Image filtering in the frequency domain

Václav Hlaváč

Czech Technical University in Prague

Czech Institute of Informatics, Robotics and Cybernetics

160 00 Prague 6, Jugoslávských partyzánů 1580/3, Czech Republic

http://people.ciirc.cvut.cz/hlavac, vaclav.hlavac@cvut.cz

also Center for Machine Perception, http://cmp.felk.cvut.cz

Outline of the talk:

- Convolution as filtration in frequency domain.
- Low pass filtering examples, sharp cut off, smooth Gaussian.
- High pass filtering examples, sharp cut off, smooth Gaussian.
- Butterworth filter.
- Homomorphic filter separating illumination and reflectance.
- Systematic design of 2D FIR filters.

Filtration in the frequency domain

1. $F(u,v) = \mathcal{F}\{f(x,y)\}$
2. $G(u,v) = H(u,v) \,.*\, F(u,v)$,

   where $.*$ denotes element-wise multiplication of matrices.
3. $g(x,y) = \mathcal{F}^{-1}\{G(u,v)\}$

Note for lab exercises: We usually use $\ln \|F(u,v)\|$ *for visualization purposes. The original spectrum* $F(u,v)$ *has to be used in the actual filtration.*

Convolution as Fourier spectrum frequency filtration

- Matrix element by element multiplication.
- The speed of operations is determined by the (high) speed of FFT.

Consider a matrix a with dimensions $M \times N$ and a matrix b with dimensions $P \times Q$.

The convolution $c = a * b$ can be calculated as follows:

1. Fill in matrices a, b by zeroes to have dimensions $M + P - 1$, $N + Q - 1$ (usually up to the order of 2 to ease FFT).
2. Calculate 2D FFT matic of matrices a, b (in MATLAB, using `fft2`). The outcome are matrices A, B.
3. Multiply complex Fourier spectra element-wise, $C = A .* B$.
4. The result of the convolution c is obtained by the inverse Fourier transformation (in MATLAB using `ifft2`).

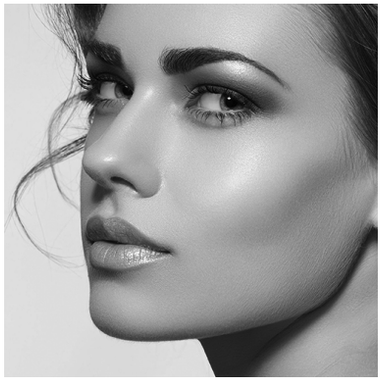
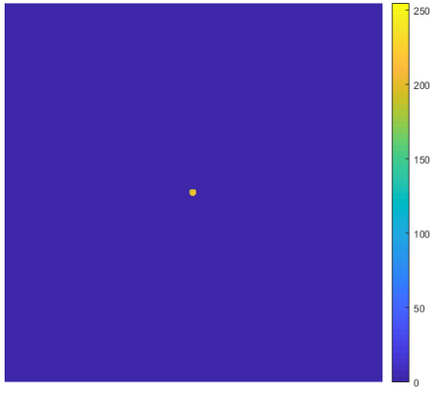
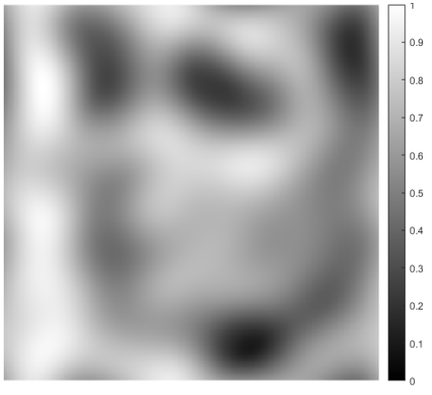
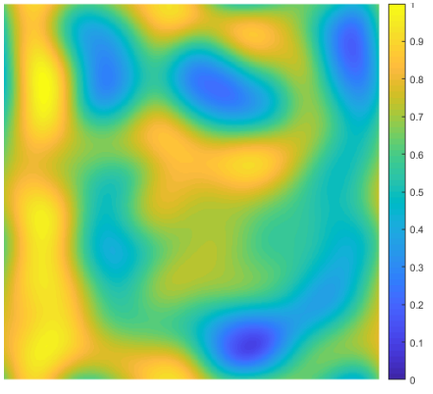
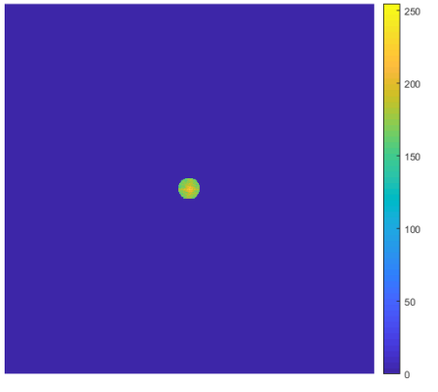
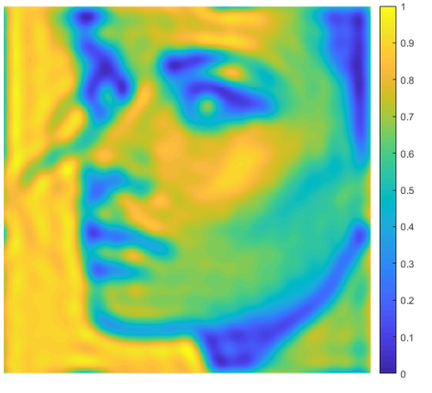
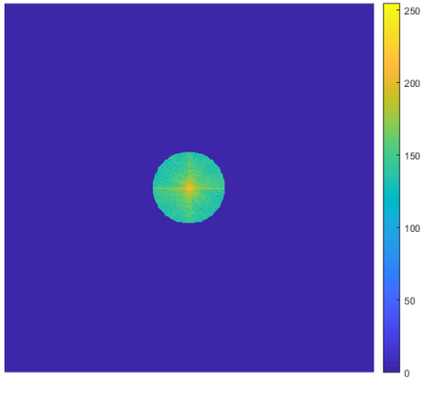
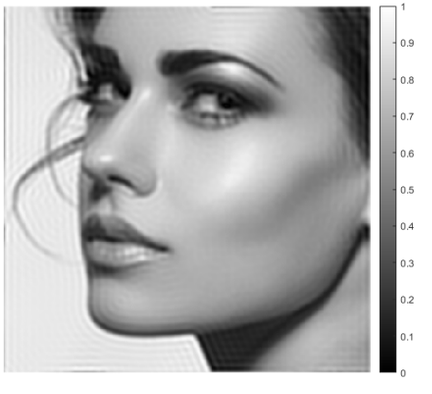
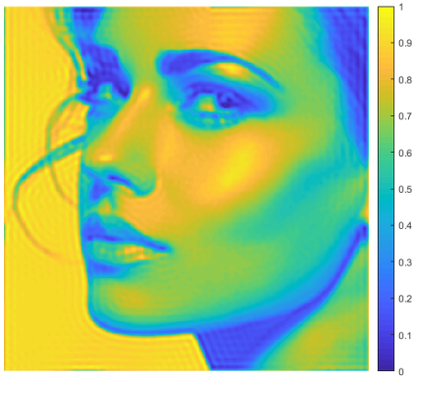
2D convolution in frequency domain in MATLAB


```
A = magic(3);

B = ones(3);

A(8,8) = 0;

B(8,8) = 0;

C = ifft2(fft2(A).*fft2(B));

C = C(1:5,1:5);

C = real(C)
```

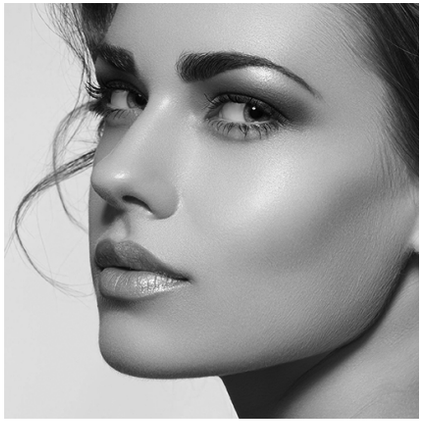
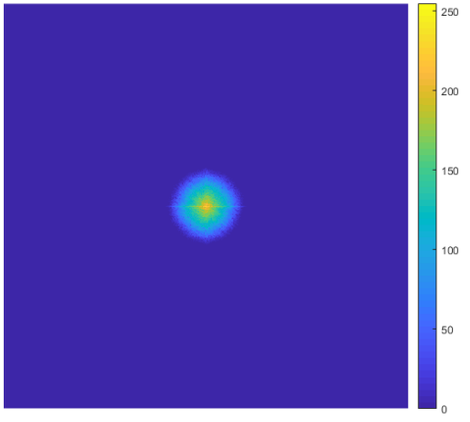
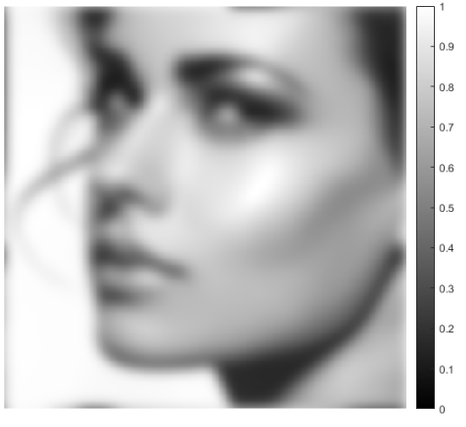
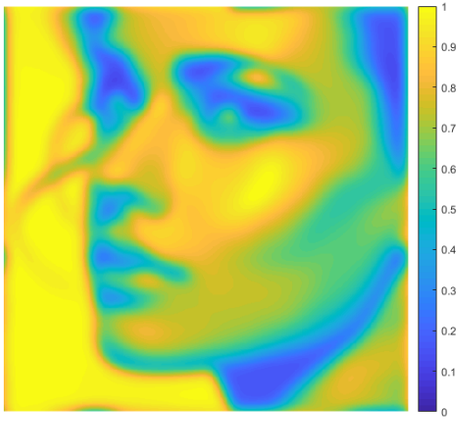
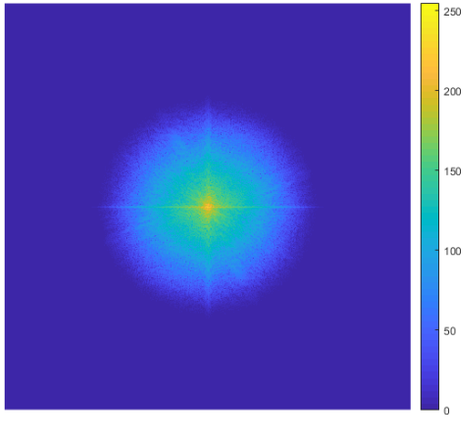
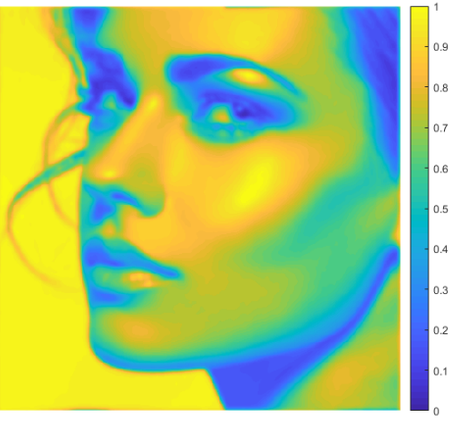
$$C = \begin{bmatrix} 8.0000 & 9.0000 & 15.0000 & 7.0000 & 6.0000 \\ 11.0000 & 17.0000 & 30.0000 & 19.0000 & 13.0000 \\ 15.0000 & 30.0000 & 45.0000 & 30.0000 & 15.0000 \\ 7.0000 & 21.0000 & 30.0000 & 23.0000 & 9.0000 \\ 4.0000 & 13.0000 & 15.0000 & 11.0000 & 2.0000 \end{bmatrix}$$

Note: the convolution calculated via spectra is faster in MATLAB for large matrices. The calculation via `conv2, filt2` is faster for small matrices.

Low pass filter, circle sharp cut-off, r=5, 15, 50

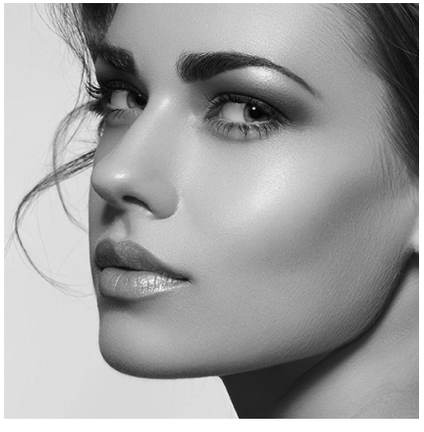
original	filter	output in gray	output in pseudocolor
			
			
			

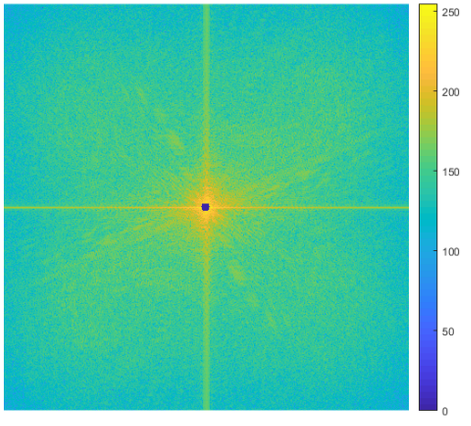
Low pass Gaussian filter, $\sigma = 10, 30$

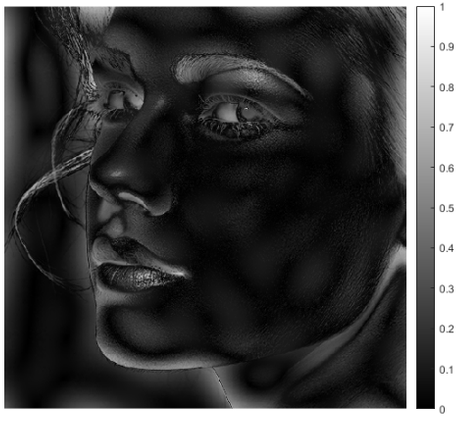
original	filter	output in gray	output in pseudocolor
			
			

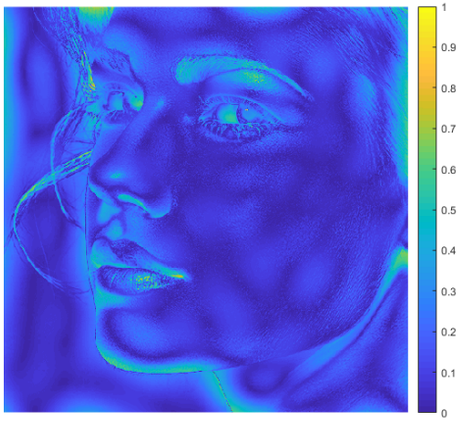
High pass circle sharp cut-off, r=5, 15

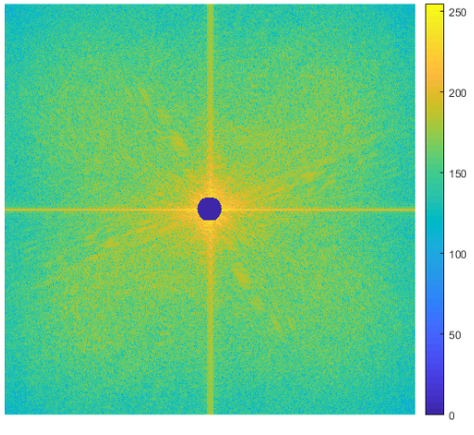
original | filter | output in gray | output in pseudocolor

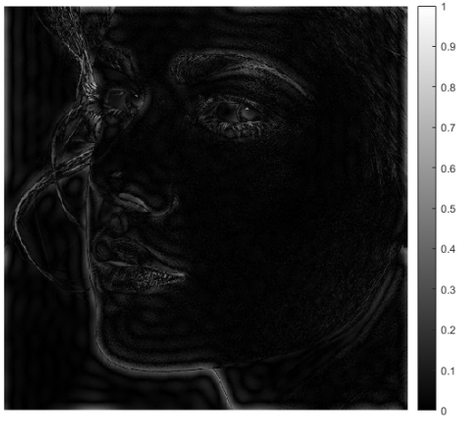


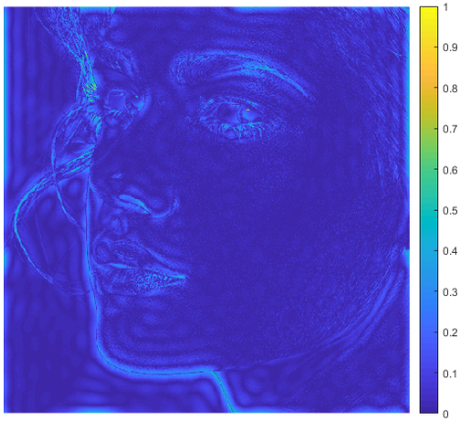




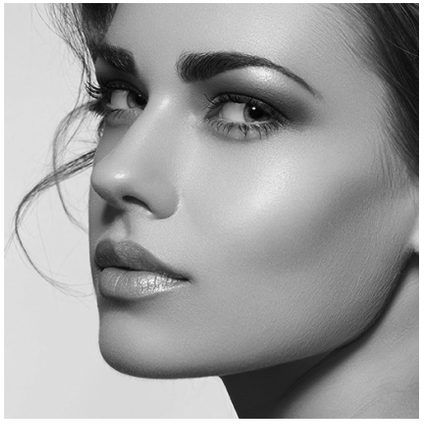
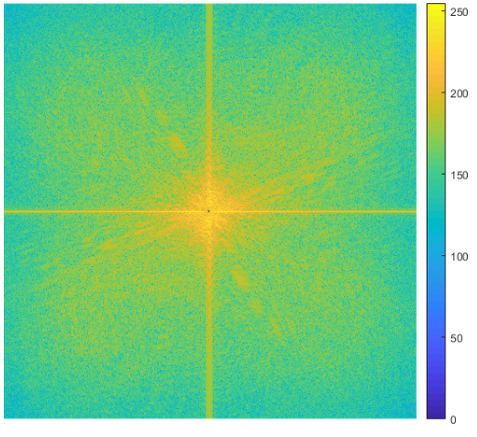
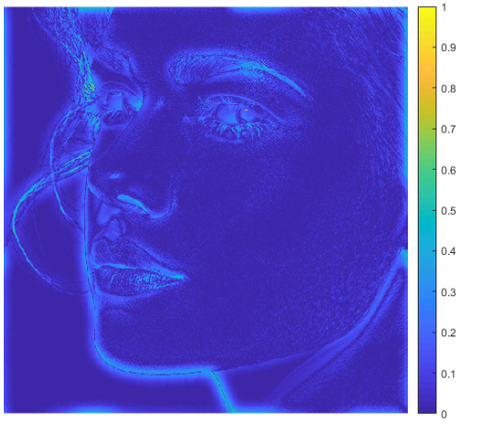
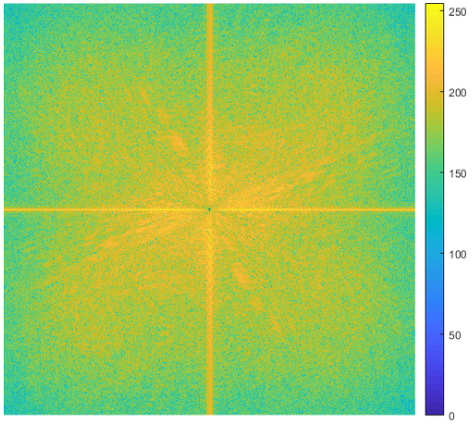
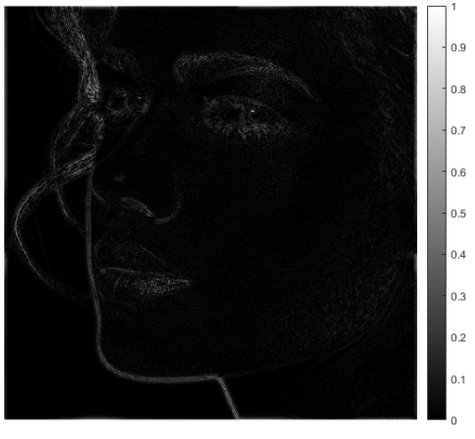
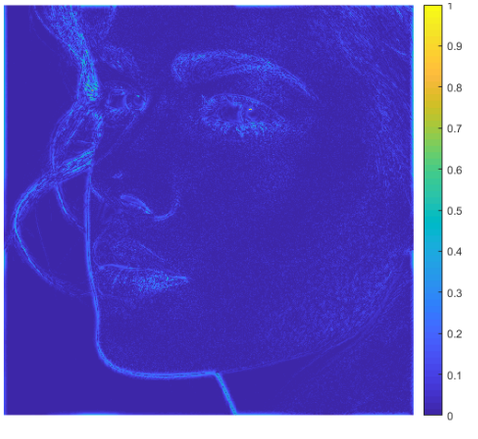








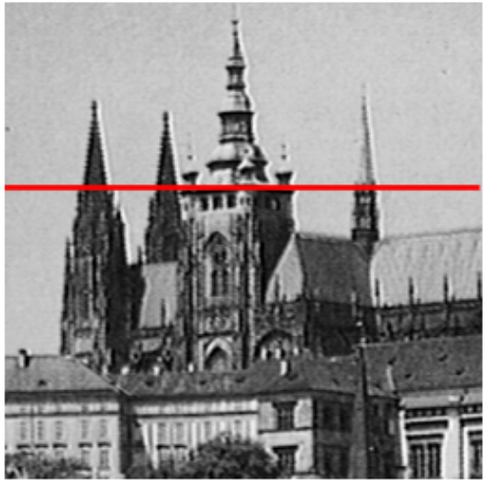
High pass Gaussian filter, σ=10, 30

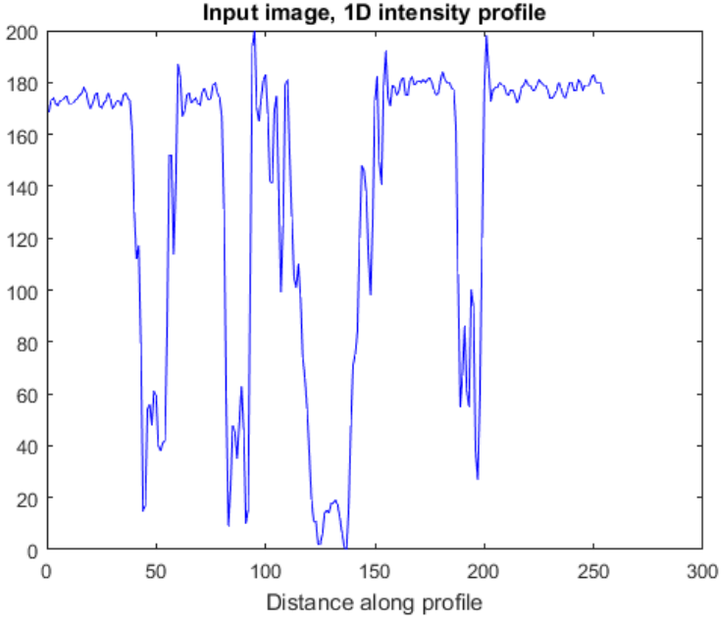
original	filter	output in gray	output in pseudocolor
			
			

Example, Fourier filtration, Gaussian filter

Input image

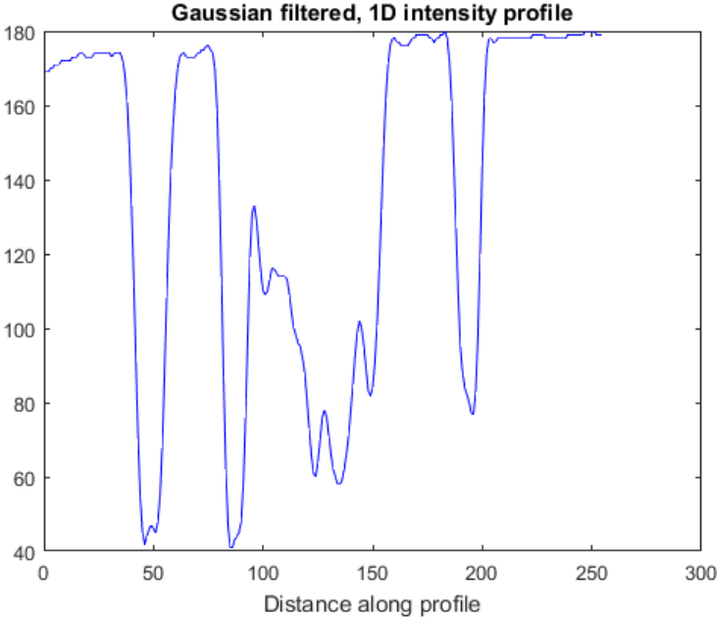
Input image, line for profile in red





Gaussian filtered

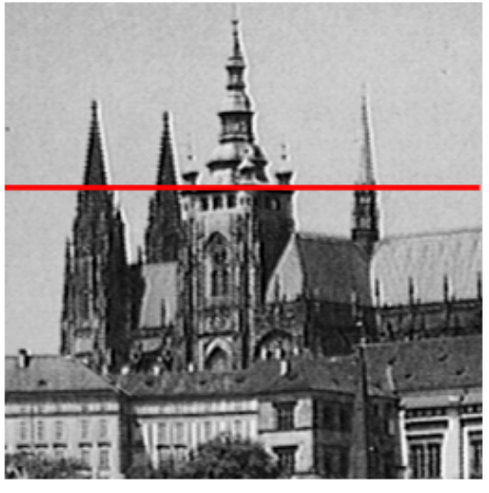
Gaussian filtered, line for profile in red

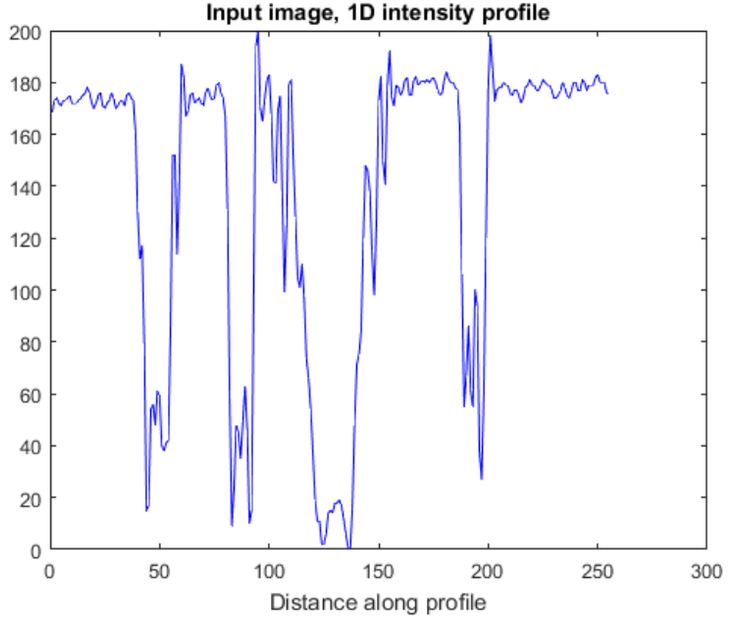


Example, Fourier filtration, Sobel filter

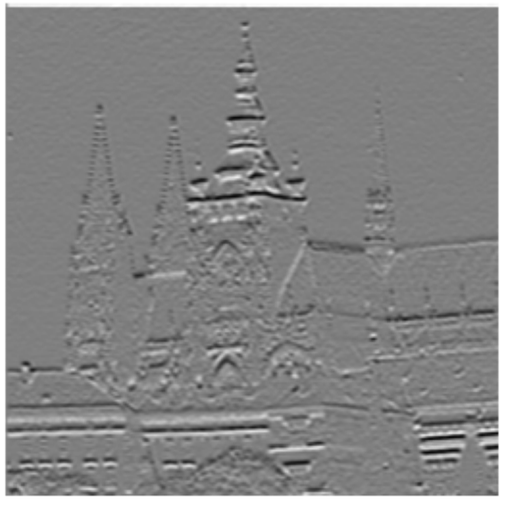
Input image

Input image, line for profile in red

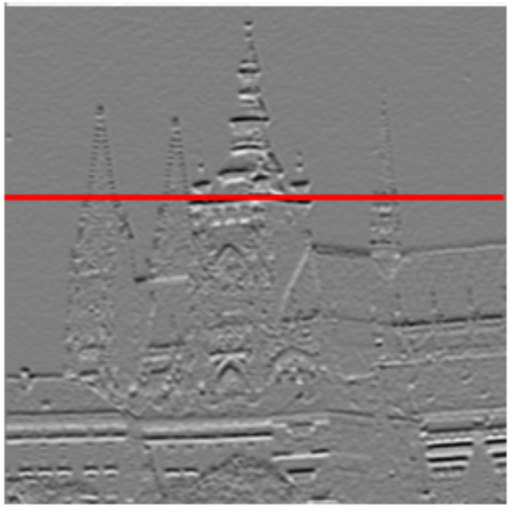


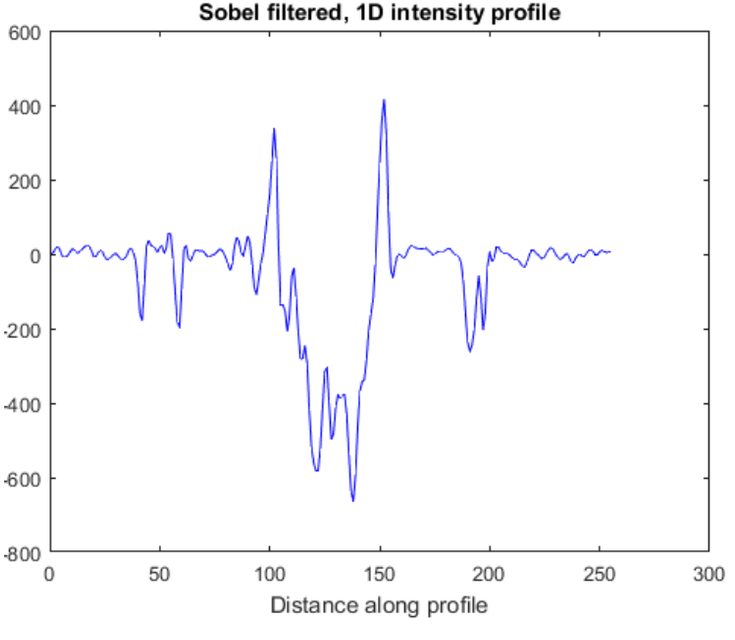


Sobel filtered



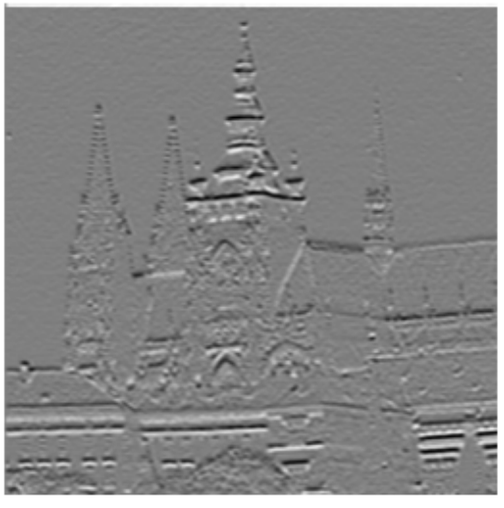
Sobel filtered, line for profile in red



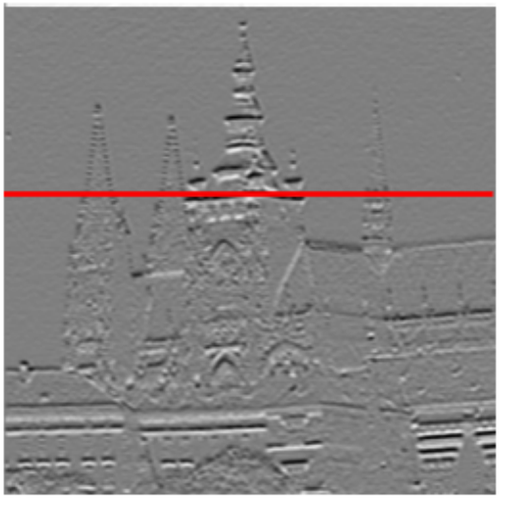


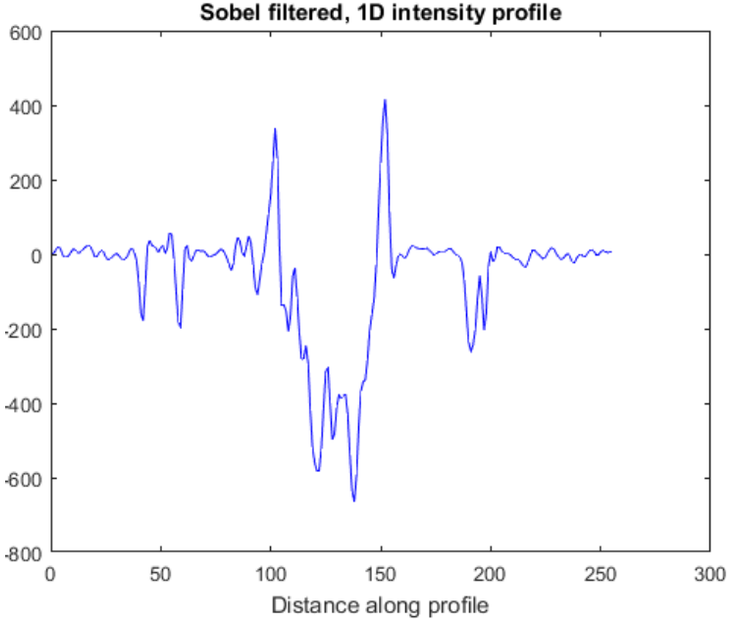
Example, Fourier filtration, Abs(Sobel filter)

Sobel filtered



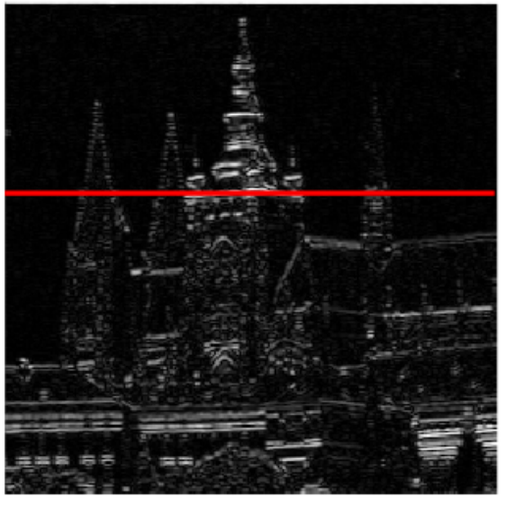
Sobel filtered, line for profile in red

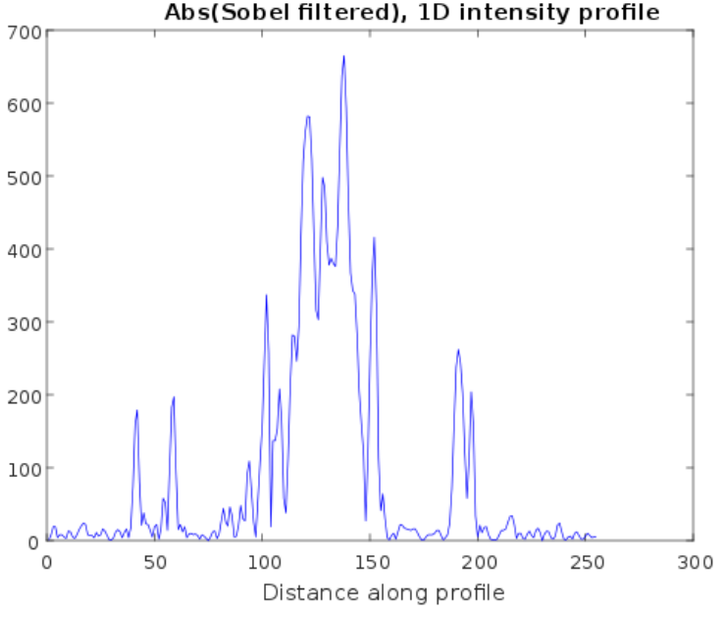




Abs(Sobel filtered)

Abs(Sobel filtered)

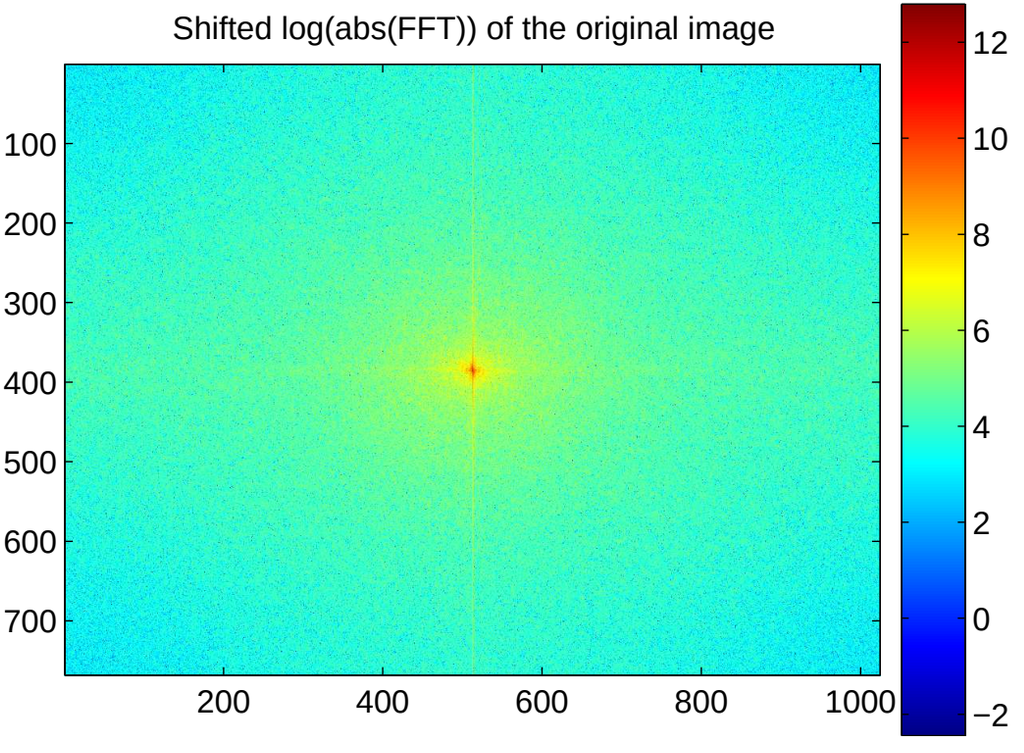




Low pass filter, Butterworth (1)

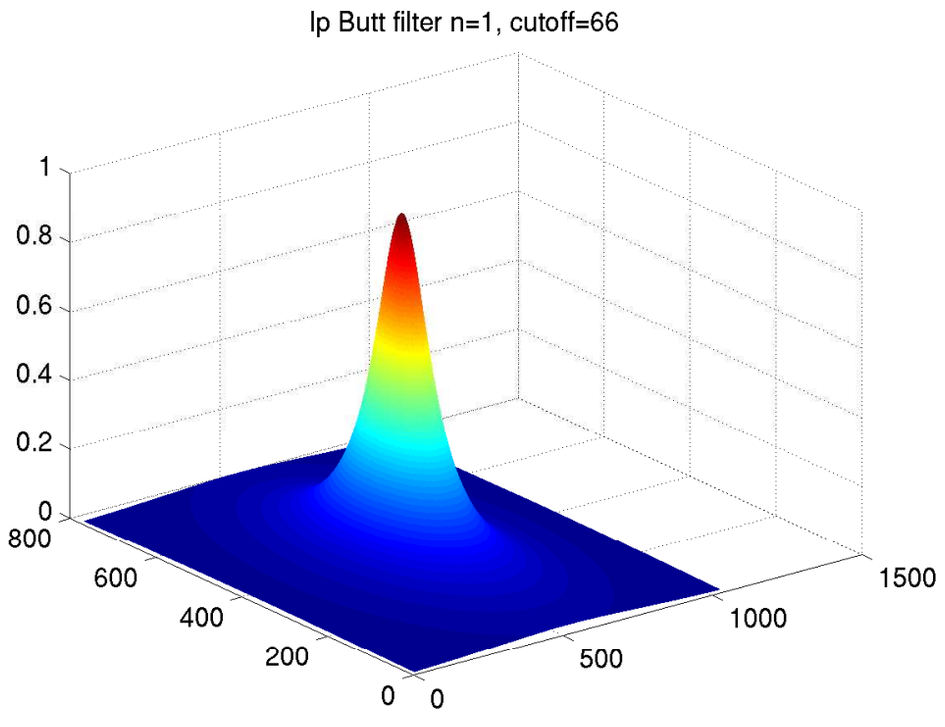
Butterworth filter has the frequency spectrum with the smallest rippling, which converges to zero for maximal frequences (S. Buttherworth, 1930).

Input image

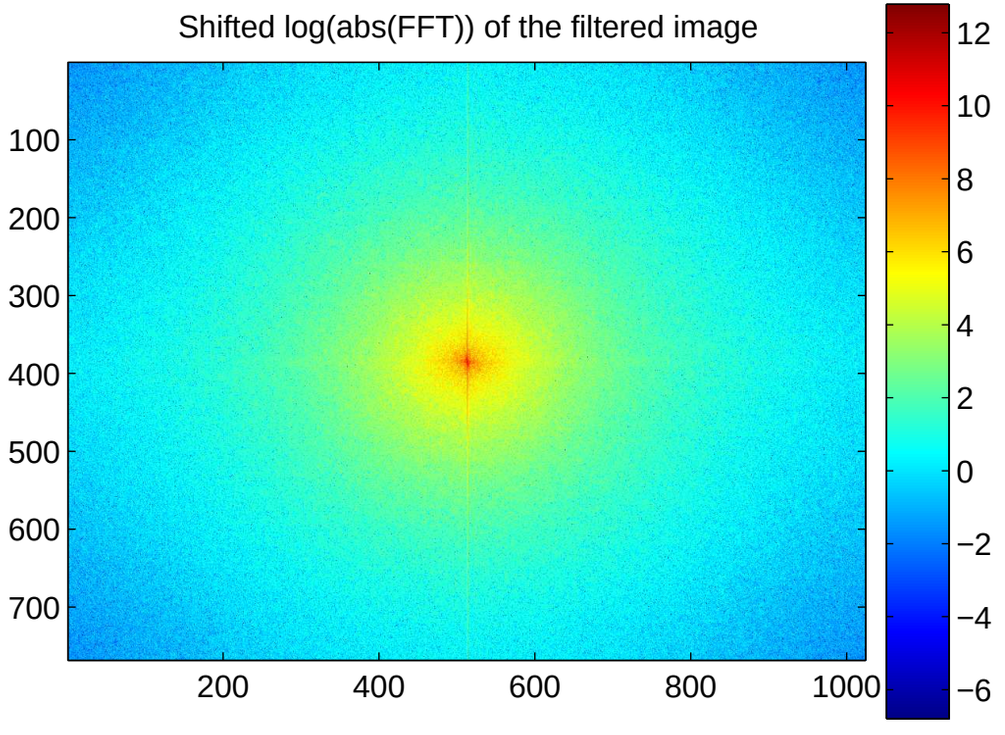


Its frequency spectrum

Low pass filter, Butterworth (2)



Butterworth low pass filter



FFT of the filtered image

$H(u,v) = \frac{1}{1+\left(\frac{D(u,v)}{D_0}\right)^{\frac{2}{n}}}$, where $D(u,v) = \sqrt{u^2+v^2}$. n is the filter degree.

D_0 is the frequency corresponding to the decrease of intensity by 3dB.

Low pass filter, Butterworth (3)

Input image

Its frequency spectrum

Homomorphic filter (1)

The aim: to normalize the intensity across the entire image and to increase contrast.

The method is based on the following assumptions:

- Illumination i changes in the image very slowly (low frequencies),
- Reflectance r changes in a more fast fashion, because the scene is usually rather diverse.
- The image can be decomposed (factorized) in each pixel into a product of two components – illumination i and reflectance r:
  $f(x, y) = i(x, y)\, r(x, y)$.

The key idea: the logarithm function can be used to separate the illumination and the reflectance components.

Homomorphic filter (2)

$$z(x,y) = \ln f(x,y) = \ln i(x,y) + \ln r(x,y)$$

Fourier spectrum

$$Z(u,v) = I(u,v) + R(u,v)$$

Filtering in the frequency domain

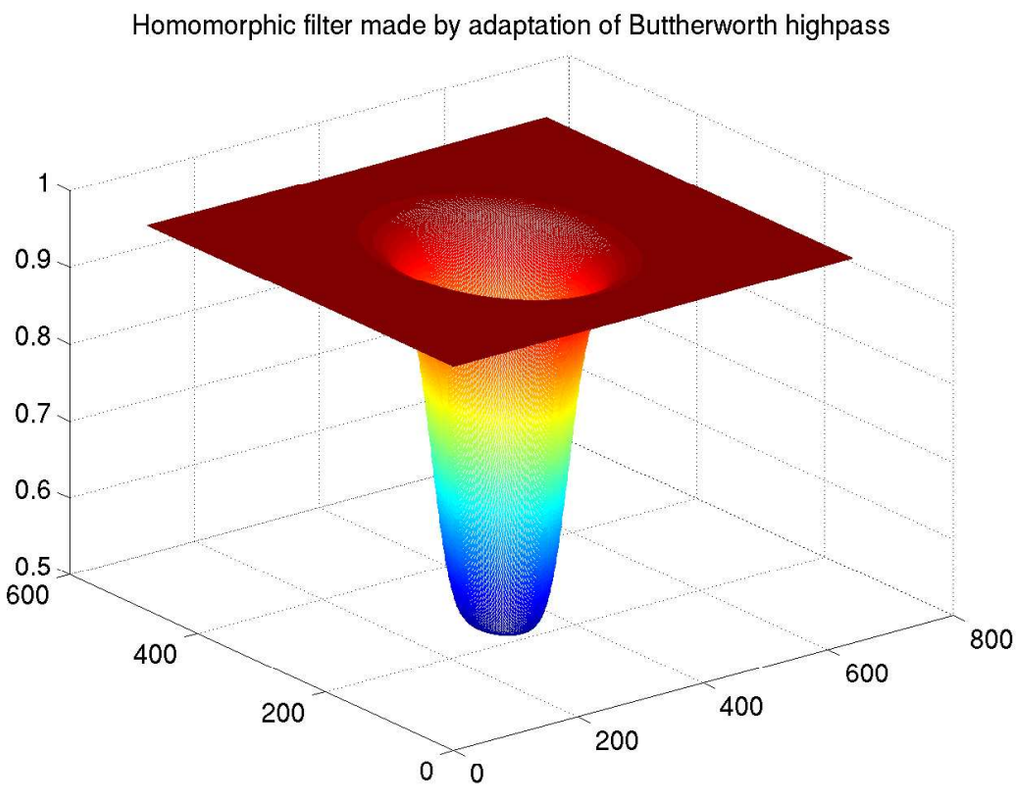
$$S(u,v) = H(u,v)Z(u,v) = H(u,v)I(u,v) + H(u,v)R(u,v)$$

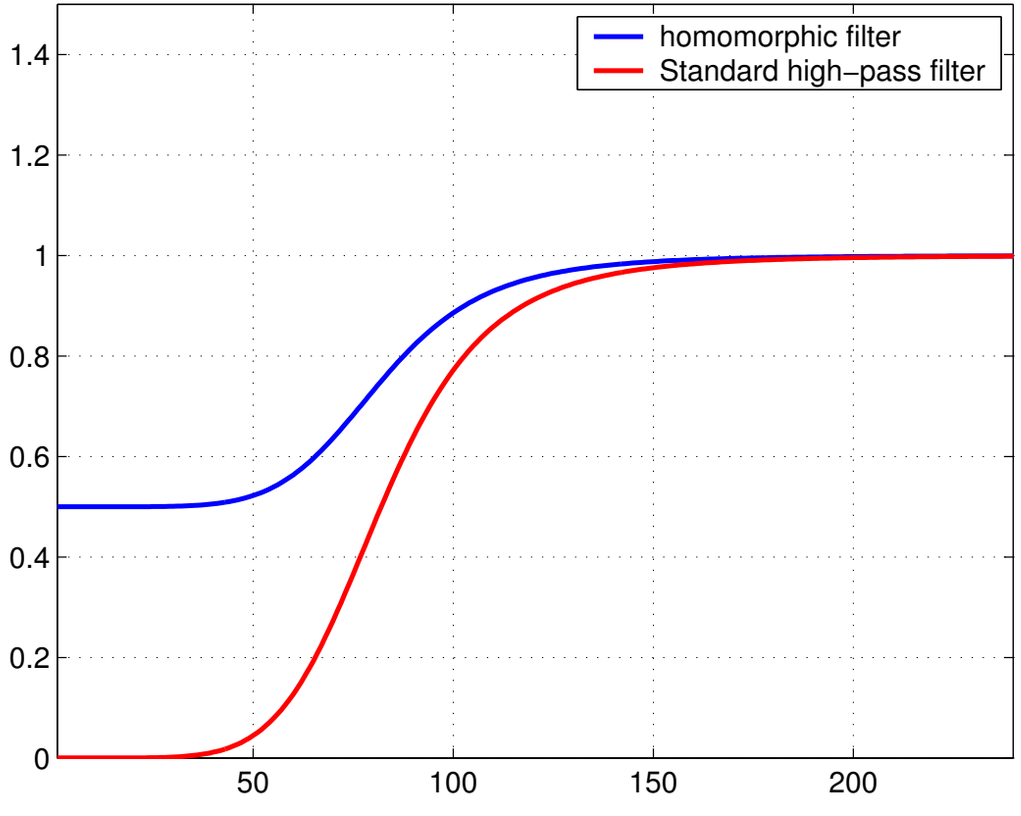
Inverse transformation back into spatial coordinates $s(x,y) = \mathcal{F}^{-1}\{S(u,v)\}$ and return to original gray scale from the logarithmic one

$$g(x,y) = \exp\left(s(x,y)\right)$$

The outcome is the suppression in the illumination changes in the scene and the improvement of the reflectance component.

Homomorphic filters





Note: The filter is used to modify $Z(u, v)$, not the original spectrum $F(u, v)$!

Outcome of homomorphic filtration

Original image.

Filtered image.

Design of 2D FIR filters

- The 2D Infinite Impulse Response (IIR) filters are not used because of their instability. (causality is not secured).
- Finite Impulse Response (FIR) filters can be easily represented as a matrix of coefficients. The implementation is easy.
- 2D FIR filters are natural generalization of 1D FIR filters.
- FIR filters can be designed to have linear phase, which reduces distortions.
- Three design methods are usually used:
  1. Frequency transformation method transforms a 1D filter into 2D.
  2. Frequency sampling method creates the filter according to the desired frequency response.
  3. Windowing method composes the filter from the ideal impulse response and the smoothing window.

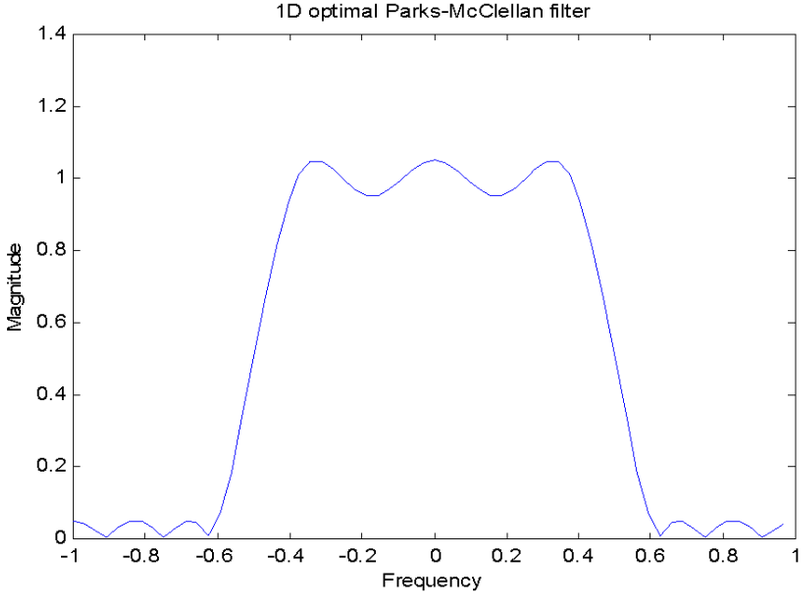
Frequency transformation method

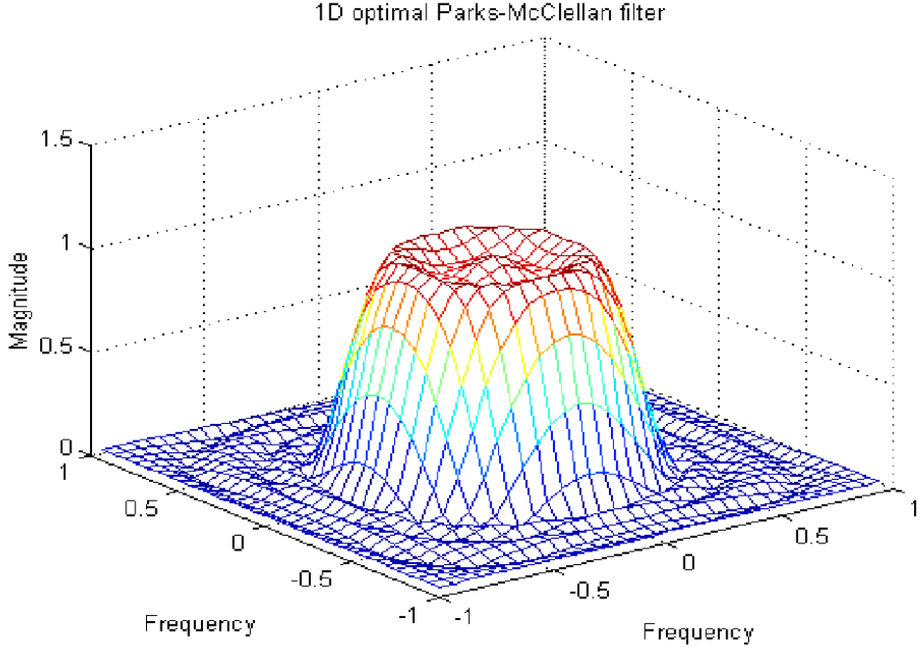
The established methods for designing 1D filters can be used. The 1D filter is converted into 2D by making the filter center symmetric. A good method.

A MATLAB example (Parks-McClellan optimal design):

```
b = remez(10,[0 0.4 0.6 1],[1 1 0 0]);

h = ftrans2(b);

[H,w] = freqz(b,1,64,'whole');

colormap(jet(64))

plot(w/pi-1,fftshift(abs(H))) figure, freqz2(h,[32 32])
```

2D Parks-McClellan Filter
Example continuation





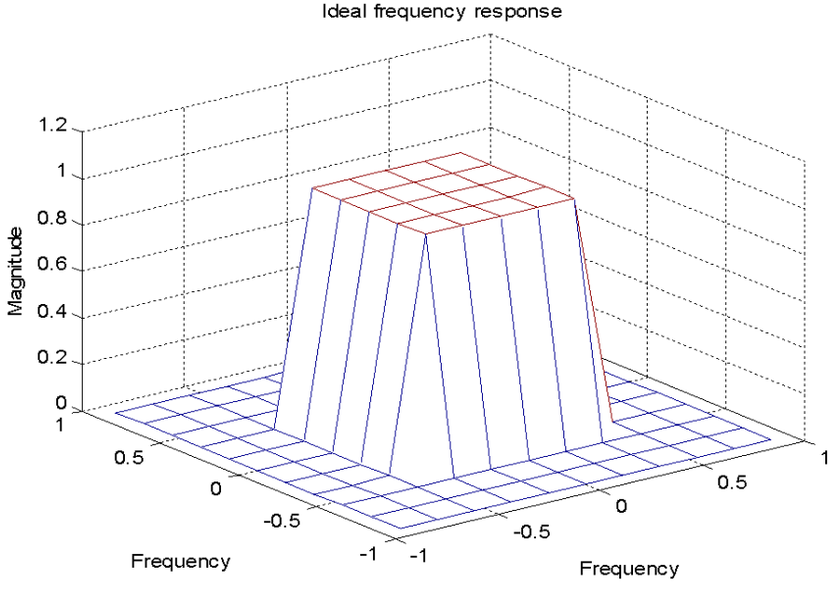
Frequency sampling method

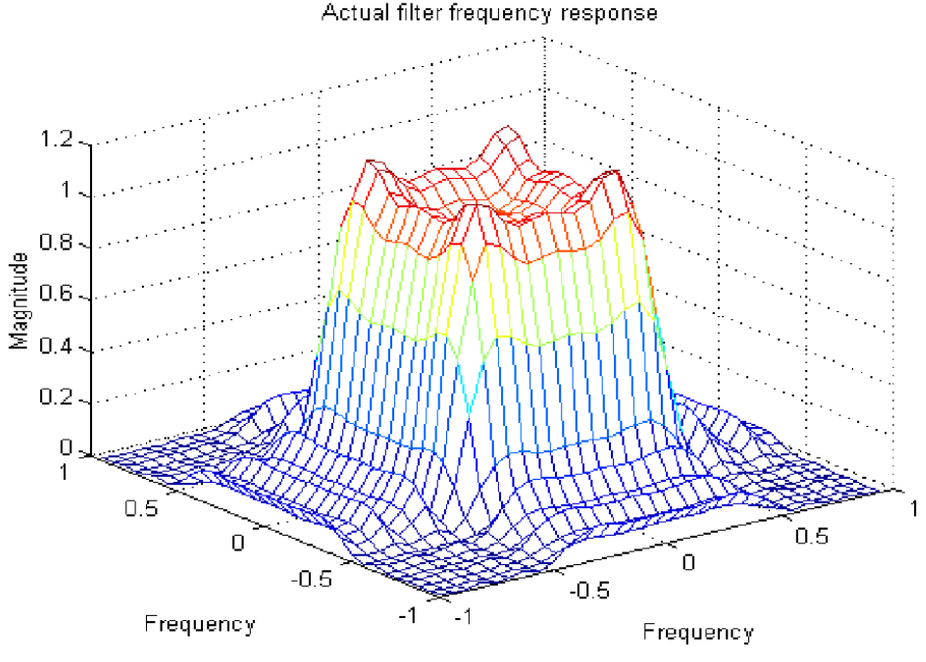
The desired frequency response is given. The filter is created in the matrix form securing that the response passes given frequency response points. The behavior can be arbitrary outside the given points. Oscillations are common.

MATLAB example (design of the 11×11 filter)

```
Hd = zeros(11,11); Hd(4:8,4:8) = 1;

[f1,f2] = freqspace(11,'meshgrid');

mesh(f1,f2,Hd), axis([-1 1 -1 1 0 1.2]), colormap(jet(64))

h = fsamp2(Hd);

figure, freqz2(h,[32 32]), axis([-1 1 -1 1 0 1.2])
```

Frequency sampling method
Example continuation





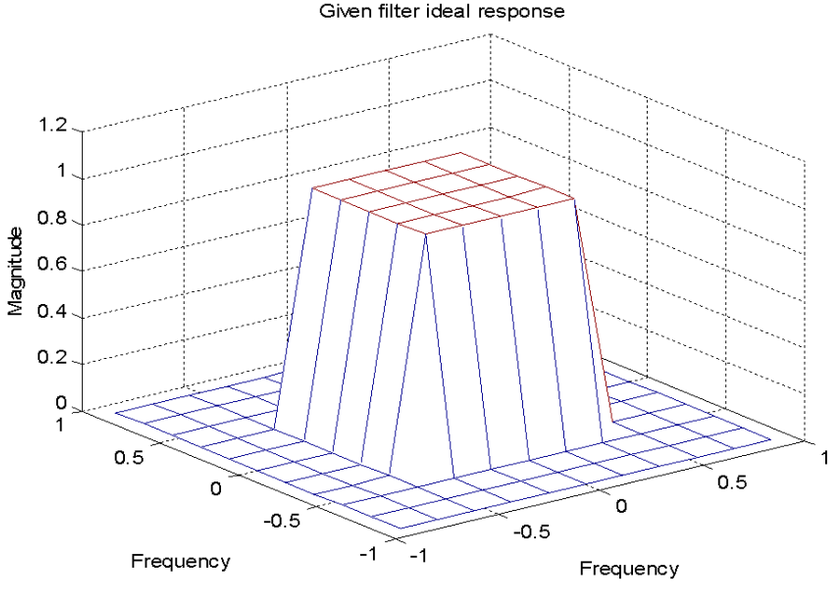
Windowing method

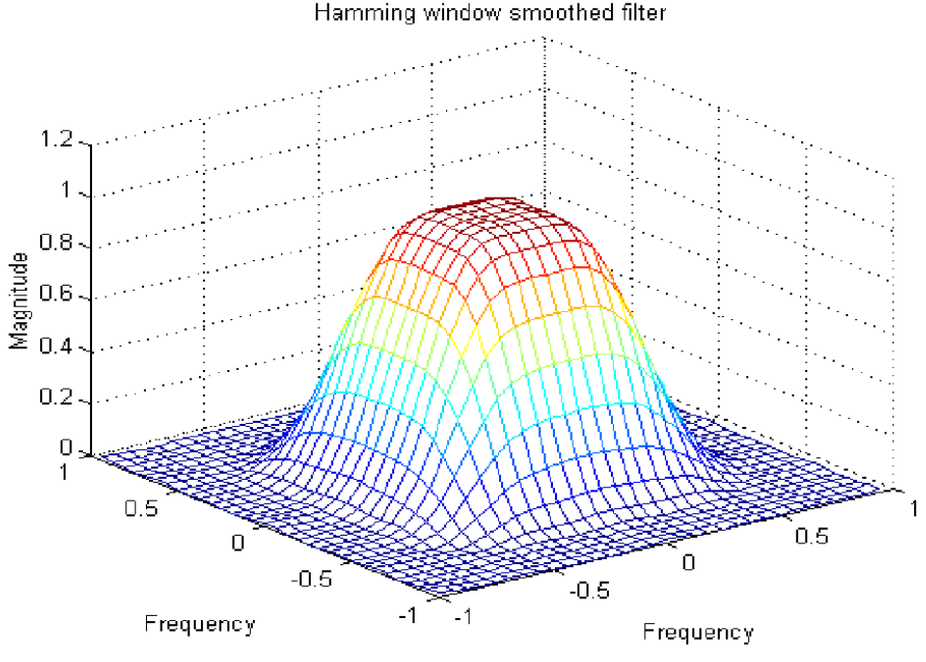
The ideal response of the filter smoothes the coefficients in the windos. The ideal filter is approximated.

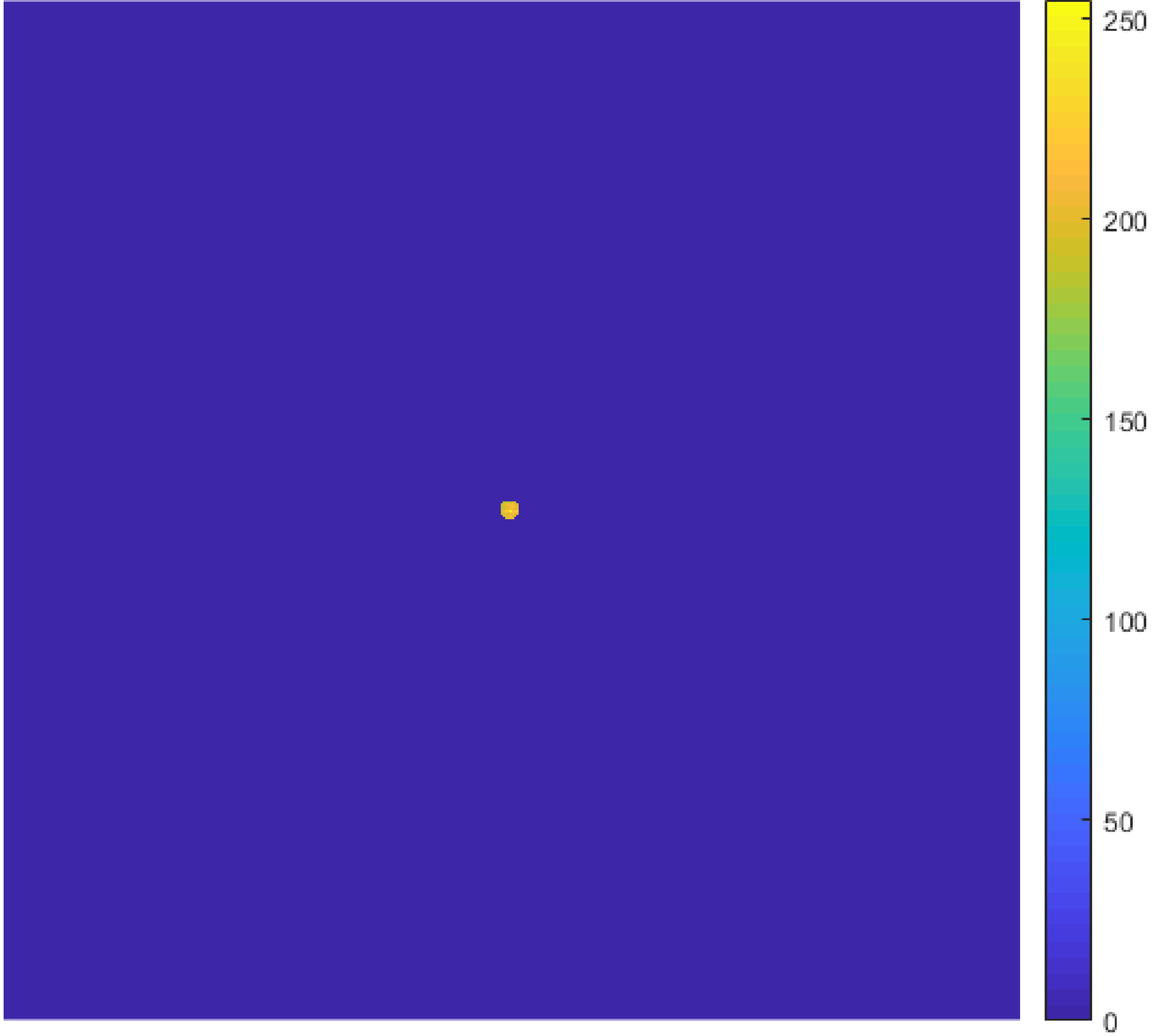
The results are usually better than the results of the Frequency Sampling Method.

```
Hd = zeros(11,11); Hd(4:8,4:8) = 1;

[f1,f2] = freqspace(11,'meshgrid');

mesh(f1,f2,Hd), axis([-1 1 -1 1 0 1.2]), colormap(jet(64))

h = fwind1(Hd,hamming(11));

figure, freqz2(h,[32 32]), axis([-1 1 -1 1 0 1.2])
```

Windowing method
Example continuation



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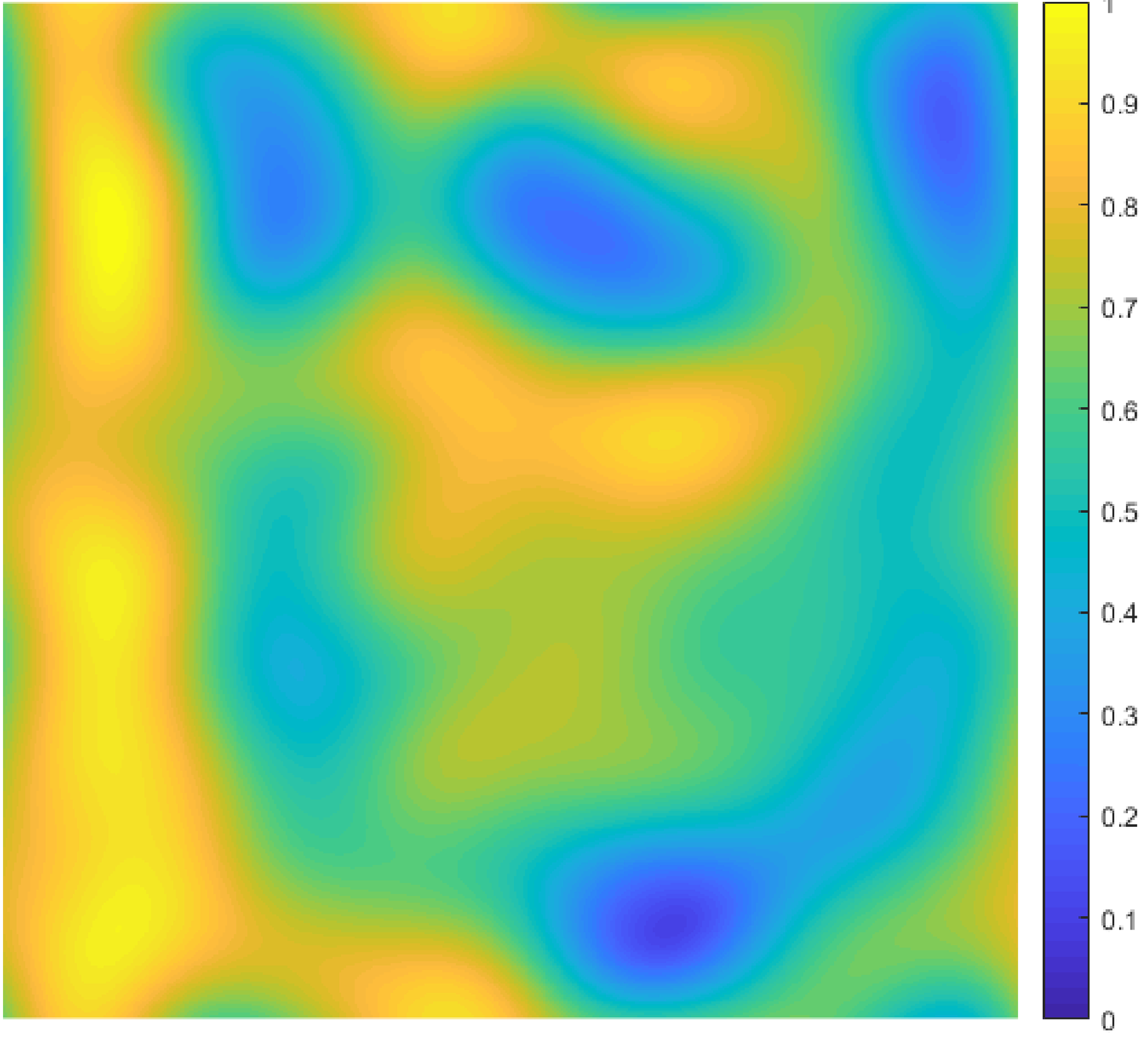
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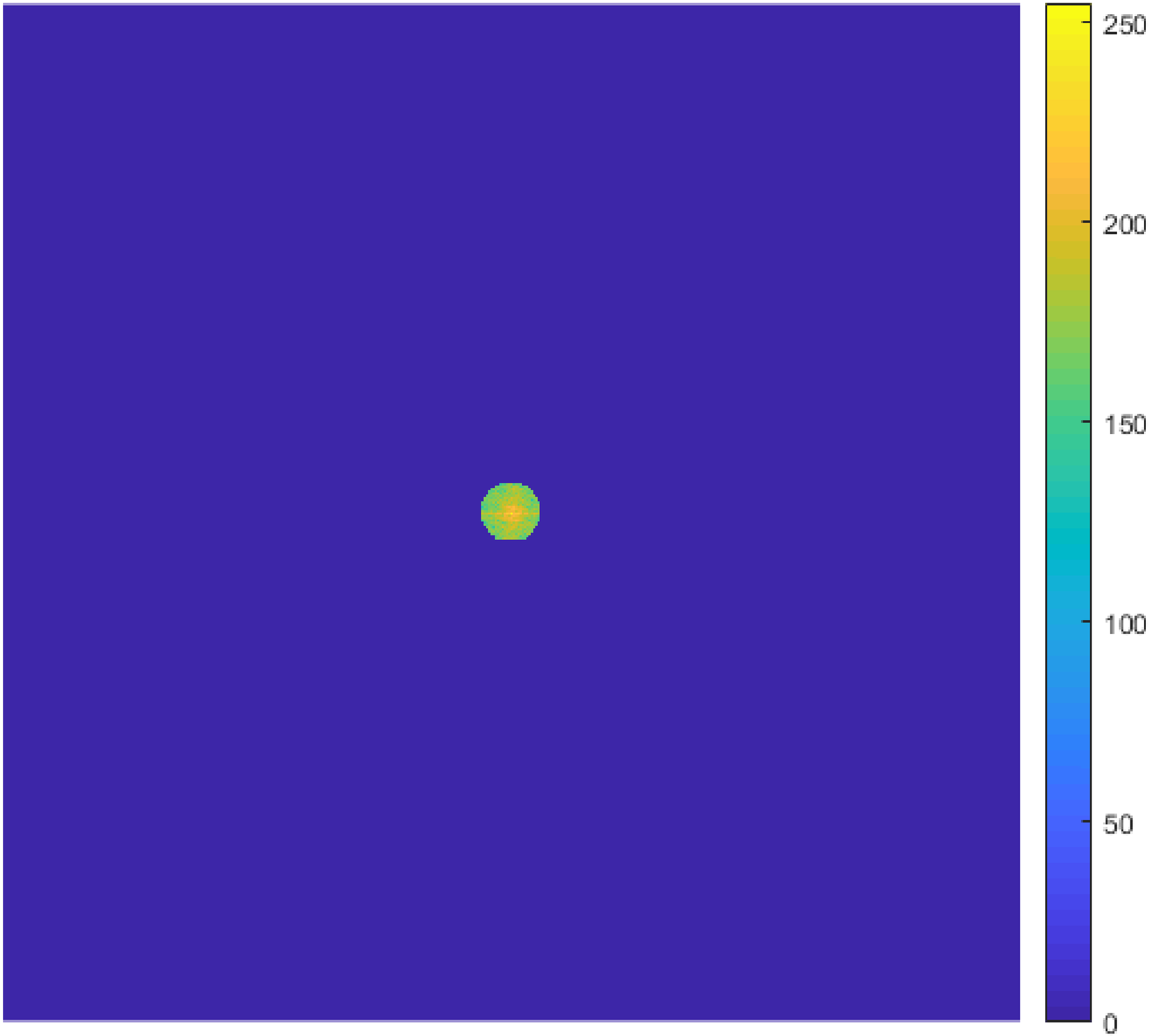
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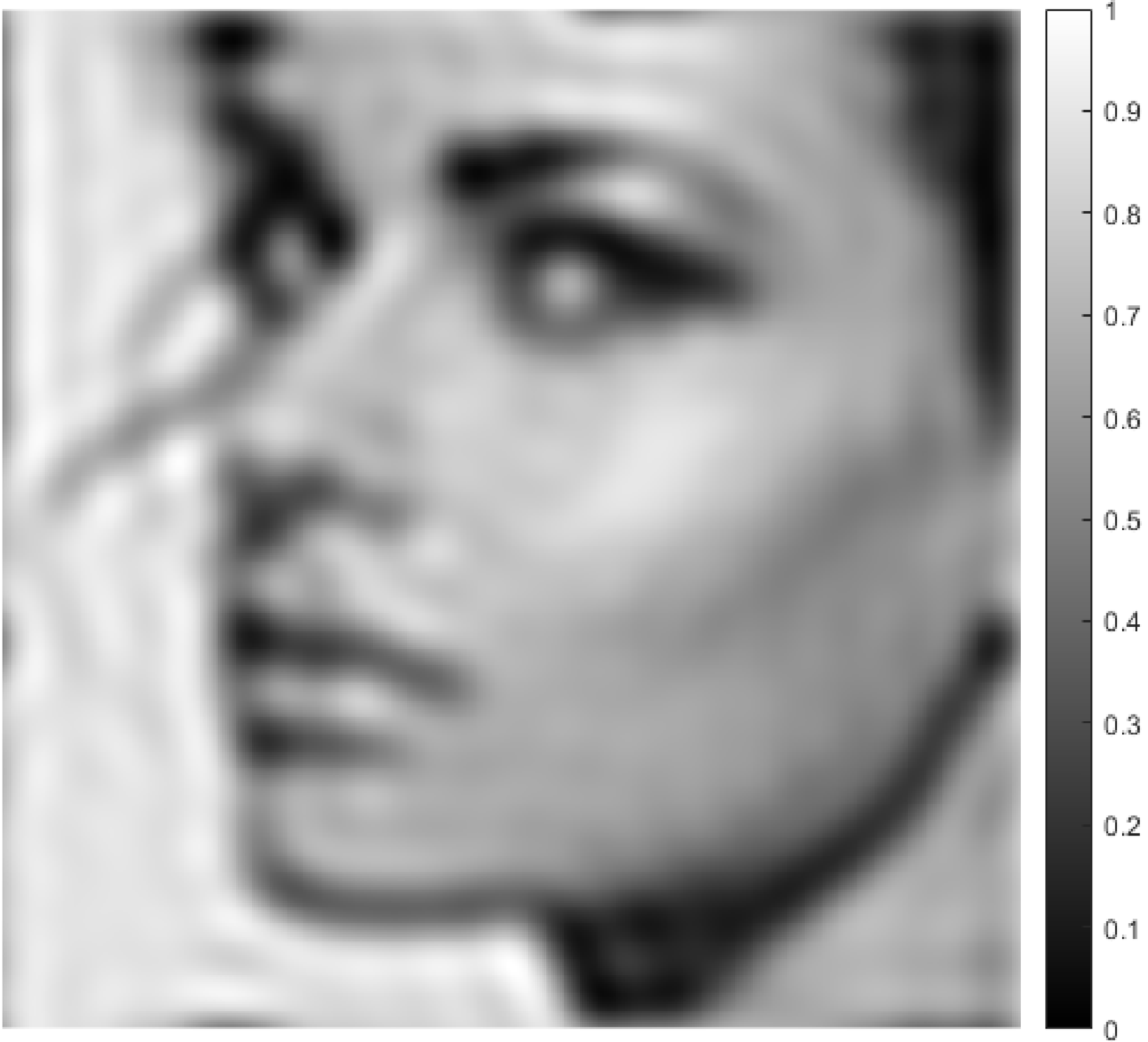
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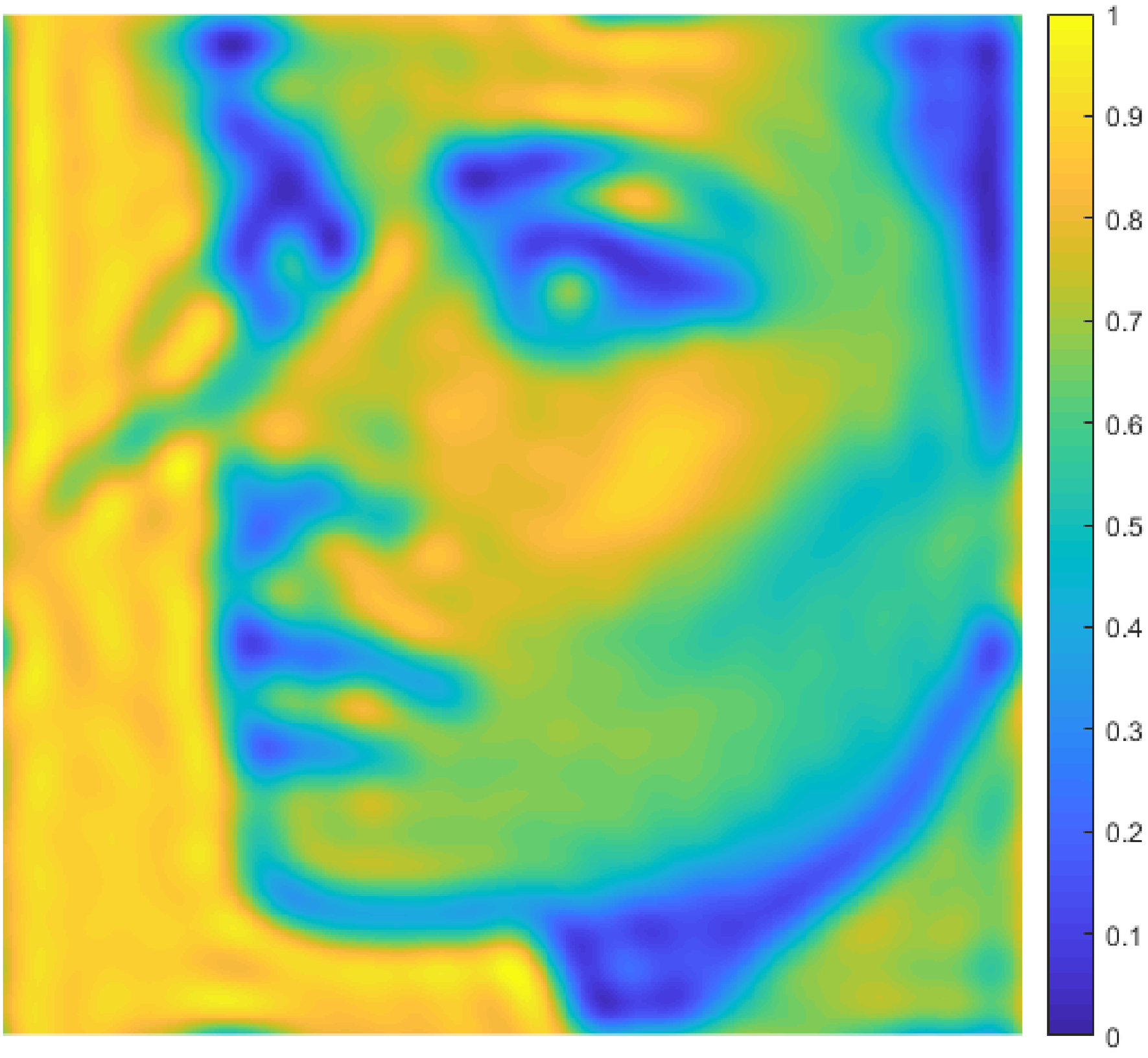
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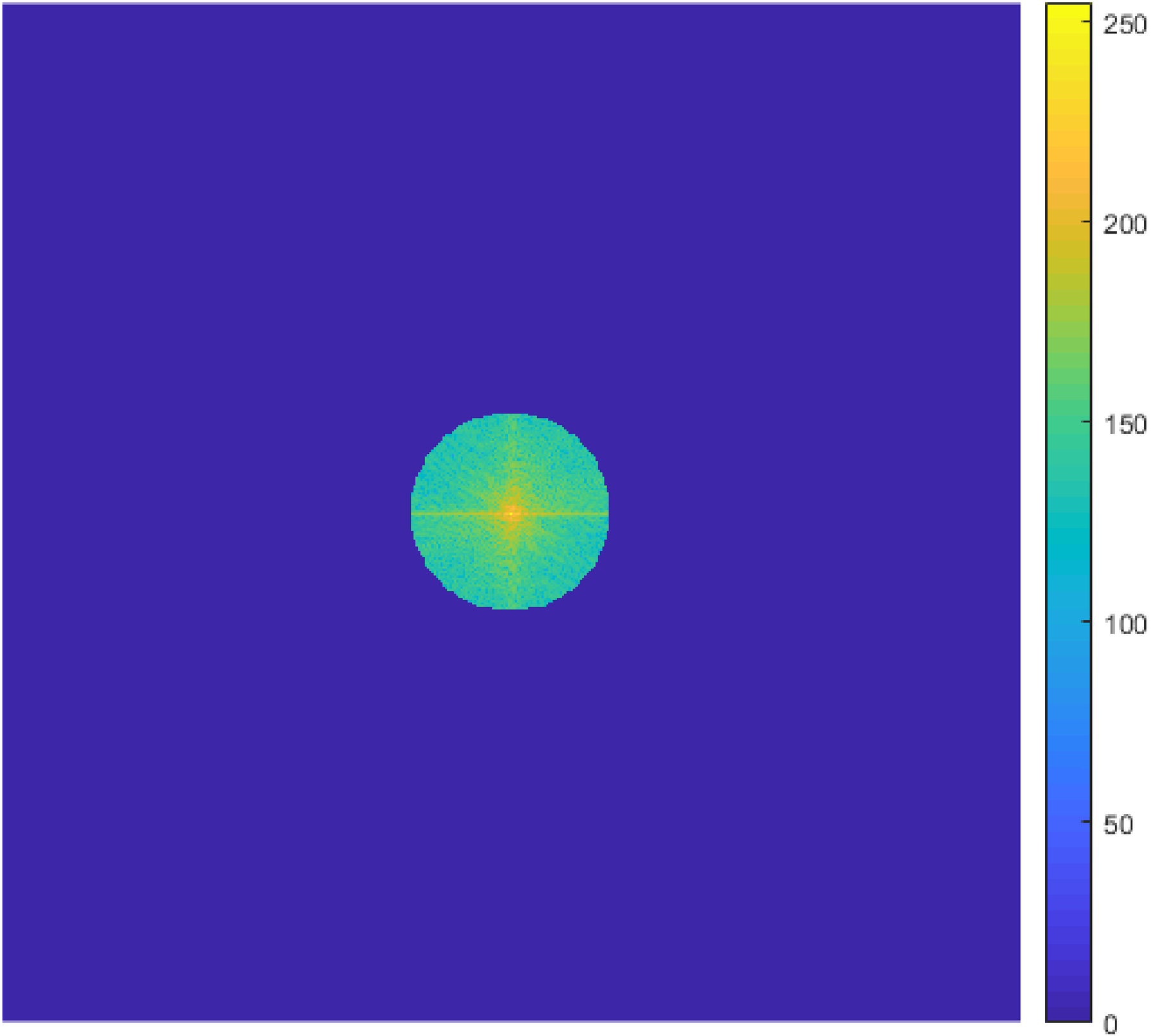
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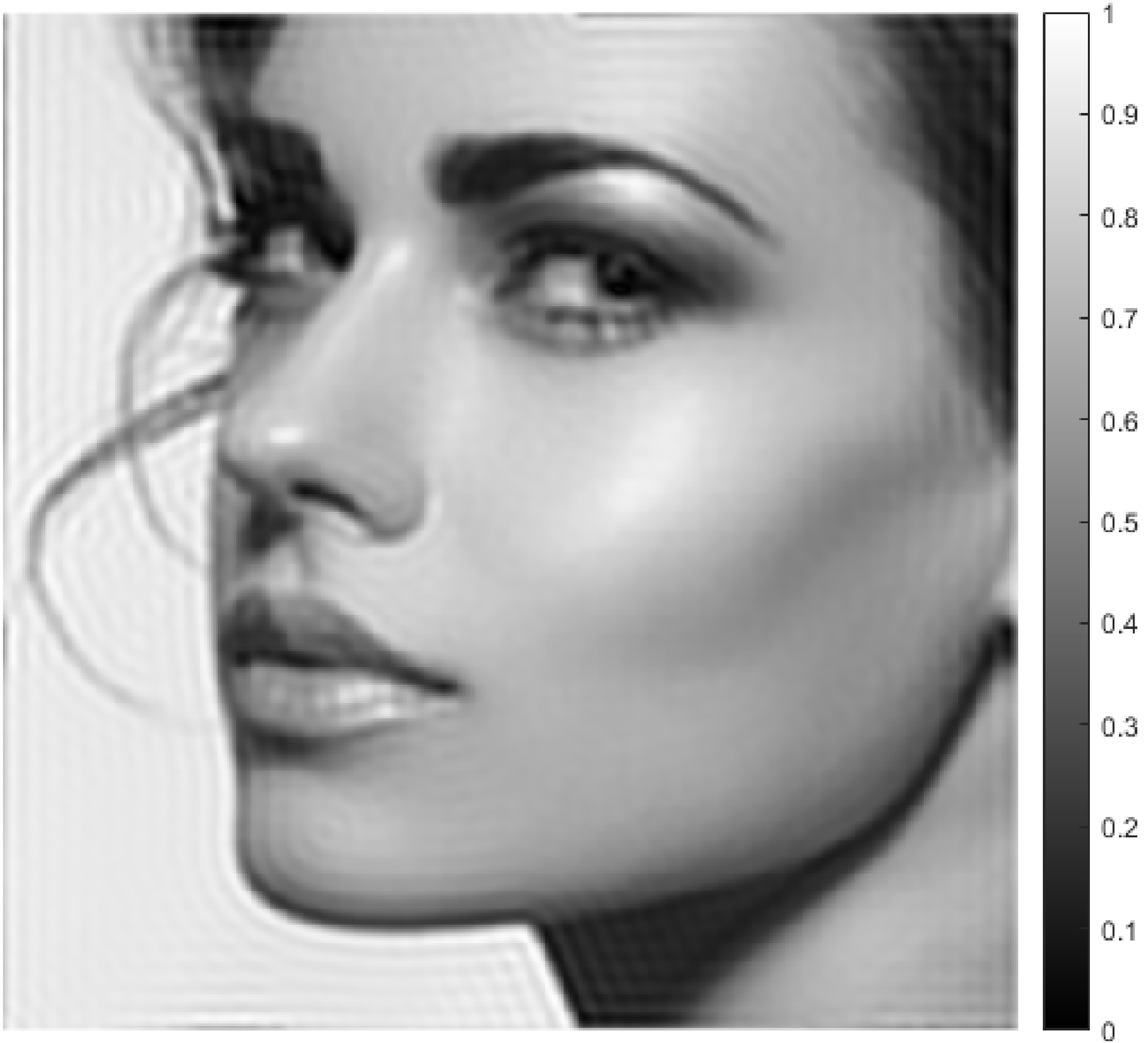
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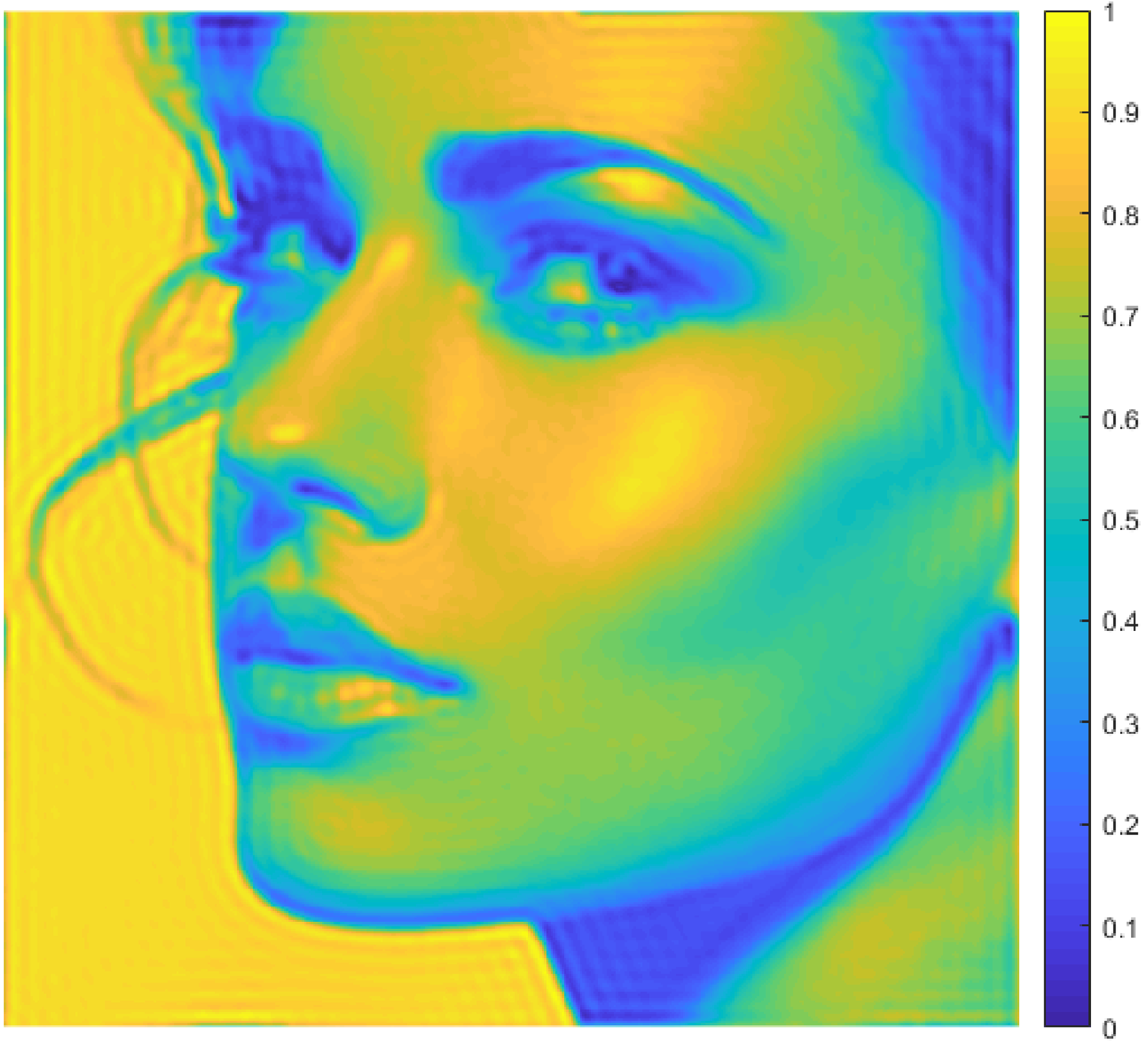
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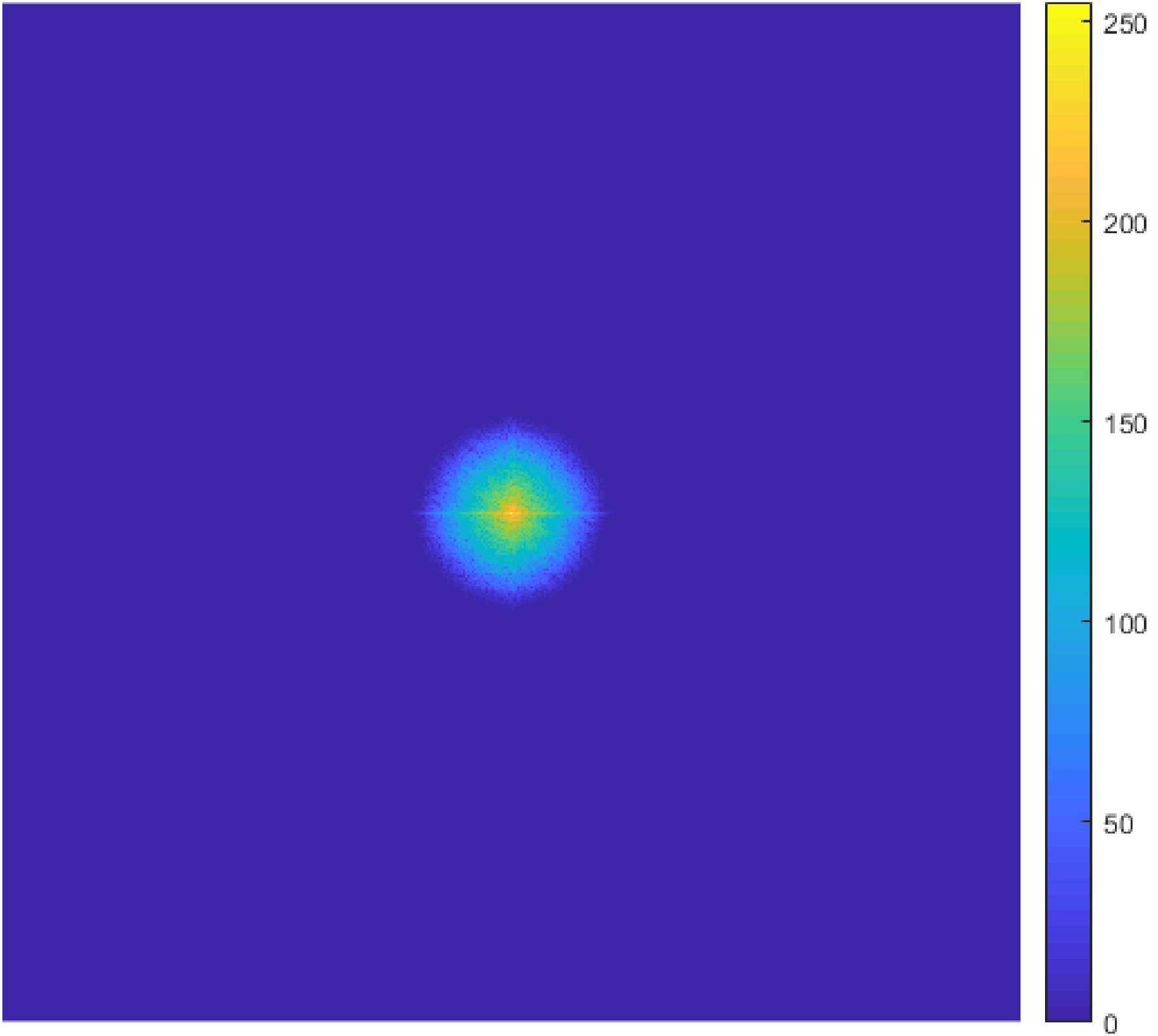
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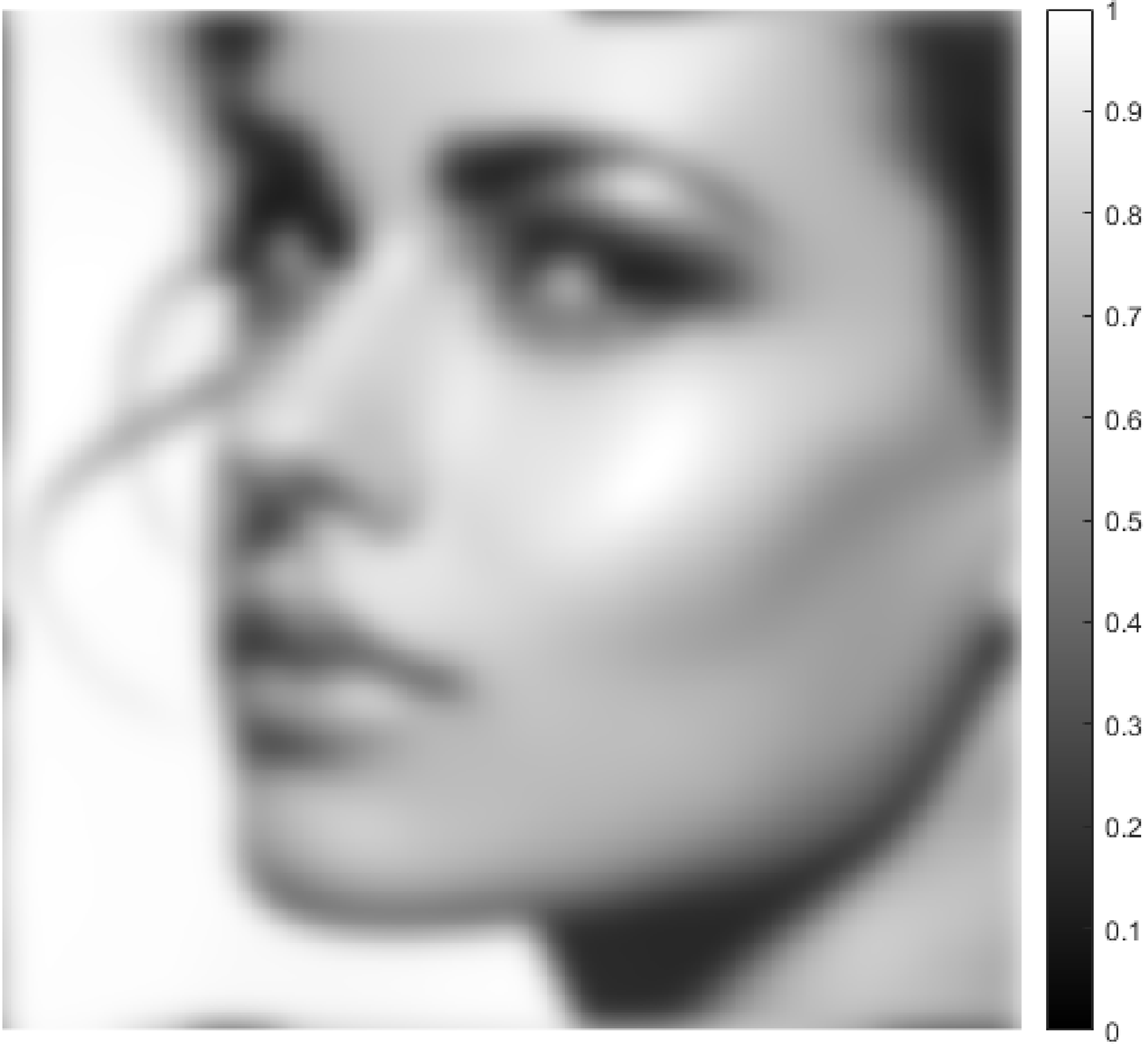
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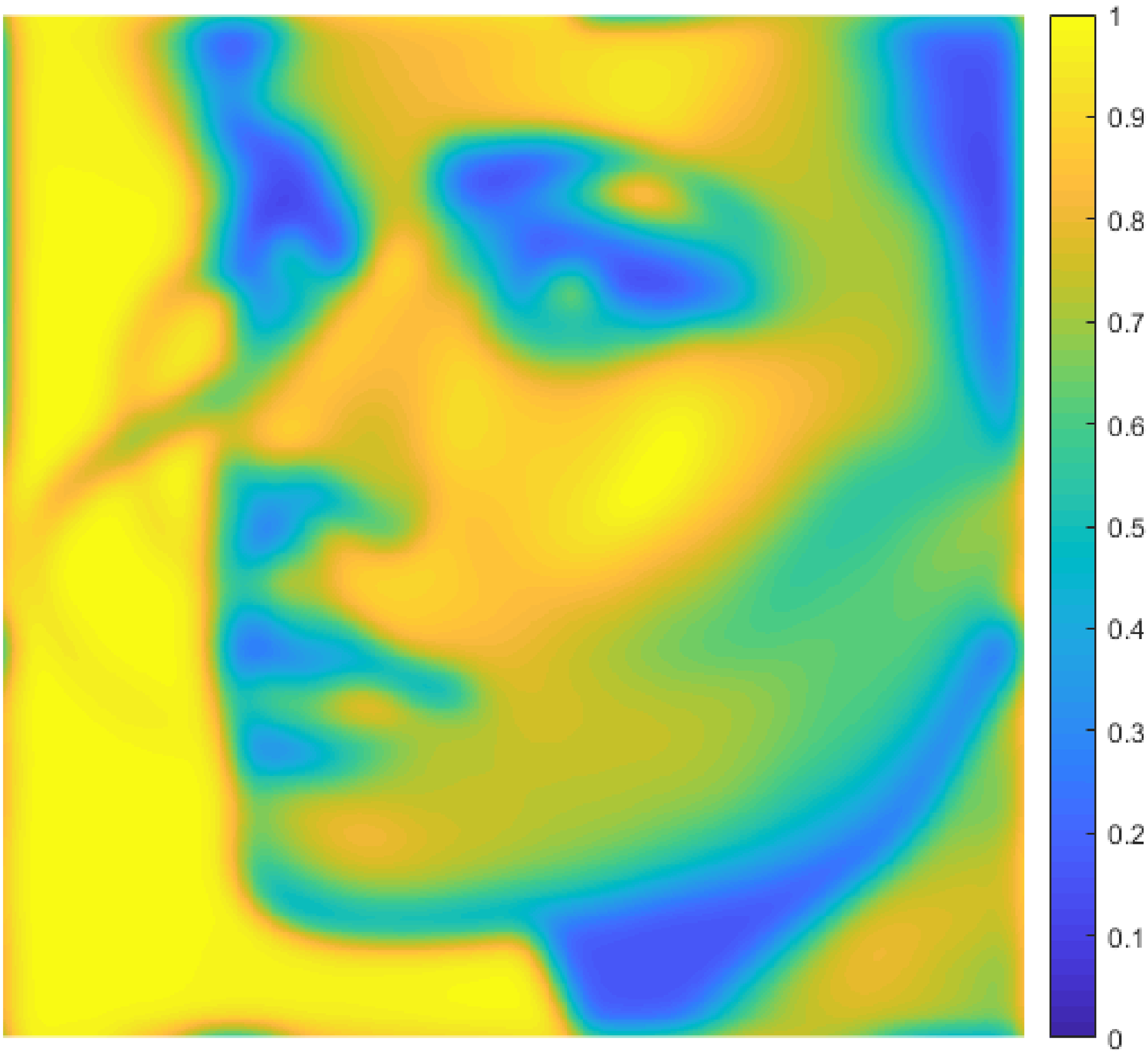
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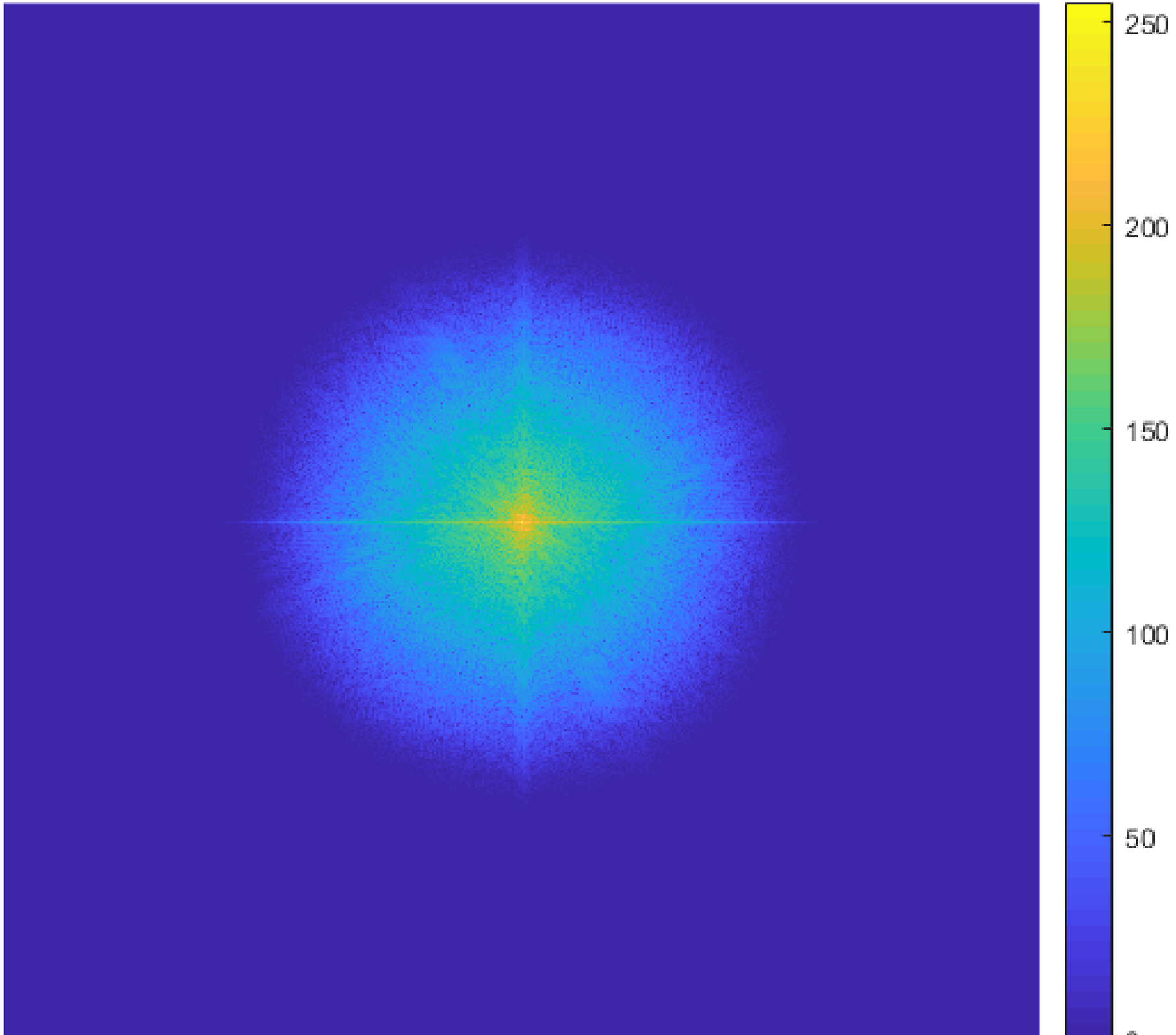
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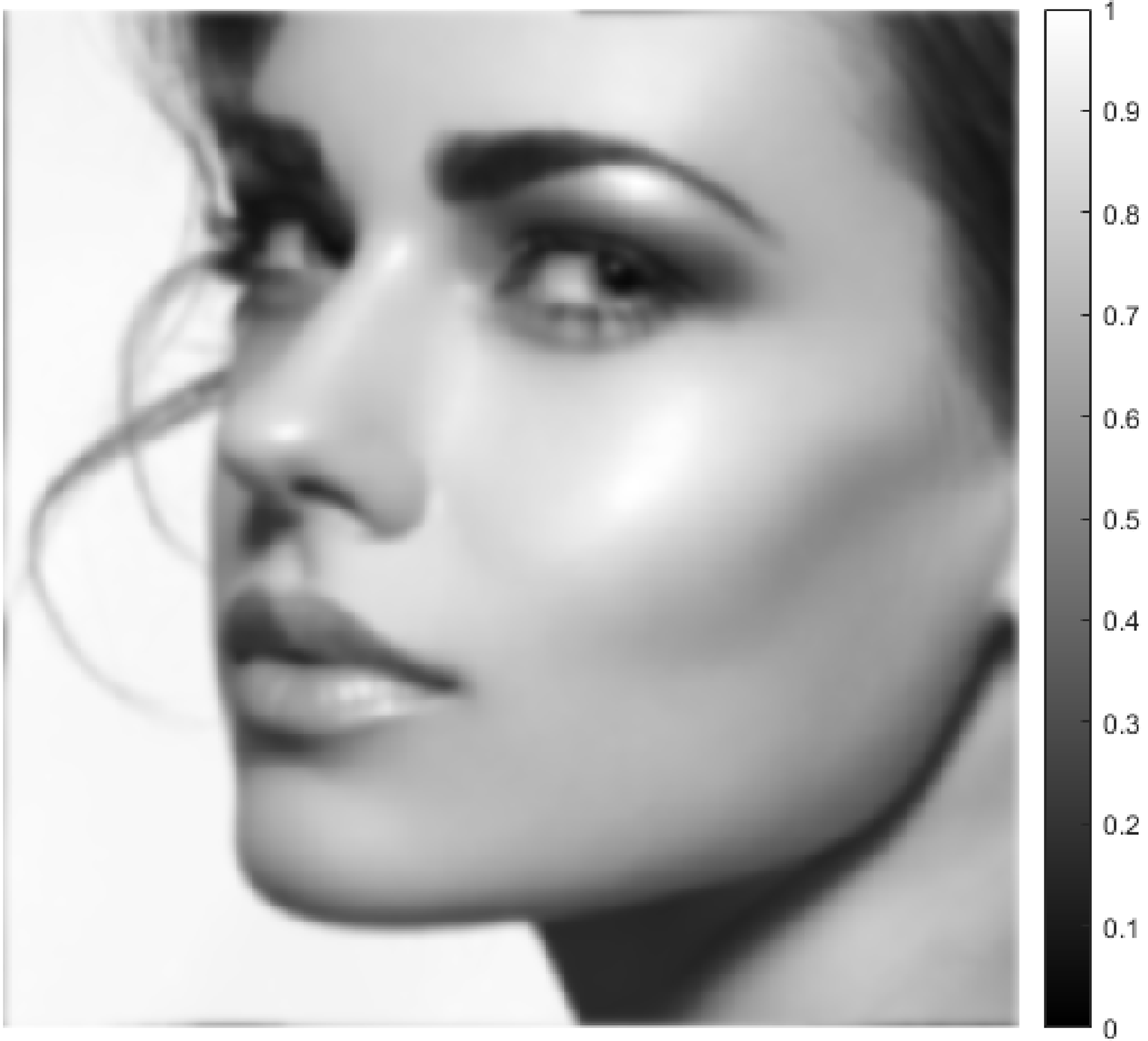
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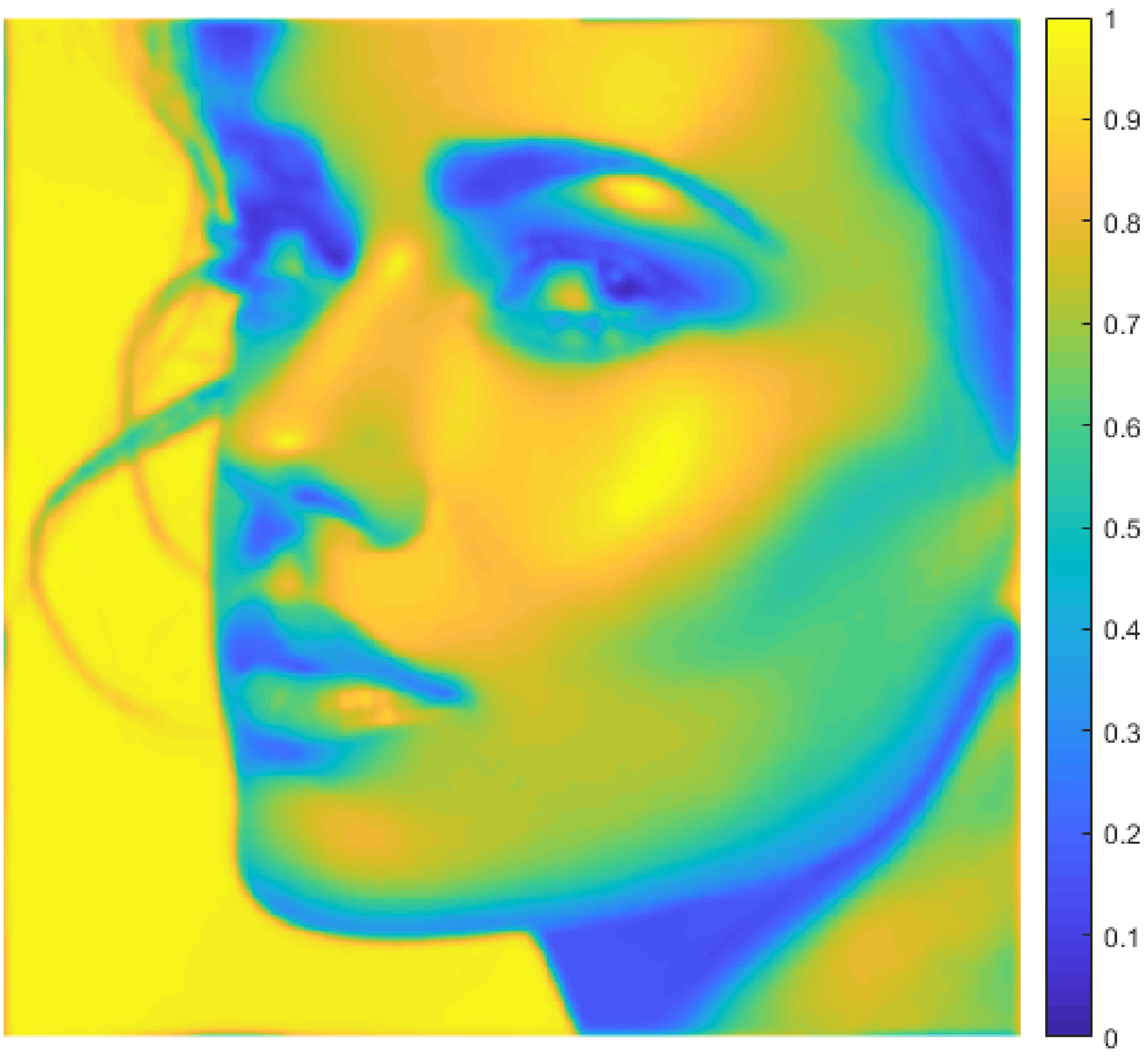
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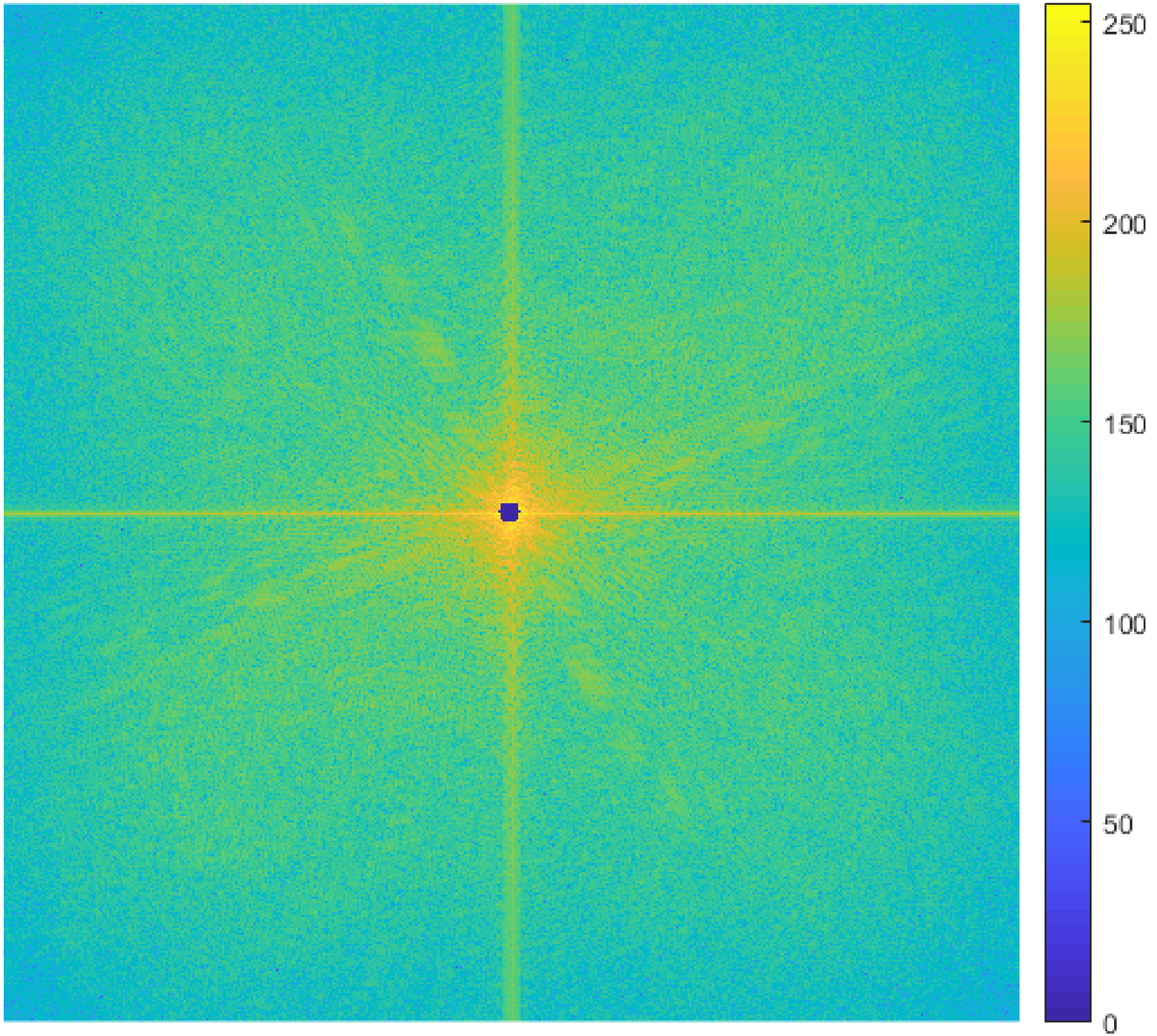
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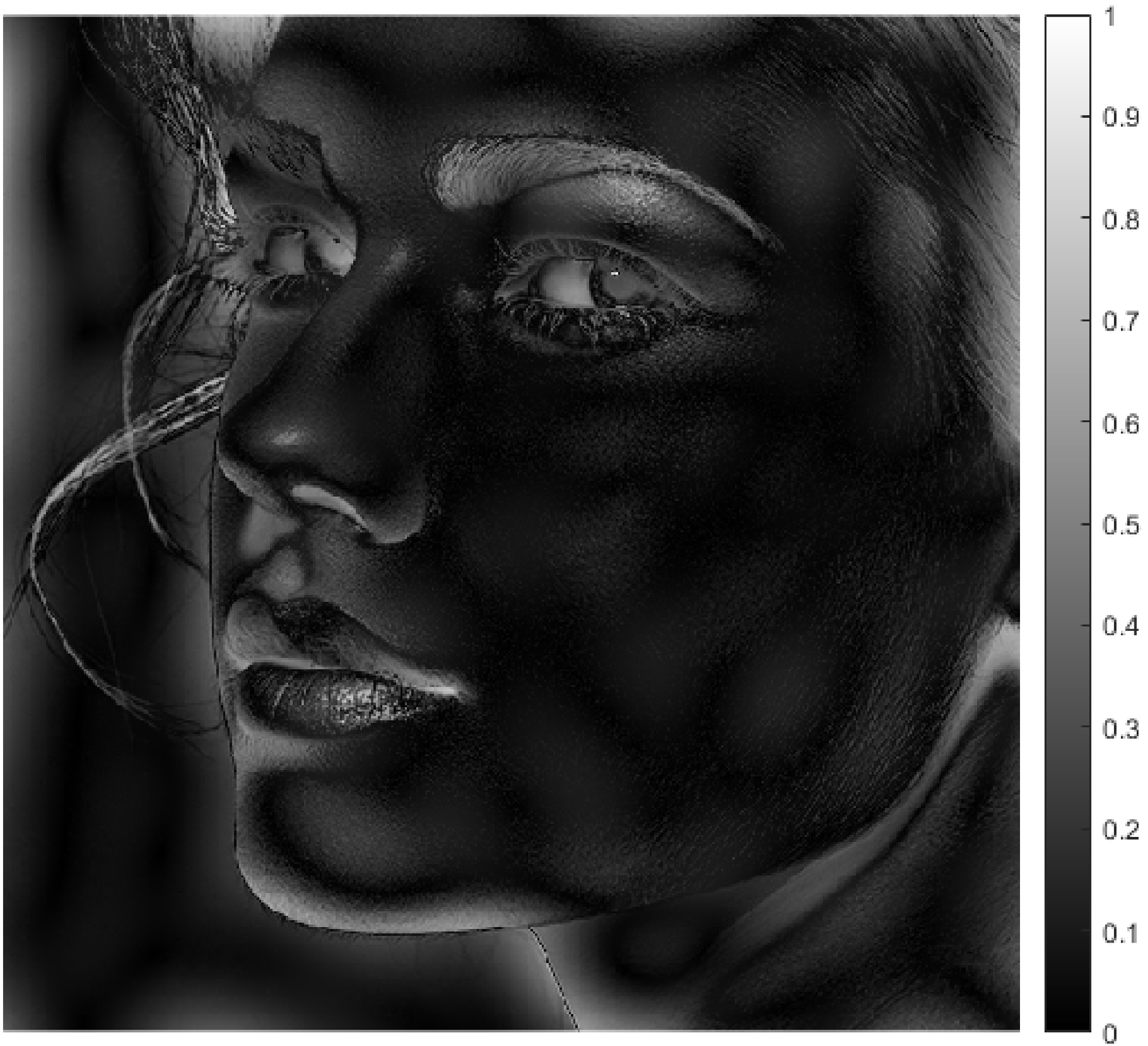
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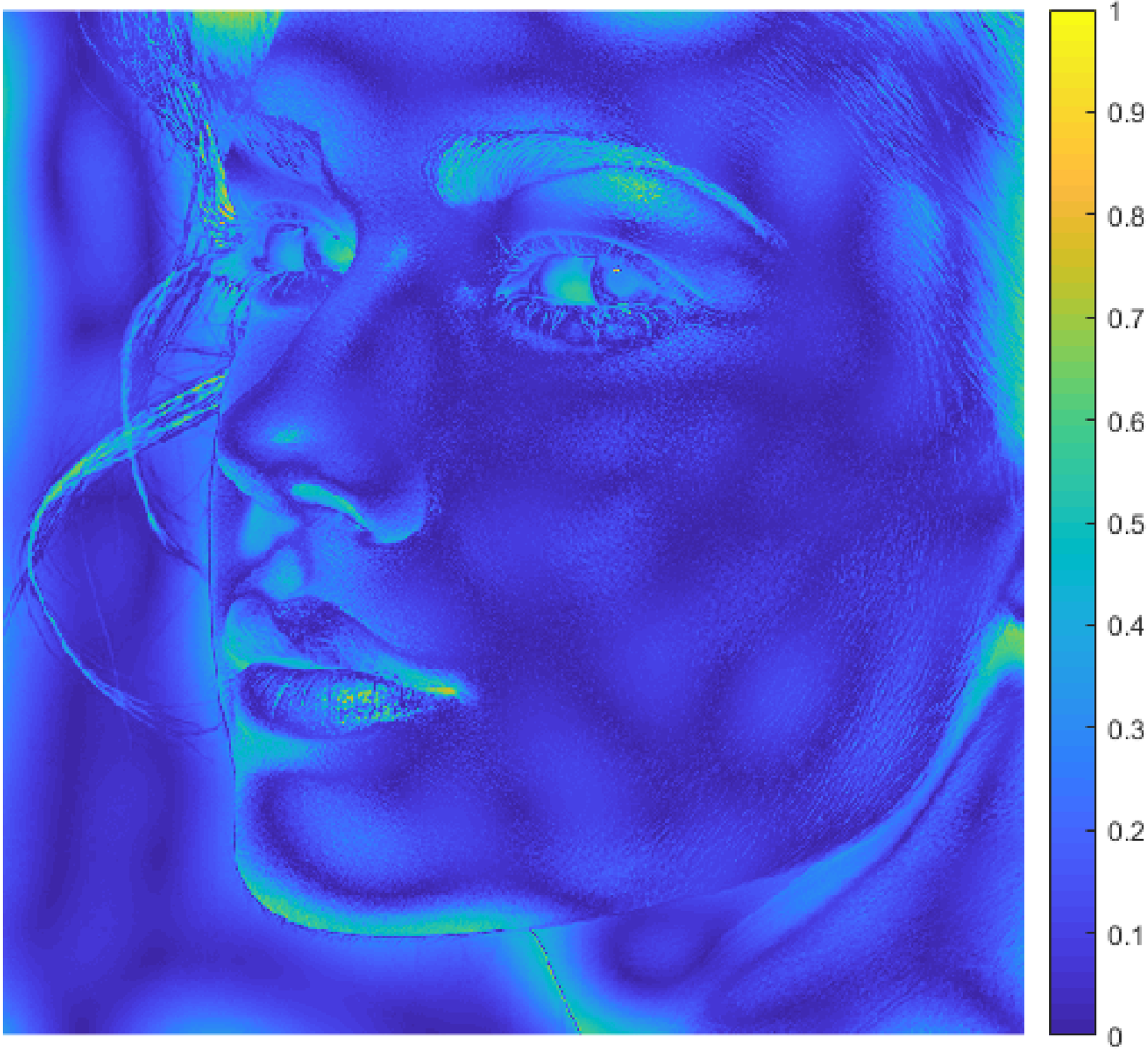
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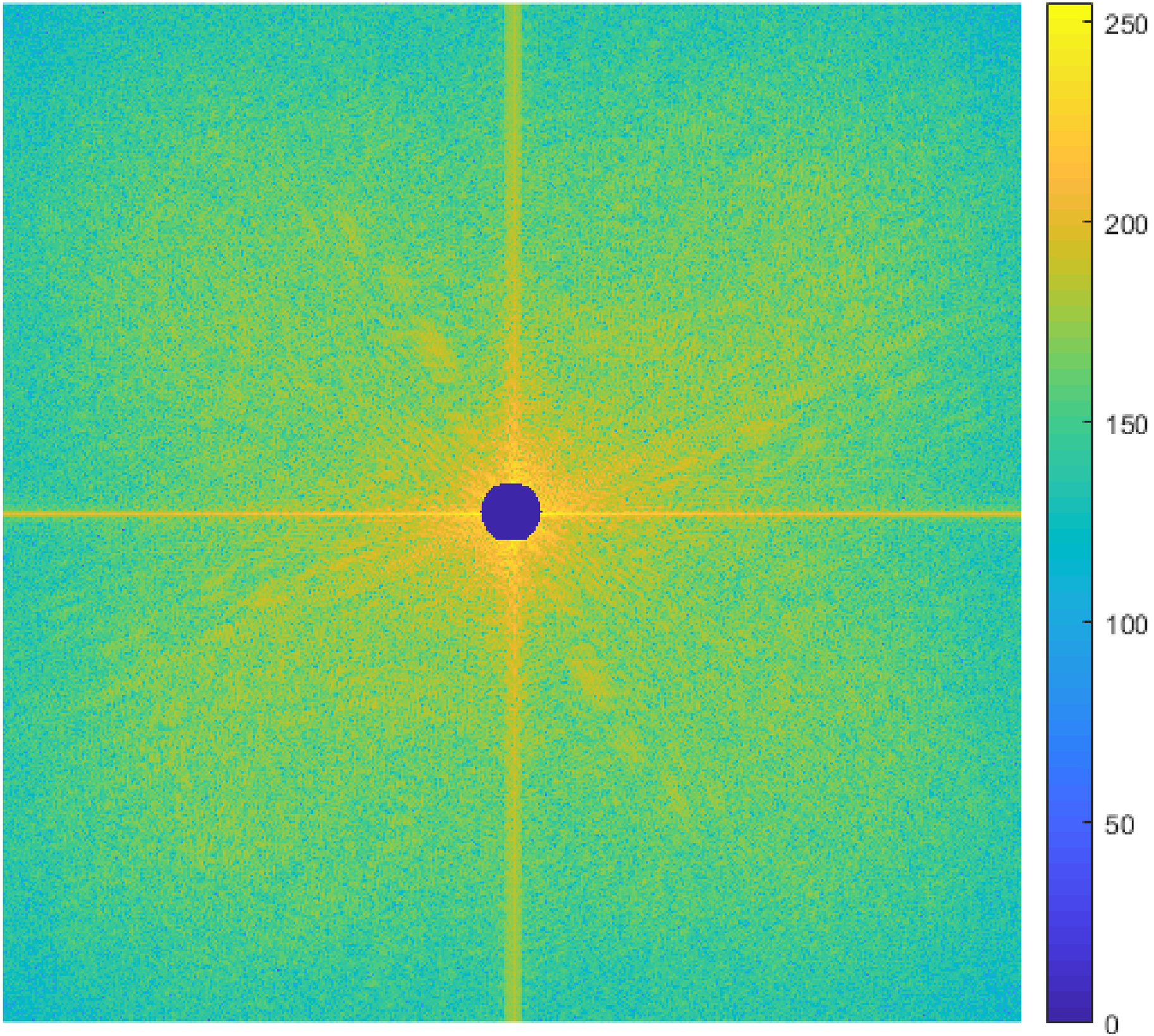
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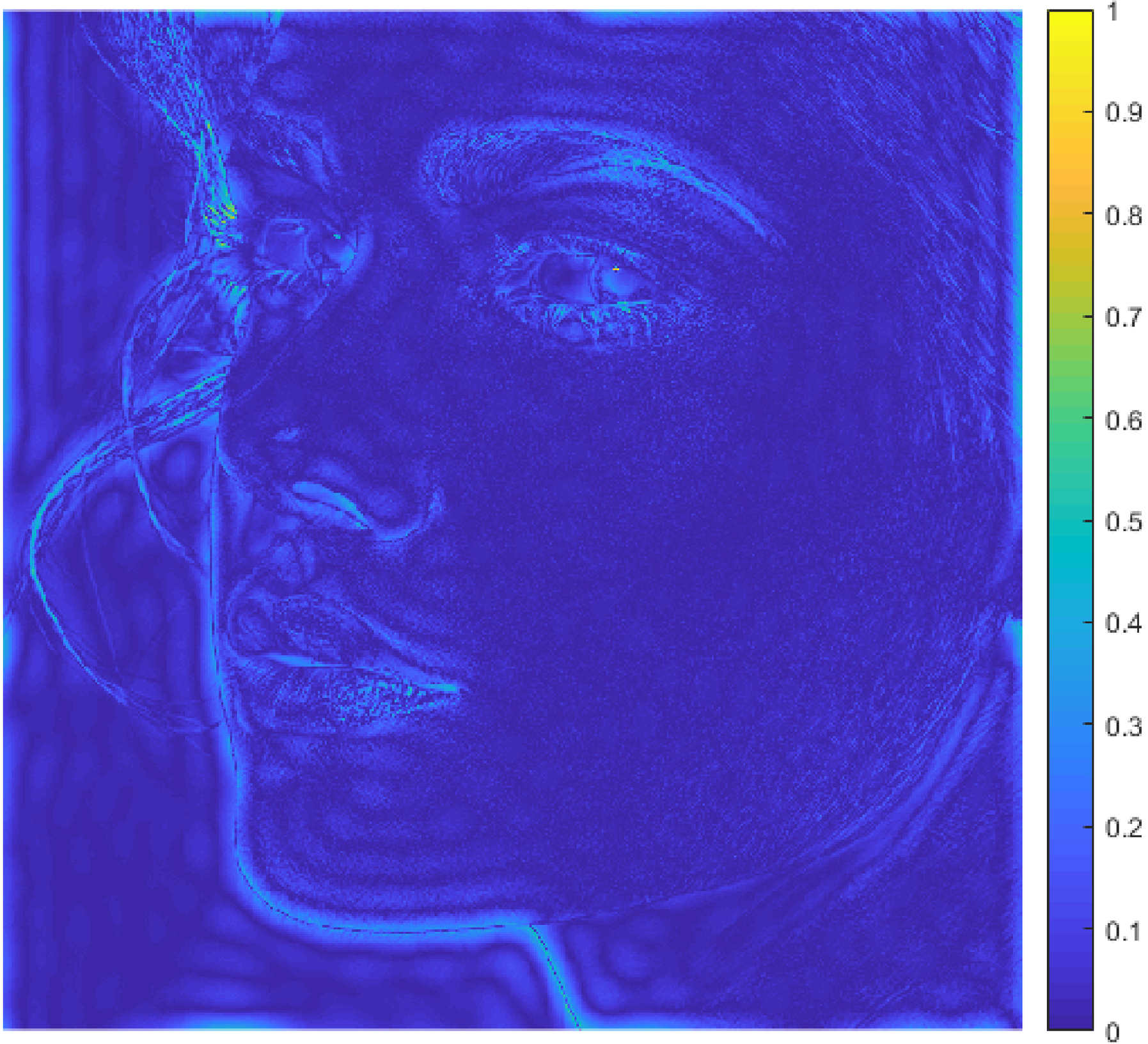
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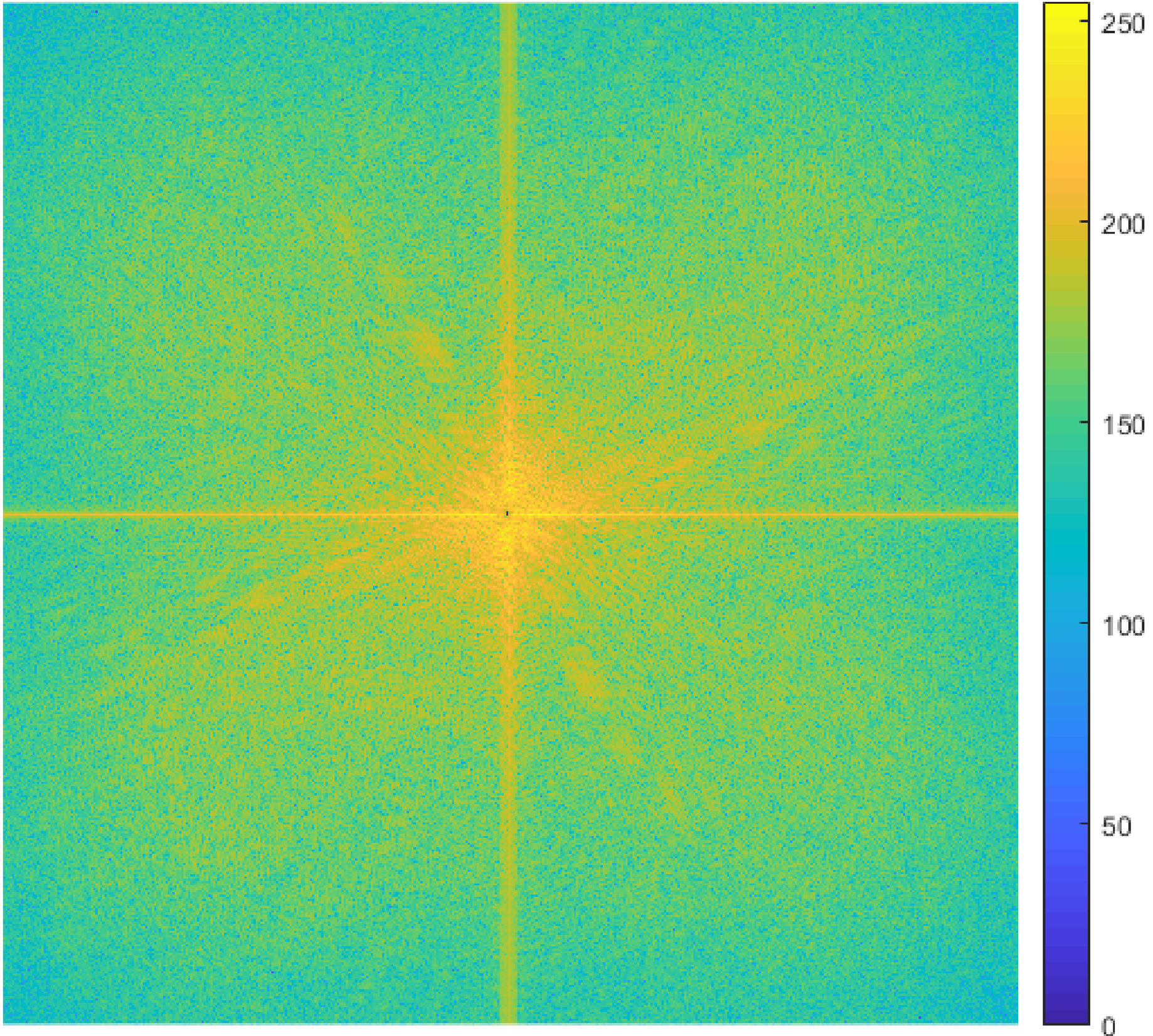
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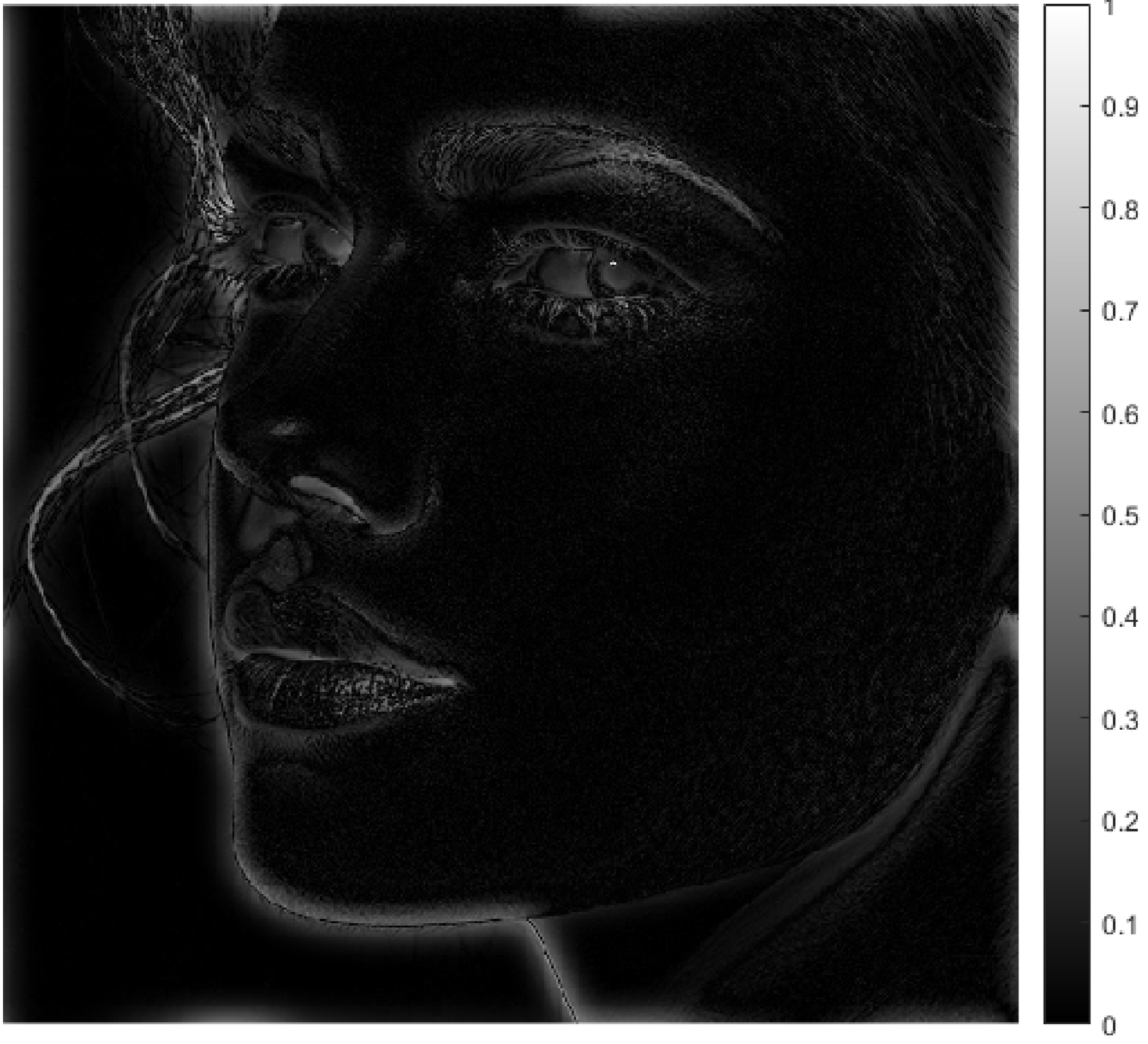
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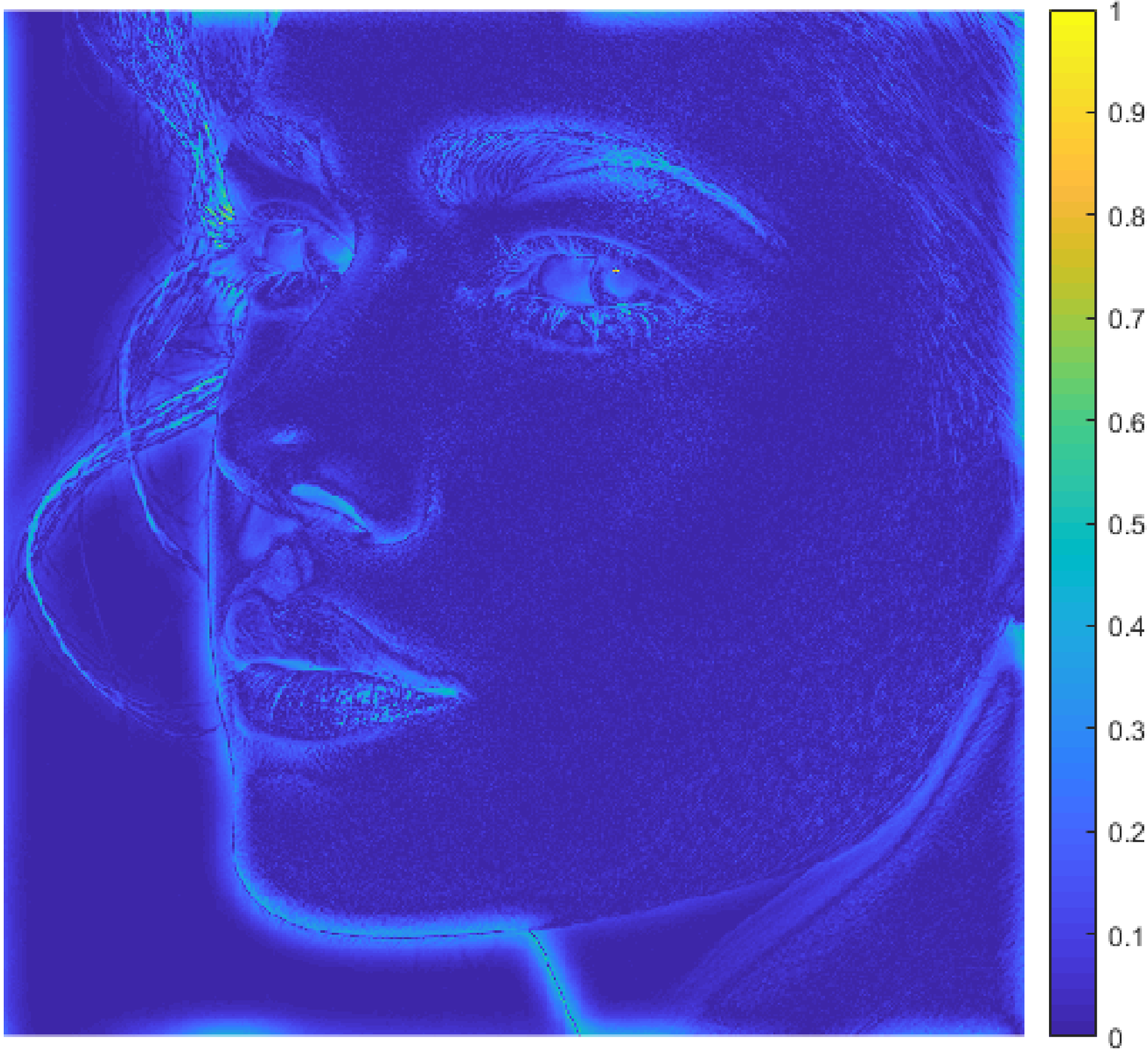
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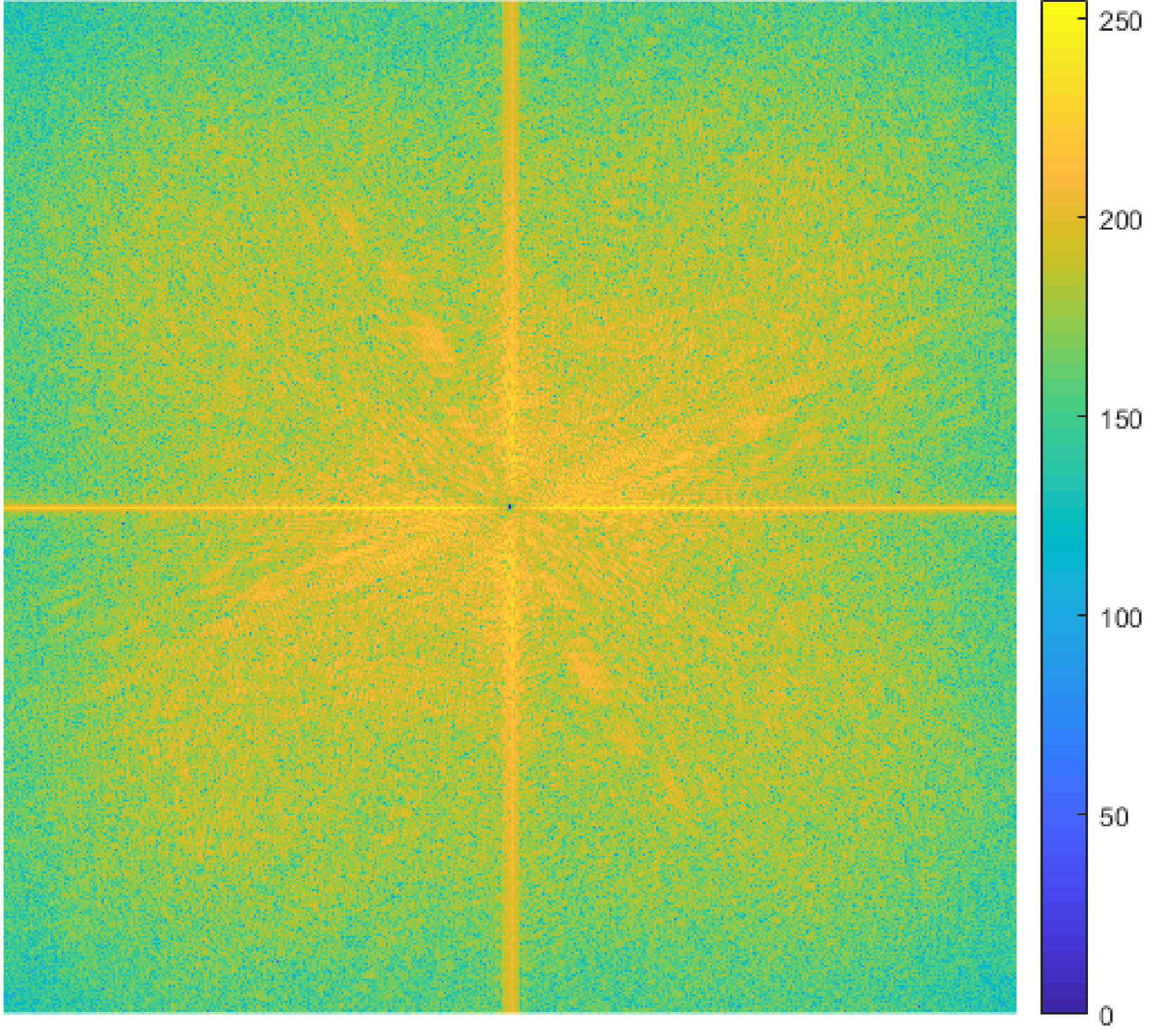
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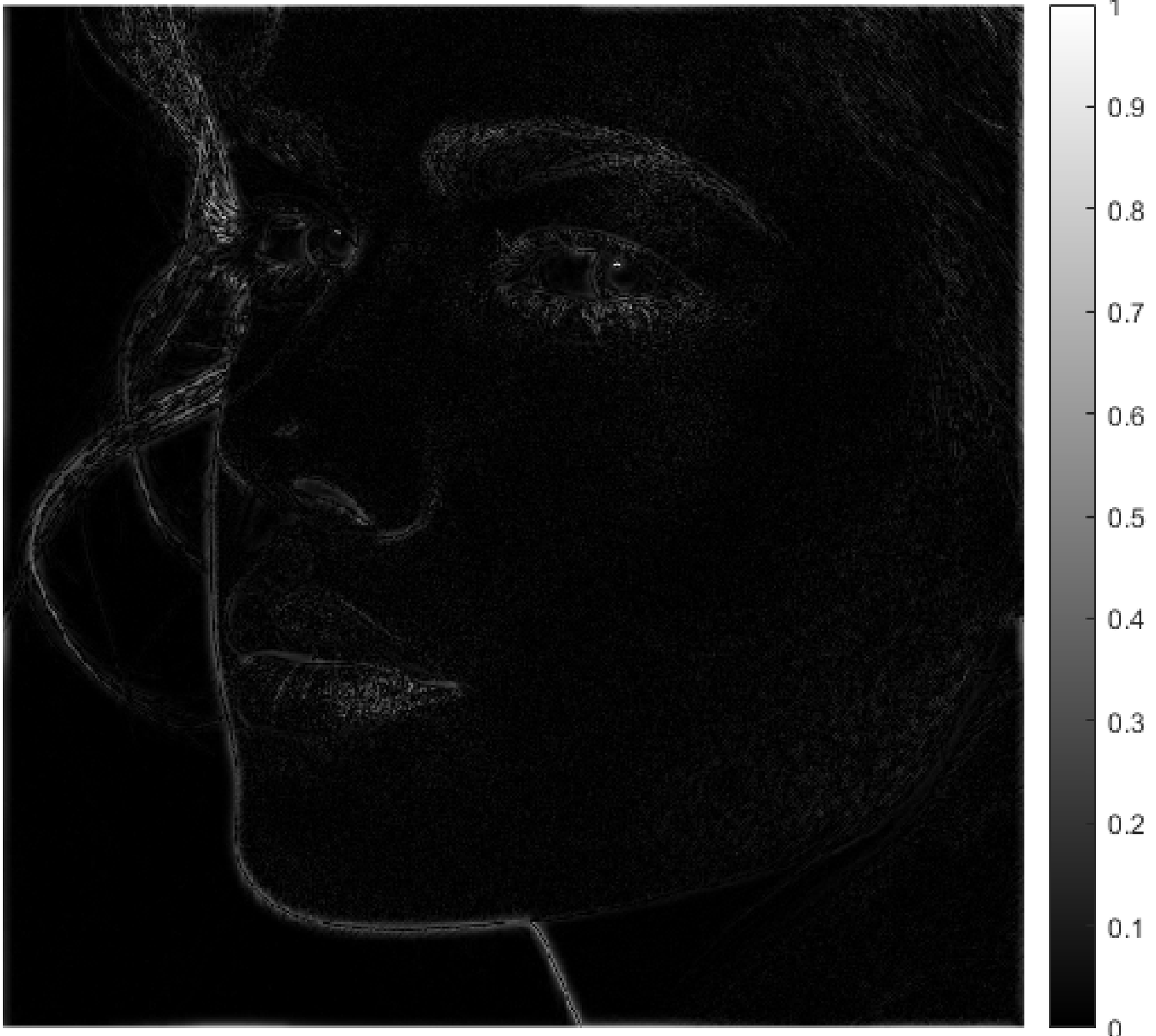
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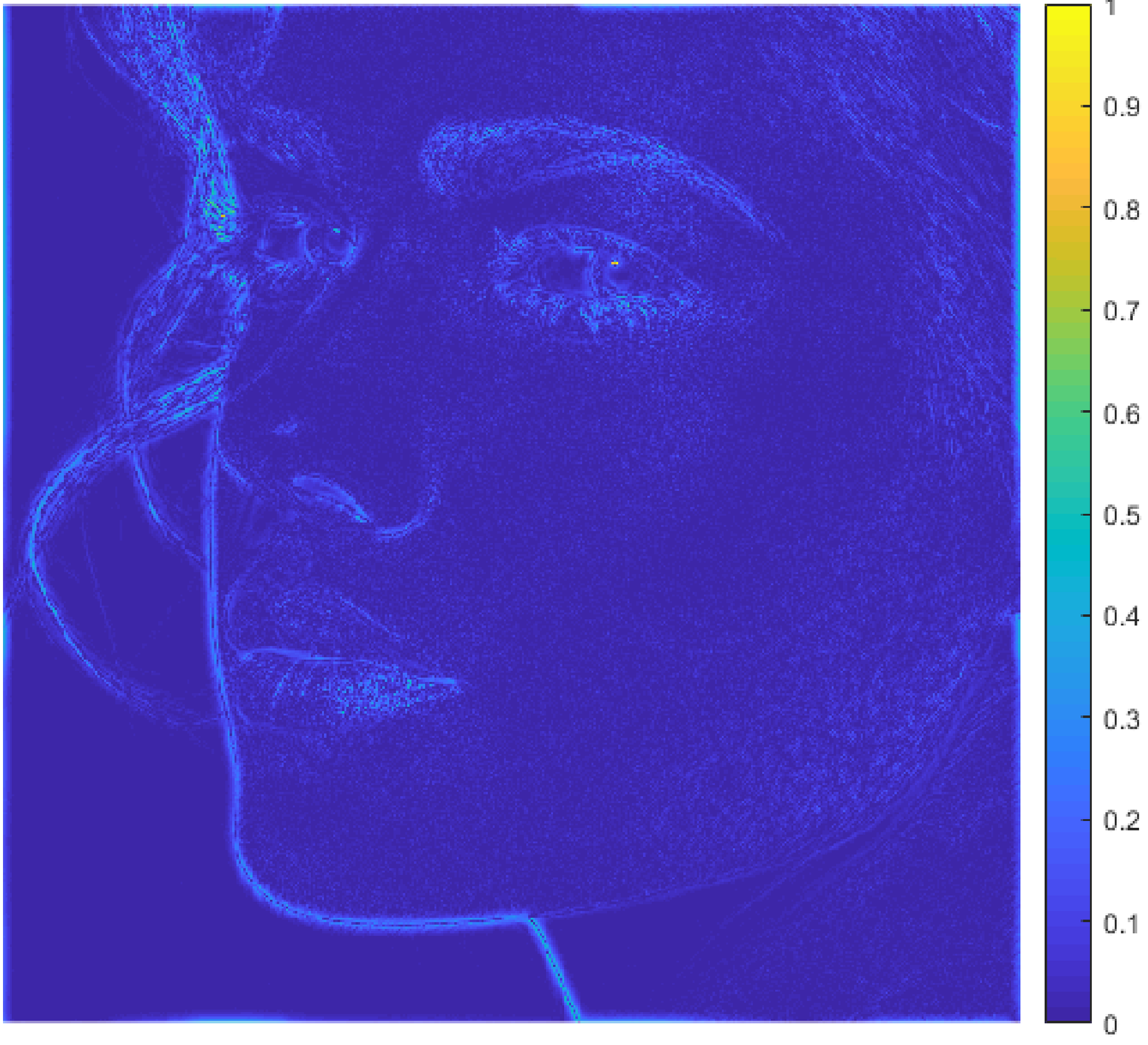
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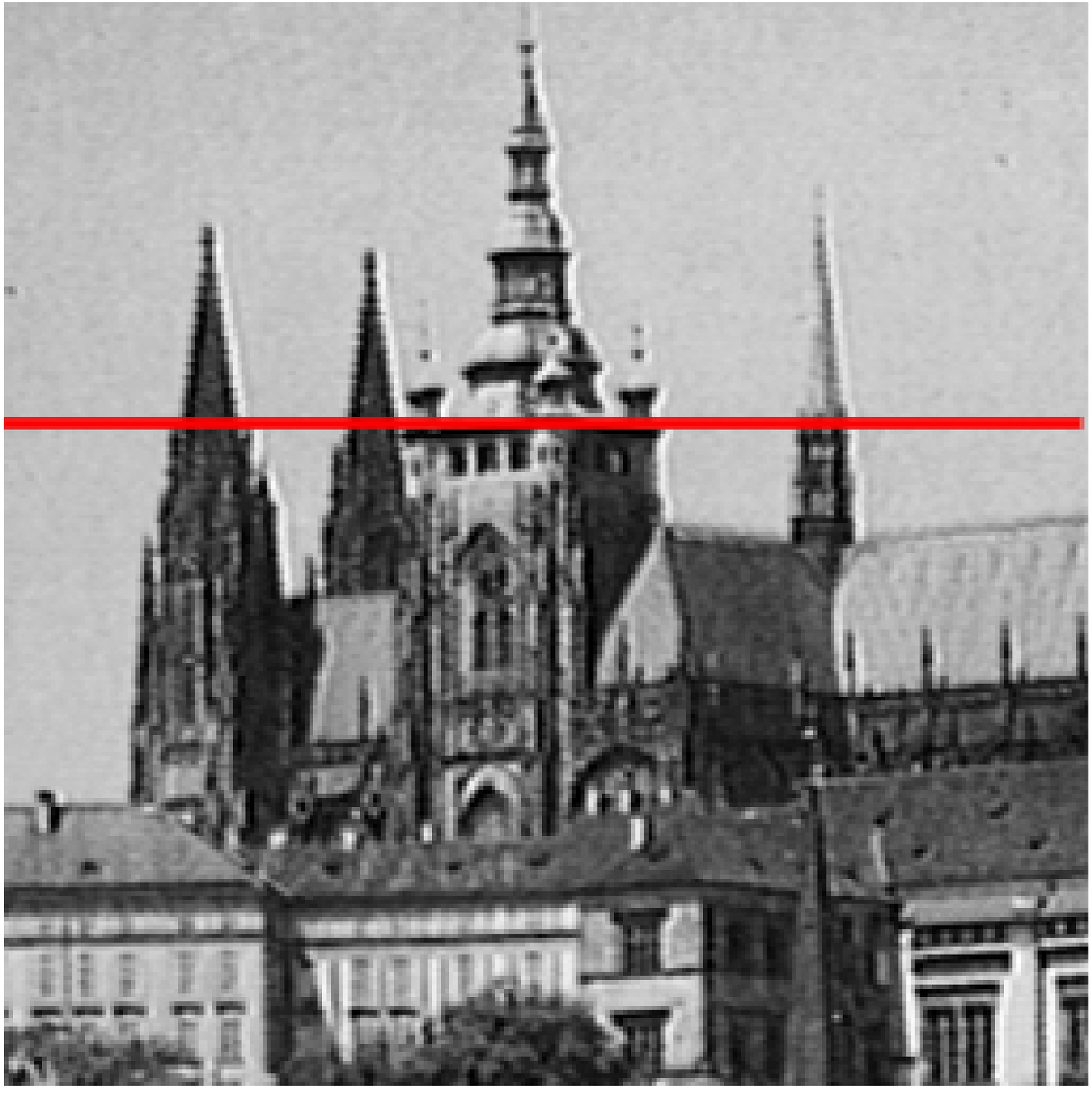
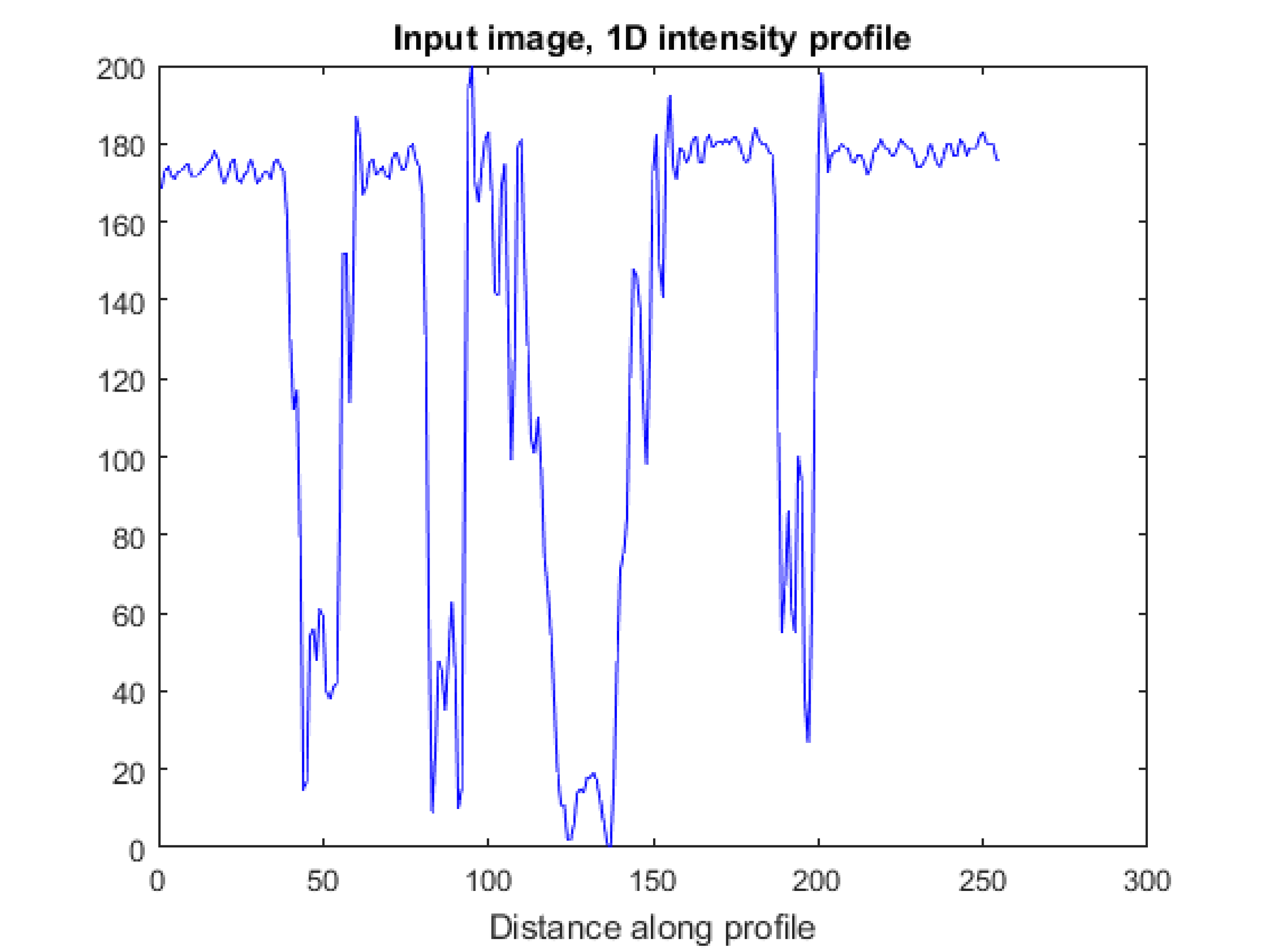
Input image

Input image, line for profile in red




Input image, 1D intensity profile
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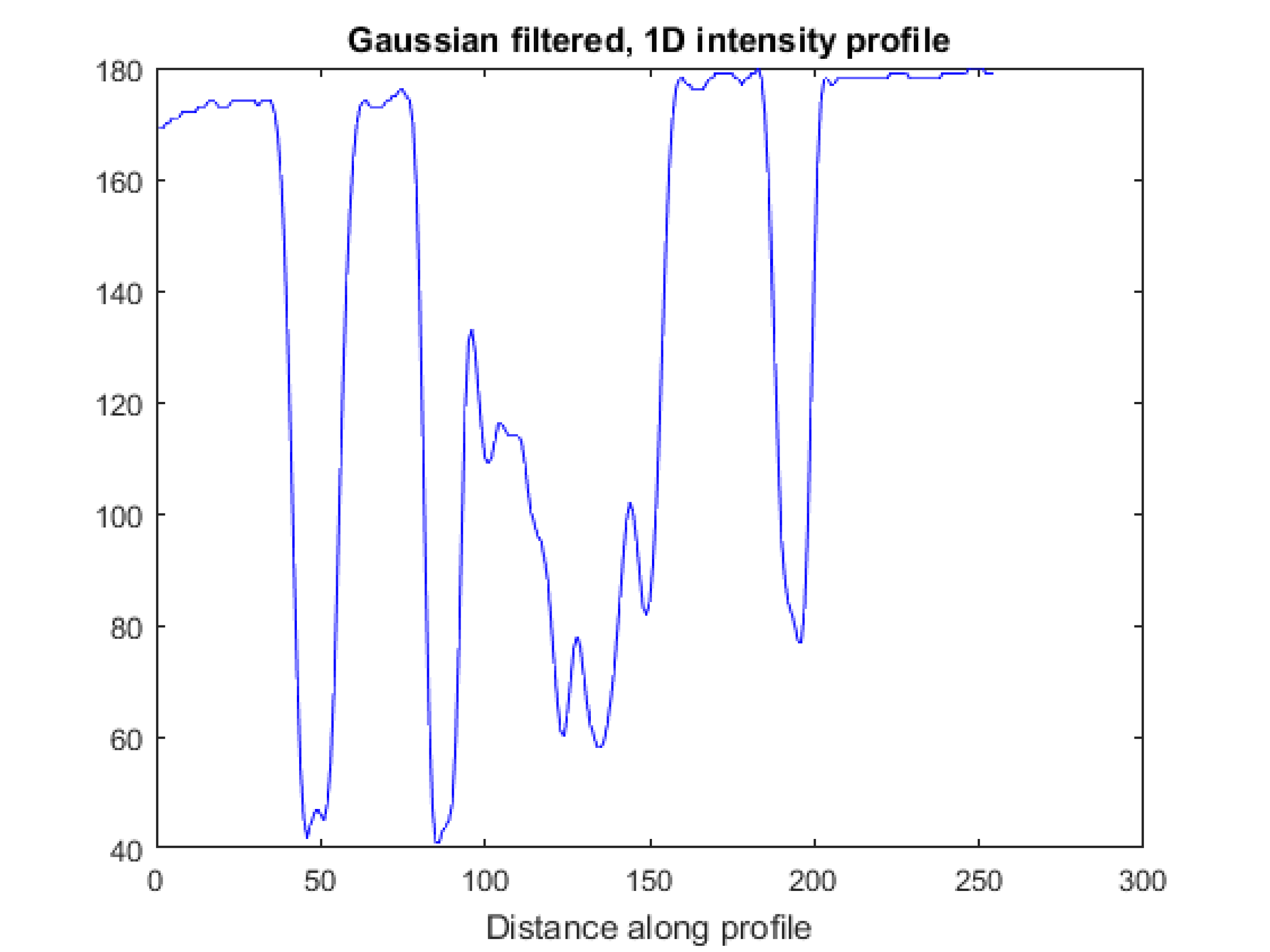
Distance along profile

Gaussian filtered

Gaussian filtered, line for profile in red


Gaussian filtered, 1D intensity profile
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Distance along profile

Input image

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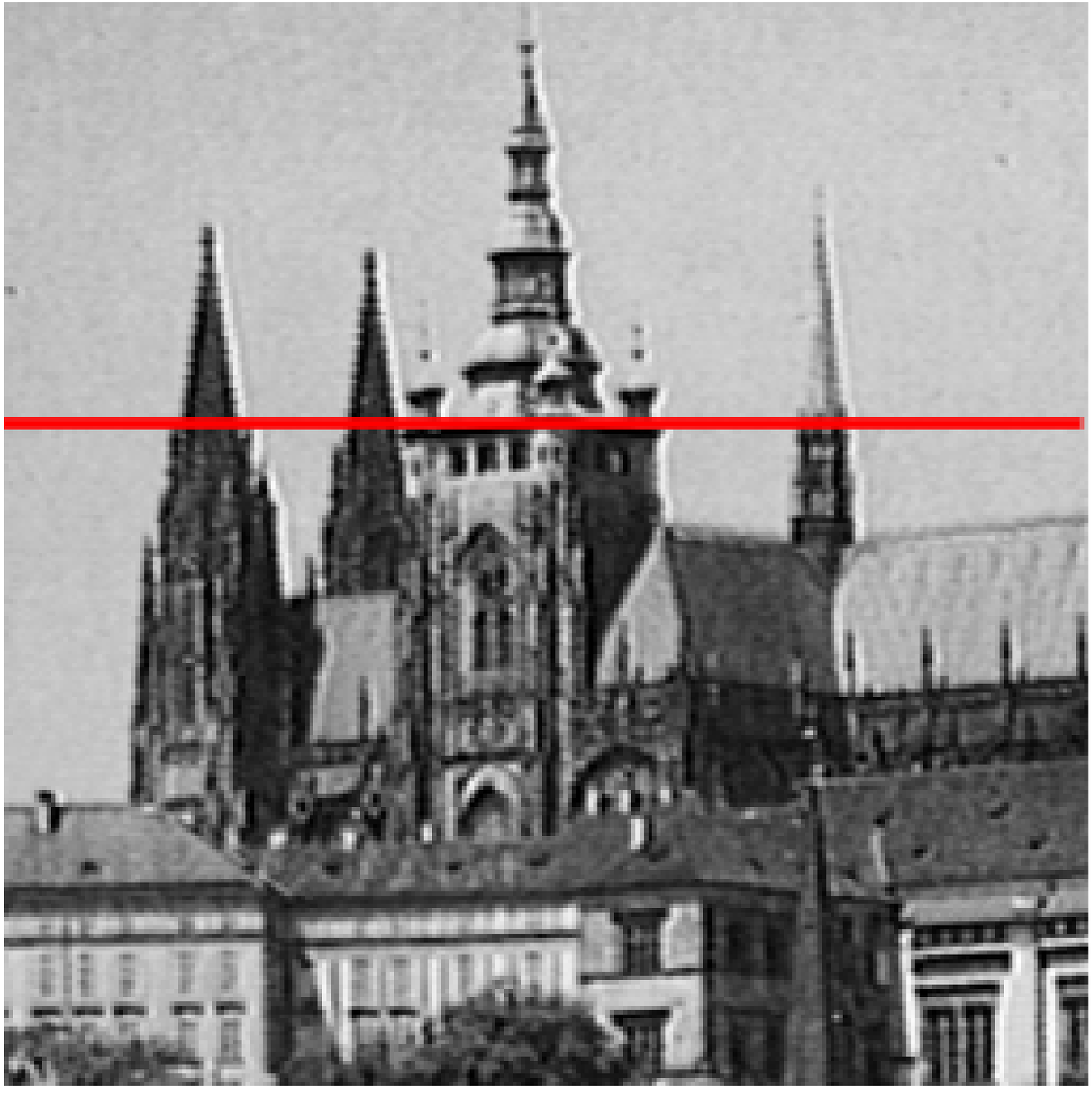

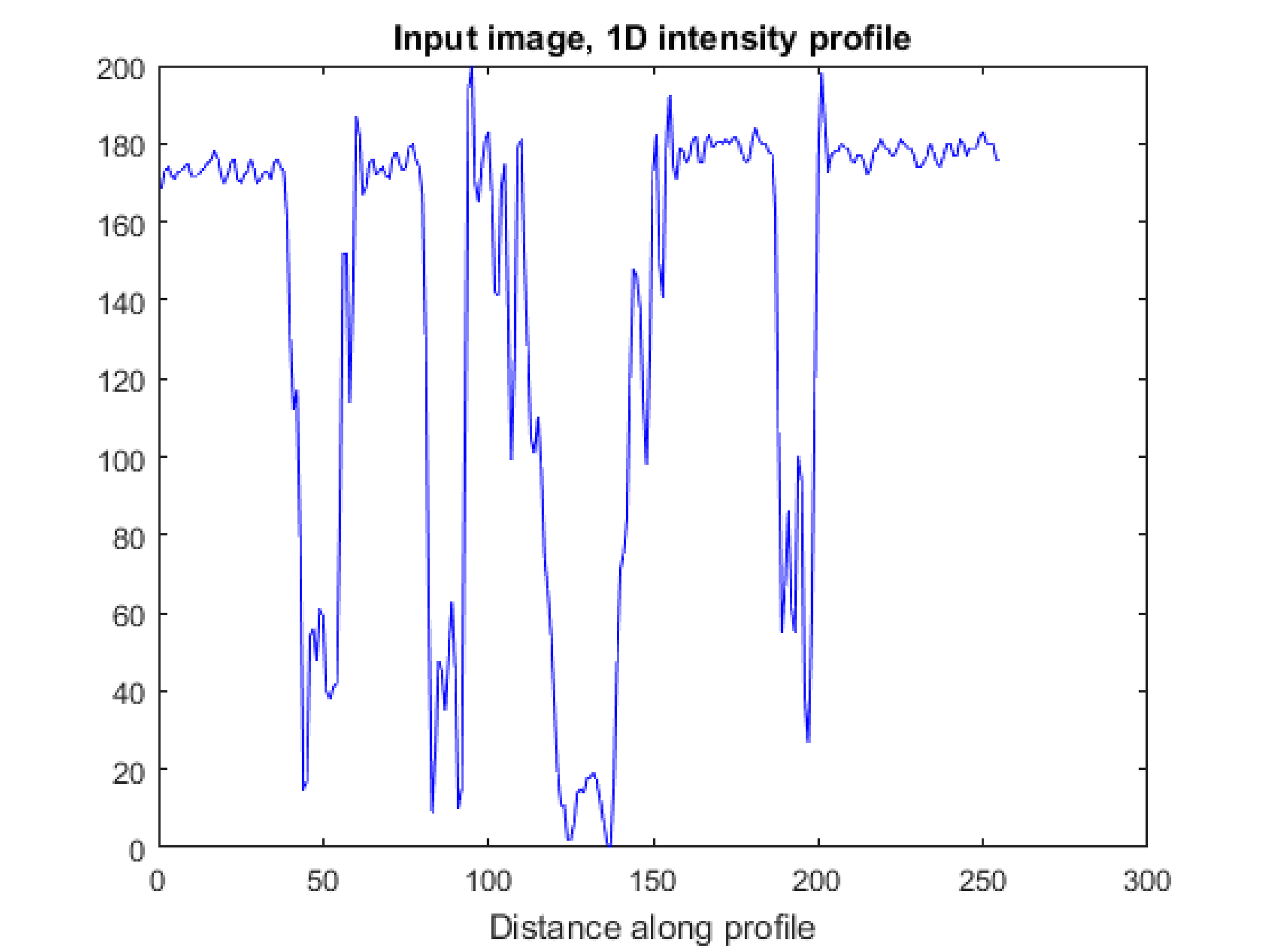
Input image, 1D intensity profile
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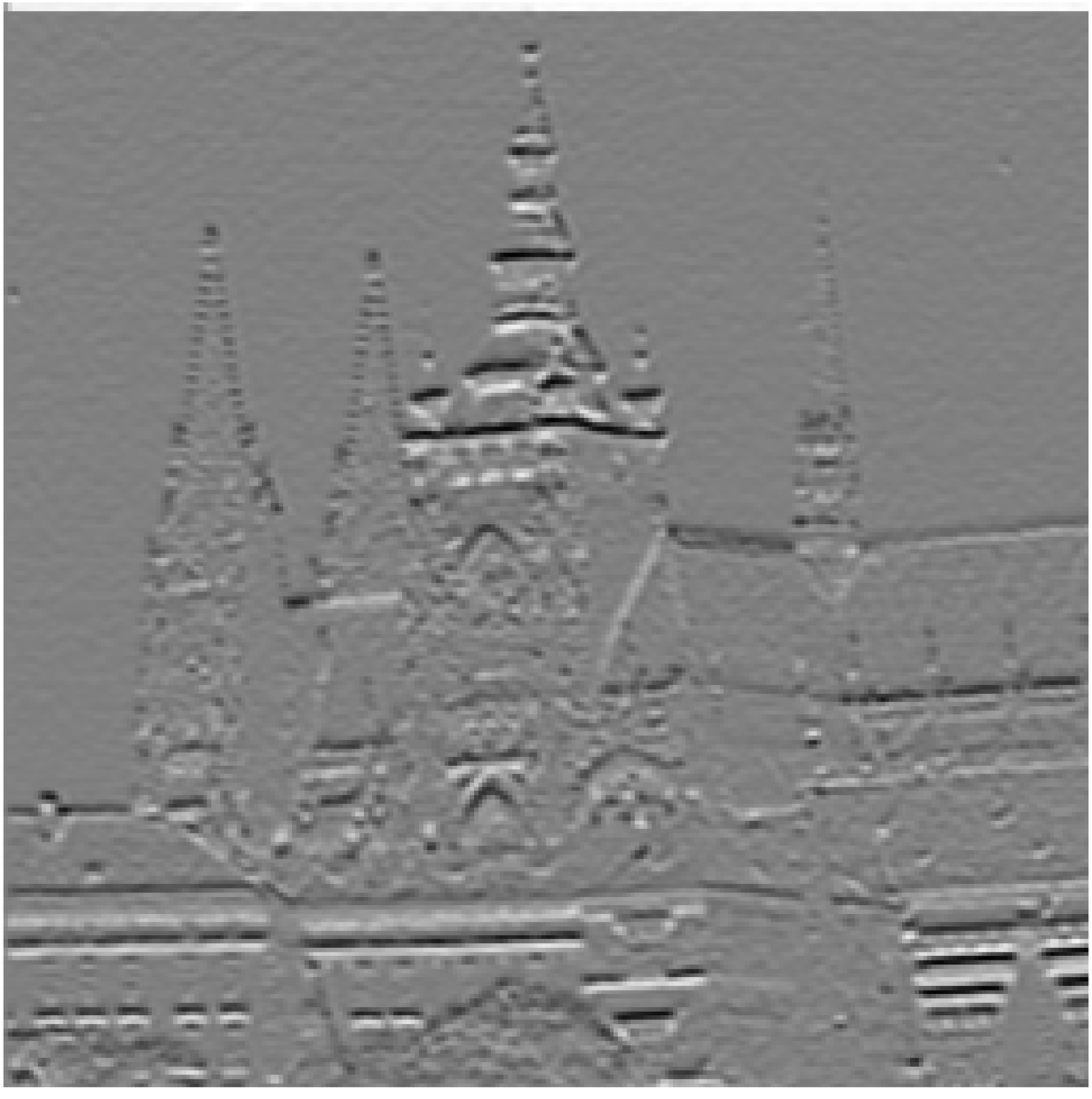
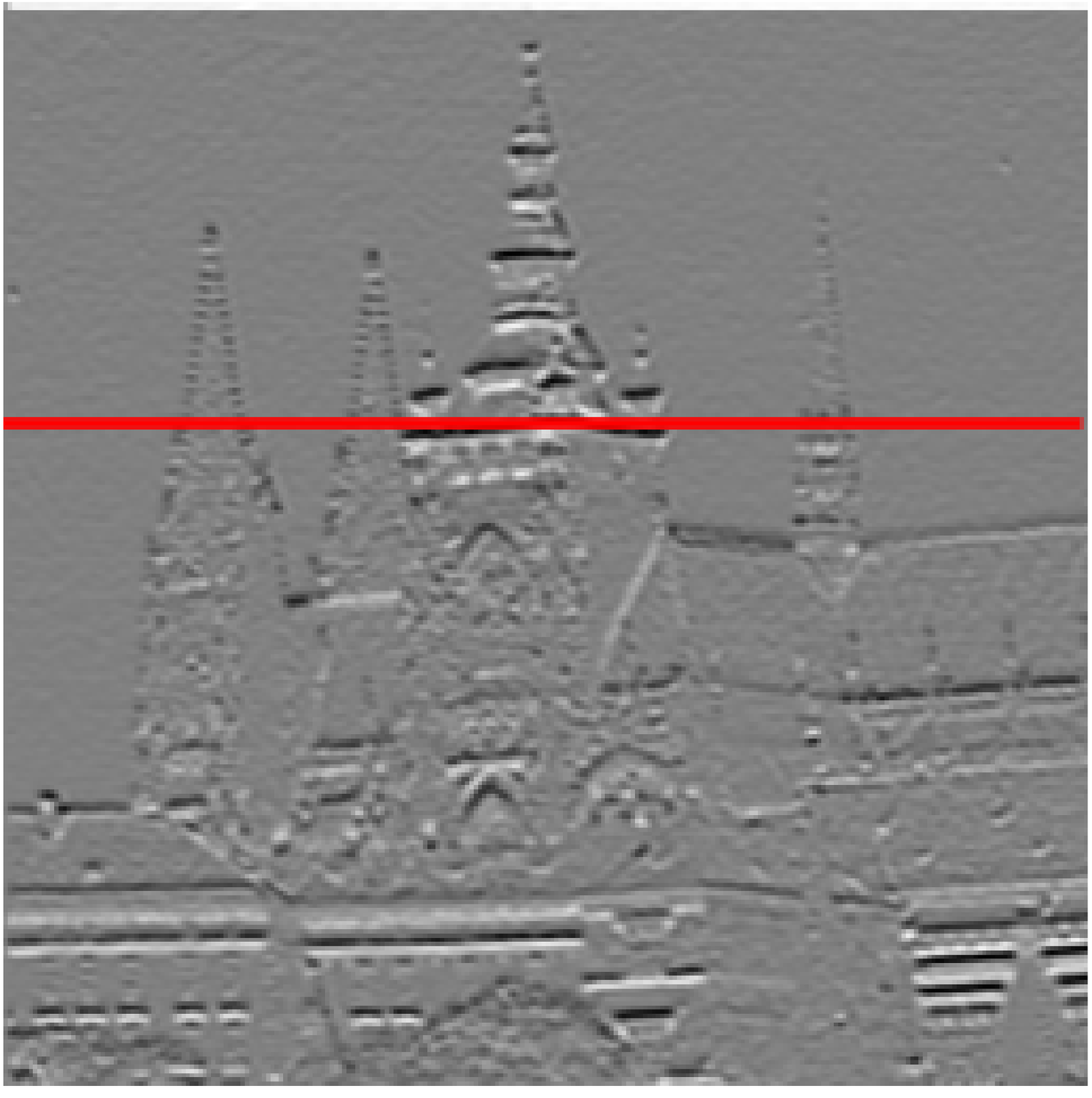
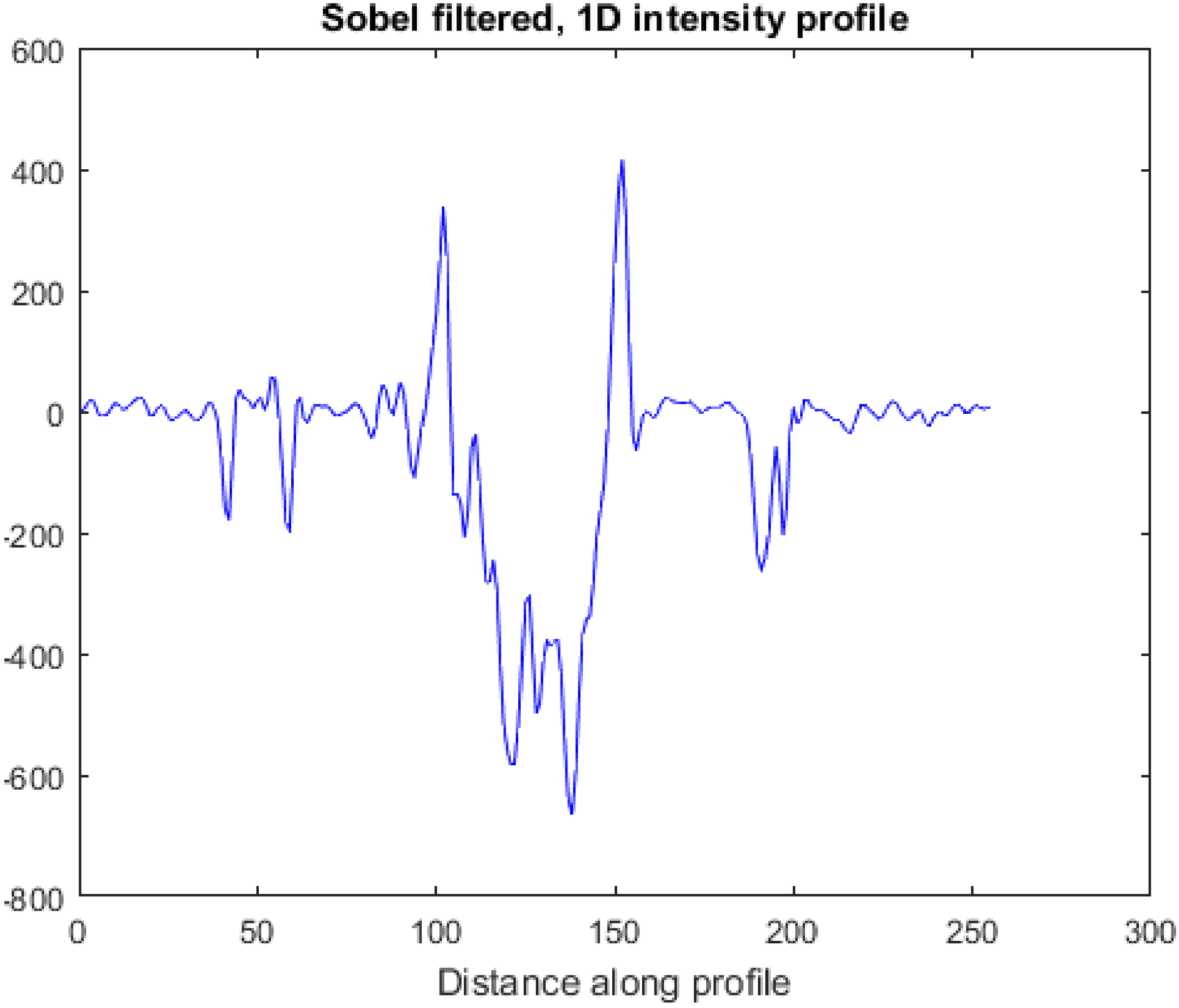
Sobel filtered

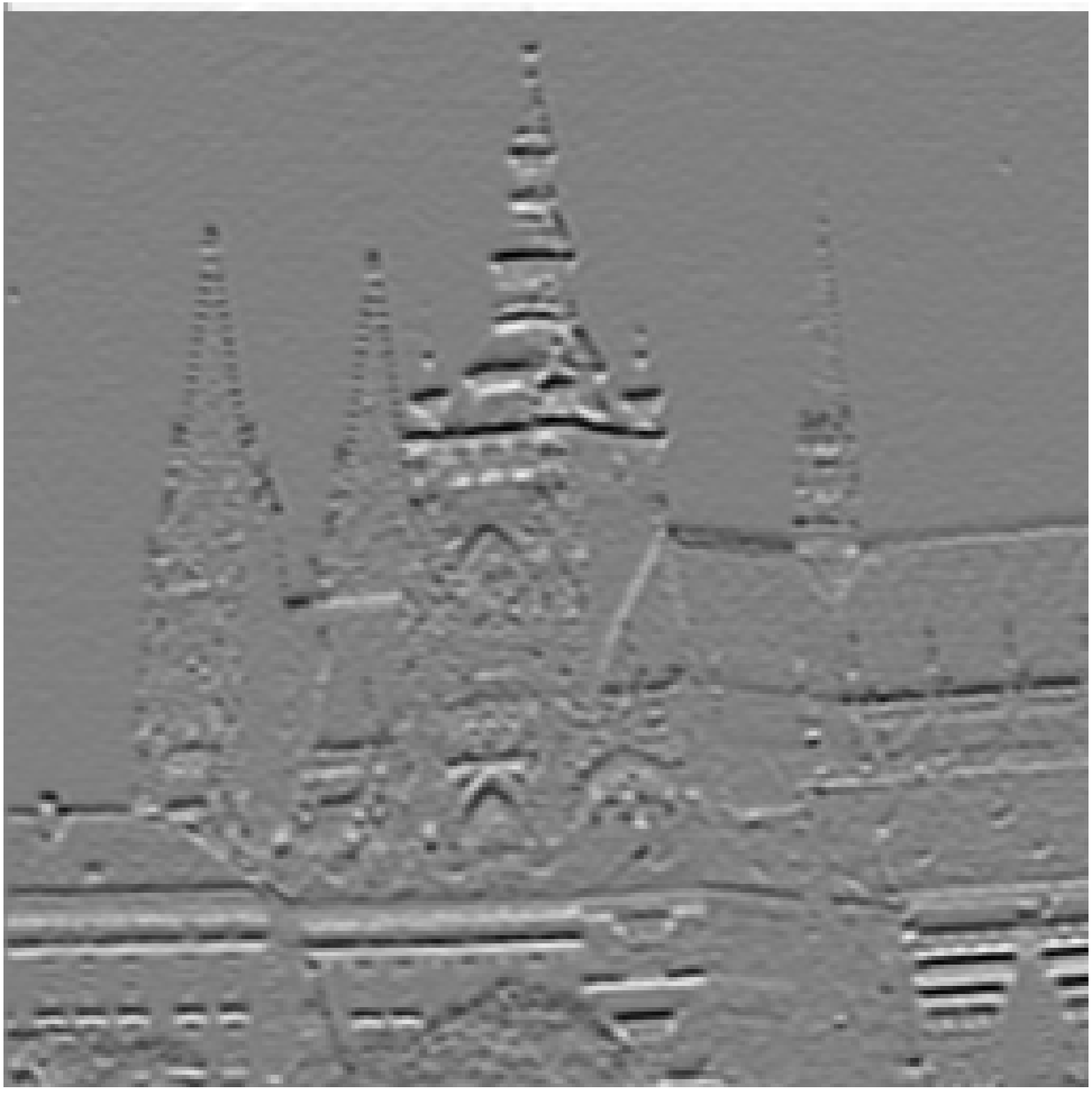


Sobel filtered, line for profile in red

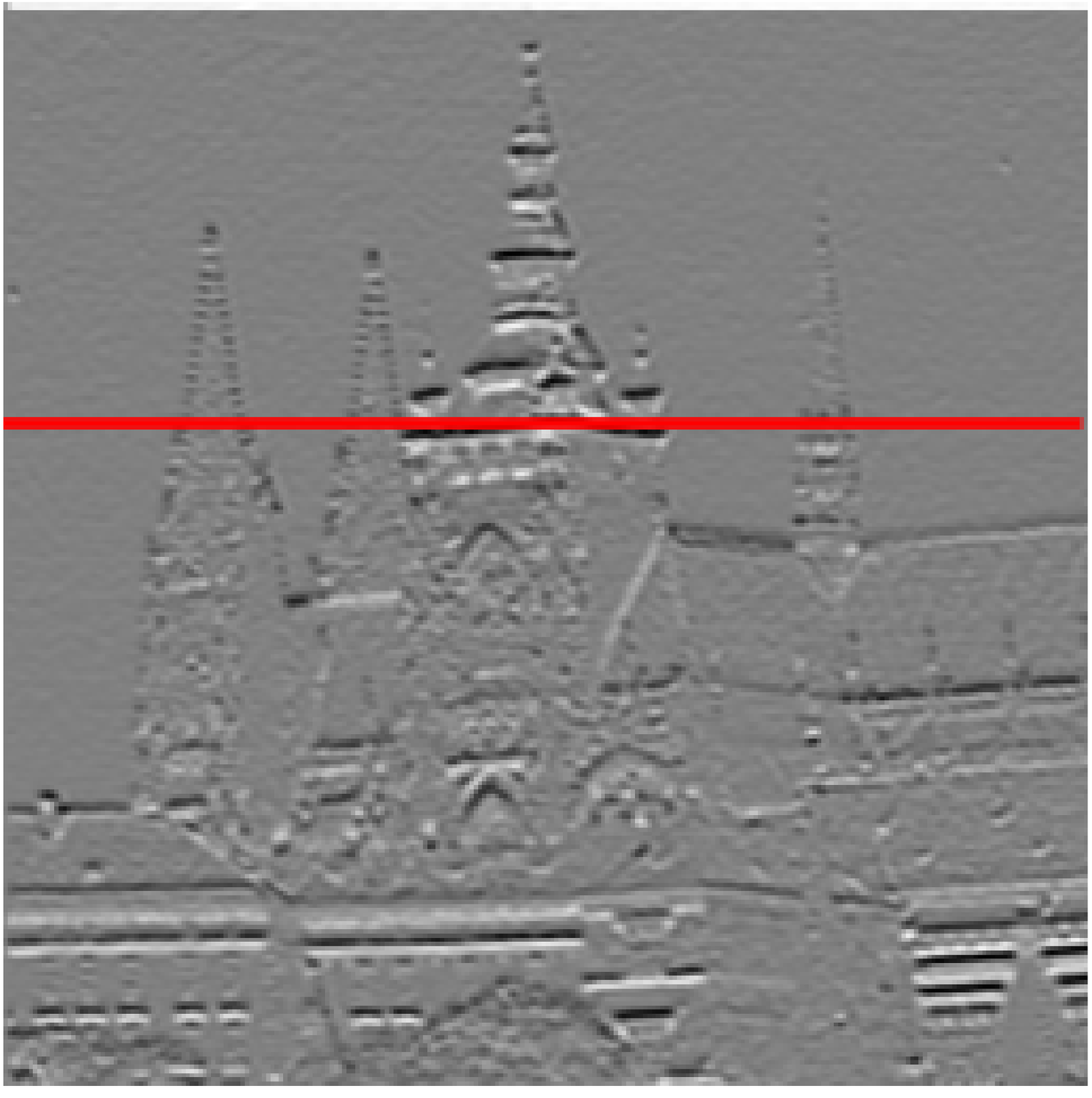


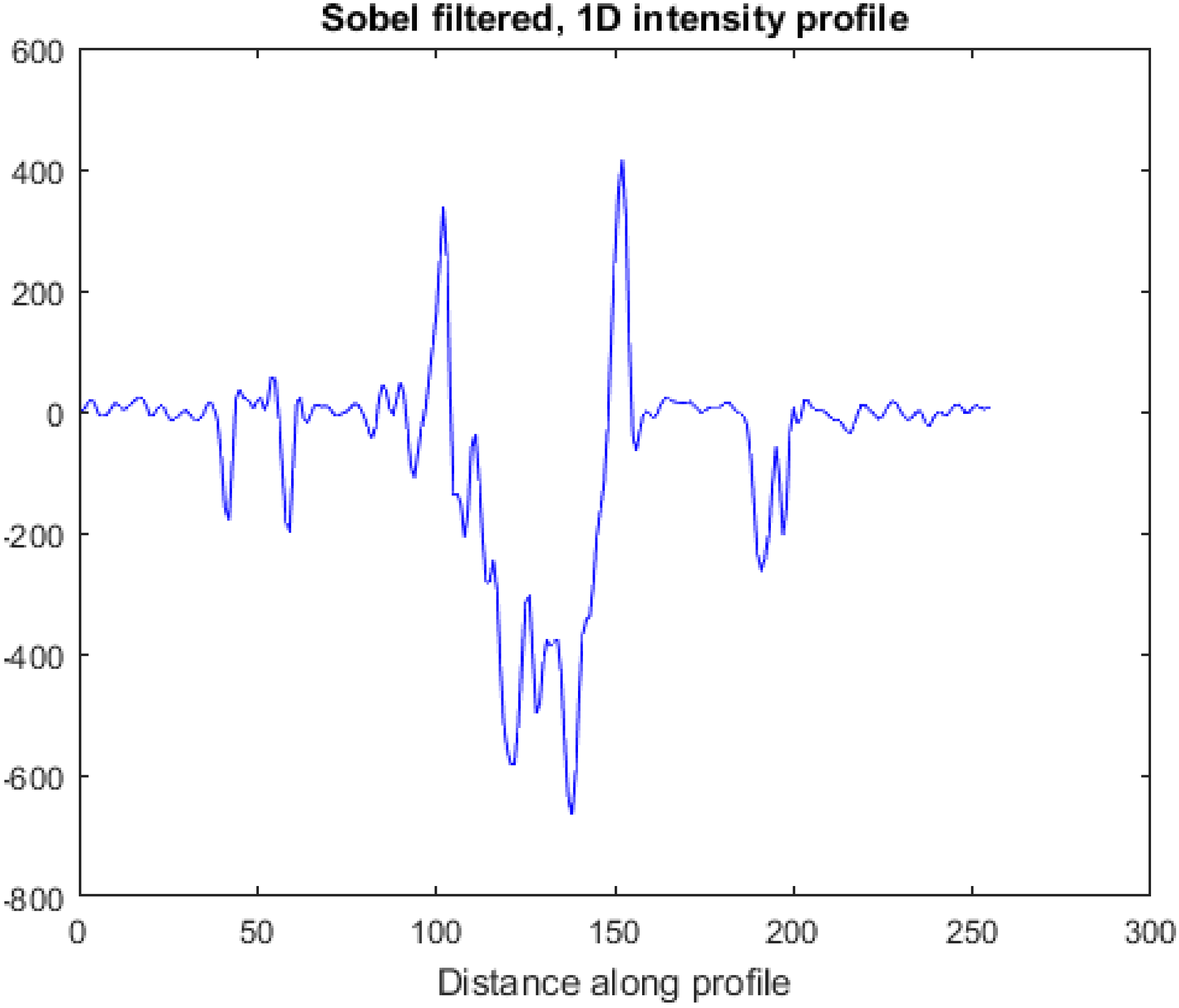

Sobel filtered, 1D intensity profile
600
400
200
0
-200
-400
-600
-800
0
50
100
150
200
250
300
Distance along profile

Sobel filtered



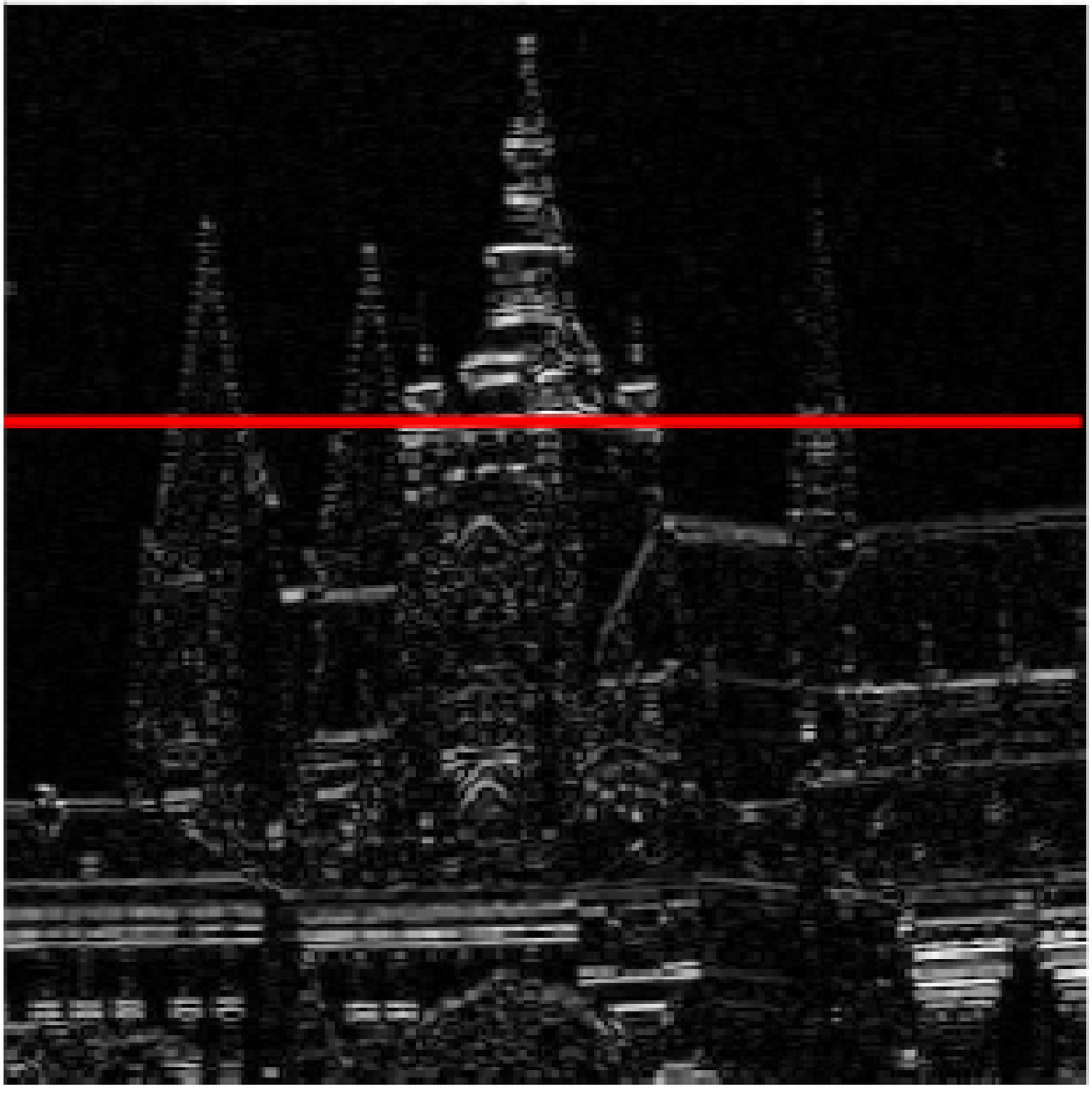
Sobel filtered, line for profile in red

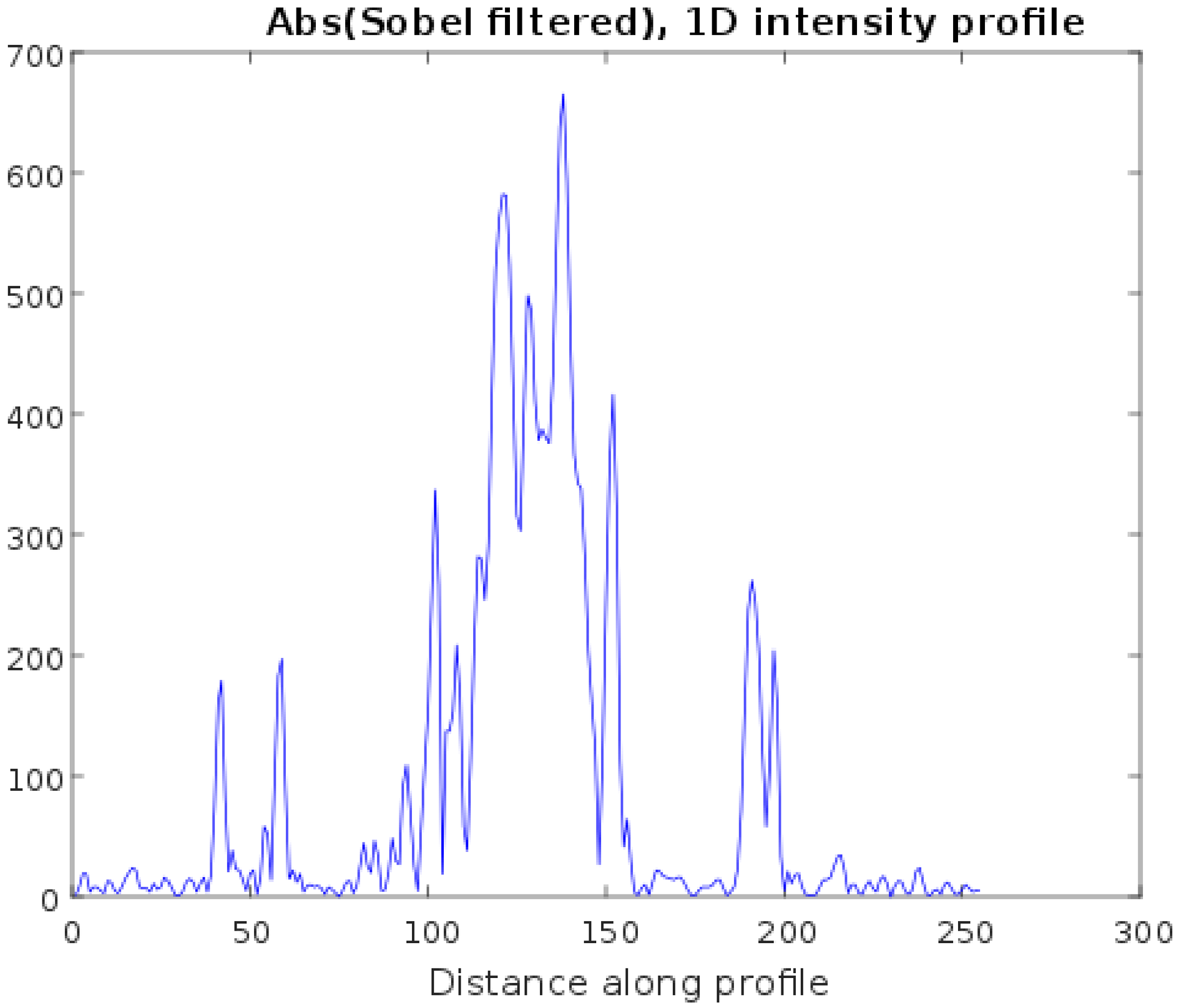


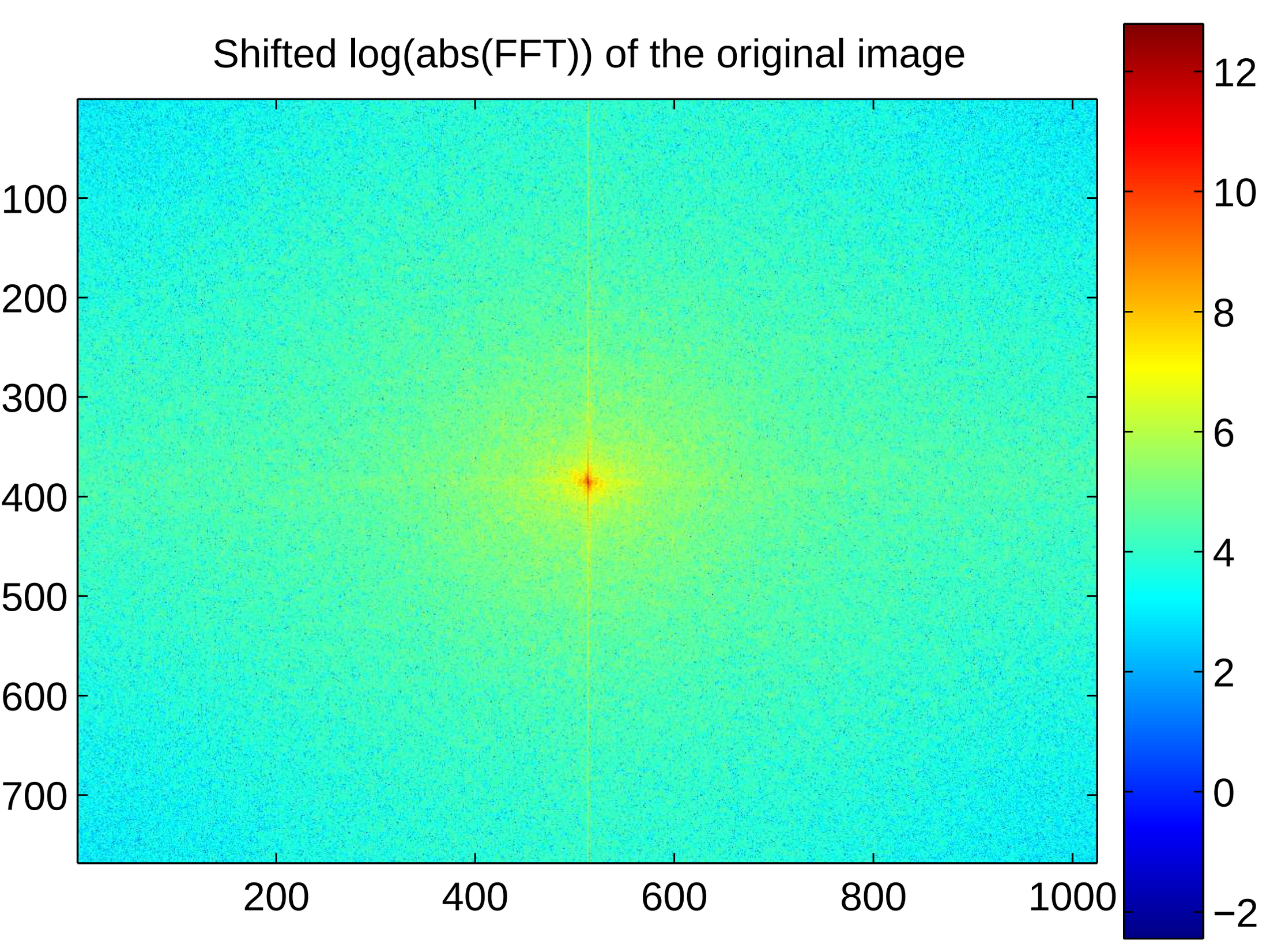

Sobel filtered, 1D intensity profile
600
400
200
0
-200
-400
-600
-800
0
50
100
150
200
250
300
Distance along profile

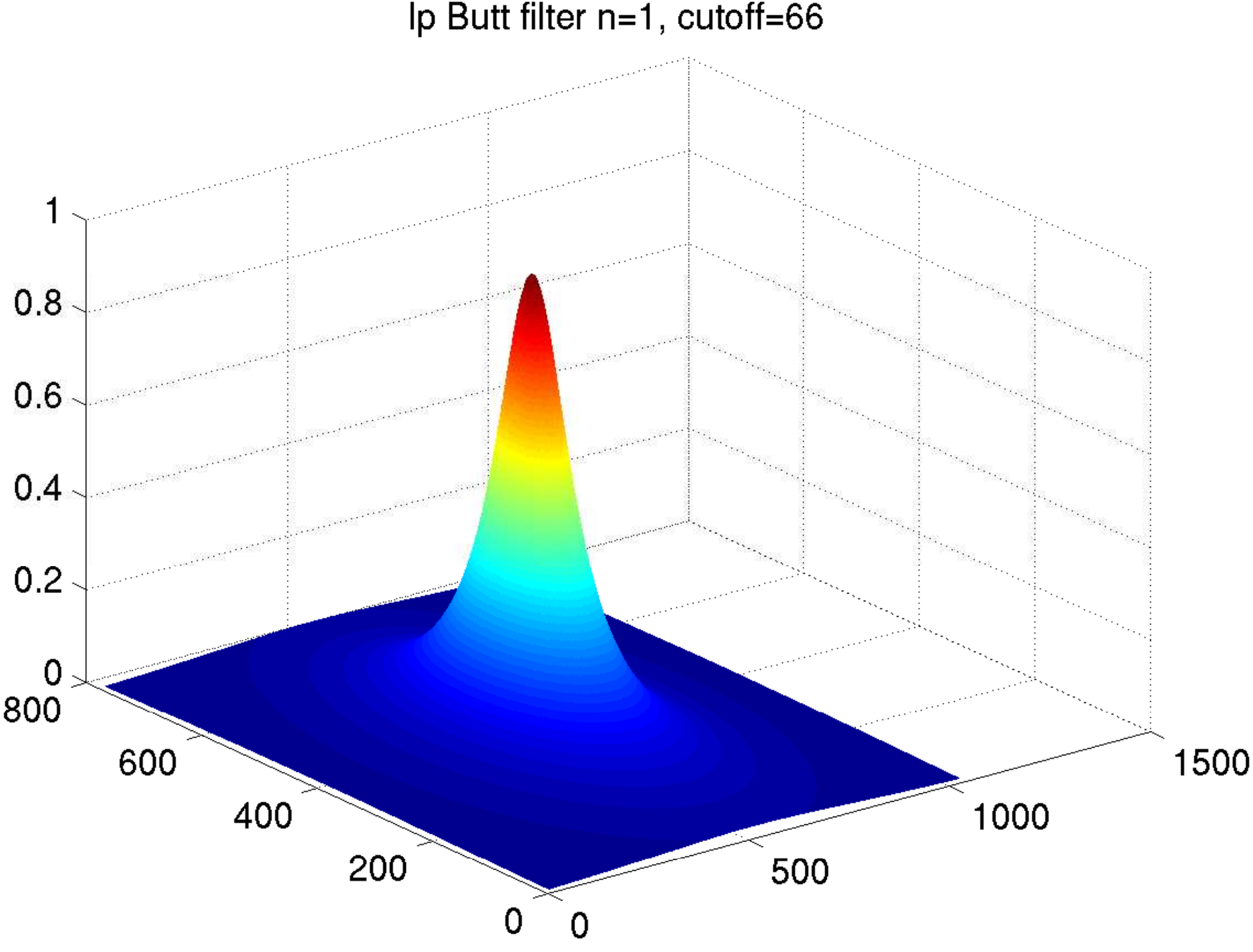
Abs(Sobel filtered)

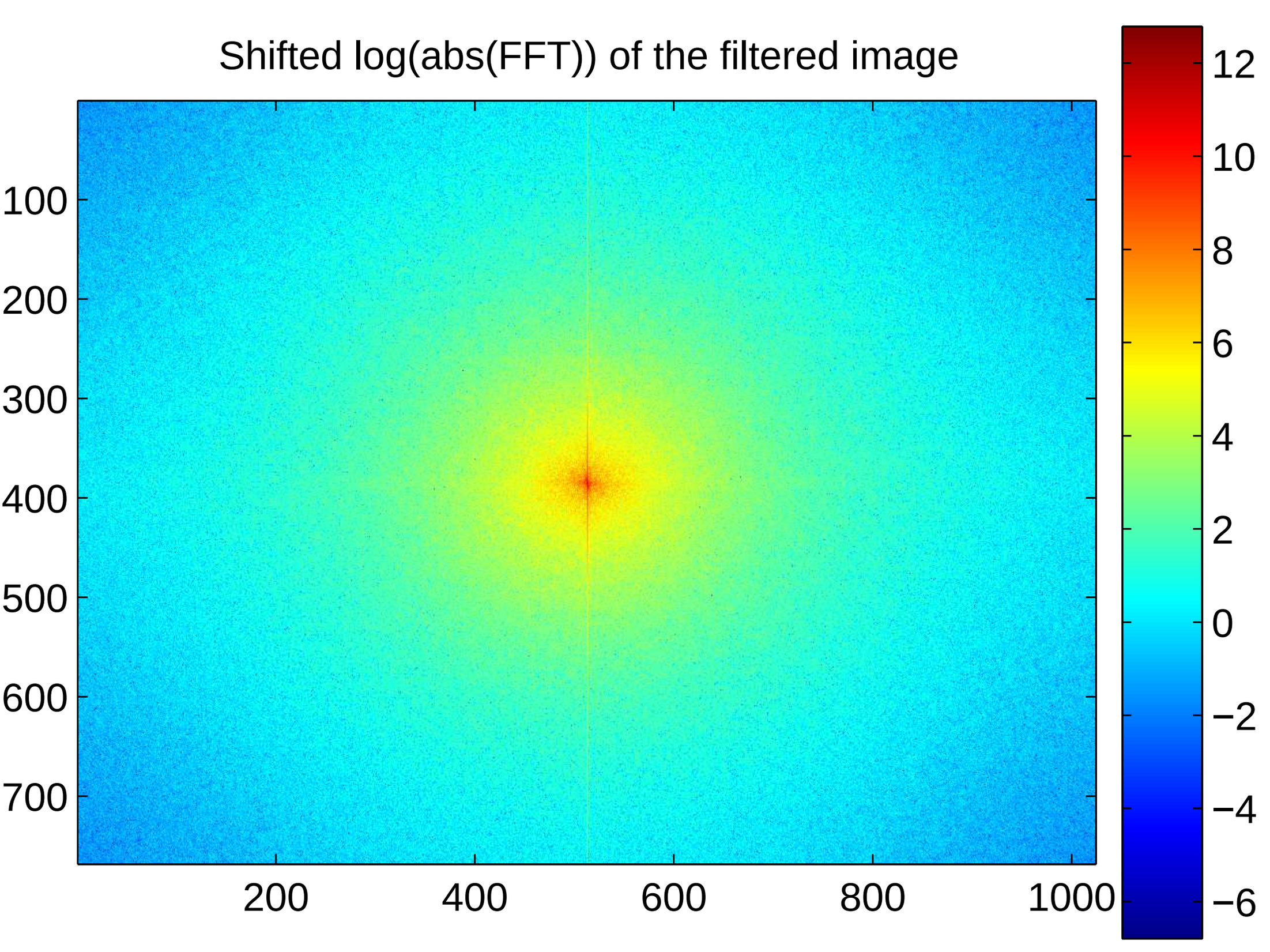
Abs(Sobel filtered)

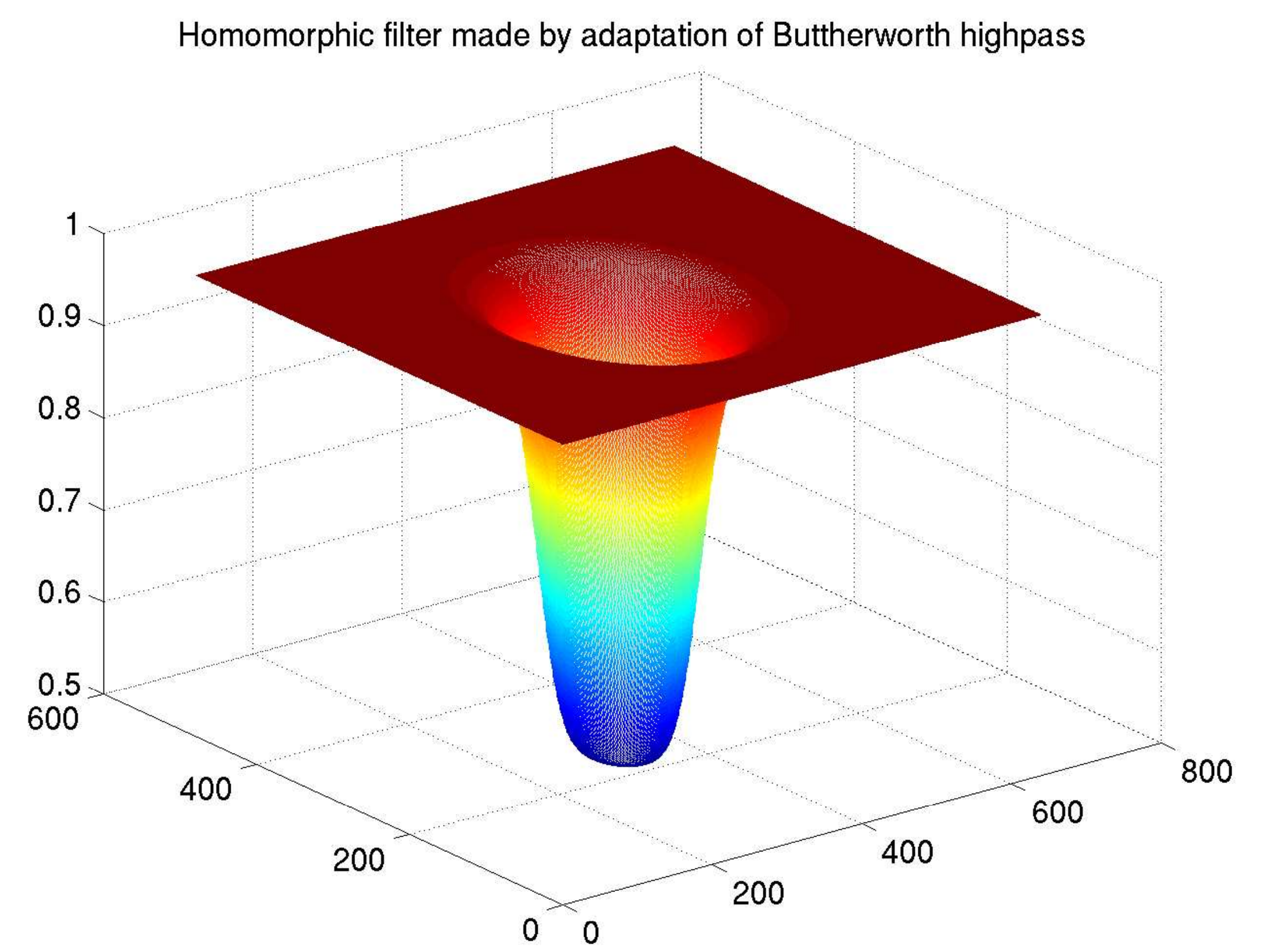


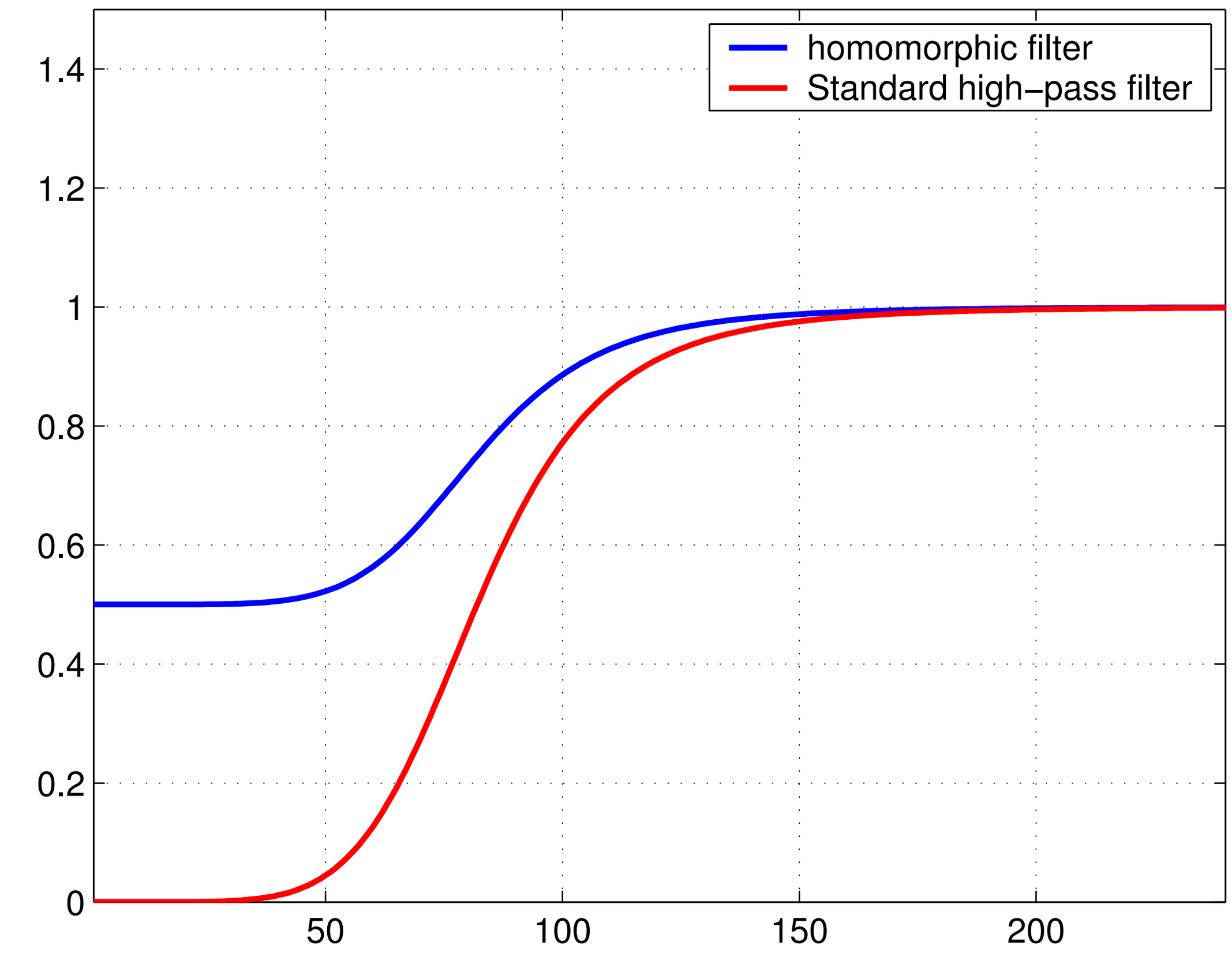

Abs(Sobel filtered), 1D intensity profile
700
600
500
400
300
200
100
0
0
50
100
150
200
250
300
Distance along profile

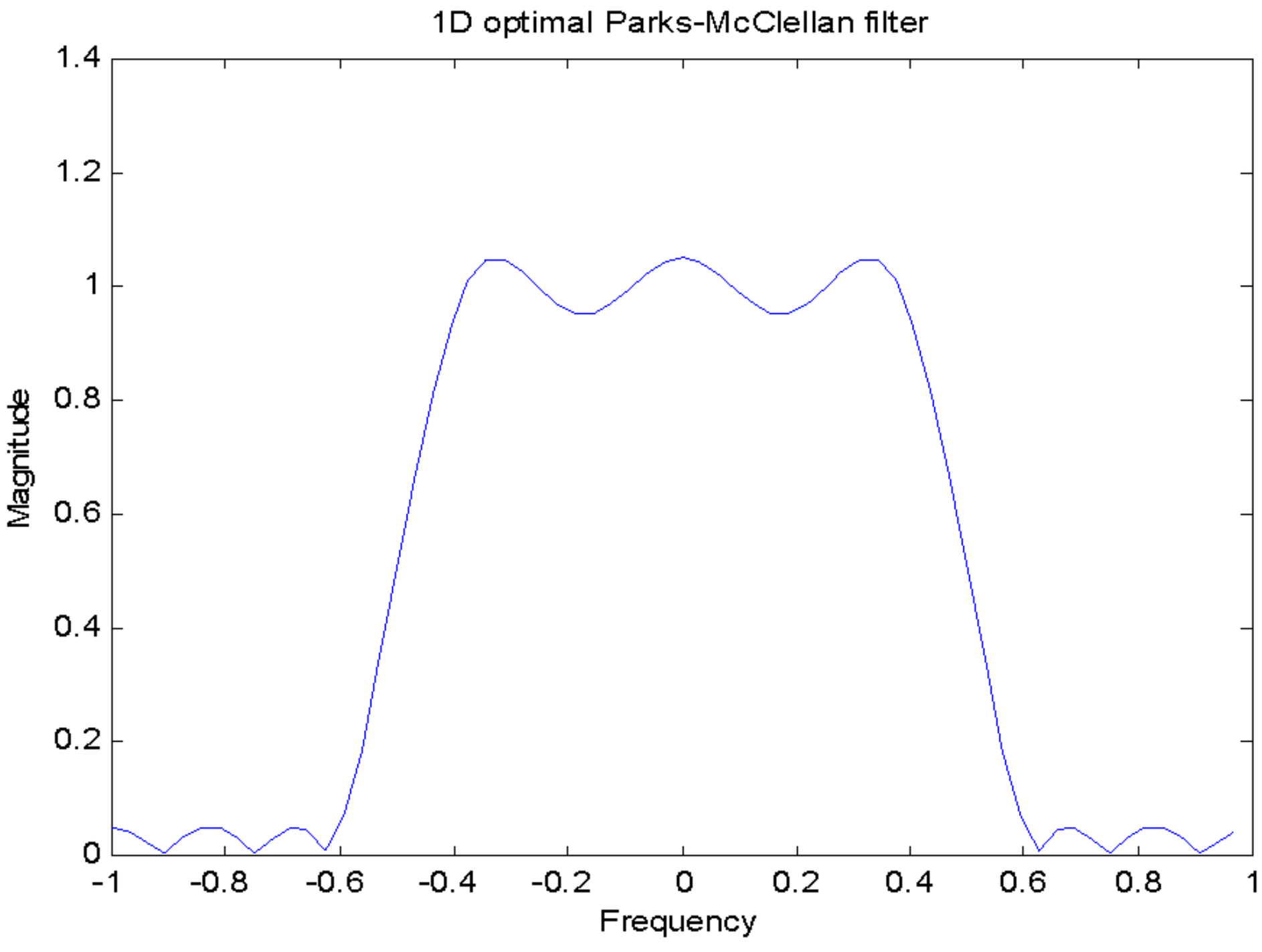

Shifted log(abs(FFT)) of the original image
100
200
300
400
500
600
700
200
400
600
800
1000
12
10
8
6
4
2
0
−2

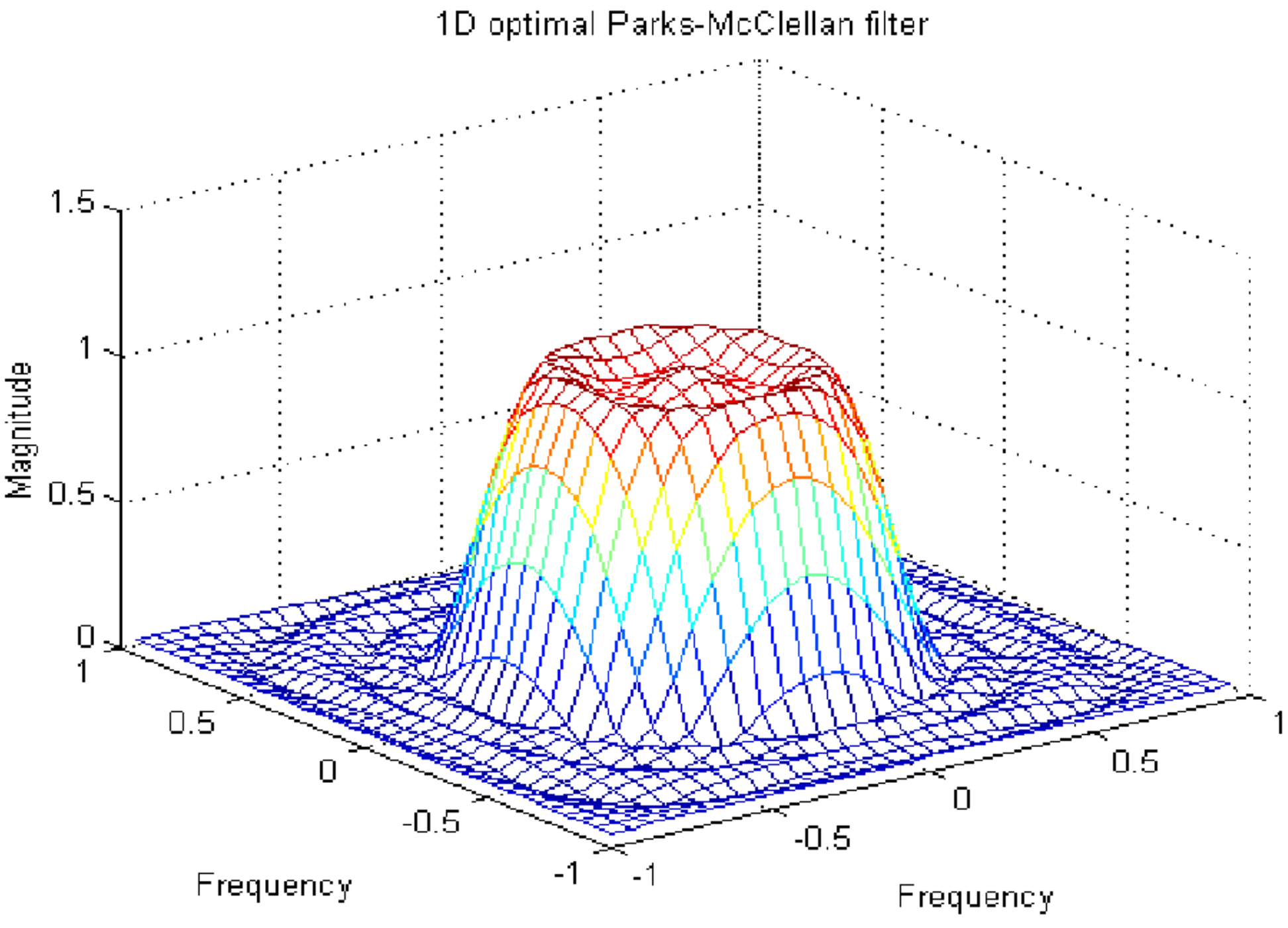

lp Butt filter n=1, cutoff=66
1
0.8
0.6
0.4
0.2
0
800
600
400
200
0
0
500
1000
1500

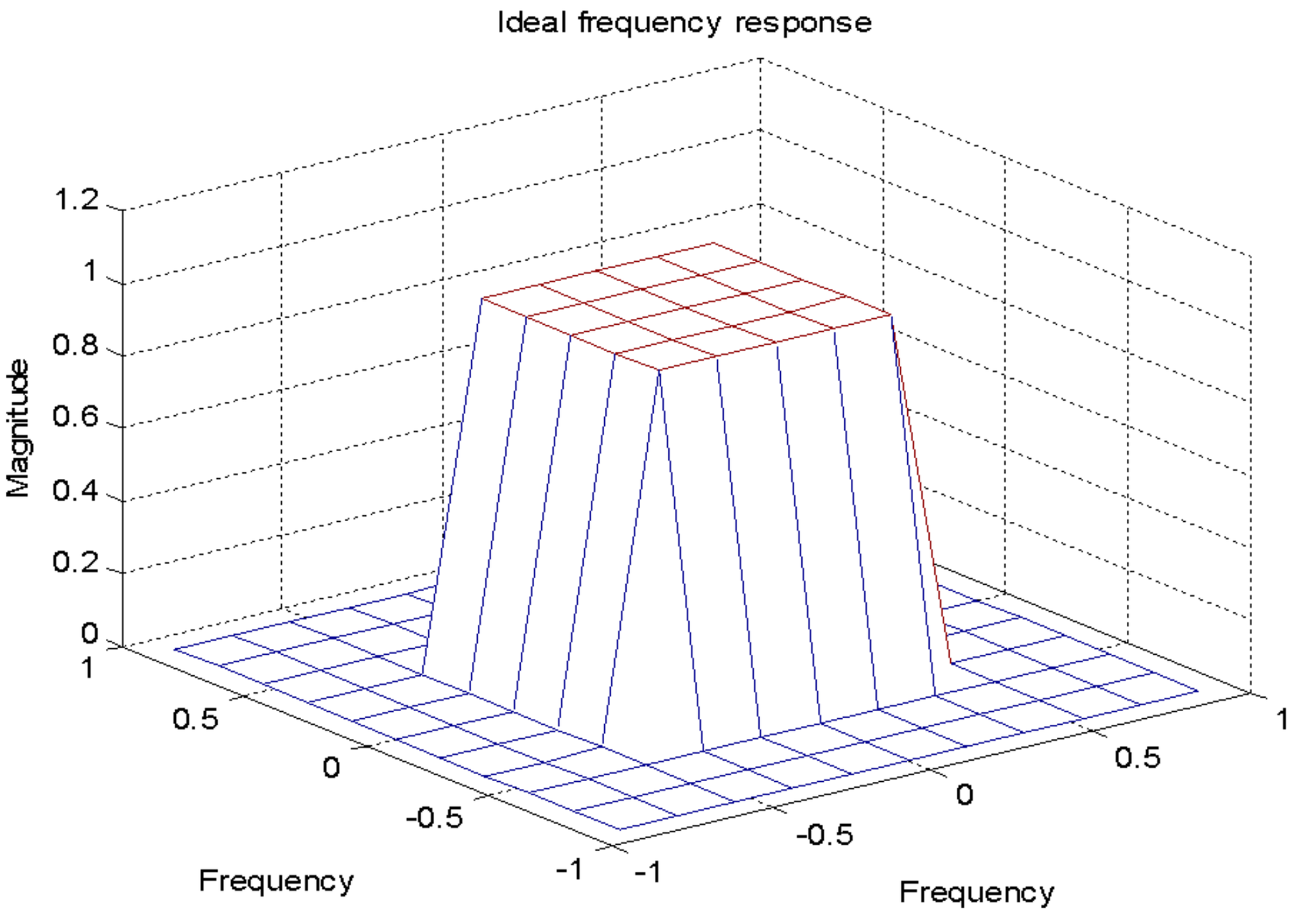

Shifted log(abs(FFT)) of the filtered image
100
200
300
400
500
600
700
200
400
600
800
1000
12
10
8
6
4
2
0
−2
−4
−6

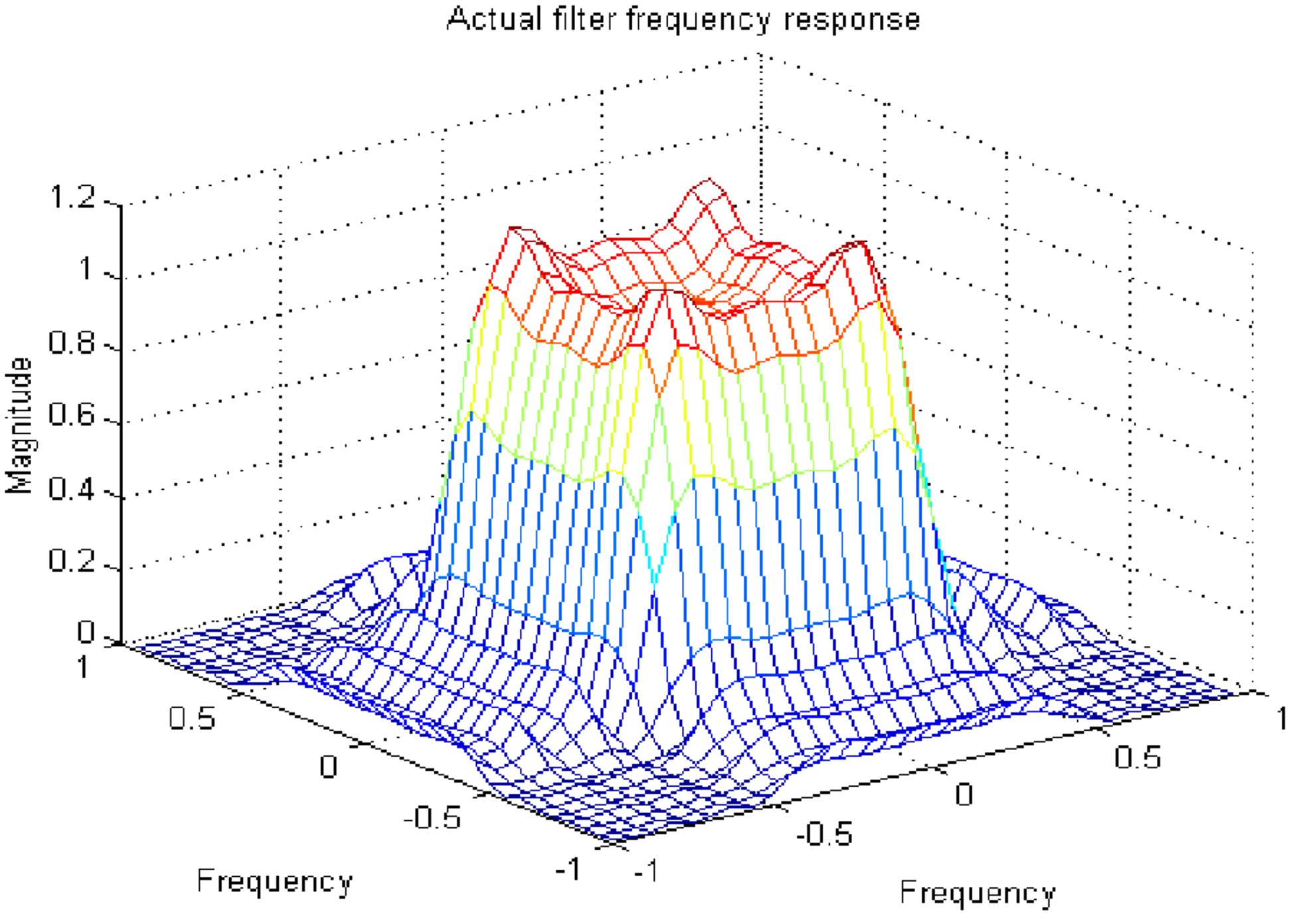

Homomorphic filter made by adaptation of Buttherworth highpass
1
0.9
0.8
0.7
0.6
0.5
600
400
200
0
0
200
400
600
800

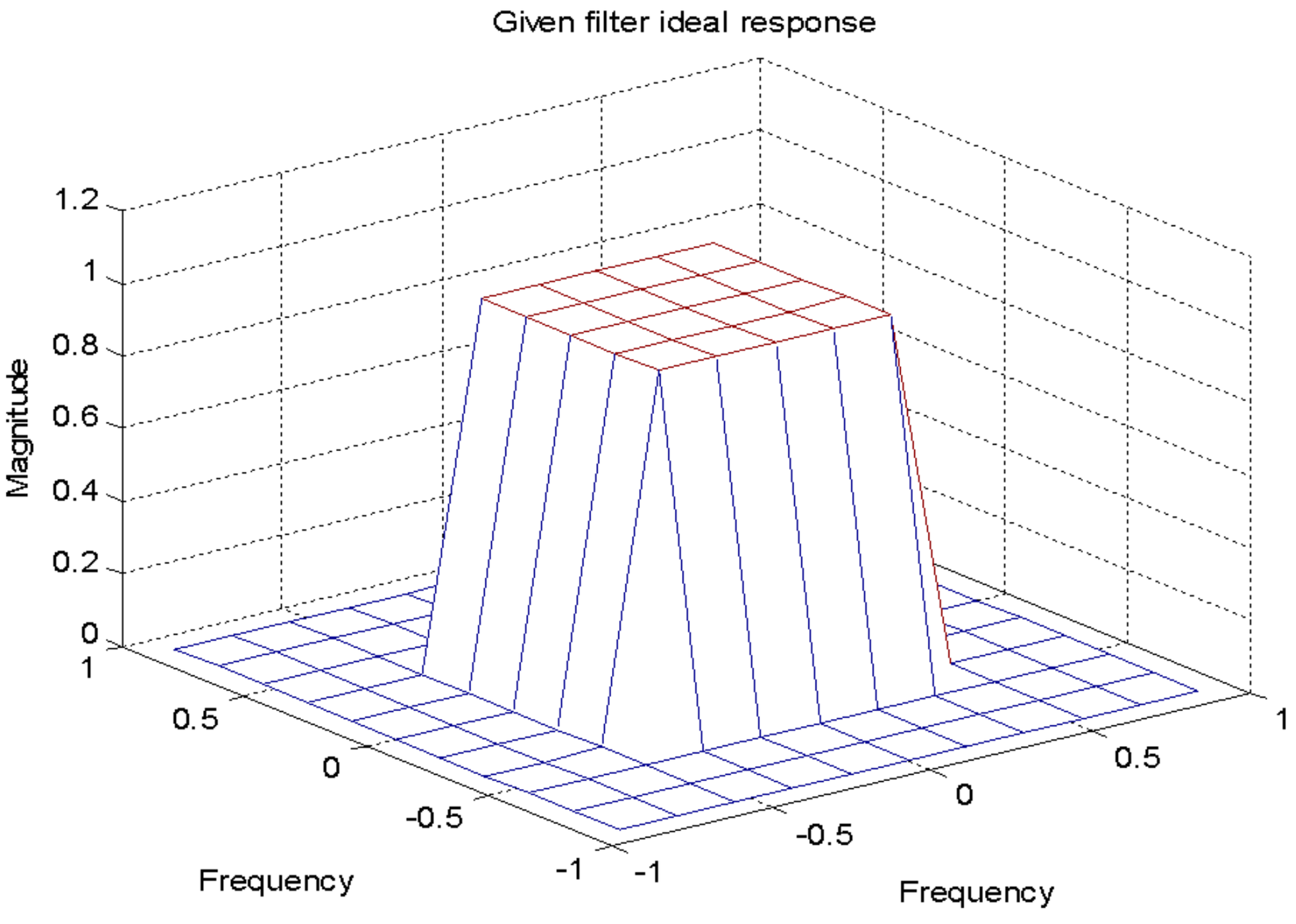

homomorphic filter
Standard high−pass filter
1.4
1.2
1
0.8
0.6
0.4
0.2
0
50
100
150
200

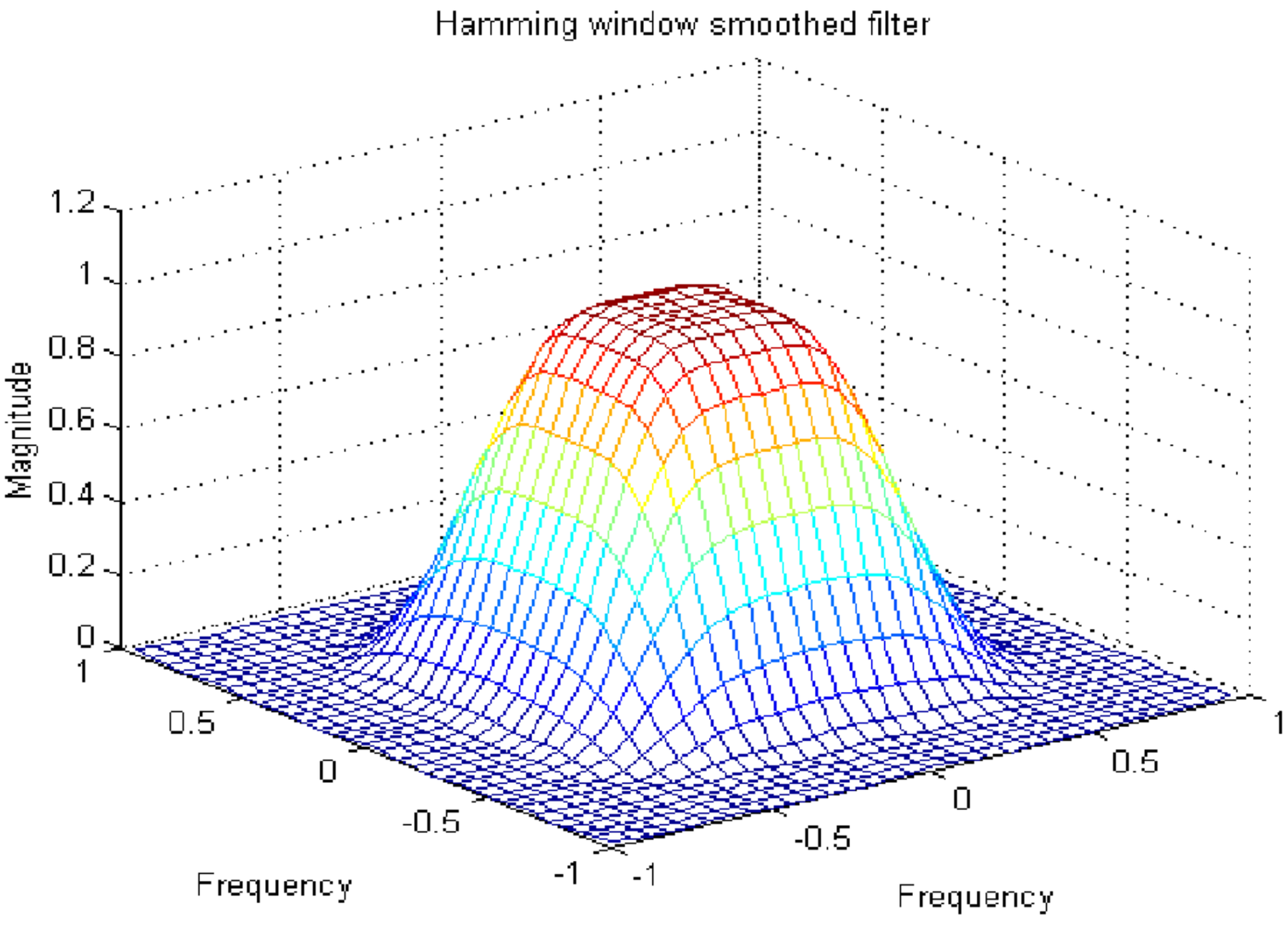

1D optimal Parks-McClellan filter
Magnitude
Frequency
1.4
1.2
1
0.8
0.6
0.4
0.2
0
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1


1D optimal Parks-McClellan filter
Magnitude
1.5
1
0.5
0
1
0.5
0
-0.5
-1
Frequency
-1
-0.5
0
0.5
1
Frequency


Ideal frequency response
Magnitude
1.2
1
0.8
0.6
0.4
0.2
0
1
0.5
0
-0.5
-1
Frequency
-1
-0.5
0
0.5
1
Frequency


Actual filter frequency response
Magnitude
1.2
1
0.8
0.6
0.4
0.2
0
1
0.5
0
-0.5
-1
Frequency
-1
-0.5
0
0.5
1
Frequency


Given filter ideal response
Magnitude
1.2
1
0.8
0.6
0.4
0.2
0
1
0.5
0
-0.5
-1
Frequency
-1
-0.5
0
0.5
1
Frequency


Hamming window smoothed filter
Magnitude
1.2
1
0.8
0.6
0.4
0.2
0
1
0.5
0
-0.5
-1
Frequency
-1
-0.5
0
0.5
1
Frequency