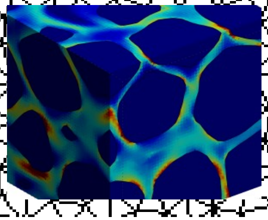

Structural Analysis and Finite Elements

Computer labworks – TETRA4 finite element

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Part I – MatLab FE code

Part 0 – create input file → Read the input file

↓

Construct the external force vector — $\{F_P\}$ nodal point loads

↓

Assemble the structural stiffness matrix — $[K_e] \rightarrow [K_{sys}]$

↓

Apply the boundary conditions — $\left\{ \begin{array}{l} \text{penalty method} \\ \text{direct method} \end{array} \right.$

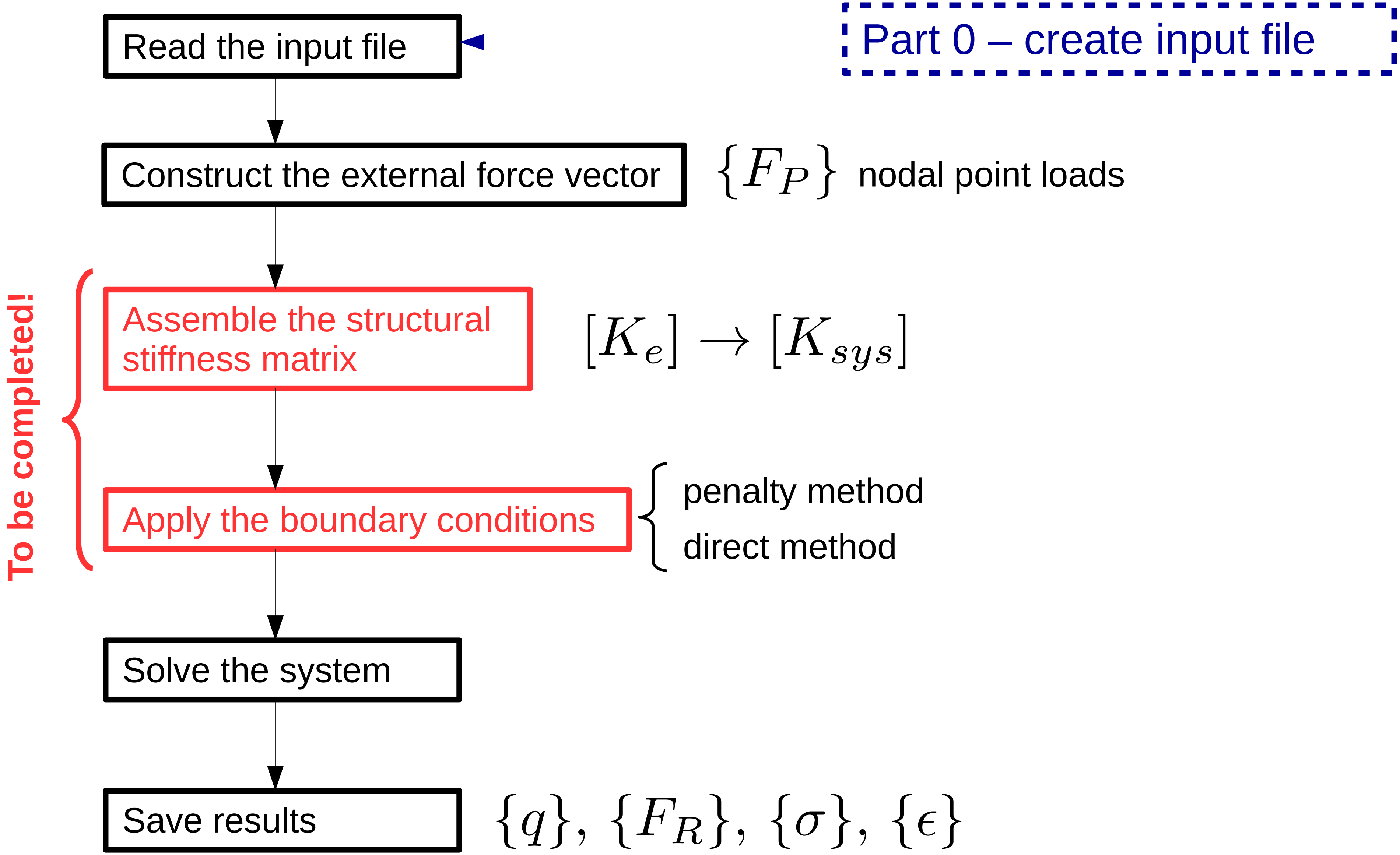
↓

Solve the system

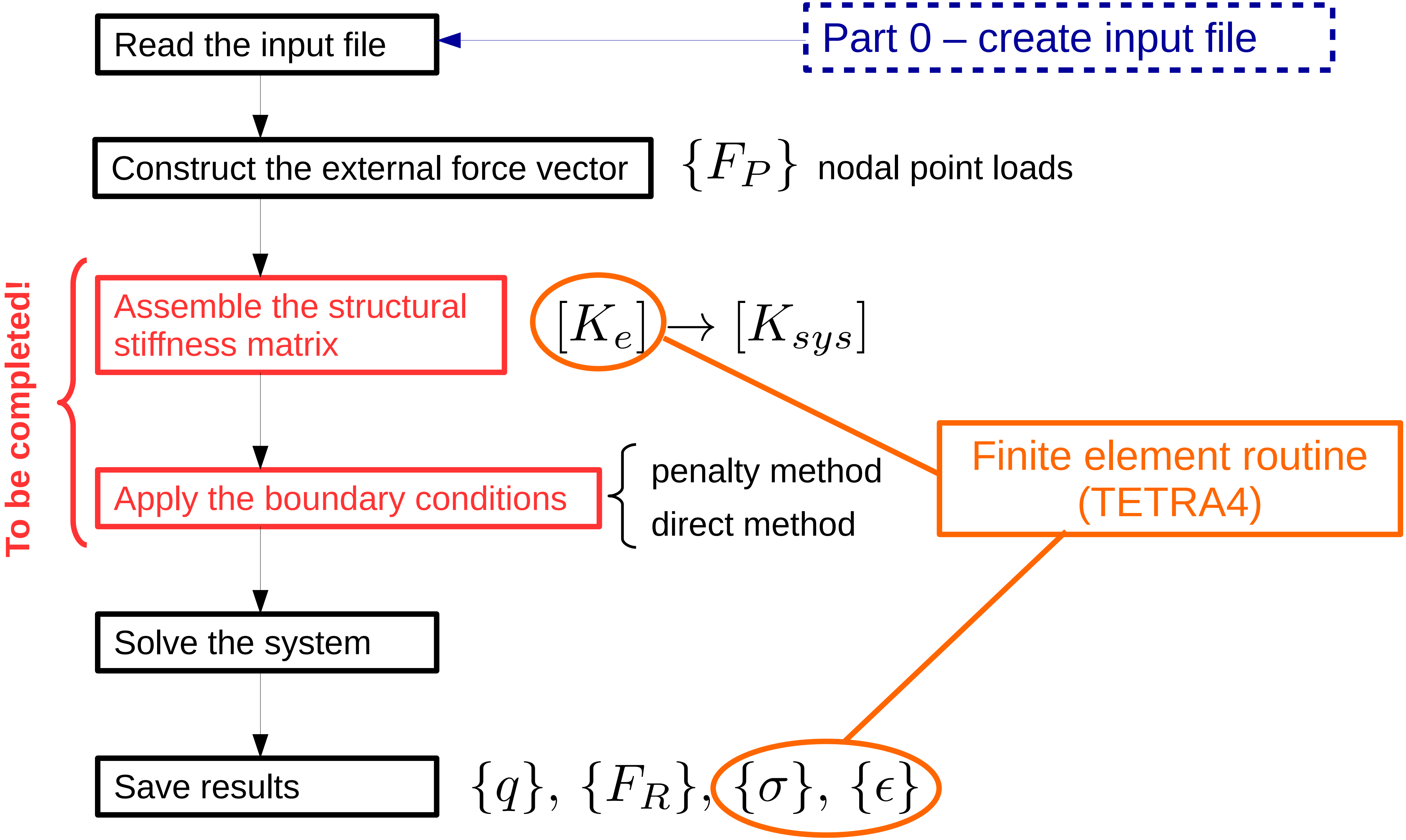
↓

Save results — $\{q\},\ \{F_R\},\ \{\sigma\},\ \{\epsilon\}$

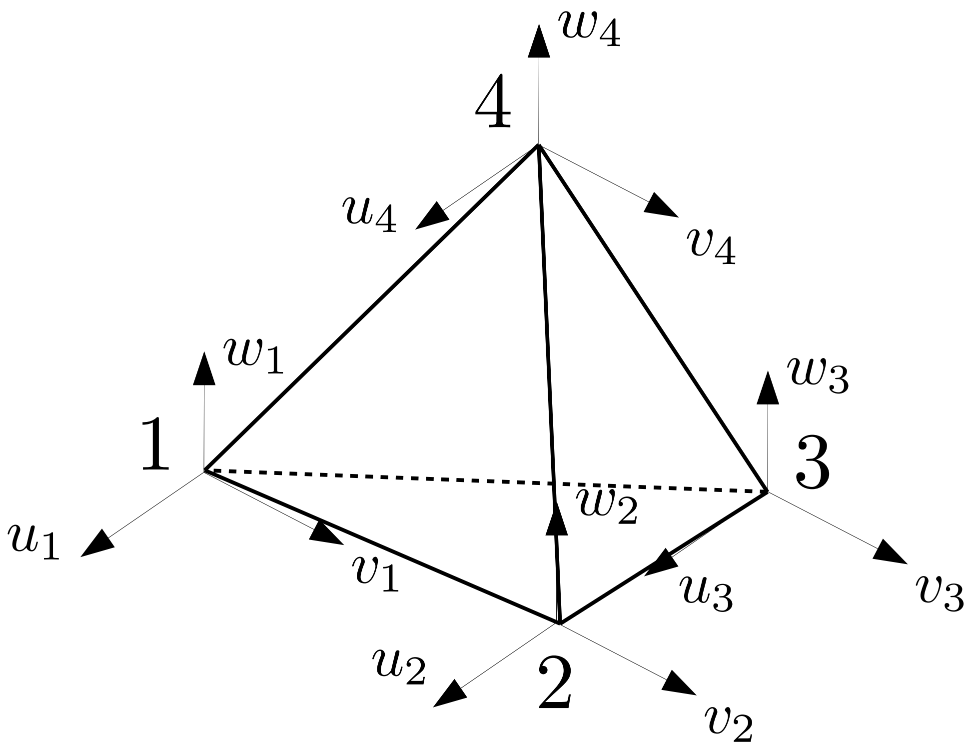
Part I – MatLab FE code



Part I – MatLab FE code



TETRA4

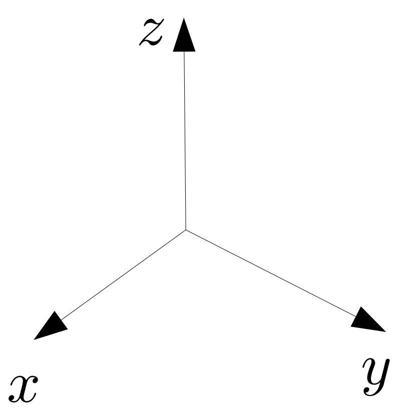


Displacement vector of the FE

$$\mathbf{q}_e = \{u_1,\, v_1,\, w_1, ...u_4,\, v_4,\, w_4\}^T$$

External force vector of the FE

$$\mathbf{f}_e = \{f_{x\,1},\, f_{y\,1},\, f_{z\,1}, ...f_{x\,4},\, f_{y\,4},\, f_{z\,4}\}^T$$

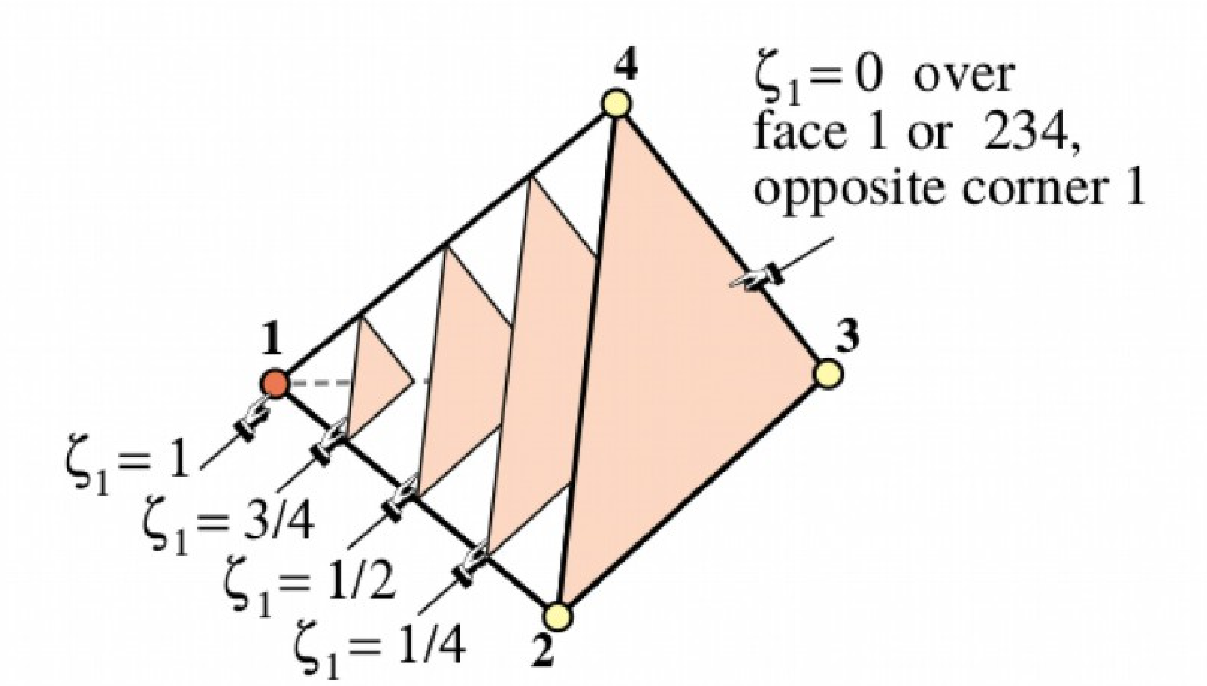


Node numbering convention:

- choose a first node: #1
- choose the face that will contain nodes #1 #2 #3 (counterclockwise numbering)
- the opposite node is #4

Shape functions in tetrahedral coordinates

$(\xi_1,\, \xi_2,\, \xi_3,\, \xi_4)$ analogous to the 2D triangular coordinates for T3



$$\xi_i = \delta_{ij} = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases}$$

$$\xi_1 + \xi_2 + \xi_3 + \xi_4 = 1$$

Shape functions in tetrahedral coordinates

Coordinate transformation Cartesian vs. tetrahedral

$$\left\{ \begin{array}{c} 1 \\ x \\ y \\ z \end{array} \right\} = \left[\begin{array}{cccc} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{array} \right] \left\{ \begin{array}{c} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \end{array} \right\}$$

Which yields after transformation, supposing $V \neq 0$ $\quad x_{ij} = x_i - x_j$

$$\left\{ \begin{array}{c} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \end{array} \right\} = \frac{1}{6V} \left[\begin{array}{cccc} 6V_{n1} & y_{42}z_{32} - y_{32}z_{42} & x_{32}z_{42} - x_{42}z_{32} & x_{42}y_{32} - x_{32}y_{42} \\ 6V_{n2} & y_{31}z_{43} - y_{34}z_{13} & x_{43}z_{31} - x_{13}z_{34} & x_{31}y_{43} - x_{34}y_{13} \\ 6V_{n3} & y_{24}z_{14} - y_{14}z_{24} & x_{14}z_{24} - x_{24}z_{14} & x_{24}y_{14} - x_{14}y_{24} \\ 6V_{n4} & y_{13}z_{21} - y_{12}z_{31} & x_{21}z_{13} - x_{31}z_{12} & x_{13}y_{21} - x_{12}y_{31} \end{array} \right] \left\{ \begin{array}{c} 1 \\ x \\ y \\ z \end{array} \right\}$$

$$\left\{ \begin{array}{c} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \end{array} \right\} = \frac{1}{6V} \left[\begin{array}{cccc} 6V_{n1} & a_1 & b_1 & c_1 \\ 6V_{n2} & a_2 & b_2 & c_2 \\ 6V_{n3} & a_3 & b_3 & c_3 \\ 6V_{n4} & a_4 & b_4 & c_4 \end{array} \right] \left\{ \begin{array}{c} 1 \\ x \\ y \\ z \end{array} \right\}$$

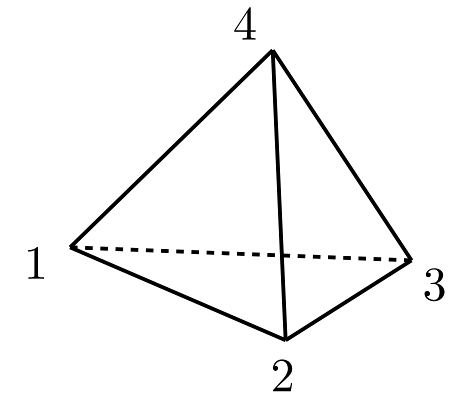
TETRA4 – partial derivatives

$$\begin{Bmatrix} 1 \\ x \\ y \\ z \end{Bmatrix} = \underbrace{\begin{bmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{bmatrix}}_{\frac{\partial x}{\partial \xi_i} = x_i \quad \frac{\partial y}{\partial \xi_i} = y_i \quad \frac{\partial z}{\partial \xi_i} = z_i} \begin{Bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \end{Bmatrix} \qquad \begin{Bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \end{Bmatrix} = \underbrace{\frac{1}{6V} \begin{bmatrix} 6V_{n1} & a_1 & b_1 & c_1 \\ 6V_{n2} & a_2 & b_2 & c_2 \\ 6V_{n3} & a_3 & b_3 & c_3 \\ 6V_{n4} & a_4 & b_4 & c_4 \end{bmatrix}}_{6V\frac{\partial \xi_i}{\partial x} = a_i \quad 6V\frac{\partial \xi_i}{\partial y} = b_i \quad 6V\frac{\partial \xi_i}{\partial z} = c_i} \begin{Bmatrix} 1 \\ x \\ y \\ z \end{Bmatrix}$$

Partial derivative of a function wrt structural (x, y, z) axes:

$$\frac{\partial F}{\partial x} = \frac{\partial F}{\partial \xi_i}\frac{\partial \xi_i}{\partial x} = \frac{1}{6V}\left(\frac{\partial F}{\partial \xi_1}a_1 + \frac{\partial F}{\partial \xi_2}a_2 + \frac{\partial F}{\partial \xi_3}a_3 + \frac{\partial F}{\partial \xi_4}a_4\right)$$

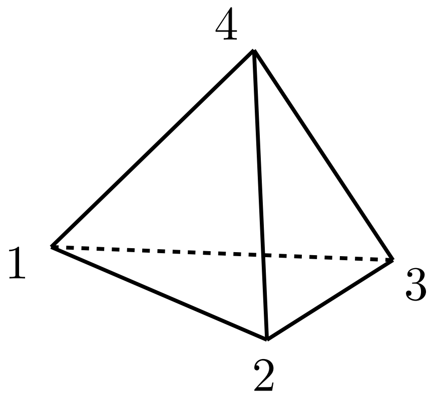
TETRA4 – strains



$$\boldsymbol{\epsilon}^e(\mathbf{x}) = \mathbf{D}\ \mathbf{u}^e(\mathbf{x}) = \mathbf{D}\ \mathbf{N}^e(\mathbf{x})\ \mathbf{q}^e = \mathbf{B}(\mathbf{x})\ \mathbf{q}^e$$

$$\boldsymbol{\epsilon}^e(\mathbf{x}) = \left\{ \begin{array}{c} \epsilon_x \\ \epsilon_y \\ \epsilon_z \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{array} \right\} \quad \mathbf{D} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \\ 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} \end{bmatrix} \quad \mathbf{q}^e = \left\{ \begin{array}{c} u_1 \\ v_1 \\ w_1 \\ u_2 \\ v_2 \\ w_2 \\ u_3 \\ v_3 \\ w_3 \\ u_4 \\ v_4 \\ w_4 \end{array} \right\}$$

TETRA4 – strains

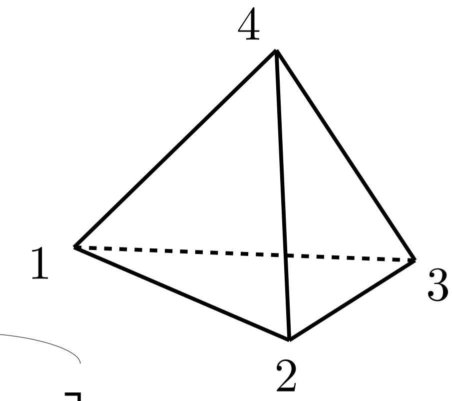


$$\boldsymbol{\epsilon}^e(\mathbf{x}) = \mathbf{D}\ \mathbf{u}^e(\mathbf{x}) = \mathbf{D}\ \mathbf{N}^e(\mathbf{x})\ \mathbf{q}^e = \mathbf{B}(\mathbf{x})\ \mathbf{q}^e$$

$$\mathbf{B} = \frac{1}{6V}\begin{bmatrix} a_1 & 0 & 0 & a_2 & 0 & 0 & a_3 & 0 & 0 & a_4 & 0 & 0 \\ 0 & b_1 & 0 & 0 & b_2 & 0 & 0 & b_3 & 0 & 0 & b_4 & 0 \\ 0 & 0 & c_1 & 0 & 0 & c_2 & 0 & 0 & c_3 & 0 & 0 & c_4 \\ b_1 & a_1 & 0 & b_2 & a_2 & 0 & b_3 & a_3 & 0 & b_4 & a_4 & 0 \\ c_1 & 0 & a_1 & c_2 & 0 & a_2 & c_3 & 0 & a_3 & c_4 & 0 & a_4 \\ 0 & c_1 & b_1 & 0 & c_2 & b_2 & 0 & c_3 & b_3 & 0 & c_4 & b_4 \end{bmatrix}$$

Constant strain a TETRA4!

TETRA4 – stresses

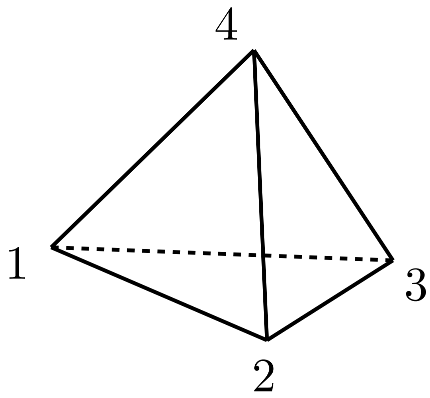


$$\boldsymbol{\sigma}^e(\mathbf{x}) = \mathbf{H}_{3D}\boldsymbol{\epsilon}^e(\mathbf{x}) = \boxed{\mathbf{H}_{3D}}\mathbf{B}(\mathbf{x})\,\mathbf{q}^e$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{xz} \\ \tau_{yz} \end{Bmatrix} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1 & \frac{\nu}{1-\nu} & \frac{\nu}{1-\nu} & 0 & 0 & 0 \\ \frac{\nu}{1-\nu} & 1 & \frac{\nu}{1-\nu} & 0 & 0 & 0 \\ \frac{\nu}{1-\nu} & \frac{\nu}{1-\nu} & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2\nu}{2(1-\nu)} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2\nu}{2(1-\nu)} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2\nu}{2(1-\nu)} \end{bmatrix} \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_z \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix}$$

Constant stress in a TETRA4!

TETRA4 – stiffness matrix



$$\mathbf{K}^e = \int_{V^e} \mathbf{B}^T \; \mathbf{H} \; \mathbf{B} \; dV = \mathbf{B}^T \; \mathbf{H} \; \mathbf{B} \; V^e$$

$$V^e = \frac{1}{6}\det \begin{bmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{bmatrix}$$

- the simplest 3D solid FE
- results in constant strain and stress within a FE
- very fine meshes are required where strain gradients are large, therefore not often used (poor performance)