : [x, y] \to \mathbb{C}$ be a continuously differentiable function. Then we have*

$$\sum_{x<n\leqslant y} a(n)f(n) = A(y)f(y) - A(x)f(x) - \int_x^y A(u)f'(u)\,du,$$

where $A(u) = \sum_{n\leqslant u} a(n)$.

Proof. Let $m = \lfloor x \rfloor$ and $M = \lfloor y \rfloor$. We can rewrite the weighted sum as

$$\sum_{x<n\leqslant y} a(n)f(n) = \sum_{n=m+1}^{M} a(n)f(n).$$

By definition $a(n) = A(n) - A(n-1)$ so we can replace $a(n)$ to get

$$\begin{aligned}
\sum_{n=m+1}^{M} a(n)f(n) &= \sum_{n=m+1}^{M} (A(n) - A(n-1))f(n) \\
&= \sum_{n=m+1}^{M} A(n)f(n) - \sum_{n=m}^{M-1} A(n)f(n+1) \\
&= A(M)f(M) - A(m)f(m+1) - \sum_{n=m+1}^{M-1} A(n)(f(n+1) - f(n)) \qquad (1.1)
\end{aligned}$$

Since $f(n+1)-f(n)=\int_n^{n+1} f'(u)\,du$ and $A(u)=A(n)$ for all $u\in[n,n+1)$, we get

$$\begin{aligned}\sum_{m+1}^{M-1} A(n)(f(n+1)-f(n)) &= \sum_{m+1}^{M-1} A(n)\int_n^{n+1} f'(u)\,du = \sum_{m+1}^{M-1}\int_n^{n+1} A(u)f'(u)\,du\\ &= \int_{m+1}^{M} A(u)f'(u)\,du. \end{aligned}\tag{1.2}$$

Substituting (1.1) into (1.2), we get

$$\sum_{n=m+1}^{M} a(n)f(n) = A(M)f(M)-A(m)f(m+1)-\int_{m+1}^{M} A(u)f'(u)\,du. \tag{1.3}$$

Using Fundamental Theorem of Calculus and observing that $A(u)=A(x)$ for $u\in[x,m+1)$, we get

$$\begin{aligned}\int_x^{m+1} A(u)f'(u)\,du &= A(x)f(m+1)-A(x)f(x)\\ &= A(m)f(m+1)-A(x)f(x).\end{aligned}\tag{1.4}$$

Doing a similar calculation for $\int_M^y A(u)f'(u)\,du$ yields

$$\int_M^y A(u)f'(u)\,du = A(y)f(y)-A(M)f(M). \tag{1.5}$$

Using (1.4) and (1.5), one can easily turn (1.3) into the required form. $\square$

Corollary 16. *Let $a:\mathbb{N}\to\mathbb{C}$ be an arithmetic function and let $f:[1,x]\to\mathbb{C}$ be a continuously differentiable function where $x\geqslant 1$. Then we have*

$$\sum_{n\leqslant x} a(n)f(n) = A(x)f(x)-\int_1^x A(u)f'(u)\,du.$$

We now employ the Abel's summation to estimate partial sums of harmonic series.

Proposition 17. *If $x\geqslant 1$, then we have*

$$\sum_{n\leqslant x}\frac{1}{n} = \log x+\gamma+O\left(\frac{1}{x}\right), \tag{1.6}$$

where γ is the Euler-Mascheroni constant.

Proof. Taking $a(n)=1$ and $f(x)=1/x$ in the summation by parts formula, we get

$$\sum_{n\leqslant x}\frac{1}{n} = \frac{\lfloor x\rfloor}{x}+\int_1^x\frac{\lfloor u\rfloor}{u^2}\,du. \tag{1.7}$$

Substituting $\lfloor x \rfloor = x - \{x\}$ in (1.7), we get

$$\begin{aligned}\sum_{n\leqslant x}\frac{1}{n} &= 1 - \frac{\{x\}}{x} + \int_1^x \frac{du}{u} - \int_1^x \frac{\{u\}}{u^2}\,du \\ &= 1 + O\left(\frac{1}{x}\right) + \log x - \int_1^x \frac{\{u\}}{u^2}\,du \\ &= 1 + O\left(\frac{1}{x}\right) + \log x - \int_1^\infty \frac{\{u\}}{u^2}\,du + \int_x^\infty \frac{\{u\}}{u^2}\,du.\end{aligned}$$

Taking $C = 1 - \int_1^\infty \{u\}u^{-2}\,du$, we obtain

$$\sum_{n\leqslant x}\frac{1}{n} = \log x + C + O\left(\frac{1}{x}\right) + \int_x^\infty \frac{\{u\}}{u^2}\,du.$$

We can bound the improper integral as

$$\int_x^\infty \frac{\{u\}}{u^2}\,du \leqslant \int_x^\infty \frac{du}{u^2} = \frac{1}{x}$$

and so

$$\int_x^\infty \frac{\{u\}}{u^2}\,du = O\left(\frac{1}{x}\right).$$

It thus follows that

$$\sum_{n\leqslant x}\frac{1}{n} = \log x + C + O\left(\frac{1}{x}\right).$$

It can be easily seen by taking limit as x approaches ∞ that $C = \gamma$. ☐

Note that in the above proof we obtained the following integral expression for γ

$$\gamma = 1 - \int_1^\infty \frac{\{u\}}{u^2}\,du.$$

Proposition 18. *If $x \geqslant 1$, then for any $s \in \mathbb{C}$ with $s \neq 1$ and $\sigma = \Re(s) > 0$ we have*

$$\sum_{n\leqslant x}\frac{1}{n^s} = \frac{x^{1-s}}{1-s} + \frac{s}{s-1} - s\int_1^\infty \frac{\{u\}}{u^{s+1}}\,du + O(x^{-\sigma}), \tag{1.8}$$

where the implicit constant depends on s.

Proof. We apply the Abel's summation by parts formula with $a(n) = 1$ and $f(x) = x^{-s}$. For $x \geqslant 1$ we then get

$$\sum_{n\leqslant x}\frac{1}{n^s} = \frac{\lfloor x \rfloor}{x^s} + s\int_1^x \frac{\lfloor u \rfloor}{u^{s+1}}\,du$$

Substituting $\lfloor x \rfloor = x - \{x\}$, we obtain

$$\begin{aligned}
\sum_{n \leqslant x} \frac{1}{n^s} &= x^{1-s} - \frac{\{x\}}{x^s} + s\int_1^x \frac{du}{u^s} - s\int_1^x \frac{\{u\}}{u^{s+1}}\,du \\
&= x^{1-s} + s\left(\frac{x^{1-s}}{1-s} - \frac{1}{1-s}\right) - s\int_1^x \frac{\{u\}}{u^{s+1}}\,du + O(x^{-\sigma}) \\
&= \frac{x^{1-s}}{1-s} + \frac{s}{s-1} - s\int_1^\infty \frac{\{u\}}{u^{s+1}}\,du + s\int_x^\infty \frac{\{u\}}{u^{s+1}}\,du + O(x^{-\sigma}). \qquad (1.9)
\end{aligned}$$

Finally note that

$$\left|\int_x^\infty \frac{\{u\}}{u^{s+1}}\,du\right| \leqslant \int_x^\infty \frac{du}{u^{\sigma+1}} = \frac{x^{-\sigma}}{\sigma}.$$

This leads to the desired estimate. □

We define the Riemann zeta function, $\zeta(s)$, for $\Re(s) > 0$ by

$$\zeta(s) = \frac{s}{s-1} - s\int_1^\infty \frac{\{u\}}{u^{s+1}}\,du$$

If $\Re(s) > 1$, then it follows from (1.8) that

$$\zeta(s) = \sum_{n=1}^\infty \frac{1}{n^s}.$$

Corollary 19. *If $x \geqslant 1$ and $\sigma > 1$, then we have*

$$\sum_{n>x} \frac{1}{n^s} = O(x^{1-\sigma}),$$

where the implicit constant depends on s.

Proposition 20. *If $x \geqslant 1$, then for any $s \in \mathbb{C}$ with $\sigma = \Re(s) \geqslant 0$ we have*

$$\sum_{n \leqslant x} n^s = \frac{x^{s+1}}{s+1} + O(x^\sigma), \qquad (1.10)$$

where the implicit constant depends on s.

Proof. We assume that $s \neq 0$. Applying the Abel summation by parts formula with $a(n) = 1$ and $f(x) = x^s$, we get

$$\sum_{n \leqslant x} n^s = \lfloor x \rfloor x^s - s\int_1^x \lfloor u \rfloor u^{s-1}\,du.$$

Substituting $\lfloor x \rfloor = x - \{x\}$, we obtain

$$\begin{aligned}
\sum_{n \leqslant x} n^s &= x^{s+1} - \{x\}x^s - s\int_1^x u^s\,du + s\int_1^x \{u\}u^{s-1}\,du \\
&= x^{s+1} + O(x^\sigma) - s\left(\frac{x^{s+1}}{s+1} - \frac{1}{s+1}\right) + s\int_1^x \{u\}u^{s-1}\,du. \qquad (1.11)
\end{aligned}$$

Now note that if $\sigma > 0$, then

$$\left|s\int_1^x \{u\}u^{s-1}\,du\right| \leqslant |s|\int_1^x u^{\sigma-1}\,du = \frac{|s|}{\sigma}(x^\sigma - 1) = O_s(x^\sigma).$$

Whereas, if $\sigma = 0$, then

$$\left|s\int_1^x \{u\}u^{s-1}\,du\right| \leqslant |s|\int_1^x \frac{du}{u} = |s|\log x = O_s(\log x).$$

Hence (1.11) simplifies to

$$\sum_{n\leqslant x} n^s = \frac{x^{s+1}}{s+1} + \frac{s}{s+1} + O(x^\sigma)$$

Observe that $O(x^\sigma)$ absorbs the constant $s/(s+1)$ and thus we get the desired result.

Finally, if $s = 0$ then we get

$$\sum_{n\leqslant x} n^s = \sum_{n\leqslant x} 1 = \lfloor x\rfloor = x - \{x\} = x + O(1).$$

But this agrees with the asymptotic formula (1.10). $\square$

We now turn our attention to the divisor function $d = 1 * 1$. We will first show that $d(n)$ behaves like $\log n$ on average. Then we will improve the error term for the partial sum of $d(n)$ using Dirichlet hyperbola method. Since $d(n) = \sum_{d\mid n} 1$, we have

$$\sum_{n\leqslant x} d(n) = \sum_{n\leqslant x}\sum_{d\mid n} 1 = \sum_{n\leqslant x}\sum_{qd=n} 1 = \sum_{qd\leqslant x} 1.$$

Thus the divisor sum can now be written as

$$\sum_{n\leqslant x} d(n) = \sum_{d\leqslant x}\sum_{q\leqslant x/d} 1 = \sum_{d\leqslant x}\left(\frac{x}{d} + O(1)\right) = x\sum_{d\leqslant x}\frac{1}{d} + O(x).$$

Now we use the asymptotic formula for the harmonic sum and obtain

$$\sum_{n\leqslant x} d(n) = x\left(\log x + \gamma + O\left(1/x\right)\right) + O(x) = x\log x + O(x)$$

Hence we get

$$\sum_{n\leqslant x} d(n) = x\log x + O(x).$$

Thus the average order of $d(n)$ is $\log n$ since

$$\sum_{n\leqslant x} d(n) \sim x\log x \quad \text{as } x\to\infty.$$

Dirichlet obtained a sharper estimate for $\sum_{d\leqslant x} d(n)$ with the error term being $O(\sqrt{x})$. We prove the Dirichlet's result below.

Theorem 21. *For all $x \geqslant 1$, we have*

$$\sum_{n\leqslant x} d(n) = x\log x + (2\gamma - 1)x + O(\sqrt{x}), \tag{1.12}$$

where γ is the Euler-Mascheroni constant.

Proof. The trick to prove this stronger estimate is to exploit the symmetry of q and d in the sum

$$\sum_{n\leqslant x} d(n) = \sum_{qd\leqslant x} 1.$$

We can split this sum as

$$\begin{aligned}
\sum_{n\leqslant x} d(n) = \sum_{qd\leqslant x} 1 = \sum_{\substack{qd\leqslant x\\ d\leqslant q}} 1 + \sum_{\substack{qd\leqslant x\\ q\leqslant d}} 1 - \sum_{\substack{qd\leqslant x\\ d=q}} 1 = 2\sum_{\substack{qd\leqslant x\\ d\leqslant q}} 1 - \sum_{d\leqslant\sqrt{x}} 1 \\
= 2\Bigg(\sum_{d\leqslant\sqrt{x}}\sum_{d\leqslant q\leqslant x/d} 1\Bigg) - \lfloor\sqrt{x}\rfloor = 2\sum_{d\leqslant\sqrt{x}}\left(\left\lfloor\frac{x}{d}\right\rfloor - d + 1\right) - \lfloor\sqrt{x}\rfloor \\
= 2\sum_{d\leqslant\sqrt{x}}\left(\left\lfloor\frac{x}{d}\right\rfloor - d\right) + \lfloor\sqrt{x}\rfloor = 2\sum_{d\leqslant\sqrt{x}}\left(\frac{x}{d} - d + O(1)\right) + O(\sqrt{x}) \\
= 2x\sum_{d\leqslant\sqrt{x}}\frac{1}{d} - 2\sum_{d\leqslant\sqrt{x}} d + O(\sqrt{x})
\end{aligned}$$

Finally using estimates in Proposition 17 and 20 we obtain

$$\begin{aligned}
\sum_{n\leqslant x} d(n) &= 2x\Bigg(\log\sqrt{x} + \gamma + O\left(\frac{1}{\sqrt{x}}\right)\Bigg) - 2\Bigg(\frac{x}{2} + O(\sqrt{x})\Bigg) + O(\sqrt{x}) \\
&= x\log x + (2\gamma - 1)x + O(\sqrt{x}),
\end{aligned}$$

which is the desired result. □

2. Chebyshev's functions

For $x > 0$, the *Chebyshev's ψ-function* and *Chebyshev's ϑ-function* are defined as

$$\psi(x) = \sum_{p^k\leqslant x}\log p, \qquad \vartheta(x) = \sum_{p\leqslant x}\log p.$$

These are weighted sum of characteristic function of primes. We can write $\psi(x)$ as

$$\psi(x) = \sum_{n\leqslant x}\Lambda(n).$$

We now state a result about the relationship between $\psi(x)$ and $\vartheta(x)$.

Proposition 22. *For $x > 0$ we have*

$$\psi(x) = \sum_{k}\vartheta(x^{1/k}), \tag{2.1}$$

$$\vartheta(x) = \sum_{k}\mu(k)\psi(x^{1/k}) \tag{2.2}$$

Proof. Note that

$$\psi(x) = \sum_{p^k\leqslant x}\log p = \sum_{k}\sum_{p\leqslant x^{1/k}}\log p = \sum_{k}\vartheta(x^{1/k}).$$

The identity (2.2) follows by substituting (2.1) in the sum;

$$\sum_k \mu(k)\psi(x^{1/k}) = \sum_k \mu(k) \sum_\ell \vartheta(x^{1/k\ell}) = \sum_{k,\ell} \mu(k)\vartheta(x^{1/k\ell}).$$

Collecting terms for which $k\ell$ is fixed we get

$$\sum_k \mu(k)\psi(x^{1/k}) = \sum_m \vartheta(x^{1/m}) \sum_{k\,|\,m} \mu(k) = \vartheta(x). \qquad \square$$

Proposition 23. *For $x \geqslant 2$ we have*

$$\psi(x) = \vartheta(x) + O(x^{1/2}\log x). \tag{2.3}$$

Proof. Note that

$$\psi(x) = \sum_k \vartheta(x^{1/k}) = \sum_{k\leqslant \log_2 x} \vartheta(x^{1/k})$$

as $x^{1/k} < 2$ for $k > \log_2 x$ in which case $\vartheta(x^{1/k}) = 0$. Now using the crude estimate $\vartheta(x) \ll x\log x$ we see that

$$\psi(x) - \vartheta(x) = \sum_{2\leqslant k\leqslant \log_2 x} \vartheta(x^{1/k}) \ll \sum_{2\leqslant k\leqslant \log_2 x} x^{1/k}\log x^{1/k} \ll x^{1/2}\log x,$$

where the last estimate follows as the first term is of order $x^{1/2}\log x$ and the sum of the rest of the terms is $\ll x^{1/3}\log^2 x \ll x^{1/2}\log x$. $\square$

Theorem 24. *We have*

$$\pi(x) \sim \frac{x}{\log x} \quad \Leftrightarrow \quad \psi(x) \sim x \quad \Leftrightarrow \quad \vartheta(x) \sim x.$$

Proof. The equivalence $\psi(x) \sim x \Leftrightarrow \vartheta(x) \sim x$ follows immediately from (2.3). Using Abel's summation by parts formula we have

$$\vartheta(x) = \sum_{p\leqslant x} \log p = \sum_{n\leqslant x} p(n)\log n = \pi(x)\log x - \int_2^x \frac{\pi(u)}{u}\,du,$$

where $p(n)$ is the characteristic function of primes. Now suppose that $\pi(x) \sim x/\log x$. Then we have

$$\int_2^x \frac{\pi(u)}{u}\,du \ll \int_2^x \frac{du}{\log u} = \left(\int_2^{\sqrt{x}} + \int_{\sqrt{x}}^x\right)\frac{du}{\log u} \leqslant \frac{\sqrt{x}}{\log 2} + \frac{x}{\log\sqrt{x}} \ll \frac{x}{\log x}$$

Thus we have

$$\frac{\vartheta(x)}{x} = \frac{\pi(x)\log x}{x} - \frac{1}{x}\int_2^x \frac{\pi(u)}{u}\,du = \frac{\pi(x)\log x}{x} + O\left(\frac{1}{\log x}\right)$$

It immediately follows from this that $\vartheta(x) \sim x$.

Now suppose that $\vartheta(x) \sim x$. Again using Abel's summation by parts formula we have

$$\pi(x) = \sum_{2\leqslant n\leqslant x} \frac{p(n)\log n}{\log n} = \frac{\vartheta(x)}{\log x} + \int_2^x \frac{\vartheta(u)}{u(\log u)^2}\,du.$$

Since $\vartheta(x) \ll x$ we have

$$\int_2^x \frac{\vartheta(u)}{u(\log u)^2}\, du \ll \int_2^x \frac{du}{(\log u)^2} = \left(\int_2^{\sqrt{x}} + \int_{\sqrt{x}}^x\right) \frac{du}{(\log u)^2} \leqslant \frac{\sqrt{x}}{(\log 2)^2} + \frac{x}{(\log \sqrt{x})^2} \ll \frac{x}{\log^2 x}.$$

Hence we have

$$\frac{\pi(x)\log x}{x} = \frac{\vartheta(x)}{x} + \frac{\log x}{x}\int_2^x \frac{du}{u(\log u)^2} = \frac{\vartheta(x)}{x} + O\left(\frac{1}{\log x}\right).$$

Since $\vartheta(x) \sim x$, we immediately obtain $\pi(x) \sim x/\log x$. □

It can be shown using properties of the middle binomial coefficient that $\vartheta(x) \ll x$, which in turn implies $\psi(x) \ll x$ due to estimate (2.3).

Proposition 25. *For $x \geqslant 1$ we have*

(i)

$$\sum_{n \leqslant x} \log n = x\log x - x + O(\log x),$$

(ii)

$$\sum_{n \leqslant x} \frac{\Lambda(n)}{n} = \log x + O(1),$$

(iii)

$$\sum_{p \leqslant x} \frac{\log p}{p} = \log x + O(1).$$

Proof. By Abel's summation by parts formula we have

$$\begin{aligned}\sum_{n \leqslant x} \log n &= \lfloor x \rfloor \log x - \int_1^x \frac{\lfloor u \rfloor}{u}\, du \\ &= x\log x - \{x\}\log x - (x-1) + \int_1^x \frac{\{u\}}{u}\, du \\ &= x\log x - x + O(\log x).\end{aligned}$$

(ii): We note that

$$\begin{aligned}\sum_{n \leqslant x} \log n &= \sum_{n \leqslant x}\sum_{d \mid n} \Lambda(d) = \sum_{d \leqslant x} \Lambda(d) \sum_{\substack{n \leqslant x \\ d \mid n}} 1 \\ &= \sum_{d \leqslant x} \Lambda(d) \left\lfloor \frac{x}{d} \right\rfloor = x\sum_{d \leqslant x} \frac{\Lambda(d)}{d} + O\left(\sum_{d \leqslant x} \Lambda(d)\right)\end{aligned}$$

By Chebyshev estimate we have $\psi(x) = \sum_{d \leqslant x} \Lambda(d) \ll x$. Thus it follows that

$$\sum_{n \leqslant x} \log n = x\sum_{d \leqslant x} \frac{\Lambda(d)}{d} + O(x).$$

Now using the estimate in part (i) we immediately obtain the desired estimate.

For part (iii) we note that

$$\sum_{n\leqslant x}\frac{\Lambda(n)}{n}=\sum_{p^k\leqslant x}\frac{\log p}{p^k}=\sum_{p\leqslant x}\frac{\log p}{p}+\sum_{\substack{p^k\leqslant x\\ k\geqslant 2}}\frac{\log p}{p^k}.$$

We now observe that the contribution of prime powers with exponent greater than 1 is very small. We find that

$$\sum_{\substack{p^k\leqslant x\\ k\geqslant 2}}\frac{\log p}{p^k}\leqslant\sum_{p\leqslant x}\log p\sum_{k=2}^{\infty}\frac{1}{p^k}=\sum_{p\leqslant x}\frac{\log p}{p(p-1)}\leqslant\sum_{n=2}^{\infty}\frac{\log n}{n(n-1)}<\infty.$$

Thus we have obtained that

$$\sum_{\substack{p^k\leqslant x\\ k\geqslant 2}}\frac{\log p}{p^k}\ll 1$$

and so we have

$$\sum_{n\leqslant x}\frac{\Lambda(n)}{n}=\sum_{p\leqslant x}\frac{\log p}{p}+O(1).$$

Using the estimate in part (ii) we readily get the desired estimate. □

Let $M(x)=\sum_{n\leqslant x}\mu(x)$. Next we show that prime number theorem is equivalent to $M(x)=o(x)$.

Theorem 26. *We have $\psi(x)\sim x$ if and only if $M(x)=o(x)$.*

Proof. Assume that $\psi(x)\sim x$ and let $H(x)=\sum_{n\leqslant x}\mu(n)\log n$. By Abel's summation by parts formula we have

$$H(x)=M(x)\log x-\int_1^x\frac{M(u)}{u}\,du.$$

and so using the trivial estimate $M(u)\ll u$ we obtain

$$\frac{H(x)}{x\log x}=\frac{M(x)}{x}+O\left(\frac{1}{\log x}\right)$$

Hence, it suffices to show that $H(x)=o(x\log x)$ as $x\to\infty$. It now follows by Corollary 10 that

$$\mu(n)\log n=-\sum_{qd=n}\mu(d)\Lambda(q).$$

Using this we get that

$$H(x)=-\sum_{n\leqslant x}\sum_{qd=n}\mu(d)\Lambda(q)=-\sum_{qd\leqslant x}\mu(d)\Lambda(q)=-\sum_{d\leqslant x}\mu(d)\psi\left(\frac{x}{d}\right).$$

By the assumption there is a function $f(x)=o(1)$ such that $\psi(x)-x=xf(x)$. We take $1<y<x$ to be a quantity that depends on x and the choice of which will be made later. We split the sum as

$$H(x)=-\sum_{n\leqslant y}\mu(n)\psi\left(\frac{x}{d}\right)-\sum_{y<n\leqslant x}\mu(n)\psi\left(\frac{x}{d}\right)$$

and bound each term separately. First note that

$$\left|\sum_{n\leqslant y}\mu(n)\psi\left(\frac{x}{n}\right)\right| = \left|\sum_{n\leqslant y}\mu(n)\left(\frac{x}{n}+\psi\left(\frac{x}{n}\right)-\frac{x}{n}\right)\right| \leqslant x\left|\sum_{n\leqslant y}\frac{\mu(n)}{n}\right| + \sum_{n\leqslant y}\left|\psi\left(\frac{x}{n}\right)-\frac{x}{n}\right|.$$

Let $\overline{f}(z)=\sup_{x\geqslant z}|f(x)|$. Note that $\overline{f}(x)=o(1)$. By Corollary 2 we obtain

$$\sum_{n\leqslant y}\mu(n)\psi\left(\frac{x}{n}\right) \ll x + x\sum_{n\leqslant y}\frac{1}{n}\left|f\left(\frac{x}{n}\right)\right| \ll x + x\log y\,\overline{f}\left(\frac{x}{y}\right).$$

For the other sum we observe that

$$\sum_{y<n\leqslant x}\mu(n)\psi\left(\frac{x}{n}\right) \ll x\psi\left(\frac{x}{y}\right).$$

Taking $y=x/\log\log x$ we see that

$$\sum_{n\leqslant y}\mu(n)\psi\left(\frac{x}{n}\right) \ll x + x\log x\overline{f}(\log\log x).$$

which implies that

$$\sum_{n\leqslant y}\mu(n)\psi\left(\frac{x}{n}\right) = o(x\log x)$$

as $\overline{f}(\log\log x)=o(1)$. For the other sum using the estimate $\psi(x)\ll x$ which follows from assumption we find that

$$\sum_{y<n\leqslant x}\mu(n)\psi\left(\frac{x}{n}\right) \ll x\log\log x$$

and so this sum is also $o(x\log x)$. Hence, we have $H(x)=o(x\log x)$.

Now suppose that $M(x)=o(x)$. Note that

$$\psi(x)-\lfloor x\rfloor+2\gamma = \sum_{n\leqslant x}(\Lambda(n)-1(n)+2\gamma e(n)).$$

Using the identities $\Lambda=\mu*\log$, $1=\mu*d$, and $e=1*\mu$ we get

$$\begin{aligned}
\psi(x)-\lfloor x\rfloor+2\gamma &= \sum_{n\leqslant x}\sum_{qd=n}\mu(d)(\log q-d(q)+2\gamma)\\
&= \sum_{dq\leqslant x}\mu(d)\left(\log q-d(q)+2\gamma\right)\\
&= \sum_{q\leqslant x}M\left(\frac{x}{q}\right)(\log q-d(q)+2\gamma).
\end{aligned}$$

By assumption there is a function $g(x)=o(1)$ such that $M(x)=xg(x)$. Again we take $1<y<x$ to be some quantity depending on x which will be chosen later. We now split the sum as

$$\psi(x)-\lfloor x\rfloor+2\gamma = \sum_{q\leqslant y}M\left(\frac{x}{q}\right)(\log q-d(q)+2\gamma) + \sum_{\substack{d\leqslant x/y\\ y<q\leqslant x/d}}\mu(d)(\log q-d(q)+2\gamma).$$

Note that we can bound the first sum as

$$\left|\sum_{q\leqslant y} M\left(\frac{x}{q}\right)(\log q - d(q) + 2\gamma)\right| \leqslant x\sum_{q\leqslant y}\frac{1}{q}\left|g\left(\frac{x}{q}\right)\right| |\log q - d(q) + 2\gamma|.$$

Let $\overline{g}(z) = \sup_{x\geqslant z}|g(z)|$. Note that $\overline{g}(x) = o(1)$. It then follows that the first sum is

$$\ll xy\log y\,\overline{g}\left(\frac{x}{y}\right) \ll xy^2\,\overline{g}\left(\frac{x}{y}\right)$$

Let $\Delta(x) = \sum_{n\leqslant x}(d(n) - \log n + 2\gamma)$. Observe that for the other sum we have

$$\begin{aligned}\left|\sum_{\substack{d\leqslant x/y\\ y<q\leqslant x/d}} \mu(d)(\log q - d(q) + 2\gamma)\right| &\leqslant \sum_{d\leqslant x/y}\left|\sum_{y<q\leqslant x/d} \log q - d(q) + 2\gamma\right| \\ &= \sum_{d\leqslant x/y} |\Delta(x/d) - \Delta(y)| \\ &\ll \sum_{d\leqslant x/y}\left(\frac{\sqrt{x}}{\sqrt{d}} + \sqrt{y}\right) \ll \frac{x}{\sqrt{y}}.\end{aligned}$$

It is easy to see that $\overline{g}$ is a decreasing function and $\overline{g}(z) > 0$ for every $z \geqslant 1$ for if $\overline{g}(z_0) = 0$ for some $z_0 \geqslant 1$, then $g(x) = 0$ eventually which in turn implies that $\mu(n) = 0$ eventually, a contradiction. We now take $y = \min\left(\sqrt{x}, \overline{g}(\sqrt{x})^{-1/3}\right)$ and note that

$$xy^2\,\overline{g}\left(\frac{x}{y}\right) \leqslant xy^2\,\overline{g}(\sqrt{x}) = x\,\overline{g}(\sqrt{x})^{1/3} = o(x).$$

Since $\overline{g}(\sqrt{x}) = o(1)$ we obtain that $y \to \infty$ as $x \to \infty$ and so $x/\sqrt{y} = o(x)$. Thus we have shown that

$$\psi(x) - \lfloor x\rfloor + 2\gamma = o(x),$$

which implies $\psi(x) \sim x$. □

Exercises

1. Let $p_1, \ldots, p_k$ be distinct primes. Show that the number of positive integers with prime factorization of the form $p_1^{a_1}\cdots p_k^{a_k}$ is $\ll (\log x)^k$. Conclude that primes cannot be finite.

2. Show that for every $k \in \mathbb{N}$ there are infinitely many n such that

$$\mu(n+1) = \cdots = \mu(n+k).$$

3. Show that

$$\sum_{d^k\,|\,n}\mu(d) = \begin{cases}1 & \text{if } n \text{ is } k\text{th power-free},\\ 0 & \text{otherwise.}\end{cases}$$

4. Show that

$$\sum_{d\,|\,n} \lambda(d) = \begin{cases} 1 & \text{if } n \text{ is a square,} \\ 0 & \text{otherwise.} \end{cases}$$

5. Show that $\omega * \mu$ is the characteristic function of primes whereas $\Omega * \mu$ is the characteristic function of prime powers.

6. Show that $\prod_{t\,|\,n} t = n^{d(n)/2}$.

7. Show that

$$\sum_{t\,|\,n} d(t)^3 = \left(\sum_{t\,|\,n} d(t) \right)^2.$$

8. Show that $\varphi(n) \to \infty$ as $n \to \infty$.

9. Let us denote $e(\alpha) = e^{2\pi i \alpha}$. Show that

$$\frac{1}{q} \sum_{a=1}^{q} e\left(\frac{an}{q} \right) = \begin{cases} 1 & \text{if } q\,|\,n, \\ 0 & \text{otherwise.} \end{cases}$$

10. The Ramanujan's sum $c_q(n)$ is defined as

$$c_q(n) = \sum_{\substack{a=1 \\ (a,q)=1}}^{q} e\left(\frac{an}{q} \right).$$

Show that

$$c_q(n) = \sum_{d\,|\,(n,q)} d\mu\left(\frac{q}{d} \right).$$

Deduce that

$$\mu(q) = \sum_{\substack{a=1 \\ (a,q)=1}}^{q} e\left(\frac{a}{q} \right).$$

11. Let $\ell = (n, q)$. Show that

$$c_q(n) = \mu\left(\frac{q}{\ell} \right) \varphi(q) \varphi\left(\frac{q}{\ell} \right)^{-1}.$$

12. Prove that

$$\sigma(n) = \frac{\pi^2 n}{6} \sum_{q=1}^{\infty} \frac{c_q(n)}{q^2}.$$

13. Show that $d(n) \ll_\epsilon n^\epsilon$ for every $\epsilon > 0$.

14. Show that $\varphi(n) \gg_\epsilon n^{1-\epsilon}$ for every $\epsilon > 0$.

15. (Pollock, 2011) Let f and g be arithmetic functions such that $f = g * 1$.

(i) Show that if f and g both have finite support, then f and g must be identically zero.

(ii) Use part (i) to give a proof of infinitude of primes.

16. Show that $\sigma(n) \leqslant n(1 + \log n)$ for every $n \in \mathbb{N}$.

17. Show that

$$\frac{6n^2}{\pi^2} \leqslant \varphi(n)\sigma(n) \leqslant n^2$$

for every $n \in \mathbb{N}$.

18. Assuming primes are finite show that $\varphi(n) \asymp n$ and obtain a contradiction.

19. Let $\gamma(n) = \prod_{p|n} p$. Show that

$$\sum_{n=1}^{\infty} \frac{1}{n\gamma(n)} < \infty.$$

20. Show that

$$\sum_{n \leqslant x} \frac{n}{\varphi(n)} \ll x.$$

21. Show that

$$\sum_{n \leqslant x} \frac{1}{\varphi(n)} \ll \log x.$$

22. Prove that

$$\sum_{n \leqslant x} \varphi(n) = \frac{3x^2}{\pi^2} + O(x \log x).$$

23. Prove that

$$\sum_{n \leqslant x} \frac{\varphi(n)}{n} = \frac{6x}{\pi^2} + O(\log^2 x).$$

24.

(i) Show that

$$\frac{n}{\varphi(n)} = \sum_{d|n} \frac{\mu(d)^2}{\varphi(d)}.$$

(ii) Using the identity in part (i) prove that

$$\sum_{n \leqslant x} \frac{1}{\varphi(n)} = A \log x + B + O\left(\frac{\log^2 x}{x}\right)$$

for some constants A and B.

25. Let $Q(x)$ denote the number of squarefree integers $\leqslant x$.

(i) Prove that

$$Q(x) = \frac{6}{\pi^2}x + O(\sqrt{x}).$$

(ii) Prove that for any $n \in \mathbb{N}$

$$Q(n) \geqslant n - \sum_p \left\lfloor \frac{n}{p^2} \right\rfloor .$$

(iii) Prove that

$$\sum_p \frac{1}{p^2} < \frac{1}{2}$$

and conclude that $Q(n) > n/2$ for all $n \in \mathbb{N}$.

(iv) Prove that every integer $n > 1$ can be written as a sum of two squarefree numbers.

26. In this exercise we show that $\vartheta(x) \ll x$.

(i) Let m, n be integers with $m \geqslant 2n > 0$. Show that

$$\prod_{m-n<p\leqslant m} p \,\Big|\, \binom{m}{n}.$$

(ii) Show that $\vartheta(2n) - \vartheta(n) \leqslant 2n \log 2$.

(iii) Prove that $\vartheta(x) \leqslant 4x \log 2$ for every $x \geqslant 1$.

27. For a prime p let $\mathcal{A}_p$ denote the set of all positive integers n such that either $n + 1 \equiv 0 \pmod{p^2}$ or $n - 1 \equiv 0 \pmod{p^2}$.

(i) If $[N] := \{1, \ldots, N\}$, then show that

$$\left| [N] \cap \bigcup_p \mathcal{A}_p \right| \leqslant 2N \sum_{p\leqslant N} \frac{1}{p^2} + 2\pi(N).$$

(ii) Prove that

$$\sum_p \frac{1}{p^2} < \frac{1}{4}$$

and conclude that the set of twin squarefree integers have positive density.

28. Using Abel's summation for the partial sum $\sum_{p\leqslant x} (\log p)/p$ show that if the limit

$$\lim_{x\to\infty} \frac{\pi(x) \log x}{x},$$

exists then it must be 1.

1. If $n = p_1^{a_1} \cdots p_k^{a_k}$ and $n \leqslant x$, then $a_i \leqslant \log_2 x$ for every i. Thus the number of integers up to x must be $\leqslant (\lfloor \log_2 x \rfloor + 1)^k \ll (\log x)^k$. But this is clearly a contradiction as $x \not\ll (\log x)^k$.

2. Let $p_1, \dots, p_k$ be distinct primes. Then by the Chinese Remainder Theorem there exist infinitely many positive integers n such that $n \equiv -j \mod p_j^2$ for every $1 \leqslant j \leqslant k$. Thus $p_j^2 \,|\, (n+j)$ for every $1 \leqslant j \leqslant k$ and so $n+j$ is not squarefree, i.e., $\mu(n+1) = \cdots = \mu(n+k) = 0$.

3. Let $n = m^k q$, where q is kth power-free. Then observe that $d^k \,|\, n$ if and only if $d \,|\, m$. Thus

$$\sum_{d^k \,|\, n} \mu(d) = \sum_{d \,|\, m} \mu(d) = e(m).$$

The assertion now follows as n is kth power-free if and only if $m = 1$.

4. Let $f(n) = \sum_{d \,|\, n} \lambda(d)$. Then $f = \lambda * 1$ is multiplicative and so it suffices to evaluate f at prime powers. Note that

$$f(p^a) = \lambda(1) + \lambda(p) + \cdots + \lambda(p^a) = 1 - 1 + \cdots + (-1)^a = \begin{cases} 1 & \text{if } a \text{ is even,} \\ 0 & \text{otherwise.} \end{cases}$$

The assertion now immediately follows from this.

5. Let $1_{\mathbb{P}}$ denote the characteristic function of primes. Then note that $\omega(n) = \sum_{d \,|\, n} 1_{\mathbb{P}}(d)$, i.e., $\omega = 1_{\mathbb{P}} * 1$ which can be written as $\omega * \mu = 1_{\mathbb{P}}$ by Möbius inversion. Similarly, if $1_{\mathcal{P}}$ is the characteristic function of prime powers, then $\Omega(n) = \sum_{d \,|\, n} 1_{\mathcal{P}}(d)$, i.e., $\Omega = 1_{\mathcal{P}} * 1$ and so we have $\Omega * \mu = 1_{\mathcal{P}}$.

6. Follows by observing that

$$\left(\prod_{t \,|\, n} t \right)^2 = \left(\prod_{t \,|\, n} \frac{n}{t} \right) \left(\prod_{t \,|\, n} t \right) = \prod_{t \,|\, n} n = n^{d(n)}.$$

7. Denote the left-hand side by $f(n)$ and right-hand side by $g(n)$. Note that both f and g are multiplicative and so it suffices to show that two coincide at prime powers. Observe that

$$f(p^a) = \sum_{t \,|\, p^a} d(t)^3 = \sum_{k=1}^{a+1} k^3 = \left(\sum_{k=1}^{a+1} k \right)^2 = \left(\sum_{t \,|\, p^a} d(t) \right)^2 = g(p^a).$$

8. Let $M > 0$ and let $\varphi(n) \leqslant M$. Take $n = \prod_{i=1}^k p_i^{a_i}$. Then we have

$$\varphi(n) = \prod_{i=1}^k p_i^{a_i - 1}(p_i - 1) \leqslant M.$$

This shows that $p_i - 1 \leqslant M$ and $2^{a_i - 1} \leqslant M$ for every i. Hence we have $p_i \leqslant M + 1$ and $2^{a_i} \leqslant 2M$ for every i. Thus the exponents a_i are bounded by $\log_2(2M)$. This shows that there are only finitely many positive integers n with $\varphi(n) \leqslant M$. Thus we conclude that $\varphi(n) \to \infty$ as $n \to \infty$.

9. Note that if $q \,|\, n$, then $e(an/q) = 1$ for every $1 \leqslant a \leqslant q$ and so we have

$$\frac{1}{q} \sum_{a=1}^{q} e(an/q) = \frac{1}{q} \sum_{a=1}^{q} 1 = 1.$$

Now suppose that $q \nmid n$. Then we have $e(n/q) \neq 1$ and so

$$\frac{1}{q} \sum_{a=1}^{q} e(an/q) = \frac{1}{q} \sum_{a=1}^{q} e(n/q)^{a} = \frac{1}{q} \left(\frac{e(n/q)^{q+1} - 1}{e(n/q) - 1} - 1 \right) = 0$$

as $e(n/q)^{q+1} = e(n/q)$.

10. Observe that

$$c_q(n) = \sum_{\substack{a=1 \\ (a,q)=1}}^{q} e\left(\frac{an}{q}\right) = \sum_{a=1}^{q} e\left(\frac{an}{q}\right) \sum_{d \,|\, (a,q)} \mu(d) = \sum_{a=1}^{q} e\left(\frac{an}{q}\right) \sum_{\substack{d \,|\, a \\ d \,|\, q}} \mu(d).$$

Changing the order of summation we get

$$c_q(n) = \sum_{d \,|\, q} \mu(d) \sum_{\substack{a=1 \\ d \,|\, a}}^{q} e\left(\frac{an}{q}\right) = \sum_{d \,|\, q} \mu(d) \sum_{r=1}^{q/d} e\left(\frac{rdn}{q}\right) = \sum_{d \,|\, q} \mu(d) \sum_{r=1}^{q/d} e\left(\frac{rn}{q/d}\right).$$

Thus we can rewrite $c_q(n)$ as

$$c_q(n) = \sum_{d \,|\, q} \mu\left(\frac{q}{d}\right) \sum_{r=1}^{d} e\left(\frac{rn}{d}\right).$$

Using the identity $\frac{1}{q} \sum_{a=1}^{q} e(an/q) = 1_{q \,|\, n}$ we end up with

$$c_q(n) = \sum_{\substack{d \,|\, q \\ d \,|\, n}} d\mu\left(\frac{q}{d}\right) = \sum_{d \,|\, (q,n)} d\mu\left(\frac{q}{d}\right).$$

Finally, note that

$$\mu(q) = c_q(1) = \sum_{\substack{a=1 \\ (a,q)=1}}^{q} e\left(\frac{a}{q}\right).$$

11. Note that

$$c_q(n) = \sum_{d \,|\, \ell} d\mu\left(\frac{q}{d}\right) = \sum_{de=\ell} d\mu\left(\frac{qe}{\ell}\right).$$

Let $q_1 = q/\ell$. Then observe that $\mu(q_1 e) = 0$ if $(q_1, e) \neq 1$ and $\mu(q_1 e) = \mu(q_1)\mu(e)$ if $(q_1, e) = 1$. Using this we get

$$c_q(n) = \sum_{de=\ell} d\mu(q_1 e) = \sum_{\substack{de=\ell \\ (q_1,e)=1}} d\mu(q_1)\mu(e) = \mu(q_1) \sum_{\substack{de=\ell \\ (q_1,e)=1}} d\mu(e) = \mu(q_1)\ell \sum_{\substack{de=\ell \\ (q_1,e)=1}} \frac{\mu(e)}{e}.$$

It is easy to see that

$$\sum_{\substack{e\,|\,\ell\\(e,q_1)=1}}\frac{\mu(e)}{e}=\prod_{\substack{p\,|\,\ell\\p\nmid q_1}}\left(1-\frac{1}{p}\right)=\prod_{\substack{p\,|\,q\\p\nmid q_1}}\left(1-\frac{1}{p}\right)=\prod_{p\,|\,q}\left(1-\frac{1}{p}\right)\cdot\prod_{p\,|\,q_1}\left(1-\frac{1}{p}\right)^{-1}=\frac{\varphi(q)}{q}\cdot\frac{q_1}{\varphi(q_1)}=\frac{\varphi(q)}{\ell\varphi(q_1)}.$$

Hence we have

$$c_q(n)=\mu(q_1)\ell\cdot\frac{\varphi(q)}{\ell\varphi(q_1)}=\mu(q_1)\varphi(q)\varphi(q_1)^{-1}.$$

12. We rewrite $\sigma(n)$ as

$$\sigma(n)=n\sum_{d\,|\,n}\frac{1}{d}=n\sum_{d=1}^{n}\frac{1}{d}\left(\frac{1}{d}\sum_{a=1}^{d}e(an/d)\right)=n\sum_{d=1}^{n}\frac{1}{d^2}\sum_{a=1}^{d}e(an/d).$$

since $\frac{1}{d}\sum_{a=1}^{d}e(an/d)$ is the characteristic function of the divisors of n. Since the factor $\sum_{a=1}^{d}e(an/d)=0$ for $d>n$, we can extend the above finite sum to an infinite sum as

$$\sigma(n)=n\sum_{d=1}^{\infty}\frac{1}{d^2}\sum_{a=1}^{d}e(an/d). \tag{2.4}$$

Observe that

$$\begin{aligned}\sum_{a=1}^{d}e(an/d)&=\sum_{q\,|\,d}\sum_{\substack{a=1\\(a,d)=q}}^{d}e(an/d)=\sum_{q\,|\,d}\sum_{\substack{r=1\\(r,d/q)=1}}^{d/q}e(rqn/d)\\&=\sum_{q\,|\,d}\sum_{\substack{r=1\\(r,d/q)=1}}^{d/q}e(rn/(d/q))=\sum_{q\,|\,d}c_{d/q}(n)=\sum_{q\,|\,d}c_q(n).\end{aligned}$$

Substituting this into (2.4) we obtain

$$\sigma(n)=n\sum_{d=1}^{\infty}\frac{1}{d^2}\sum_{q\,|\,d}c_q(n)=n\sum_{d=1}^{\infty}\sum_{q\,|\,d}\frac{c_q(n)}{d^2}.$$

Changing the order of summation, we get

$$\begin{aligned}\sigma(n)&=n\sum_{q=1}^{\infty}\sum_{\substack{d=1\\q\,|\,d}}^{\infty}\frac{1}{d^2}c_q(n)=n\sum_{q=1}^{\infty}\sum_{\ell=1}^{\infty}\frac{c_q(n)}{(q\ell)^2}\\&=n\sum_{q=1}^{\infty}\sum_{\ell=1}^{\infty}\frac{1}{\ell^2}\frac{c_q(n)}{q^2}=n\left(\sum_{\ell=1}^{\infty}\frac{1}{\ell^2}\right)\sum_{q=1}^{\infty}\frac{c_q(n)}{q^2}\\&=\frac{n\pi^2}{6}\sum_{q=1}^{\infty}\frac{c_q(n)}{q^2},\end{aligned}$$

where we use $\sum_{k=1}^{\infty}1/k^2=\pi^2/6$ in the final equality. The change of order of summation is justified by the fact that $c_q(n)\ll_n 1$ and $d(n)\leqslant\sqrt{n}$.

13. Let $n = p_1^{a_1} \cdots p_k^{a_k}$. Then we have

$$\frac{d(n)}{n^\epsilon} = \prod_{p^a || n} \frac{a+1}{p^{a\epsilon}} \leqslant \prod_{\substack{p^a || n \\ p < 2^{1/\epsilon}}} \frac{a+1}{p^{a\epsilon}}$$

for if $p \geqslant 2^{1/\epsilon}$, then $p^\epsilon \geqslant 2$ and so $p^{a\epsilon} \geqslant 2^a \geqslant a+1$ which gives $(a+1)/p^{a\epsilon} \leqslant 1$. Now observe that

$$\frac{d(n)}{n^\epsilon} \leqslant \prod_{\substack{p^a || n \\ p < 2^{1/\epsilon}}} \frac{a+1}{2^{a\epsilon}} \leqslant \prod_{\substack{p^a || n \\ p < 2^{1/\epsilon}}} \frac{a+1}{a\epsilon \log 2}$$

as $2^{a\epsilon} = e^{a\epsilon \log 2} \geqslant a\epsilon \log 2$. Finally we have

$$\frac{d(n)}{n^\epsilon} \leqslant \prod_{p < 2^{1/\epsilon}} \frac{2}{\epsilon \log 2} \leqslant \left(\frac{2}{\epsilon \log 2} \right)^{\pi(2^{1/\epsilon})}.$$

This shows that $d(n) \ll_\epsilon n^\epsilon$.

14. Note that

$$\varphi(n) = n \prod_{p \,|\, n} \left(1 - \frac{1}{p}\right) \geqslant \frac{n}{2^{\omega(n)}} \geqslant \frac{n}{d(n)} \gg n^{1-\epsilon}$$

as $1 - 1/p \geqslant 1/2$ for every prime p and $d(n) \geqslant 2^{\omega(n)}$ for every $n \in \mathbb{N}$.

We present another solution which involves bounding the product $\prod_{p|n}(1 - 1/p)$ from below. Let n be a positive integer. Then we have

$$\log \prod_{p \,|\, n} \left(1 - \frac{1}{p}\right) = \sum_{p \,|\, n} \log\left(1 - \frac{1}{p}\right) = -\sum_{p \,|\, n} \sum_{m=1}^{\infty} \frac{1}{mp^m} = -\sum_{m=1}^{\infty} \frac{1}{m} \sum_{p \,|\, n} \frac{1}{p^m}.$$

Let $p_1 < p_2 < \cdots$ be the enumeration of primes. Then $K > 1$ we have

$$\sum_{p \,|\, n} \frac{1}{p^m} \leqslant \sum_{k < K} \frac{1}{p_k^m} + \frac{\omega(n)}{p_K^m}.$$

This then leads to

$$\begin{aligned} \sum_{m=1}^{\infty} \frac{1}{m} \sum_{p \,|\, n} \frac{1}{p^m} &\leqslant \sum_{m=1}^{\infty} \frac{1}{m} \sum_{k<K} \frac{1}{p_k^m} + \omega(n) \sum_{m=1}^{\infty} \frac{1}{mp_K^m} \\ &= -\sum_{k<K} \log\left(1 - \frac{1}{p_k}\right) - \omega(n) \log\left(1 - \frac{1}{p_K}\right). \end{aligned}$$

Using the inequality $\omega(n) \leqslant \log_2 n$ we obtain

$$\log \prod_{p \,|\, n} \left(1 - \frac{1}{p}\right) \geqslant \sum_{k<K} \log\left(1 - \frac{1}{p_k}\right) + \frac{\log n}{\log 2} \log\left(1 - \frac{1}{p_K}\right).$$

If we denote

$$c_K = \sum_{k<K} \log\left(1 - \frac{1}{p_k}\right) \quad \text{and} \quad \epsilon_K = -\frac{1}{\log 2} \log\left(1 - \frac{1}{p_K}\right),$$

then we have

$$\log\varphi(n) = \log n + \log\prod_{p\,|\,n}\left(1-\frac{1}{p}\right) \geqslant c_K + (1-\epsilon_K)\log n.$$

Hence we conclude that

$$\varphi(n) \geqslant e^{c_K} n^{1-\epsilon_K}.$$

Since $\epsilon_K \to 0$ as $K \to \infty$ we conclude that $\varphi(n) \gg_\epsilon n^{1-\epsilon}$ for every $\epsilon > 0$.

15. For part (i) suppose for the sake of contradiction that f and g both have finite support. Now consider the entire function $F(z)$ defined as

$$F(z) = \sum_{n=1}^{\infty} f(n)z^n.$$

The function $F(z)$ is a polynomial in z. It then follows that

$$F(z) = \sum_{n=1}^{\infty}\sum_{d\,|\,n} g(d)z^n = \sum_{d} g(d)\sum_{\substack{n=1\\ d\,|\,n}}^{\infty} z^n = \sum_{d=1}^{\infty} g(d)\frac{z^d}{1-z^d}$$

for $|z| < 1$. The change of order of summation is justified by observing that

$$\sum_{n=1}^{\infty}\sum_{d\,|\,n} |g(d)||z|^n \leqslant A\sum_{n=1}^{\infty} n|z|^n = A\frac{|z|}{(1-|z|^2)} < \infty,$$

where $A = \max_{d\in\mathbb{N}} |g(d)|$. It then follows by principle of analytic continuation that

$$F(z) = \sum_{d=1}^{\infty} g(d)\frac{z^d}{1-z^d}$$

for every z with $z^d \neq 1$. Let $D = \max\{d \in \mathbb{N} : g(d) \neq 0\}$. Then the sum on the right hand side has a pole at $z = e^{2\pi i/D}$, a contradiction. Thus f and g must both be identically zero.

If there are only finitely many primes, then μ has a finite support as there would then be only finitely many squarefree integers. But $\mu * 1 = e$ has finite support as well. This contradicts part (i) as μ and e are not identically zero.

16. Note that

$$\sigma(n) = n\sum_{d\,|\,n}\frac{1}{d} \leqslant n\sum_{k=1}^{n}\frac{1}{k} = n(\log n + 1).$$

17. Observe that

$$\begin{aligned}\varphi(n)\sigma(n) &= \prod_{p^a||n}\varphi(p^a)\sigma(p^a) = \prod_{p^a||n} p^{a-1}(p-1)\left(\frac{p^{a+1}-1}{p-1}\right)\\ &= \prod_{p^a||n} p^{a-1}(p^{a+1}-1) = n^2\prod_{p^a||n}\left(1-\frac{1}{p^{a+1}}\right).\end{aligned}$$

Because the product is $\leqslant 1$ we get the inequality $\varphi(n)\sigma(n) \leqslant 1$. As for the other inequality note that

$$\prod_{p^a || n}\left(1-\frac{1}{p^{a+1}}\right) \geqslant \prod_{p}\left(1-\frac{1}{p^2}\right) = \frac{1}{\zeta(2)} = \frac{6}{\pi^2}.$$

18. If primes are finite then

$$\varphi(n) = n\prod_{p\,|\,n}\left(1-\frac{1}{p}\right) \geqslant n\prod_{p}\left(1-\frac{1}{p}\right).$$

Let $c = \prod_p (1-1/p)$. Then we have $cn^2\sum_{d\,|\,n} 1/d \leqslant \varphi(n)\sigma(n) \leqslant n^2$ and so $\sum_{d\,|\,n} 1/d = O(1)$, a contradiction as $\sum_{d\,|\,m!} 1/d \geqslant \sum_{k=1}^{m} 1/k$ and harmonic sums are unbounded.

19. First note that the map $n \mapsto n\gamma(n)$ is injective due to unique factorization. Now observe that

$$\begin{aligned}\sum_{n\leqslant x}\frac{1}{n\gamma(n)} &\leqslant \prod_{p\leqslant x}\left(1+\frac{1}{p^2}+\frac{1}{p^3}+\cdots\right)\\ &= \prod_{p\leqslant x}\left(1+\frac{1}{p^2}\frac{1}{1-1/p}\right)\\ &= \prod_{p\leqslant x}\left(1+\frac{1}{p(p-1)}\right),\end{aligned}$$

where the first inequality follows since every prime occurs in $n\gamma(n)$ occurs with power at least 2. Because the infinite product $\prod_p (1+1/(p(p-1)))$ converges, the conclusion follows.

20. Note that

$$\frac{n}{\varphi(n)} = \prod_{p\,|\,n}\left(1-\frac{1}{p}\right)^{-1} = \prod_{p\,|\,n}\left(1+\frac{1}{p}+\frac{1}{p^2}+\cdots\right) = \sum_{\substack{m\\ p\,|\,m \Rightarrow p\,|\,n}}\frac{1}{m} = \sum_{\substack{m\\ \gamma(m)|\gamma(n)}}\frac{1}{m}.$$

Using this we get

$$\sum_{n\leqslant x}\frac{n}{\varphi(n)} = \sum_{n\leqslant x}\sum_{\substack{m\\ \gamma(m)|\gamma(n)}}\frac{1}{m} = \sum_{m}\frac{1}{m}\sum_{\substack{n\leqslant x\\ \gamma(m)|\gamma(n)}} 1 \leqslant \sum_{m}\frac{1}{m}\sum_{\substack{n\leqslant x\\ \gamma(m)|n}} 1 \leqslant x\sum_{m}\frac{1}{m\gamma(m)}.$$

21. Let $A(x) = \sum_{n\leqslant x} n/\varphi(n)$. Using Abel's summation we obtain

$$\sum_{n\leqslant x}\frac{1}{\varphi(n)} = \frac{A(x)}{x} + \int_1^x \frac{A(u)}{u^2}\,du.$$

Using the estimate $A(x) \ll x$ we have $A(x)/x = O(1)$ and

$$\int_1^x \frac{A(u)}{u^2}\,du \ll \int_1^x \frac{du}{u} = \log x.$$

This readily implies the desired result.

22. Note that

$$\sum_{n\leqslant x}\varphi(n)=\sum_{n\leqslant x}\sum_{d\,|\,n}\mu(d)\frac{n}{d}=\sum_{d\leqslant x}\frac{\mu(d)}{d}\sum_{\substack{n\leqslant x\\ d\,|\,n}}n=\sum_{d\leqslant x}\mu(d)\sum_{q\leqslant x/d}q.$$

Using the estimate $\sum_{n\leqslant x}n^{\alpha}=x^{1+\alpha}/(1+\alpha)+O(x^{\alpha})$ we get

$$\begin{aligned}\sum_{n\leqslant x}\varphi(n)=\frac{x^2}{2}\sum_{d\leqslant x}\frac{\mu(d)}{d^2}+O\left(x\sum_{d\leqslant x}\frac{|\mu(d)|}{d}\right)&=\frac{x^2}{2}\left(\frac{6}{\pi^2}+O\left(\frac{1}{x}\right)\right)+O\left(x\log x\right)\\ &=\frac{3x^2}{\pi^2}+O(x\log x),\end{aligned}$$

where we use $\sum_{n=1}^{\infty}\mu(n)n^{-2}=1/\zeta(2)=6/\pi^2$ and the bounds

$$\sum_{n>x}\frac{1}{n^2}=O\left(\frac{1}{x}\right),\qquad \sum_{n\leqslant x}\frac{1}{n}=\log x+\gamma+O\left(\frac{1}{x}\right).$$

in the penultimate step.

23. Let $A(x)=\sum_{n\leqslant x}\varphi(n)$. Using Abel's summation by parts formula we have

$$\begin{aligned}\sum_{n\leqslant x}\frac{\varphi(n)}{n}&=\frac{A(x)}{x}+\int_1^x\frac{A(u)}{u^2}\,du\\ &=\frac{3x}{\pi^2}+O(\log x)+\int_1^x\left(\frac{3}{\pi^2}+O\left(\frac{\log u}{u}\right)\right)\,du\\ &=\frac{6x}{\pi^2}+O\left(\log x\right)+O\left(\int_1^x\frac{\log u}{u}\,du\right)\\ &=\frac{6x}{\pi^2}+O\left(\log^2 x\right).\end{aligned}$$

A slightly better estimate can be obtained by noting that

$$\begin{aligned}\sum_{n\leqslant x}\frac{\varphi(n)}{n}&=\sum_{n\leqslant x}\sum_{d\,|\,n}\frac{\mu(d)}{d}=\sum_{d\leqslant x}\frac{\mu(d)}{d}\sum_{\substack{n\leqslant x\\ d\,|\,n}}1\\ &=x\sum_{d\leqslant x}\frac{\mu(d)}{d^2}+O\left(\sum_{d\leqslant x}\frac{|\mu(d)|}{d}\right)\\ &=\frac{6x}{\pi^2}+O(\log x).\end{aligned}$$

24. For part (i) note that

$$\frac{n}{\varphi(n)} = \prod_{p\,|\,n}\left(1-\frac{1}{p}\right)^{-1} = \prod_{p\,|\,n}\frac{p}{p-1} = \prod_{p\,|\,n}\left(1+\frac{1}{p-1}\right)$$
$$= \sum_{I\subset\{p\,|\,n\}}\frac{1}{\prod_{p\in I}(p-1)} = \sum_{I\subset\{p\,|\,n\}}\frac{1}{\varphi(\prod_{p\in I}p)}$$
$$= \sum_{\substack{d\,|\,n\\ d\,\text{sq. free}}}\frac{1}{\varphi(d)} = \sum_{d\,|\,n}\frac{\mu(d)^2}{\varphi(d)}.$$

For part (ii) observe that

$$\sum_{n\leqslant x}\frac{1}{\varphi(n)} = \sum_{n\leqslant x}\frac{1}{n}\sum_{d\,|\,n}\frac{\mu(d)^2}{\varphi(d)}.$$

Changing the order of summation the above sum becomes

$$\sum_{d\leqslant x}\frac{\mu(d)^2}{\varphi(d)}\sum_{\substack{n\leqslant x\\ d\,|\,n}}\frac{1}{n} = \sum_{d\leqslant x}\frac{\mu(d)^2}{d\varphi(d)}\sum_{q\leqslant x/d}\frac{1}{q}.$$

Incorporating the estimate for harmonic sums we get

$$\sum_{n\leqslant x}\frac{1}{\varphi(n)} = \sum_{d\leqslant x}\frac{\mu(d)^2}{d\varphi(d)}\left(\log\frac{x}{d}+\gamma+O\left(\frac{d}{x}\right)\right)$$
$$= \log x\sum_{d\leqslant x}\frac{\mu(d)^2}{d\varphi(d)} - \sum_{d\leqslant x}\frac{\mu(d)^2\log d}{d\varphi(d)} + \gamma\sum_{d\leqslant x}\frac{\mu(d)^2}{d\varphi(d)} + O\left(\frac{1}{x}\sum_{d\leqslant x}\frac{\mu(d)^2}{\varphi(d)}\right).$$

Since $\varphi(n) \gg n/\log n$ we have

$$\sum_{d\leqslant x}\frac{\mu(d)^2}{d\varphi(d)} = \sum_{d=1}^{\infty}\frac{\mu(d)^2}{d\varphi(d)} + O\left(\sum_{d>x}\frac{\log d}{d^2}\right).$$

By comparison the error term is

$$\ll \int_x^\infty \frac{\log u}{u^2}\,du \ll \frac{\log x}{x}$$

by integral test and integration by parts. Similarly we have

$$\sum_{d\leqslant x}\frac{\mu(d)^2\log d}{d\varphi(d)} = \sum_{d=1}^{\infty}\frac{\mu(d)^2\log d}{d\varphi(d)} + O\left(\frac{\log^2 x}{x}\right). \tag{2.5}$$

Finally,

$$\sum_{d\leqslant x}\frac{\mu(d)^2}{\varphi(d)} \ll \sum_{d\leqslant x}\frac{\log d}{d} \ll \log x\sum_{d\leqslant x}\frac{1}{d} \ll \log^2 x.$$

In fact, the above sum is $\ll \log x$ as $\sum_{n\leqslant x} 1/\varphi(n) \ll \log x$ but we can afford to lose a log power because of (2.5). Putting everything together we conclude

$$\sum_{n\leqslant x}\frac{1}{\varphi(n)} = \log x\sum_{d=1}^{\infty}\frac{\mu(d)^2}{d\varphi(d)} + \gamma\sum_{d=1}^{\infty}\frac{\mu(d)^2}{d\varphi(d)} - \sum_{d=1}^{\infty}\frac{\mu(d)^2}{d\varphi(d)} + O\left(\frac{\log^2 x}{x}\right).$$

25. (i): Using the identity $\sum_{d^2\,|\,n}\mu(d) = \mu(n)^2$ we obtain

$$\begin{aligned}
Q(x) &= \sum_{n\leqslant x}\mu(n)^2 = \sum_{n\leqslant x}\sum_{d^2\,|\,n}\mu(d) = \sum_{d\leqslant\sqrt{x}}\sum_{\substack{n\leqslant x\\ d^2\,|\,n}}\mu(d)\\
&= \sum_{d\leqslant\sqrt{x}}\mu(d)\sum_{\substack{n\leqslant x\\ d^2\,|\,n}}1 = \sum_{d\leqslant\sqrt{x}}\mu(d)\left\lfloor\frac{x}{d^2}\right\rfloor\\
&= x\sum_{d\leqslant\sqrt{x}}\frac{\mu(d)}{d^2} + O\left(\sum_{d\leqslant\sqrt{x}}|\mu(d)|\right) = x\sum_{d\leqslant\sqrt{x}}\frac{\mu(d)}{d^2} + O(\sqrt{x}) \qquad (2.6)
\end{aligned}$$

We now push d off to ∞ in the above sum and incur an error due to the tail which fortunately is only $O(\sqrt{x})$ as seen can be seen by

$$\sum_{d\leqslant\sqrt{x}}\frac{\mu(d)}{d^2} = \sum_{d=1}^{\infty}\frac{\mu(d)}{d^2} + O\left(\sum_{d>\sqrt{x}}\frac{1}{d^2}\right) = \frac{6}{\pi^2} + O\left(\frac{1}{\sqrt{x}}\right).$$

Using this estimate in (2.6), we obtain the desired result.

(ii): Observe that for an integer n we have

$$\{1\leqslant m\leqslant n : m \text{ not squarefree}\} \subset \bigcup_p\{1\leqslant m\leqslant n : p^2\,|\,m\},$$

where p runs over all primes. It thus follows that

$$n - Q(n) \leqslant \sum_p\left\lfloor\frac{n}{p^2}\right\rfloor,$$

which is the desired inequality.

(iii): Note that

$$\sum_p\frac{1}{p^2} < \frac{1}{4} + \sum_{k=1}^{\infty}\frac{1}{(2k+1)^2} < \frac{1}{4} + \sum_{k=1}^{\infty}\frac{1}{4k(k+1)} = \frac{1}{2}.$$

It now follows by part (ii) that

$$Q(n) \geqslant n - \sum_p\left\lfloor\frac{n}{p^2}\right\rfloor \geqslant n - n\sum_p\frac{1}{p^2} > \frac{n}{2}.$$

for every $n\in\mathbb{N}$.

(iv): Suppose for the sake of contradiction that $n > 1$ and n cannot be expressed as a sum of two squarefree integers. Then for every $a, b\in\mathbb{N}$ satisfying $a + b = n$ either a or b is not squarefree. It follows that there are at least $(n-1)/2$ (this is the number of ways n can be written as a sum of two positive integers without regard for order) integers up to $n-1$ that are not square free. Hence, we must have

$$Q(n-1) \leqslant (n-1) - \frac{(n-1)}{2} = \frac{n-1}{2}.$$

But this contradicts the inequality in part (iii).

26. (i): Note that

$$\binom{m}{n} = \frac{m(m-1)\cdots(m-n+1)}{n!}.$$

If $m-n < p \leqslant m$, then p divides the numerator but not the denominator as $m-n+1 > n$. This implies that $\prod_{m-n<p\leqslant m} p$ divides $\binom{m}{n}$.

(ii): Observe that $\binom{2n}{n} \leqslant 2^{2n}$ for every $n \in \mathbb{N}$. Thus taking logarithm we deduce that

$$\vartheta(2n) - \vartheta(n) \leqslant 2n \log 2.$$

(iii): Now suppose that $2^m \leqslant n < 2^{m+1}$. Then we have

$$\vartheta(n) \leqslant \vartheta(2^{m+1}) = \sum_{k=0}^{m}(\vartheta(2^{k+1}) - \vartheta(2^k)) \leqslant \sum_{k=0}^{m} 2^{k+1}\log 2 = (2^{m+2}-2)\log 2.$$

Finally note that $2^{m+2} = 4\cdot 2^m \leqslant 4n$. Let $x \geqslant 1$ with $n = \lfloor x \rfloor$. Then we have $\vartheta(x) = \vartheta(n) \leqslant 4n\log 2 \leqslant 4x \log 2$.

27. Observe that if $n \in \mathcal{A}_p$, then $p^2 \leqslant n+1$ and so $p \leqslant n$. Consequently $[N] \cap \mathcal{A}_p = \emptyset$ for $p > N$ and so we have

$$\left|[N] \cap \bigcup_p \mathcal{A}_p\right| = \left|\bigcup_{p\leqslant N} [N] \cap \mathcal{A}_p\right| \leqslant \sum_{p\leqslant N} |[N]\cap\mathcal{A}_p|.$$

Since there are exactly two elements of $\mathcal{A}_p$ in any set of p^2 consecutive integers we obtain that $|[N]\cap\mathcal{A}_p| \leqslant 2\lceil N/p^2\rceil$ and so

$$\left|[N] \cap \bigcup_p \mathcal{A}_p\right| \leqslant 2\sum_{p\leqslant N}\left\lceil \frac{N}{p^2}\right\rceil \leqslant 2N\sum_{p\leqslant N}\frac{1}{p^2} + 2\pi(N).$$

Note that

$$\sum_p \frac{1}{p^2} \leqslant \frac{1}{2^2} + \sum_{k=1}^{\infty}\frac{1}{(2k+1)^2} \leqslant \frac{1}{4} + \frac{1}{4}\sum_{k=1}^{\infty}\frac{1}{k(k+1)} = \frac{1}{2}.$$

Let $C = \sum_p p^{-2}$. We have shown above that $C < \frac{1}{2}$. Thus

$$\frac{1}{N}\left|[N] \cap \bigcup_p \mathcal{A}_p\right| \leqslant 2C + 2\frac{\pi(N)}{N}.$$

Hence we have

$$\limsup_{N\to\infty}\frac{1}{N}\left|[N] \cap \bigcup_p \mathcal{A}_p\right| \leqslant 2C < 1.$$

as $\pi(N)/N \to \infty$, i.e., the set of primes has density 0. Since $[N] \cap \bigcup_p \mathcal{A}_p$ is the set of all positive integers n up to N such that either $n-1$ or $n+1$ is not squarefree, $\bigcap_p [N] \backslash \mathcal{A}_p = [N] \backslash \left([N] \cap \bigcup_p \mathcal{A}_p \right)$ is the set of all twin squarefree integers. Finally,

$$\liminf_{N\to\infty} \frac{1}{N} \left| \bigcap_p [N] \backslash \mathcal{A}_p \right| = 1 - \limsup_{N\to\infty} \frac{1}{N} \left| [N] \cap \bigcup_p \mathcal{A}_p \right| = 1 - 2C > 0.$$

This shows that the set of twin squarefree integers has positive (lower) density.

28. Using Abel's summation by parts formula we obtain

$$\sum_{p \leqslant x} \frac{\log p}{p} = \frac{\pi(x)\log x}{x} - \int_2^x \pi(u) \left(\frac{1 - \log u}{u^2} \right) du.$$

Using the estimate for $\sum_{p\leqslant x} (\log p)/p$ we see that

$$\log x + O(1) = \frac{\pi(x)\log x}{x} + \int_2^x \frac{\pi(u)\log u}{u} \left(\frac{1}{u} - \frac{1}{u \log u} \right) du.$$

Hence we obtain

$$\int_2^x \frac{\pi(u)\log u}{u} \left(\frac{1}{u} - \frac{1}{u \log u} \right) du = \log x + O(1) \tag{2.7}$$

as $\pi(x)(\log x)/x \ll 1$. Now let L be the limit of $\pi(x)(\log x)/x$ as $x \to \infty$ and let $\epsilon > 0$ be fixed. Let $x_0 \geqslant 1$ be such that $\pi(x)(\log x)/x < L + \epsilon$ for $x \geqslant x_0$. Then we get

$$\begin{aligned} \int_2^x \frac{\pi(u)\log u}{u} \left(\frac{1}{u} - \frac{1}{u \log u} \right) du &= \left(\int_2^{x_0} + \int_{x_0}^x \right) \frac{\pi(u)\log u}{u} \left(\frac{1}{u} - \frac{1}{u \log u} \right) du \\ &\leqslant (L+\epsilon)(\log x - \log\log x) + O(1) \\ &\leqslant (L+\epsilon)\log x + O(1), \end{aligned}$$

where the implicit constant depends on ϵ. Combining this with (2.7) we immediately deduce that $L \geqslant 1$. Similarly we have

$$\int_2^x \frac{\pi(u)\log u}{u} \left(\frac{1}{u} - \frac{1}{u \log u} \right) du \geqslant (L-\epsilon)\log x + O(1).$$

Combining this with (2.7) it follows that $L \leqslant 1$. Thus we must have $L = 1$.

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