Finite difference simulation of 2D waves

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Introduction

In this project we are trying to implement a simulation of a two dimensional wave using the finite difference methods. Then using the simulation to study the behavior of waves as they pass through different medium with different velocities.

Mathematical Problem

A standard linear 2D wave with damping effect can be represented as:

$$\frac{\partial^2 u}{\partial t^2} + b\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}(q(x,y)\frac{\partial u}{\partial x}) + \frac{\partial}{\partial y}(q(x,y)\frac{\partial u}{\partial y}) + f(x,y,t) \tag{1}$$

The boundary condition is given as:

$$\frac{\partial u}{\partial n} = 0 \tag{2}$$

with initial conditions:

$$u(x,y,0) = I(x,y) \tag{3}$$

$$u_t(x,y,0) = V(x,y) \tag{4}$$

I have used SymPy to help me solve the equation and simplify the results, but for the first step, I'm only trying to solve the left side of the equation, and call the right hand side "F".

I have used the estimations:

$$\frac{\partial^2 u}{\partial t^2} = \frac{u^{n+1} - 2u^n + u^{n-1}}{\Delta t^2}$$

And

$$\frac{\partial u}{\partial t} = \frac{u^{n+1} - u^{n-1}}{2\Delta t}$$

Solving the following equation for u^{n+1}

$$\frac{\partial^2 u}{\partial t^2} + b\frac{\partial u}{\partial t} = F$$

We get the result:

$$u^{n+1} = \frac{1}{b\Delta t + 2}(u^{n-1}(b\Delta t - 2) + 2\Delta t^2 F + 4u^n) \tag{5}$$

Now to get the scheme for the first step, using the equations 3, 4 and the estimate:

$$u^{-1} = u^1 - 2\Delta t u^0$$

We have

$$u^1 = \frac{\Delta t^2}{2}(F - bV(i,j)) + \Delta t I(i,j) + I(i,j) \tag{6}$$

We have used the approximations:

$$[D_x q D_x u]^n_{i,j} = \frac{1}{\Delta x^2}(q_{i+\frac{1}{2}}(u_{i+1} - u_i) - q_{i-\frac{1}{2}}(u_i - u_{i-1})) \tag{7}$$

$$[D_y q D_y u]^n_{j,j} = \frac{1}{\Delta y^2}(q_{j+\frac{1}{2}}(u_{j+1} - u_j) - q_{j-\frac{1}{2}}(u_j - u_{j-1})) \tag{8}$$