ECE 468: Digital Image Processing

Lecture 15

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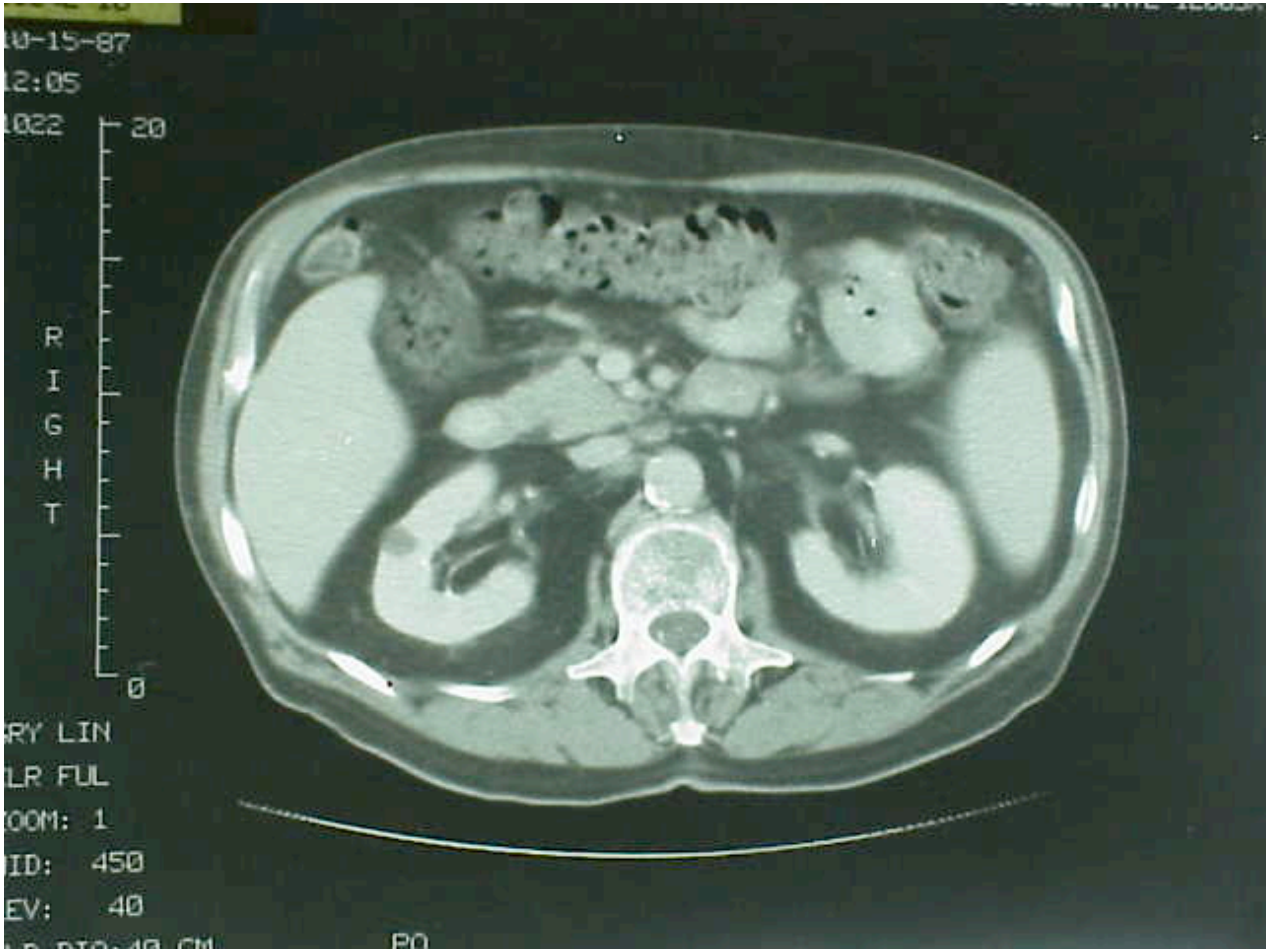
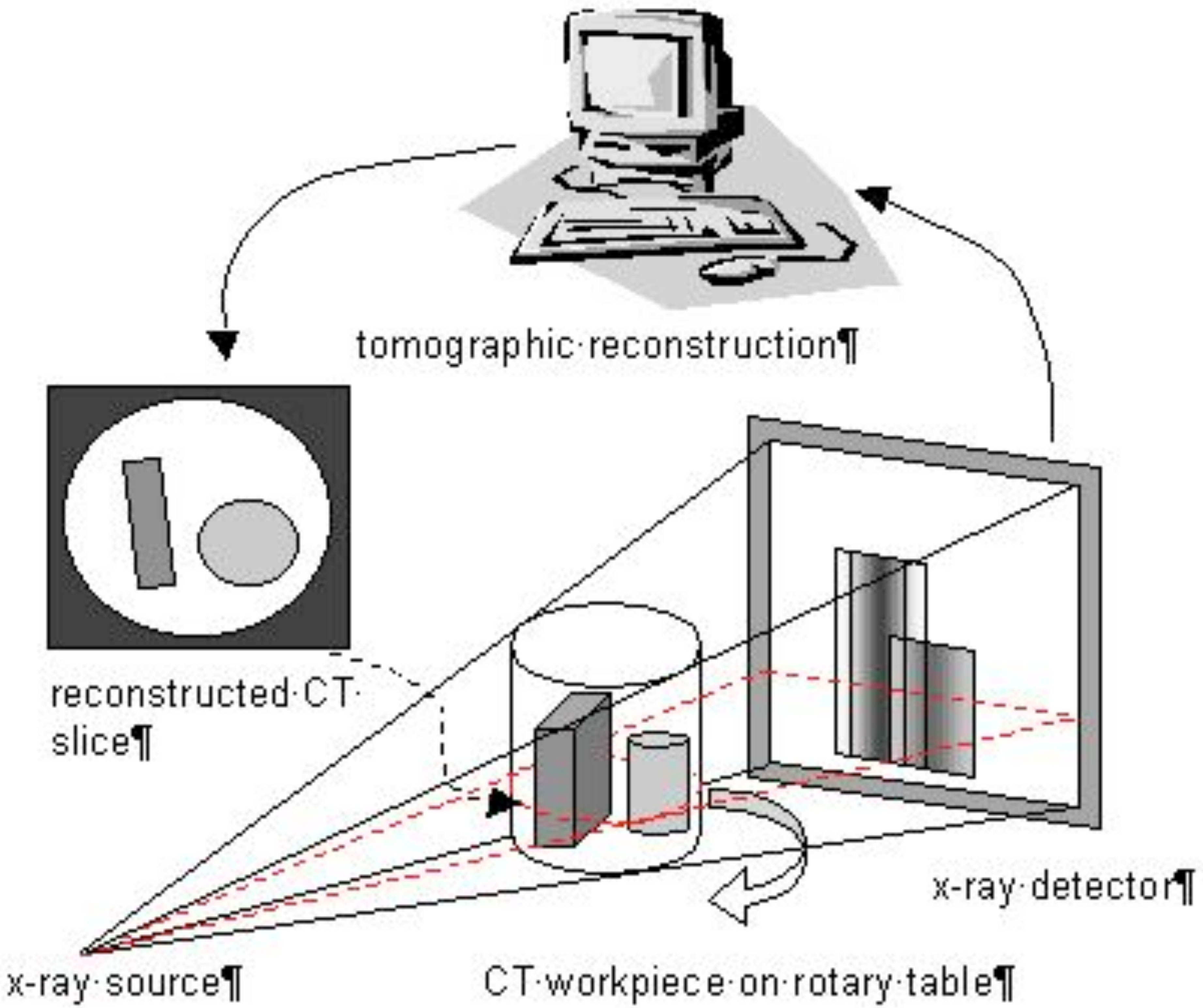
Outline

- Image reconstruction from projections (Textbook 5.11)
- Radon Transform (Textbook 5.11.3)
- Fourier-Slice Theorem (Textbook 5.11.4)

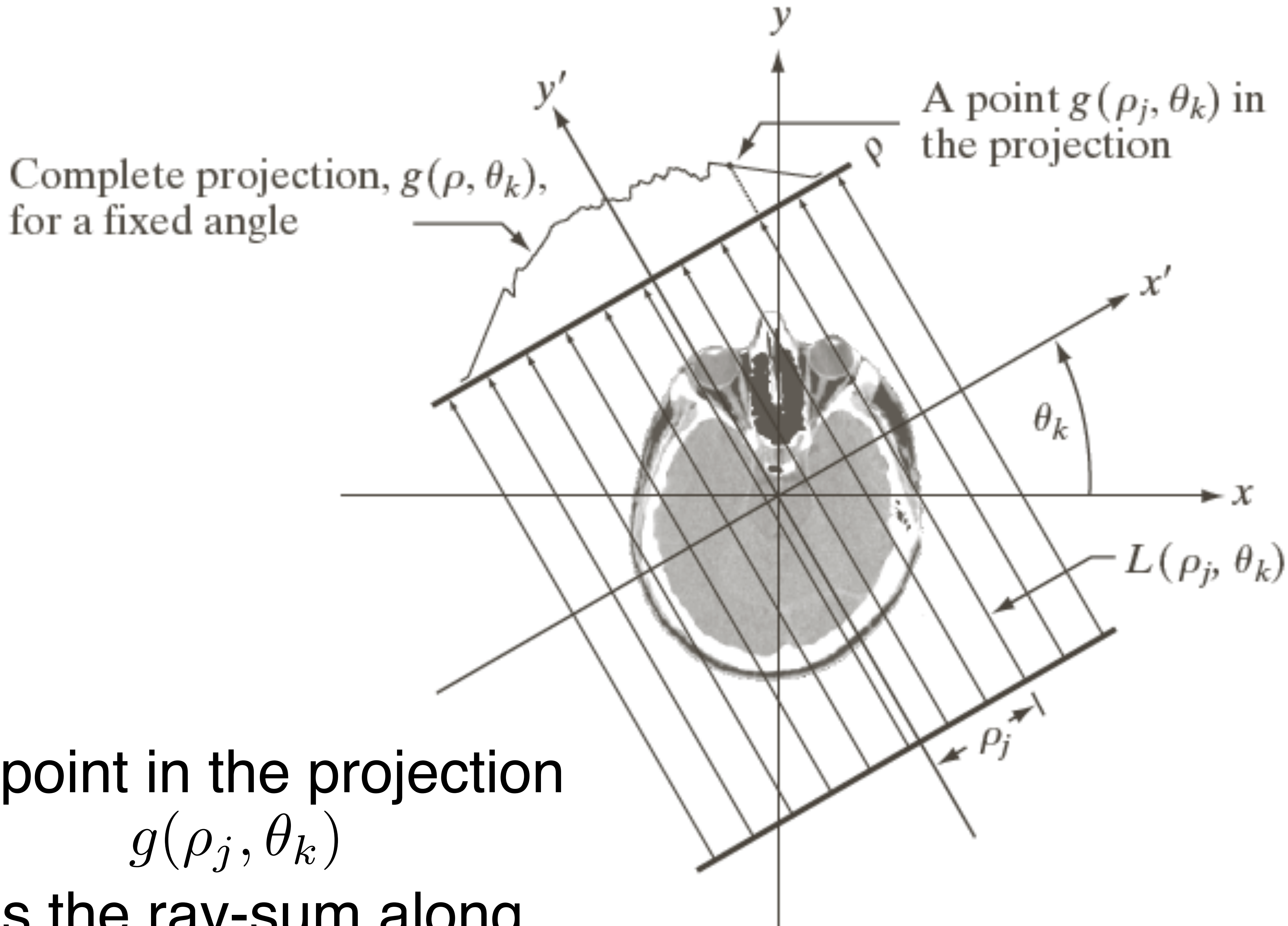
Computed Tomography



Computed Tomography



Radon Transform



A point in the projection

$$g(\rho_j, \theta_k)$$

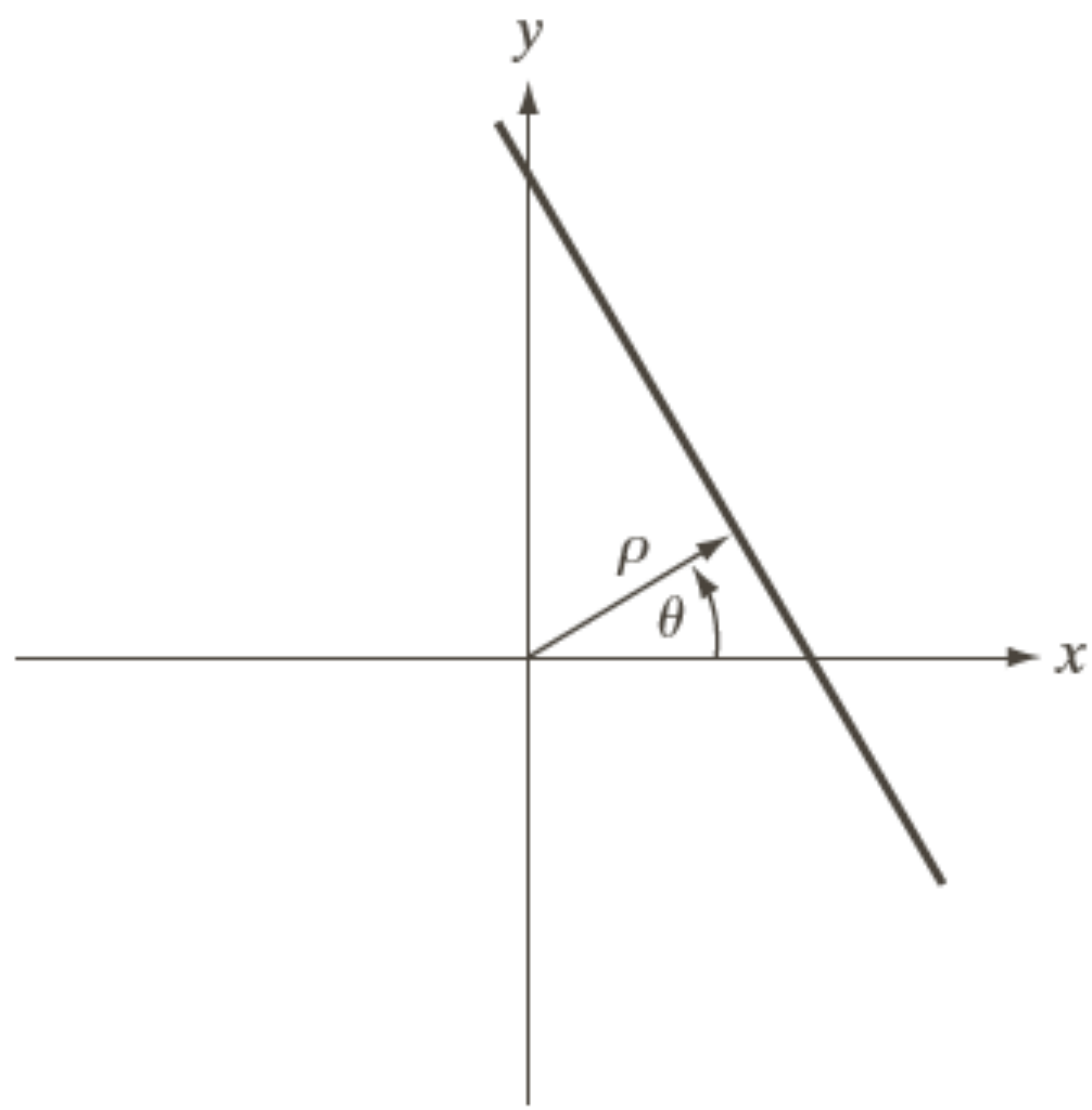
is the ray-sum along

$$x \cos\theta_k + y \sin\theta_k = \rho_j$$

Two Equivalent Definitions of the Line

$$y = ax + b$$

$$x\cos\theta + y\sin\theta = \rho$$



Radon Transform

$$g(\rho, \theta) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \delta(x \cos\theta + y \sin\theta - \rho) dx dy$$

continuous space coordinates

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continuous space coordinates

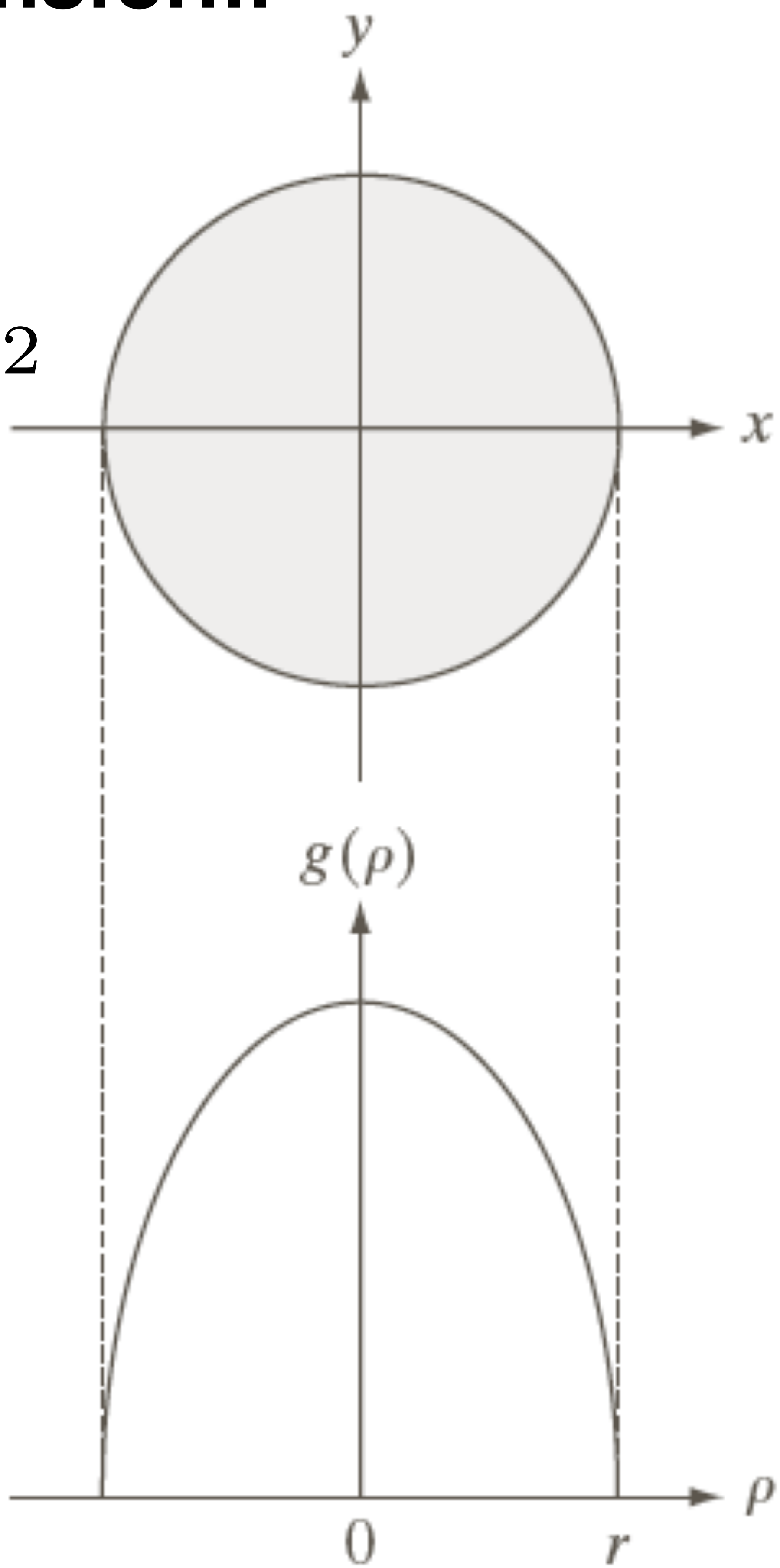
$$g(\rho, \theta) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y) \delta(x \cos \theta + y \sin \theta - \rho)$$

discrete space coordinates

Example: Radon Transform

$$f(x,y) = \begin{cases} A & , \quad x^2+y^2 \le r^2 \\ 0 & , \quad \text{o.w} \end{cases}$$

$$g(\rho,\theta) = ?$$



Example: Radon Transform

$$\theta = 0$$

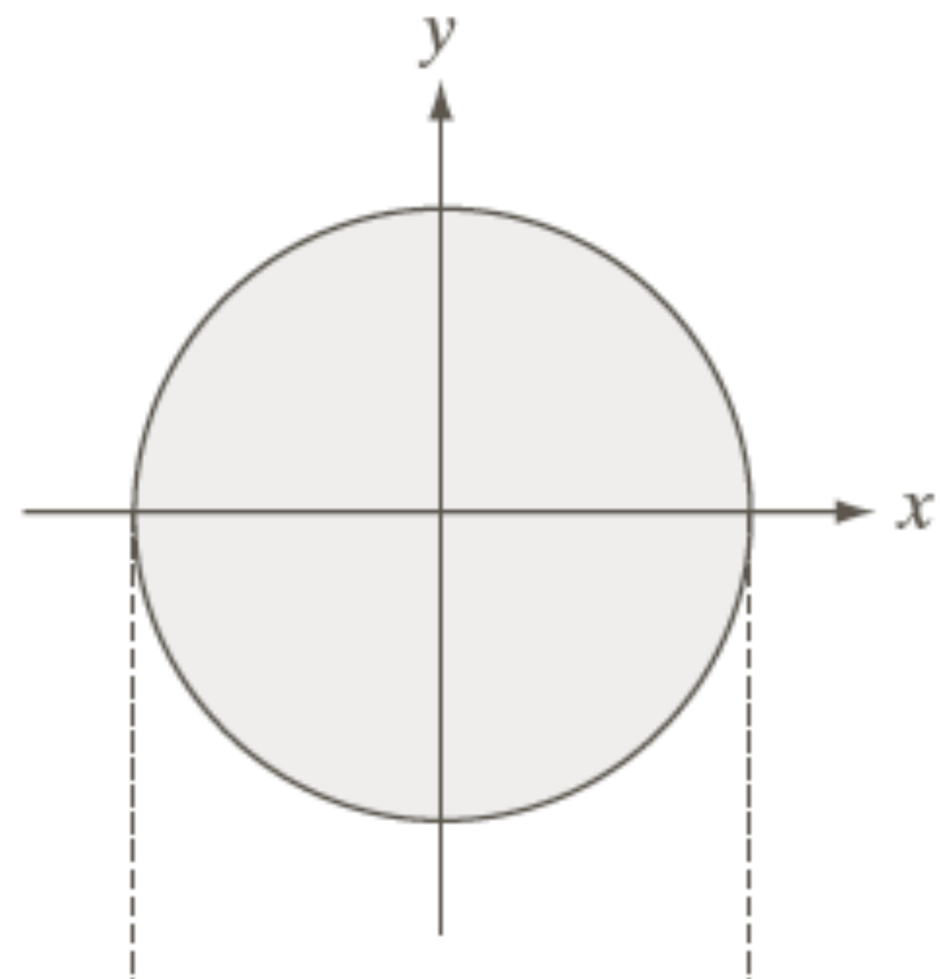
$$g(\rho, \theta) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y)\delta(x - \rho) dx dy$$

$$= \int_{-\infty}^{\infty} f(\rho, y) dy$$

Example: Radon Transform

$$g(\rho, \theta) = \int_{-\sqrt{r^2-\rho^2}}^{\sqrt{r^2-\rho^2}} f(\rho, y) dy$$

$$= \int_{-\sqrt{r^2-\rho^2}}^{\sqrt{r^2-\rho^2}} A dy$$



Properties of the Radon Transform

$$g(\rho, \theta + 180^\circ) \quad =$$

Properties of the Radon Transform

$$g(\rho, \theta + 180^\circ) \quad = \quad \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \delta(x \cos(\theta + 180^\circ) + y \sin(\theta + 180^\circ) - \rho) \; dx \; dy$$

Properties of the Radon Transform

$$
\begin{aligned}
g(\rho, \theta + 180^\circ) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\delta(x\cos(\theta+180^\circ) + y\sin(\theta+180^\circ) - \rho)\ dx\ dy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\delta(-x\cos\theta - y\sin\theta - \rho)\ dx\ dy \\
&= g(-\rho, \theta)
\end{aligned}
$$

Sinogram = Image of Radon Transform

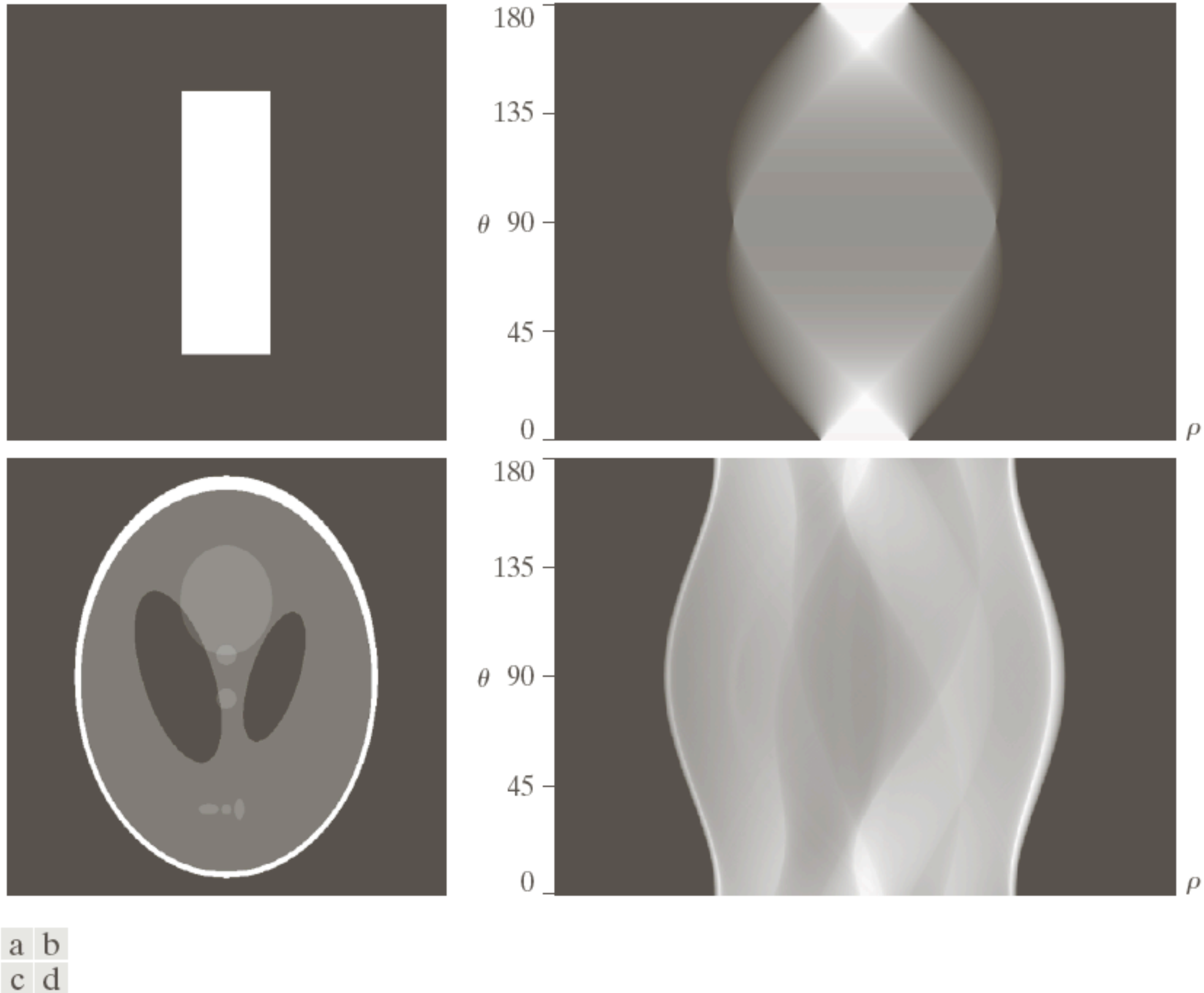


FIGURE 5.39 Two images and their sinograms (Radon transforms). Each row of a sinogram is a projection along the corresponding angle on the vertical axis. Image (c) is called the *Shepp-Logan phantom*. In its original form, the contrast of the phantom is quite low. It is shown enhanced here to facilitate viewing.

Properties of Objects from Sinogram

- Sinogram symmetric = Object symmetric

- Sinogram symmetric about image center = Object symmetric and parallel to x and y axes

- Sinogram smooth = Object has uniform intensity

Outline

- Fourier-Slice Theorem (Textbook 5.11.4)

Computed Tomography (CT)

Key objective: Reconstruct $f(x,y)$

from its projections $g(\rho,\theta)$

Fourier Slice Theorem

relates

1D Fourier Transform of the projection

with

2D Fourier Transform of the original image

1D Fourier Transform of the Projection

$$G(\omega, \theta) = \int_{-\infty}^{\infty} g(\rho, \theta) \mathrm{e}^{-j2\pi\omega\rho} d\rho$$

1D FT of the Projection -- Properties

$$G(\omega, \theta + 180^\circ) = ?$$

1D FT of the Projection -- Properties

$$G(\omega, \theta + 180^\circ) = ?$$

$$\begin{aligned} G(\omega, \theta + 180^\circ) &= \int_{-\infty}^{\infty} g(\rho, \theta + 180^\circ) \mathrm{e}^{-j2\pi\omega\rho} d\rho \\ &= \int_{-\infty}^{\infty} g(-\rho, \theta) \mathrm{e}^{-j2\pi\omega\rho} d\rho \end{aligned}$$

1D FT of the Projection -- Properties

$$G(\omega, \theta + 180^\circ) = ?$$

1D FT of the Projection -- Properties

$$G(\omega, \theta + 180^\circ) = ?$$

$$= \quad -\int_{\infty}^{-\infty} g(\rho, \theta) \mathrm{e}^{j2\pi\omega\rho} d\rho$$

$$= \quad \int_{-\infty}^{\infty} g(\rho, \theta) \mathrm{e}^{-j2\pi(-\omega)\rho} d\rho$$

$$= \quad G(-\omega, \theta)$$

Fourier Slice Theorem

$$G(\omega,\theta) = \int_{-\infty}^{\infty} g(\rho,\theta)\mathrm{e}^{-j2\pi\omega\rho} d\rho$$

Fourier Slice Theorem

$$G(\omega, \theta) = \int_{-\infty}^{\infty} g(\rho, \theta) \mathrm{e}^{-j2\pi\omega\rho} d\rho$$

by definition

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \delta(x \cos\theta + y \sin\theta - \rho) \mathrm{e}^{-j2\pi\omega\rho} dx\ dy\ d\rho$$

Fourier Slice Theorem

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$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \mathrm{e}^{-j2\pi\omega(x\cos\theta + y\sin\theta)} dx\ dy$$

Fourier Slice Theorem

$$G(\omega, \theta) = \int_{-\infty}^{\infty} g(\rho, \theta) \mathrm{e}^{-j2\pi\omega\rho} d\rho$$

by definition

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \mathrm{e}^{-j2\pi\omega(x\cos\theta + y\sin\theta)} dx\ dy$$

$$= F(\omega\cos\theta, \omega\sin\theta)$$

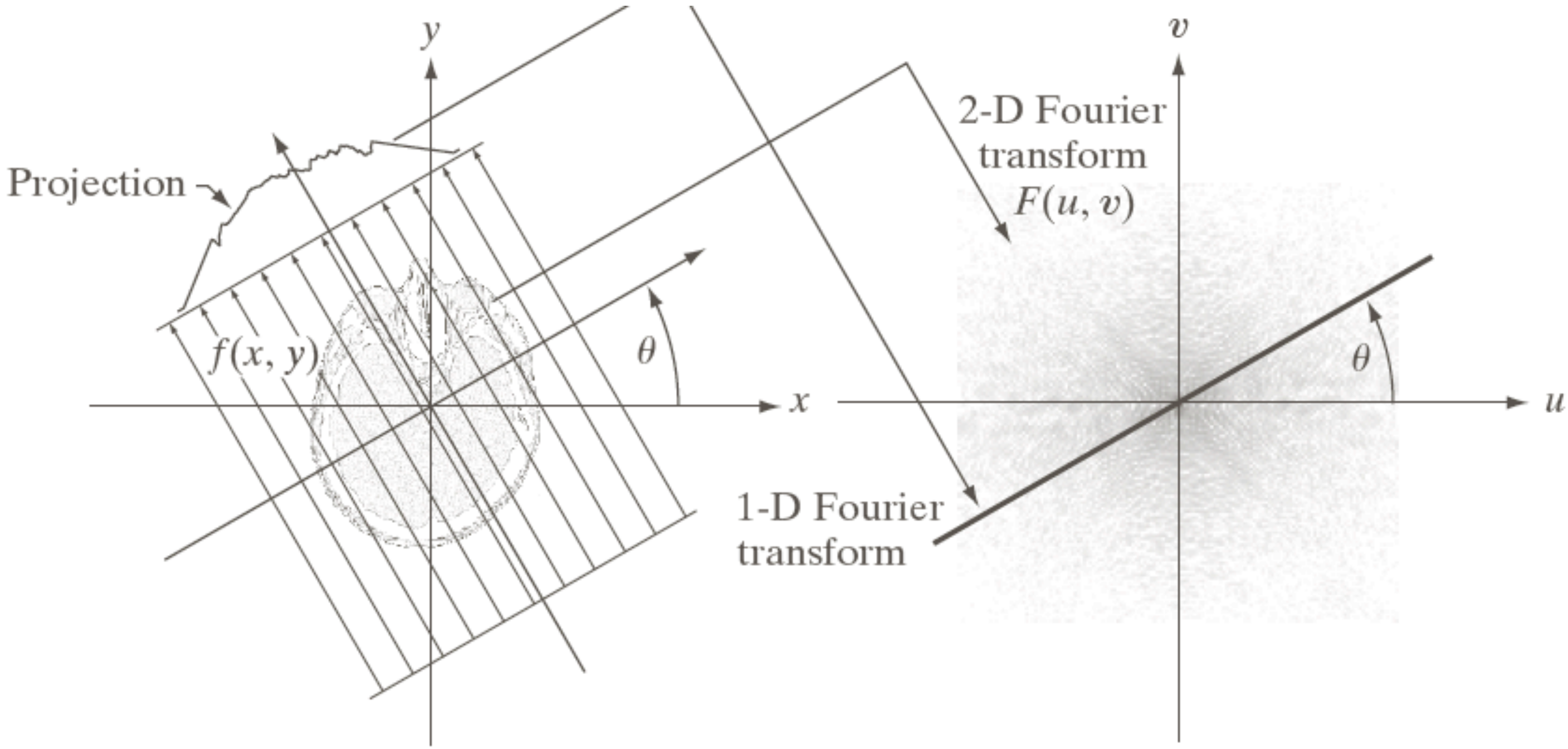
Fourier Slice Theorem relates

1D Fourier Transform of the projection

with

2D Fourier Transform of the original image

Fourier Slice Theorem



1D FT = a slice of 2D FT

Reconstruction Using Backprojections

Given $g(\rho, \theta)$, that is $G(\omega, \theta)$

find $f(x, y)$

Reconstruction Using Backprojections

by definition

$$f(x,y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(u,v) \mathrm{e}^{j2\pi(ux+vy)} du\ dv$$

Reconstruction Using Backprojections

by definition

$$f(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v)\mathrm{e}^{j2\pi(ux+vy)}du\ dv$$

$\Downarrow$ polar coordinates in the frequency domain

$$u = \omega\cos\theta,\ \ v = \omega\sin\theta, \quad \Rightarrow \ \ dudv = \omega d\omega d\theta$$

$$f(x,y) = \int_{0}^{2\pi}\int_{0}^{\infty} F(\omega\cos\theta, \omega\sin\theta)\mathrm{e}^{j2\pi\omega(x\cos\theta + y\sin\theta)}\omega\ d\omega\ d\theta$$

Reconstruction Using Backprojections

$$f(x,y) = \int_0^{2\pi} \int_0^{\infty} F(\omega\cos\theta, \omega\sin\theta) \mathrm{e}^{j2\pi\omega(x\cos\theta + y\sin\theta)} \omega \; d\omega \; d\theta$$

Reconstruction Using Backprojections

$$f(x,y) = \int_0^{2\pi} \int_0^{\infty} F(\omega \cos\theta, \omega \sin\theta) \mathrm{e}^{j2\pi\omega(x\cos\theta + y\sin\theta)} \omega \; d\omega \; d\theta$$

by Fourier Slice Theorem

$$f(x,y) = \int_0^{2\pi} \int_0^{\infty} G(\omega, \theta) \mathrm{e}^{j2\pi\omega(x\cos\theta + y\sin\theta)} \omega \; d\omega \; d\theta$$

Reconstruction Using Backprojections

$$f(x,y) = \int_0^{2\pi} \int_0^{\infty} G(\omega,\theta) \mathrm{e}^{j2\pi\omega(x\cos\theta + y\sin\theta)} \omega \; d\omega \; d\theta$$

Reconstruction Using Backprojections

$$f(x,y) = \int_0^{2\pi} \int_0^{\infty} G(\omega,\theta) \mathrm{e}^{j2\pi\omega(x\cos\theta + y\sin\theta)} \omega \; d\omega \; d\theta$$

$$\Downarrow \qquad G(\omega, \theta + 180^\circ) = G(-\omega, \theta)$$

$$f(x,y) = \int_0^{\pi} \int_0^{\infty} |\omega| G(\omega,\theta) \mathrm{e}^{j2\pi\omega(x\cos\theta + y\sin\theta)} \; d\omega \; d\theta$$

Reconstruction Using Backprojections

Reconstruction Using Backprojections

$$f(x,y) = \int_0^{\pi} \left[\int_0^{\infty} |\omega| G(\omega,\theta) \mathrm{e}^{j2\pi\omega\rho} \, d\omega \right]_{\rho = x\cos\theta + y\sin\theta} d\theta$$

Reconstruction Using Backprojections

$$f(x,y) = \int_0^{\pi} \left[\int_0^{\infty} |\omega| G(\omega,\theta) \mathrm{e}^{j2\pi\omega\rho} \; d\omega \right]_{\rho = x\cos\theta + y\sin\theta} d\theta$$

1D filtering

Box + Ramp Filter

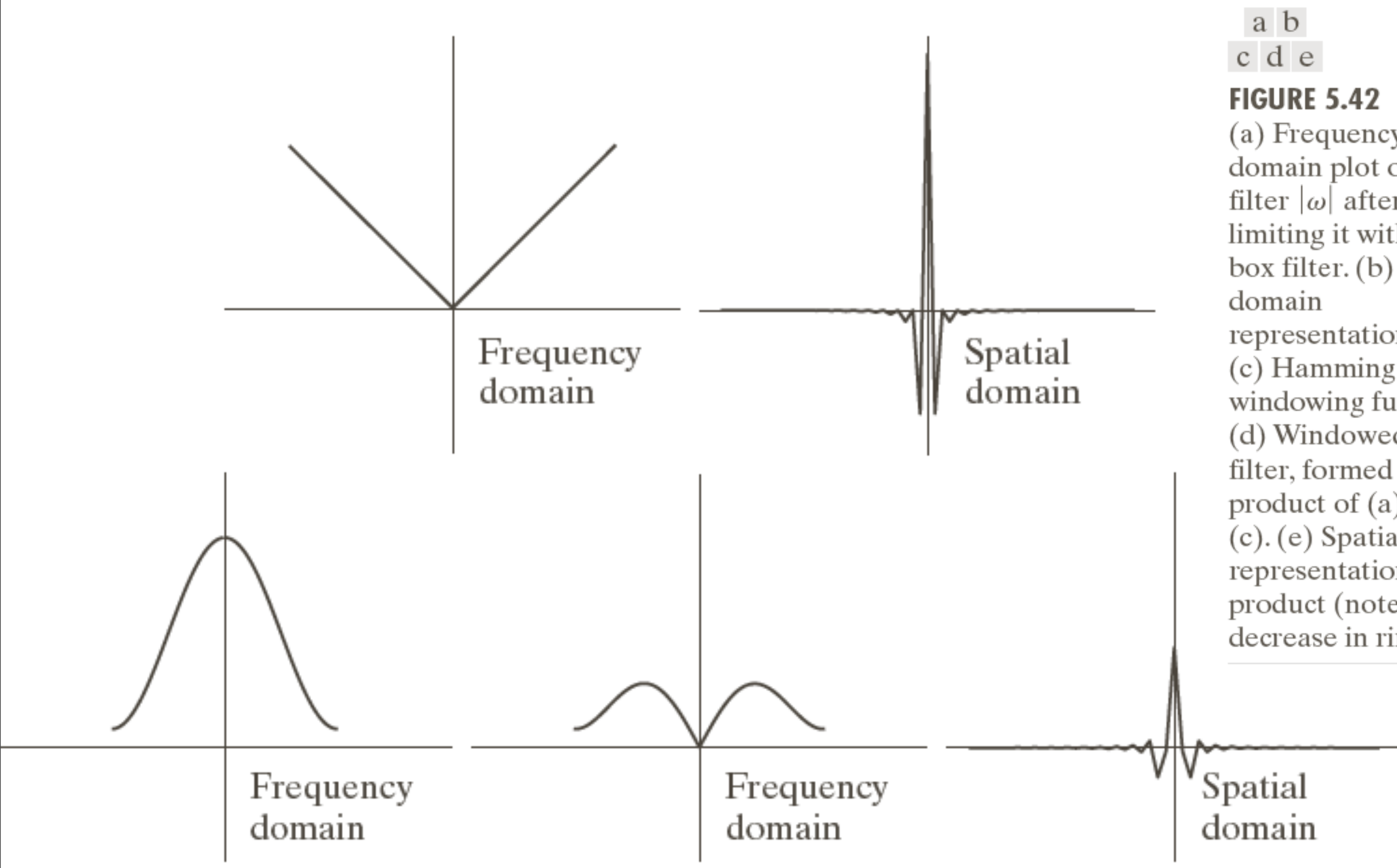


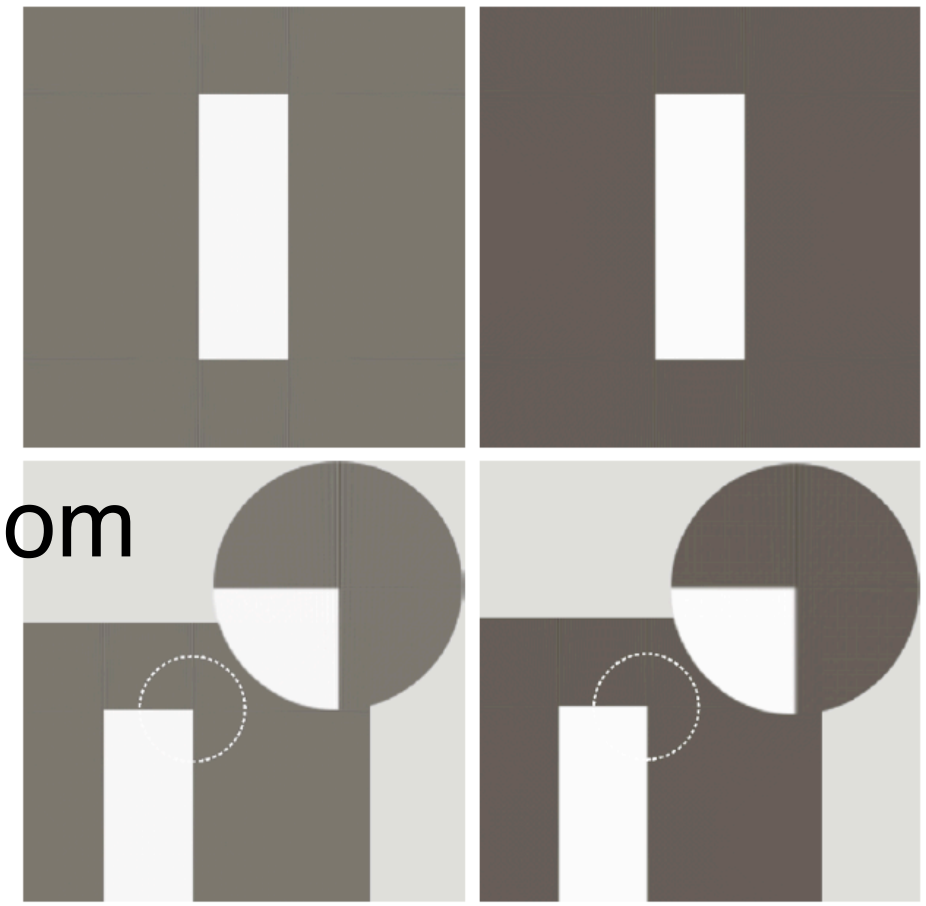
FIGURE 5.42 (a) Frequency domain plot of the filter $|\omega|$ after band-limiting it with a box filter. (b) Spatial domain representation. (c) Hamming windowing function. (d) Windowed ramp filter, formed as the product of (a) and (c). (e) Spatial representation of the product (note the decrease in ringing).

Algorithm for Filtered Backprojection

1. Given projections $g(\rho,\theta)$ obtained at each fixed angle θ
2. Compute $G(\omega,\theta)$ = 1D Fourier Transform of each projection $g(\rho,\theta)$
3. Multiply $G(\omega,\theta)$ by the filter function $|\omega|$ modified by Hamming window
4. Compute the inverse of the results from 3.
5. Integrate (sum) over θ all results from 4.

Examples

zoom



ramp filter

windowed ramp filter

ramp filter

windowed ramp filter