

Structural Analysis and Finite Elements

Computer labworks – TETRA4 finite element

Péter Z. Berke

Professor

Building, Architecture and Town Planning (BATir) Dept. CP194/2
Université libre de Bruxelles
E-mail: peter.berke@ulb.be

Louis Remes

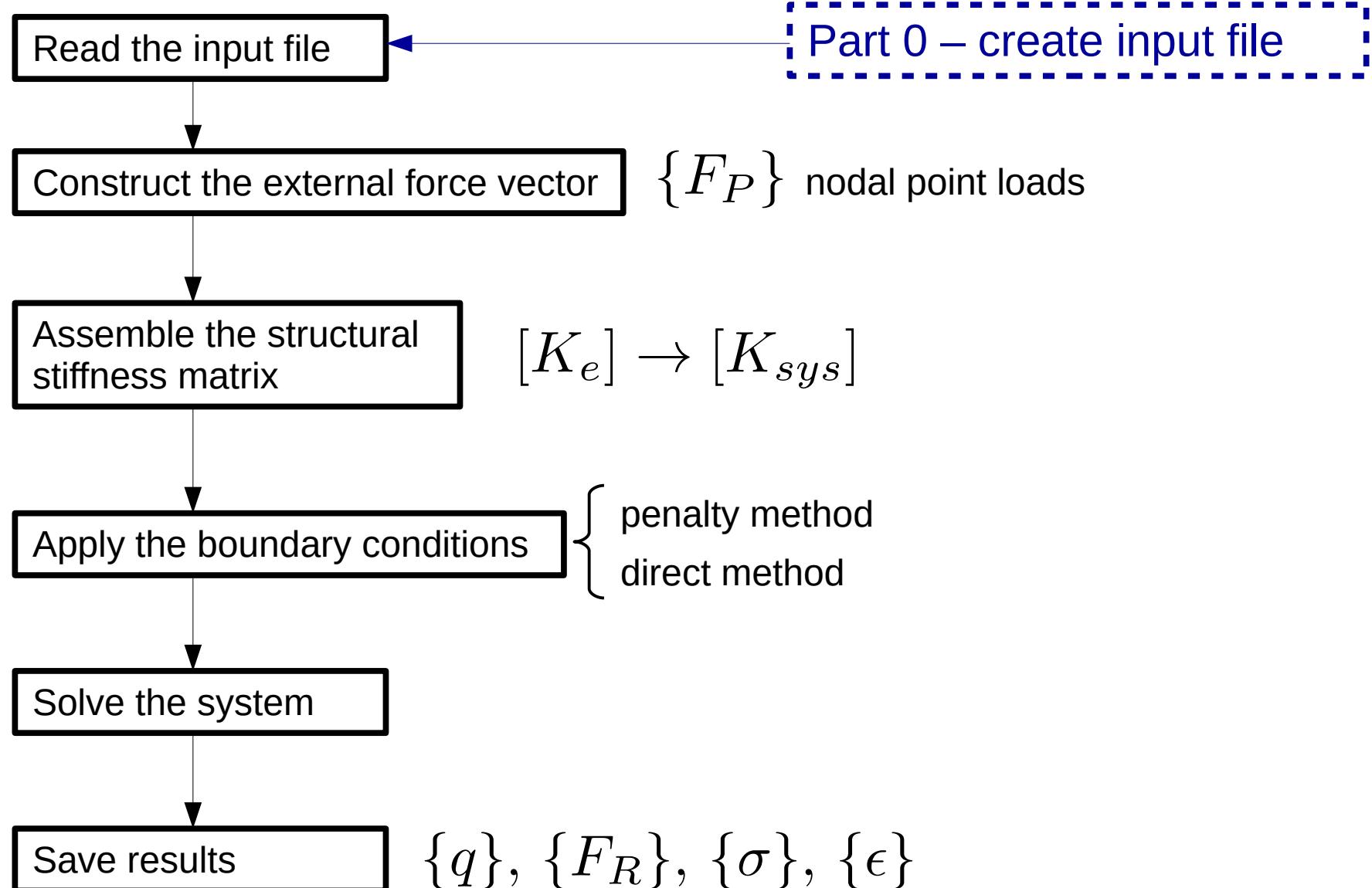
Teaching Assistant
BATir Dept.
Université libre de Bruxelles
E-mail: louis.remes@ulb.be

Noémi Mertens

Teaching Assistant
BATir Dept.
Université libre de Bruxelles
E-mail: Noemi.Mertens@ulb.be

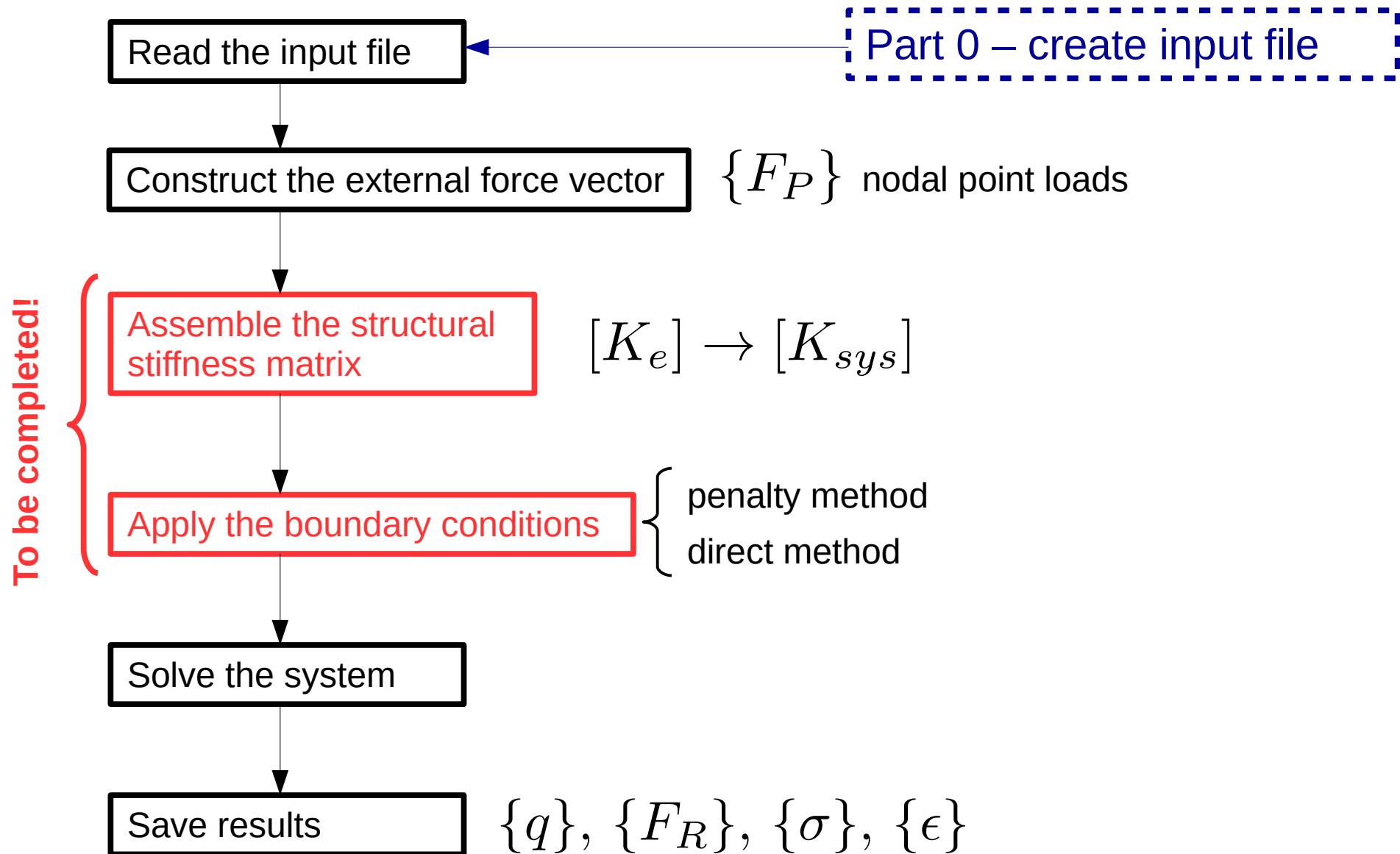
Part I – MatLab FE code

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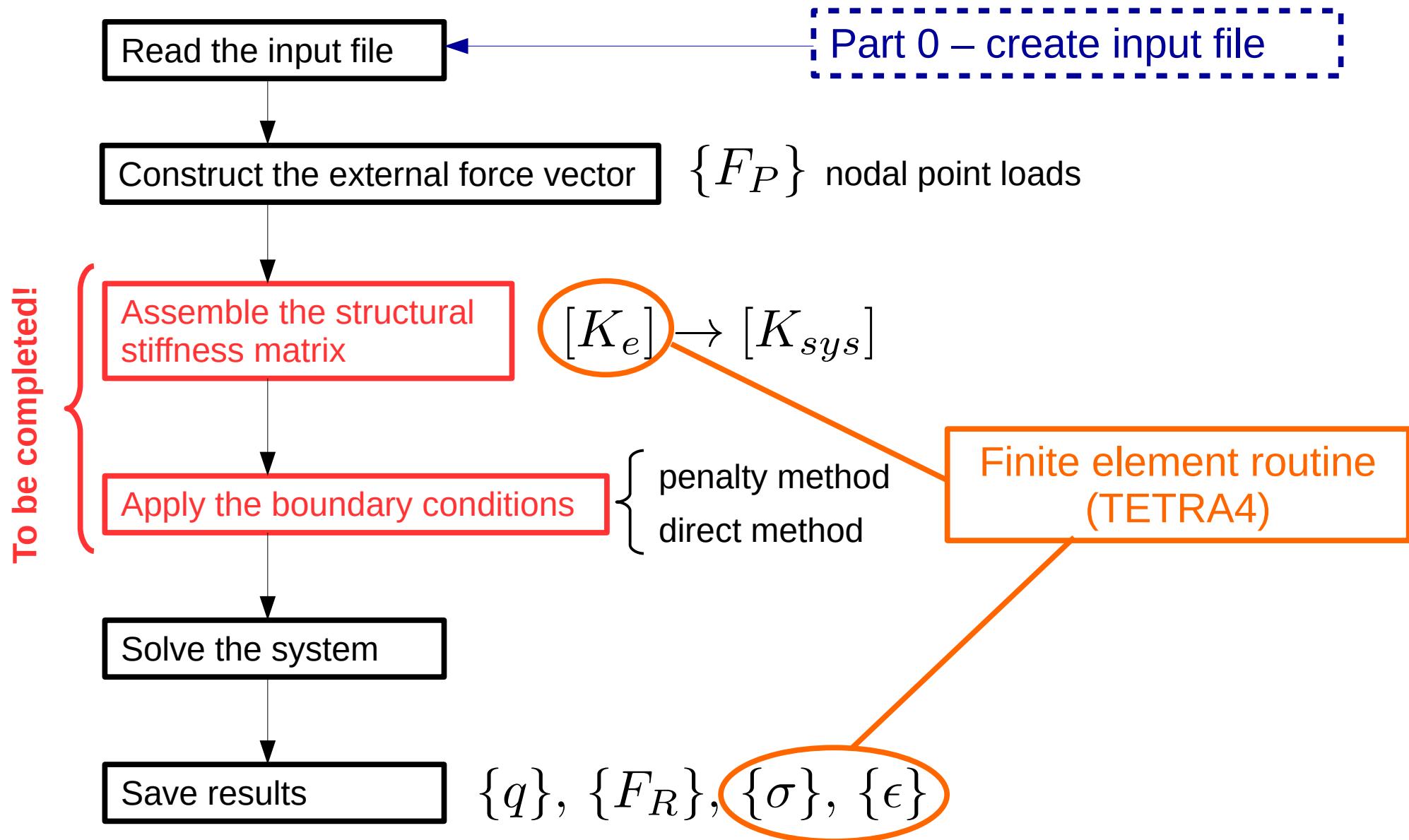
Part I – MatLab FE code

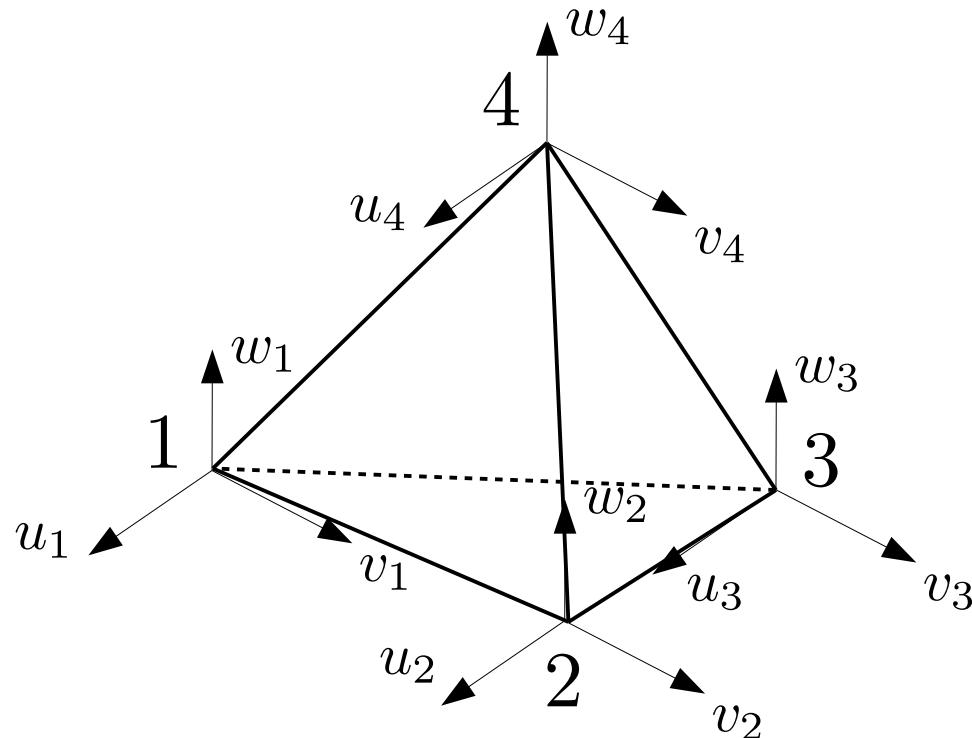
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Part I – MatLab FE code

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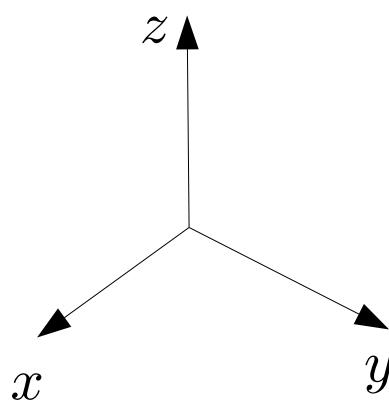


Displacement vector of the FE

$$\mathbf{q}_e = \{u_1, v_1, w_1, \dots, u_4, v_4, w_4\}^T$$

External force vector of the FE

$$\mathbf{f}_e = \{f_{x1}, f_{y1}, f_{z1}, \dots, f_{x4}, f_{y4}, f_{z4}\}^T$$

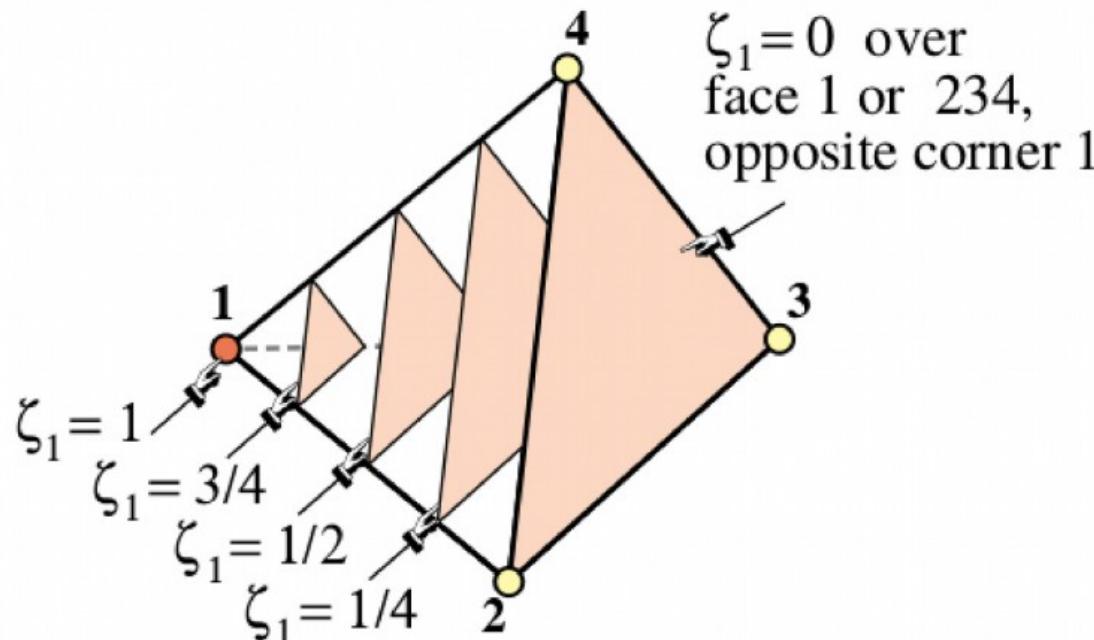


Node numbering convention:

- choose a first node: #1
- choose the face that will contain nodes #1 #2 #3
(counterclockwise numbering)
- the opposite node is #4

Shape functions in tetrahedral coordinates

$(\xi_1, \xi_2, \xi_3, \xi_4)$ analogous to the 2D triangular coordinates for T3



$$\xi_i = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

$$\xi_1 + \xi_2 + \xi_3 + \xi_4 = 1$$

Shape functions in tetrahedral coordinates

Coordinate transformation Cartesian vs. tetrahedral

$$\begin{Bmatrix} 1 \\ x \\ y \\ z \end{Bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{bmatrix} \begin{Bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \end{Bmatrix}$$

Which yields after transformation, supposing $V \neq 0$ $x_{ij} = x_i - x_j$

$$\begin{Bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \end{Bmatrix} = \frac{1}{6V} \begin{bmatrix} 6V_{n1} & y_{42}z_{32} - y_{32}z_{42} & x_{32}z_{42} - x_{42}z_{32} & x_{42}y_{32} - x_{32}y_{42} \\ 6V_{n2} & y_{31}z_{43} - y_{34}z_{13} & x_{43}z_{31} - x_{13}z_{34} & x_{31}y_{43} - x_{34}y_{13} \\ 6V_{n3} & y_{24}z_{14} - y_{14}z_{24} & x_{14}z_{24} - x_{24}z_{14} & x_{24}y_{14} - x_{14}y_{24} \\ 6V_{n4} & y_{13}z_{21} - y_{12}z_{31} & x_{21}z_{13} - x_{31}z_{12} & x_{13}y_{21} - x_{12}y_{31} \end{bmatrix} \begin{Bmatrix} 1 \\ x \\ y \\ z \end{Bmatrix}$$

$$\begin{Bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \end{Bmatrix} = \frac{1}{6V} \begin{bmatrix} 6V_{n1} & a_1 & b_1 & c_1 \\ 6V_{n2} & a_2 & b_2 & c_2 \\ 6V_{n3} & a_3 & b_3 & c_3 \\ 6V_{n4} & a_4 & b_4 & c_4 \end{bmatrix} \begin{Bmatrix} 1 \\ x \\ y \\ z \end{Bmatrix}$$

TETRA4 – partial derivatives

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$$\begin{Bmatrix} 1 \\ x \\ y \\ z \end{Bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{bmatrix} \begin{Bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \end{Bmatrix} \quad \begin{Bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \end{Bmatrix} = \frac{1}{6V} \begin{bmatrix} 6V_{n1} & a_1 & b_1 & c_1 \\ 6V_{n2} & a_2 & b_2 & c_2 \\ 6V_{n3} & a_3 & b_3 & c_3 \\ 6V_{n4} & a_4 & b_4 & c_4 \end{bmatrix} \begin{Bmatrix} 1 \\ x \\ y \\ z \end{Bmatrix}$$

$$\overbrace{\frac{\partial x}{\partial \xi_i}}^{} = x_i \quad \overbrace{\frac{\partial y}{\partial \xi_i}}^{} = y_i \quad \overbrace{\frac{\partial z}{\partial \xi_i}}^{} = z_i$$

$$6V \overbrace{\frac{\partial \xi_i}{\partial x}}^{} = a_i \quad 6V \overbrace{\frac{\partial \xi_i}{\partial y}}^{} = b_i \quad 6V \overbrace{\frac{\partial \xi_i}{\partial z}}^{} = c_i$$

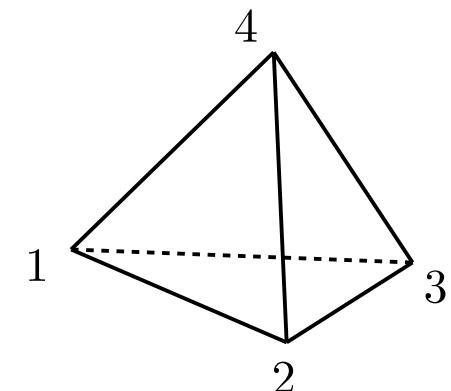
Partial derivative of a function wrt structural (x, y, z) axes:

$$\frac{\partial F}{\partial x} = \frac{\partial F}{\partial \xi_i} \frac{\partial \xi_i}{\partial x} = \frac{1}{6V} \left(\frac{\partial F}{\partial \xi_1} a_1 + \frac{\partial F}{\partial \xi_2} a_2 + \frac{\partial F}{\partial \xi_3} a_3 + \frac{\partial F}{\partial \xi_4} a_4 \right)$$

TETRA4 – strains

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$$\boldsymbol{\epsilon}^e(\mathbf{x}) = \mathbf{D} \ \mathbf{u}^e(\mathbf{x}) = \mathbf{D} \ \mathbf{N}^e(\mathbf{x}) \ \mathbf{q}^e = \mathbf{B}(\mathbf{x}) \ \mathbf{q}^e$$

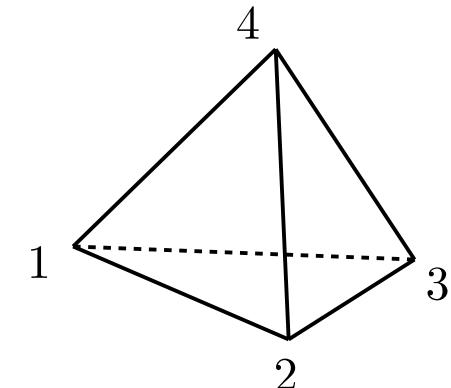


$$\boldsymbol{\epsilon}^e(\mathbf{x}) = \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_z \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix} \quad \mathbf{D} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \\ 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} \end{bmatrix} \quad \mathbf{q}^e = \begin{Bmatrix} u_1 \\ v_1 \\ w_1 \\ u_2 \\ v_2 \\ w_2 \\ u_3 \\ v_3 \\ w_3 \\ u_4 \\ v_4 \\ w_4 \end{Bmatrix}$$

TETRA4 – strains

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$$\boldsymbol{\epsilon}^e(\mathbf{x}) = \mathbf{D} \ \mathbf{u}^e(\mathbf{x}) = \mathbf{D} \ \mathbf{N}^e(\mathbf{x}) \ \mathbf{q}^e = \mathbf{B}(\mathbf{x}) \ \mathbf{q}^e$$



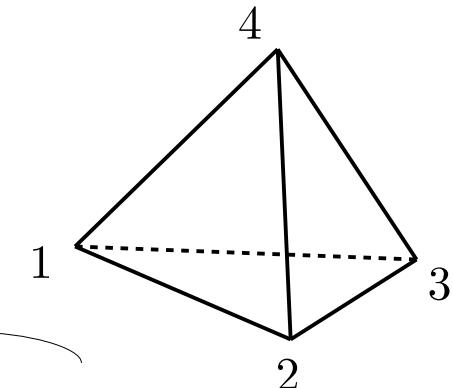
$$\mathbf{B} = \frac{1}{6V} \begin{bmatrix} a_1 & 0 & 0 & a_2 & 0 & 0 & a_3 & 0 & 0 & a_4 & 0 & 0 \\ 0 & b_1 & 0 & 0 & b_2 & 0 & 0 & b_3 & 0 & 0 & b_4 & 0 \\ 0 & 0 & c_1 & 0 & 0 & c_2 & 0 & 0 & c_3 & 0 & 0 & c_4 \\ b_1 & a_1 & 0 & b_2 & a_2 & 0 & b_3 & a_3 & 0 & b_4 & a_4 & 0 \\ c_1 & 0 & a_1 & c_2 & 0 & a_2 & c_3 & 0 & a_3 & c_4 & 0 & a_4 \\ 0 & c_1 & b_1 & 0 & c_2 & b_2 & 0 & c_3 & b_3 & 0 & c_4 & b_4 \end{bmatrix}$$

Constant strain a TETRA4!

TETRA4 – stresses

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$$\boldsymbol{\sigma}^e(\mathbf{x}) = \mathbf{H}_{3D} \boldsymbol{\epsilon}^e(\mathbf{x}) = \boxed{\mathbf{H}_{3D}} \mathbf{B}(\mathbf{x}) \mathbf{q}^e$$



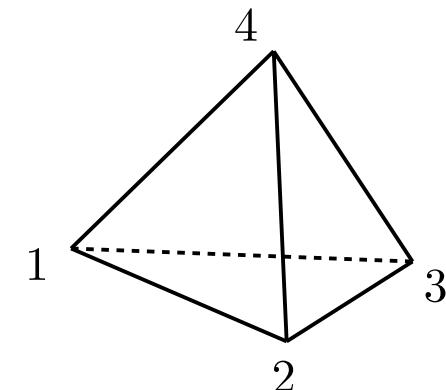
$$\left\{ \begin{array}{l} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{xz} \\ \tau_{yz} \end{array} \right\} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} \left[\begin{array}{cccccc} 1 & \frac{\nu}{1-\nu} & \frac{\nu}{1-\nu} & 0 & 0 & 0 \\ \frac{\nu}{1-\nu} & 1 & \frac{\nu}{1-\nu} & 0 & 0 & 0 \\ \frac{\nu}{1-\nu} & \frac{\nu}{1-\nu} & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2\nu}{2(1-\nu)} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2\nu}{2(1-\nu)} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2\nu}{2(1-\nu)} \end{array} \right] \left\{ \begin{array}{l} \epsilon_x \\ \epsilon_y \\ \epsilon_z \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{array} \right\}$$

Constant stress in a TETRA4!

TETRA4 – stiffness matrix

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$$\mathbf{K}^e = \int_{V^e} \mathbf{B}^T \mathbf{H} \mathbf{B} dV = \mathbf{B}^T \mathbf{H} \mathbf{B} V^e$$



$$V^e = \frac{1}{6} \det \begin{bmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{bmatrix}$$

- the simplest 3D solid FE
- results in constant strain and stress within a FE
- very fine meshes are required where strain gradients are large, therefore not often used (poor performance)