

Practice Problems

Chapter-6: Least-Squares Approximations

1. Let's say the following matrix below is an orthonormal matrix:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

Compute the value of $(A^T A)^{-1}$

2. For $f(2) = 3, f(4) = 5, f(5) = 8$, prepare an overdetermined system and then solve the system using,
- Least Square Method
 - QR Decomposition
3. For $f(18) = 22, f(19) = 35, f(20) = 60, f(25) = 80$, prepare an overdetermined system and then solve the system using,
- Least Square Method
 - QR Decomposition
4. Consider a set of four data points: $f(0) = 3, f(2) = -2, f(-1) = 2, f(1) = 1$. Find the best-fit polynomial of degree two, $p_2(x)$, for the above data points using the least-squares method by answering the following:
- Write down the matrices, A and b, for the above data.
 - Compute the normal matrix $A^T A$ and $A^T b$.
 - Use the results of the previous part to compute the column matrix $x = (a_0 \ a_1 \ a_2)^T$, where a_0, a_1, a_2 are the coefficients of the polynomial p_2 , and then write the expression of the polynomial p_2 .
5. Consider a set of four data points: $f(0) = 1, f(2) = -2, f(-1) = 2, f(1) = 1$. Find the best-fit polynomial of degree two, $p_2(x)$, for the above data points using the least-squares method by answering the following:
- Write down the matrices, A and b, for the above data.
 - Compute the normal matrix $A^T A$ and $A^T b$.

- c. Use the results of the previous part to compute the column matrix, x and write the expression of the polynomial p_2 .

6. Consider a set of four data points: $f(0) = 0, f(2) = -1, f(-1) = 2, f(1) = 2$.

We now find the solution by the QR-decomposition method using these four data points by answering the following:

- a. Write down the matrices A and b. Also identify the linearly independent column vectors u_1, u_2, u_3 from the matrix A.
- b. Using the Gram-Schmidt process, construct the orthonormal column matrices (or vectors) q_1, q_2, q_3 from the linearly independent column vectors u_1, u_2, u_3 obtained in the previous part, and then write down the Q matrix.
- c. Calculate the matrix elements of R and write down the matrix R.
- d. Compute Rx and $Q^T b$, where $x = (a_0 \ a_1 \ a_2)^T$ is a column vector with a_0, a_1, a_2 which are the coefficients of the polynomial p_2 .
- e. Using the above result, find the values of a_0, a_1, a_2 and write the expression of the polynomial $p_2(x)$.

7. Consider a set of four data points: $f(0) = 1, f(0.5) = 1.4, f(1) = 1.7, f(1.5) = 2$.

We now find the solution by the QR-decomposition method using these four data points by answering the following:

- a. Write down the matrices A and b. Also identify the linearly independent column vectors u_1, u_2, u_3 from the matrix A.
- b. Using the Gram-Schmidt process, construct the orthonormal column matrices (or vectors) q_1, q_2, q_3 from the linearly independent column vectors u_1, u_2, u_3 obtained in the previous part, and then write down the Q matrix.
- c. Calculate the matrix elements of R and write down the matrix R.
- d. Compute Rx and $Q^T b$, where $x = (a_0 \ a_1 \ a_2)^T$ is a column vector with a_0, a_1, a_2 which are the coefficients of the polynomial p_2 .
- e. Using the above result, find the values of a_0, a_1, a_2 and write the expression of the polynomial $p_2(x)$.