Eigen decomposition of a matrix

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What is Eigen?

Eigen means own or self. In linear algebra, eigenvalue, eigenvector and Eigen decomposition are terms that are intrinsically related. Eigen decomposition is the method to decompose a square matrix into its eigenvalues and eigenvectors. For a matrix A, if.

$$Av=\lambda v \quad (1)$$

Then v is an eigenvector of matrix A and λ is the corresponding eigenvalue. That is, if matrix A is multiplied by a vector and the result is a scaled version of the same vector, then it is an eigenvector of A and the scaling factor is its eigenvalue.

Eigendecomposition

So how do we find the eigenvectors of a matrix? From (1):

$$Av-\lambda Iv=0$$
$$(A-\lambda I)v=0 \quad (2)$$

Where I is the identity matrix. The values of λ where (2) holds are the eigenvalues of A. It turns out that this equation is equivalent to:

$$\det(A-\lambda I)=0 \quad (3)$$

Where det() is the determinant of a matrix.

Why is eigendecomposition useful?

we can join both eigenvectors and eigenvalues in a single matrix equation

$$A\left[\mathbf{v_1}\mathbf{v_2}\right] = \begin{bmatrix} v_{11} & v_{21} \\ v_{12} & v_{22} \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \left[\mathbf{v_1}\mathbf{v_2}\right] \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

If we replace:

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

$$V = \left[\mathbf{v_1}\mathbf{v_2}\right]$$

it is also true that:

$$AV = V\Lambda$$
$$A = V\Lambda V^{-1} \quad (4)$$

Eigen decomposition decomposes a matrix A into a multiplication of a matrix of eigenvectors V and a diagonal matrix of eigenvalues Λ. This can only be done if a matrix is diagonalizable. In fact, the definition of a diagonalizable matrix $A \in R_{n\times n}$ is that it can be Eigen decomposed into n eigenvectors, so that $V^{-1}AV=\Lambda$.

Power of a matrix with Eigen decomposition

From (4):

$$A^2=V\Lambda V^{-1}V\Lambda V^{-1}=V\Lambda^2 V^{-1}$$
$$A^n=V\Lambda^n V^{-1}$$

The power of Λ is just the power of each diagonal element. This becomes much simpler than multiplications of A.

Properties of Eigen decomposition

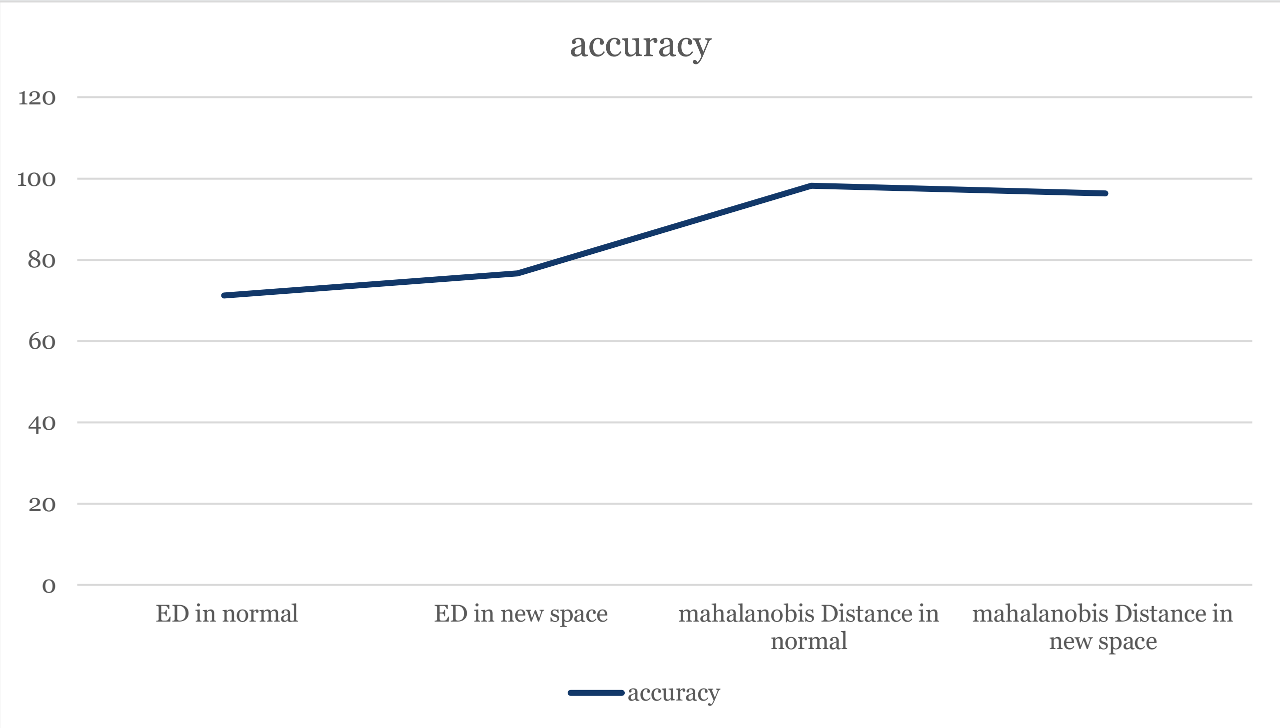
- $det(A) = \prod_{i=1}^{n} \lambda_i$ (the determinant of A is equal to the product of its eigenvalues)
- $tr(A) = \sum_{i=1}^{n} \lambda_i$ (the trace of A is equal to the sum of its eigenvalues)
- The eigenvalues of A^{-1} are λ_i^{-1}
- The eigenvalues of A^n are λ_i^n
- In general, the eigenvalues of $f(A)$ are $f(\lambda_i)$
- The eigenvectors of A^{-1} are the same as the eigenvectors of A.
- if A is hermitian (its conjugate transpose is equal to itself) and full-rank (all rows or columns are linearly independent), then the eigenvectors are mutually orthogonal (the dot-product between any two eigenvectors is zero) and the eigenvalues are real.
- A is invertible if all its eigenvalues are different from zero and vice-versa.
- If the eigenvalues of matrix A are distinct (not repeated), then A can be eigendecomposed.

More details can find https://en.wikipedia.org/wiki/QR_decomposition
And https://medium.com/machine-learning-mindset/matrix-eigendecomposition-its-importance-and-the-applications-26c92f76731

The Dataset : wine

Classes : 3
Samples per class : [59,71,48]
Samples total : 178
Dimensionality : 13
Features : real, positive

We change the space of data by Φ(x) = Q'*landa^(1/2)*x and use ED distance to see the effect of this change on our Dataset:



It's totally like using Mahalonobis distance.The Mahalanobis distance is a measure of the distance between a point P and a distribution D. It is a multi-dimensional generalization of the idea of measuring how many standard deviations away P is from the mean of D. This distance is zero if P is at the mean of D, and grows as P moves away from the mean along each principal component axis. If each of these axes is re-scaled to have unit variance, then the Mahalanobis distance corresponds to standard Euclidean distance in the transformed space. The Mahalanobis distance is thus unit less and scale-invariant, and takes into account the correlations of the data set.

Thank you :)