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OBLIG 1 — Obligatorisk oppgave 1 av 2

> *Skriv det komplekse tallet $z = \frac{6}{\sqrt{3}+3i}$ først på formen $a + ib$ også på polarformen $re^{i\theta}$.*

$$z = \frac{6}{\sqrt{3} + 3i}$$

$$z = \frac{2}{\sqrt{\frac{1}{3}} + i}$$

$$z = \frac{2 \times (\sqrt{\frac{1}{3}} - i)}{(\sqrt{\frac{1}{3}} + i) \times (\sqrt{\frac{1}{3}} - i)}$$

$$z = \frac{2 \times (\sqrt{\frac{1}{3}} - i)}{\frac{1}{3} - i^2}$$

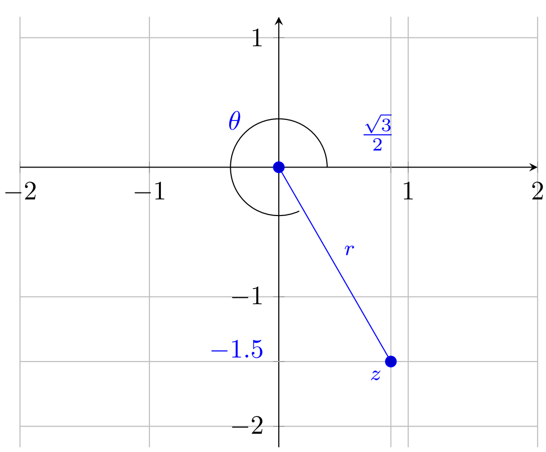
$$z = \frac{2 \times (\sqrt{\frac{1}{3}} - i)}{\frac{4}{3}}$$

$$z = \frac{3 \times (\sqrt{\frac{1}{3}} - i)}{2}$$

$$z = \frac{\sqrt{3}}{2} - \frac{3}{2}i$$

□

Using $z = \frac{\sqrt{3}}{2} - \frac{3}{2}i$:



With Pythagora's Theorem

$$r = \sqrt{(\frac{\sqrt{3}}{2})^2 + (\frac{3}{2})^2}$$

$$r = \sqrt{\frac{3}{4} + \frac{9}{4}}$$

$$r = \sqrt{3}$$

■

By the laws of trigonometry

$$\theta' := 2\pi - \theta$$

$$\sin(\theta') = \frac{-1.5}{\sqrt{3}}$$

$$\theta' = \arcsin(\frac{-1.5\sqrt{3}}{3})$$

$$\theta' = \arcsin(\frac{-\sqrt{3}}{2})$$

$$\theta' = -\frac{\pi}{3} \equiv \frac{5\pi}{3}$$

$$\therefore \theta = \frac{\pi}{3}$$

■

And since z is in the fourth quadrant, as $Re(z) > 0 \wedge Im(z) < 0$ we know to use θ':

$$r : \sqrt{3} \wedge \theta : \frac{5\pi}{3} \therefore z = \sqrt{3}e^{\frac{-i\pi}{3}} \equiv \sqrt{3}e^{\frac{5i\pi}{3}}$$

□

“ *Finn de to løsningene til likningen $w^2 - w + 1 = 0$, og bruk disse til å finne alle komplekse løsninger til likningen $z^4 - z^2 + 1 = 0$. Gi en faktorisering av $z^4 - z^2 + 1$, først i komplekse førstegradspolynomer og så i reelle andregradspolynomer.* ”

Assuming $cis(x) \equiv cos(x) + i \cdot sin(x)$, and $i^2 = -1$.

$$\boldsymbol{a = 1, b = -1, c = 1} \tag{2-1}$$
$$\boldsymbol{w = \tfrac{1 \pm \sqrt{1-4}}{2} = \tfrac{1}{2} \pm \tfrac{\sqrt{3}}{2} i} \tag{2-2}$$
$$\tag{2-3}$$

With $\boldsymbol{t = z^2}$:

$$\boldsymbol{z^4 - z^2 + 1 = t^2 - t^2 + 1 = 0} \tag{2-4}$$
$$\boldsymbol{t = \tfrac{1}{2} \pm \tfrac{\sqrt{3}}{2} i} \tag{2-5}$$
$$\tag{2-6}$$

With $\boldsymbol{t_0 = \tfrac{1}{2} + \tfrac{\sqrt{3}}{2} i = cis(\tfrac{\pi}{3})}$:

$$\boldsymbol{\sqrt{cis(\tfrac{\pi}{3})}^4 - \sqrt{cis(\tfrac{\pi}{3})}^2 + 1 = 0} \tag{2-7}$$
$$\boldsymbol{cis^2(\tfrac{\pi}{3}) - cis(\tfrac{\pi}{3}) + 1 = 0} \tag{2-8}$$
$$\tag{2-9}$$

With De Moivres formula:

$$\boldsymbol{cis(\tfrac{2\pi}{3}) - cis(\tfrac{\pi}{3}) + 1 = 0} \tag{2-10}$$
$$\boldsymbol{cos(\tfrac{2\pi}{3}) + i \cdot sin(\tfrac{2\pi}{3}) - cos(\tfrac{\pi}{3}) - i \cdot sin(\tfrac{\pi}{3}) + 1 = 0} \tag{2-11}$$
$$\boldsymbol{-\tfrac{1}{2} + \tfrac{\sqrt{3}}{2} i - \tfrac{1}{2} - \tfrac{\sqrt{3}}{2} i + 1 = 0} \tag{2-12}$$
$$\boldsymbol{-\tfrac{1}{2} - \tfrac{1}{2} + 1 = 0} \tag{2-13}$$
$$\boldsymbol{-1 + 1 = 0} \tag{2-14}$$
$$\tag{2-15}$$

Therefore, $\boldsymbol{z = t_0^{\frac{1}{2}} = cis^{\frac{1}{2}}(\tfrac{\pi}{3}) = cis(\tfrac{\pi}{6})}$ is a root $\boldsymbol{z_0}$.
By the complex conjugate root theorem, $\boldsymbol{z = t_1^{\frac{1}{2}} = cis^{\frac{1}{2}}(-\tfrac{\pi}{3}) = cis(-\tfrac{\pi}{6})}$ is a root $\boldsymbol{z_0'}$.

■

By dividing by the original polynom by $(z - z_0)(z - z_0')$:

$$\frac{z^4 - z^2 + 1}{(z - \frac{\sqrt{3}}{2} - \frac{1}{2}i) \cdot (z - \frac{\sqrt{3}}{2} + \frac{1}{2}i)} = 0 \tag{2-16}$$

$$\frac{z^4 - z^2 + 1}{(z - \frac{\sqrt{3}}{2})^2 - (\frac{1}{2}i)^2} = 0 \tag{2-17}$$

$$\frac{z^4 - z^2 + 1}{z^2 - \sqrt{3}z + \frac{3}{4} + \frac{1}{4}} = 0 \tag{2-18}$$

$$\frac{z^4 - z^2 + 1}{z^2 - \sqrt{3}z + 1} = 0 \tag{2-19}$$

$$\text{By polynomial division:} \qquad z^2 + \sqrt{3}z + 1 = 0 \tag{2-20}$$

Thus the original equation is equivalent to the following:

$$(z^2 + \sqrt{3}z + 1) \cdot (z - cis(\frac{\pi}{6})) \cdot (z - cis(\frac{-\pi}{6})) = 0 \tag{2-21}$$

Solving for $(z^2 + \sqrt{3}z + 1) = 0$, viz. $(z - z_1)(z - z_1')$:

$$a = 1, b = \sqrt{3}, c = 1 \tag{2-22}$$

$$z = \frac{-\sqrt{3} \pm \sqrt{3 - 4}}{2} \tag{2-23}$$

$$z = \frac{-\sqrt{3}}{2} \pm \frac{i}{2} = cis(\pi \pm \frac{\pi}{6}) \tag{2-24}$$

$$z_1 = cis(\frac{5\pi}{6}), \qquad z_1' = cis(-\frac{5\pi}{6}) \tag{2-25}$$

■

By having z_0, z_0', z_1, and z_1', we can factorize as $(z - z_0) \cdot (z - z_0') \cdot (z - z_1) \cdot (z - z_1') = 0$, so $z^4 - z^2 + 1 = 0$ is equivalent in complex roots to

$(z - cis(\frac{\pi}{6})) \cdot (z - cis(-\frac{\pi}{6})) \cdot (z - cis(\frac{5\pi}{6})) \cdot (z - cis(-\frac{5\pi}{6})) = 0$. □

By the denominator of 2-19 we know $(z - z_0)(z - z_0') = z^2 + \sqrt{3}z + 1$,

so $z^4 - z^2 + 1 = 0$ expressed with second degree polynoms is $(z^2 - \sqrt{3}z + 1)(z^2 + \sqrt{3}z + 1)$.

□

“ *Finn grensene* $\lim_{n\to\infty}\frac{3n+2}{\sqrt{4n^2-1}}$ *og* $\lim_{n\to\infty}(\sqrt{n^2-5n}-n)$. ”

$$\lim_{n\to\infty}\frac{3n+2}{\sqrt{4n^2-1}} =$$

$$\lim_{n\to\infty}\frac{(3n+2)\frac{1}{n}}{\sqrt{4n^2-1}\frac{1}{n}} =$$

$$\lim_{n\to\infty}\frac{3+\frac{2}{n}}{\sqrt{\frac{4n^2-1}{n^2}}} =$$

$$\lim_{n\to\infty}\frac{3+\frac{2}{n}}{\sqrt{4-\frac{1}{n^2}}} =$$

$$\frac{3+\not 0}{\sqrt{4-\not 0}} =$$

$$\frac{3}{2}$$

$\square$

$$\lim_{n\to\infty}(\sqrt{n^2-5n}-n)$$

$$\lim_{n\to\infty}\frac{(\sqrt{n^2-5n}-n)(\sqrt{n^2-5n}+n)}{\sqrt{n^2-5n}+n}$$

$$\lim_{n\to\infty}\frac{n^2-5n-n^2}{\sqrt{n^2-5n}+n}$$

$$\lim_{n\to\infty}\frac{-5n}{\sqrt{n^2-5n}+n}$$

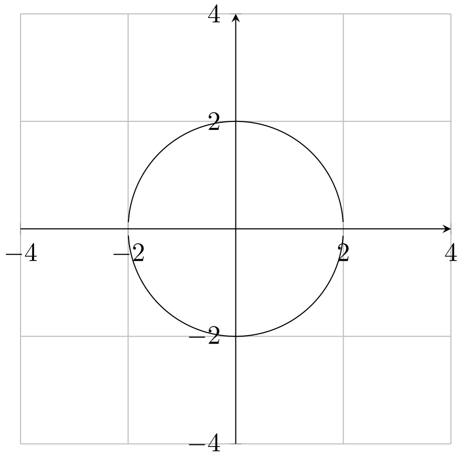
$$\lim_{n\to\infty}\frac{-5}{\sqrt{1-\frac{5}{n}}+1}$$

$$\frac{-5}{\sqrt{1-\not 0}+1}$$

$$\frac{-5}{2}$$

$\square$

> *Finn de komplekse tallene z som oppfyller likningen $2|z-1| = |z-4|$ og skisser løsningsmengden i det komplekse planet.(Hint: Sett inn $z = x+iy$ og finn en polynomlikning ix og y forløsningsmengden.)*



$$2|z-1| = |z-4| \equiv$$

$$4|a+ib-1| = |a+ib-4|$$

$$4((a-1)^2+(b)^2) = (a-4)^2+(b)^2$$

$$4(a-1)^2+4b^2 = a^2-8a+16+b^2$$

$$4(a^2-2a+1)+4b^2 = a^2-8a+16+b^2$$

$$4a^2-8a+4+4b^2 = a^2-8a+16+b^2$$

$$3a^2z+3b^2 = 12$$

$$b^2 = 4-a^2$$

$$b = \pm\sqrt{4-a^2}$$

$$\therefore Im(z) = \pm\sqrt{4-Re(z)^2}, \exists z \forall \{a \in \mathbb{R} | -2 < a < 2\}$$

Since a circle's point is defined by all the points with radius r from its center, and we can recognize the modulus of the radius to be defined by Pythagora's theorem, as b^2+a^2, we know that the equation defines a circle of radius $\mathbf{2}$. It's position is the Origin, as there is no transition on any of the direction vectors. $\square$

“ *En følge $\{a_n\}$ er definert ved $a_1 = 3, a_{n+1} = 3\sqrt{a_n}$ for $n \geq 1$. Vis at $a_n < 9$ og at $a_{n+1} > a_n$ for alle n. Forklar hvorfor følgen konvergerer og finn $\lim_{n\to\infty} a_n$* ”

$$\begin{aligned} \boldsymbol{a_1} &:= \boldsymbol{3,} \qquad (5\text{-}1) \\ \boldsymbol{a_{n+1}} &:= \boldsymbol{3\sqrt{a_n}} \qquad (5\text{-}2) \end{aligned}$$

To be proven by induction:

$$\begin{aligned} \boldsymbol{P_n} &:= \boldsymbol{a_n < 9} \qquad (5\text{-}3) \\ \boldsymbol{Q_n} &:= \boldsymbol{a_{n+1} > a_n} \qquad (5\text{-}4) \end{aligned}$$

Prove for n=1:

$$\begin{aligned} \boldsymbol{a_1} &= \boldsymbol{3 \therefore a_2 = 3\sqrt{3}} \qquad (5\text{-}5) \\ \boldsymbol{P_1} &:= \boldsymbol{a_1 < 9 \equiv 3 < 9 \therefore P_1} \qquad (5\text{-}6) \\ \boldsymbol{Q_1} &:= \boldsymbol{a_2 > a_1 \equiv 3\sqrt{3} > 3 \therefore Q_1} \qquad (5\text{-}7) \end{aligned}$$

Assume $\boldsymbol{Q_k}$, and prove $(\boldsymbol{Q_k \implies Q_{k+1}})$:

$$\begin{aligned} \boldsymbol{Q_k} &:= \boldsymbol{a_{k+1} > a_k} \qquad (5\text{-}8) \\ \boldsymbol{Q_{k+1}} &:= \boldsymbol{a_{k+2} > 3\sqrt{a_k}} \qquad (5\text{-}9) \\ \boldsymbol{Q_{k+1}} &:= \boldsymbol{3\sqrt{a_{k+1}} > 3\sqrt{a_k}} \qquad (5\text{-}10) \\ \boldsymbol{Q_{k+1}} &:= \boldsymbol{\sqrt{a_{k+1}} > \sqrt{a_k}} \qquad (5\text{-}11) \\ \boldsymbol{Q_{k+1}} &:= \boldsymbol{a_{k+1} > a_k} \qquad (5\text{-}12) \end{aligned}$$

□

Assume $\boldsymbol{P_{k+1}}$, and prove $(\boldsymbol{P_k \implies P_{k+1}})$

$$\begin{aligned} \boldsymbol{P_k} &:= \boldsymbol{a_k < 9} \qquad (5\text{-}13) \\ \boldsymbol{P_{k+1}} &:= \boldsymbol{3\sqrt{a_k} < 9} \qquad (5\text{-}14) \\ \boldsymbol{P_{k+1}} &:= \boldsymbol{\sqrt{a_k} < 3} \qquad (5\text{-}15) \\ \boldsymbol{P_{k+1}} &:= \boldsymbol{a_k < 9} \qquad (5\text{-}16) \end{aligned}$$

□

The sequence converges since all values increment as $\boldsymbol{n}$ increments, but is bounded by $\mathbf{9}$. We also know by this that the sequence's infinum is 3, as $\boldsymbol{a_1 = 3}$ $\square$

Since the sequence is convergent, we know its limit exsits:

$$\boldsymbol{L = \lim_{n\to\infty} a_n} \tag{5-17}$$

$$\boldsymbol{L = \lim_{n\to\infty} 3\sqrt{a_{n-1}}} \tag{5-18}$$

Since all $\boldsymbol{a_n > 1}$:

$$\boldsymbol{L = 3\sqrt{\lim_{n\to\infty} a_{n-1}}} \tag{5-19}$$

By the monotone convergence theorem:

$$\boldsymbol{L = 3\sqrt{\lim_{n\to\infty} a_n}} \tag{5-20}$$

$$\boldsymbol{L = 3\sqrt{L}} \tag{5-21}$$

$$\boldsymbol{L^2 = 9L} \tag{5-22}$$

$$\boldsymbol{L = 9} \tag{5-23}$$

Thus we know the supremum of the sequence is $\mathbf{9}$. $\square$.

Submitted by Rolf Vidar Hoksaas *on September 13, 2019.*