DEMYSTIFYING DEEP LEARNING THROUGH HIGH-DIMENSIONAL STATISTICS

JEFFREY PENNINGTON

GOOGLE BRAIN

CU BOULDER

2-19-21

OUTLINE

1. Background
2. Problem Setup and Motivation
3. Part I: What Causes Double Descent?
4. Part II: Beyond Double Descent

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SUPERVISED LEARNING

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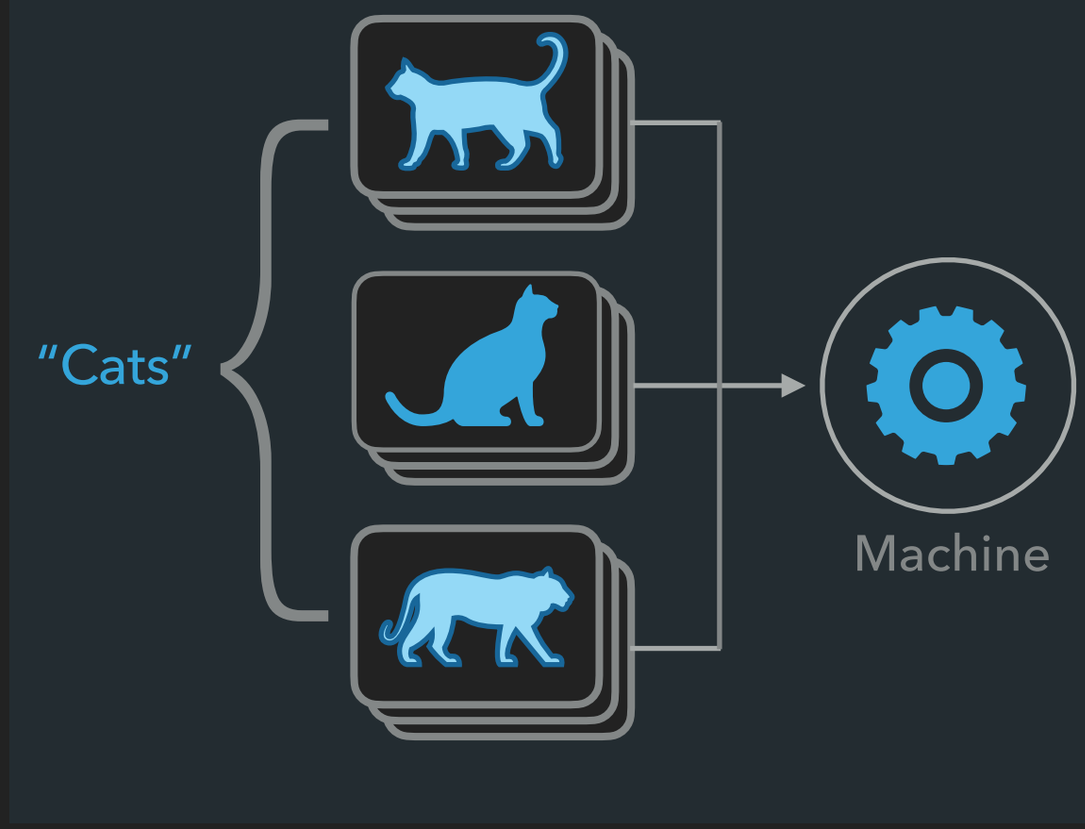

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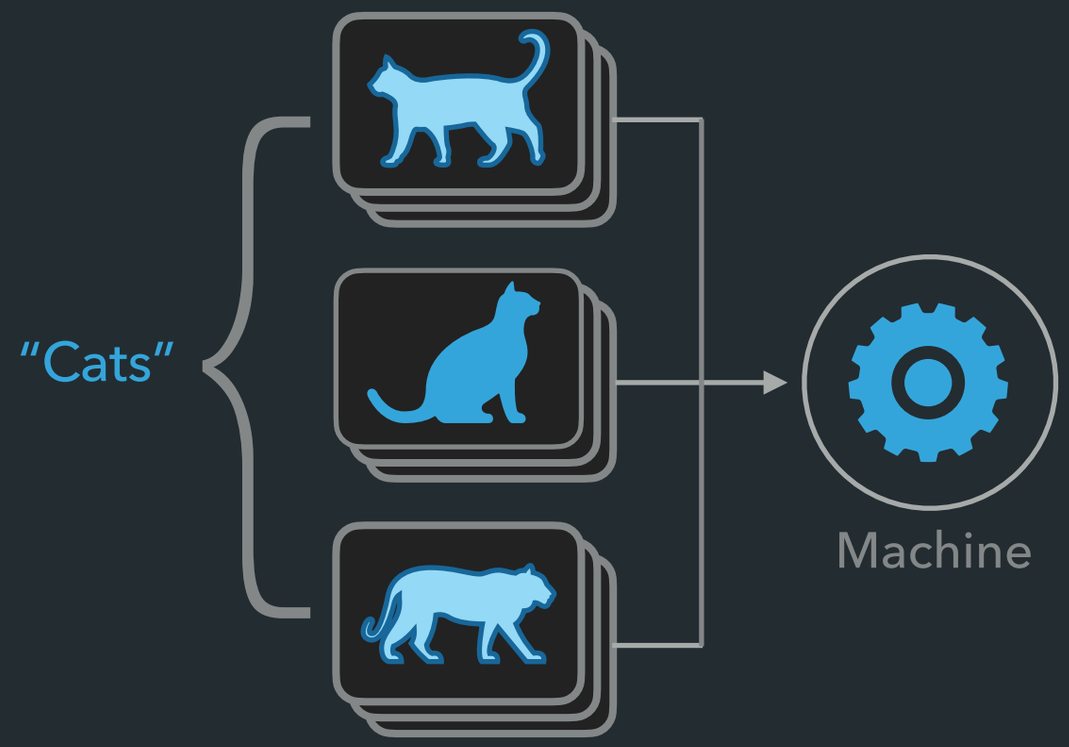
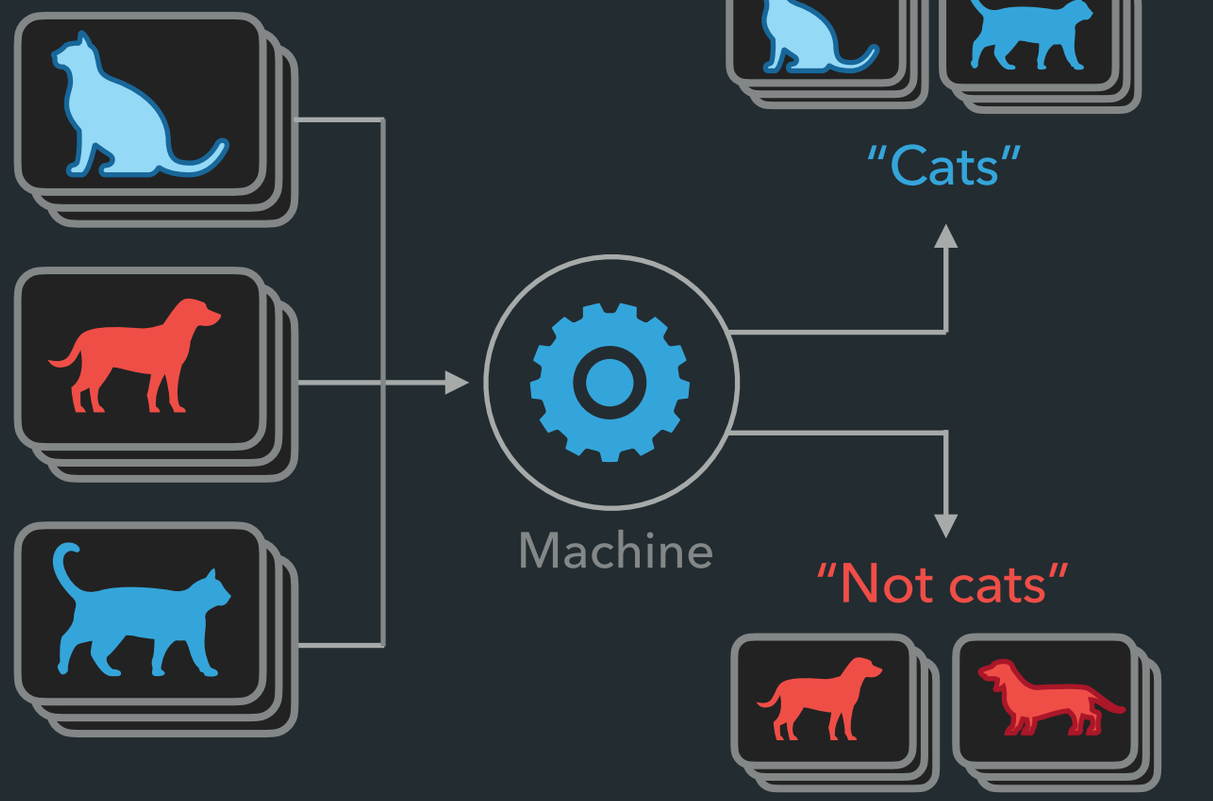


3. Provide new, unlabeled datapoints for the machine to automatically label



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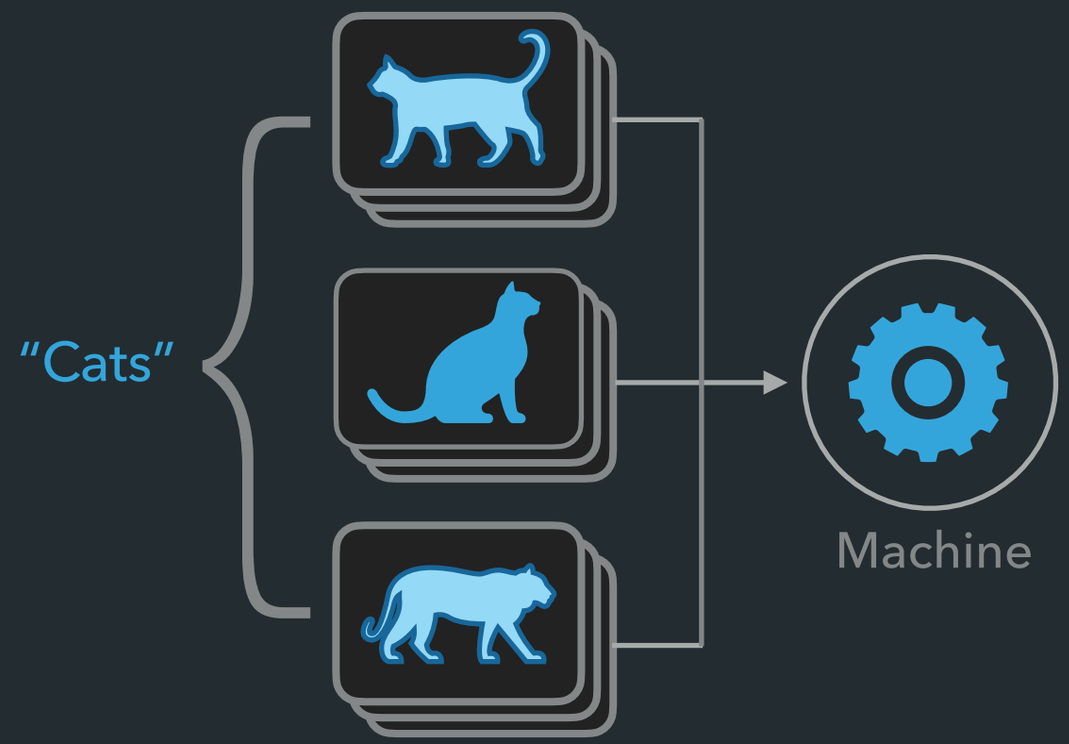

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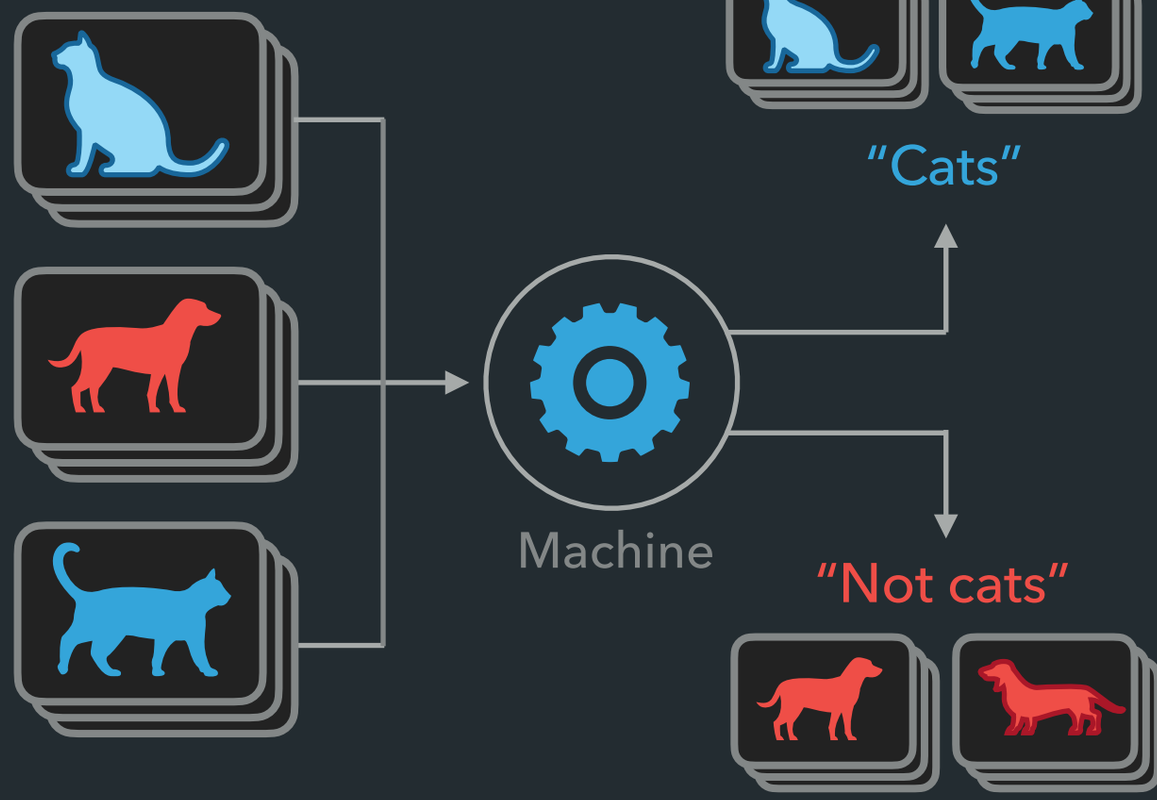
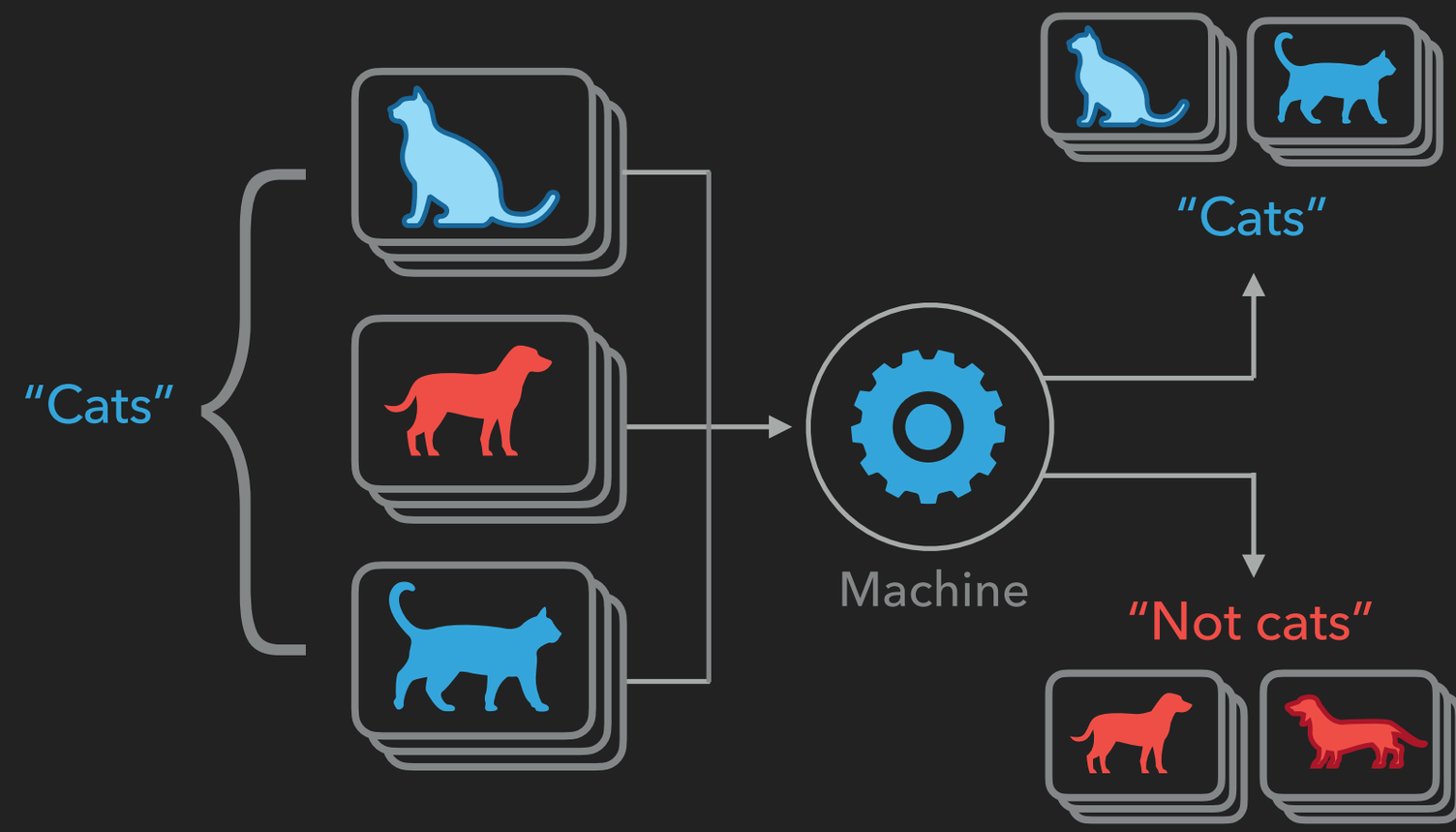


WHAT ABOUT STEP 2? HOW TO TRAIN THE MACHINE?

FORMALIZING THE PROBLEM



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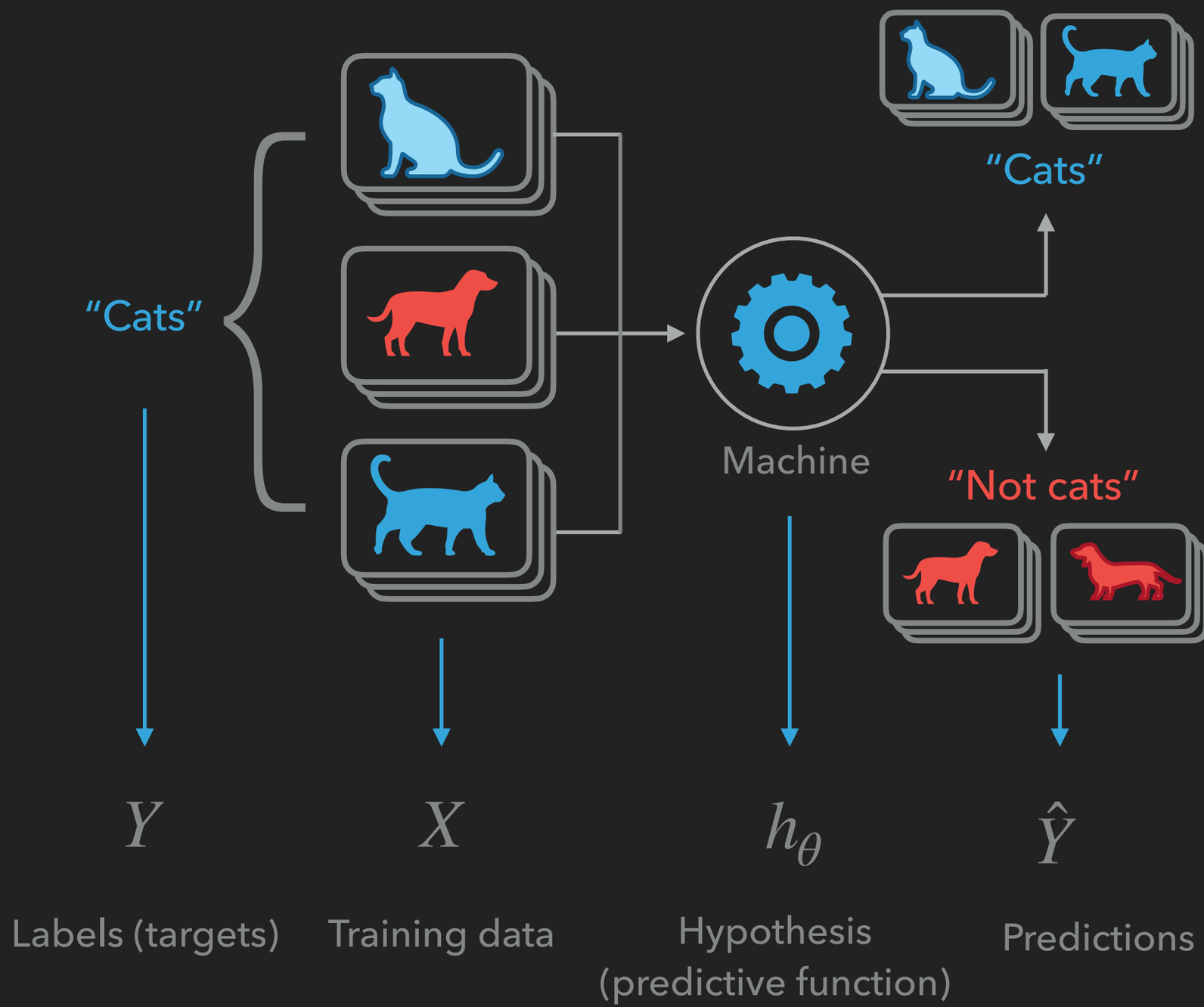


STEP 2: LEARNING A PREDICTIVE FUNCTION

Strategy: Introduce a loss function $L(\hat{Y}, Y)$ whose minimizer gives a good predictive function

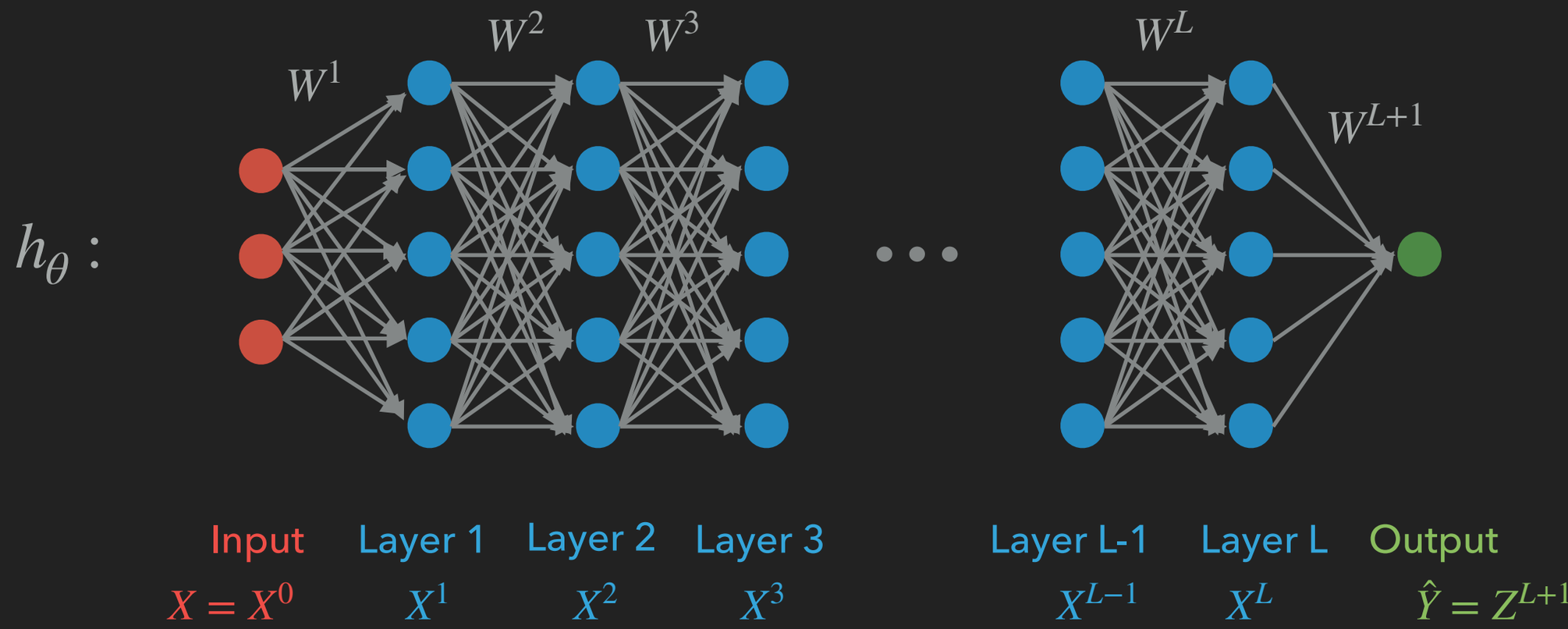
Ex: Least squares regression

$$
\begin{aligned}
L &= \sum_{i\mu} (Y_{i\mu} - \hat{Y}_{i\mu})^2 \\
&= \|Y - \hat{Y}\|_F^2 \\
&= \|Y - h_\theta(X)\|_F^2
\end{aligned}
$$

Goal is to find $\text{argmin}\{L \,|\, h_\theta \in \mathscr{H}\}$ for a well-specified hypothesis class $\mathscr{H}$.

DEEP NEURAL NETWORKS

In practice, one common and effective hypothesis class is defined by deep neural networks



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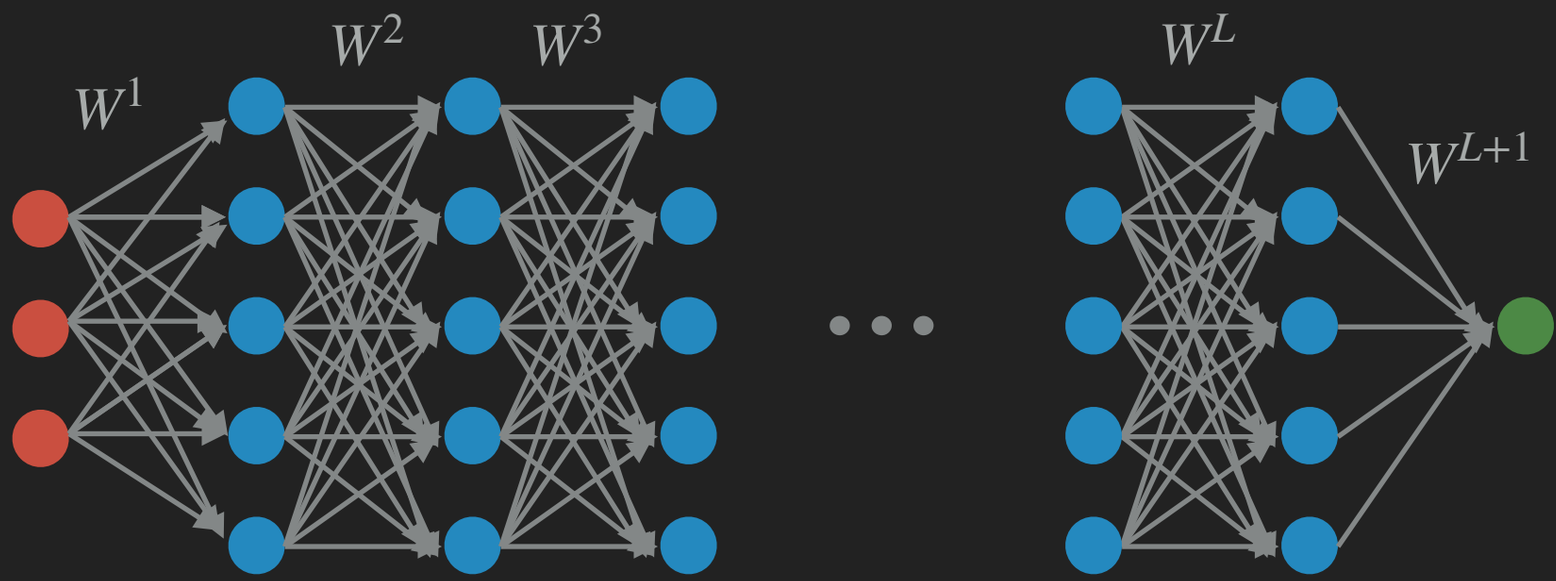

Input $X = X^0$ | Layer 1 X^1 | Layer 2 X^2 | Layer 3 X^3 | Layer L-1 X^{L-1} | Layer L X^L | Output $\hat{Y} = Z^{L+1}$

$$X^l_{i\mu} = f(Z^l_{i\mu}), \quad Z^{l+1} = W^{l+1}X^l + B^{l+1}$$

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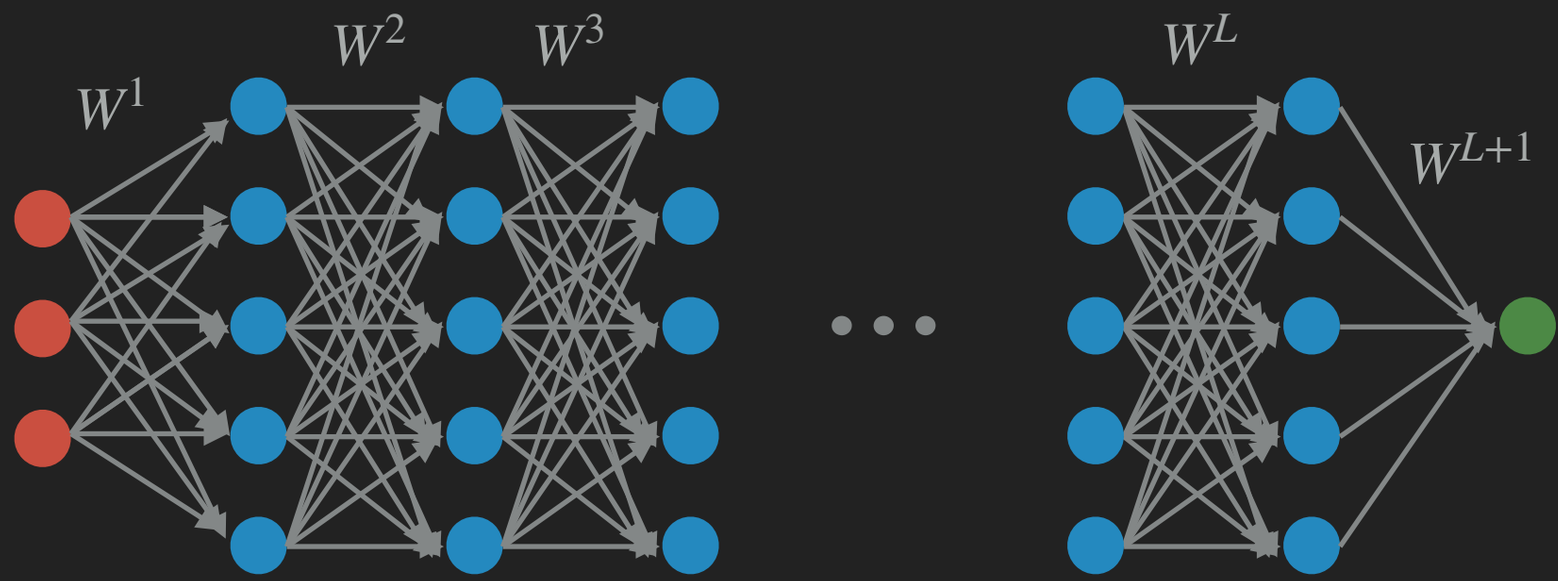

Input	Layer 1	Layer 2	Layer 3	Layer L-1	Layer L	Output
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(Post) activation neurons

Activation function

(Pre) activation neurons

Weights

Biases

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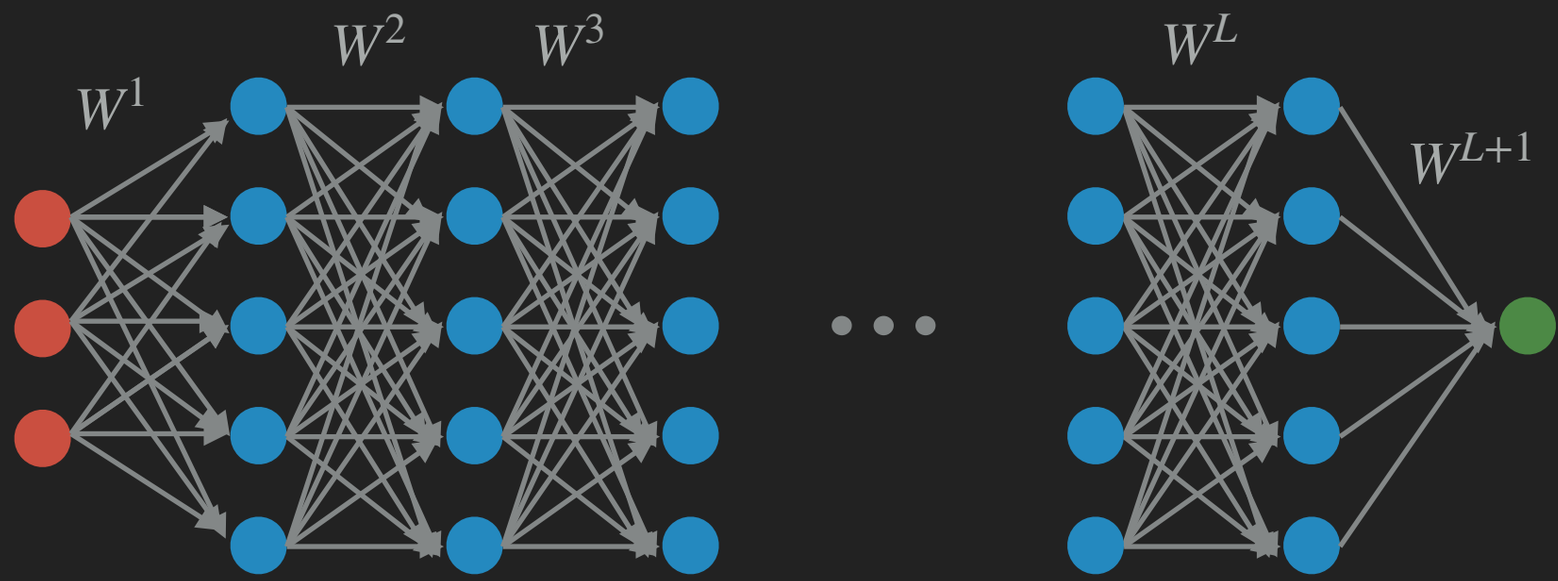

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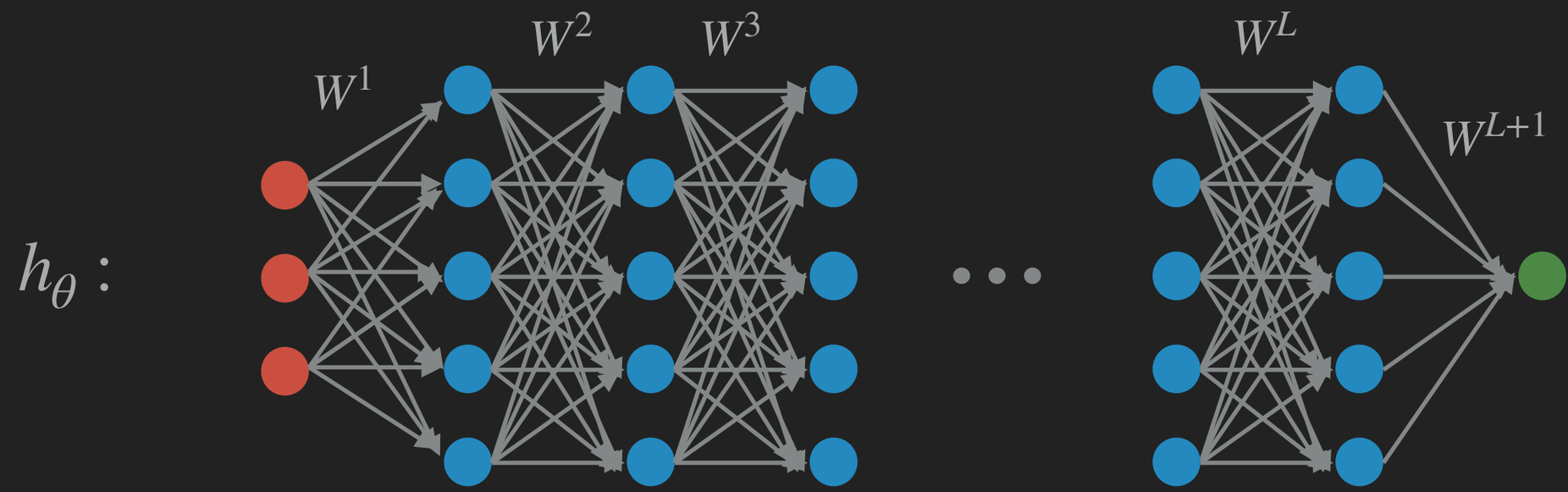

m - # samples

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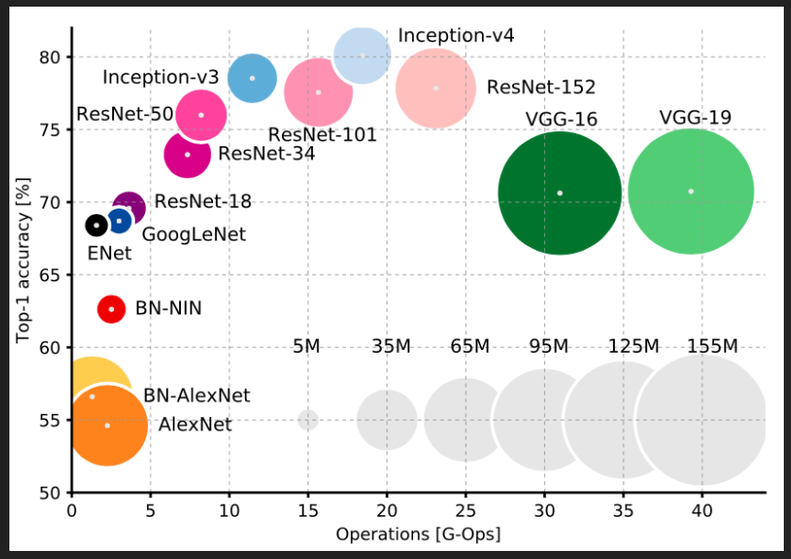
WHY DOES DEEP LEARNING WORK?

Deep neural networks define a very flexible and expressive class of functions.

State-of-the-art models have millions or billions of parameters

- Meena: 2.6 billion
- Turing NLG: 17 billion
- GPT-3: 175 billion



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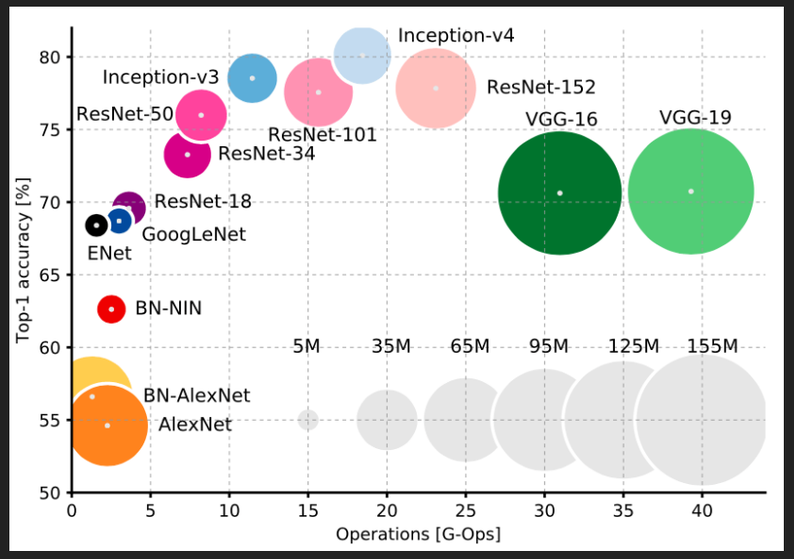
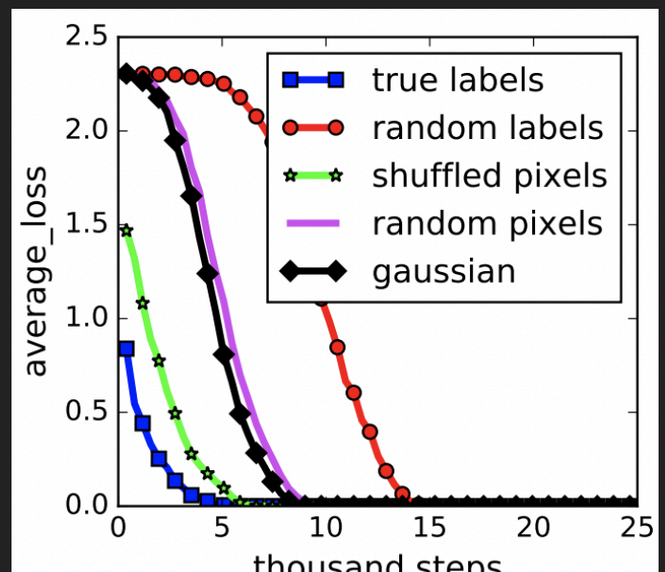
Models that perform well on real data can easily memorize noise (Zhang et al., 2017)





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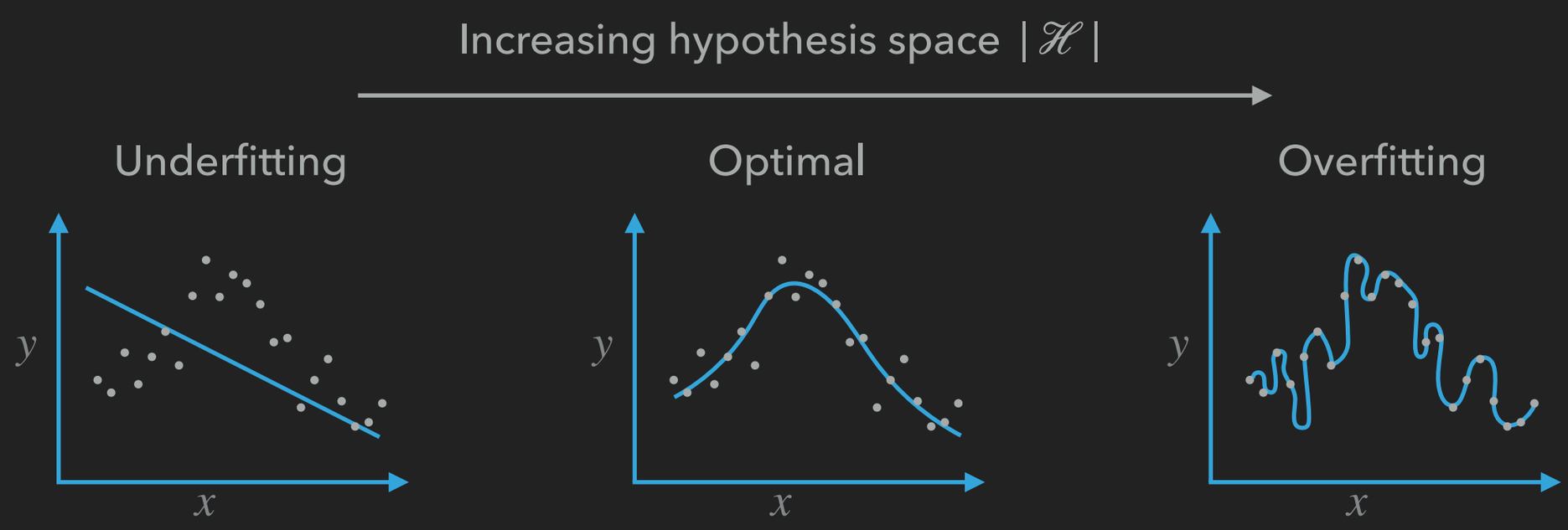
Deep neural networks define a very flexible and expressive class of functions.

➡ Standard wisdom suggests they should overfit, i.e.

$$E_{\text{train}} \equiv \frac{1}{m}\sum_{\mu=1}^{m} L\big(h_\theta(X_\mu), Y_\mu\big) \to 0 \quad \text{but} \quad E_{\text{test}} \equiv \mathbb{E}_{(x,y)} L\big(h_\theta(x), y\big) \gg 0$$



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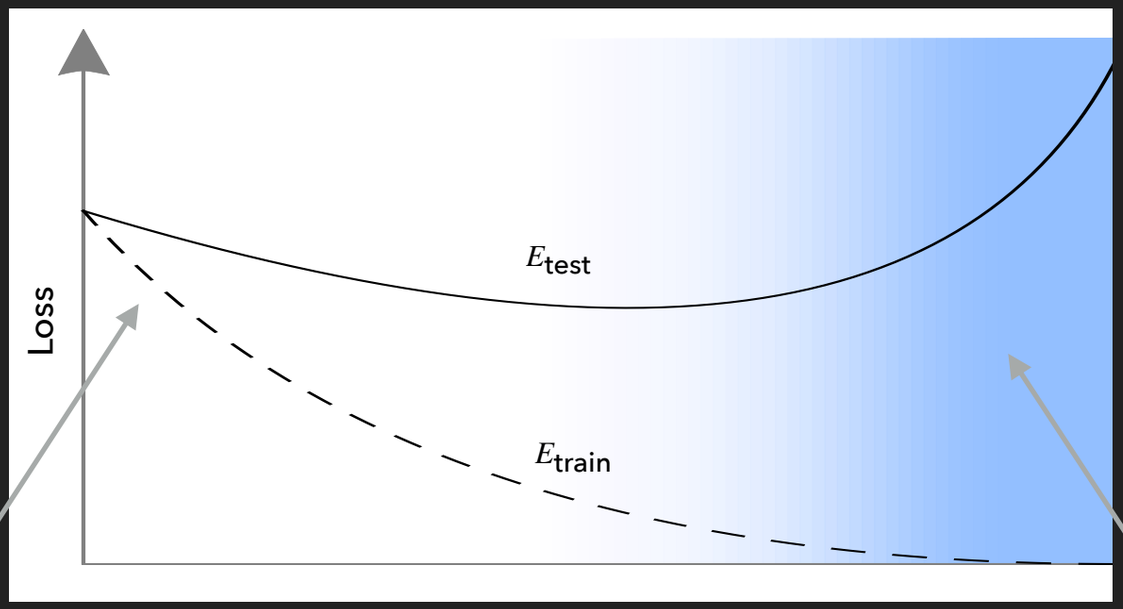
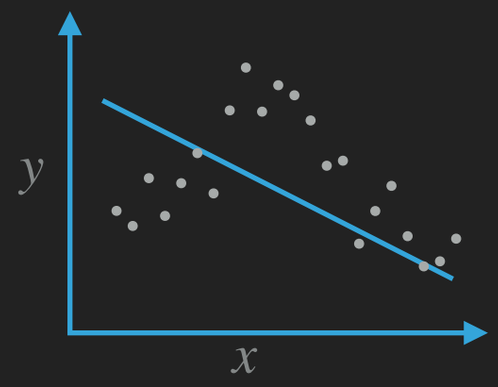
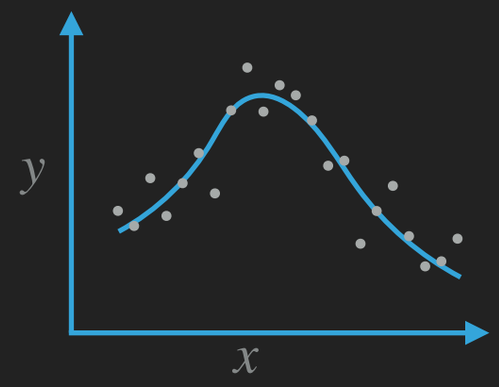


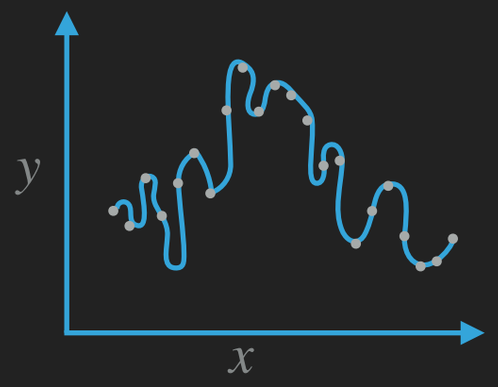
Underfitting



Optimal



Overfitting



BIAS-VARIANCE DECOMPOSITION

Condition on a test point x, then decompose the per-sample loss

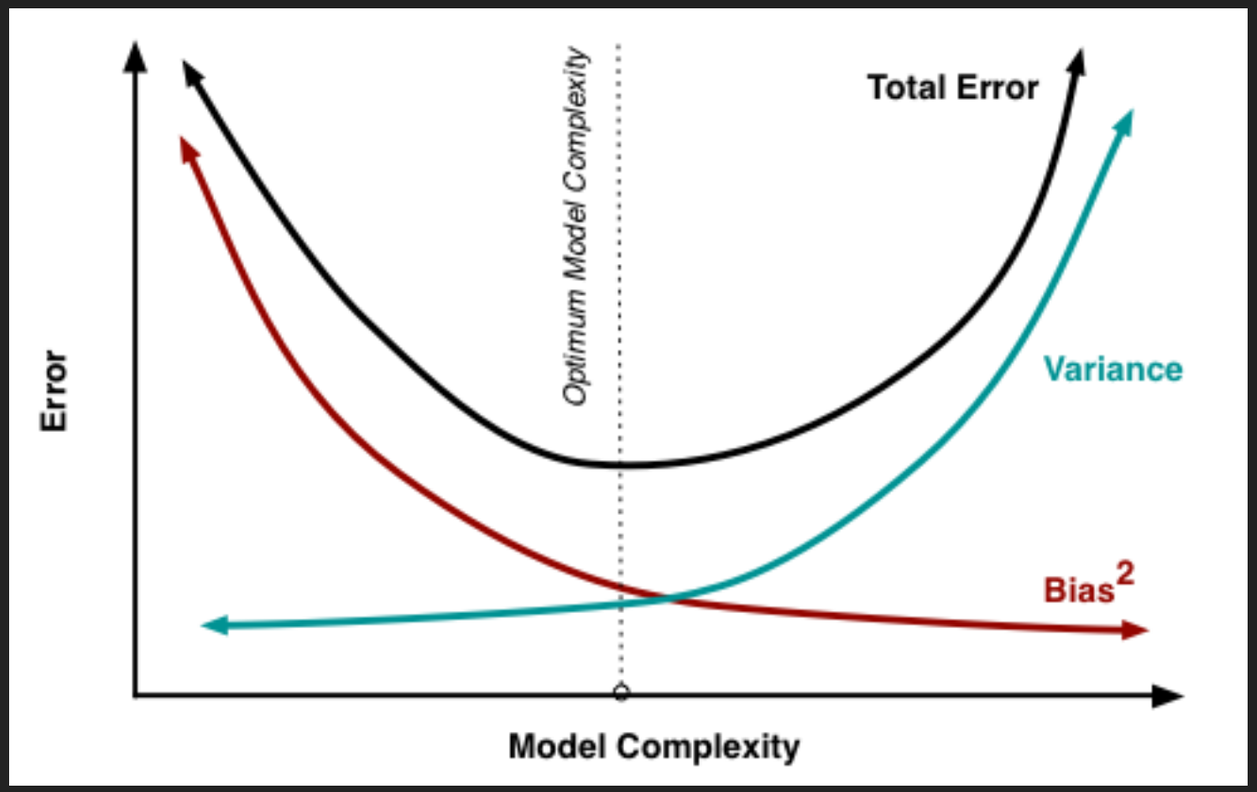
$$\mathbb{E}[(\hat{y}(x) - y(x))^2] = (\mathbb{E}\hat{y}(x) - \mathbb{E}y(x))^2 + \mathbb{V}[\hat{y}(x)] + \mathbb{V}[y(x)]$$

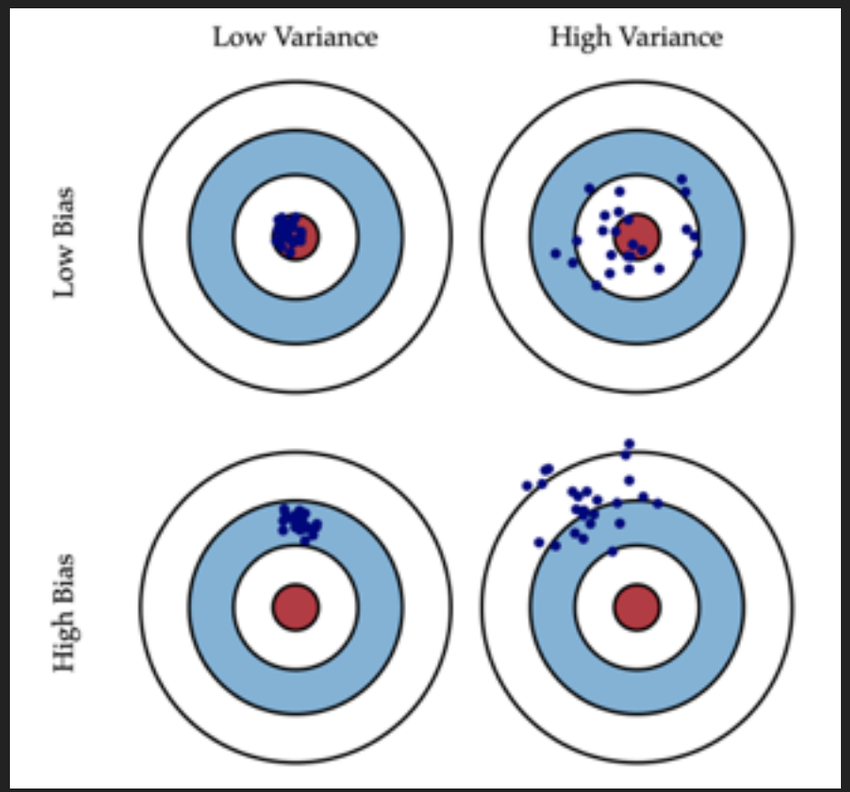
Averaging over x gives:

Bias

Variance

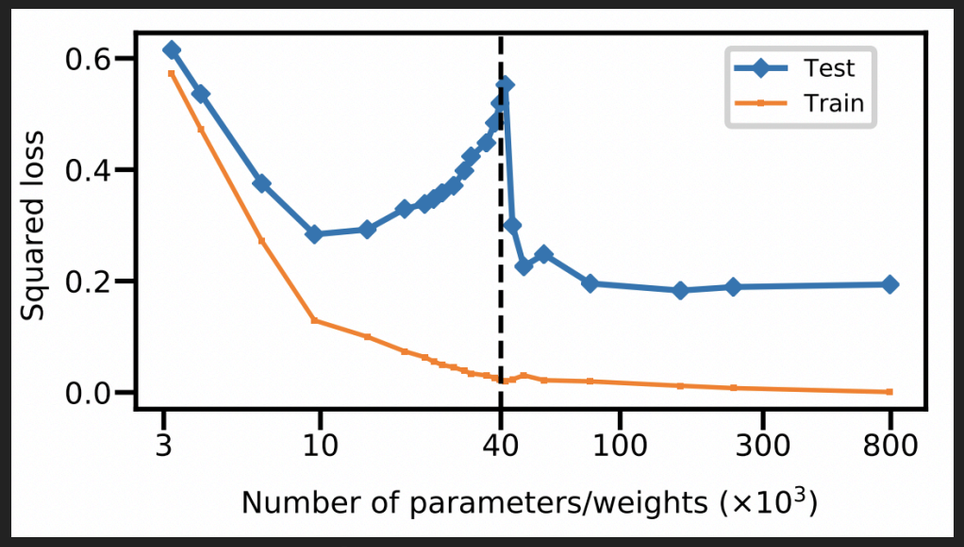
Noise



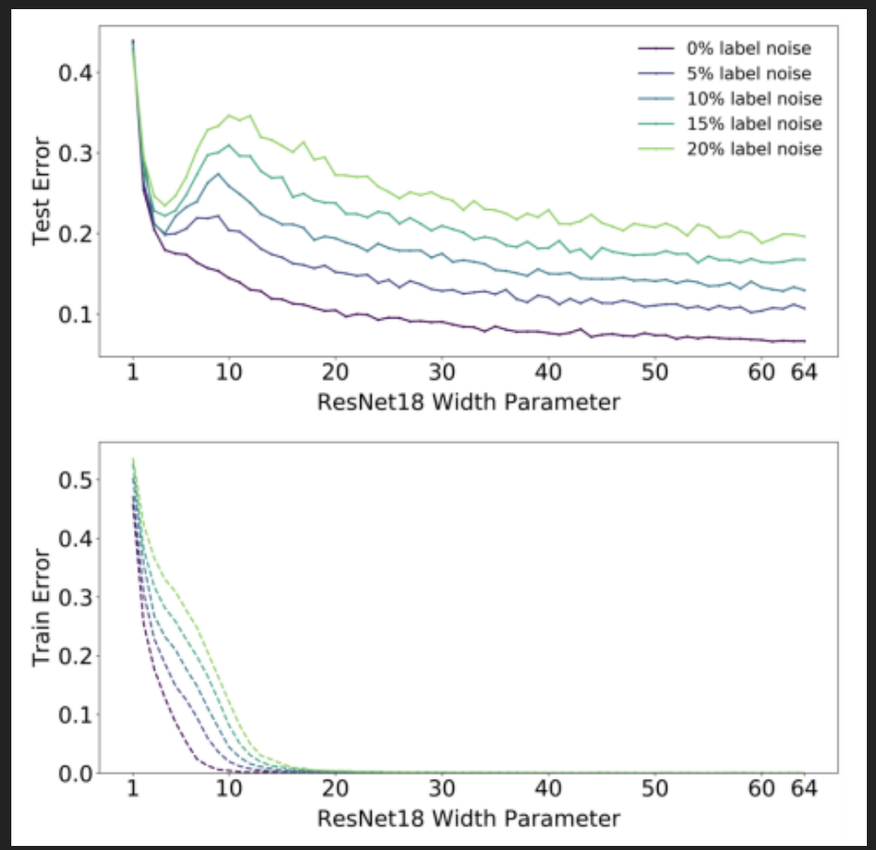


DOUBLE DESCENT

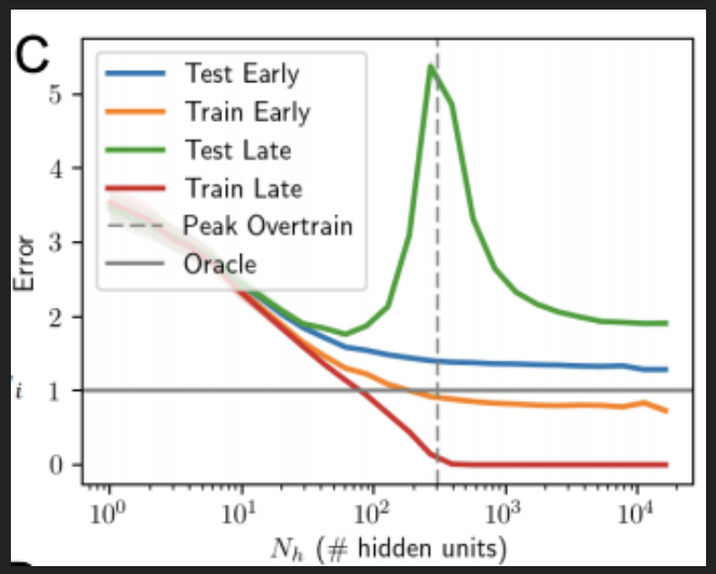
However, large NNs do not seem to obey the classical theory:



Belkin *et al.*, 2018



Nakkiran *et al.*, 2019



Advani and Saxe, 2017

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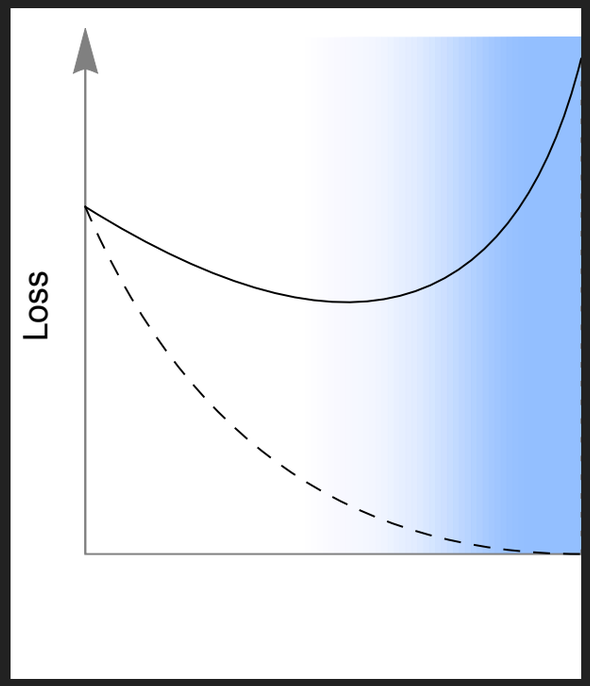
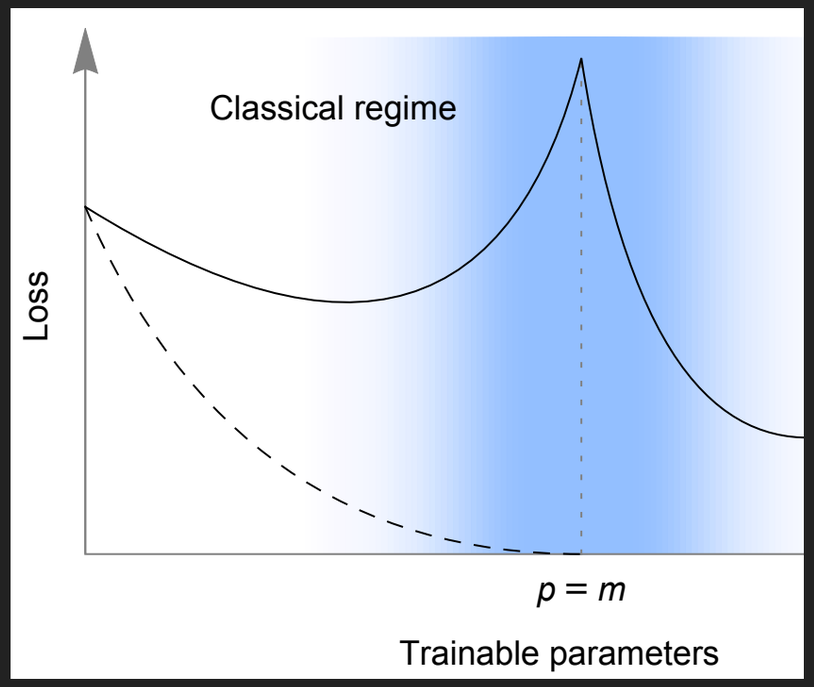


DOUBLE DESCENT

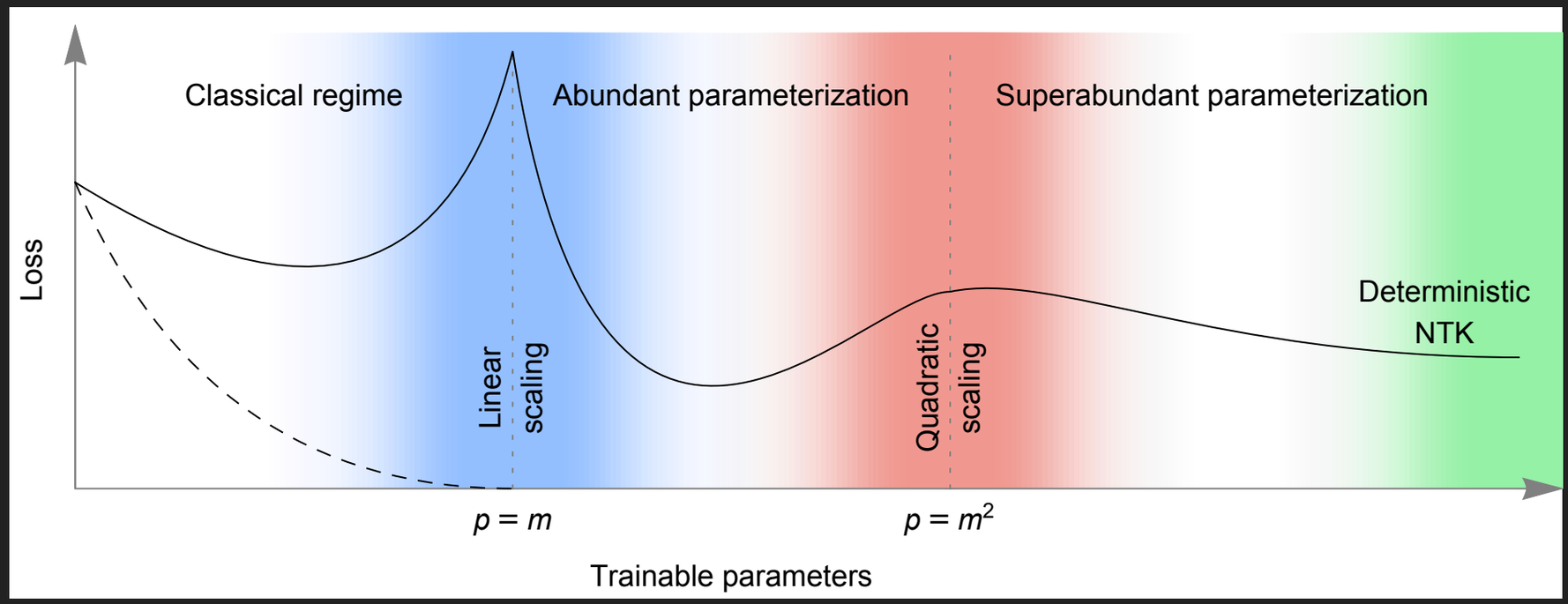
The emerging paradigm of *double descent* offers some insight:



Part I: What is causing the peak? What about the bias/variance tradeoff?

TRIPLE DESCENT

In some scenarios, NNs can exhibit non-monotonic behavior deep in the overparameterized regime



Part II: Triple descent and multi-scale theory of generalization

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DEEP LEARNING MODELS ARE HIGH-DIMENSIONAL

Deep learning models employ large numbers of parameters.
At least two practically-relevant high-dimensional regimes:

1. Quadratic overparameterization ($n_l \sim m$, i.e. $p \sim m^2$)
2. Linear overparameterization ($p \sim m$)

Examples:

	Width n_l	# Samples m	# Parameters p
FC/ CIFAR-10	10^3	10^4	10^6
ResNet/ ImageNet	10^3	10^7	10^8

HIGH-DIMENSIONAL SCALING LIMITS

We will focus on the following high-dimensional asymptotics of one hidden-layer networks:

1. Dataset size $m \to \infty$
2. Input dimensionality $n_0 \to \infty$
3. Hidden-layer size $n_1 \to \infty$

with the ratios $\phi = n_0/m$ and $\psi = n_0/n_1$ held constant

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I) For random features F, $p = n_1$, i.e. linear overparameterization

II) For NTK features J, $p = n_1(n_0 + 1)$, i.e. quadratic overparameterization

RANDOM FEATURE KERNEL RIDGE REGRESSION

Basic linear regression cannot capture double descent

- Data dimensionality tied to number of parameters ($p = n_0$)

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  - $W_1 \in \mathbb{R}^{n_1 \times n_0}$, $[W_1]_{ij} \sim \mathcal{N}(0, 1/n_0)$

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  - Equivalent to training only the top layer of single-layer NN
  - For simple data, test error can computed asymptotically:
    - $X_{ij} \sim \mathcal{N}(0,1)$, $Y = \beta^\top X + \varepsilon$
    - $n_0, n_1, m \to \infty$ with $\phi \equiv n_0/m$, $\psi = n_0/n_1$ held constant

KERNEL RIDGE REGRESSION IN HIGH DIMENSIONS

$$L = \|WF - Y\|_F^2 + \gamma \|W\|_F^2, \qquad Y = \beta^T X + \epsilon \qquad F = f(W_1 X)$$

$$W^* = YQF^T, \qquad Q = (K + \gamma I)^{-1}, \qquad K = \frac{1}{n_1} F^T F$$

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Consider an unseen test point $\tilde{x}$, with random features $\tilde{f} = f(W_1\tilde{x})$ and targets $\tilde{y} = \beta^T\tilde{x}$.

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$$\begin{aligned} E_{test} &= \mathbb{E}_{\tilde{x}} \|W^*\tilde{f} - \tilde{y}\|_F^2 \\ &= \mathbb{E}_{\tilde{x}}\, tr[(YQF^T\tilde{f} - \tilde{y})^T(YQF^T\tilde{f} - \tilde{y})] \\ &= \mathbb{E}_{\tilde{x}}\, tr[\tilde{f}^T FQY^T YQF^T\tilde{f}] - 2\mathbb{E}_{\tilde{x}}\, tr[\tilde{f}^T FQY^T\tilde{y}] + \mathbb{E}_{\tilde{x}}\, tr[\tilde{y}^T\tilde{y}] \end{aligned}$$

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$$W^* = YQF^T, \qquad Q = (K + \gamma I)^{-1}, \qquad K = \frac{1}{n_1} F^T F$$

Consider an unseen test point $\tilde{x}$, with random features $\tilde{f} = f(W_1\tilde{x})$ and targets $\tilde{y} = \beta^T\tilde{x}$.

$$\begin{aligned} E_{test} &= \mathbb{E}_{\tilde{x}} \|W^*\tilde{f} - \tilde{y}\|_F^2 \\ &= \mathbb{E}_{\tilde{x}}\, tr[(YQF^T\tilde{f} - \tilde{y})^T(YQF^T\tilde{f} - \tilde{y})] \\ &= \mathbb{E}_{\tilde{x}}\, tr[\tilde{f}^T FQY^T YQF^T\tilde{f}] - 2\mathbb{E}_{\tilde{x}}\, tr[\tilde{f}^T FQY^T\tilde{y}] + \mathbb{E}_{\tilde{x}}\, tr[\tilde{y}^T\tilde{y}] \end{aligned}$$

How to simplify these terms?

- Complex dependence on W_1, X, F

GAUSSIAN EQUIVALENTS

Asymptotic universality – can replace $F = f(WX)$ by another matrix such that all relevant first and second moments are preserved

$$F \simeq F^{lin} \equiv \sqrt{\zeta} WX + \sqrt{\eta - \zeta}\, A$$

$$\eta = \int dz\, \frac{e^{-z^2/2}}{\sqrt{2\pi}} f(\sigma_w \sigma_x z)^2 \qquad \zeta = \left[\sigma_w \sigma_x \int dz\, \frac{e^{-z^2/2}}{\sqrt{2\pi}} f'(\sigma_w \sigma_x z) \right]^2 \qquad A_{ij} \sim \mathcal{N}(0,1)$$

KERNEL RIDGE REGRESSION IN HIGH DIMENSIONS

$$E_{test} = \mathbb{E}_{\tilde{x}}\, tr[\tilde{f}^T F Q Y^T Y Q F^T \tilde{f}] - 2\mathbb{E}_{\tilde{x}}\, tr[\tilde{f}^T F Q Y^T \tilde{y}] + \mathbb{E}_{\tilde{x}}\, tr[\tilde{y}^T \tilde{y}]$$

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To simplify, apply the linearizations:

$$F \to F^{lin} \equiv \sqrt{\zeta} W_1 X + \sqrt{\eta - \zeta}\, A \qquad \tilde{f} \to \tilde{f}^{lin} \equiv \sqrt{\zeta} W_1 \tilde{x} + \sqrt{\eta - \zeta}\, \tilde{a}$$

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The expectations over $\beta, \varepsilon, \tilde{a}, \tilde{x}$ are trivial because

$$Q \to ((F^{lin})^T F^{lin} + \gamma I)^{-1} = \left((\sqrt{\zeta} W_1 X + \sqrt{\eta - \zeta}\, A)^T (\sqrt{\zeta} W_1 X + \sqrt{\eta - \zeta}\, A)\right)^{-1}$$

depends only on W_1, X, A.

KERNEL RIDGE REGRESSION IN HIGH DIMENSIONS

After applying linearization and performing the trivial expectations, the result can be written as

$$E_{test} = \sum_i tr[R_i Q S_i Q] + \sum_i tr[T_i Q]$$

where R_i, S_i, T_i are low-order polynomials in W_1, X, A, and

$$Q = \left((\sqrt{\zeta} W_1 X + \sqrt{\eta - \zeta}\, A)^T (\sqrt{\zeta} W_1 X + \sqrt{\eta - \zeta}\, A) \right)^{-1}$$

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Q: How to evaluate the trace of a *rational function* of random matrices?

A: Linear pencil + operator-valued free probability

GENERALIZATION ERROR

$$K = \frac{1}{n_1} F^\top F + \gamma I$$

Lemma 1. *Let* $\eta = \mathbb{E}[\sigma(g)^2]$ *and* $\zeta = (\mathbb{E}[g\sigma(g)])^2$ *for* $g \sim \mathcal{N}(0,1)$. *Then, in the high-dimensional asymptotics defined above, the traces* $\tau_1(\gamma) = \frac{1}{m}\mathbb{E}\,\mathrm{tr}(K^{-1})$ *and* $\tau_2(\gamma) = \frac{1}{m}\mathbb{E}\,\mathrm{tr}(\frac{1}{n_0} X^\top X K^{-1})$ *are given by the unique solutions to the coupled polynomial equations,*

$$\zeta\tau_1\tau_2\left(1-\gamma\tau_1\right) = \phi/\psi\left(\zeta\tau_1\tau_2 + \phi(\tau_2-\tau_1)\right) = \left(\tau_1-\tau_2\right)\phi\left((\eta-\zeta)\tau_1 + \zeta\tau_2\right)\,, \tag{21}$$

such that $\tau_1, \tau_2 \in \mathbb{C}^+$ *for* $\gamma \in \mathbb{C}^+$.

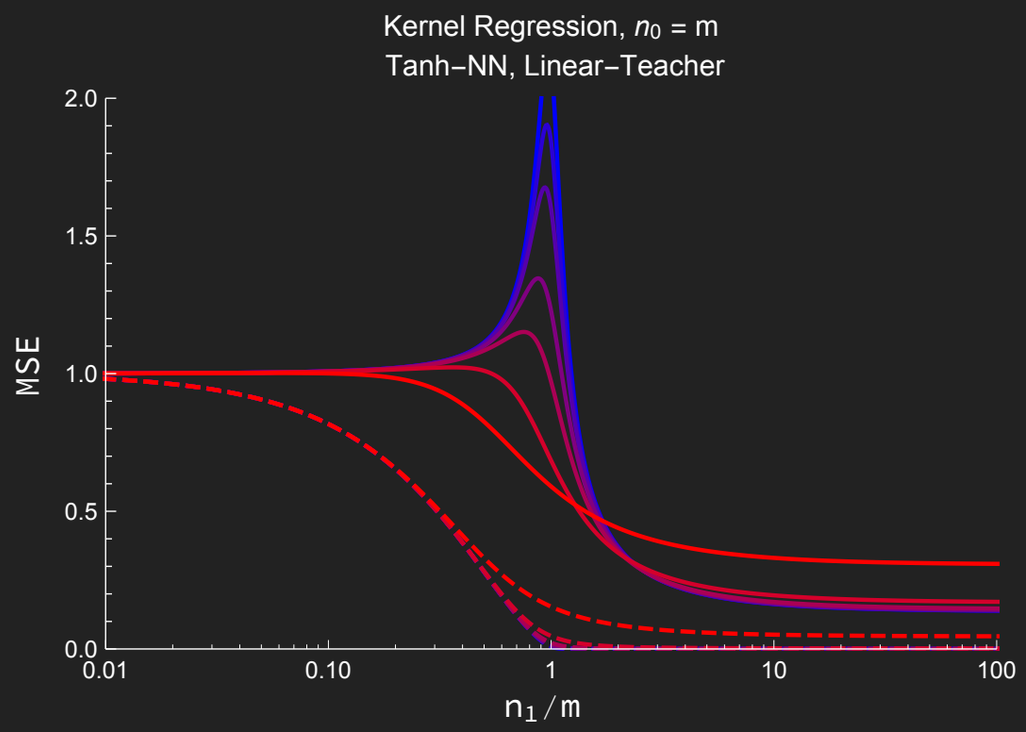
Theorem 1. *Let* $\gamma = Re(z)$ *and let* τ_1 *and* τ_2 *be defined as in Lemma 1 with* $Im(z) \to 0^+$. *Then the asymptotic training error* $E_{train} = \frac{1}{m}\mathbb{E}\|Y - \hat{y}(X)\|_F^2$ *is given by,*

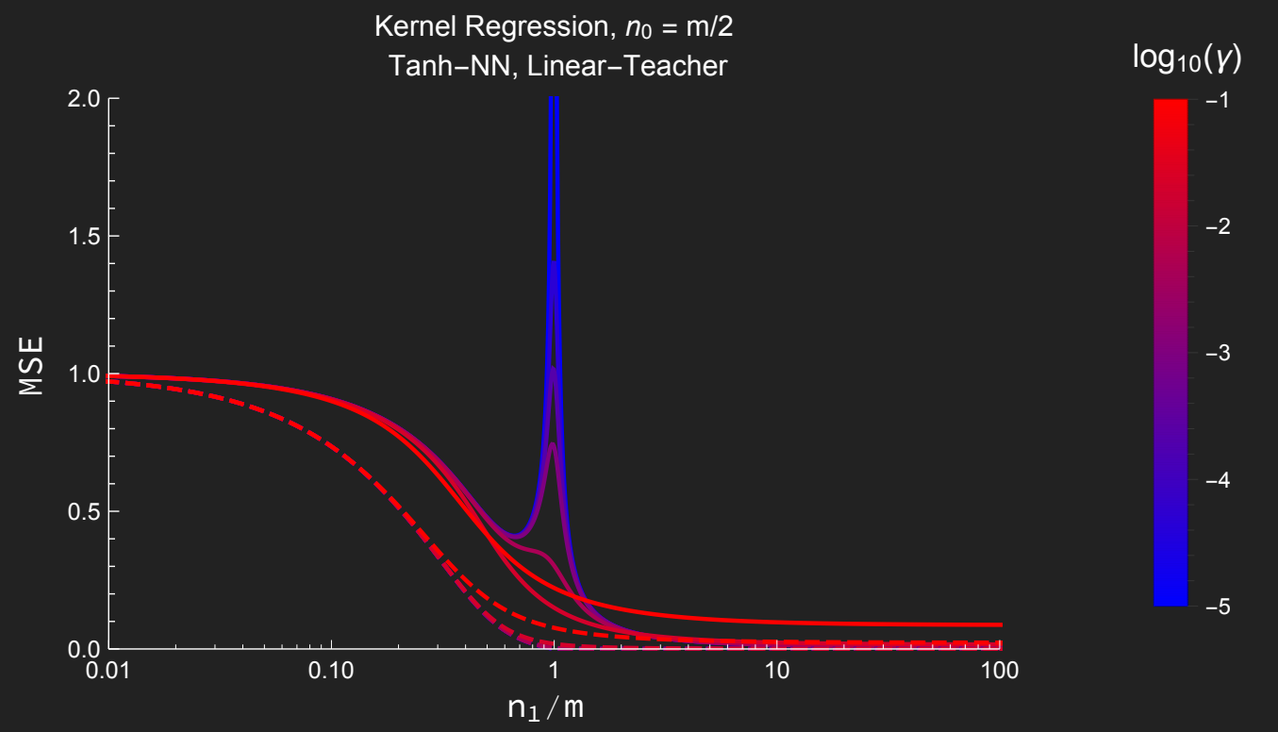
$$E_{train} = -\gamma^2(\sigma_\varepsilon^2\tau_1' + \tau_2')\,, \tag{22}$$

and the asymptotic test error $E_{test} = \mathbb{E}(y - \hat{y}(\mathbf{x}))^2$ *is given by*

$$E_{test} = (\gamma\tau_1)^{-2}E_{train} - \sigma_\varepsilon^2\,. \tag{23}$$

GENERALIZATION ERROR





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BIAS–VARIANCE DECOMPOSITION

Recall that the per-sample loss can be decomposed as

$$\mathbb{E}[(\hat{y}(x) - y(x))^2] = (\mathbb{E}\hat{y}(x) - \mathbb{E}y(x))^2 + \mathbb{V}[\hat{y}(x)] + \mathbb{V}[y(x)]$$

If the additive label noise ε on the training points is the only randomness, then we can use the classical decomposition:

$$E_{test} = \mathbb{E}_x\mathbb{E}_\varepsilon\left[\hat{y}(x) - y(x)\right]^2 = \underbrace{\mathbb{E}_x\mathbb{V}_\varepsilon\left[\hat{y}(x)\right]}_{\text{Variance}} + \underbrace{\mathbb{E}_x(\mathbb{E}_\varepsilon[\hat{y}(x)] - y(x)^2)}_{\text{Bias}}$$

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But what about any other sources of randomness ($\Theta = \{W_1, X\}$)?

MULTIVARIATE APPROACH: SYMMETRIC DECOMPOSITION

There is a unique way to symmetrically decompose the variance:

Proposition 1. *Let $X_1, \ldots, X_K$, and Y be random variables and $\mathcal{X} := \{X_1, \ldots, X_K\}$. We define a* variance decomposition *of Y to be a multiset $\{V_1, \ldots, V_N\}$ of nonnegative real numbers such that $\mathbb{V}[Y] = \sum_i V_i$. Then there exists a unique variance decomposition $\mathcal{V} := \{V_s : s \subseteq \mathcal{X}\}$ such that $\mathcal{V}$ is invariant under permutations of $\mathcal{X}$, and such that for all $S \subseteq \mathcal{X}$ the marginal variances satisfy the subset-sum relation,*

$$\mathbb{VE}[Y|X_j \text{ for } j \in S] = \sum_{s \subseteq S} V_s \,. \tag{5}$$

Example 1. *Consider the case of two random variables, the parameters P and the data D. Then $\mathcal{X} = \{P, D\}$ and the decomposition satisfying Proposition 1 is given by,*

$$V_P := \mathbb{E}_{\mathbf{x}}\mathbb{VE}[\hat{y}|P] \tag{6}$$

$$V_D := \mathbb{E}_{\mathbf{x}}\mathbb{VE}[\hat{y}|D] \tag{7}$$

$$V_{PD} := \mathbb{E}_{\mathbf{x}}\mathbb{VE}[\hat{y}|P, D] - \mathbb{E}_{\mathbf{x}}\mathbb{VE}[\hat{y}|P] - \mathbb{E}_{\mathbf{x}}\mathbb{VE}[\hat{y}|D] \,. \tag{8}$$

We can interpret V_{PD} as the variance explained by the parameters and data together beyond what they explain individually.

MULTIVARIATE APPROACH: SYMMETRIC DECOMPOSITION

Example 2. *Further decomposing D into randomness from sampling the inputs X and label noise $\boldsymbol{\epsilon}$, we can write $\mathcal{X} = \{P, X, \boldsymbol{\epsilon}\}$ and the decomposition satisfying Proposition 1 is given by,*

$$V_X := \mathbb{E}_{\mathbf{x}}\mathbb{V}\mathbb{E}[\hat{y}|X], \tag{9}$$

$$V_{\boldsymbol{\epsilon}} := \mathbb{E}_{\mathbf{x}}\mathbb{V}\mathbb{E}[\hat{y}|\boldsymbol{\epsilon}], \tag{10}$$

$$V_P := \mathbb{E}_{\mathbf{x}}\mathbb{V}\mathbb{E}[\hat{y}|P], \tag{11}$$

$$V_{X\boldsymbol{\epsilon}} := \mathbb{E}_{\mathbf{x}}\mathbb{V}\mathbb{E}[\hat{y}|X, \boldsymbol{\epsilon}] - \mathbb{E}_{\mathbf{x}}\mathbb{V}\mathbb{E}[\hat{y}|X] - \mathbb{E}_{\mathbf{x}}\mathbb{V}\mathbb{E}[\hat{y}|\boldsymbol{\epsilon}], \tag{12}$$

$$V_{PX} := \mathbb{E}_{\mathbf{x}}\mathbb{V}\mathbb{E}[\hat{y}|P, X] - \mathbb{E}_{\mathbf{x}}\mathbb{V}\mathbb{E}[\hat{y}|X] - \mathbb{E}_{\mathbf{x}}\mathbb{V}\mathbb{E}[\hat{y}|P], \tag{13}$$

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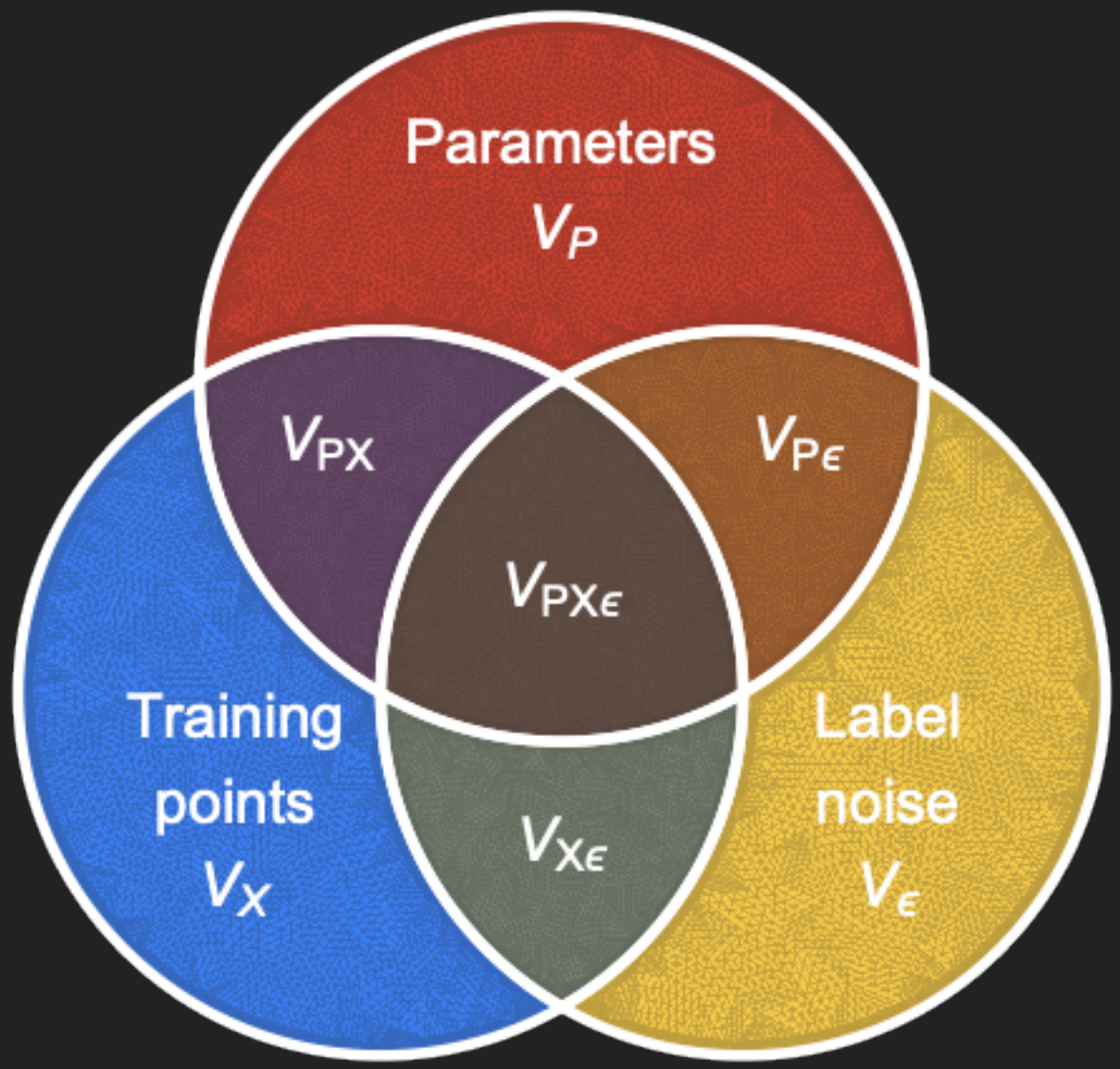
MULTIVARIATE APPROACH: SYMMETRIC DECOMPOSITION

Remark 1. *Because* $V_s \geq 0$ *and* $V = \mathbb{V}[\hat{y}] = \sum_s V_s$, *the subset-sum relation (5) yields an interpretation of* V *as the union of disjoint areas, forming a Venn diagram.*



MULTIVARIATE APPROACH: SYMMETRIC DECOMPOSITION

Lemma 1. *Let* $\eta = \mathbb{E}[\sigma(g)^2]$ *and* $\zeta = (\mathbb{E}[g\sigma(g)])^2$ *for* $g \sim \mathcal{N}(0,1)$. *Then, in the high-dimensional asymptotics defined above, the traces* $\tau_1(\gamma) = \frac{1}{m}\operatorname{tr}(K^{-1})$ *and* $\tau_2(\gamma) = \frac{1}{m}\operatorname{tr}(\frac{1}{n_0}X^\top X K^{-1})$ *are given by the unique solutions to the coupled polynomial equations,*

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such that $\tau_1, \tau_2 \in \mathbb{C}^+$ *for* $\gamma \in \mathbb{C}^+$.

Theorem 1. *Let* τ_1 *and* τ_2 *be defined as in Lemma 1. Then the asymptotic bias and variance terms of eqns.* (9)-(15) *are given by,*

$$\begin{aligned}
B &= \tau_2^2/\tau_1^2 & V_{PX} &= -\tau_2'/\tau_1^2 - B - V_P - V_X \\
V_P &= \tau_2'/\tau_1' - B & V_{P\boldsymbol{\varepsilon}} &= 0 \\
V_X &= \phi B(\tau_1-\tau_2)^2/(\tau_1^2 - \phi(\tau_1-\tau_2)^2) & V_{X\boldsymbol{\varepsilon}} &= \sigma_{\boldsymbol{\varepsilon}}^2 V_X/B \\
V_{\boldsymbol{\varepsilon}} &= 0 & V_{PX\boldsymbol{\varepsilon}} &= \sigma_{\boldsymbol{\varepsilon}}^2(-\tau_1'/\tau_1^2 - 1) - V_{X\boldsymbol{\varepsilon}}\,.
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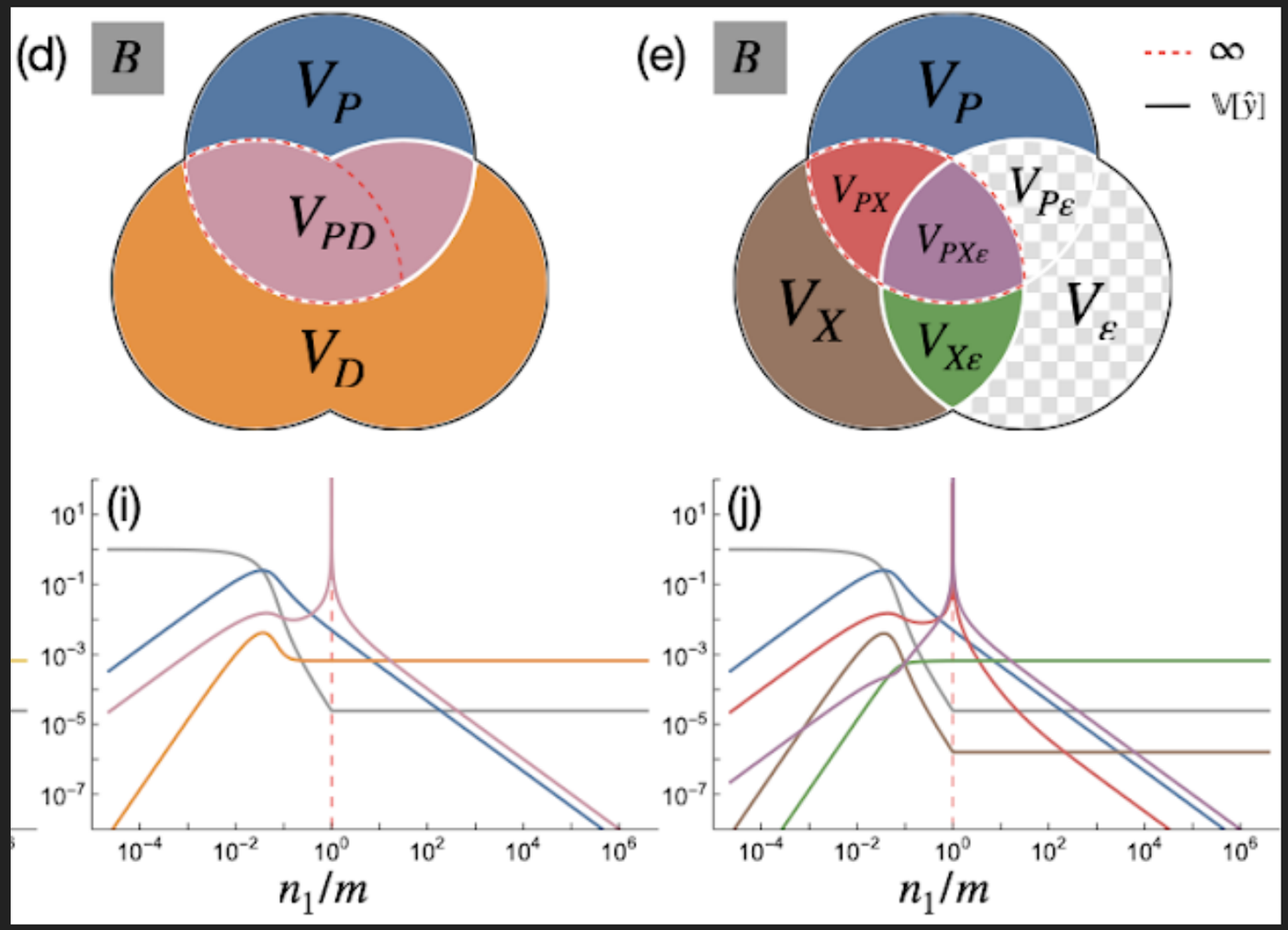
A UNIFYING PERSPECTIVE ON BIAS–VARIANCE DECOMPOSITIONS



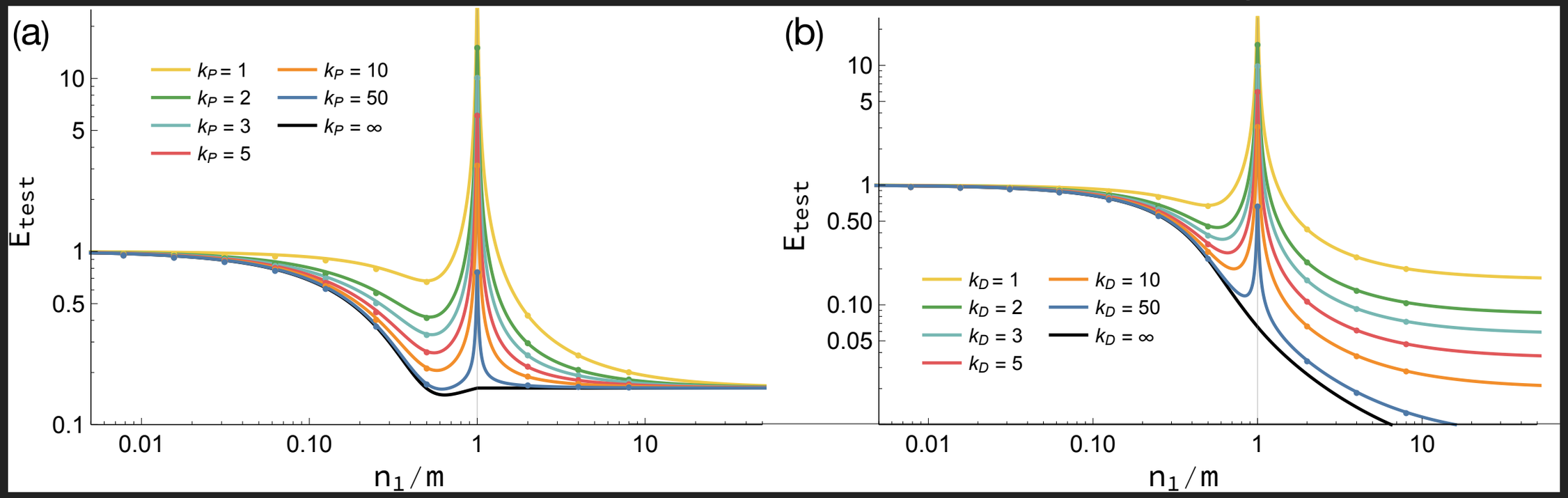
The symmetric decomposition reveals that the divergence comes from $V_{PD} = V_{PX} + V_{PX\varepsilon}$

- This the variance that is explained by the *interaction of the* parameters and the data
- Predicts that ensembling over parameters or samples will eliminate divergence

THE ORIGIN OF DOUBLE DESCENT

Parameter-ensemble of k_P different models with same data

Data-ensemble of k_D different models with same parameters



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FROM KERNELS TO NEURAL NETWORKS AND BACK AGAIN

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This motivates a linear approximation

$$N(x; \theta(t)) \approx N(x; \theta(0)) + \left.\frac{\partial N}{\partial \theta}\right|_{\theta=\theta(0)} (\theta(t) - \theta(0)) + \mathcal{O}(\theta(t) - \theta(0))^2$$

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The dynamics are determined by the Neural Tangent Kernel Θ

$$\Theta = J_0^T J_0$$

NEURAL TANGENT KERNEL

The offset N_0 contributes unnecessary variance. Can set $N_0 = 0$ by subtracting two copies of the model with same initialization

$$N^{VR}(x; \{\theta_1, \theta_2\}) = \frac{1}{\sqrt{2}}\big(N(x;\theta_1) - N(x;\theta_2)\big) \qquad \theta_1(0) = \theta_2(0)$$

$$N_0^{VR} = 0, \quad \Theta^{VR} = \Theta$$

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- $\Theta_1 = (F')^T D_{W_2} F' \odot X^T X, \qquad F' = f'(W_1 X)$

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- $\Theta_1 = (F')^T D_{W_2} F' \odot X^T X, \qquad F' = f'(W_1 X)$
- Only the mean of $(F')^T D_{W_2} F'$ contributes at leading order
- $\Theta_1 \simeq c_1 I + c_2 X^T X$

QUADRATIC OVERPARAMETERIZATION

Proposition 1. *As $n_0, n_1, m \to \infty$ with $\phi = n_0/m$ and $\psi = n_0/n_1$ fixed, the traces $\tau_1(z) := \frac{1}{m}\mathbb{E}\,\mathrm{tr}(K(z)^{-1})$ and $\tau_2(z) := \frac{1}{m}\mathbb{E}\,\mathrm{tr}(\frac{1}{n_0}X^\top X K(z)^{-1})$ are given by the unique solutions to the coupled polynomial equations,*

$$\begin{aligned}
&\phi\left(\zeta\tau_2\tau_1 + \phi(\tau_2-\tau_1)\right) + \zeta\tau_1\tau_2\psi\left(z\tau_1 - 1\right) \\
&\quad = -\zeta\tau_1\tau_2\sigma_{W_2}^2\left(\zeta\left(\tau_2-\tau_1\right)\psi + \tau_1\psi\eta' + \phi\right) \\
&\zeta\tau_1^2\tau_2\left(\eta'-\eta\right)\sigma_{W_2}^2 + \zeta\tau_1\tau_2\left(z\tau_1 - 1\right) \\
&\quad = \left(\tau_2-\tau_1\right)\phi\left(\zeta\left(\tau_2-\tau_1\right) + \eta\tau_1\right),
\end{aligned} \tag{25}$$

such that $\tau_1, \tau_2 \in \mathbb{C}^+$ for $z \in \mathbb{C}^+$.

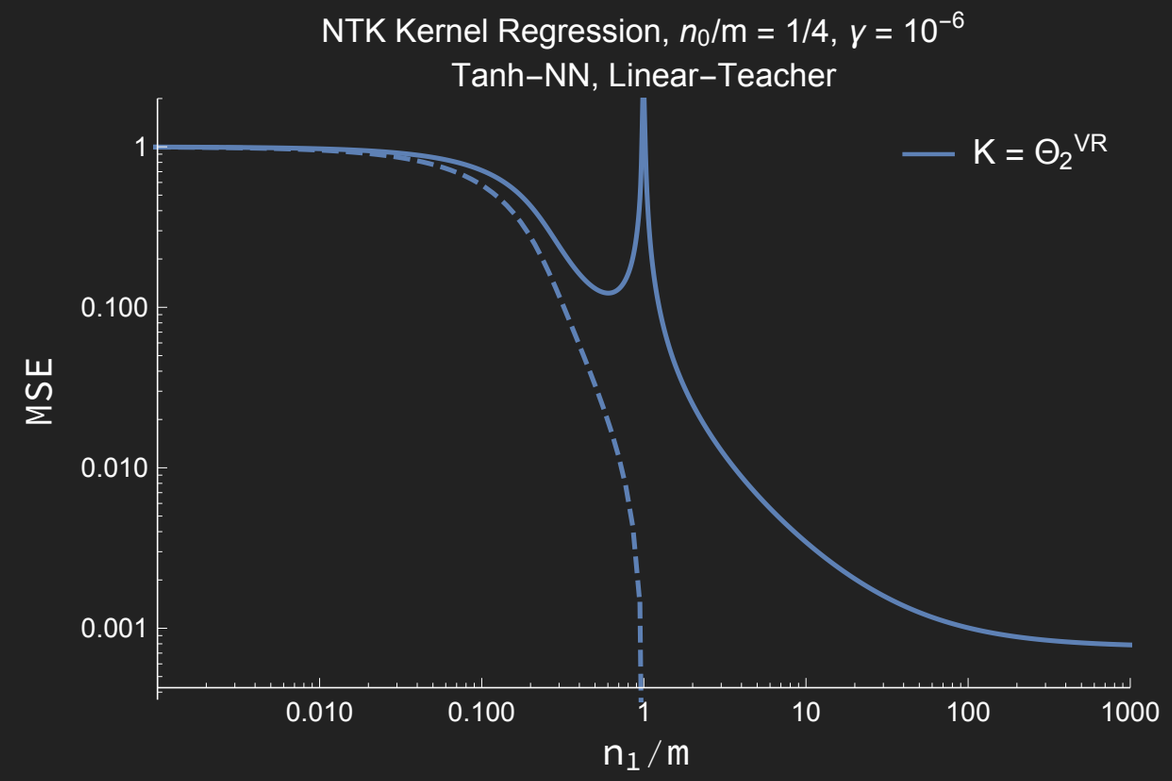
Theorem 1. *Let $\gamma = Re(z)$ and let τ_1 and τ_2 be defined as in Proposition 1 with $Im(z) \to 0^+$. Then the asymptotic training error $E_{train} = \frac{1}{m}\mathbb{E}\|Y - \hat{y}(X)\|_F^2$ is given by,*

$$\begin{aligned}
E_{train} = &-\gamma^2(\sigma_\varepsilon^2\tau_1' + \tau_2') + \nu\sigma_{W_2}^2\gamma^2(\tau_1 + \gamma\tau_1') \\
&+ \nu\sigma_{W_2}^4\gamma^2\left((\eta'-\zeta)\tau_1' + \zeta\tau_2'\right),
\end{aligned} \tag{26}$$

and the asymptotic test error $E_{test} = \mathbb{E}(y - \hat{y}(\mathbf{x}))^2$ is given by

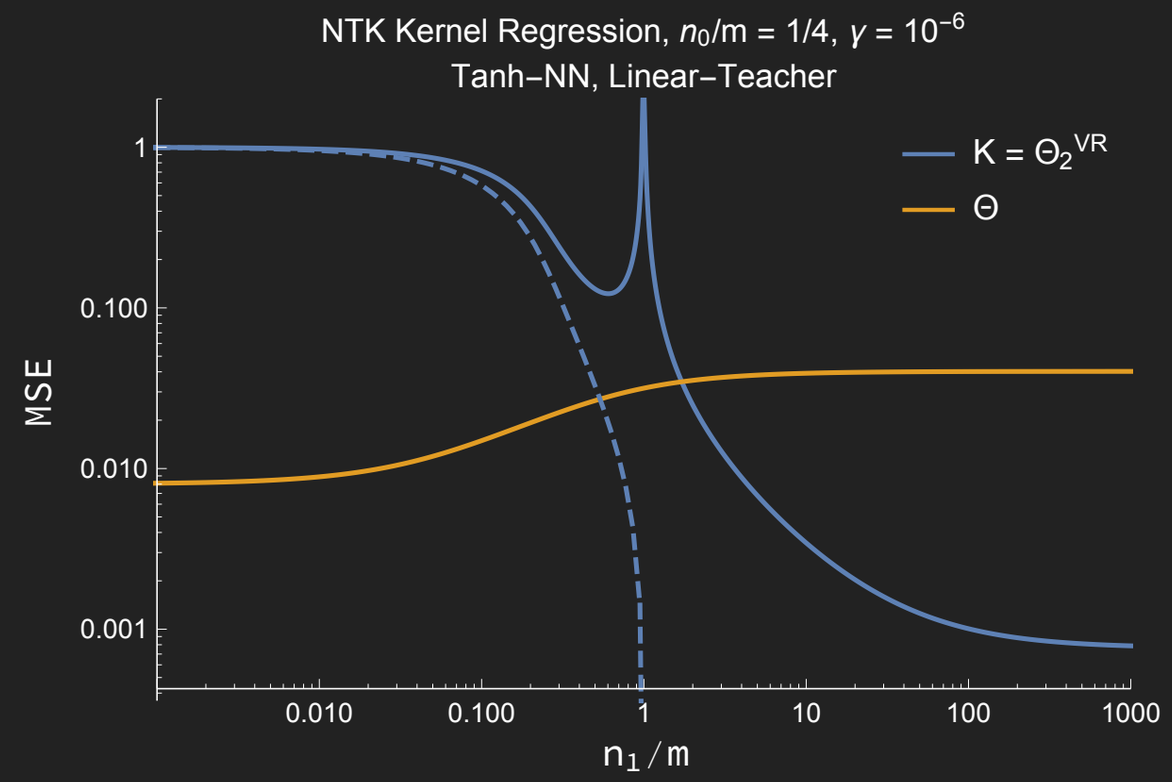
$$E_{test} = (\gamma\tau_1)^{-2}E_{train} - \sigma_\varepsilon^2\,. \tag{27}$$

QUADRATIC OVERPARAMETERIZATION



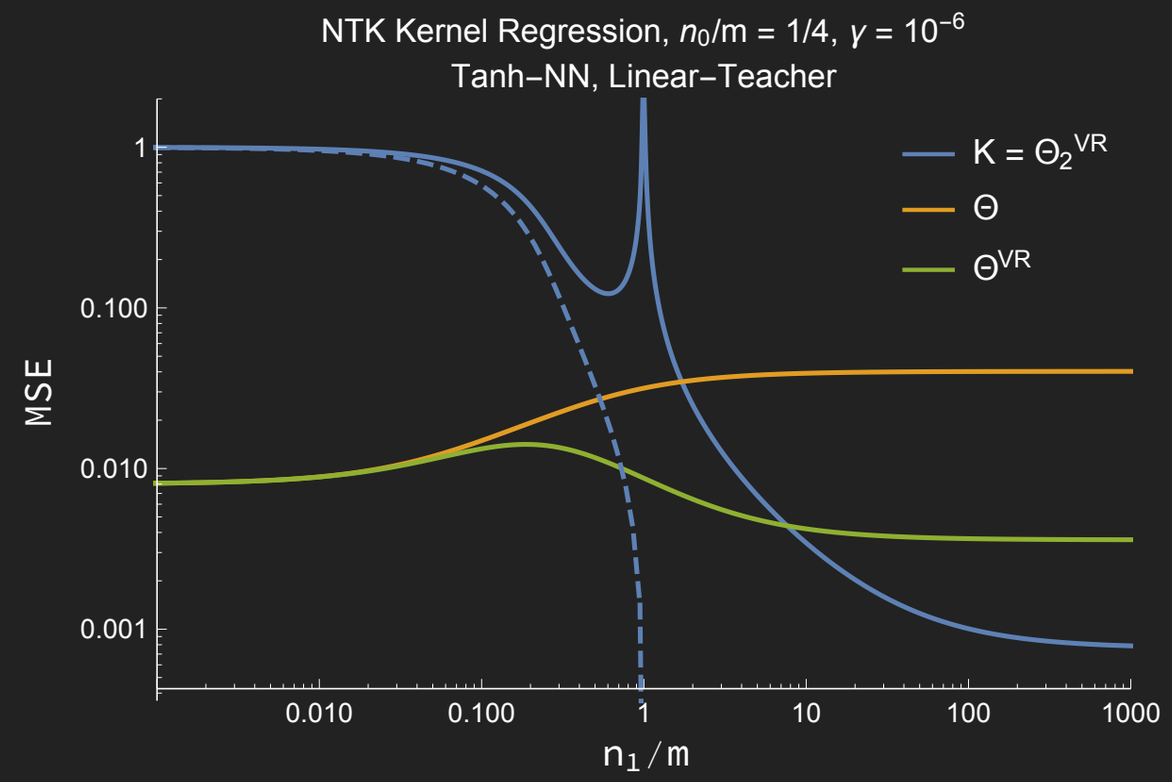
QUADRATIC OVERPARAMETERIZATION

The network can be too overparametrized

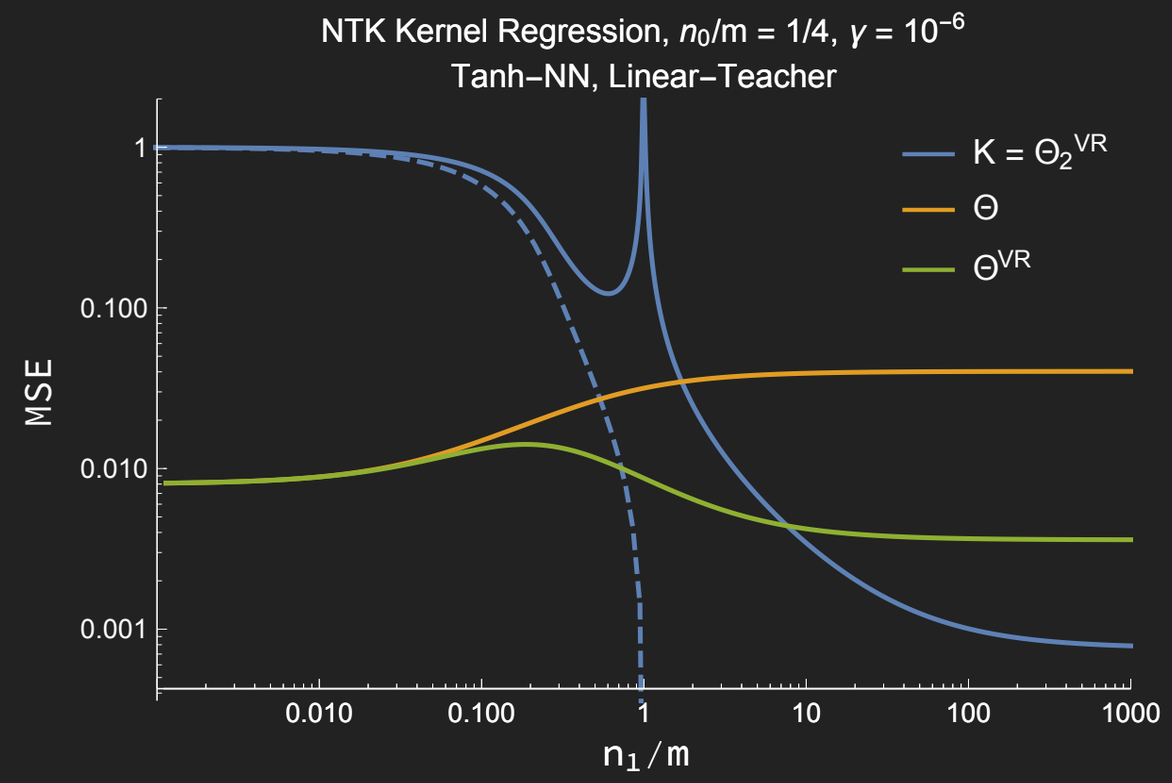


QUADRATIC OVERPARAMETERIZATION

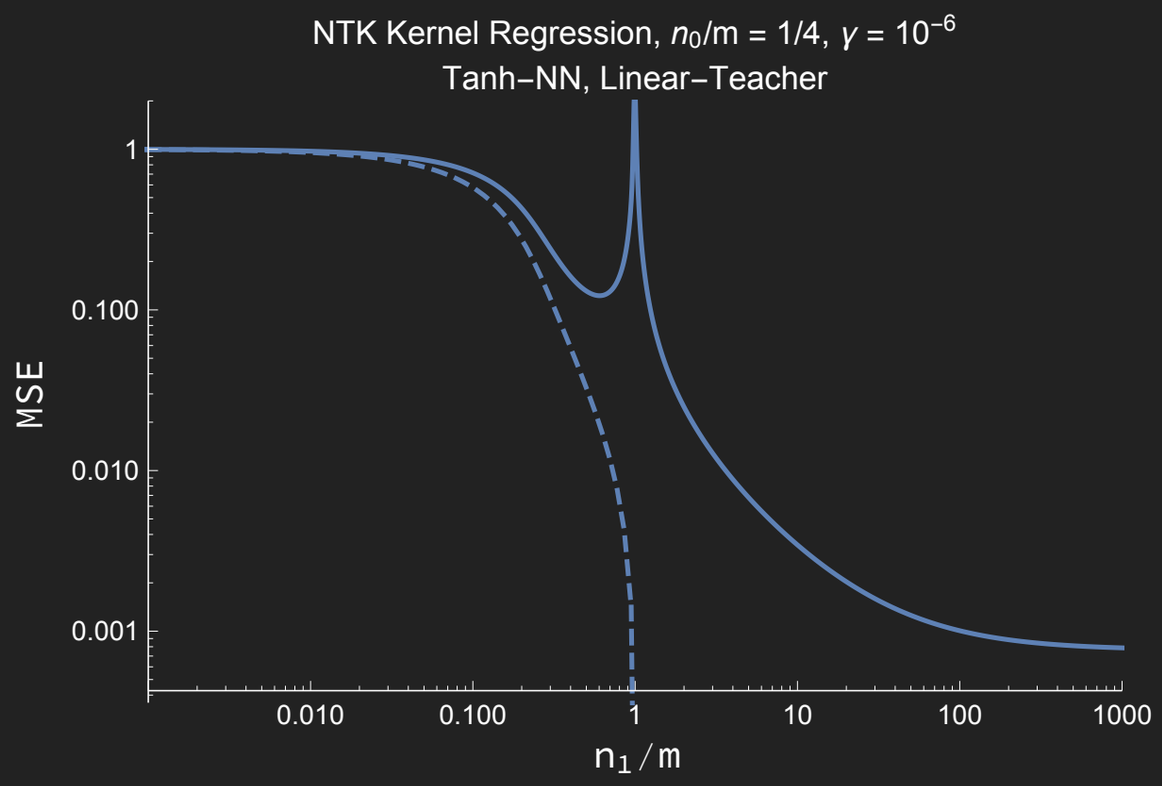
Reducing the variance helps, but a peak emerges

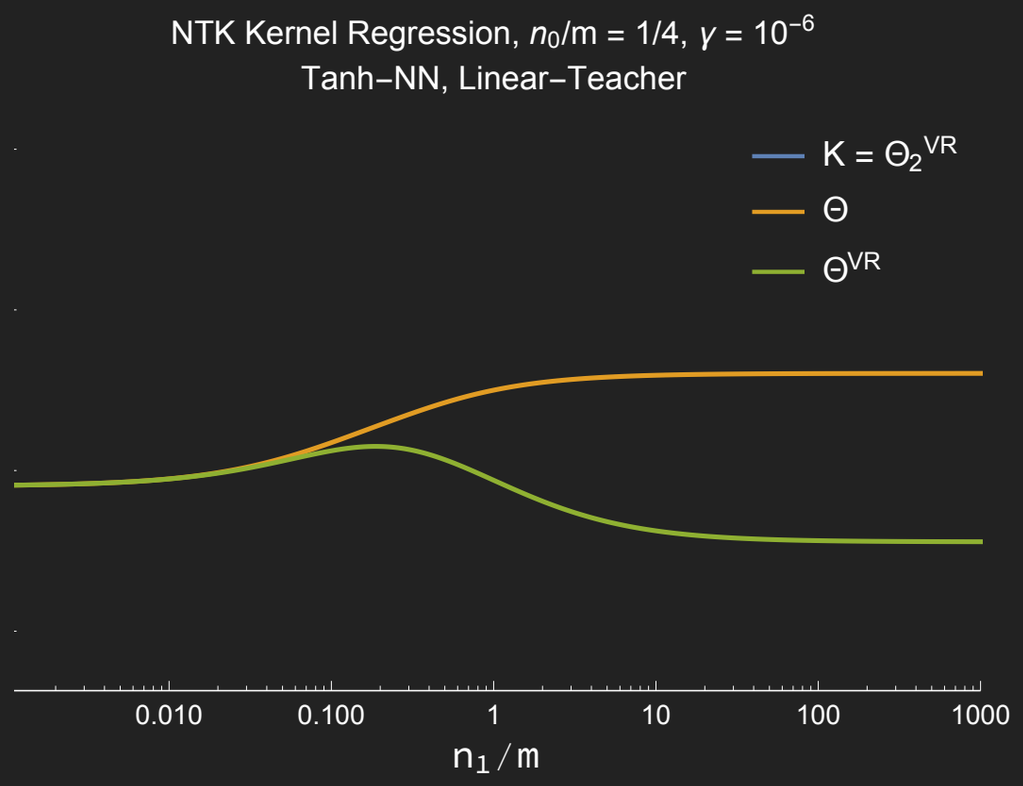


TWO OVERPARAMETERIZATION SCALES



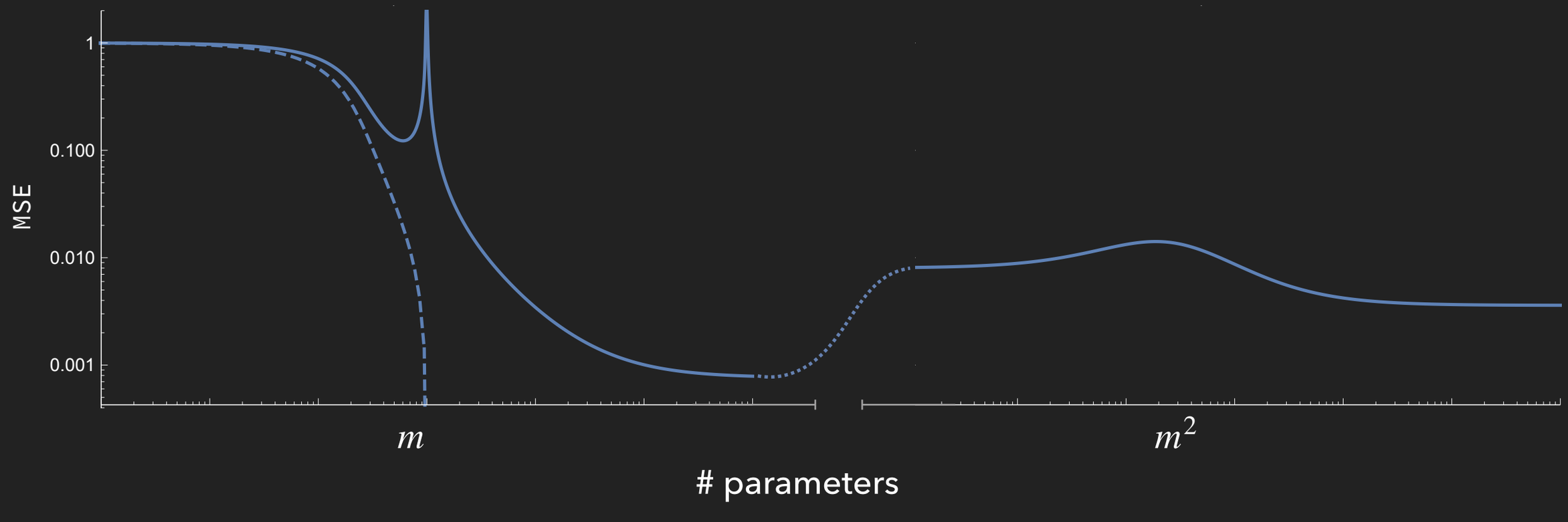
TWO OVERPARAMETERIZATION SCALES





TWO OVERPARAMETERIZATION SCALES

NTK Kernel Regression, $n_0/m = 1/4$, $\gamma = 10^{-6}$
Tanh–NN, Linear–Teacher

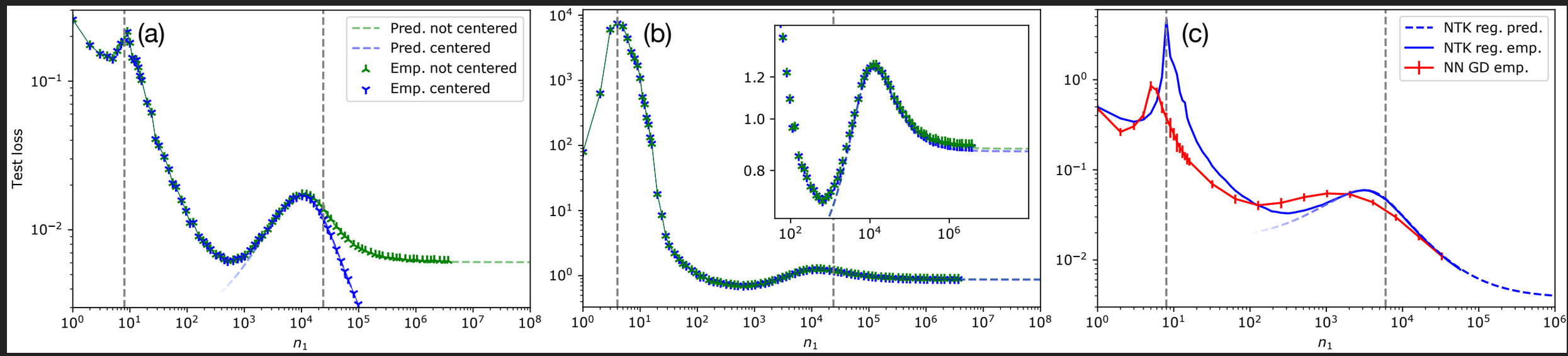


TWO OVERPARAMETERIZATION SCALES

Triple descent in finite-sized kernel regression models

Global minimum need not be at infinity

Qualitative picture holds for gradient descent as well



TWO OVERPARAMETERIZATION SCALES