*Rolf Vidar Hoksaas*
*Institutt for Informatikk*
*Universitetet i Oslo*

**September 13, 2019**

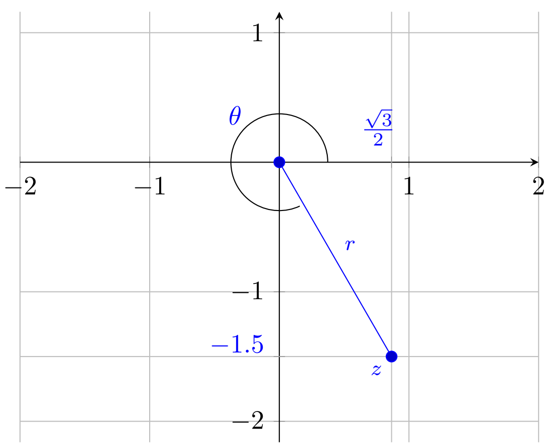
## OBLIG 1 — Obligatorisk oppgave 1 av 2

> *“Skriv det komplekse tallet $z = \frac{6}{\sqrt{3}+3i}$ først på formen $a + ib$ også på polarformen $re^{i\theta}$.”*

$$
\begin{aligned}
z &= \frac{6}{\sqrt{3}+3i} \\
z &= \frac{2}{\sqrt{\frac{1}{3}}+i} \\
z &= \frac{2 \times (\sqrt{\frac{1}{3}} - i)}{(\sqrt{\frac{1}{3}}+i) \times (\sqrt{\frac{1}{3}} - i)} \\
z &= \frac{2 \times (\sqrt{\frac{1}{3}} - i)}{\frac{1}{3} - i^2} \\
z &= \frac{2 \times (\sqrt{\frac{1}{3}} - i)}{\frac{4}{3}} \\
z &= \frac{3 \times (\sqrt{\frac{1}{3}} - i)}{2} \\
z &= \frac{\sqrt{3}}{2} - \frac{3}{2}i
\end{aligned}
$$

□

Using $z = \frac{\sqrt{3}}{2} - \frac{3}{2}i$:




*With Pythagora's Theorem*

$$
\begin{aligned}
r &= \sqrt{(\frac{\sqrt{3}}{2})^2 + (\frac{3}{2})^2} \\
r &= \sqrt{\frac{3}{4} + \frac{9}{4}} \\
r &= \sqrt{3}
\end{aligned}
$$

■

*By the laws of trigonometry*

$$\theta' := 2\pi - \theta$$

$$\sin(\theta') = \frac{-1.5}{\sqrt{3}}$$

$$
\begin{aligned}
\theta' &= \arcsin(\frac{-1.5\sqrt{3}}{3}) \\
\theta' &= \arcsin(\frac{-\sqrt{3}}{2}) \\
\theta' &= -\frac{\pi}{3} \equiv \frac{5\pi}{3} \\
\therefore \theta &= \frac{\pi}{3}
\end{aligned}
$$

■

*And since $z$ is in the fourth quadrant, as $Re(z) > 0 \wedge Im(z) < 0$ we know to use $\theta'$:*

$$r : \sqrt{3} \wedge \theta : \frac{5\pi}{3} \therefore z = \sqrt{3}e^{\frac{-i\pi}{3}} \equiv \sqrt{3}e^{\frac{5i\pi}{3}}$$

□

*“Finn de to løsningene til likningen $w^2 - w + 1 = 0$, og bruk disse til å finne alle komplekse løsninger til likningen $z^4 - z^2 + 1 = 0$. Gi en faktorisering av $z^4 - z^2 + 1$, først i komplekse førstegradspolynomer og så i reelle andregradspolynomer.”*

Assuming $cis(x) \equiv cos(x) + i \cdot sin(x)$, and $i^2 = -1$.

$$\boldsymbol{a = 1, b = -1, c = 1} \tag{2-1}$$
$$\boldsymbol{w = \frac{1 \pm \sqrt{1-4}}{2} = \frac{1}{2} \pm \frac{\sqrt{3}}{2} i} \tag{2-2}$$
$$\tag{2-3}$$

With $\boldsymbol{t = z^2}$:

$$\boldsymbol{z^4 - z^2 + 1 = t^2 - t^2 + 1 = 0} \tag{2-4}$$
$$\boldsymbol{t = \frac{1}{2} \pm \frac{\sqrt{3}}{2} i} \tag{2-5}$$
$$\tag{2-6}$$

With $\boldsymbol{t_0 = \frac{1}{2} + \frac{\sqrt{3}}{2} i = cis(\frac{\pi}{3})}$:

$$\boldsymbol{\sqrt{cis(\frac{\pi}{3})}^4 - \sqrt{cis(\frac{\pi}{3})}^2 + 1 = 0} \tag{2-7}$$
$$\boldsymbol{cis^2(\frac{\pi}{3}) - cis(\frac{\pi}{3}) + 1 = 0} \tag{2-8}$$
$$\tag{2-9}$$

With De Moivres formula:

$$\boldsymbol{cis(\frac{2\pi}{3}) - cis(\frac{\pi}{3}) + 1 = 0} \tag{2-10}$$
$$\boldsymbol{cos(\frac{2\pi}{3}) + i \cdot sin(\frac{2\pi}{3}) - cos(\frac{\pi}{3}) - i \cdot sin(\frac{\pi}{3}) + 1 = 0} \tag{2-11}$$
$$\boldsymbol{-\frac{1}{2} + \frac{\sqrt{3}}{2} i - \frac{1}{2} - \frac{\sqrt{3}}{2} i + 1 = 0} \tag{2-12}$$
$$\boldsymbol{-\frac{1}{2} - \frac{1}{2} + 1 = 0} \tag{2-13}$$
$$\boldsymbol{-1 + 1 = 0} \tag{2-14}$$
$$\tag{2-15}$$

Therefore, $\boldsymbol{z = t_0^{\frac{1}{2}} = cis^{\frac{1}{2}}(\frac{\pi}{3}) = cis(\frac{\pi}{6})}$ is a root $\boldsymbol{z_0}$.
By the complex conjugate root theorem, $\boldsymbol{z = t_1^{\frac{1}{2}} = cis^{\frac{1}{2}}(-\frac{\pi}{3}) = cis(-\frac{\pi}{6})}$ is a root $\boldsymbol{z_0'}$.

■

By dividing by the original polynom by $(z - z_0)(z - z_0')$:

$$\frac{z^4 - z^2 + 1}{(z - \frac{\sqrt{3}}{2} - \frac{1}{2}i) \cdot (z - \frac{\sqrt{3}}{2} + \frac{1}{2}i)} = 0 \tag{2-16}$$

$$\frac{z^4 - z^2 + 1}{(z - \frac{\sqrt{3}}{2})^2 - (\frac{1}{2}i)^2} = 0 \tag{2-17}$$

$$\frac{z^4 - z^2 + 1}{z^2 - \sqrt{3}z + \frac{3}{4} + \frac{1}{4}} = 0 \tag{2-18}$$

$$\frac{z^4 - z^2 + 1}{z^2 - \sqrt{3}z + 1} = 0 \tag{2-19}$$

By polynomial division:
$$z^2 + \sqrt{3}z + 1 = 0 \tag{2-20}$$

Thus the original equation is equivalent to the following:

$$(z^2 + \sqrt{3}z + 1) \cdot (z - cis(\frac{\pi}{6})) \cdot (z - cis(\frac{-\pi}{6})) = 0 \tag{2-21}$$

Solving for $(z^2 + \sqrt{3}z + 1) = 0$, viz. $(z - z_1)(z - z_1')$:

$$a = 1, b = \sqrt{3}, c = 1 \tag{2-22}$$

$$z = \frac{-\sqrt{3} \pm \sqrt{3 - 4}}{2} \tag{2-23}$$

$$z = \frac{-\sqrt{3}}{2} \pm \frac{i}{2} = cis(\pi \pm \frac{\pi}{6}) \tag{2-24}$$

$$z_1 = cis(\frac{5\pi}{6}), \qquad z_1' = cis(-\frac{5\pi}{6}) \tag{2-25}$$

■

By having $z_0, z_0', z_1,$ and $z_1'$, we can factorize as $(z - z_0) \cdot (z - z_0') \cdot (z - z_1) \cdot (z - z_1') = 0$, so $z^4 - z^2 + 1 = 0$ is equivalent in complex roots to

$(z - cis(\frac{\pi}{6})) \cdot (z - cis(-\frac{\pi}{6})) \cdot (z - cis(\frac{5\pi}{6})) \cdot (z - cis(-\frac{5\pi}{6})) = 0$. □

By the denominator of 2-19 we know $(z - z_0)(z - z_0') = z^2 + \sqrt{3}z + 1$,

so $z^4 - z^2 + 1 = 0$ expressed with second degree polynoms is $(z^2 - \sqrt{3}z + 1)(z^2 + \sqrt{3}z + 1)$.

□

“ *Finn grensene* $\lim_{n\to\infty} \frac{3n+2}{\sqrt{4n^2-1}}$ *og* $\lim_{n\to\infty}(\sqrt{n^2-5n}-n)$. ”

$$\lim_{n\to\infty} \frac{3n+2}{\sqrt{4n^2-1}} =$$

$$\lim_{n\to\infty} \frac{(3n+2)\frac{1}{n}}{\sqrt{4n^2-1}\frac{1}{n}} =$$

$$\lim_{n\to\infty} \frac{3+\frac{2}{n}}{\sqrt{\frac{4n^2-1}{n^2}}} =$$

$$\lim_{n\to\infty} \frac{3+\frac{2}{n}}{\sqrt{4-\frac{1}{n^2}}} =$$

$$\frac{3+\cancel{0}}{\sqrt{4-\cancel{0}}} =$$

$$\frac{3}{2}$$

$\square$

$$\lim_{n\to\infty} (\sqrt{n^2-5n}-n)$$

$$\lim_{n\to\infty} \frac{(\sqrt{n^2-5n}-n)(\sqrt{n^2-5n}+n)}{\sqrt{n^2-5n}+n}$$

$$\lim_{n\to\infty} \frac{n^2-5n-n^2}{\sqrt{n^2-5n}+n}$$

$$\lim_{n\to\infty} \frac{-5n}{\sqrt{n^2-5n}+n}$$

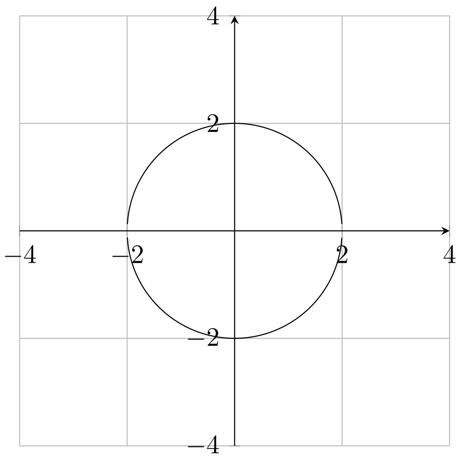
$$\lim_{n\to\infty} \frac{-5}{\sqrt{1-\frac{5}{n}}+1}$$

$$\frac{-5}{\sqrt{1-\cancel{0}}+1}$$

$$\frac{-5}{2}$$

$\square$

“ *Finn de komplekse tallene $z$ som oppfyller likningen $2|z-1| = |z-4|$ og skisser løsningsmengden i det komplekse planet.(Hint: Sett inn $z = x+iy$ og finn en polynomlikning i$x$ og $y$ forløsningsmengden.)* ”




$$\mathbf{2|z-1| = |z-4|} \equiv$$

$$\mathbf{4|x+iy-1| = |x+iy-4|}$$

$$\mathbf{4((x-1)^2+(y)^2) = (x-4)^2+(y)^2}$$

$$\mathbf{4(x-1)^2+4y^2 = x^2-8x+16+y^2}$$

$$\mathbf{4(x^2-2x+1)+4y^2 = x^2-8x+16+y^2}$$

$$\mathbf{4x^2-8x+4+4y^2 = x^2-8x+16+y^2}$$

$$\mathbf{3x^2+3y^2 = 12}$$

$$\mathbf{y^2 = 4-x^2}$$

$$\mathbf{y = \pm\sqrt{4-x^2}}$$

$$\therefore \boldsymbol{Im(z) = \pm\sqrt{4-Re(z)^2}, \exists z \forall \{x \in Re(z) | -2 < x < 2\}}$$

$\square$

“ *En følge $\{a_n\}$ er definert ved $a_1 = 3, a_n+1 = 3\sqrt{a_n}$ for $n \geq 1$. Vis at $a_n < 9$ og at $a_n+1 > a_n$ for alle $n$. Forklar hvorfor følgen konvergerer og finn $\lim_{n\to\infty} a_n$* ”

$\square$.

*Submitted by* Rolf Vidar Hoksaas *on September 13, 2019.*