

September 13, 2019

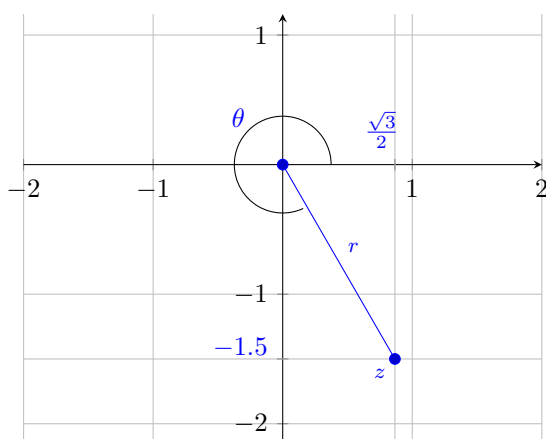
OBLIG 1 — Obligatorisk oppgave 1 av 2

“Skriv det komplekse tallet $z = \frac{6}{\sqrt{3}+3i}$ først på formen $a + ib$ også på polarformen $re^{i\theta}$.”

$$\begin{aligned} z &= \frac{6}{\sqrt{3} + 3i} \\ z &= \frac{2}{\sqrt{\frac{1}{3}} + i} \\ z &= \frac{2 \times (\sqrt{\frac{1}{3}} - i)}{(\sqrt{\frac{1}{3}} + i) \times (\sqrt{\frac{1}{3}} - i)} \\ z &= \frac{2 \times (\sqrt{\frac{1}{3}} - i)}{\frac{1}{3} - i^2} \\ z &= \frac{2 \times (\sqrt{\frac{1}{3}} - i)}{\frac{4}{3}} \\ z &= \frac{3 \times (\sqrt{\frac{1}{3}} - i)}{2} \\ z &= \frac{\sqrt{3}}{2} - \frac{3}{2}i \end{aligned}$$

□

Using $z = \frac{\sqrt{3}}{2} - \frac{3}{2}i$:



With Pythagora's Theorem

$$\begin{aligned} r &= \sqrt{\left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{3}{2}\right)^2} \\ r &= \sqrt{\frac{3}{4} + \frac{9}{4}} \\ r &= \sqrt{3} \end{aligned}$$

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By the laws of trigonometry

$$\begin{aligned} \theta' &:= 2\pi - \theta \\ \sin(\theta') &= \frac{-1.5}{\sqrt{3}} \\ \theta' &= \arcsin\left(\frac{-1.5\sqrt{3}}{3}\right) \\ \theta' &= \arcsin\left(\frac{-\sqrt{3}}{2}\right) \\ \theta' &= -\frac{\pi}{3} \equiv \frac{5\pi}{3} \\ \therefore \theta &= \frac{\pi}{3} \end{aligned}$$

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And since z is in the fourth quadrant, as $\text{Re}(z) > 0 \wedge \text{Im}(z) < 0$ we know to use θ' :

$$r : \sqrt{3} \wedge \theta : \frac{5\pi}{3} \therefore z = \sqrt{3}e^{\frac{-i\pi}{3}} \equiv \sqrt{3}e^{\frac{5i\pi}{3}}$$

□

“
 Finn de to løsningene til likningen $w^2 - w + 1 = 0$, og bruk disse til å finne alle komplekse løsninger til likningen $z^4 - z^2 + 1 = 0$. Gi en faktorisering av $z^4 - z^2 + 1$, først i komplekse førstegradspolynomer og så i reelle andregradspolynomer.
 ”

Assuming $\text{cis}(x) \equiv \cos(x) + i \cdot \sin(x)$, and $i^2 = -1$.

$$a = 1, b = -1, c = 1 \quad (2-1)$$

$$w = \frac{1 \pm \sqrt{1-4}}{2} = \frac{1}{2} \pm \frac{\sqrt{3}}{2}i \quad (2-2)$$

$$(2-3)$$

With $t = z^2$:

$$z^4 - z^2 + 1 = t^2 - t + 1 = 0 \quad (2-4)$$

$$t = \frac{1}{2} \pm \frac{\sqrt{3}}{2}i \quad (2-5)$$

$$(2-6)$$

With $t_0 = \frac{1}{2} + \frac{\sqrt{3}}{2}i = \text{cis}(\frac{\pi}{3})$:

$$\sqrt{\text{cis}(\frac{\pi}{3})^4} - \sqrt{\text{cis}(\frac{\pi}{3})^2} + 1 = 0 \quad (2-7)$$

$$\text{cis}^2(\frac{\pi}{3}) - \text{cis}(\frac{\pi}{3}) + 1 = 0 \quad (2-8)$$

$$(2-9)$$

With De Moivre's formula:

$$\text{cis}(\frac{2\pi}{3}) - \text{cis}(\frac{\pi}{3}) + 1 = 0 \quad (2-10)$$

$$\cos(\frac{2\pi}{3}) + i \cdot \sin(\frac{2\pi}{3}) - \cos(\frac{\pi}{3}) - i \cdot \sin(\frac{\pi}{3}) + 1 = 0 \quad (2-11)$$

$$-\frac{1}{2} + \frac{\sqrt{3}}{2}i - \frac{1}{2} - \frac{\sqrt{3}}{2}i + 1 = 0 \quad (2-12)$$

$$-\frac{1}{2} - \frac{1}{2} + 1 = 0 \quad (2-13)$$

$$-1 + 1 = 0 \quad (2-14)$$

$$(2-15)$$

Therefore, $z = t_0^{\frac{1}{2}} = \text{cis}^{\frac{1}{2}}(\frac{\pi}{3}) = \text{cis}(\frac{\pi}{6})$ is a root z_0 .

By the complex conjugate root theorem, $z = t_1^{\frac{1}{2}} = \text{cis}^{\frac{1}{2}}(-\frac{\pi}{3}) = \text{cis}(-\frac{\pi}{6})$ is a root z'_0 .

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By dividing by the original polynom by $(z - z_0)(z - z'_0)$:

$$\frac{z^4 - z^2 + 1}{(z - \frac{\sqrt{3}}{2} - \frac{1}{2}i) \cdot (z - \frac{\sqrt{3}}{2} + \frac{1}{2}i)} = 0 \quad (2-16)$$

$$\frac{z^4 - z^2 + 1}{(z - \frac{\sqrt{3}}{2})^2 - (\frac{1}{2}i)^2} = 0 \quad (2-17)$$

$$\frac{z^4 - z^2 + 1}{z^2 - \sqrt{3}z + \frac{3}{4} + \frac{1}{4}} = 0 \quad (2-18)$$

$$\frac{z^4 - z^2 + 1}{z^2 - \sqrt{3}z + 1} = 0 \quad (2-19)$$

$$\text{By polynomial division:} \quad z^2 + \sqrt{3}z + 1 = 0 \quad (2-20)$$

Thus the original equation is equivalent to the following:

$$(z^2 + \sqrt{3}z + 1) \cdot (z - \text{cis}(\frac{\pi}{6})) \cdot (z - \text{cis}(\frac{-\pi}{6})) = 0 \quad (2-21)$$

Solving for $(z^2 + \sqrt{3}z + 1) = 0$, viz. $(z - z_1)(z - z'_1)$:

$$a = 1, b = \sqrt{3}, c = 1 \quad (2-22)$$

$$z = \frac{-\sqrt{3} \pm \sqrt{3-4}}{2} \quad (2-23)$$

$$z = \frac{-\sqrt{3}}{2} \pm \frac{i}{2} = \text{cis}(\pi \pm \frac{\pi}{6}) \quad (2-24)$$

$$z_1 = \text{cis}(\frac{5\pi}{6}), \quad z'_1 = \text{cis}(-\frac{5\pi}{6}) \quad (2-25)$$

■

By having z_0, z'_0, z_1 , and z'_1 , we can factorize as $(z - z_0) \cdot (z - z'_0) \cdot (z - z_1) \cdot (z - z'_1) = 0$,

so $z^4 - z^2 + 1 = 0$ is equivalent in complex roots to

$$(z - \text{cis}(\frac{\pi}{6})) \cdot (z - \text{cis}(-\frac{\pi}{6})) \cdot (z - \text{cis}(\frac{5\pi}{6})) \cdot (z - \text{cis}(-\frac{5\pi}{6})) = 0. \quad \square$$

By the denominator of 2-19 we know $(z - z_0)(z - z'_0) = z^2 + \sqrt{3}z + 1$,

so $z^4 - z^2 + 1 = 0$ expressed with second degree polynoms is $(z^2 - \sqrt{3}z + 1)(z^2 + \sqrt{3}z + 1)$.

□

“ Finn grensene $\lim_{n \rightarrow \infty} \frac{3n+2}{\sqrt{4n^2-1}}$ og $\lim_{n \rightarrow \infty} (\sqrt{n^2-5n} - n)$. ”

$$\lim_{n \rightarrow \infty} \frac{3n+2}{\sqrt{4n^2-1}} =$$

$$\lim_{n \rightarrow \infty} \frac{(3n+2) \frac{1}{n}}{\sqrt{4n^2-1} \frac{1}{n}} =$$

$$\lim_{n \rightarrow \infty} \frac{3 + \frac{2}{n}}{\sqrt{\frac{4n^2-1}{n^2}}} =$$

$$\lim_{n \rightarrow \infty} \frac{3 + \frac{2}{n}}{\sqrt{4 - \frac{1}{n^2}}} =$$

$$\frac{3 + \emptyset}{\sqrt{4 - \emptyset}} =$$

$$\frac{3}{2}$$

$$\lim_{n \rightarrow \infty} (\sqrt{n^2-5n} - n)$$

$$\lim_{n \rightarrow \infty} \frac{(\sqrt{n^2-5n} - n)(\sqrt{n^2-5n} + n)}{\sqrt{n^2-5n} + n}$$

$$\lim_{n \rightarrow \infty} \frac{n^2 - 5n - n^2}{\sqrt{n^2-5n} + n}$$

$$\lim_{n \rightarrow \infty} \frac{-5n}{\sqrt{n^2-5n} + n}$$

$$\lim_{n \rightarrow \infty} \frac{-5}{\sqrt{1 - \frac{5}{n}} + 1}$$

$$\frac{-5}{\sqrt{1 - \emptyset} + 1}$$

$$\frac{-5}{2}$$

□

□

“ Finn de komplekse tallene z som oppfyller likningen $2|z-1| = |z-4|$ og skisser løsningsmengden i det komplekse planet. (Hint: Sett inn $z = x+iy$ og finn en polynomlikning i x og y for løsningsmengden.) ”

$$2|z - 1| = |z - 4| \equiv$$

$$4|x + iy - 1| = |x + iy - 4|$$

$$4((x - 1)^2 + (y)^2) = (x - 4)^2 + (y)^2$$

$$4(x - 1)^2 + 4y^2 = x^2 - 8x + 16 + y^2$$

$$4(x^2 - 2x + 1) + 4y^2 = x^2 - 8x + 16 + y^2$$

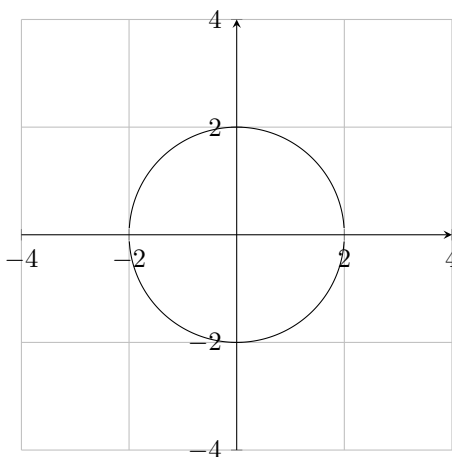
$$4x^2 - 8x + 4 + 4y^2 = x^2 - 8x + 16 + y^2$$

$$3x^2 + 3y^2 = 12$$

$$y^2 = 4 - x^2$$

$$y = \pm\sqrt{4 - x^2}$$

$$\therefore \operatorname{Im}(z) = \pm\sqrt{4 - \operatorname{Re}(z)^2}, \exists z \forall \{x \in \operatorname{Re}(z) \mid -2 < x < 2\}$$



□

“ En følge $\{a_n\}$ er definert ved $a_1 = 3, a_{n+1} = 3\sqrt{a_n}$ for $n \geq 1$. Vis at $a_n < 9$ og at $a_{n+1} > a_n$ for alle n . Forklar hvorfor følgen konvergerer og finn $\lim_{n \rightarrow \infty} a_n$ ”

$$a_1 := 3, \quad (5-1)$$

$$a_{n+1} := 3\sqrt{a_n} \quad (5-2)$$

To be proven by induction:

$$P_n := a_n < 9 \quad (5-3)$$

$$Q_n := a_{n+1} > a_n \quad (5-4)$$

Prove for $n=1$:

$$a_1 = 3 \therefore a_2 = 3\sqrt{3} \quad (5-5)$$

$$P_1 := a_1 < 9 \equiv 3 < 9 \therefore P_1 \quad (5-6)$$

$$Q_1 := a_2 > a_1 \equiv 3\sqrt{3} > 3 \therefore Q_1 \quad (5-7)$$

Assume Q_k , and prove $(Q_k \implies Q_{k+1})$:

$$Q_k := a_{k+1} > a_k \quad (5-8)$$

$$Q_{k+1} := a_{k+2} > 3\sqrt{a_k} \quad (5-9)$$

$$Q_{k+1} := 3\sqrt{a_{k+1}} > 3\sqrt{a_k} \quad (5-10)$$

$$Q_{k+1} := \sqrt{a_{k+1}} > \sqrt{a_k} \quad (5-11)$$

$$Q_{k+1} := a_{k+1} > a_k \quad (5-12)$$

□

Assume P_{k+1} , and prove $(P_k \implies P_{k+1})$

$$P_k := a_k < 9 \quad (5-13)$$

$$P_{k+1} := 3\sqrt{a_k} < 9 \quad (5-14)$$

$$P_{k+1} := \sqrt{a_k} < 3 \quad (5-15)$$

$$P_{k+1} := a_k < 9 \quad (5-16)$$

□

The sequence converges since all values increment as n increments, but is bounded by 9 . We also know by this that the sequence's infimum is 3 , as $a_1 = 3$ \square

Since the sequence is convergent, we know its limit exists:

$$L = \lim_{n \rightarrow \infty} a_n \quad (5-17)$$

$$L = \lim_{n \rightarrow \infty} 3\sqrt{a_{n-1}} \quad (5-18)$$

Since all $a_n > 1$:

$$L = 3\sqrt{\lim_{n \rightarrow \infty} a_{n-1}} \quad (5-19)$$

By the monotone convergence theorem:

$$L = 3\sqrt{\lim_{n \rightarrow \infty} a_n} \quad (5-20)$$

$$L = 3\sqrt{L} \quad (5-21)$$

$$L^2 = 9L \quad (5-22)$$

$$L = 9 \quad (5-23)$$

Thus we know the supremum of the sequence is 9 . \square .