Bilinear Isoparametric Mapping

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1 1D Mapping

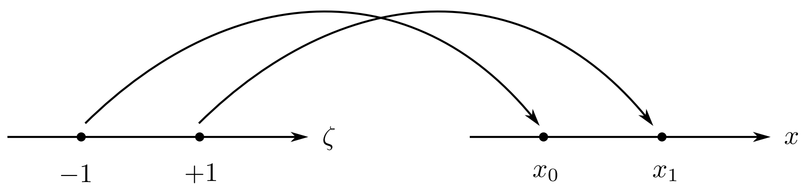


Figure 1: Illustration of one dimension transformation (mapping) between parent and global domains, $\zeta(x)$ and $x(\zeta)$, respectively.

In this document, we briefly discuss the derivation of isoparametric transformation for several useful shapes. First, we start with the 1D transformation shown in Figure 1. Here, we seek a relation $x(\zeta)$ assuming linear transformation. For this case, it is appropriate to postulate

$$x(\zeta) = C_0 + C_1\zeta \tag{1}$$

The constants can be found by substituting the corresponding values of ζ and x for the known points. This results in the relation

$$\begin{bmatrix} C_0 \\ C_1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} +1 & +1 \\ -1 & +1 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \end{bmatrix} \tag{2}$$

Thus, one can write the transformation as

$$x(\zeta) = \frac{1}{2}\big[(1-\zeta)x_0 + (1+\zeta)x_1\big] \tag{3}$$

1.1 Application to Numerical Integral

Considering the canonical integral

$$\int_{-1}^{1} f(\zeta)d\zeta \approx \sum_{n=1}^{N} w_n f(\zeta_n) \tag{4}$$

where ζ_n and w_n are the nodes and weights for N order quadrature rule, respectively. Now consider a general integral on the global domain described by the region between points x_0 and x_1

$$I = \int_{x_0}^{x_1} f(x)dx = \int_{-1}^{1} f(x(\zeta))|\boldsymbol{J}(\zeta)|d\zeta \tag{5}$$

The Jacobian is easily found as

$$|\boldsymbol{J}(\zeta)| = \left|\frac{\partial(x)}{\partial(\zeta)}\right| = \frac{\partial x}{\partial \zeta} = \frac{1}{2}(x_1 - x_0) \tag{6}$$

Using the previous results, the integral can evaluated as

$$I \approx h^- \sum_{n=1}^{N} w_n f(h^- \zeta_n + h^+) \tag{7}$$

where $h^\pm = (x_1 \pm x_0)/2$.

2 2D Mapping

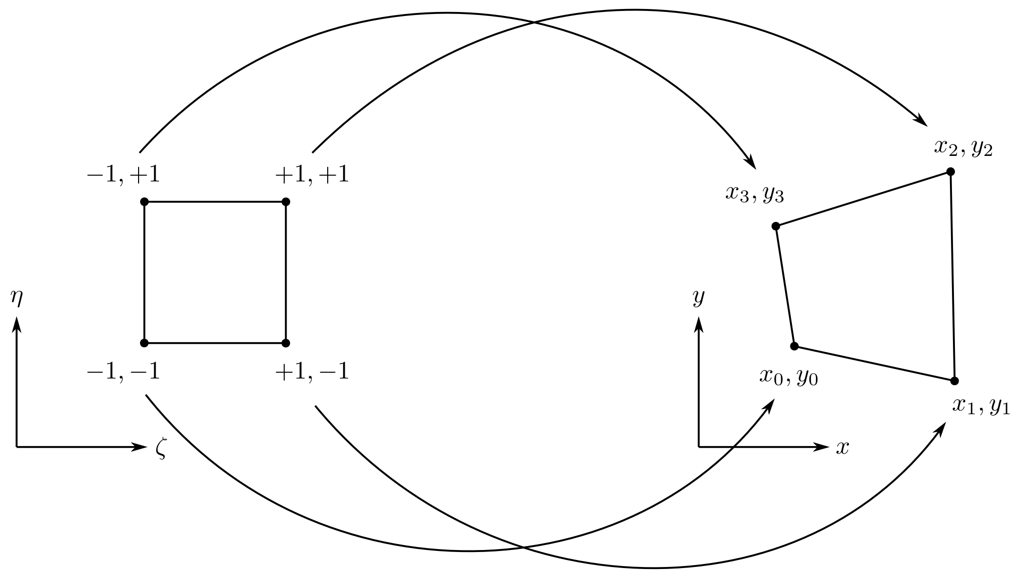


Figure 2: Illustration of two dimension transformation (mapping) between parent and global domains, $\big(\zeta(x,y), \eta(x,y)\big)$ and $\big(x(\zeta,\eta), y(\zeta,\eta)\big)$, respectively.

The 2D transformation example is shown in Figure 2. Here, we propose the bilinear relations

$$\begin{aligned} x(\zeta,\eta) &= A_0 + A_1\zeta + A_2\eta + A_3\zeta\eta \\ y(\zeta,\eta) &= B_0 + B_1\zeta + B_2\eta + B_3\zeta\eta \end{aligned} \tag{8}$$

Notice the bilinear term $\zeta\eta$. Similarly, after substitution the corresponding points from both domains we can write the relation

$$\begin{bmatrix} A_0 \\ A_1 \\ A_2 \\ A_3 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} +1 & +1 & +1 & +1 \\ -1 & +1 & +1 & -1 \\ -1 & -1 & +1 & +1 \\ +1 & -1 & +1 & -1 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix} \tag{9}$$

and

$$\begin{bmatrix} B_0 \\ B_1 \\ B_2 \\ B_3 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} +1 & +1 & +1 & +1 \\ -1 & +1 & +1 & -1 \\ -1 & -1 & +1 & +1 \\ +1 & -1 & +1 & -1 \end{bmatrix} \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{bmatrix} \tag{10}$$

Thus, one can write the transformation as

$$\begin{aligned} x(\zeta,\eta) &= \frac{1}{4}\sum_{n=0}^{3}\big(1+\zeta_n\zeta\big)\big(1+\eta_n\eta\big)x_n \\ y(\zeta,\eta) &= \frac{1}{4}\sum_{n=0}^{3}\big(1+\zeta_n\zeta\big)\big(1+\eta_n\eta\big)y_n \end{aligned} \tag{11}$$

where ζ_n and η_n values are summarized in Table 1.

Table 1: ζ_n and η_n values.

n	ζ_n	η_n
0	-1	-1
1	$+1$	-1
2	$+1$	$+1$
3	-1	$+1$

2.1 Application to Numerical Integral

Considering the canonical integral

$$\int_{-1}^{1}\int_{-1}^{1} f(\zeta,\eta)d\zeta d\eta \approx \sum_{m=1}^{N}\sum_{n=1}^{N} w_m w_n f(\zeta_m,\eta_n) \tag{12}$$

Now consider a general integral over region R on the global domain described by the points (x_0, y_0), (x_1, y_1), (x_2, y_2), and (x_3, y_3)

$$I = \iint_R f(x,y)dxdy = \int_{-1}^{1}\int_{-1}^{1} f(x(\zeta,\eta), y(\zeta,\eta))|\boldsymbol{J}(\zeta,\eta)|d\zeta d\eta \tag{13}$$

The Jacobian in this case is found as

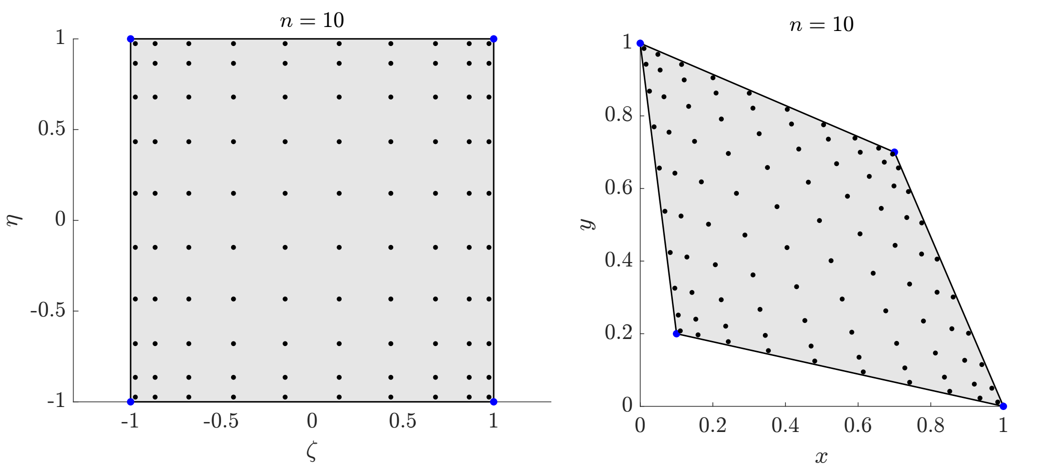
$$|\boldsymbol{J}(\zeta,\eta)| = \left|\frac{\partial(x,y)}{\partial(\zeta,\eta)}\right| = \begin{vmatrix} \frac{\partial x}{\partial \zeta} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \zeta} & \frac{\partial y}{\partial \eta} \end{vmatrix} = \frac{\partial x}{\partial \zeta}\frac{\partial y}{\partial \eta} - \frac{\partial x}{\partial \eta}\frac{\partial y}{\partial \zeta} \tag{14}$$

And the partial derivatives can be found as

$$
\begin{aligned}
\frac{\partial x}{\partial \zeta} &= \frac{1}{4}\sum_{n=0}^{3} \zeta_n \big(1+\eta_n \eta\big) x_n \\
\frac{\partial x}{\partial \eta} &= \frac{1}{4}\sum_{n=0}^{3} \eta_n \big(1+\zeta_n \zeta\big) x_n \\
\frac{\partial y}{\partial \zeta} &= \frac{1}{4}\sum_{n=0}^{3} \zeta_n \big(1+\eta_n \eta\big) y_n \\
\frac{\partial y}{\partial \eta} &= \frac{1}{4}\sum_{n=0}^{3} \eta_n \big(1+\zeta_n \zeta\big) y_n
\end{aligned} \tag{15}
$$

Thus, the integral can be evaluated as

$$
I \approx \sum_{m=1}^{N}\sum_{n=1}^{N} w_m w_n f(x(\zeta_m,\eta_n), y(\zeta_m,\eta_n))|\boldsymbol{J}(\zeta_m,\eta_n)| \tag{16}
$$



(a) (b)

Figure 3: An example of Gauss-Legendre quadrature nodes (order $n = 10$) mapped into arbitrary x-y global domain. (a) Parent domain (b) Global domain.

2.2 Square to Triangle Mapping Example

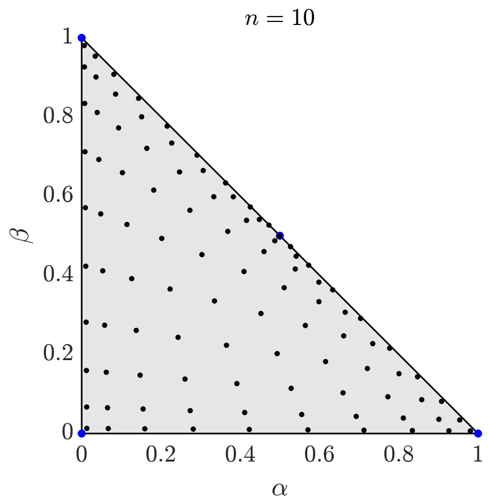


Figure 4: Illustration Gauss-quadrature mapped into a triangle domain. The triangle was defined by the 4 nodes $(0, 0)$, $(1, 0)$, $(\frac{1}{2}, \frac{1}{2})$, and $(0, 1)$.

A useful application can be demonstrated in the example of square quadrature mapped into a triangle domain as shown in Figure 1. Since our formulation was designed for 4 nodes shapes, we defined triangle using 4 nodes in the α-β domain. In this case, we can write explicit expressions for the mapping and the Jacobian as follows

$$\begin{aligned} \alpha(\zeta, \eta) &= \frac{1}{8}\big[3 + 3\zeta - \eta - \zeta\eta\big] \\ \beta(\zeta, \eta) &= \frac{1}{8}\big[3 + 3\eta - \zeta - \zeta\eta\big] \\ |\boldsymbol{J}(\zeta, \eta)| &= \frac{1}{16}\big[2 - \zeta - \eta\big] \end{aligned} \tag{17}$$

One can easily write a customized Gauss-quadrature for triangular domain integrals as

$$I = \int_0^1 \int_0^{1-\alpha} f(\alpha, \beta) d\beta d\alpha \approx \sum_{m=1}^{N} \sum_{n=1}^{N} w_{mn} f(\alpha_{mn}, \beta_{mn}) \tag{18}$$

where

$$
\begin{aligned}
\alpha_{mn} &= \frac{1}{8}\left[3 + 3\zeta_m - \eta_n - \zeta_m\eta_n\right] \\
\beta_{mn} &= \frac{1}{8}\left[3 + 3\eta_n - \zeta_m - \zeta_m\eta_n\right] \\
w_{mn} &= \frac{1}{16}\left[2 - \zeta_m - \eta_n\right] w_m w_n
\end{aligned} \tag{19}
$$

and ζ_m, η_n, w_m, and w_n are the nodes and weights for N order quadrature rule. Note that ζ_m and η_n are simply the nodes of indices m and n for a 1D Gauss-quadrature. Similarly, w_m and w_n are the weights of indices m and n for the same 1D Gauss-quadrature. A numerical example for the convergence of this quadrature is illustrated in Figure 1.

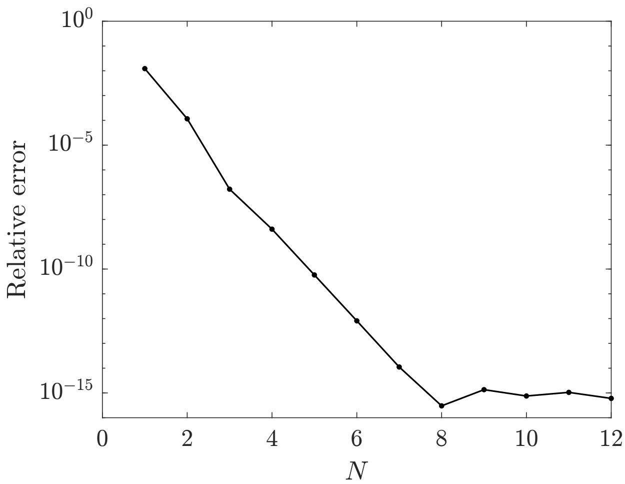


Figure 5: Convergence of the triangle Gauss-Legendre quadrature for the integral $I = \int_0^1 \int_0^{1-\alpha} \left[\alpha \log(2-\beta)\right] d\beta d\alpha \approx 0.0921601712977595$.