ANALYTIC NUMBER THEORY

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Contents

1. Arithmetical functions

1.1. Some basic arithmetical functions.

Definition 1.1. The *Möbius function*, denoted μ, is defined as

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1, \\ (-1)^k & \text{if } n = p_1 \dots p_k, \text{ where } p_1, \dots, p_k \text{ are distinct primes,} \\ 0 & \text{otherwise.} \end{cases}$$

Note that μ is the signed characteristic function of the squarefree integers. The definition of μ seems to be unmotivated right now but later we will see that μ is the inverse of the unit function in some group of arithmetical functions. Knowing that the inverse of unit function exists one can recover this definition.

Proposition 1.2. If $n \geqslant 1$, then

$$\sum_{d|n} \mu(d) = e(n).$$

Proof. If $n = 1$, then the formula clearly holds as $\mu(1) = 1$. Now suppose that $n = \prod_{i=1}^k p_i^{a_i}$. Because $\mu(d)$ is nonzero if and only if d is squarefree, we can restrict the sum to divisors of the form $\prod_{i\in I} p_i$, where I is a subset of $\{1,\dots,k\}$. Hence, we get

$$\sum_{d\,|\,n}\mu(d) = \sum_{I\subset\{1,\dots,n\}}\mu\left(\prod_{i\in I}p_i\right) = \sum_{I\subset\{1,\dots,n\}}(-1)^{|I|}.$$

Since for each $0 \leqslant r \leqslant k$ there are precisely $\binom{k}{r}$ subsets of $\{1,\dots,k\}$ containing r elements, we therefore deduce that

$$\sum_{d\,|\,n}\mu(d) = \sum_{r=0}^{k}\binom{k}{r}(-1)^r = (-1+1)^k = 0. \qquad \square$$

Problem 1.1. Show that for every $k \in \mathbb{N}$ there are infinitely many n such that

$$\mu(n+1) = \cdots = \mu(n+k).$$

Solution. Let $p_1,\dots,p_k$ be distinct primes. Then by Chinese remainder theorem there are infinitely many n with $n \equiv -j \pmod{p_j^2}$ for $1 \leqslant j \leqslant k$ and so $p_j^2|(n+j)$ and $\mu(n+j) = 0$.

Definition 1.3. The *Euler's totient function*, denoted φ, is defined to be the number of positive integers not exceeding n which are relatively prime to n, i.e.,

$$\varphi(n) = |\{1 \leqslant k \leqslant n : (k,n) = 1\}|$$

We can rewrite $\varphi(n)$ in the summation notation as

$$\varphi(n) = \sum_{\substack{k=1\\(k,n)=1}}^{n} 1 = \sum_{k=1}^{n} e((k,n))$$

Proposition 1.4. If $n \geqslant 1$, then

$$\sum_{d|n}\varphi(d) = n.$$

Proof. The key idea behind the proof is to partition the set $\{1,\dots,n\}$ into sets $A_d = \{1 \leqslant k \leqslant n : (k,n) = d\}$, where d is a divisor of n, and to note that there is a one-to-one bijection between elements of A_d and integers $1 \leqslant r \leqslant n/d$ satisfying $(r, n/d) = 1$. This then implies that

$$n = \sum_{d|n}|A_d| = \sum_{d|n}\varphi(n/d) = \sum_{d|n}\varphi(d), \qquad \square$$

where the last equality follows by the bijection $d \mapsto n/d$ between the divisors of n.

Proposition 1.5. If $n \geqslant 1$, then we have

$$\varphi(n) = \sum_{d|n}\mu(d)\frac{n}{d}. \tag{1.1}$$

Proof. We use the formula for the divisor sum of μ to obtain

$$\varphi(n) = \sum_{k=1}^{n} e((k,n)) = \sum_{k=1}^{n}\sum_{d|(n,k)}\mu(d) = \sum_{k=1}^{n}\sum_{\substack{d|n\\d|k}}\mu(d).$$

Changing the order of summation we get

$$\varphi(n) = \sum_{d|n} \sum_{\substack{k=1 \\ d|k}}^{n} \mu(d) = \sum_{d|n} \mu(d) \sum_{\substack{k=1 \\ d|k}}^{n} 1 = \sum_{d|n} \mu(d) \frac{n}{d},$$

completing the proof. □

Proposition 1.6. For $n \geqslant 1$ we have

$$\varphi(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right),$$

Proof. If $n = 1$, then the product on the right hand side is empty and so the formula trivially holds. Now let $p_1, \ldots, p_k$ be the prime divisors of n let $[k] := \{1, \ldots, k\}$. Then expanding the product, we get

$$\prod_{i=1}^{k} \left(1 - \frac{1}{p_i}\right) = \sum_{I \subset [k]} \prod_{i \in I} \left(-\frac{1}{p_i}\right) = \sum_{I \subset [k]} \frac{(-1)^{|I|}}{\prod_{i \in I} p_i} = \sum_{d|n} \frac{\mu(d)}{d} = \frac{\varphi(n)}{n}.$$ □

We now obtain some interesting properties of φ.

Proposition 1.7. The Euler's totient function has the following properties:

(a) $\varphi(p^a) = p^a - p^{a-1}$ for prime p and $a \geqslant 1$.
(b) $\varphi(mn) = \varphi(m)\varphi(n)(d/\varphi(d))$, where $d = (m, n)$.
(c) $\varphi(mn) = \varphi(m)\varphi(n)$ if $(m, n) = 1$.
(d) $n|m$ implies $\varphi(n)|\varphi(m)$.
(e) $\varphi(n)$ is even for $n \geqslant 3$. Moreover, if n has r distinct odd prime factors, then $2^r|\varphi(n)$.

Proof. (a): Follows immediately from the product formula.

(b): Note that

$$\begin{aligned}
\frac{\varphi(mn)}{mn} = \prod_{p|mn} \left(1 - \frac{1}{p}\right) &= \prod_{p|m} \left(1 - \frac{1}{p}\right) \prod_{\substack{p|n \\ p\nmid m}} \left(1 - \frac{1}{p}\right) \\
&= \prod_{p|m} \left(1 - \frac{1}{p}\right) \prod_{p|n} \left(1 - \frac{1}{p}\right) \prod_{\substack{p|n \\ p|m}} \left(1 - \frac{1}{p}\right)^{-1} \\
&= \frac{\varphi(m)}{m} \frac{\varphi(n)}{n} \prod_{p|(n,m)} \left(1 - \frac{1}{p}\right)^{-1} \\
&= \frac{\varphi(m)}{m} \frac{\varphi(n)}{n} \frac{d}{\varphi(d)},
\end{aligned}$$

where $d = (m, n)$.

(c): Follows immediately from part (b).

(d): Let $n = p_1^{a_1} \cdots p_k^{a_k}$ and $m = p_1^{b_1} \cdots p_k^{b_k}$, where a_i are nonnegative. Because $a_i \leqslant b_i$, we have $\varphi(p_i^{a_i})|\varphi(p_i^{b_i})$ due to part (i). This coupled with the fact that φ is multiplicative (due to part (c)) gives us the desired result.

(e): Observe that if $n \geqslant 3$ and $n = 2^a$ for some positive integer a then a must be at least 2 and so $\varphi(2^a) = 2^a - 2^{a-1} = 2(2^{a-1} - 2^{a-2})$ is even. Now note that

$$\varphi(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right) = \frac{n}{\prod_{p|n} p} \prod_{p|n} (p-1),$$

where the factor $n(\prod_{p|n} p)^{-1}$ is an integer. If n is not of the form 2^a, then an odd prime p divides n, and so the factor on the right must be even which implies that $\varphi(n)$ is even. Finally, if n has r distinct odd prime factors then $2^r | \prod_{p|n}(p-1)$ and hence $2^r | \varphi(n)$. $\square$

Definition 1.8. The *von-Mangoldt function* (usually referred to as simply Mangoldt function), denoted Λ, is defined as

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^a \text{ for some prime } p \text{ and integer } a \geqslant 1, \\ 0 & \text{otherwise.} \end{cases}$$

Claim 1.9. If $n \geqslant 1$, then we have

$$\log n = \sum_{d|n} \Lambda(d).$$

1.2. Dirichlet multiplication.

Definition 1.10. If f and g are two arithmetical functions we define their *Dirichlet product* (or *Dirichlet convolution*) to be the arithmetical function $f * g$ defined by

$$(f * g)(n) = \sum_{d|n} f(d) g\left(\frac{n}{d}\right).$$

Claim 1.11. Dirichlet multiplication is commutative and associative, i.e., for any arithmetical functions f, g, h we have

$$f * g = g * f \quad \text{and} \quad (f * g) * h = f * (g * h).$$

Claim 1.12. For any arithmetical function f, we have $e * f = f * e = f$.

Claim 1.13. If f is an arithmetical function with $f(1) \neq 0$, then there is a unique arithmetical function g such that

$$g * f = f * g = e.$$

The function g is given by

$$g(1) = \frac{1}{f(1)}, \qquad g(n) = -\frac{1}{f(1)} \sum_{\substack{d|n \\ d<n}} g(d) f\left(\frac{n}{d}\right) \quad \text{for } n > 1.$$

The above results show that the set of all arithmetical functions f satisfying $f(1) \neq 0$ form an abelian group under Dirichlet multiplication.

Using the notation of Dirichlet product, we can write the identities in Proposition 1.2 and Proposition 1.4 in compact form as

$$\mu * 1 = e \quad \text{and} \quad \varphi * 1 = N.$$

Thus μ and 1 are Dirichlet inverses of each other. Also note that the identity (1.1) follows seamlessly from $\varphi * 1 = N$ by multiplying by μ on both sides; $\varphi = N * \mu$.

Proposition 1.14 (Möbius inversion formula)**.** Let f and g be arithmetical functions. Then

$$f(n) = \sum_{d|n} g(d)$$

if and only if

$$g(n) = \sum_{d|n} f(d)\mu\left(\frac{n}{d}\right).$$

The Möbius inversion formula has already been illustrated by a pair of identities in Proposition 1.4 and Proposition 1.5:

$$n = \sum_{d\,|\,n} \varphi(d), \qquad \varphi(n) = \sum_{d\,|\,n} d\mu\left(\frac{n}{d}\right).$$

1.3. **Multiplicative functions.**

Definition 1.15. An arithmetical function f is called *multiplicative* if $f \not\equiv 0$ and

$$f(mn) = f(m)f(n) \quad \text{whenever } (m,n) = 1.$$

A multiplicative function f is called *completely multiplicative* (or *totally multiplicative*) if $f \not\equiv 0$ and

$$f(mn) = f(m)f(n) \quad \text{for all } m, n.$$

Example 1.16. We note some common examples of multiplicative functions.

(a) The power function N^α is completely multiplicative.
(b) The identity function E is completely multiplicative.
(c) The Möbius function μ is multiplicative. However, it is not completely multiplicative as $\mu(4) = 0 \neq 1 = \mu(2)^2$.
(d) The Euler totient function φ is multiplicative. However, it is not completely multiplicative as $\varphi(4) = 2 \neq 1 = \varphi(2)^2$.

Claim 1.17. If f is multiplicative, then $f(1) = 1$.

From this property of multiplicative functions it immediately follows that Λ is not multiplicative.

Proposition 1.18. Let f be an arithmetical function with $f(1) = 1$.

(a) f is multiplicative if and only if

$$f(p_1^{a_1} \cdots p_k^{a_k}) = f(p_1^{a_1}) \cdots f(p_k^{a_k}),$$

where $p_1, \ldots, p_k$ are distinct primes.

(b) If f is multiplicative, then f is completely multiplicative if and only if

$$f(p^a) = f(p)^a$$

for all primes p and all integers $a \geqslant 1$.

The above result shows that a multiplicative function is uniquely determined by its values on prime powers, and a completely multiplicative function is uniquely determined by its values on primes.

Claim 1.19. If f and g are multiplicative, then so is their Dirichlet product $f * g$.

Proof. Let m and n be relatively prime integers. Then observe that

$$(f * g)(mn) = \sum_{d|mn} f(d) g\left(\frac{mn}{d}\right) = \sum_{\substack{a|m \\ b|n}} f(ab) g\left(\frac{mn}{ab}\right)$$

as every divisor of mn can be uniquely written as ab, where $a|m$ and $b|n$. Using the multiplicativity of f and g we obtain

$$\begin{aligned}(f * g)(mn) &= \sum_{\substack{a|m \\ b|n}} f(a)f(b)g\left(\frac{m}{a}\right)g\left(\frac{n}{b}\right) = \sum_{a|m}\sum_{b|n} f(a)f(b)g\left(\frac{m}{a}\right)g\left(\frac{n}{b}\right) \\ &= \sum_{a|m} f(a)g\left(\frac{m}{a}\right)\sum_{b|n} f(b)g\left(\frac{n}{b}\right) = (f * g)(m)(f * g)(n).\end{aligned}$$

□

This completes the proof.

The Dirichlet product of two completely multiplicative functions need not be completely multiplicative. For instance, the divisor function $d = 1 * 1$ is not completely multiplicative as $d(4) = 3 \neq 4 = d(2)^2$ whereas 1 clearly is.

Claim 1.20. If f is multiplicative, then so is it's Dirichlet inverse f^{-1}.

Proof. Suppose for the sake of contradiction that f^{-1} is not multiplicative. Then there exist positive integers m and n with $(m, n) = 1$ such that

$$f^{-1}(mn) \neq f^{-1}(m)f^{-1}(n).$$

We choose such a pair m and n for which the product mn is the smallest. Since f is multiplicative therefore $f^{-1}(1) = 1/f(1) = 1$ and hence neither m nor n can be 1. In particular, $mn > 1$. By the construction of the product mn, $f(ab) = f(a)f(b)$ for all positive integers a and b with $(a,b)=1$ and $ab<mn$. It now follows that

$$\begin{aligned}
f^{-1}(mn) &= -\sum_{\substack{a\mid m\\ b\mid n\\ ab<mn}} f^{-1}(ab) f\left(\frac{mn}{ab}\right)\\
&= -\sum_{\substack{a\mid m\\ b\mid n\\ ab<mn}} f^{-1}(a)f^{-1}(b) f\left(\frac{m}{a}\right) f\left(\frac{n}{b}\right)\\
&= -f^{-1}(n)\sum_{\substack{a\mid m\\ a<m}} f^{-1}(a) f\left(\frac{m}{a}\right) - f^{-1}(m)\sum_{\substack{b\mid n\\ b<n}} f^{-1}(b) f\left(\frac{n}{b}\right)\\
&\quad -\sum_{\substack{a\mid m\\ a<m}}\sum_{\substack{b\mid n\\ b<n}} f^{-1}(a)f^{-1}(b) f\left(\frac{m}{a}\right) f\left(\frac{n}{b}\right)\\
&= f^{-1}(n)f^{-1}(m) + f^{-1}(m)f^{-1}(n) - f^{-1}(m)f^{-1}(n)\\
&= f^{-1}(m)f^{-1}(n).
\end{aligned}$$

This contradiction proves the result.

Second Proof. Let g be an arithmetical function defined as

$$g(n) = \prod_{p^a\| n} f^{-1}(p^a).$$

Then g is a multiplicative function by definition and so it suffices to show that $f^{-1}=g$. Note that

$$\begin{aligned}
(g*f)(p^k) &= \sum_{d\mid p^k} g(d) f(p^k/d) = \sum_{i=0}^{k} g(p^i) f(p^{k-i})\\
&= \sum_{i=0}^{k} f^{-1}(p^i) f(p^{k-i}) = \sum_{d\mid p^k} f^{-1}(d) f(p^k/d) = (f^{-1}*f)(p^k) = E(p^k).
\end{aligned}$$

Because $g*f$ and E are both multiplicative functions and agree on prime powers, it follows that $g*f=E$ and so $g=f^{-1}$. ☐

Proposition 1.21. Let f be multiplicative. Then f is completely multiplicative if and only if $f^{-1}=\mu f$.

Proof. Suppose f is completely multiplicative. Then observe that

$$(f*\mu f)(n) = \sum_{d\mid n} \mu(d) f(d) f\left(\frac{n}{d}\right) = f(n)\sum_{d\mid n}\mu(d) = f(n)E(n) = E(n).$$

Conversely, assume that $f^{-1}=\mu f$. Then observe that

$$\sum_{d\mid n} \mu(d) f(d) f\left(\frac{n}{d}\right) = 0$$

for $n > 1$. Let $n = p^a$, where $a \geqslant 1$. Then, we get

$$\mu(1)f(1)f(p^a) + \mu(p)f(p)f(p^{a-1}) = 0.$$

It then follows that

$$f(p^a) = f(p)f(p^{a-1}).$$

This implies that $f(p^a) = f(p)^a$. Thus f is completely multiplicative. □

Example 1.22. Since $\varphi = \mu * N$ we have $\varphi^{-1} = \mu^{-1} * N^{-1}$. But $N^{-1} = \mu N$ since N is completely multiplicative, so

$$\varphi^{-1} = \mu^{-1} * \mu N = 1 * \mu N.$$

Thus we have

$$\varphi^{-1}(n) = \sum_{d|n} d\mu(d).$$

Proposition 1.23. If f is a multiplicative arithmetical function then

$$\sum_{d|n} \mu(d)f(d) = \prod_{p|n}(1 - f(p)).$$

Proof. Let $g = 1 * \mu f$. Then g is a multiplicative function. Thus, it suffices to know the value of g at prime powers. We observe that

$$g(p^a) = \sum_{d|p^a} \mu(d)f(d) = \mu(1)f(1) + \mu(p)f(p) = 1 - f(p).$$

Hence, we obtain

$$g(n) = \prod_{p|n} g(p^a) = \prod_{p|n}(1 - f(p)).$$ □

as desired.

We earlier gave a product formula for $\varphi(n)$ in Proposition 1.6. This formula also follows from Proposition 1.5 and above proposition by taking $f(n) = 1/n$.

2. Estimates of arithmetical functions

2.1. **Summation by parts.** The Abel's summation by parts formula is an extremely important and ubiquitous formula in analytic number theory which is frequently employed to estimate the partial sums of an arithmetical function $a : \mathbb{N} \to \mathbb{C}$ weighted by some smooth function f.

Theorem 2.1. Let $a : \mathbb{N} \to \mathbb{C}$ be an arithmetical function, let $0 < x < y$ be real numbers and $f : [x, y] \to \mathbb{C}$ be a continuously differentiable function. Then we have

$$\sum_{x<n\leqslant y} a(n)f(n) = A(y)f(y) - A(x)f(x) - \int_x^y A(t)f'(t)\, dt,$$

where $A(t) = \sum_{n\leqslant t} a(n)$.

Proof. Let $m_1 = \lfloor x \rfloor$ and $m_2 = \lfloor y \rfloor$. We can rewrite the weighted sum as

$$\sum_{x<n\leqslant y} a(n)f(n) = \sum_{n=m_1+1}^{m_2} a(n)f(n).$$

By definition $a(n) = A(n) - A(n-1)$ so we can replace $a(n)$ to get

$$\begin{aligned}\sum_{n=m_1+1}^{m_2} a(n)f(n) &= \sum_{n=m_1+1}^{m_2} (A(n) - A(n-1))f(n) \\ &= \sum_{n=m_1+1}^{m_2} A(n)f(n) - \sum_{n=m_1}^{m_2-1} A(n)f(n+1) \\ &= A(m_2)f(m_2) - A(m_1)f(m_1+1) \\ &- \sum_{n=m_1+1}^{m_2-1} A(n)(f(n+1) - f(n)) \end{aligned} \tag{2.1}$$

Since $f(n+1) - f(n) = \int_n^{n+1} f'(t)\,dt$ and $A(t) = A(n)$ for all $t \in [n, n+1)$, we get

$$\begin{aligned}\sum_{m_1+1}^{m_2-1} A(n)(f(n+1) - f(n)) &= \sum_{m_1+1}^{m_2-1} A(n) \int_n^{n+1} f'(t)\,dt \\ &= \sum_{m_1+1}^{m_2-1} \int_n^{n+1} A(t)f'(t)\,dt \\ &= \int_{m_1+1}^{m_2} A(t)f'(t)\,dt. \end{aligned} \tag{2.2}$$

Substituting (2.1) into (2.2), we get

$$\sum_{n=m_1+1}^{m_2} a(n)f(n) = A(m_2)f(m_2) - A(m_1)f(m_1+1) - \int_{m_1+1}^{m_2} A(t)f'(t)\,dt. \tag{2.3}$$

Because we are missing the terms $\int_x^{m_1+1} A(t)f'(t)\,dt$ and $\int_{m_2}^{y} A(t)f'(t)\,dt$ parts, we try to evaluate what they look like. Using Fundamental Theorem of Calculus and the fact that $A(t) = A(x)$ for $t \in [x, m_1+1)$, we get

$$\begin{aligned}\int_x^{m_1+1} A(t)f'(t)\,dt &= A(x)f(m_1+1) - A(x)f(x) \\ &= A(m_1)f(m_1+1) - A(x)f(x). \end{aligned} \tag{2.4}$$

Doing a similar calculation for $\int_{m_2}^{y} A(t)f'(t)\,dt$ yields

$$\int_{m_2}^{y} A(t)f'(t)\,dt = A(y)f(y) - A(m_2)f(m_2). \tag{2.5}$$

Using (2.4) and (2.5), one can easily turn (2.3) into the required form. $\square$

Corollary 2.2. Let $a : \mathbb{N} \to \mathbb{C}$ be an arithmetical function and let $f : [1, x] \to \mathbb{C}$ be a continuously differentiable function where $x \geqslant 1$. Then we have

$$\sum_{n \leqslant x} a(n) f(n) = A(x) f(x) - \int_1^x A(t) f'(t)\, dt.$$

Proposition 2.3. If $x \geqslant 1$, then we have

$$\sum_{n \leqslant x} \frac{1}{n} = \log x + \gamma + O\left(\frac{1}{x}\right), \tag{2.6}$$

where γ is the Euler-Mascheroni constant.

Proof. Taking $a(n) = 1$ and $f(x) = 1/x$ in the summation by parts formula, we get

$$\sum_{n \leqslant x} \frac{1}{n} = \frac{\lfloor x \rfloor}{x} + \int_1^x \frac{\lfloor t \rfloor}{t^2}\, dt. \tag{2.7}$$

Substituting $\lfloor x \rfloor = x - \{x\}$ in (2.7), we get

$$\begin{aligned}
\sum_{n \leqslant x} \frac{1}{n} &= 1 - \frac{\{x\}}{x} + \int_1^x \frac{1}{t}\, dt - \int_1^x \frac{\{t\}}{t^2}\, dt \\
&= 1 + O\left(\frac{1}{x}\right) + \log x - \int_1^x \frac{\{t\}}{t^2}\, dt \\
&= 1 + O\left(\frac{1}{x}\right) + \log x - \int_1^\infty \frac{\{t\}}{t^2}\, dt + \int_x^\infty \frac{\{t\}}{t^2}\, dt.
\end{aligned}$$

Letting $C = 1 - \int_1^\infty \{t\} t^{-2}\, dt$, we obtain

$$\sum_{n \leqslant x} \frac{1}{n} = \log x + C + O\left(\frac{1}{x}\right) + \int_x^\infty \frac{\{t\}}{t^2}\, dt.$$

But by the comparison test for improper integrals, we have

$$\int_x^\infty \frac{\{t\}}{t^2}\, dt \leqslant \int_x^\infty \frac{1}{t^2}\, dt = \frac{1}{x}$$

and so

$$\int_x^\infty \frac{\{t\}}{t^2}\, dt = O\left(\frac{1}{x}\right).$$

It thus follows that

$$\sum_{n \leqslant x} \frac{1}{n} = \log x + C + O\left(\frac{1}{x}\right).$$

It can be easily seen by taking limit as x approaches ∞ that $C = \gamma$. $\square$

Note that in the above proof, we obtained an expression for γ in terms of integral; that is,

$$\gamma = 1 - \int_1^\infty \frac{\{t\}}{t^2}\, dt.$$

Definition 2.4. The *Riemann zeta function* is defined as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \quad \text{if } s > 1,$$

and as

$$\zeta(s) = \lim_{x\to\infty} \left(\sum_{n\leqslant x} \frac{1}{n^s} - \frac{x^{1-s}}{1-s} \right) \quad \text{if } 0 < s < 1. \tag{2.8}$$

In order for the above definition to make sense we need to prove that the limit in (2.8) indeed exists. This will be done in the course of proving the following result.

Proposition 2.5. If $x \geqslant 1$, then for $s > 0$ and $s \neq 1$ we have

$$\sum_{n\leqslant x} \frac{1}{n^s} = \frac{x^{1-s}}{1-s} + \zeta(s) + O(x^{-s}). \tag{2.9}$$

Proof. We apply the Abel summation by parts formula with $a(n) = 1$ and $f(x) = x^{-s}$. For $x \geqslant 1$, we then get

$$\sum_{n\leqslant x} \frac{1}{n^s} = \frac{\lfloor x \rfloor}{x^s} + s \int_1^x \frac{\lfloor t \rfloor}{t^{s+1}}$$

Substituting $\lfloor x \rfloor = x - \{x\}$, we obtain

$$\begin{aligned} \sum_{n\leqslant x} \frac{1}{n^s} &= x^{1-s} - \frac{\{x\}}{x^s} + s\int_1^x \frac{1}{t^s}\,dt - s\int_1^x \frac{\{t\}}{t^{s+1}}\,dt \\ &= x^{1-s} + s\left(\frac{x^{1-s}}{1-s} - \frac{1}{1-s}\right) - s\int_1^x \frac{\{t\}}{t^{s+1}}\,dt + O(x^{-s}) \\ &= \frac{x^{1-s}}{1-s} + \frac{s}{s-1} - s\int_1^\infty \frac{\{t\}}{t^{s+1}}\,dt + s\int_x^\infty \frac{\{t\}}{t^{s+1}}\,dt + O(x^{-s}). \end{aligned} \tag{2.10}$$

By the comparison test for improper integrals, we have

$$\int_x^\infty \frac{\{t\}}{t^{s+1}}\,dt \leqslant \int_x^\infty \frac{1}{t^{s+1}}\,dt = \frac{x^{-s}}{s}.$$

Thus, the equation (2.10) turns into

$$\sum_{n\leqslant x} \frac{1}{n^s} = \frac{x^{1-s}}{1-s} + \frac{s}{s-1} - s\int_1^\infty \frac{\{t\}}{t^{s+1}}\,dt + O(x^{-s}). \tag{2.11}$$

Taking the limit as x approaches ∞, we get

$$\begin{aligned} \lim_{x\to\infty}\left(\sum_{n\leqslant x} \frac{1}{n^s} - \frac{x^{1-s}}{1-s}\right) &= \lim_{x\to\infty}\left(\frac{s}{s-1} - s\int_1^\infty \frac{\{t\}}{t^{s+1}}\,dt + O(x^{-s})\right) \\ &= \frac{s}{s-1} - s\int_1^\infty \frac{\{t\}}{t^{s+1}}\,dt \end{aligned}$$

Note that if $s > 1$ then

$$\lim_{x\to\infty}\left(\sum_{n\leqslant x} \frac{1}{n^s} - \frac{x^{1-s}}{1-s}\right) = \lim_{x\to\infty} \sum_{n\leqslant x} \frac{1}{n^s} = \sum_{n=1}^{\infty} \frac{1}{n^s} = \zeta(s).$$

Consequently, for every $s > 0$ with $s \neq 1$ we have

$$\zeta(s) = \frac{s}{s-1} - s\int_1^\infty \frac{\{t\}}{t^{s+1}}\,dt. \tag{2.12}$$

Finally, using this identity along with (2.11) we get the desired result. □

Corollary 2.6. If $x \geqslant 1$ and $s > 1$, we have

$$\sum_{n>x} \frac{1}{n^s} = O(x^{1-s}).$$

Looking at the asymptotic formulae (2.6) and (2.9), one might suspect that

$$\lim_{s\to 1}\left(\zeta(s) + \frac{x^{1-s}}{1-s}\right) = \log x + \gamma.$$

Problem 2.1. Show that for every $x > 0$

$$\lim_{s\to 1}\left(\zeta(s) + \frac{x^{1-s}}{1-s}\right) = \log x + \gamma.$$

Solution. Note that

$$\begin{aligned}\zeta(s) + \frac{x^{1-s}}{1-s} &= \frac{x^{1-s}}{1-s} + \frac{s}{s-1} - s\int_1^\infty \frac{\{t\}}{t^{s+1}}\,dt \\ &= -\left(\frac{x^{1-s}-1}{s-1}\right) + 1 - s\int_1^\infty \frac{\{t\}}{t^{s+1}}\,dt.\end{aligned}$$

Since the derivative of x^{1-s} at $s = 1$ is $-\log x$ and

$$\lim_{s\to 1}\int_1^\infty \frac{\{t\}}{t^{s+1}}\,dt = \int_1^\infty \frac{\{t\}}{t^2}\,dt = 1 - \gamma$$

we obtain the desired result.

Proposition 2.7. If $x \geqslant 1$, then for any $\alpha \geqslant 0$ we have

$$\sum_{n\leqslant x} n^\alpha = \frac{x^{1+\alpha}}{1+\alpha} + O(x^\alpha). \tag{2.13}$$

Proof. We assume that $\alpha > 0$. Applying the Abel summation by parts formula with $a(n) = 1$ and $f(x) = x^\alpha$, we get

$$\sum_{n\leqslant x} n^\alpha = \lfloor x\rfloor x^\alpha - \alpha\int_1^x \lfloor t\rfloor t^{\alpha-1}\,dt.$$

Substituting $\lfloor x\rfloor = x - \{x\}$, we obtain

$$\begin{aligned}\sum_{n\leqslant x} n^\alpha &= x^{\alpha+1} - \{x\}x^\alpha - \alpha\int_1^x t^\alpha\,dt + \alpha\int_1^x \{t\}t^{\alpha-1}\,dt \\ &= x^{1+\alpha} + O(x^\alpha) - \alpha\left(\frac{x^{1+\alpha}}{1+\alpha} - \frac{1}{1+\alpha}\right) + \alpha\int_1^x \{t\}t^{\alpha-1}\,dt. \end{aligned} \tag{2.14}$$

Note that

$$\alpha\int_1^x \{t\}t^{\alpha-1}\,dt \leqslant \alpha\int_1^x t^{\alpha-1}\,dt = x^\alpha - 1 = O(x^\alpha).$$

Hence (2.14) simplifies to

$$\sum_{n\leqslant x} n^\alpha = \frac{x^{1+\alpha}}{1+\alpha} + \frac{\alpha}{1+\alpha} + O(x^\alpha)$$

Observe that $O(x^\alpha)$ absorbs the constant $\alpha/(1+\alpha)$ and thus we get the desired result.

If $\alpha = 0$ then we get

$$\sum_{n\leqslant x} n^\alpha = \sum_{n\leqslant x} 1 = \lfloor x\rfloor = x - \{x\} = x + O(1).$$

But this agrees with the asymptotic formula (2.13). $\square$

2.2. **Estimate of the average of divisor function.** We now turn our attention to the divisor function $d = 1 * 1$. We will see that on average $d(n)$ behaves like $\log n$. Since $d(n) = \sum_{d|n} 1$, we have

$$\sum_{n\leqslant x} d(n) = \sum_{n\leqslant x}\sum_{d|n} 1 = \sum_{n\leqslant x}\sum_{qd=n} 1 = \sum_{qd\leqslant x} 1.$$

Thus the divisor sum can now be written as

$$\sum_{n\leqslant x} d(n) = \sum_{d\leqslant x}\sum_{q\leqslant x/d} 1 = \sum_{d\leqslant x}\left(\frac{x}{d} + O(1)\right) = x\sum_{d\leqslant x}\frac{1}{d} + O(x).$$

Now we use the asymptotic formula for the Harmonic sum and obtain

$$\begin{aligned}\sum_{n\leqslant x} d(n) &= x\left(\log x + \gamma + O\left(\frac{1}{x}\right)\right) + O(x)\\ &= x\log x + \gamma x + O(1) + O(x)\\ &= x\log x + O(x)\end{aligned}$$

Note that the term γx does not give us any information as error is of the order x thus it gets engulfed by the big error term $O(x)$. Hence we get

$$\sum_{n\leqslant x} d(n) = x\log x + O(x).$$

Thus the average order of $d(n)$ is $\log n$ since

$$\sum_{n\leqslant x} d(n) \sim x\log x \quad \text{as } x\to\infty.$$

Dirichlet obtained sharper estimate for $\sum_{d\leqslant x} d(n)$ with the error term being $O(\sqrt{x})$. We prove the Dirichlet's result below.

Theorem 2.8. For all $x \geqslant 1$, we have

$$\sum_{n\leqslant x} d(n) = x\log x + (2\gamma - 1)x + O(\sqrt{x}), \tag{2.15}$$

where γ is the Euler-Mascheroni constant.

Proof. The trick to prove this stronger estimate is to exploit the symmetry of q and d in the sum

$$\sum_{n \leqslant x} d(n) = \sum_{qd \leqslant x} 1.$$

We can split this sum into sums as

$$\begin{aligned}
\sum_{n \leqslant x} d(n) = \sum_{qd \leqslant x} 1 &= \sum_{\substack{qd \leqslant x \\ d \leqslant q}} 1 + \sum_{\substack{qd \leqslant x \\ q \leqslant d}} 1 - \sum_{\substack{qd \leqslant x \\ d = q}} 1 \\
&= 2 \sum_{\substack{qd \leqslant x \\ d \leqslant q}} 1 - \sum_{d \leqslant \sqrt{x}} 1 \\
&= 2 \Bigg(\sum_{d \leqslant \sqrt{x}} \sum_{d \leqslant q \leqslant x/d} 1 \Bigg) - \lfloor \sqrt{x} \rfloor \\
&= 2 \sum_{d \leqslant \sqrt{x}} \left(\left\lfloor \frac{x}{d} \right\rfloor - d + 1 \right) - \lfloor \sqrt{x} \rfloor \\
&= 2 \sum_{d \leqslant \sqrt{x}} \left(\left\lfloor \frac{x}{d} \right\rfloor - d \right) + \lfloor \sqrt{x} \rfloor \\
&= 2 \sum_{d \leqslant \sqrt{x}} \left(\frac{x}{d} - d + O(1) \right) + O(\sqrt{x}) \\
&= 2x \sum_{d \leqslant \sqrt{x}} \frac{1}{d} - 2 \sum_{d \leqslant \sqrt{x}} d + O(\sqrt{x}) \\
&= 2x \Bigg(\log \sqrt{x} + \gamma + O\left(\frac{1}{\sqrt{x}} \right) \Bigg) - 2 \Bigg(\frac{x}{2} + O(\sqrt{x}) \Bigg) + O(\sqrt{x}) \\
&= x \log x + (2\gamma - 1)x + O(\sqrt{x}).
\end{aligned}$$

□

Proposition 2.9. For all $x \geqslant 1$ we have

$$\sum_{n \leqslant x} \sigma_1(n) = \frac{1}{2}\zeta(2)x^2 + O(x \log x).$$

Proof. We have

$$\begin{aligned}
\sum_{n \leqslant x} \sigma_1(n) &= \sum_{n \leqslant x} \sum_{d \mid n} d = \sum_{qd \leqslant x} d = \sum_{q \leqslant x} \sum_{d \leqslant x/q} d \\
&= \sum_{q \leqslant x} \left(\frac{1}{2} \left(\frac{x}{q} \right)^2 + O\left(\frac{x}{q} \right) \right) = \frac{x^2}{2} \sum_{q \leqslant x} \frac{1}{q^2} + O\Bigg(x \sum_{q \leqslant x} \frac{1}{q} \Bigg) \\
&= \frac{x^2}{2} \left(\zeta(2) + O\left(\frac{1}{x} \right) \right) + O(x \log x) = \frac{1}{2}\zeta(2)x^2 + O(x \log x)
\end{aligned}$$

by Proposition 2.3 and 2.5. □

2.3. **Problems.**

Problem 2.2. Let $S(x)$ denote the number of squarefree integers not exceeding x.

(a) Show that
$$\mu^2(n) = \sum_{d^2\,|\,n} \mu(d).$$
(b) Show that
$$S(x) = \frac{6}{\pi^2}x + O(\sqrt{x}).$$
(c) Show that
$$S(n) \geqslant n - \sum_p \left\lfloor \frac{n}{p^2} \right\rfloor.$$
(d) Show that
$$\sum_p \frac{1}{p^2} < \frac{1}{2}$$
and conclude that $S(n) > n/2$ for all $n \in \mathbb{N}$.

(e) Show that every integer $n > 1$ can be written as a sum of two squarefree numbers.

Solution. (a): Let $n = m^2 k$, where k is squarefree. Then observe that $d^2|n$ if and only if $d|m$. Hence, we have
$$\sum_{d^2\,|\,n} \mu(d) = \sum_{d\,|\,m} \mu(d) = E(m).$$
Note that n is squarefree if and only if $m = 1$ and so we have
$$\sum_{d^2\,|\,n} \mu(d) = \begin{cases} 1 & \text{if } n \text{ is squarefree,} \\ 0 & \text{otherwise.} \end{cases}$$
Because $\mu^2(n)$ is the characteristic function of squarefree integers, the identity follows.

(b): We use the key relation in part (i) to write $S(x)$ as
$$\begin{aligned} S(x) &= \sum_{n\leqslant x} \mu^2(n) = \sum_{n\leqslant x}\sum_{d^2\,|\,n} \mu(d) = \sum_{d\leqslant\sqrt{x}} \sum_{\substack{n\leqslant x \\ d^2\,|\,n}} \mu(d) \\ &= \sum_{d\leqslant\sqrt{x}} \mu(d) \sum_{\substack{n\leqslant x \\ d^2\,|\,n}} 1 = \sum_{d\leqslant\sqrt{x}} \mu(d) \left\lfloor \frac{x}{d^2} \right\rfloor \\ &= x \sum_{d\leqslant\sqrt{x}} \frac{\mu(d)}{d^2} + O\left(\sum_{d\leqslant\sqrt{x}} \mu(d) \right) = x \sum_{d\leqslant\sqrt{x}} \frac{\mu(d)}{d^2} + O(\sqrt{x}) \end{aligned} \tag{2.16}$$
We now push d off to ∞ in the above sum and incur an error due to the tail which fortunately is only $O(\sqrt{x})$ as seen can be seen by
$$\sum_{d\leqslant\sqrt{x}} \frac{\mu(d)}{d^2} = \sum_{d=1}^{\infty} \frac{\mu(d)}{d^2} + O\left(\sum_{d>\sqrt{x}} \frac{1}{d^2} \right) = \frac{6}{\pi^2} + O\left(\frac{1}{\sqrt{x}}\right)$$
Using this estimate in (2.16), we obtain the desired result.

(c): Observe that for an integer n we have

$$\{1 \leqslant m \leqslant n : m \text{ not squarefree}\} \subset \bigcup_p \{1 \leqslant m \leqslant n : p^2 | m\},$$

where p runs over all primes. It thus follows that

$$n - S(n) \leqslant \sum_p \left\lfloor \frac{n}{p^2} \right\rfloor,$$

which is the desired inequality.

(d): Note that

$$\sum_p \frac{1}{p^2} < \frac{1}{4} + \sum_{k=1}^{\infty} \frac{1}{(2k+1)^2} < \frac{1}{4} + \sum_{k=1}^{\infty} \frac{1}{4k(k+1)} = \frac{1}{2}.$$

It now follows by part (c) that

$$S(n) \geqslant n - \sum_p \left\lfloor \frac{n}{p^2} \right\rfloor \geqslant n - n \sum_p \frac{1}{p^2} > \frac{n}{2}.$$

for every $n \in \mathbb{N}$.

(e): Suppose for the sake of contradiction that $n > 1$ and n cannot be expressed as a sum of two squarefree integers. Then for every $a, b \in \mathbb{N}$ satisfying $a + b = n$ either a or b is not squarefree. It follows that there are at least $(n-1)/2$ (this is the number of ways n can be written as a sum of two positive integers without regard for order) integers up to $n - 1$ that are not square free. Hence, we must have

$$S(n-1) \leqslant (n-1) - \frac{(n-1)}{2} = \frac{n-1}{2}.$$

But this contradicts the inequality in part (d).

3. Dirichlet series

3.1. Basic definitions and examples.

Definition 3.1. The *Dirichlet series* of an arithmetical function $f : \mathbb{N} \to \mathbb{C}$ is defined as

$$L_f(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}.$$

Example 3.2. Note that $L_1(s) = \zeta(s)$ for $\Re(s) > 1$.

Proposition 3.3 (region of absolute convergence of $L_f(s)$)**.** Let f be an arithmetical function. Then there exists a number $\sigma_a(f) \in [-\infty, \infty]$ such that $L_f(s)$ converges absolutely in the half plane $\sigma > \sigma_a(f)$ and diverges absolutely[1] for $\sigma < \sigma_a(f)$.

Proof. Let $\sigma_a(f)$ be the infimum of all $\sigma \in \mathbb{R}$ such that $L_f(\sigma)$ converges. The conclusion then follows simply by the comparison test. □

[1]By this we mean that it does not converge absolutely.

Example 3.4. If f is an arithmetical function such that $|f(n)| \leqslant 1$ for all $n \in \mathbb{N}$, then $L_f(s)$ converges absolutely for $\sigma > 1$, i.e., $\sigma_a(f) \leqslant 1$.

The next lemma although simple to prove gives us important information about the decay of the tail of Dirichlet series.

Lemma 3.5. If f is an arithmetical function and $N \geqslant 1$ then for any fixed $c > \sigma_a(f)$ and $\sigma \geqslant c$ we have

$$\left| \sum_{n=N}^{\infty} \frac{f(n)}{n^s} \right| \leqslant N^{-(\sigma-c)} \sum_{n=N}^{\infty} \frac{|f(n)|}{n^c}.$$

Proof. The proof is very easy and follows by noting that

$$\left| \sum_{n=N}^{\infty} \frac{f(n)}{n^s} \right| \leqslant \sum_{n=N}^{\infty} \frac{|f(n)|}{n^\sigma} = \sum_{n=N}^{\infty} \frac{|f(n)|}{n^{\sigma-c} n^c} \leqslant N^{-(\sigma-c)} \sum_{n=N}^{\infty} \frac{|f(n)|}{n^c}.$$ □

An interesting consequence of the above lemma is the following limit

$$\lim_{\sigma \to \infty} L_f(\sigma + it) = f(1).$$

This shows that value of an arithmetical function f at 1 is uniquely determined by its Dirichlet series $L_f(s)$. In fact, using the above bound one can shows that an arithmetical function is completely and uniquely determined by its Dirichlet series. For instance,

$$f(N) = \lim_{\sigma \to \infty} N^s \left(L_f(s) - \sum_{n=1}^{N-1} \frac{f(n)}{n^s} \right)$$

for $N > 1$.

Theorem 3.6. Let f and g be arithmetical functions such that Dirichlet series $L_f(s)$ and $L_g(s)$ both converge absolutely in the half plane $\sigma > \sigma_0$. Let $\{s_k\}_{k=1}^{\infty}$ be a sequence such that $\sigma_k = \Re(s_k) > \sigma_0$ for all k and $\sigma_k \to \infty$ as $k \to \infty$. If $L_f(s_k) = L_g(s_k)$ for all k then $f = g$.

Proof. Let $h(n) = f(n) - g(n)$ for all n. Then note that $L_h(s)$ converges absolutely in the half plane $\sigma > \sigma_0$ we have $L_h(s_k) = 0$ for all k. Our aim is to show that $h(n) = 0$ for all n. Suppose for the sake of contradiction that there exists an $n \in \mathbb{N}$ such that $h(n) \neq 0$. Then let N be the smallest such integer. Now note that if $L_h(s) = 0$, then

$$h(N) = -N^s \sum_{n=N+1}^{\infty} \frac{h(n)}{n^s}.$$

Due to Lemma 3.5, it follows that

$$|h(N)| = \left| N^s \sum_{n=N+1}^{\infty} \frac{h(n)}{n^s} \right| \leqslant N^\sigma (N+1)^{-(\sigma-c)} \sum_{n=N+1}^{\infty} \frac{|h(n)|}{n^c} = A(c) \left(\frac{N}{N+1} \right)^\sigma,$$

where $c > \sigma_0$ is fixed and $A(c)$ is a constant depending on c only. Substituting $\sigma = \sigma_k$ and taking the limit as $k \to \infty$ we get $h(N) = 0$, which is a contradiction. □

Note that it follows by the above theorem that for a Dirichlet series $L_f(s) \not\equiv 0$, there exists a half plane $\sigma > c$ in which the series never vanishes.

Theorem 3.7. Let f and g be arithmetical functions such that Dirichlet series $L_f(s)$ and $L_g(s)$ both converge absolutely in the half plane $\sigma > \sigma_0$. Then for any $\sigma > \sigma_0$ we have

$$L_f(s)L_g(s) = L_{f*g}(s).$$

Proof. Let s be such that $\sigma > \sigma_0$. Then we have

$$L_f(s)L_g(s) = \left(\sum_{n=1}^{\infty} \frac{f(n)}{n^s}\right)\left(\sum_{m=1}^{\infty} \frac{g(m)}{m^s}\right) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{f(n)g(m)}{(nm)^s}.$$

Since the double sum converges absolutely, we can make this double sum into a single sum and rearrange the terms as we wish without changing the value (See Terence Tao *Analysis I*, pg. 188, 189). Thus it follows that

$$L_f(s)L_g(s) = \sum_{k=1}^{\infty}\sum_{nm=k} \frac{f(n)g(m)}{k^s} = \sum_{k=1}^{\infty} \frac{(f*g)(k)}{k^s} = L_{f*g}(s),$$

which is the desired result. □

Example 3.8. If f is an arithmetical function with $f(1) \neq 0$ and such that both $L_f(s)$ and $L_{f^{-1}}(s)$ converge absolutely in some half plane $\sigma > \sigma_0$, then $L_f(s)L_{f^{-1}}(s) = 1$ in the half plane $\sigma > \sigma_0$ and so in particular we have $L_f(s) \neq 0$ for $\sigma > \sigma_0$.

Example 3.9. If f is a completely multiplicative arithmetical function then $f^{-1} = \mu f$ and so the Dirichlet series $L_{f^{-1}}(s)$ converges in the half plane of absolute convergence of $L_f(s)$. It thus follows that

$$L_{\mu f}(s) = \sum_{n=1}^{\infty} \frac{\mu(n)f(n)}{n^s} = \frac{1}{L_f(s)}.$$

for $\sigma > \sigma_a(f)$. Hence, the Dirichlet series of a completely multiplicative function does not vanish in its half plane of absolute convergence.

In particular we have $\zeta(s)L_\mu(s) = L_1(s)L_\mu(s) = 1$ for $\sigma > 1$ and so $\zeta(s)$ does not vanish in the half plane $\sigma > 1$;

$$L_\mu(s) = \frac{1}{\zeta(s)}$$

Claim 3.10. The Dirichlet series $L_f(s)$ of an arithmetical function f is analytic in the half plane $\sigma > \sigma_a(f)$ and

$$L'_f(s) = -\sum_{n=1}^{\infty} \frac{f(n)\log n}{n^s}.$$

Proof. Note that the Dirichlet series $L_f(s)$ converges uniformly in the shifted half plane $\sigma > \sigma_a(f) + \delta$ for any $\delta > 0$. Thus $L_f(s)$ is analytic in the half plane $\sigma > \sigma_a(f) + \delta$ for any $\delta > 0$ and satisfies

$$L'_f(s) = -\sum_{n=1}^{\infty} \frac{f(n)\log n}{n^s}$$

(see Theorem 5.2 and 5.3 in [1]). Since δ is arbitrary, the above formula holds in the half plane $\sigma > \sigma_a(f)$. □

Example 3.11. We now record some examples of Dirichlet series of familiar arithmetical functions.

(a) We have
$$\sum_{n=1}^{\infty} \frac{e(n)}{n^s} = 1$$
for every $s \in \mathbb{C}$ and so $\sigma_a(e) = -\infty$.

(b) Since we have $\mu * 1 = E$, it follows that
$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \frac{1}{\zeta(s)}$$
for $\sigma > 1$.

(c) Simply differentiating the Dirichlet series of $\zeta(s)$, we get
$$\zeta'(s) = -\sum_{n=1}^{\infty} \frac{\log n}{n^s}.$$
for $\sigma > 1$.

(d) Since we have $\Lambda = \log * \mu$, it follows that
$$\frac{\zeta'(s)}{\zeta(s)} = -\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}.$$
for $\sigma > 1$.

(e) Since we have $d = 1 * 1$, it follows that
$$\sum_{n=1}^{\infty} \frac{d(n)}{n^s} = \zeta(s)^2$$
for $\sigma > 1$.

(f) Since $\lambda^{-1} = \mu\lambda = \mu^2$ and $s = 1 * \lambda$ is the characteristic function of the squares, it follows that $\lambda * \mu^2 = E$ and so $s * \mu^2 = (1 * \lambda) * \mu^2 = 1 * (\lambda * \mu^2) = 1$. Thus we have
$$\sum_{n=1}^{\infty} \frac{\mu(n)^2}{n^s} = \frac{\zeta(s)}{\zeta(2s)}$$
for $\sigma > 1$.

(g) Due to the identity $\varphi = \mu * N$, it follows that
$$\sum_{n=1}^{\infty} \frac{\varphi(n)}{n^s} = \frac{\zeta(s-1)}{\zeta(s)}$$
for $\sigma > 2$.

Proposition 3.12 (Kronecker's lemma)**.** Let f be an arithmetical function and let s be a complex number such that $\sigma > 0$ and the Dirichlet series $\sum_{n=1}^{\infty} f(n)n^{-s}$ converges. Then
$$\lim_{n\to\infty} \frac{1}{x^s} \sum_{n \leqslant x} f(n) = 0.$$

My Proof. Let $0 < \delta < 1$ be sufficiently small so that the series $\sum_{n=1}^{\infty} |f(n)| n^{-\sigma/2\delta}$ converges. We now split the sum as

$$\frac{1}{x^s} \sum_{n \leqslant x} f(n) = \sum_{n \leqslant x^\delta} \frac{f(n)}{x^s} + \sum_{x^\delta < n \leqslant x} \frac{f(n)}{x^s}.$$

We now show that both the sums on the right hand side go to 0 as $x \to \infty$. For the first sum note that

$$\left| \sum_{n \leqslant x^\delta} \frac{f(n)}{x^s} \right| \leqslant \sum_{n \leqslant x^\delta} \frac{|f(n)|}{x^\sigma} = \sum_{n \leqslant x^\delta} \frac{|f(n)|}{n^{\sigma/2\delta}} \frac{1}{(x/n^{1/2\delta})^\sigma}.$$

If $n \leqslant x^\delta$, then

$$\left(\frac{x}{n^{1/2\delta}} \right)^\sigma = \frac{x^\sigma}{n^{\sigma/2\delta}} \geqslant \frac{x^\sigma}{x^{\sigma/2}} = x^{\sigma/2}.$$

Thus we have

$$\left| \sum_{n \leqslant x^\delta} \frac{f(n)}{x^s} \right| \leqslant \frac{1}{x^{\sigma/2}} \sum_{n \leqslant x^\delta} \frac{|f(n)|}{n^{\sigma/2\delta}}.$$

Because the series $\sum_{n=1}^{\infty} |f(n)| n^{-\sigma/2\delta}$ converges and $\sigma > 0$, it follows that

$$\lim_{x \to \infty} \sum_{n \leqslant x^\delta} \frac{f(n)}{x^s} = 0.$$

For the other sum we will apply the Abel's summation formula as

$$\begin{aligned} \sum_{x^\delta < n \leqslant x} \frac{f(n)}{x^s} &= \sum_{x^\delta < n \leqslant x} \frac{f(n)}{n^s} \frac{1}{(x/n)^s} \\ &= \sum_{n \leqslant x} \frac{f(n)}{n^s} - \frac{1}{x^{s(1-\delta)}} \sum_{n \leqslant x^\delta} \frac{f(n)}{n^s} - \frac{s}{x^s} \int_{x^\delta}^{x} \left(\sum_{n \leqslant t} \frac{f(n)}{n^s} \right) t^{s-1}\, dt. \qquad (3.1) \end{aligned}$$

We split the integral as

$$\begin{aligned} \int_{x^\delta}^{x} \left(\sum_{n \leqslant t} \frac{f(n)}{n^s} \right) t^{s-1}\, dt &= \int_{x^\delta}^{x} \left(\sum_{n \leqslant x^\delta} \frac{f(n)}{n^s} \right) t^{s-1}\, dt + \int_{x^\delta}^{x} \left(\sum_{x^\delta < n \leqslant t} \frac{f(n)}{n^s} \right) t^{s-1}\, dt \\ &= \left(\frac{x^s}{s} - \frac{x^{\delta s}}{s} \right) \sum_{n \leqslant x^\delta} \frac{f(n)}{n^s} + \int_{x^\delta}^{x} \left(\sum_{x^\delta < n \leqslant t} \frac{f(n)}{n^s} \right) t^{s-1}\, dt. \end{aligned}$$

Substituting this into (3.1) we obtain

$$\begin{aligned}\sum_{x^\delta<n\leqslant x}\frac{f(n)}{x^s} &= \sum_{n\leqslant x}\frac{f(n)}{n^s} - \frac{1}{x^{s(1-\delta)}}\sum_{n\leqslant x^\delta}\frac{f(n)}{n^s} - \sum_{n\leqslant x^\delta}\frac{f(n)}{n^s}\\ &\quad + \frac{1}{x^{s(1-\delta)}}\sum_{n\leqslant x^\delta}\frac{f(n)}{n^s} - \frac{s}{x^\delta}\int_{x^\delta}^{x}\left(\sum_{x^\delta<n\leqslant t}\frac{f(n)}{n^s}\right)t^{s-1}\,dt\\ &= \sum_{x^\delta<n\leqslant x}\frac{f(n)}{n^s} - \frac{s}{x^\delta}\int_{x^\delta}^{x}\left(\sum_{x^\delta<n\leqslant t}\frac{f(n)}{n^s}\right)t^{s-1}\,dt\end{aligned}$$

Note that $\sum_{x^\delta<n\leqslant t} f(n)n^{-s} = o(x)$ irrespective of t and so we get

$$\sum_{x^\delta<n\leqslant x}\frac{f(n)}{x^s} = o(1) + \frac{s}{x^\delta}\int_{x^\delta}^{x} o(1)t^{s-1}\,dt.$$

Hence we have

$$\begin{aligned}\left|\sum_{x^\delta<n\leqslant x}\frac{f(n)}{x^s}\right| &\leqslant o(1) + \frac{o(1)}{x^\delta}\int_{x^\delta}^{x} t^{\delta-1}\,dt\\ &= o(1) + \frac{o(1)}{\sigma x^\delta}(x^\sigma - x^{\sigma\delta}) = o(1)\end{aligned}$$

This completes the proof. □

3.2. **Euler's product formula.** We now present a very important formula which represents a Dirichlet series as a product over primes.

Theorem 3.13 (Euler's product formula)**.** Let f be a multiplicative arithmetical function such that the series $\sum_{n=1}^{\infty} f(n)$ is absolutely convergent. Then the series can be expressed as

$$\sum_{n=1}^{\infty} f(n) = \prod_p (1 + f(p) + f(p^2) + \cdots)$$

and the above product converges absolutely. Moreover, the series converges absolutely if and only if

$$\sum_{p^m} |f(p^m)| < \infty. \tag{3.2}$$

Proof. The absolute convergence of the product follows trivially. Now consider the finite product

$$P(x) = \prod_{p\leqslant x}(1 + f(p) + f(p^s) + \cdots)$$

extended over all primes $\leqslant x$. Since this is a product of a finite number of absolutely convergent series we can multiply the series and rearrange the terms in any fashion without altering the sum. A typical term is of the form

$$f(p_1^{a_1})\cdots f(p_r^{a_r}) = f(p_1^{a_1}\cdots p_r^{a_r})$$

since f is multiplicative. By the Fundamental Theorem of Arithmetic we can write

$$P(x) = \sum_{n \in A(x)} f(n)$$

where $A(x)$ consists of integers having their primes factors $\leqslant x$. Therefore

$$\sum_{n=1}^{\infty} f(n) - P(x) = \sum_{n \in B(x)} f(n),$$

where $B(x)$ is the set of integers having at least one prime factor $> x$. Therefore we have

$$\left|\sum_{n=1}^{\infty} f(n) - P(x)\right| \leqslant \sum_{n \in B(x)} |f(n)| \leqslant \sum_{n > x} |f(n)|.$$

Note that the sum on the right goes to 0 as $x \to \infty$ since the series $\sum_{n=1}^{\infty} |f(n)|$ is convergent. Thus the product formula follows.

Note that the convergence of series in (3.2) follows immediately from the convergence of the series $\sum_{n=1}^{\infty} |f(n)|$. To prove the converse, observe that the product

$$P = \prod_{p} \left(1 + \sum_{m=1}^{\infty} |f(p^m)|\right)$$

converges and we have

$$P \geqslant \prod_{p \leqslant x} \left(1 + \sum_{m=1}^{\infty} |f(p^m)|\right) = \sum_{n \in A(x)} |f(n)| \geqslant \sum_{n \leqslant x} |f(n)|.$$

Thus we have $\sum_{n=1}^{\infty} |f(n)| < \infty$ as desired. □

Corollary 3.14. Let f be a completely multiplicative arithmetical function such that the series $\sum_{n=1}^{\infty} f(n)$ is absolutely convergent. Then

$$\sum_{n=1}^{\infty} f(n) = \prod_{p} (1 - f(p))^{-1}.$$

Remark 3.15. Note that nowhere in the proof of above theorem did we make use of the fact that the primes are infinite. We did apply the fundamental theorem of arithmetic but it does not assume the infinitude of primes.

Example 3.16. The Euler's product for $\zeta(s) = L_1(s)$ is given as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p} \left(1 - \frac{1}{p^s}\right)^{-1}.$$

We will now use this formula to give yet another proof of the infinitude of primes. We restrict s to be a real number $\sigma > 1$. Now note that $\zeta(\sigma)$ increases as $\sigma \to 1^+$. Hence for any $N \in \mathbb{N}$ it follows that

$$\lim_{\sigma \to 1^+} \zeta(\sigma) \geqslant \lim_{\sigma \to 1^+} \sum_{n=1}^{N} \frac{1}{n^\sigma} = \sum_{n=1}^{N} \frac{1}{n}.$$

Thus we have

$$\lim_{\sigma\to 1^+} \zeta(\sigma) = \infty.$$

This shows that primes cannot be finite lest we would have

$$\lim_{\sigma\to 1^+} \zeta(\sigma) = \lim_{\sigma\to 1^+} \prod_p \left(1-\frac{1}{p^\sigma}\right)^{-1} = \prod_p \left(1-\frac{1}{p}\right)^{-1} < \infty.$$

Working a little harder we can glean more information than just the infinitude of primes.

Observe that for $\sigma > 1$ we have

$$\log\zeta(\sigma) = -\sum_p \log(1-p^{-\sigma}).$$

Since $p^{-\sigma} \leqslant \frac{1}{2}$, we can use the series expression of log and rewrite the above formula as

$$\log\zeta(\sigma) = \sum_p \sum_{m=1}^{\infty} \frac{1}{mp^{\sigma m}} = \sum_p \frac{1}{p^\sigma} + \sum_p \sum_{m=2}^{\infty} \frac{1}{mp^{\sigma m}}.$$

Note that

$$\sum_p \sum_{m=2}^{\infty} \frac{1}{mp^{\sigma m}} \leqslant \sum_p \sum_{m=2}^{\infty} \frac{1}{p^m} = \sum_p \frac{1}{p(p-1)} < \infty.$$

Hence we have

$$\log\zeta(\sigma) = \sum_p \frac{1}{p^\sigma} + O(1) \leqslant \sum_p \frac{1}{p} + O(1).$$

Since $\zeta(\sigma) \to \infty$ as $\sigma \to 1^+$, it follows that $\sum \frac{1}{p} = \infty$. This result shows that the primes in some sense are not very sparse unlike the sequence of perfect squares.

3.3. **Analytic properties of Dirichlet series.**

Theorem 3.17. Let f be an arithmetical function. Suppose that the the Dirichlet series $L_f(s)$ converges at the point $s = s_0 = \sigma_0 + it_0$. Then for every constant $H > 0$ the Dirichlet series $F(s)$ converges uniformly in the sector $S = \{s \in \mathbb{C} : \sigma \geqslant \sigma_0, |t-t_0| \leqslant H(\sigma-\sigma_0)\}$.

Proof. The proof is quite easy when the convergence at s_0 is absolute, so we first prove this case. Note that $|f(n)n^{-s}| = |f(n)|n^{-\sigma} \leqslant |f(n)|n^{-\sigma_0}$ whenever $\sigma \geqslant \sigma_0$. It thus follows by the Weierstrass' theorem that the Dirichlet series $L_f(s)$ converges absolutely and uniformly for $\sigma \geqslant \sigma_0$.

We now treat the general case which is more subtle. Let $1 \leqslant x < y$. Applying the Abel's summation formula we obtain

$$\sum_{x<n\leqslant y} \frac{f(n)}{n^s} = \frac{1}{y^{s-s_0}} \sum_{x<n\leqslant y} f(n)n^{-s_0} + (s-s_0)\int_x^y \left(\sum_{x<n\leqslant t} f(n)n^{-s_0}\right)\frac{dt}{t^{s-s_0+1}} \tag{3.3}$$

Take $\epsilon > 0$ be fixed and take x to be sufficiently large so that $\left|\sum_{x<n\leqslant y} f(n)n^{-s_0}\right| < \epsilon$ for every $y > x$. Now assume that $s \in S$ and $s \neq s_0$. Then we have $\sigma > \sigma_0$. Applying the triangle inequality to (3.3), we get

$$\left|\sum_{x<n\leqslant y}\frac{f(n)}{n^s}\right| \leqslant \epsilon+\epsilon\frac{|s-s_0|}{\sigma-\sigma_0} \leqslant \epsilon+\epsilon\left(\frac{(\sigma-\sigma_0)+|t-t_0|}{\sigma-\sigma_0}\right) = (2+H)\epsilon.$$

Note that the inequality also holds if $s = s_0$. Hence the Dirichlet series $F(s)$ converges uniformly in the sector S. □

Corollary 3.18. Let f be an arithmetical function. If the Dirichlet series $L_f(s)$ converges at the point $s = s_0$. Then $L_f(s)$ converges uniformly in every compact subset of the half plane $\sigma > \sigma_0$. Moreover, $L_f(s)$ defines a holomorphic function in the half plane $\sigma > \sigma_0$.

References

[1] Elias M Stein and Rami Shakarchi. *Complex analysis*, volume 2. Princeton University Press, 2010.