# NORMED SPACES

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## 1. Basic definitions and examples

A normed space is simply a vector space with the notion of length just as in Euclidean spaces. Precisely speaking, a *norm* on a vector space $X$ is a map $||\cdot|| : X \times X \to \mathbb{R}_{\geq 0}$ that satisfies the following properties:

(1) (positive definiteness) $||x|| = 0$ if and only if $x = 0$.
(2) (absolute homogeneity) $||\lambda x|| = |\lambda| \, ||x||$ for every $x \in X$ and $\lambda \in \mathbb{K}$.
(3) (triangle inequality) $||x + y|| \leq ||x|| + ||y||$ for every $x, y \in X$.

In this case the pair $(X, ||\cdot||)$ is called a *normed space*.

A $||\cdot||$ on a vector space $X$ induces a metric on $X$ defined as

$$d(x, y) = ||x - y|| \, .$$

This metric satisfies the additional properties of translation invariance ($d(x + z, y + z) = d(x, y)$) and absolute homogeneity ($d(\lambda x, \lambda y) = |\lambda| d(x, y)$). Due to this not every metric on a vector space is induced by a norm. For instance, the discrete metric does not satisfy the absolute homogeneity property and hence is not induced by a norm.

As in the case of Euclidean space open/closed balls are convex in a normed space. In some sense this is the characteristic property of a norm. For instance, one can show that if $||\cdot|| : X \times X \to \mathbb{R}_{\geq 0}$ is a map which satisfies positive definiteness and absolute homogeneity, then $||\cdot||$ is a norm if and only if the closed unit ball $\overline{B_1}$ is convex.

**Exercise 1.1.** Let $X$ be a vector space over $\mathbb{K}$ and let $||\cdot|| : X \to \mathbb{R}_{\geq 0}$ be a map that satisfies properties (1) and (2) of norm. Show that $||\cdot||$ is a norm if and only if the closed unit ball is convex. (Hint: The triangle inequality is equivalent to the inequality $||\lambda x + (1 - \lambda) y|| \leq 1$, where $x, y$ are unit vectors and $0 \leq \lambda \leq 1$.)

We now present some examples of normed spaces.

**Example 1.2.** Let $1 \leq p < \infty$ and let $n \in \mathbb{N}$. Then for $x \in \mathbb{K}^n$ we define the *p-norm* of $x$ as

$$||x||_p = \left( \sum_{k=1}^{n} |x_k|^p \right)^{1/p} . \tag{1.1}$$

It is easy to see that $||\cdot||_p$ satisfies positive definiteness and absolute homogeneity. However, the triangle inequality is not obvious except for the case $p = 1$. We have to show that for any $x, y \in \mathbb{K}^n$ we have the inequality

$$||x + y||_p \leq ||x||_p + ||y||_p \,.$$

This is equivalent to the inequality

$$||\lambda x + (1 - \lambda)y||_p \leq 1,$$

where $||x||_p = ||y||_p = 1$ and $0 \leq \lambda \leq 1$. Because the function $t \mapsto t^p$ is convex (as $\dfrac{d^2 t^p}{dt^2} = p(p-1)t^{p-2} \geq 0$) for $t \in [0, \infty)$, we have the convexity bound

$$(\lambda a + (1 - \lambda)b)^p \leq \lambda a^p + (1 - \lambda)b^p$$

for $a, b \geq 0$ and $0 \leq \lambda \leq 1$. Applying this we obtain

$$\begin{aligned} ||\lambda x + (1 - \lambda)y||_p^p &= \sum_{k=1}^{n} |\lambda x_k + (1 - \lambda)y_k|^p \\ &\leq \sum_{k=1}^{n} (\lambda |x_k| + (1 - \lambda)|y_k|)^p \\ &\leq \lambda \sum_{k=1}^{n} |x_k|^p + (1 - \lambda) \sum_{k=1}^{n} |y_k|^p \\ &= \lambda + (1 - \lambda) = 1. \end{aligned}$$

This shows that $||\cdot||_p$ satisfies the triangle inequality and hence is a norm.

**Example 1.3.** The $\infty$-*norm* on $\mathbb{K}^n$ is defined as

$$||x||_\infty = \max_{1 \leq k \leq \infty} |x_k|.$$

It is easy to verify that $||\cdot||_\infty$ is indeed a norm on $\mathbb{K}^n$. The triangle inequality is particularly easy to verify. It is easy to see that

$$||x||_\infty \leq ||x||_p \leq n^{1/p} \, ||x||_\infty \,.$$

It follows from this that

$$||x||_\infty = \lim_{p \to \infty} ||x||_p \,.$$

**Example 1.4.** Let $1 \leq p < \infty$. We define the $\ell^p$ space as

$$\ell^p = \left\{ \{a_n\}_{n \in \mathbb{N}} \subset \mathbb{K} : \sum_{n=1}^{\infty} |a_n|^p < \infty \right\},$$

i.e., $\ell^p$ is the collection of all $p$-summable sequences. It can be easily seen that $\ell^p$ is a subspace of $\mathbb{K}^{\mathbb{N}}$. To see why it is closed under addition note that

$$(a + b)^p \leq (2 \max\{a, b\})^p \leq 2^p(a^p + b^p)$$

for any nonnegative $a, b$. We define the *p-norm* on $\ell^p$ as

$$||a||_p = \left( \sum_{n=1}^{\infty} |a_n|^p \right)^{1/p},$$

where $a = \{a_n\}_{n\in\mathbb{N}}$. It is easy to see that $||\cdot||_p$ satisfies positive definiteness and absolute homogeneity. Let $a = \{a_n\}_{n\in\mathbb{N}} \in \ell^p$, $b = \{b_n\}_{n\in\mathbb{N}} \in \ell^p$. Then for a fixed $N \in \mathbb{N}$ we have

$$\left( \sum_{n=1}^{N} |a_n + b_n|^p \right)^{1/p} \leq \left( \sum_{n=1}^{N} |a_n|^p \right)^{1/p} + \left( \sum_{n=1}^{N} |b_n|^p \right)^{1/p} \leq ||a||_p + ||b||_p$$

due to the triangle inequality for $\mathbb{K}^N$. Now taking the limit as $N \to \infty$ we get

$$||a+b||_p = \lim_{N\to\infty} \left( \sum_{n=1}^{N} |a_n + b_n|^p \right)^{1/p} \leq ||a||_p + ||b||_p .$$

Thus $||\cdot||_p$ is a norm on $\ell^p$.

**Example 1.5.** Let $\ell^\infty$ denote the space of all bounded sequences, i.e.,

$$\ell^\infty = \left\{ \{a_n\}_{n\in\mathbb{N}} \subset \mathbb{K} : \sup_{k\in\mathbb{N}} |a_k| < \infty \right\}$$

Then we define the $\infty$*-norm* on $\ell^\infty$ as

$$||a||_\infty = \sup_{k\in\mathbb{N}} |a_k|,$$

where $a = \{a_n\}_{n\in\mathbb{N}} \in \ell^\infty$. It is easy to verify that $||\cdot||_\infty$ is indeed a norm on $\ell^\infty$.

**Example 1.6.** Let $X$ be a measure space with measure $\mu$ and let $1 \leq p < \infty$. The space $L^p(X)$ of all equivalence classes of $p$-integrable functions ($f$ is equivalent to $g$ if $f = g$ a.e.) is a normed space under the norm

$$||f||_p = \left( \int_X |f|^p \, d\mu \right)^{1/p}$$

Moreover, for a measurable function $f : X \to \overline{\mathbb{R}}$ (or $\mathbb{C}$) we define $||f||_\infty$ to be the essential supremum of $|f|$, i.e.,

$$||f||_\infty = \inf\{\alpha \in [0, \infty] : |f| \leq \alpha \text{ a.e.}\}$$

The space $L^\infty(X)$ is defined to be the collection of equivalence classes of essentially bounded functions, i.e., $||f||_\infty < \infty$. It can be easily seen that $L^\infty(X)$ is a normed space.

In fact, the $p$-norm on $\mathbb{K}^n$ and $\ell^p$ are particular instances of this example with the measure $\mu$ being the counting measure.

Two norms $||\cdot||_1$ and $||\cdot||_2$ on a vector space $X$ are said to be *equivalent* if there are positive constants $A$ and $B$ such that

$$A\,||x||_2 \leq ||x||_1 \leq B\,||x||_2\,.$$

It can be easily verified that the equivalence of norms in indeed an equivalence relation. Moreover, if $||\cdot||_1$ and $||\cdot||_2$ are equivalent norms on $X$ and $\{x_n\}_{n\in\mathbb{N}}$ is a sequence in $X$, then $\{x_n\}_{n\in\mathbb{N}}$ is Cauchy with respect to $||\cdot||_1$ if and only if $\{x_n\}_{n\in\mathbb{N}}$ is Cauchy with respect to $||\cdot||_2$. Similarly, $\{x_n\}_{n\in\mathbb{N}}$ converges to $x$ with respect to the norm $||\cdot||_1$ if and only if $\{x_n\}_{n\in\mathbb{N}}$ converges to $x$ with respect to the norm $||\cdot||_2$.

All $p$-norms on $\mathbb{K}^n$ are equivalent as

$$||x||_\infty \leq ||x||_p \leq n^{1/p}\,||x||_\infty$$

for every $x \in \mathbb{K}^n$ and $1 \leq p < \infty$.

## 2. Bounded operators

A linear operator $\Lambda : X \to Y$ between two normed spaces is said to be *bounded* if there is a $C > 0$ such that

$$||\Lambda x|| \leq C\,||x||$$

for every $x \in X$.

The *norm* of a linear operator $\Lambda : X \to Y$ between two normed spaces is defined as

$$||\Lambda|| = \sup_{\substack{x\in X\\ ||x||=1}} ||\Lambda x||\,.$$

It can be easily seen that a linear operator $\Lambda : X \to Y$ is bounded if and only if $||\Lambda|| < \infty$.

**Exercise 2.1.** Show that if $X, Y$ are normed spaces, then a linear operator $\Lambda : X \to Y$ is bounded if and only if it maps bounded sets to bounded sets. Also show that composition of bounded linear operators is also bounded.

**Proposition 2.2.** *Let $\Lambda : X \to Y$ be a linear operator. Then $\Lambda$ is bounded if and only if it is continuous.*

*Proof.* Note that if a linear operator $\Lambda : X \to Y$ between normed spaces is bounded, then it Lipschitz continuous due to linearity and hence is continuous in particular.

Now suppose that $\Lambda : X \to Y$ is a continuous linear operator. Then there is a $\delta > 0$ such that $||\Lambda(x)||_Y \leq 1$ whenever $||x||_X \leq \delta$. Now if $||x||_X = 1$, then $||\delta x||_X = \delta$ and so we have

$$||\Lambda(\delta x)||_Y \leq 1.$$

This implies that

$$||\Lambda(x)||_Y \leq \frac{1}{\delta}$$

for every $x \in X$ with $||x||_X = 1$. Hence, we have $||\Lambda|| < \infty$, i.e., $\Lambda$ is bounded. $\square$

**Theorem 2.3.** *Let $X$ be a finite-dimensional normed space over the field $\mathbb{K}$ of real or complex numbers. If $\{v_1, \ldots, v_N\}$ is a basis of $X$, then the linear operator $\Lambda : \mathbb{K}^N \to X$ defined as*

$$\Lambda(\alpha) = \alpha_1 v_1 + \cdots + \alpha_N v_N$$

*is bounded. Moreover, $\Lambda$ is a bijection and $\Lambda^{-1}$ is also bounded,*

*Proof.* First observe that

$$||\Lambda(\alpha)||_X = \left|\left|\sum_{i=1}^{n} \alpha_i v_i\right|\right|_X \leq \sum_{i=1}^{n} |\alpha_i| \, ||v_i||_X \leq ||\alpha||_1 \sum_{i=1}^{n} ||v_i||_X \, .$$

This shows that $\Lambda$ is bounded and hence continuous.

The boundedness of $\Lambda^{-1}$ requires a little effort. We suppose for the sake of contradiction that $\Lambda^{-1}$ is not bounded. Then there is a sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$ such that $||x_n||_X = 1$ for every $n \in \mathbb{N}$ and

$$\lim_{n\to\infty} \left|\left|\Lambda^{-1}(x_n)\right|\right|_1 = \infty.$$

Now let

$$\tilde{x}_n = \frac{x_n}{||\Lambda^{-1}(x_n)||_1} \quad \text{and} \quad \beta_n = \frac{\Lambda^{-1}(x_n)}{||\Lambda^{-1}(x_n)||_1}$$

for $n \in \mathbb{N}$. Then we have $\Lambda(\beta_n) = \tilde{x}_n$. Moreover, we have

$$\lim_{n\to\infty} ||\tilde{x}_n||_X = \lim_{n\to\infty} \frac{1}{||\Lambda^{-1}(x_n)||_1} = 0. \tag{2.1}$$

Since $\{\beta_n\}_{n\in\mathbb{N}}$ is a bounded sequence (as $||\beta_n||_1 = 1$), it has a convergent subsequence $\{\beta_{n_k}\}_{k\in\mathbb{N}}$ (as every component sequence is bounded and $\mathbb{K}$ has the Heini-Borel property). Let $\beta = \lim_{k\to\infty} \beta_{n_k}$. Then we have $||\beta||_1 = 1$ and in particular $\beta \neq 0$. As $\Lambda$ is continuous, we have

$$\lim_{k\to\infty} \Lambda(\beta_{n_k}) = \Lambda(\beta).$$

But we also have

$$\lim_{k\to\infty} \Lambda(\beta_{n_k}) = \lim_{k\to\infty} \tilde{x}_{n_k} = 0$$

due to (2.1). This shows that $\Lambda(\beta) = 0$ and so $\beta = 0$, a contradiction. $\square$

**Corollary 2.4.** *In a finite-dimensional space, all norms are equivalent.*

*Proof.* Let $||\cdot||_\triangle$ and $||\cdot||_\Box$ be norms on $X$. Suppose dim $X = N$ and let $\Lambda : \mathbb{K}^N \to X$ be the bijective linear operator defined in the Theorem 2.3. Then we know that $\Lambda$ is a homeomorphism between $(\mathbb{K}^N, ||\cdot||_1)$ and $(X, ||\cdot||_\triangle)$ and also between $(\mathbb{K}^N, ||\cdot||_1)$ and $(X, ||\cdot||_\Box)$. Because of the boundedness of $\Lambda$ and $\Lambda^{-1}$ (both with respect to $||\cdot||_\triangle$ and $||\cdot||_\Box$) there are positive constants $A, B, C, D$ such that

$$A\left|\left|\Lambda^{-1}(x)\right|\right|_1 \leq ||x||_\triangle \leq B\left|\left|\Lambda^{-1}(x)\right|\right|_1 \quad \text{and} \quad C\left|\left|\Lambda^{-1}(x)\right|\right|_1 \leq ||x||_\Box \leq D\left|\left|\Lambda^{-1}(x)\right|\right|_1 .$$

From this we get

$$\frac{A}{D}\,||x||_\Box \leq ||x||_\triangle \leq \frac{B}{C}\,||x||_\Box\,,$$

i.e., $||\cdot||_\triangle$ and $||\cdot||_\Box$ are equivalent.

$\Box$

**Corollary 2.5.** *Every linear operator on a finite-dimensional normed space is bounded.*

*Proof.* Let $\Lambda_0 : \mathbb{K}^N \to X$ be a linear homeomorphism. Then note that $\Lambda = (\Lambda \circ \Lambda_0) \circ \Lambda_0^{-1}$. Because every linear operator on $\mathbb{K}^N$ is bounded it follows that $\Lambda \circ \Lambda_0 : \mathbb{K}^N \to Y$ is bounded. Now since $\Lambda_0^{-1}$ and $\Lambda \circ \Lambda_0$ are both bounded we conclude that the composition $\Lambda$ is bounded as well. $\Box$

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