NYU COURANT INSTITUTE OF MATHEMATICAL SCIENCES
NYU CENTER FOR DATA SCIENCE
FLATIRON INSTITUTE

JOAN BRUNA

MATHEMATICAL ASPECTS OF NEURAL NETWORK APPROXIMATION AND LEARNING

joint work with


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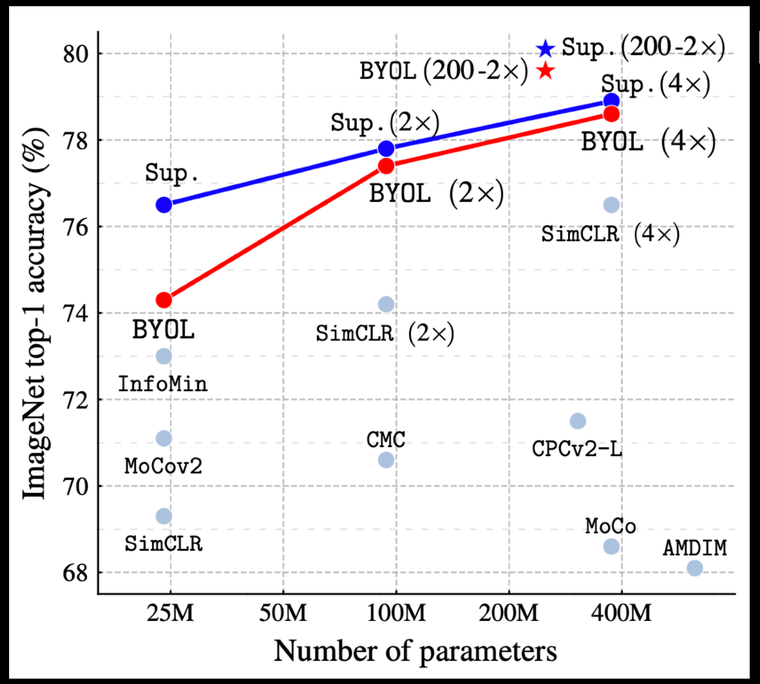
E.Vanden-Eijnden


Luca Venturi

DEEP LEARNING TODAY: EXPERIMENTAL REVOLUTION

[v.d. Oord et al.'19]

Gatys et al'14



[Grill et al'20]

[He et al.'17]

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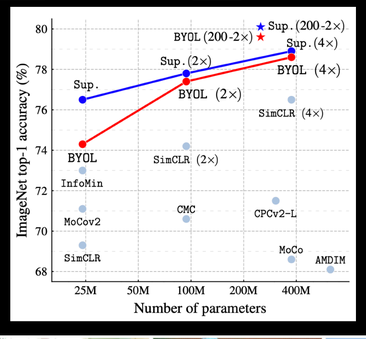
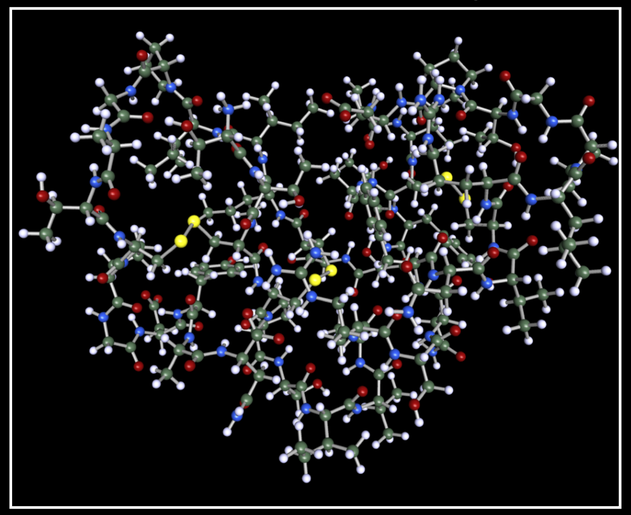


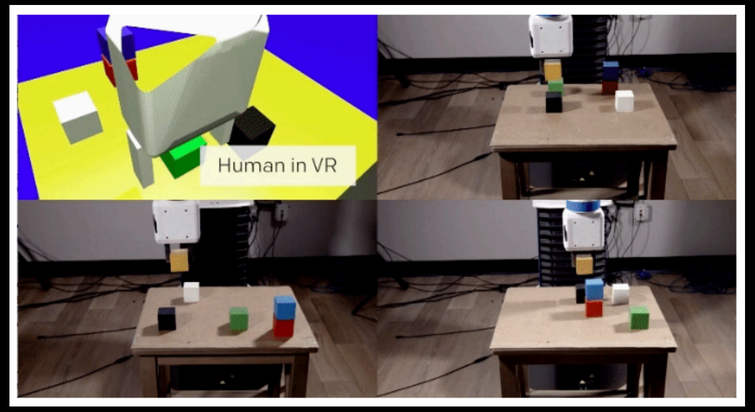
Computational Biology

Quantum Chemistry

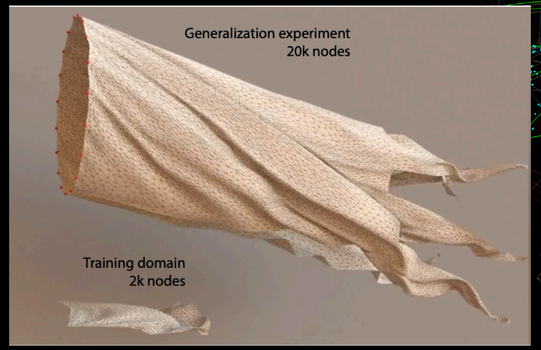


Games

Robotics

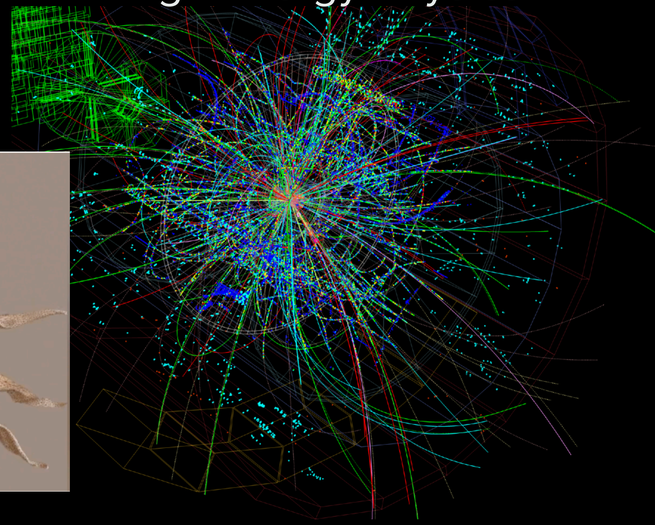


Scientific Computing



[Pfaff et al]

High Energy Physics



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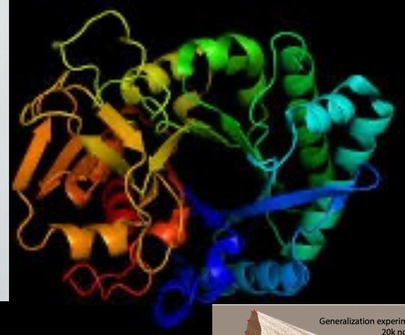

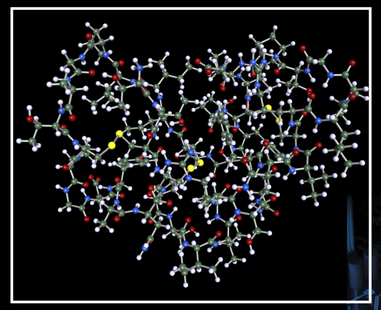

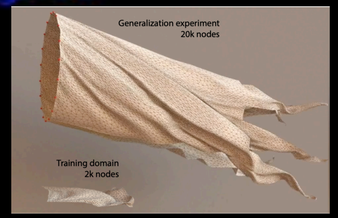

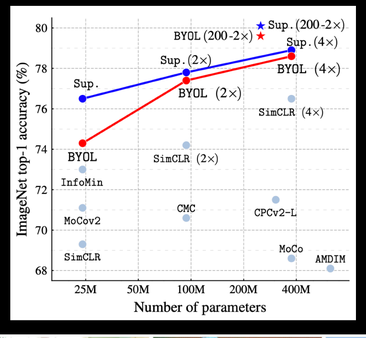

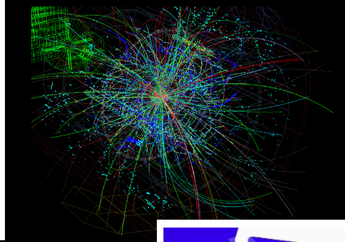

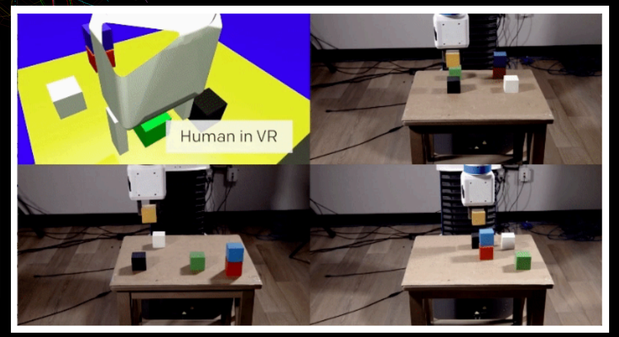

- Phenomenal ability to extract information from ***high-dimensional*** observations $x \in \Omega$
- In essence: ***non-linear***, compositional ***representation learning.***

$$f(x) = \theta^\top \Phi(x) \qquad \Phi : \Omega \to \mathbb{R}^K$$

  - Φ and θ are both ***learnt*** from data.
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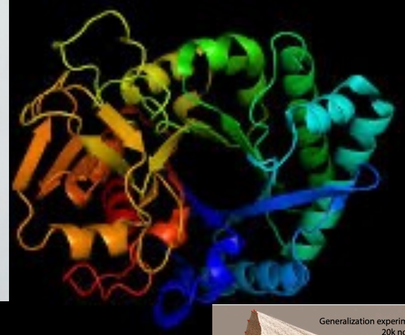

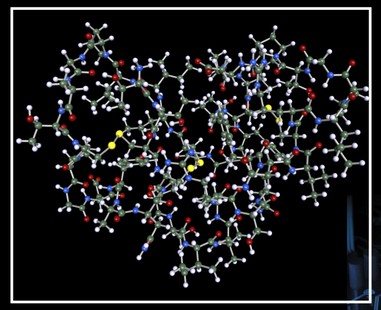

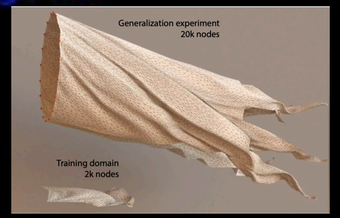

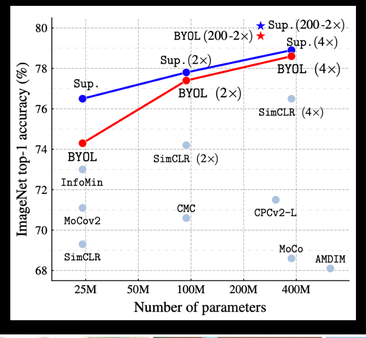

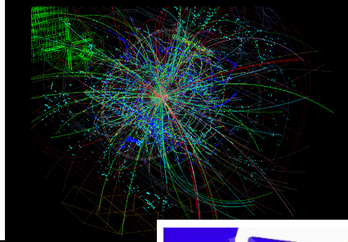

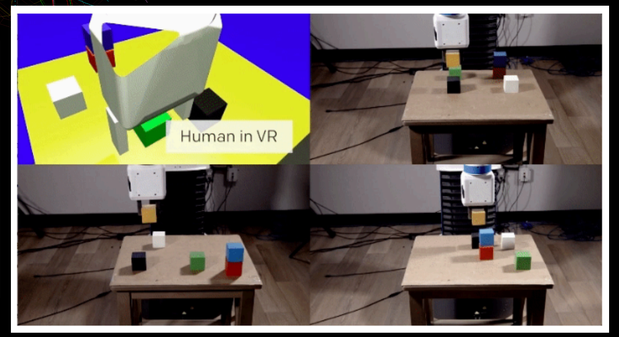

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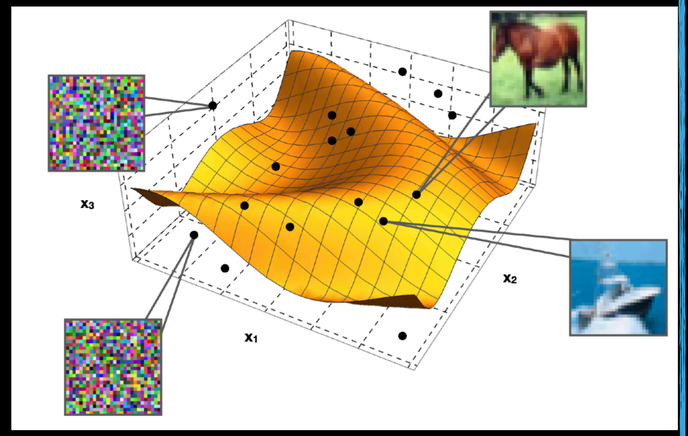
How, when, and why can DL systems approximate high-dimensional functions from data?

SUPERVISED LEARNING BASIC SETUP

DATA

$$x_i \sim \nu : \text{ data distribution in } \Omega.$$
$$y_i = f^*(x_i) \text{ for some } f^* \in L^2(\mathbb{R}^m, \mathrm{d}\nu).$$

$$(x_i, y_i) \in \Omega \times \mathbb{R}$$



[Goldt, Zdeborova, Krzakala et al]

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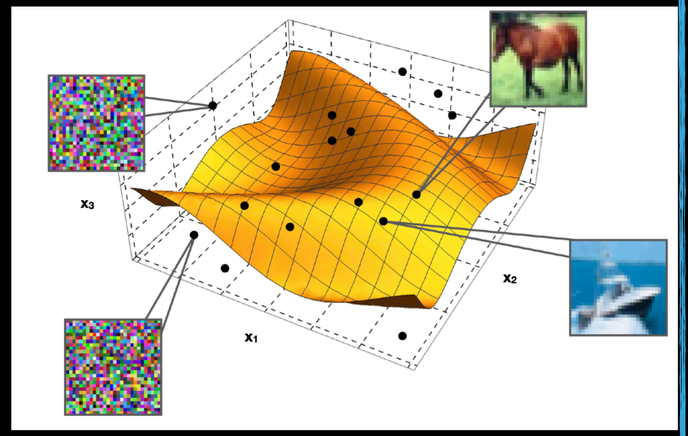

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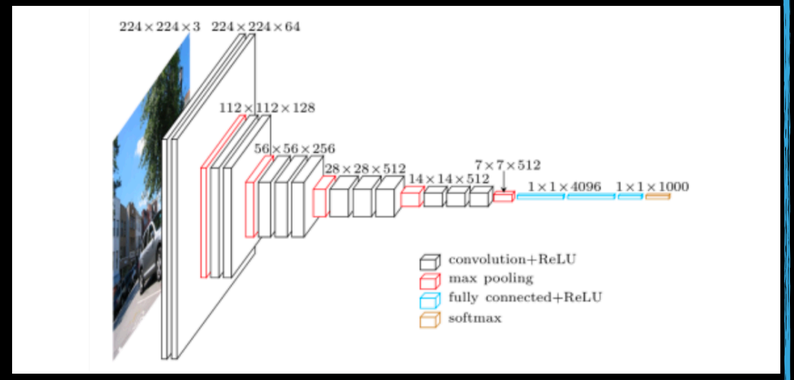
MODEL

$\mathcal{F} \subset \{f : \Omega \to \mathbb{R}\}$ e.g. $f(x; \Theta)$, $\Theta \in \mathcal{D}$.
Normed space: $\gamma(f)$, $f \in \mathcal{F}$.



Vgg

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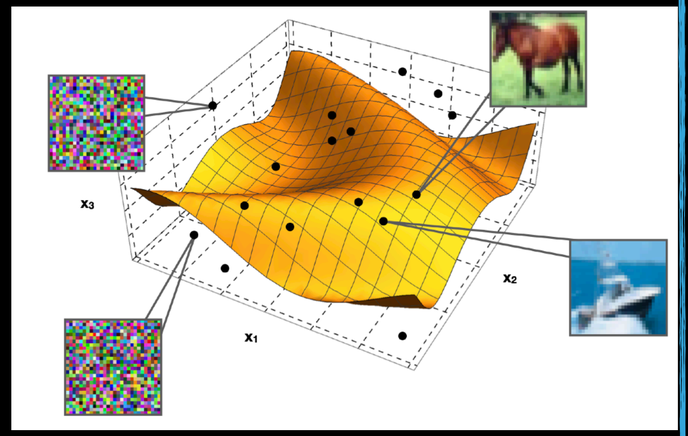

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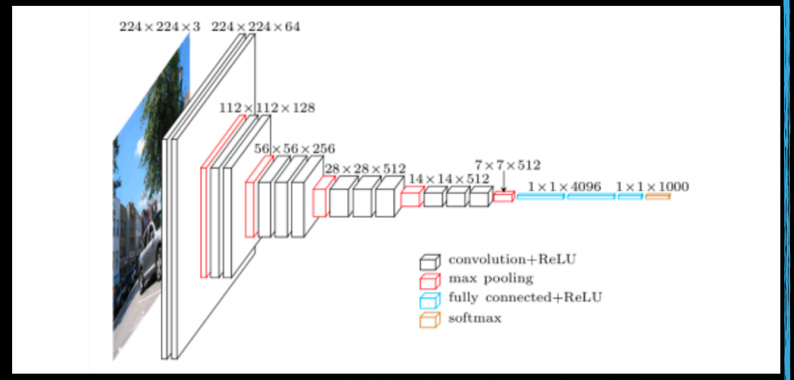

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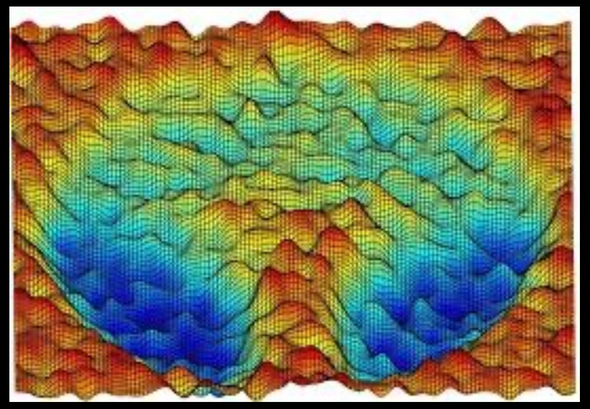
ERROR METRIC

$\mathcal{R}(f)$ convex, e.g.
$\mathcal{R}(f) = \mathbb{E}_\nu |f(x) - f^*(x)|^2$



[fig credit E. Vanden-Eijnden]

Empirical Loss:

$$\widehat{\mathcal{R}}(f) = \frac{1}{L} \sum_{i=1}^{L} |f(x_i) - f^*(x_i)|^2.$$

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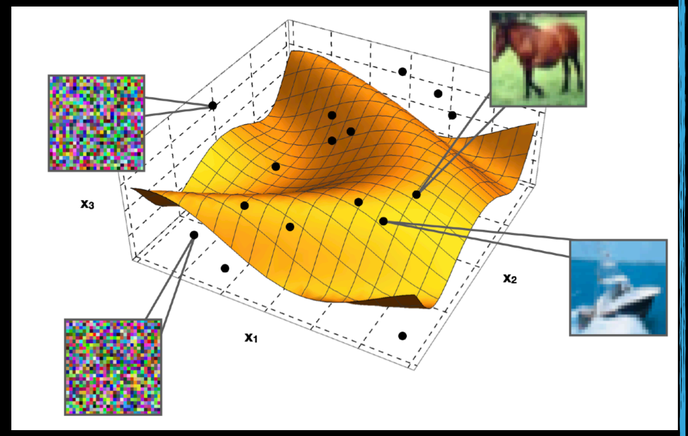

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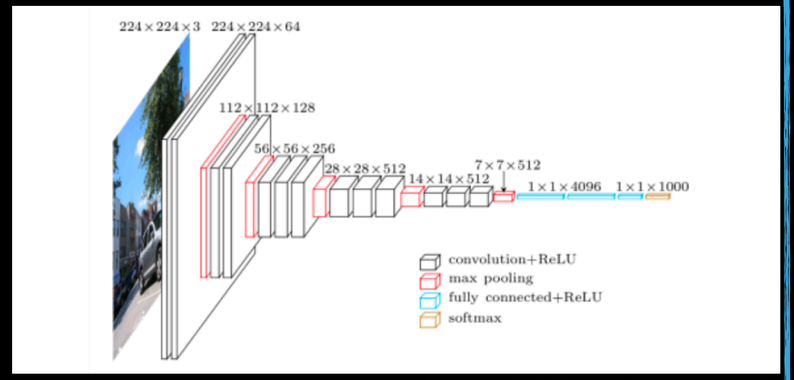

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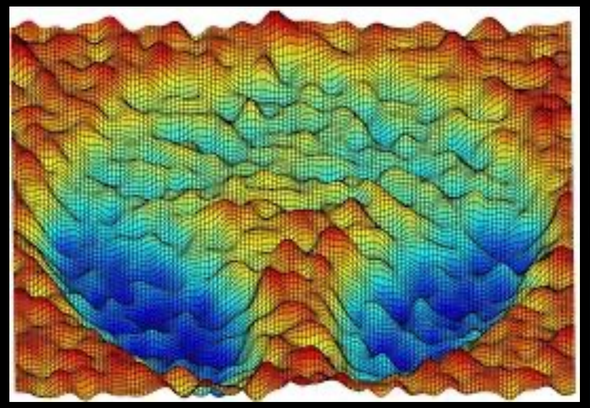

[fig credit E. Vanden-Eijnden]

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ALGORITHM

Empirical Risk Minimization

Ball $\mathcal{F}_\delta = \{f \in \mathcal{F}; \gamma(f) \leq \delta\}$.

Find $\hat{f}$ such that $\widehat{R}(\hat{f}) \leq \min_{f \in \mathcal{F}_\delta} \widehat{R}(f) + \epsilon$.

(constrained)

$\hat{f}$ such that $\widehat{\mathcal{R}} \leq \min_f \left(\widehat{\mathcal{R}}(f) + \lambda \gamma(f) \right) + \epsilon$.

(penalized)

CHALLENGES OF HIGH-DIMENSIONAL LEARNING

- Basic decomposition of error:

$$\text{Find } \hat{f} \text{ such that } \widehat{R}(\hat{f}) \leq \min_{f \in \mathcal{F}_\delta} \widehat{R}(f) + \epsilon.$$

$$\mathcal{R}(\hat{f}) - \inf_{f \in \mathcal{F}} \mathcal{R}(f) \leq \underbrace{\inf_{f \in \mathcal{F}_\delta} \mathcal{R}(f) - \inf_{f \in \mathcal{F}} \mathcal{R}(f)}_{\text{approx error}} + \underbrace{2 \sup_{\mathcal{F}_\delta} |\mathcal{R}(f) - \widehat{\mathcal{R}}(f)|}_{\text{statistical error}} + \underbrace{\epsilon}_{\text{optim. error}}.$$

[Bottou & Bousquet]

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[Bottou & Bousquet]

APPROXIMATION

Functional Approximation that is not cursed by input dimensionality, capturing right notion of regularity

STATISTICAL

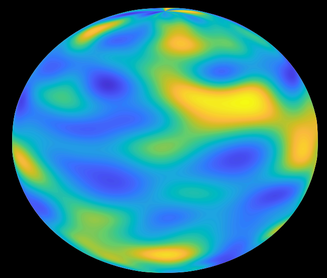
Concentration Bounds: Requires spaces $\mathcal{F}_\delta$ not to grow too quickly with dimension

COMPUTATIONAL

How to solve the ERM efficiently in the high-dimensional regime for the chosen hypothesis space?

THE CURSE OF DIMENSIONALITY

- "Classic" functional spaces do not play well with this tradeoff:
  - $\mathcal{F} = \{f : \mathbb{R}^d \to \mathbb{R} \text{ is } \mathrm{Lipschitz}\}$ is too big: the number of samples required to identify $f^* \in \mathcal{F}$ up to error ϵ is $\Omega(\epsilon^{-d})$ [von Luxburg & Bousquet].



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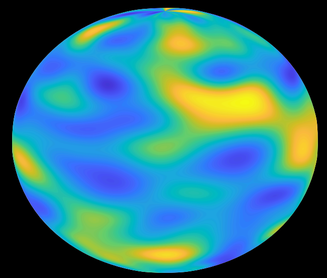
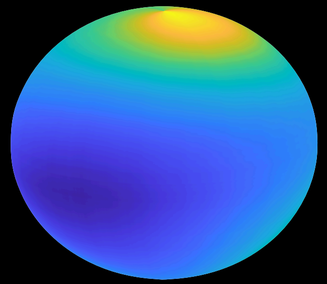
  - $\mathcal{F} = \mathcal{H}^{s,p}$: $\mathrm{Sobolev\ spaces}$. Minimax rate of approximation is cursed unless $s \geq d/2$ [Tsybakov]: only very smooth functions are allowed!





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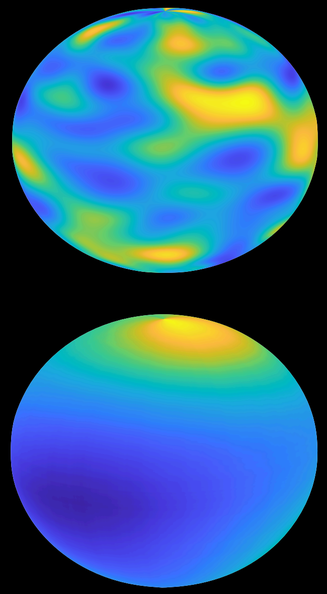

Which functions can be learnt efficiently in the high-dimensional regime?

...with ***neural networks*** and using ***gradient descent***?

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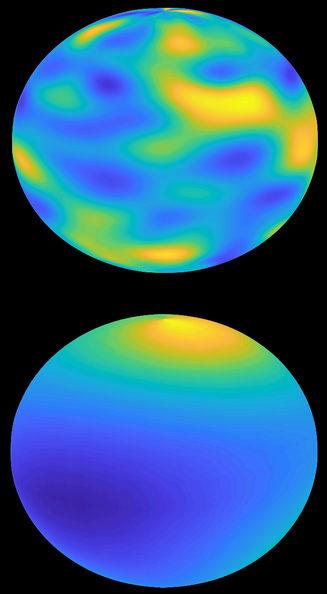

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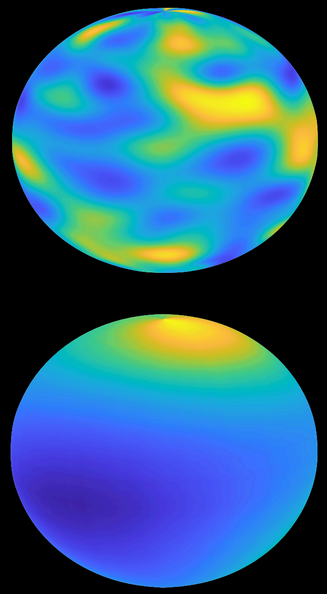

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...with ***deep*** neural networks?

...with ***deep structured*** neural networks?

THIS TALK

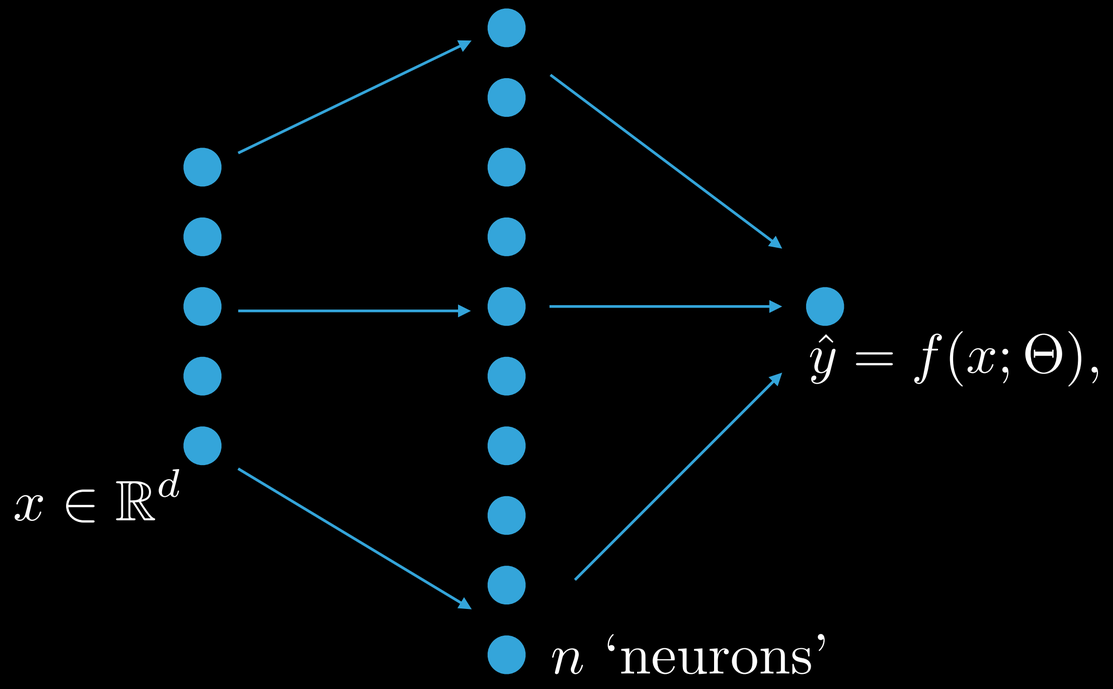
- Simplest instance of nonlinear feature learning: shallow NNs.
  - Gradient-descent Optimization analyzed as measure dynamics. Retains non-linear essence with Mean-field global convergence guarantees.
  - Towards Finite-width guarantees by CLT and fine-grained analysis of ReLU activations.

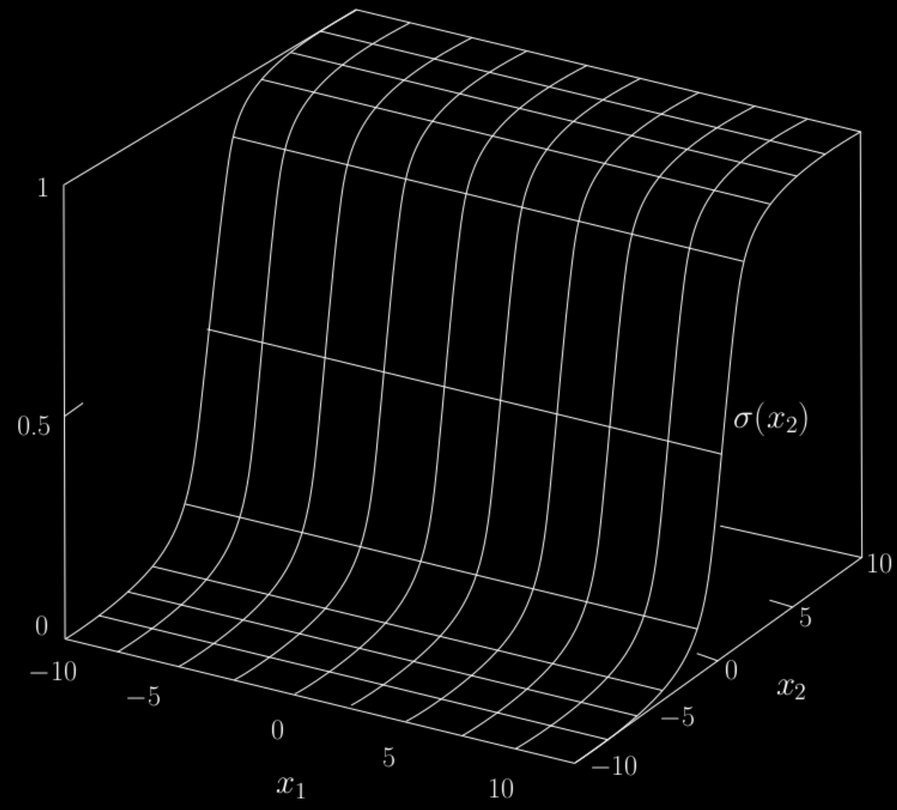
- Beyond Shallow Learning
  - Depth-Separation for ReLU networks
  - Depth-Separation and Learning for Symmetric Functions
  - [Mean-Field Dynamics on zero-sum two-player games].

SINGLE HIDDEN-LAYER NEURAL NETWORK

- $f(x;\Theta) = \sum_{j \le n} \tilde{\varphi}(x;\theta_j)$ is a sum of ridge functions:

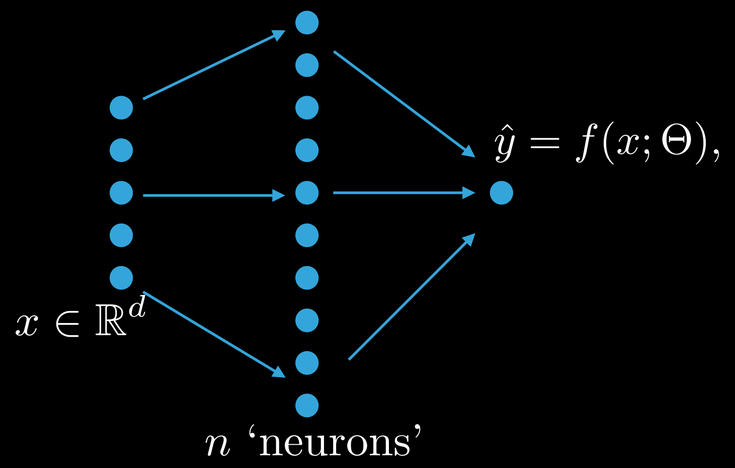
$$\tilde{\varphi}(x;\theta) = a\varphi(x;z),$$
$$\varphi(x;z) = \sigma(\langle x, w\rangle + b),$$
$$\theta = \{a, z\} \in \mathbb{R} \times \mathcal{D}.$$





- Three basic scaling quantities:
  - L datapoints, d input dimensions, n neurons.

SINGLE HIDDEN-LAYER NEURAL NETWORK



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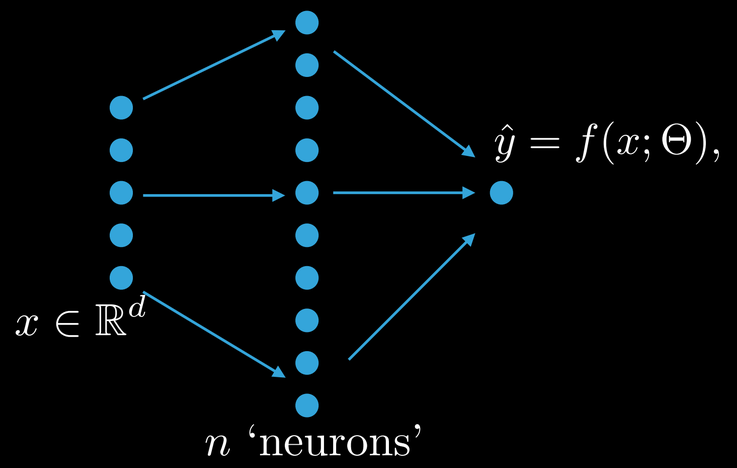

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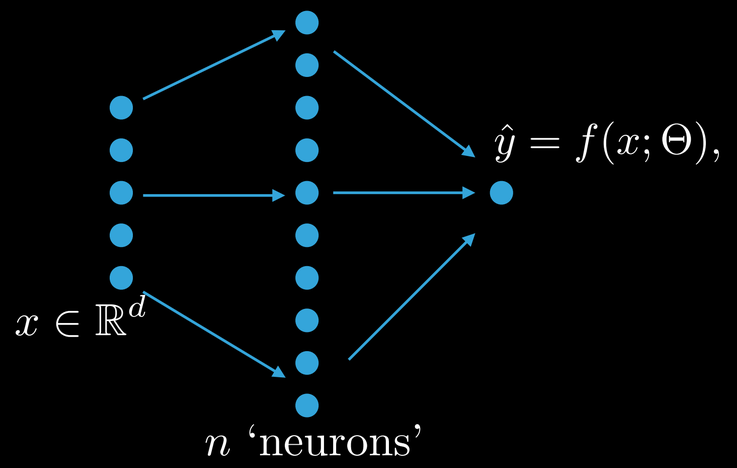

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- What are the associated functional spaces?

REPRODUCING KERNEL HILBERT SPACES

- Consider first γ_0 to be a fixed probability measure on $\mathcal{D}$.

$$\mathcal{F}_2 = \left\{ f : \mathbb{R}^d \to \mathbb{R} \,;\, f(x) = \int_{\mathcal{D}} \varphi(x,z) g(z) \mu_0(dz) \text{ and } g \in L^2(\mathcal{D}, d\mu_0) \right\}.$$

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- However, they are cursed by dimensionality: only contain very smooth functions (derivatives of order $O(d)$ must exist).
  - Kernels arising from linearizing NNs recently studied [***NTK***, Jacot et al, Arora et al., Mei et al. Tibshirani, Belkin, Bietti & Mairal, Bietti & Bach].

VARIATION-NORM SPACES

[Bengio et al'06, Rosset et al.'07, Bach'17]

- Alternatively, we can consider

$$\mathcal{F}_1 = \left\{ f : \mathbb{R}^d \to \mathbb{R} \,;\, f(x) = \int_{\mathcal{D}} \varphi(x,z)\mu(dz); \|\mu\|_{TV} < \infty. \right\}.$$

- $\mathcal{F}_1$ is a Banach space, with norm $\|f\|_{\mathcal{F}_1} := \inf \left\{ \|\mu\|_{TV}; f = \int \varphi d\mu \right\}.$
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- How to perform optimization and approximation in these spaces?

NEURAL NETWORKS AS PARTICLE INTERACTION SYSTEMS

- Regression on noise-less targets: $f^* \in L_2(\mathbb{R}^d, d\nu)$: target function.
- Single-hidden layer architecture and associated ERM:

Lifting

$$\Theta = (\theta_1 \dots \theta_n)\,, f(x;\Theta) = \frac{1}{n}\sum_{j\le n} a_j \varphi(x;z_j) = \frac{1}{n}\sum_{j\le n} \phi(x;\theta_j)\,, \theta_j = (a_j, z_j) \in \mathbb{R}\times\mathcal{D}.$$

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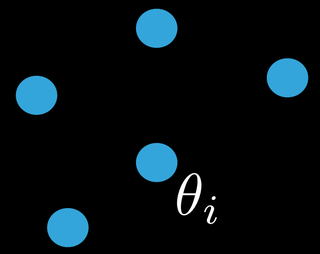

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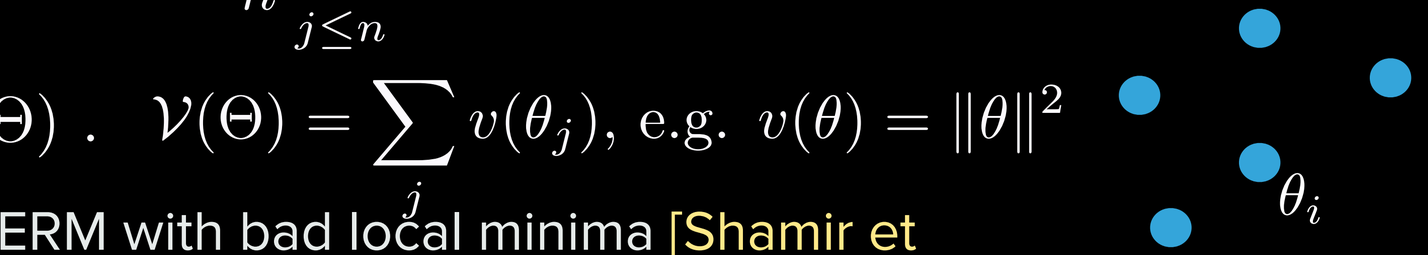

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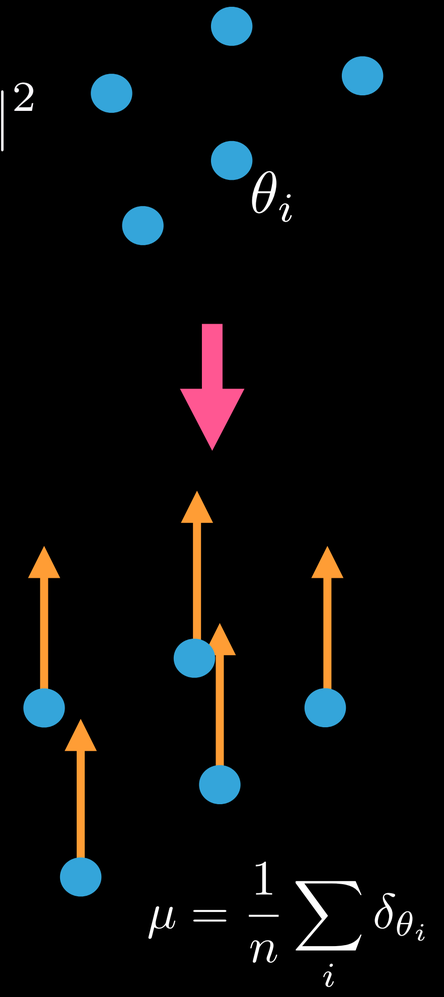

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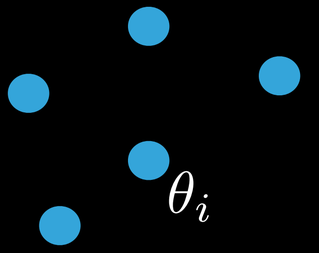

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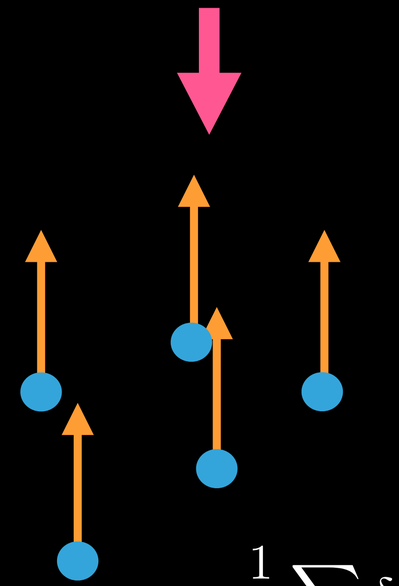

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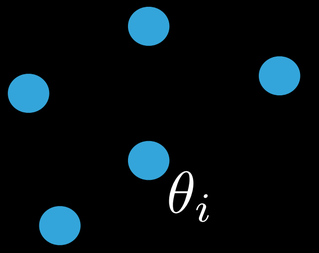

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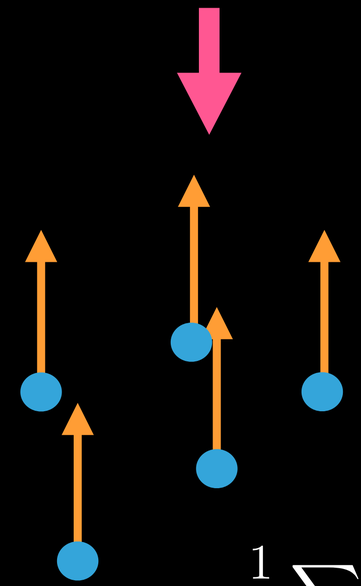
- Training dynamics expressed in the measure domain?



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CONTINUITY EQUATION

[Mei, Montanari, Nguyen, PNAS'18] [Sirignano, Spiliopoulos,'18]
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- Particle gradients correspond to evaluating a scaled velocity field:

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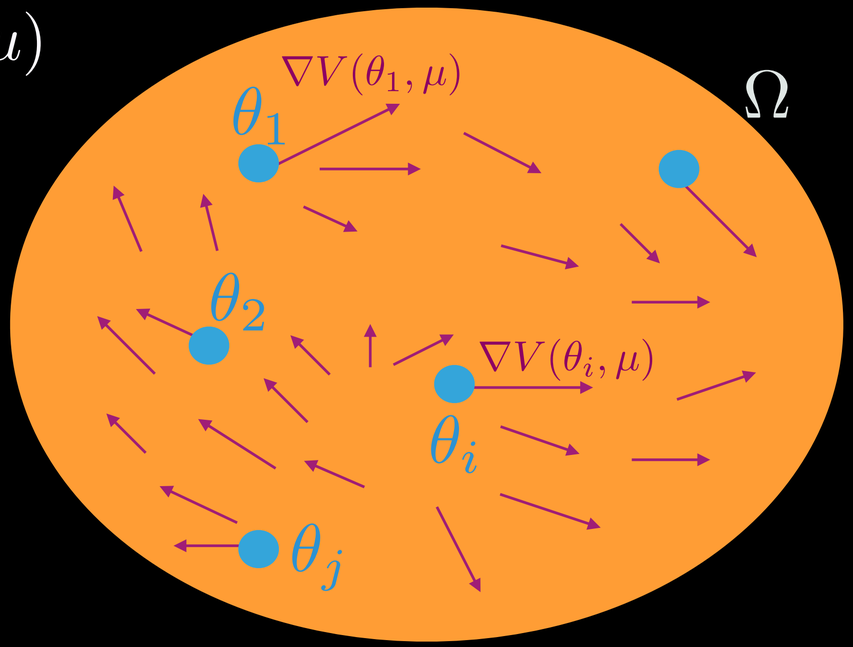

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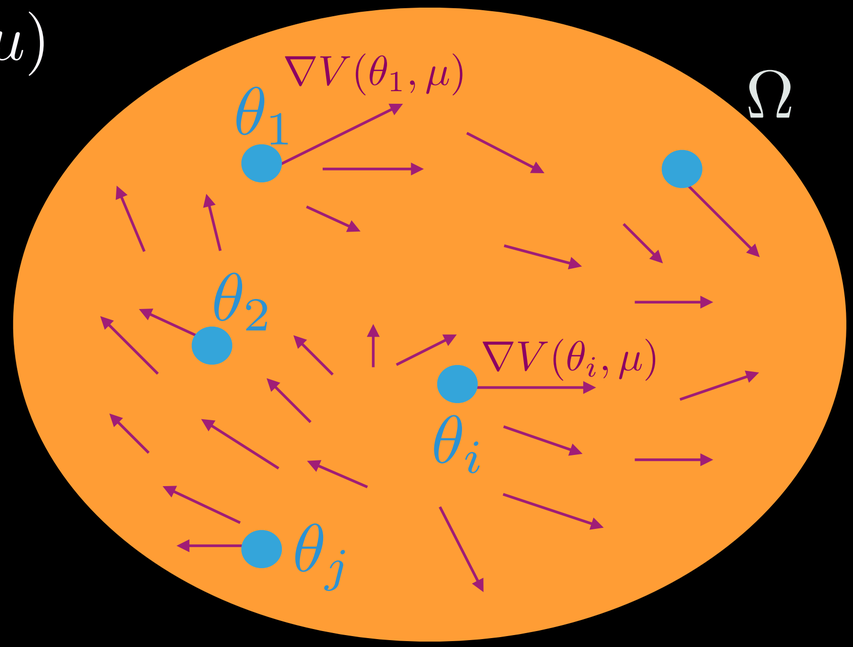

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  - Gradient flow of $\hat{\mathcal{R}}$ for the Wasserstein metric W_2 in $\mathcal{P}(\Omega)$
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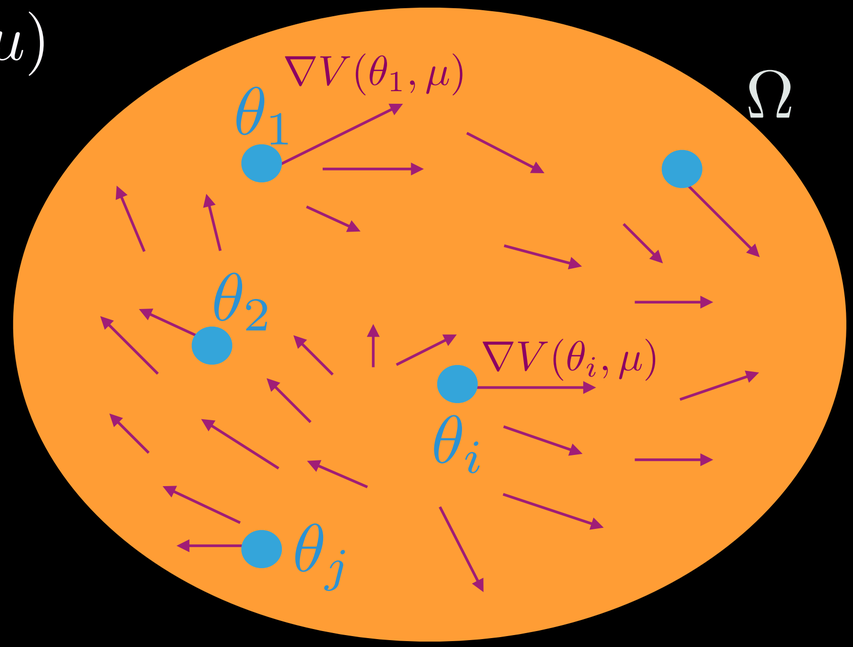

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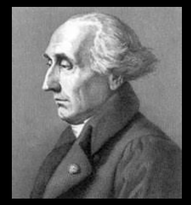
  - ***Exact description*** of particle gradient for atomic measures.



LAGRANGIAN
Non-Convexity
Euclidean Dynamics
Finite-dimensional

EULERIAN
Convexity
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MEAN-FIELD LIMIT

[Mei, Montanari, Nguyen, PNAS'18] [Sirignano, Spiliopoulos,'18]
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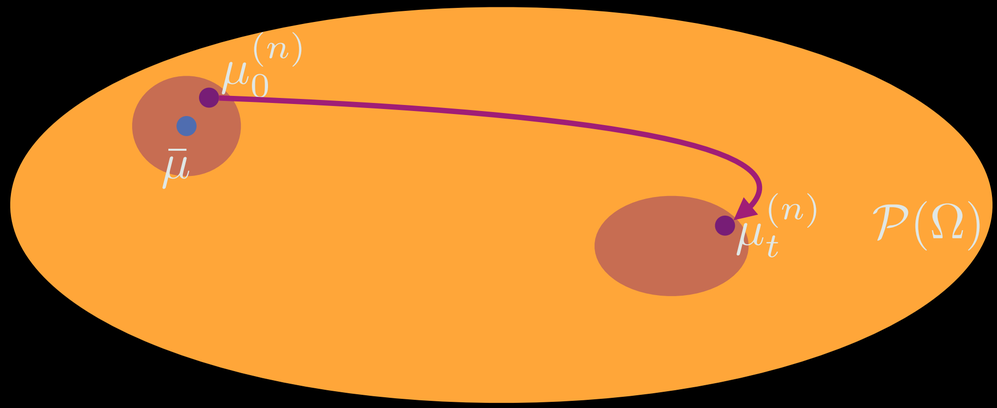
- Consider the evolution of the particle system as n grows.
- $\mu_t^{(n)}$: state of the system after time t, with $\theta_i(0) \sim \bar{\mu}$ iid.



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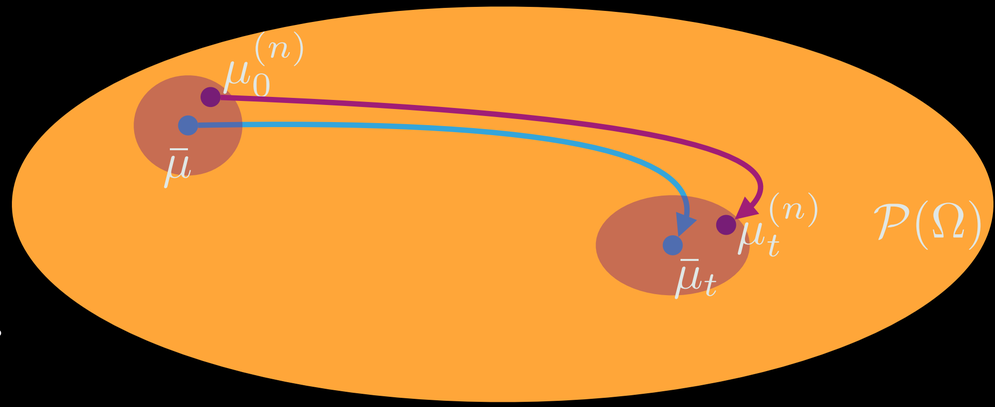

Theorem: [R,EVE'18], [CB'18], [MMN'18],[SS'18]
For any fixed $T > 0$, $\mu_T^{(n)}$ converges weakly to μ_T as $n \to \infty$, which solves $\partial_t \mu_t = \mathrm{div}(\nabla V \mu_t)$ with $\mu_0 = \bar{\mu}$.

- Dynamics and sampling commute in the limit (when it exists).

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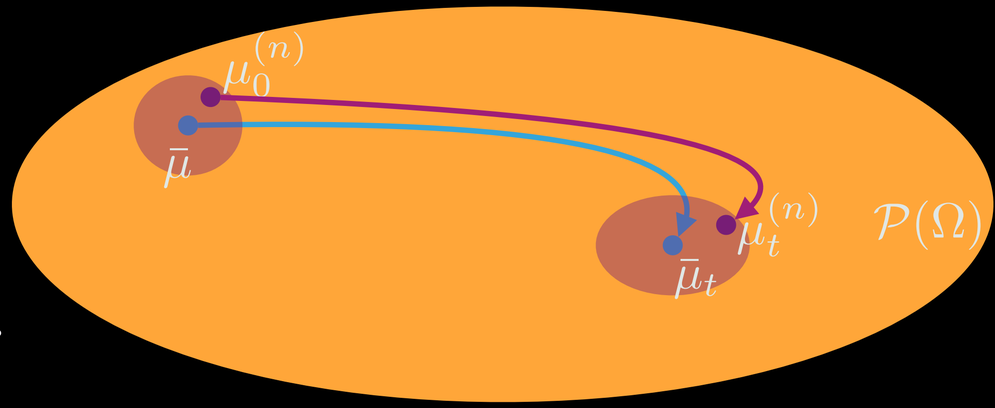


Theorem: [R,EVE'18], [CB'18], [MMN'18],[SS'18]
For any fixed $T > 0$, $\mu_T^{(n)}$ converges weakly to μ_T as $n \to \infty$, which solves $\partial_t \mu_t = \mathrm{div}(\nabla V \mu_t)$ with $\mu_0 = \bar{\mu}$.

- Dynamics and sampling commute in the limit (when it exists).

- Key convergence questions:

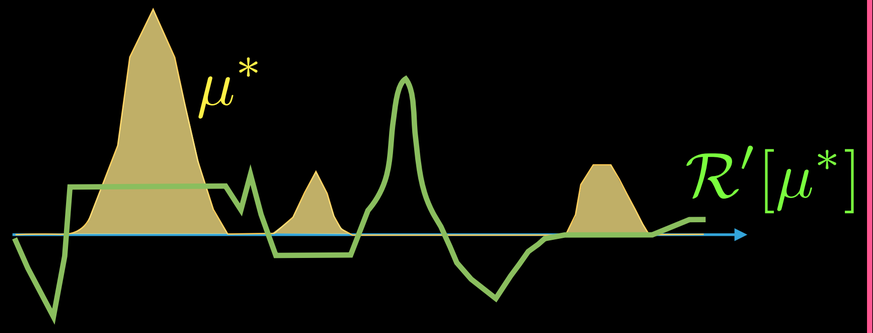
In t : Convergence properties of this PDE towards minimisers of $\mathcal{R}$?

In n: Beyond Mean-Field limit: What is the scale of the fluctuations?

GLOBAL CONVERGENCE IN THE MEAN-FIELD LIMIT

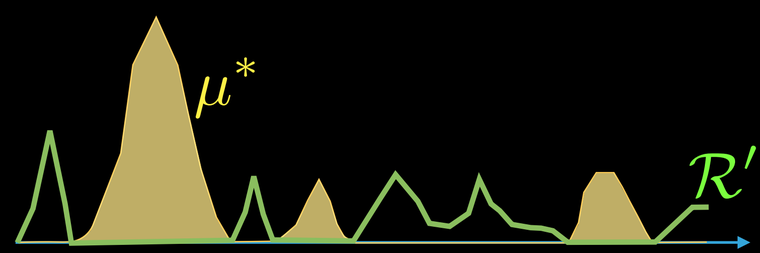
- Stationary points of Wasserstein gradient flow of $\mathcal{R}$ $\neq$global minimisers of $\mathcal{R}$ (lack of displacement convexity).

STATIONARY POINTS



$$\nabla \mathcal{R}'[\mu^*](\theta) = 0 \text{ for } \theta \in \text{supp}(\mu^*)$$

GLOBAL MINIMISERS



$$\mathcal{R}'[\mu^*](\theta) \geq 0 \text{ and}$$
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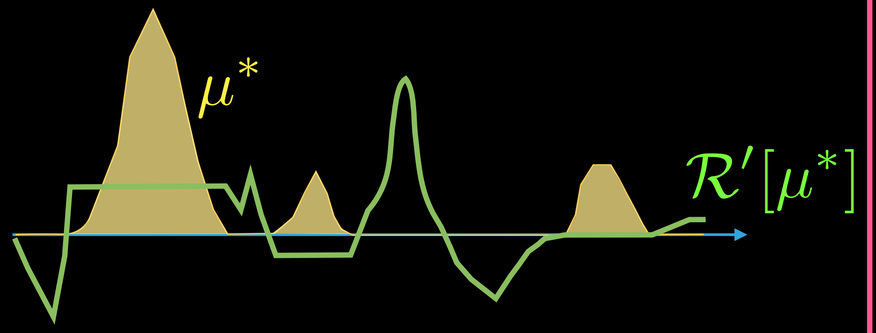

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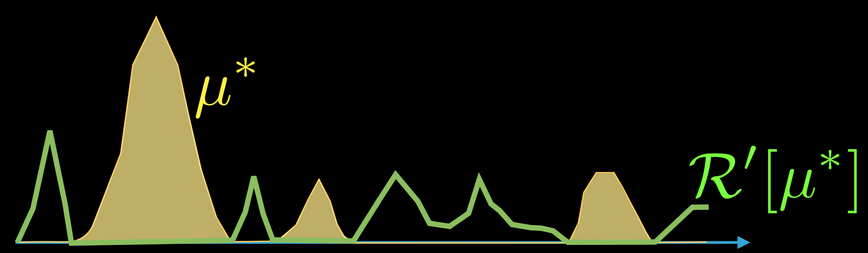

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- Existing results:
  - Additive noise in the dynamics [Mei et al.] enforces smoothness (and full support) through an elliptic Euler-Lagrange problem: $\partial_t \mu_t = \text{div}(\nabla V \mu_t) + \beta^{-1} \Delta \mu_t$
  - Full initial support of μ_0 and homogeneity of the neuron $\phi(\lambda\theta, x) = \lambda^p \phi(\theta, x)$ [Chizat & Bach]. Captures qualitative behavior of deterministic gradient flow.
  - General unbalanced transport through extra lifting stage [Jelassi, Rotskoff, B, EVE]

TOWARDS FINITE-WIDTH GUARANTEES

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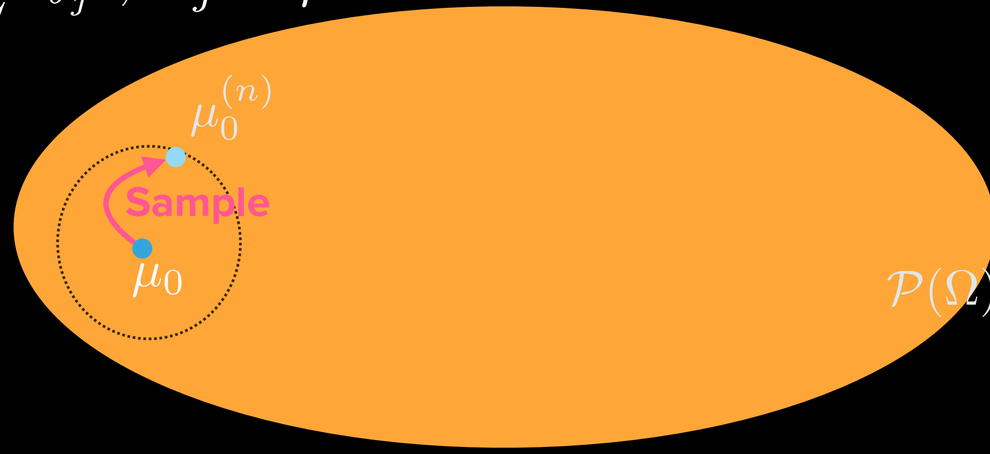
Recall $\mathcal{F} = \left\{ f(x) = \int_{\tilde{\mathcal{D}}} \phi(x,\theta)\mu(d\theta);\ \int_{\tilde{\mathcal{D}}} \|\theta\|^2 \mu(d\theta) < \infty \right\}$ (ϕ Lipschitz)

- Monte-Carlo estimate of $f(x) = \mathbb{E}_\mu[\phi(x,\theta)]$:

$$f^{(n)}(x) = \frac{1}{n}\sum_{j \le n} \phi(x,\theta_j) = \mathbb{E}_{\mu^{(n)}}[\phi(x,\theta)]\ ,\ \mu^{(n)} = \frac{1}{n}\sum_j \delta_{\theta_j}\ ,\ \theta_j \sim \mu$$



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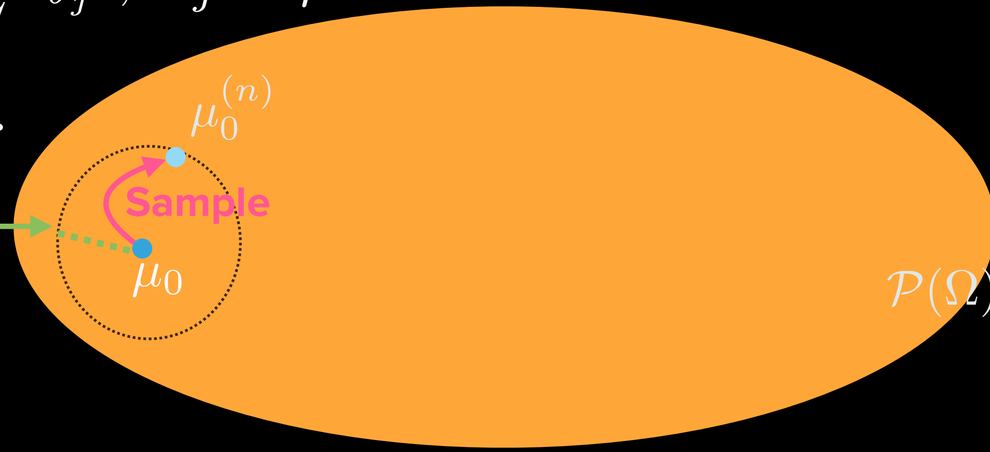
- Statistical Rate: $\mathcal{K}(\mu) := \mathbb{E}_\mu \|\phi(\cdot,\theta)\|_\nu^2 - \|\mathbb{E}_\mu \phi(\cdot,\theta)\|_\nu^2$.

$$\mathbb{E}_\theta \|f - f^{(n)}\|_\nu^2 = \frac{\mathcal{K}(\mu)}{n} \le \frac{C\|f\|_{\mathcal{F}}^2}{n}\ .$$



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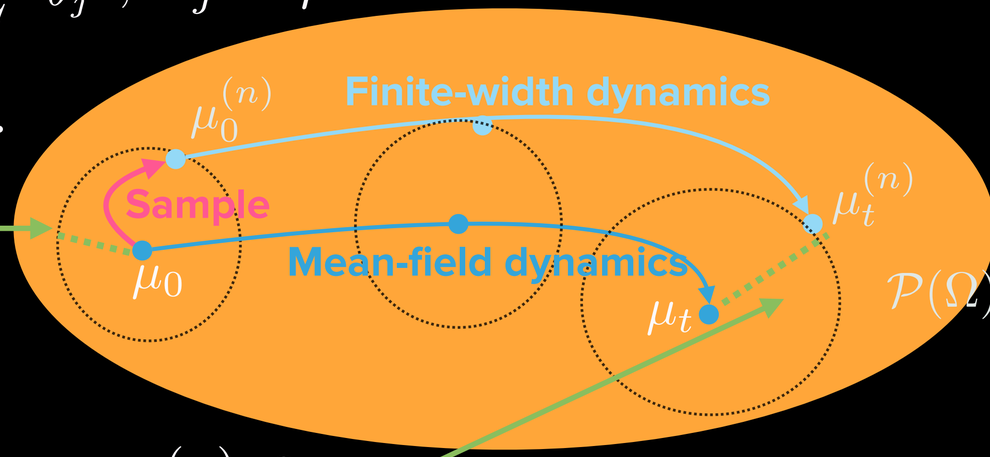

- How does this sampling error evolve with training?

$$f_t^{(n)} = \frac{1}{n}\sum_j \phi(x,\theta_j(t)) = \mathbb{E}_{\mu_t}[\phi(x,\theta)]$$

$$\mathbb{E}_\theta \|f_t - f_t^{(n)}\|_\nu^2$$

DYNAMIC CLT FOR SHALLOW NEURAL NETWORKS

[NeurIPS'20]

- A natural baseline is the MC error associated with μ_t :

$$\tilde{\mu}_t^{(n)} = \frac{1}{n}\sum_{j\le n}\delta_{\theta_j}\ ,\ \theta_j \sim \mu_t \quad \tilde{f}_t^{(n)}(x) = \int \phi(x,\theta)\tilde{\mu}_t^{(n)}(d\theta)$$

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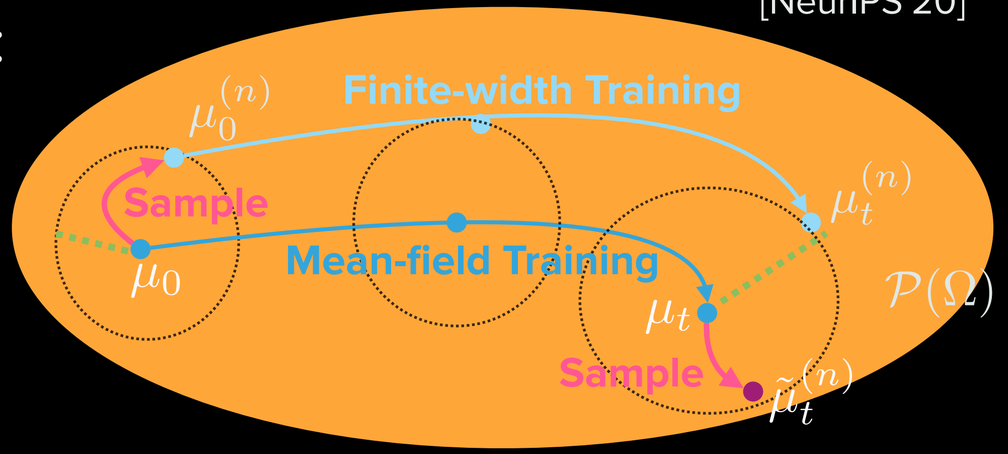

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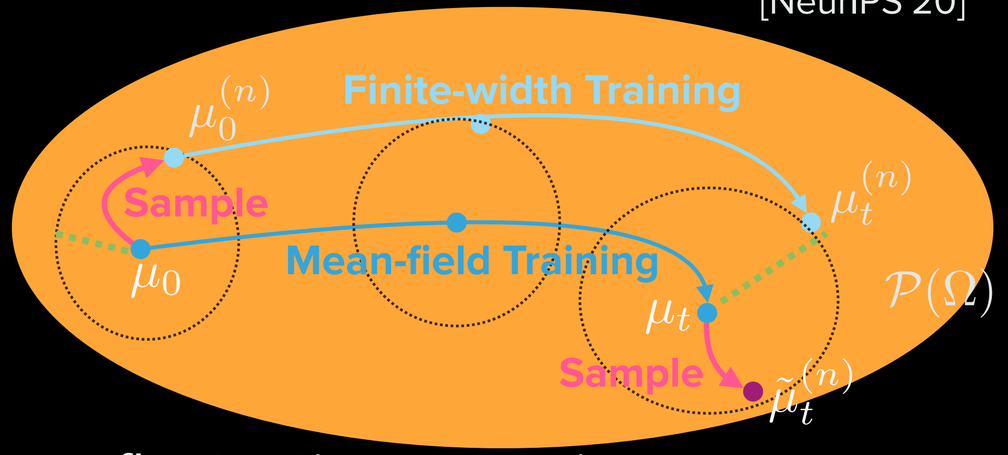


- Under appropriate mean-field convergence assumptions, fluctuations remain asymptotically bounded by this Monte-Carlo rate:

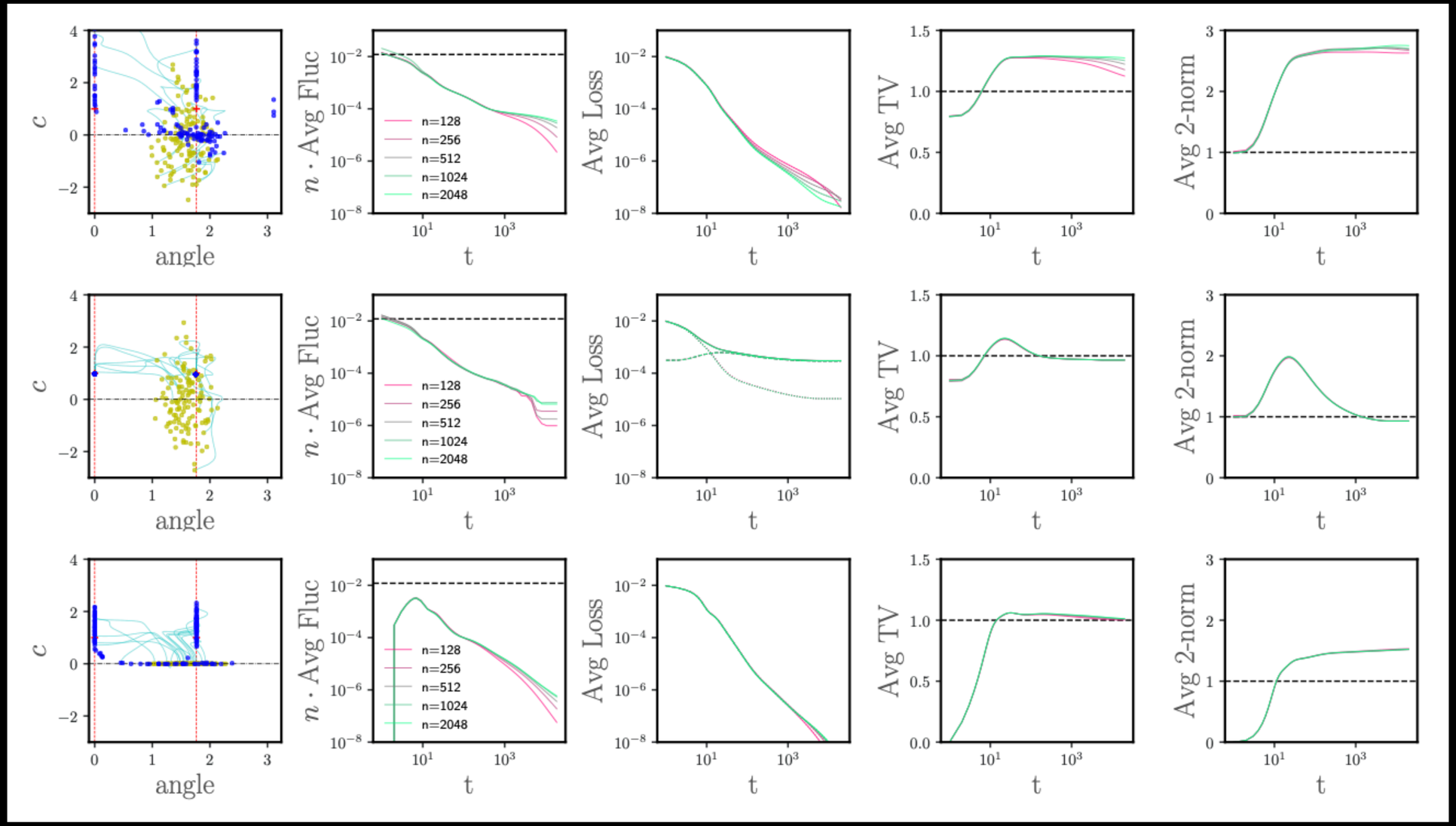
Theorem [CRBV'20]: Under appropriate smoothness and mean-field convergence assumptions, it holds

$$\lim_{t\to\infty}\lim_{n\to\infty} n\mathbb{E}_\theta \| f_t^{(n)} - f_t \|_\nu^2 \le \mathcal{K}(\mu_\infty)\ .$$

- Extends finite horizon CLT bounds from [Braun & Hepp,'70s] (also [Spilopoulos'19, De Bortoli et al.'20]). [Chizat'19] establishes zero fluctuations on sparse well-conditioned recovery problems.
- In the unregularised interpolating regime, fluctuations vanish at the MC scale.
- Proof idea: Baseline fluctuations dominate dynamic fluctuations through a Volterra Kernel.

NUMERICAL EXPERIMENTS: TEACHER-STUDENT SETUP

[NeurIPS'20]



- We verify scale of fluctuations at or below MC.

CURRENT: TOWARDS QUANTITATIVE GUARANTEES

Jon Niles-Weed

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- Super-Polynomial Statistical-Query lower bounds for learning certain planted shallow models [Goel et al. ‘20, Diakonikolas et al.’20, Vempala’18, Kearns’94]
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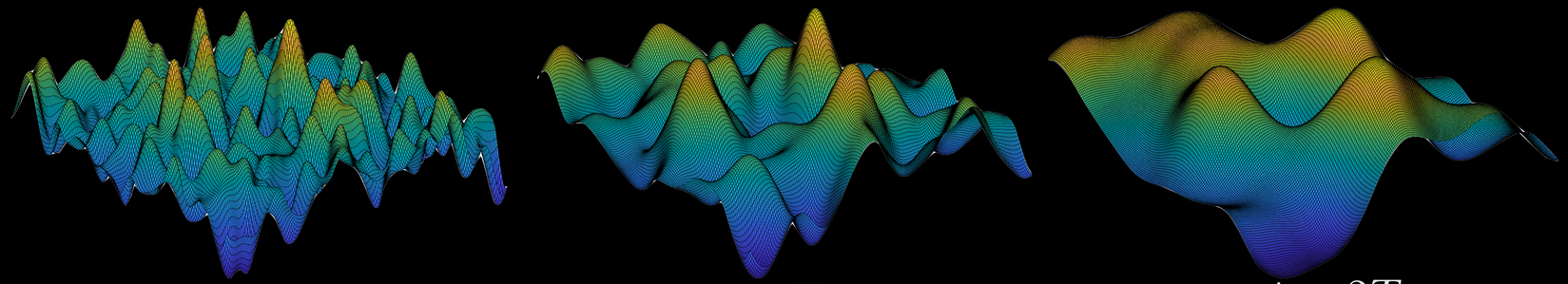
- Generic planted model studied by lifting optimization landscape $\mathcal{R}(\Theta)$ using a diffusion semigroup (e.g heat kernel): $\widetilde{\mathcal{R}}(\Theta, t)$ solves $\partial_t \widetilde{\mathcal{R}}(\Theta, t) = \Delta \widetilde{\mathcal{R}}(\Theta, t)$, $\widetilde{\mathcal{R}}(\Theta, 0) = \mathcal{R}(\Theta)$.



$t = 0$ $t = T$ $t = 2T$

- Preliminary results: provably learning random planted shallow models with $K = O(d^\alpha), \alpha < 1$ using $O(d)$ Frank-Wolfe smoothed gradient descent [Jaggi, Bach, Zhong et al., Ma, Lee & Ge]

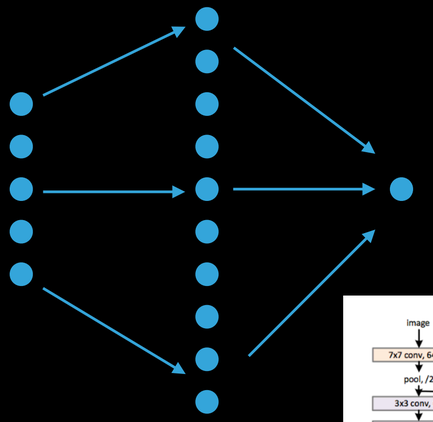
BEYOND VARIATION-NORM SPACES: DEPTH SEPARATION

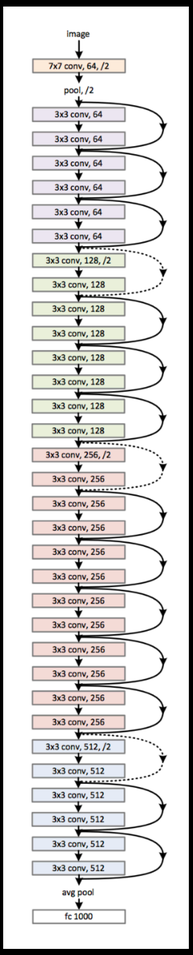
[submitted]

- Functions in $\mathcal{F}_1$ are expressed as sparse sums of ridge functions.
- We just saw they admit statistical rates of approximation using Monte-Carlo estimates: if $f \in \mathcal{F}$, there exists n-term approximation f_n such that

$$\|f - f_n\|_{L^2(\Omega)} \leq n^{-1/2} \|f\|_{\mathcal{F}_1} \sup_{z \in D} \|\varphi(., z)\|$$

  - Rates can be slightly improved in certain regimes [Bach'17]
- Explicit control using Fourier moments, e.g. $\|f\|_{\mathcal{F}_1} \lesssim \int \|\omega\|_1^2 |\hat{f}(\omega)| d\omega$ for the ReLU and compact domain Ω [Barron'93, Klusowski, Barron,'18, Ongie et al.'19]





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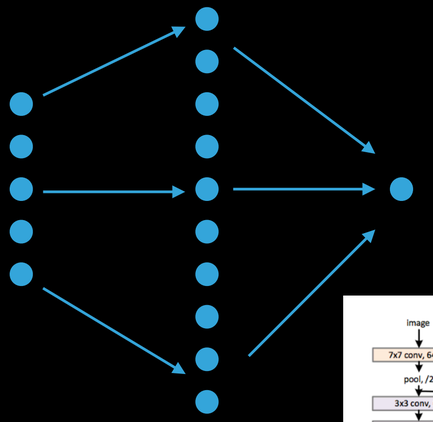
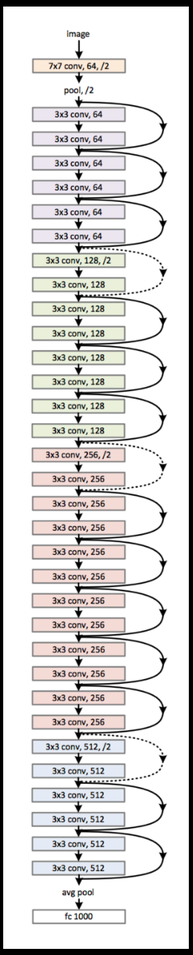
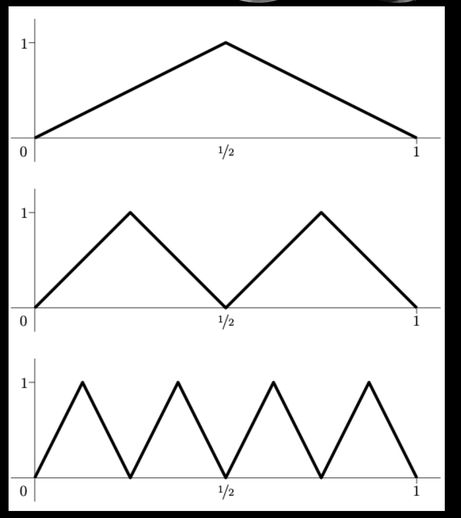




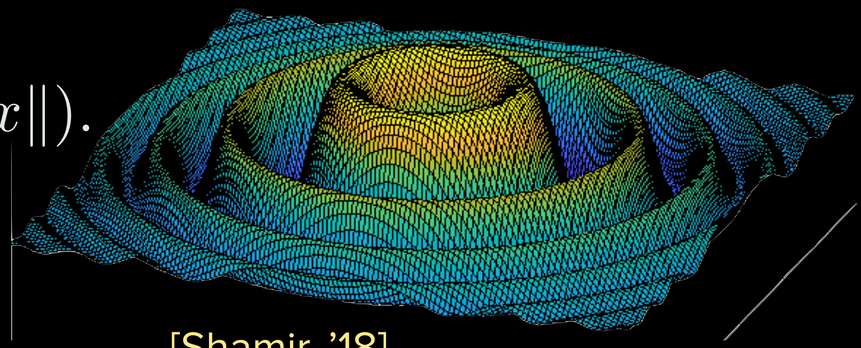
- Which function classes are not well approximated in $\mathcal{F}_1$, but are approximable/learnable by deeper architectures efficiently?
- How necessary are these (strong) Fourier decay conditions?
- Depth Separation in presence of input structure, eg images?

DEPTH SEPARATION PRIOR WORK

- Rich literature in boolean [Rossman, Hastad'68] or threshold [Hajnal'93] circuit lower bounds.
- [Martens et al'13] shows lower bounds for RBMs.
- [Telgarsky'15] Exploits combinatorial limitations of shallow networks
  - Refined periodicity analysis in [Chatziafratis et al'20].
- [Montufar et al.] bound number of linear regions of deep ReLU nets.
- [Eldan, Shamir, Safran, Daniely] construct oscillatory functions with depth-separation. Provably require $\exp(d)$ width for shallow model, but $\mathrm{poly}(d)$ for deeper neural network.
  - Constructions are inherently low-dimensional, e.g. $f(x) = g(\|x\|)$.
- Towards more general function separations?



[Telgarsky, '15]



[Shamir, '18]

LOWER BOUNDS FOR PIECE-WISE OSCILLATORY FUNCTIONS


- Towards more expressive example functions, we consider functions with piece-wise structure.

- Approximation lower bound for shallow neural networks with efficient approximation with depth-3 networks:

$$D_\mu(f,g) = \mathbb{E}_\mu |f(x) - g(x)|^2$$

Theorem [BJV'20]: Let $f^*(x) = \exp\{i\langle \omega_d, \rho(Ux+b)\rangle\}$ with $U \in \mathbb{R}^{d\times d}$, $\|\omega_d\| = \Omega(d^3)$ and $\rho(t) = \max(0,t)$. Let μ be a heavy-tailed distribution. Then

(i) f^* is not $\Omega(1)$-approximable by any shallow $\mathsf{exp}(o(d))$-wide network.

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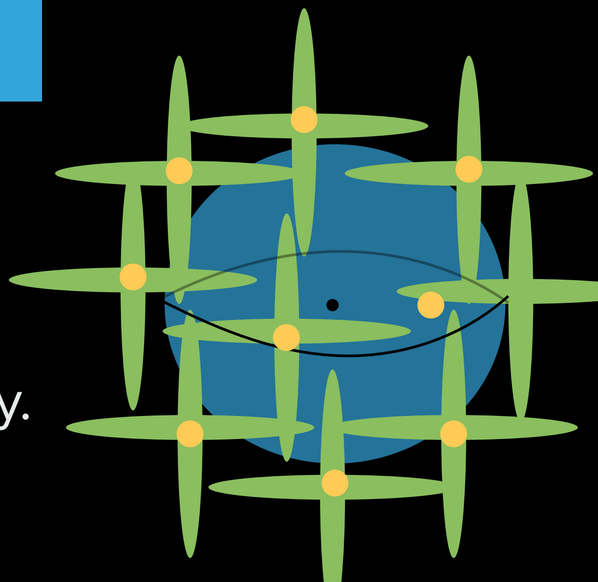
- Proof builds from [Eldan & Shamir'16], based on Fourier analysis, with tighter control on the data distribution.

- Heavy-tailed and increasing oscillation assumptions are (jointly) necessary.



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$$n \geq \left(\Theta(\exp L)(1+\epsilon^{-2})\mathsf{poly}(d)\right)^{\Omega(\epsilon^{-L})} .$$

- Extends previous results in [Safran, Eldan, Shamir'19] for radial functions.
- Rate is polynomial in d , but exponential in ϵ^{-1}.
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- Take-home: approximation gaps with depth, but no optimization guarantees (yet!)
- For ***unstructured*** inputs, depth brings unclear approximation advantages.

  Not much beyond Barron's Fourier Sparsity condition.

  Current techniques hard to extend to deeper models [Vardi & Shamir'20]

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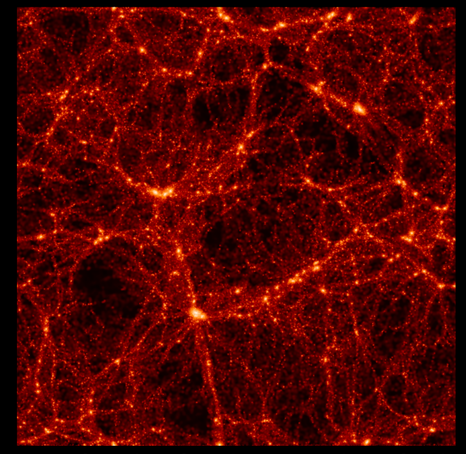
- E.g particle interaction systems, 3d point-clouds.

(Source: S. Grunewald, credit Qi et al.)



Cosmological n-body simulations
Joerg Colberg, Virgo Simulation

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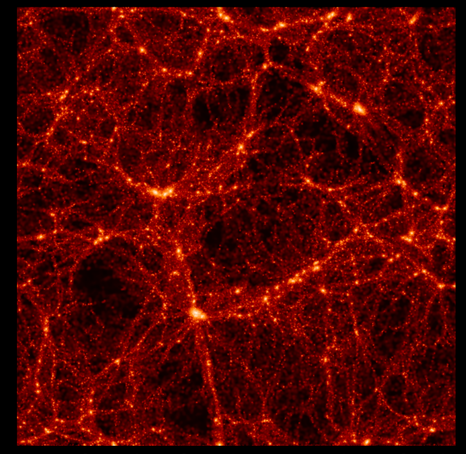
  - E.g particle interaction systems, 3d point-clouds.

(Source: S. Grunewald, credit Qi et al.)

- Input Embedding into $\mathcal{P}(\Omega)$: $(x_1, \ldots, x_k) \to \mu^{(k)} = \frac{1}{k} \sum_{j=1}^{k} \delta_{x_j}$.

[De Vie, Peyre, Cuturi]

  - Under appropriate regularity, f extended to $\overline{f} : \mathcal{P}(\Omega) \to \mathbb{R}$.
  - Input domain is not-Euclidean, infinite-dimensional.



Cosmological n-body simulations
Joerg Colberg, Virgo Simulation

- Functional neural spaces?

LEARNING UNDER SYMMETRY

- A “neuron” is now a ridge function $\varphi(\cdot, \theta) : \mathcal{P}(\Omega) \to \mathbb{R}$

$$\varphi(\mu, \theta) = a\sigma(\langle \mu, \phi \rangle),\ a \in \mathbb{R}, \boxed{\phi : \Omega \to \mathbb{R},} \langle \mu, \phi \rangle = \int_\Omega \phi(u)\mu(du).$$

  - Input “weights” ϕ are now test functions.

- Shallow invariant neural network:

$$f(\mu, \Theta) = \frac{1}{n} \sum_{i=1}^{n} a_i \varphi(\mu, \phi_i).$$

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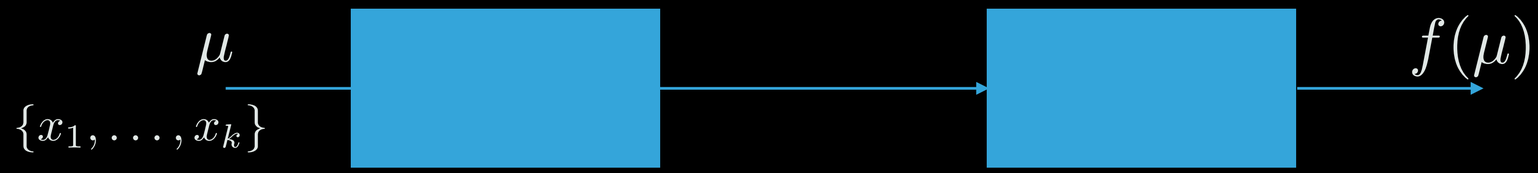
- Integral representation:

$$f(\mu, \chi) = \int_{\mathcal{D}} \varphi(\mu, \phi)\chi(d\phi)$$

$\mathcal{D} =$ domain of test functions in Ω,
$\chi \in \mathcal{M}(\mathcal{D})$ Radon Measure over $\mathcal{D}$.

- Different over-parametrised regimes as in fully connected case?



LEARNING UNDER SYMMETRY

- Hierarchy of functional spaces for learning:

$$\mathcal{S}_1 = \left\{ \mathcal{D} = \{\phi; \|\phi\|_{\mathcal{F}_1} \leq 1\}, f = \int_{\mathcal{D}} \varphi d\chi; \|\chi\|_{\mathrm{TV}} < \infty \right\}$$

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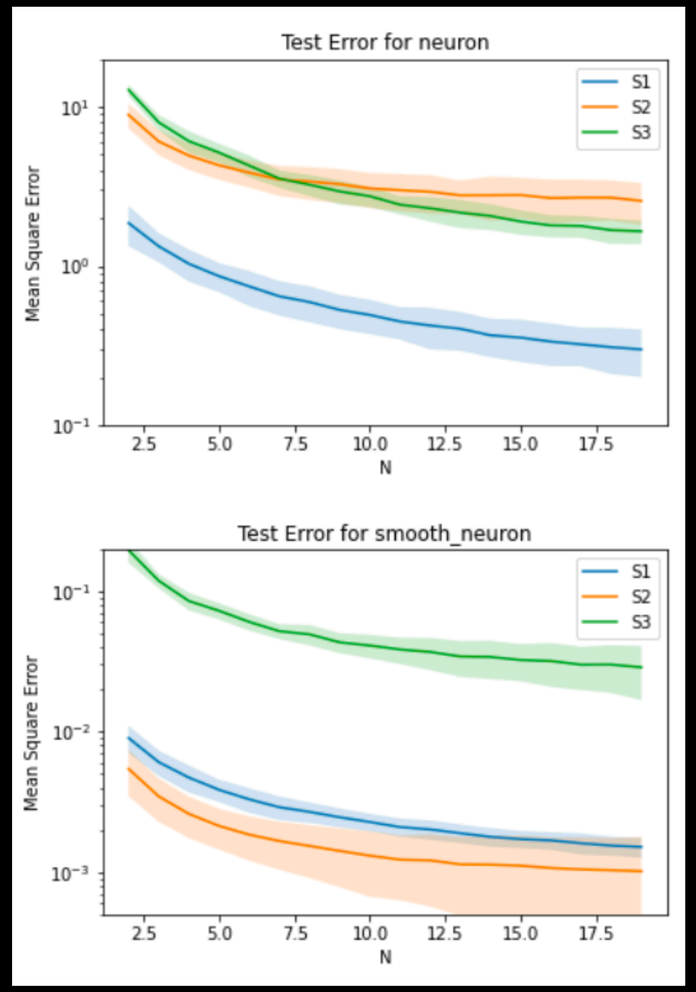
- Approximation lower bounds and generalization guarantees:

Theorem [BZ'20]: For ReLU activations, there exists f_1, f_2 with $\|f_i\|_{\mathcal{S}_i} \le 1$ such that (depth-separation)

$$\inf_{\|f\|_{\mathcal{S}_2} \le \delta} \mathrm{poly}(d) |f_1 - f|_\infty \gtrsim \left| 1 - \delta d^{-d/3} \right| , \text{ and } \inf_{\|f\|_{\mathcal{S}_3} \le \delta} \mathrm{poly}(d) \|f_2 - f\|_\infty \gtrsim \delta^{-5/d} .$$

Moreover, assuming bounded feature domain Ω, we have

$$\mathbb{E} \sup_{\|f\|_{\mathcal{S}_1} \le \delta} \left| \mathbb{E}_{\mu \sim \mathcal{D}} \ell(f^*(\mu), f(\mu)) - \frac{1}{L} \sum_{i=1}^{L} \ell(f^*(\mu_i), f(\mu_i)) \right| \lesssim \frac{\delta(1+\delta)}{\sqrt{L}} . \quad \text{(generalization bounds)}$$



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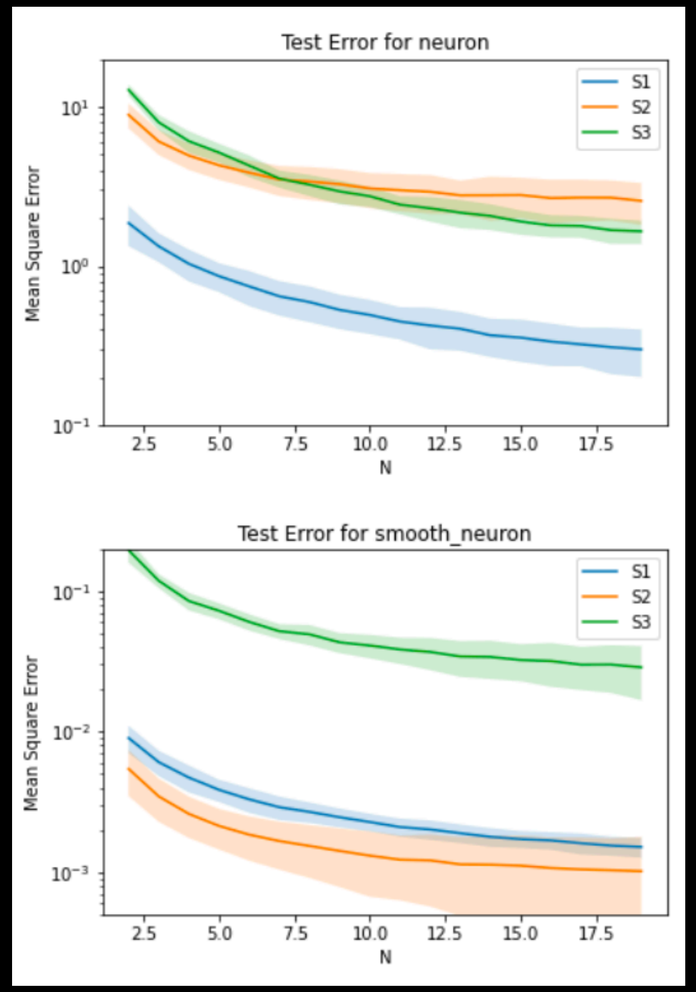

- ***Current***: extension to Fermion antisymmetry

CONCLUSIONS: APPROXIMATION VS OPTIMIZATION IN DEEP MODELS

- Two necessary ingredients for learning: good approximation and efficient optimization algorithms.
- Simplified setup: $\mathcal{N}_k = \{\text{Net of depth } k \text{ and width } \text{poly}(d)\}$
  - Given target f^*, consider $\delta_k = \inf_{f \in \mathcal{N}_k} D(f, f^*)$.

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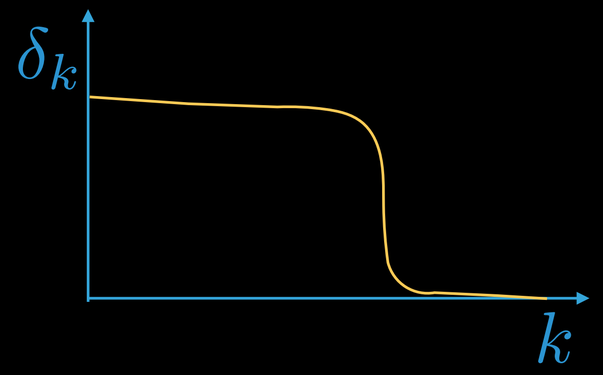
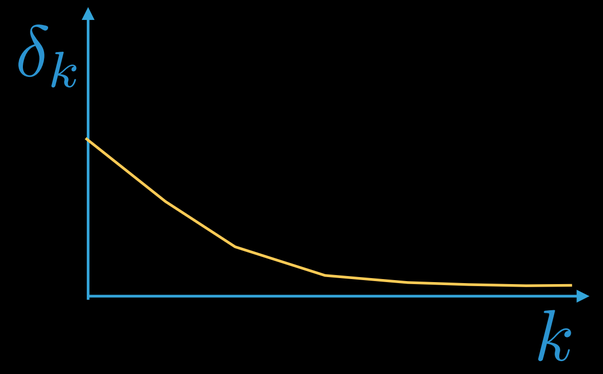
  - Given target f^*, consider $\delta_k = \inf_{f \in \mathcal{N}_k} D(f, f^*)$.
- ***Mild Depth Separation:*** necessary and sufficient for deep optimization?
  - Shown to be sufficient in the context of "lacunary" polynomials [Allen-Zhu & Li].
  - Shown to be necessary in the context of one-dimensional fractal distributions [Malach & Shalev-Schwartz].
- How to measure depth correlation in practice?
- ***Physical structure is key to uncover the practical benefits of depth***.
- In presence of physically-structured data (eg grids), how does depth correlation relate to Weak scale interactions (e.g. fast multipole, scattering representations)?



Strong depth separation: Shallow models provide no efficient approximation



Mild depth separation: Shallow models provide some approximation

THANKS!

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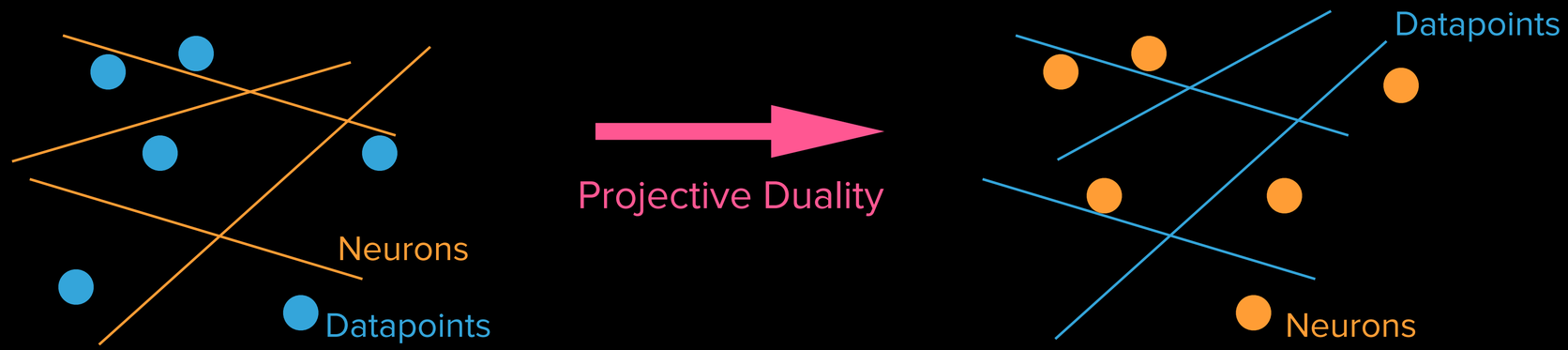
TOWARDS FINITE-WIDTH GUARANTEES

- The previous CLT results are still qualitative (limit of infinitely wide networks).
- For shallow ReLU networks, we can strengthen to finite-width guarantees by leveraging fine-grained ReLU structure.

> **Theorem** [DB'20]: The $\mathcal{F}_1$ regularised ERM using ReLU units only admits atomic minimisers, and the functional $\mathcal{E}[\mu]$ is locally strongly convex.

- Leveraging results from [Chizat'19] we can provide guarantees for finite width (albeit still exponential in dimension).
- ERM is reduced to a finite-dimensional linear program.



BEYOND SUPERVISED LEARNING: COMPETITIVE OPTIMIZATION

- Wasserstein-Fisher-Rao dynamics can also be used to study equilibria in games.
- Canonical setup: finding mixed strategies in two player zero-sum game:

$$\mathcal{L}[\mu_x, \mu_y] = \int_{\mathcal{X}\times\mathcal{Y}} \ell(x,y)\mu_x(dx)\mu_y(dy)\,.$$

μ_x, μ_y: players strategy distribution

$\mathcal{X}, \mathcal{Y}$: compact spaces

$\ell(x,y)$ smooth

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- (mixed) Nash Equilibria: (μ_x^*, μ_y^*) such that

$$\forall \mu_x \;,\; \mathcal{L}[\mu_x^*, \mu_y^*] \leq \mathcal{L}[\mu_x, \mu_y^*] \;,\quad \forall \mu_y \;,\; \mathcal{L}[\mu_x^*, \mu_y^*] \geq \mathcal{L}[\mu_x, \mu_y]\;.$$

  - Guaranteed to exist [Nash'50s]
  - Algorithms to find them in the high-dimensional setting?

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- Gradient dynamics:

$$\partial_t \mu_{x,t} = \mathrm{div}(\nabla \frac{\partial \mathcal{L}}{\partial \mu_x}) \qquad \partial_t \mu_{y,t} = -\mathrm{div}(\nabla \frac{\partial \mathcal{L}}{\partial \mu_y})$$

BEYOND SUPERVISED LEARNING: COMPETITIVE OPTIMIZATION

- Measure dynamics associated with particle gradient ascent/ descent:

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- We establish Global convergence to approximate Nash equilibria using WFR.

- Similar propagation-of-chaos and robustness in high-dimensions.