Supplementary material for "A new generalized exponentially weighted moving average quantile model and its statistical inference"

Ke Zhu

University of Hong Kong

This supplement provides some additional simulation results, when ε_t follows the standardized Pearson Type IV distribution (denoted by $P(\nu, m)$) such that its τth quantile is -1. Note that the Pearson Type IV distribution has the density given by

$$f(x; \nu, m) = K(1 + x^2)^{-m} \exp\big(-\nu \cdot \tan^{-1}(x)\big) \quad \text{with} \quad K = \frac{2^{2m-1}|\Gamma(m + i\nu/2)|^2}{\pi\Gamma(2m-1)}$$

for $m > 1/2$, where it is negatively skewed when $\nu > 0$ and it has a heavier tail when the value of m becomes smaller (see Zhu and Li (2015) for more details on this distribution).

S Additional simulations

S.1 Simulation studies for $\widehat{\theta}_n$

In this subsection, we examine the finite-sample performance of the weighted quantile estimator $\widehat{\theta}_n$ in (3.4). We generate 1000 replications of sample size $n = 500$ and 1000 from the following model:

$$y_t = q_t\varepsilon_t \text{ and } q_t = \psi_0|y_{t-1}| + 0.9q_{t-1}, \tag{S.1}$$

where ε_t follows the standardized $P(0.5, 5)$ or $P(0.5, 3)$ such that its τth quantile is -1, the values of ψ_0 are taken as the cases of $\gamma_s = 0$ for $\varepsilon_t \sim P(0.5, 5)$ and $P(0.5, 3)$ That is, when $\tau = 0.01$, we take $\psi_0 = 0.3603$ or 0.4144; and when $\tau = 0.05$, we take $\psi_0 = 0.2392$ or 0.2534.

Tables S1 and S2 report the sample bias, sample ESD, and ASD of $\widehat{\theta}_n$ based on 1000 replications for $\tau = 0.01$ and 0.05, respectively, where the ASD is calculated based on $\widehat{\Omega}_n$ in (3.5). From these two tables, we can have the similar findings as those in Tables 2 and 3.

Table S1: The results for $\widehat{\theta}_n$, when $\tau = 0.01$.

ε_t	n		$r=2$		$r=1$		$r=0$	
			$\widehat{\psi}_n$	$\widehat{\lambda}_n$	$\widehat{\psi}_n$	$\widehat{\lambda}_n$	$\widehat{\psi}_n$	$\widehat{\lambda}_n$
			Panel A: $(\psi_0, \lambda_0) = (0.3603, 0.9)$					
$P(0.5, 5)$	500	Bias	0.0045	-0.0013	0.0064	-0.0017	0.0078	-0.0020
		ESD	0.1310	0.0363	0.1335	0.0367	0.1370	0.0375
		ASD	0.1426	0.0372	0.1434	0.0373	0.1436	0.0373
	1000	Bias	0.0043	-0.0012	0.0054	-0.0015	0.0058	-0.0015
		ESD	0.1114	0.0299	0.1130	0.0304	0.1152	0.0309
		ASD	0.1092	0.0287	0.1090	0.0286	0.1093	0.0286
$P(0.5, 3)$	500	Bias	0.0001	-0.0001	0.0009	-0.0001	0.0025	-0.0004
		ESD	0.1429	0.0348	0.1465	0.0354	0.1524	0.0365
		ASD	0.1596	0.0363	0.1593	0.0362	0.1634	0.0368
	1000	Bias	0.0072	-0.0013	0.0077	-0.0014	0.0082	-0.0014
		ESD	0.1273	0.0303	0.1301	0.0309	0.1335	0.0315
		ASD	0.1293	0.0295	0.1291	0.0294	0.1286	0.0293
			Panel B: $(\psi_0, \lambda_0) = (0.4144, 0.9)$					
$P(0.5, 5)$	500	Bias	-0.0094	0.0017	-0.0088	0.0017	-0.0070	0.0014
		ESD	0.1296	0.0367	0.1331	0.0375	0.1396	0.0390
		ASD	0.1492	0.0388	0.1494	0.0388	0.1506	0.0390
	1000	Bias	-0.0011	-0.0000	0.0003	-0.0004	0.0020	-0.0008
		ESD	0.1113	0.0298	0.1137	0.0304	0.1168	0.0311
		ASD	0.1179	0.0307	0.1184	0.0308	0.1188	0.0309
$P(0.5, 3)$	500	Bias	0.0163	-0.0043	0.0200	-0.0049	0.0214	-0.0050
		ESD	0.1848	0.0439	0.1888	0.0444	0.1949	0.0457
		ASD	0.1801	0.0405	0.1861	0.0415	0.2821	0.0582
	1000	Bias	-0.0100	0.0021	-0.0065	0.0013	-0.0058	0.0013
		ESD	0.1315	0.0310	0.1344	0.0315	0.1389	0.0324
		ASD	0.1388	0.0315	0.1394	0.0316	0.1392	0.0315

Note: The distribution of ε_t is standardized such that its τth quantile is -1.

S.2 Simulation studies for S_n and M_n

In this subsection, we examine the finite-sample performance of the stability test S_n in (4.2) and the mean invariance test M_n in (4.4). We generate 1000 replications of sample size $n = 1000$ and 2000 from the following model:

$$y_t = q_t \varepsilon_t \text{ and } q_t = (\psi_0 + \zeta)|y_{t-1}| + 0.9 q_{t-1}, \tag{S.2}$$

Table S2: The results for $\widehat{\theta}_n$, when $\tau = 0.05$.

ε_t	n		$r=2$		$r=1$		$r=0$	
			$\widehat{\psi}_n$	$\widehat{\lambda}_n$	$\widehat{\psi}_n$	$\widehat{\lambda}_n$	$\widehat{\psi}_n$	$\widehat{\lambda}_n$
			Panel A: $(\psi_0, \lambda_0) = (0.2392, 0.9)$					
$P(0.5,5)$	500	Bias	0.0008	-0.0003	0.0003	-0.0001	-0.0001	0.0001
		ESD	0.0781	0.0314	0.0779	0.0314	0.0773	0.0311
		ASD	0.0752	0.0298	0.0750	0.0297	0.0750	0.0297
	1000	Bias	-0.0001	-0.0001	-0.0005	0.0001	-0.0012	0.0004
		ESD	0.0565	0.0225	0.0567	0.0225	0.0566	0.0225
		ASD	0.0553	0.0220	0.0552	0.0220	0.0551	0.0219
$P(0.5,3)$	500	Bias	0.0046	-0.0018	0.0038	-0.0015	0.0034	-0.0014
		ESD	0.0878	0.0336	0.0881	0.0338	0.0893	0.0343
		ASD	0.0807	0.0301	0.0805	0.0300	0.0805	0.0300
	1000	Bias	-0.0006	0.0003	-0.0014	0.0006	-0.0021	0.0008
		ESD	0.0614	0.0232	0.0612	0.0231	0.0618	0.0233
		ASD	0.0591	0.0221	0.0590	0.0221	0.0589	0.0220
			Panel B: $(\psi_0, \lambda_0) = (0.2534, 0.9)$					
$P(0.5,5)$	500	Bias	-0.0011	0.0006	-0.0004	0.0003	-0.0003	0.0003
		ESD	0.0761	0.0310	0.0776	0.0316	0.0788	0.0321
		ASD	0.0775	0.0307	0.0776	0.0307	0.0775	0.0307
	1000	Bias	0.0020	-0.0006	0.0020	-0.0007	0.0016	-0.0005
		ESD	0.0572	0.0228	0.0574	0.0229	0.0582	0.0232
		ASD	0.0567	0.0224	0.0567	0.0224	0.0567	0.0224
$P(0.5,3)$	500	Bias	-0.0002	0.0004	-0.0004	0.0005	-0.0003	0.0005
		ESD	0.0909	0.0340	0.0914	0.0341	0.0923	0.0344
		ASD	0.0829	0.0307	0.0829	0.0307	0.0828	0.0307
	1000	Bias	0.0005	-0.0001	0.0001	0.0001	-0.0000	0.0001
		ESD	0.0649	0.0244	0.0655	0.0246	0.0654	0.0245
		ASD	0.0623	0.0232	0.0622	0.0231	0.0622	0.0231

Note: As in Table S1.

where ε_t is chosen as in (S.1), $\zeta \in \{-0.05, ..., -0.01, 0, 0.01, ..., 0.05\}$, and the values of ψ_0 are taken with respect to $\gamma_s = 0$ (or $\gamma_m = 1$) for S_n (or M_n) so that q_t in model (S.2) is stable for S_n or mean-invariant for M_n when $\zeta = 0$. Specifically, when $\varepsilon_t \sim P(0.5, 5)$ and $\tau = 0.01$, we take $\psi_0 = 0.3603$ for S_n and $\psi_0 = 0.3484$ for M_n; when $\varepsilon_t \sim P(0.5, 5)$ and $\tau = 0.05$, we take $\psi_0 = 0.2392$ for S_n and $\psi_0 = 0.2312$ for M_n; when $\varepsilon_t \sim P(0.5, 3)$ and $\tau = 0.01$, we take

$\psi_0 = 0.4144$ for S_n and $\psi_0 = 0.3980$ for M_n; and when $\varepsilon_t \sim P(0.5, 3)$ and $\tau = 0.05$, we take $\psi_0 = 0.2534$ for S_n and $\psi_0 = 2434$ for M_n.

Since the power of S_n and M_n is invariant to the choice of r due to the adaptiveness property, we only plot the power of S_n and M_n for $r = 2$ in Figures S1 and S2, respectively, where the sizes of S_n and M_n are corresponding to the cases of $\zeta = 0$. Clearly, we can obtain the similar findings from Figures S1 and S2 as those from Figures 2 and 3.

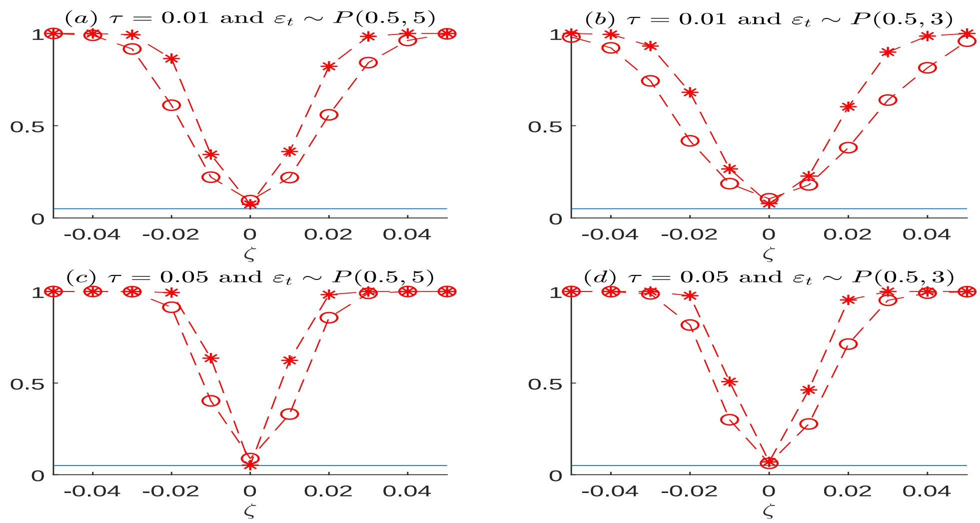

Figure S1: The power of S_n across ζ in model (S.2), where n is 1000 (dashed circle line) or 2000 (dashed star line). Here, the solid line stands for the significance level $\alpha = 5\%$.

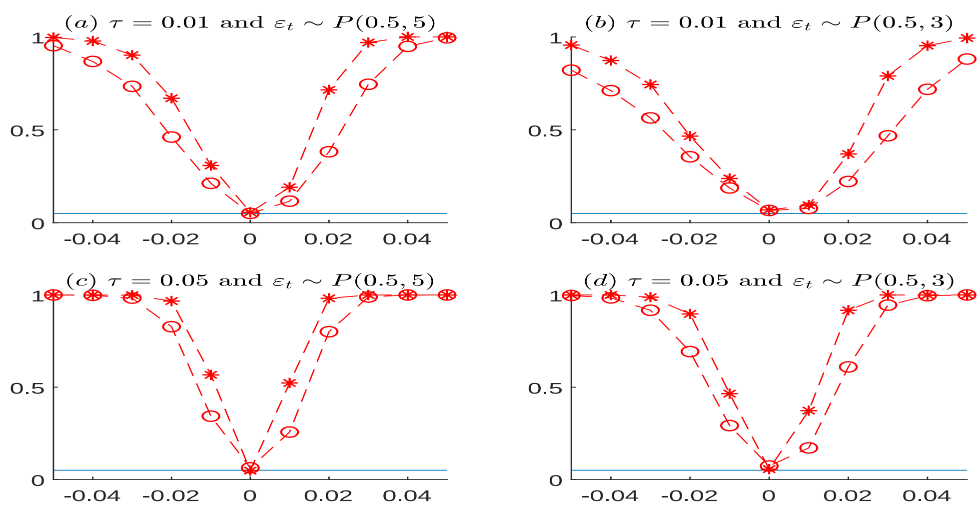

Figure S2: The power of M_n across ζ in model (S.2), where n is 1000 (dashed circle line) or 2000 (dashed star line). Here, the solid line stands for the significance level $\alpha = 5\%$.

S.3 Simulation studies for U_n

In this subsection, we examine the finite-sample performance of the unit root test U_n in (4.6). We generate 1000 replications of sample size $n = 1000$ and 2000 from the following model:

$$y_t = q_t \varepsilon_t \text{ and } q_t = \omega_0 + \psi_0 |y_{t-1}| + 0.9 q_{t-1}, \tag{S.3}$$

where ε_t is chosen as in (S.1), $\omega_0 \in \{0, 10^{-3}, 10^{-2}\}$, and three different values of ψ_0 are taken for the cases of $\gamma_s < 0$, $\gamma_s = 0$, and $\gamma_s > 0$, respectively. For each replication, we apply U_n to detect whether $\omega_0 = 0$ and $\gamma_s = 0$. Table S3 reports the power of U_n at the significance level of 5%, where the sizes of U_n are corresponding to the cases that $\omega_0 = 0$ and $\gamma_s = 0$. From this table, we can get the similar findings as those in Table 4.

S.4 Simulation studies for $D_{n,k}^{(p)}$

In this subsection, we examine the finite-sample performance of the dynamic quantile test $D_{n,k}^{(p)}$ in (5.1). We generate 1000 replications of sample size $n = 1000$ and 2000 from the following model:

$$y_t = q_t \varepsilon_t \text{ and } q_t = \psi_0 |y_{t-1}| + \zeta |y_{t-2}| + 0.9 q_{t-1}, \tag{S.4}$$

where the settings of ε_t and ψ_0 are the same as those for S_n above, and $\zeta \in \{0, 0.2, 0.4, 0.6, 0.8\}$. For each replication, we fit it by using the GEWMA quantile model, and then apply $D_{n,k}^{(p)}$ to check whether the fitted model is adequate. Below, we consider the cases that $k = 2$, 4, and 6 with $p = 1.6$ as in the paper, and only report the testing results for $r = 2$ based on 1000 replications, since the performance of $D_{n,k}^{(p)}$ is invariant to the choice of r. Table S4 reports the power of three tests $D_{n,2}^{(1.6)}$, $D_{n,4}^{(1.6)}$, and $D_{n,6}^{(1.6)}$, where the sizes correspond to the results for $\zeta = 0$. From this table, we can have the similar findings as those in Table 5.

Overall, our additional simulation studies in this supplement show that our proposed estimator and tests still perform well even when the model innovation is skewed.

References

[1] Zhu, K. and Li, W. K. (2015) A new Pearson-type QMLE for conditionally heteroskedastic models. *Journal of Business & Economic Statistics* **33**, 552-565.

Table S3: The power of U_n across ω_0, based on model (S.3).

				$r=2$			$r=1$			$r=0$		
				ω_0			ω_0			ω_0		
τ	ε_t	n	ψ_0	0	10^{-3}	10^{-2}	0	10^{-3}	10^{-2}	0	10^{-3}	10^{-2}
0.01	$P(0.5,5)$	1000	0.3400	0.5080	0.2350	0.8140	0.5040	0.2310	0.8110	0.5020	0.2290	0.8130
			0.3603	**0.0960**	0.5860	0.1490	**0.0950**	0.5790	0.1520	**0.0940**	0.5820	0.1500
			0.3800	0.1490	0.1780	0.1180	0.1530	0.1770	0.1170	0.1450	0.1770	0.1100
		2000	0.3400	0.7610	0.1980	0.8330	0.7580	0.1980	0.8330	0.7560	0.2000	0.8320
			0.3603	**0.0590**	0.5040	0.0560	**0.0580**	0.5040	0.0570	**0.0580**	0.5020	0.0570
			0.3800	0.2460	0.3820	0.4070	0.2450	0.3830	0.3980	0.2390	0.3740	0.3870
	$P(0.5,3)$	1000	0.3900	0.4600	0.1900	0.7820	0.4490	0.2000	0.7810	0.4400	0.1900	0.7780
			0.4144	**0.1090**	0.4980	0.1720	**0.1060**	0.4950	0.1720	**0.1100**	0.4890	0.1680
			0.4300	0.1510	0.3020	0.0850	0.1510	0.2970	0.0810	0.1510	0.2910	0.0800
		2000	0.3900	0.7420	0.1640	0.8010	0.7390	0.1600	0.8010	0.7290	0.1590	0.8010
			0.4144	**0.0590**	0.4070	0.0530	**0.0610**	0.4080	0.0530	**0.0610**	0.4080	0.0530
			0.4300	0.1060	0.1890	0.1720	0.1080	0.1880	0.1710	0.1040	0.1870	0.1630
0.05	$P(0.5,5)$	1000	0.2200	0.8080	0.1440	0.9770	0.8040	0.1450	0.9770	0.8020	0.1410	0.9770
			0.2392	**0.0750**	0.6690	0.2110	**0.0750**	0.6690	0.2130	**0.0750**	0.6680	0.2120
			0.2600	0.1670	0.2510	0.2590	0.1680	0.2500	0.2570	0.1680	0.2480	0.2520
		2000	0.2200	0.9780	0.0480	0.9930	0.9780	0.0470	0.9930	0.9790	0.0510	0.9930
			0.2392	**0.0480**	0.5600	0.0900	**0.0480**	0.5580	0.0900	**0.0470**	0.5570	0.0900
			0.2600	0.6810	0.7950	0.8130	0.6800	0.7930	0.8140	0.6780	0.7930	0.8120
	$P(0.5,3)$	1000	0.2300	0.8450	0.0740	0.9780	0.8450	0.0750	0.9780	0.8460	0.0780	0.9780
			0.2534	**0.0760**	0.5950	0.2370	**0.0740**	0.5960	0.2360	**0.0740**	0.5970	0.2350
			0.2700	0.1140	0.1790	0.0990	0.1130	0.1790	0.0990	0.1110	0.1790	0.0990
		2000	0.2300	0.9900	0.0180	0.9940	0.9900	0.0180	0.9940	0.9900	0.0180	0.9940
			0.2534	**0.0580**	0.5320	0.0740	**0.0580**	0.5340	0.0740	**0.0570**	0.5330	0.0740
			0.2700	0.2680	0.4190	0.4370	0.2690	0.4190	0.4360	0.2680	0.4210	0.4350

Note: The size of U_n is in boldface.

Table S4: The power of $D_{n,k}^{(p)}$ for $p = 1.6$, based on model (S.4).

				Model (S.4)				
τ	ε_t	Tests	n	$\zeta = 0$	$\zeta = 0.2$	$\zeta = 0.4$	$\zeta = 0.6$	$\zeta = 0.8$
0.01	$P(0.5, 5)$	$D_{n,2}^{(p)}$	1000	0.0550	0.0570	0.0510	0.0680	0.0740
			2000	0.0650	0.0550	0.0960	0.1280	0.1810
		$D_{n,4}^{(p)}$	1000	0.0870	0.0690	0.0980	0.1070	0.1240
			2000	0.0670	0.0720	0.0980	0.1450	0.1940
		$D_{n,6}^{(p)}$	1000	0.1000	0.1020	0.1080	0.1140	0.1250
			2000	0.0720	0.0800	0.1190	0.1510	0.1730
	$P(0.5, 3)$	$D_{n,2}^{(p)}$	1000	0.0440	0.0410	0.0580	0.0590	0.0680
			2000	0.0560	0.0510	0.0620	0.0790	0.1010
		$D_{n,4}^{(p)}$	1000	0.0780	0.0760	0.0780	0.0800	0.1070
			2000	0.0590	0.0780	0.0880	0.0960	0.1220
		$D_{n,6}^{(p)}$	1000	0.0870	0.0970	0.0930	0.1360	0.1410
			2000	0.0900	0.0880	0.1100	0.1240	0.1440
0.05	$P(0.5, 5)$	$D_{n,2}^{(p)}$	1000	0.0450	0.0900	0.2020	0.3540	0.4960
			2000	0.0530	0.1710	0.4440	0.7140	0.8540
		$D_{n,4}^{(p)}$	1000	0.0700	0.1040	0.1800	0.2830	0.4260
			2000	0.0460	0.1410	0.3520	0.5850	0.7530
		$D_{n,6}^{(p)}$	1000	0.0840	0.1090	0.2010	0.2650	0.3880
			2000	0.0630	0.1350	0.3120	0.5120	0.7140
	$P(0.5, 3)$	$D_{n,2}^{(p)}$	1000	0.0570	0.0810	0.1320	0.2160	0.3430
			2000	0.0540	0.1270	0.2630	0.4760	0.6460
		$D_{n,4}^{(p)}$	1000	0.0760	0.0880	0.1470	0.2020	0.2610
			2000	0.0690	0.0970	0.2280	0.4230	0.5140
		$D_{n,6}^{(p)}$	1000	0.0900	0.1110	0.1610	0.1810	0.2700
			2000	0.0680	0.1180	0.2230	0.3510	0.4760