SOME SERIES RELATED TO NONTRIVIAL ZEROS OF DEDEKIND ZETA FUNCTIONS

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Abstract. In this manuscript we establish a closed-form of an infinite series involving derivatives of an analogue of the Riemann xi function for Dedekind zeta function and nontrivial zeros of Dedekind zeta functions, assuming Extended Riemann Hypothesis. Conversely, we prove that if the closed form of this series holds, then all of the zeros of Dedekind zeta function beyond a certain height lie on the critical line. This yields a large number of equivalent statements of Riemann Hypothesis.

1. Introduction

The Dedekind zeta function, $\zeta_K(s)$, associated to a number field K of degree n over $\mathbb{Q}$, is defined for $\Re(s) > 1$ as the absolutely convergent series

$$\zeta_K(s) = \sum_{\mathfrak{a}} \frac{1}{N_{K/\mathbb{Q}}(\mathfrak{a})^s},$$

where the sum runs through all nonzero ideals $\mathfrak{a}$ of the ring of integers $\mathcal{O}_K$ of K and $N(\mathfrak{a})$ denotes the norm of a nonzero ideal $\mathfrak{a}$ in $\mathcal{O}_K$, i.e., $N(\mathfrak{a}) = [\mathcal{O}_K : \mathfrak{a}]$. For the special case $K = \mathbb{Q}$, the Dedekind zeta function reduces to the classical Riemann zeta function; $\zeta_{\mathbb{Q}}(s) = \zeta(s)$.

Analogous to Riemann zeta function, Dedekind zeta functions also have Euler product representation

$$\zeta_K(s) = \prod_{\mathfrak{p}} \frac{1}{1 - N_{K/\mathbb{Q}}(\mathfrak{p})^{-s}} \tag{1.1}$$

for $\Re(s) > 1$, where the product is extended over all prime ideals $\mathfrak{p}$ of $\mathcal{O}_K$. This is the result of the uniqueness of prime factorization of nonzero ideals in $\mathcal{O}_K$. As a consequence, $\zeta_K(s) \neq 0$ for $\Re(s) > 1$.

Just as in the case of Riemann zeta function, Dedekind zeta function also admits analytic continuation to the whole complex plane except for a simple pole at $s = 1$.

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Dedekind proved the *analytic class number formula*, showing that $\zeta_K(s)$ has a simple pole at $s = 1$ with residue

$$\frac{2^{r_1+r_2}\pi^{r_2}R_K h_K}{\omega_K|\Delta_K|^{1/2}}, \tag{1.2}$$

where r_1 and r_2 are the number of real embeddings and pairs of complex embeddings respectively, R_K is the regulator of K, h_K is the class number of K, ω_K is the number of roots of unity contained in K, and Δ_K is the discriminant of the field extension $K/\mathbb{Q}$. This is an instance of the “local-global principle": the Dedekind zeta function $\zeta_K(s)$ is assembled from local components, namely the prime ideals of $\mathcal{O}_K$, as expressed by the Euler product (1.1); yet it also reflects global invariants of the field in an integrated fashion, as is revealed by the class number formula (1.2).

Dedekind zeta function satisfies the functional equation

$$\zeta_K(1-s) = |\Delta_K|^{s-1/2}2^{n(1-s)}\pi^{-ns}\left(\cos\frac{\pi s}{2}\right)^{r_1+r_2}\left(\sin\frac{\pi s}{2}\right)^{r_2}\Gamma(s)^n\zeta_K(s)$$

which was discovered by Hecke [3]. A detailed exposition of Hecke's proof of the functional equation for $\zeta_K(s)$ can be found in [4, Chap. 7, Â§5]. It reduces to the functional equation of $\zeta(s)$ for the case $K = \mathbb{Q}$. Furthermore, it can be seen from the above functional equation that $\zeta_K(s)$ has a zero of order $r_1 + r_2 - 1$ at $s = 0$ since the term involving cosine contributes a zero of order $r_1 + r_2$ at $s = 1$, $\zeta_K(s)$ has a pole of order 1 at $s = 1$ and other terms are analytic and do not vanish at $s = 1$.

2. Some preliminaries

One of the most notoriously difficult problem in all of mathematics is the Riemann Hypothesis (RH) which states that all of the nontrivial trivial zeros $\rho = \beta + i\gamma$ of $\zeta(s)$ have $\beta = 1/2$. One can conjecture a similar statement for the Dedekind zeta functions: All of the zeros $\rho_K = \beta_K + i\gamma_K$ of $\zeta_K(s)$ satisfying $0 < \beta_K < 1$ lie on the critical line, i.e., $\beta_K = 1/2$. This is usually referred as the *Extended Riemann Hypothesis* (ERH).

We consider an analogue of the Riemann xi function, sometimes referred as the completed Riemann zeta function, for Dedekind zeta function as

$$\xi_K(s) = \frac{1}{2}s(s-1)|\Delta_K|^{s/2}2^{(1-s)r_2}\pi^{-ns/2}\Gamma\left(\frac{s}{2}\right)^{r_1}\Gamma(s)^{r_2}\zeta_K(s).$$

It is an entire function of order 1 and satisfies the functional equation

$$\xi_K(s) = \xi_K(1-s). \tag{2.1}$$

Just as for $\zeta_K(s)$, $\xi_K(s)$ reduces to $\xi(s)$ for the case $K = \mathbb{Q}$. Moreover, $\xi_K(s)$ does not vanish at $s = 0$ and the zeros of $\xi_K(s)$ coincide with the zeros of $\zeta_K(s)$ in the critical strip $0 < \Im(s) < 1$.

Lemma 2.1. *For k odd, $\xi_K^{(k)}(1/2) = 0$.*

Proof. From the functional equation 2.1 we readily obtain functional equation for $\xi_K^{(k)}(s)$ by repeated differentiation as

$$\xi_K^{(k)}(s) = (-1)^k \xi_K^{(k)}(1-s).$$

The result is now immediate by plugging $s = 1/2$. □

Lemma 2.2. *If $m = \mathrm{ord}_{s=1/2}\zeta_K(s)$, then m is even.*

Proof. If $\zeta_K(1/2) \neq 0$, then $m = 0$ and so the conclusion is trivial. From the definition of $\xi_K(s)$ we have $m = \mathrm{ord}_{s=1/2}\xi_K(s)$ Now suppose that $\zeta_K(1/2) = 0$. Since $\xi_K(s)$ is entire, there is a power series expansion

$$\xi_K(s) = \sum_{k=0}^{\infty} \frac{\xi_K^{(k)}(1/2)}{k!} \left(s - \frac{1}{2}\right)^k.$$

Because $\xi_K(1/2) = 0$ and m is the smallest power k for which $\xi_K^{(k)}(1/2) \neq 0$, it follows by Lemma 2.1 that m must be even. □

We now define

$$\mathscr{X}_K(s) = \left(s - \frac{1}{2}\right)^{-m} \xi_K(s),$$

where m is the order of vanishing of $\zeta_K(s)$ at $s = 1/2$. Since m is even, $\mathscr{X}_K(s)$ satisfies the functional equation $\mathscr{X}_K(s) = \mathscr{X}_K(1-s)$

Lemma 2.3. *For k odd, $\mathscr{X}_K^{(k)}(1/2) = 0$.*

Lemma 2.4. *For k even, $(\mathscr{X}_K'/\mathscr{X}_K)^{(k)}(1/2) = 0$.*

Because $\mathscr{X}_K(s)$ is complex-conjugate symmetric, i.e., $\overline{\mathscr{X}_K(s)} = \mathscr{X}_K(\overline{s})$, so are its higher derivatives and as a result $\mathscr{X}_K^{(k)}(s)$ is real-valued on the real-axis. For a detailed proof see [5].

Since $\mathscr{X}_K(s)$ is an entire function of order 1, it can be expressed as a Hadamard product

$$\mathscr{X}_K(s) = e^{A+Bs} \prod_{\rho_K \neq \frac{1}{2}} \left(1 - \frac{s}{\rho_K}\right) e^{s/\rho_K}, \tag{2.2}$$

where A and B are constants and the product runs over the zeros of $\zeta_K(s)$ that distinct from $1/2$ and lie in the critical strip $0 < \Im(s) < 1$. This product converges uniformly on compact sets not containing any zeros $\rho_K \neq 1/2$.

3. Main result

A plethora of equivalent statements of RH have been discovered in the last century. A detailed account of both arithmetic and analytic equivalents can be found in [1] and [2] respectively. Recently, Suman and Das [8] showed that RH is equivalent to

$$\sum_{\rho} \frac{1}{|\frac{1}{2}-\rho|^2} = \frac{\xi''(\frac{1}{2})}{\xi(\frac{1}{2})}.$$

Later in [7], they showed that RH holds if and only if

$$\frac{1}{2}\frac{\xi''(\frac{1}{2})^2}{\xi(\frac{1}{2})^2} - \frac{1}{6}\frac{\xi^{(4)}(\frac{1}{2})}{\xi(\frac{1}{2})} = \sum_{\rho} \frac{1}{|\frac{1}{2}-\rho|^4}.$$

Very recently, Nguyen [5] extended Suman and Das result to Dedekind zeta functions and showed that ERH for $\zeta_K(s)$ is equivalent to

$$\sum_{\rho_K \neq \frac{1}{2}} \frac{1}{|\frac{1}{2}-\rho_K|^2} = \frac{\mathscr{X}_K''(\frac{1}{2})}{\mathscr{X}_K(\frac{1}{2})}.$$

We generalize Suman and Das result and obtain a large number of equivalent statements for RH involving closed form of sums involving $|\frac{1}{2}-\rho|^{k+1}$, where k is an odd positive integer. We do so in a general setting of Dedekind zeta functions, as is done by Nguyen for the special case $k = 1$.

Theorem 3.1. *Let K be a number field and let k be an odd positive integer. If ERH holds, then*

$$\frac{(-1)^{(k-1)/2}}{k!}\left(\frac{\mathscr{X}_K'}{\mathscr{X}_K}\right)^{(k)}\left(\frac{1}{2}\right) = \sum_{\rho_K \neq \frac{1}{2}} \frac{1}{|\frac{1}{2}-\rho_K|^{k+1}}, \tag{3.1}$$

where the sum runs over all nontrivial zeros ρ_K of $\zeta_K(s)$ distinct from $1/2$. Moreover, if the above equality holds, then $\beta_K = 1/2$ for $|\gamma_K| > \frac{1}{2}\cot\frac{2\pi}{(k+1)}$

Proof. Taking the logarithmic derivative of the Hadamard product (2.2) we obtain

$$\frac{\mathscr{X}_K'}{\mathscr{X}_K}(s) = B + \sum_{\rho_K \neq \frac{1}{2}} \left(\frac{1}{s-\rho_K} + \frac{1}{\rho_K}\right).$$

Because the sum converges uniformly on compact subsets not containing the zeros of $\zeta_K(s)$ we can differentiate term by term and deduce that

$$\left(\frac{\mathscr{X}_K'}{\mathscr{X}_K}\right)'(s) = -\sum_{\rho_K \neq \frac{1}{2}} \frac{1}{(s-\rho_K)^2}.$$

By repeated differentiation we end up with the identity

$$\frac{(-1)^k}{k!}\left(\frac{\mathscr{X}_K'}{\mathscr{X}_K}\right)^{(k)}(s)=\sum_{\rho_K\neq\frac{1}{2}}\frac{1}{(s-\rho_K)^{k+1}}$$

which holds for any positive integer k. For k even it immediately follows from Lemma 2.4 that

$$\sum_{\rho_K\neq\frac{1}{2}}\frac{1}{(\frac{1}{2}-\rho_K)^{k+1}}=0.$$

But this is well-known and follows simply due to the symmetry of zeros ρ_K under conjugation and reflection across the critical line. We now assume that k is odd. By noting that $(\mathscr{X}_K'/\mathscr{X}_K)^{(k)}(\frac{1}{2})$ is real we obtain

$$\begin{aligned}\frac{(-1)^k}{k!}\left(\frac{\mathscr{X}_K'}{\mathscr{X}_K}\right)^{(k)}\left(\frac{1}{2}\right)&=\frac{1}{2}\sum_{\rho_K\neq\frac{1}{2}}\frac{1}{(\frac{1}{2}-\rho_K)^{k+1}}+\frac{1}{2}\sum_{\rho_K\neq\frac{1}{2}}\frac{1}{(\frac{1}{2}-\overline{\rho_K})^{k+1}}\\&=\frac{1}{2}\sum_{\rho_K\neq\frac{1}{2}}\frac{(\frac{1}{2}-\rho_K)^{k+1}+(\frac{1}{2}-\overline{\rho_K})^{k+1}}{|\frac{1}{2}-\rho_K|^{2(k+1)}}\\&=\sum_{\rho_K\neq\frac{1}{2}}\frac{\Re(\frac{1}{2}-\rho_K)^{k+1}}{|\frac{1}{2}-\rho_K|^{2(k+1)}}.\end{aligned}$$

If ERH holds, then $\Re(\frac{1}{2}-\rho_K)^{k+1}=(-1)^{(k+1)/2}|\frac{1}{2}-\rho_K|^{k+1}$ and so we get (3.1).

Now suppose that (3.1) holds. Then we have

$$\sum_{\rho_K}\frac{\Re(\frac{1}{2}-\rho_K)^{k+1}-(-1)^{(k+1)/2}|\frac{1}{2}-\rho_K|^{k+1}}{|\frac{1}{2}-\rho_K|^{2(k+1)}}=0. \tag{3.2}$$

If $k\equiv 1 \pmod 4$, then the numerator in the summand is always nonnegative and so $\Re(\frac{1}{2}-\rho_K)^{k+1}=-|\frac{1}{2}-\rho_K|^{k+1}$ for every zero $\rho_K\neq\frac{1}{2}$. Let $\frac{1}{2}-\rho_K=|\frac{1}{2}-\rho_K|e^{i\phi_{\rho_K}}$, where ϕ_{ρ_K} is the principal argument of $\frac{1}{2}-\rho_K$. Then it follows that

$$\Re\left(\left|\frac{1}{2}-\rho_K\right|^{k+1}e^{i(k+1)\phi_{\rho_K}}\right)=-\left|\frac{1}{2}-\rho_K\right|^{k+1}$$

for every zero ρ_K, which in turn implies that $\cos((k+1)\phi_{\rho_K})=-1$ and so $\phi_{\rho_K}=\frac{(2n+1)\pi}{k+1}$, where $0\le n\le k$. Because $-\frac{1}{2}<\frac{1}{2}-\beta_K<\frac{1}{2}$, it follows that if $|\gamma_K|>\frac{1}{2}\cot\frac{2\pi}{k+1}$, then $\tan^{-1}(2|\gamma_K|)>\frac{\pi}{2}-\frac{2\pi}{k+1}$. This yields $|\phi_{\rho_K}-\frac{\pi}{2}|<\frac{2\pi}{k+1}$ or $|\phi_{\rho_K}-\frac{3\pi}{2}|<\frac{2\pi}{k+1}$. Consequently we must have either $\phi_{\rho_K}=\frac{\pi}{2}$ or $\phi_{\rho_K}=\frac{3\pi}{2}$ and hence $\frac{1}{2}-\beta_K=\Re(\frac{1}{2}-\rho_K)=|\frac{1}{2}-\rho_K|\cos(\phi_{\rho_K})=0$, i.e., $\beta_K=\frac{1}{2}$.

If $k \equiv 3 \pmod 4$. Then numerator in each summand in the sum (3.2) is nonpositive for every zero $\rho_K \neq \frac{1}{2}$. Thus $\Re(\frac{1}{2}-\rho_K)^{k+1} = |\frac{1}{2}-\rho_K|^{k+1}$ and so

$$\Re\left(\left|\frac{1}{2}-\rho_K\right|^{k+1} e^{i(k+1)\phi_{\rho_K}}\right) = \left|\frac{1}{2}-\rho_K\right|^{k+1}$$

for every zero ρ_K, which leads to $\cos((k+1)\phi_{\rho_K}) = 1$ and so $\phi_{\rho_K} = \frac{2n\pi}{k+1}$, where $0 \leq n \leq k$. Again it follows that if $|\gamma_K| > \frac{1}{2}\cot(\frac{2\pi}{k+1})$, then we must have $\phi_{\rho_K} = \frac{\pi}{2}, \frac{3\pi}{2}$. Hence $\frac{1}{2}-\beta_K = \Re(\frac{1}{2}-\rho_K) = |\frac{1}{2}-\rho_K|\cos(\phi_{\rho_K}) = 0$, i.e., $\beta_K = \frac{1}{2}$. This completes the proof. $\square$

Observe that the height beyond which all zeros ρ_K of $\zeta_K(s)$ lie on the critical line is uniform in K and depends only on the order k of the derivative of $\mathscr{X}_K'/\mathscr{X}_K$.

Corollary 3.2. *The statement 3.1 is equivalent to ERH for $k = 1, 3$.*

Proof. Follows simply by observing that $\cot(\frac{2\pi}{k+1}) = 0$ for $k = 1$ and 3. $\square$

Corollary 3.3. *Let k be an odd positive integer. If RH holds, then*

$$\frac{(-1)^{(k-1)/2}}{k!}\left(\frac{\xi'}{\xi}\right)^{(k)}\left(\frac{1}{2}\right) = \sum_{\rho}\frac{1}{|\frac{1}{2}-\rho|^{k+1}}, \tag{3.3}$$

where the sum is taken over all nontrivial zeros ρ of $\zeta(s)$. Moreover, if the above equality holds, then $\beta = 1/2$ for $|\gamma| > \frac{1}{2}\cot\frac{2\pi}{(k+1)}$

Corollary 3.4. *If $k < 12\pi \cdot 10^{12}$, then (3.3) is equivalent to RH.*

Proof. Note that due to inequality $\cot^{-1}(x) \leq 1/x$ for $x > 0$ we have

$$\frac{2\pi}{\cot^{-1}(6\cdot 10^{12})} \geq 12\pi\cdot 10^{12} \geq k+1$$

which in turn implies that

$$\frac{1}{2}\cot\left(\frac{2\pi}{k+1}\right) \leq 3\cdot 10^{12}.$$

Since $\beta = \frac{1}{2}$ for $|\gamma| \leq 3\cdot 10^{12}$ [6], it follows by Corollary 3.3 above that RH holds. $\square$

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