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OBLIG 1 — Obligatorisk oppgave 1 av 2

> “*Skriv det komplekse tallet* $z = \frac{6}{\sqrt{3}+3i}$ *først på formen* $a + ib$ *også på polarformen* $re^{i\theta}$.”

$$\boldsymbol{z = \frac{6}{\sqrt{3}+3i}}$$

$$\boldsymbol{z = \frac{2}{\sqrt{\frac{1}{3}}+i}}$$

$$\boldsymbol{z = \frac{2 \times (\sqrt{\frac{1}{3}} - i)}{(\sqrt{\frac{1}{3}}+i) \times (\sqrt{\frac{1}{3}} - i)}}$$

$$\boldsymbol{z = \frac{2 \times (\sqrt{\frac{1}{3}} - i)}{\frac{1}{3} - i^2}}$$

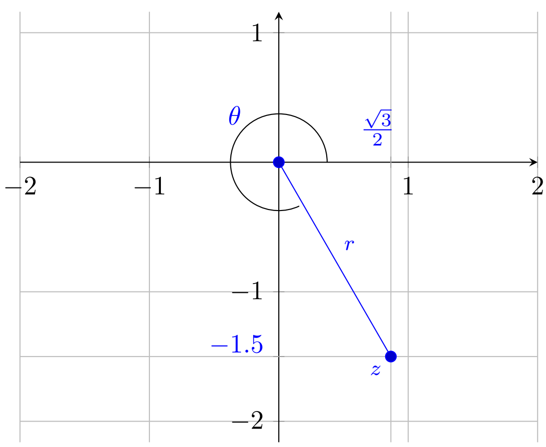
$$\boldsymbol{z = \frac{2 \times (\sqrt{\frac{1}{3}} - i)}{\frac{4}{3}}}$$

$$\boldsymbol{z = \frac{3 \times (\sqrt{\frac{1}{3}} - i)}{2}}$$

$$\boldsymbol{z = \frac{\sqrt{3}}{2} - \frac{3}{2}i}$$

□

Using $z = \frac{\sqrt{3}}{2} - \frac{3}{2}i$:



With Pythagora's Theorem

$$\boldsymbol{r = \sqrt{(\frac{\sqrt{3}}{2})^2 + (\frac{3}{2})^2}}$$

$$\boldsymbol{r = \sqrt{\frac{3}{4} + \frac{9}{4}}}$$

$$\boldsymbol{r = \sqrt{3}}$$

■

By the laws of trigonometry

$$\boldsymbol{\theta' := 2\pi - \theta}$$

$$\boldsymbol{\sin(\theta') = \frac{-1.5}{\sqrt{3}}}$$

$$\boldsymbol{\theta' = \arcsin(\frac{-1.5\sqrt{3}}{3})}$$

$$\boldsymbol{\theta' = \arcsin(\frac{-\sqrt{3}}{2})}$$

$$\boldsymbol{\theta' = -\frac{\pi}{3} \equiv \frac{5\pi}{3}}$$

$$\boldsymbol{\therefore \theta = \frac{\pi}{3}}$$

■

And since $\boldsymbol{z}$ *is in the fourth quadrant, as* $\boldsymbol{Re(z) > 0 \wedge Im(z) < 0}$ *we know to use* $\boldsymbol{\theta'}$*:*

$$\boldsymbol{r : \sqrt{3} \wedge \theta : \frac{5\pi}{3} \therefore z = \sqrt{3}e^{\frac{-i\pi}{3}} \equiv \sqrt{3}e^{\frac{5i\pi}{3}}}$$

□

“*Finn de to løsningene til likningen $w^2 - w + 1 = 0$, og bruk disse til å finne alle komplekse løsninger til likningen $z^4 - z^2 + 1 = 0$. Gi en faktorisering av $z^4 - z^2 + 1$, først i komplekse førstegradspolynomer og så i reelle andregradspolynomer.*”

Assuming $cis(x) \equiv cos(x) + i \cdot sin(x)$, and $i^2 = -1$.

$$\boldsymbol{a = 1, b = -1, c = 1} \tag{2-1}$$
$$\boldsymbol{w = \frac{1 \pm \sqrt{1-4}}{2} = \frac{1}{2} \pm \frac{\sqrt{3}}{2} i} \tag{2-2}$$
$$\tag{2-3}$$

With $\boldsymbol{t = z^2}$:

$$\boldsymbol{z^4 - z^2 + 1 = t^2 - t^2 + 1 = 0} \tag{2-4}$$
$$\boldsymbol{t = \frac{1}{2} \pm \frac{\sqrt{3}}{2} i} \tag{2-5}$$
$$\tag{2-6}$$

With $\boldsymbol{t_0 = \frac{1}{2} + \frac{\sqrt{3}}{2} i = cis(\frac{\pi}{3})}$:

$$\boldsymbol{\sqrt{cis(\frac{\pi}{3})}^4 - \sqrt{cis(\frac{\pi}{3})}^2 + 1 = 0} \tag{2-7}$$
$$\boldsymbol{cis^2(\frac{\pi}{3}) - cis(\frac{\pi}{3}) + 1 = 0} \tag{2-8}$$
$$\tag{2-9}$$

With De Moivres formula:

$$\boldsymbol{cis(\frac{2\pi}{3}) - cis(\frac{\pi}{3}) + 1 = 0} \tag{2-10}$$
$$\boldsymbol{cos(\frac{2\pi}{3}) + i \cdot sin(\frac{2\pi}{3}) - cos(\frac{\pi}{3}) - i \cdot sin(\frac{\pi}{3}) + 1 = 0} \tag{2-11}$$
$$\boldsymbol{-\frac{1}{2} + \frac{\sqrt{3}}{2} i - \frac{1}{2} - \frac{\sqrt{3}}{2} i + 1 = 0} \tag{2-12}$$
$$\boldsymbol{-\frac{1}{2} - \frac{1}{2} + 1 = 0} \tag{2-13}$$
$$\boldsymbol{-1 + 1 = 0} \tag{2-14}$$
$$\tag{2-15}$$

Therefore, $\boldsymbol{z = t_0^{\frac{1}{2}} = cis^{\frac{1}{2}}(\frac{\pi}{3}) = cis(\frac{\pi}{6})}$ is a root $\boldsymbol{z_0}$.
By the complex conjugate root theorem, $\boldsymbol{z = t_1^{\frac{1}{2}} = cis^{\frac{1}{2}}(-\frac{\pi}{3}) = cis(-\frac{\pi}{6})}$ is a root $\boldsymbol{z_0'}$.

$$(z^4 - z^2 + 1) \quad = \quad 0 \tag{2-16}$$

$$\frac{z^4 - z^2 + 1}{(z - \frac{\sqrt{3}}{2} - \frac{1}{2}i) \cdot (z - \frac{\sqrt{3}}{2} + \frac{1}{2}i)} \quad = \quad 0 \tag{2-17}$$

$$\frac{z^4 - z^2 + 1}{(z - \frac{\sqrt{3}}{2})^2 - (\frac{1}{2}i)^2} \quad = \quad 0 \tag{2-18}$$

$$\frac{z^4 - z^2 + 1}{z^2 - \sqrt{3}z + \frac{3}{4} + \frac{1}{4}} \quad = \quad 0 \tag{2-19}$$

$$\frac{z^4 - z^2 + 1}{z^2 - \sqrt{3}z + 1} \quad = \quad 0 \tag{2-20}$$

$$z^2 + \sqrt{3}z + 1 \quad = \quad 0 \tag{2-21}$$

$$\therefore (z^4 - z^2 + 1) \quad = \quad 0 \equiv \tag{2-22}$$

$$(z^2 + \sqrt{3}z + 1) \cdot (z - cis(\frac{\pi}{6})) \cdot (z - cis(\frac{-\pi}{6})) \quad = \quad 0 \tag{2-23}$$

$$a \quad = \quad 1, b = \sqrt{3}, c = 1 \tag{2-24}$$

$$z \quad = \quad \frac{-\sqrt{3} \pm \sqrt{3 - 4}}{2} \tag{2-25}$$

$$z \quad = \quad \frac{-\sqrt{3}}{2} \pm \frac{i}{2} = cis(\pi \pm \frac{\pi}{6}) \tag{2-26}$$

$$z_1 \quad = \quad cis(\frac{5\pi}{6}), \tag{2-27}$$

$$z_1' \quad = \quad cis(-\frac{5\pi}{6}) \tag{2-28}$$

By having $z_0, z_0', z_1,$ and z_1', we can factorize as $(z - z_0) \cdot (z - z_0') \cdot (z - z_1) \cdot (z - z_1') = 0$, so $z^4 - z^2 + 1 = 0$ is equivalent to $(z - cis(\frac{\pi}{6})) \cdot (z - cis(-\frac{\pi}{6})) \cdot (z - cis(\frac{5\pi}{6})) \cdot (z - cis(-\frac{5\pi}{6})) = 0$.

By 2-20, we know $(z - z_1)(z - z_1') = z^2 - \sqrt{3}z + 1$,

and by 2-21 we know $(z - z_0)(z - z_0') = z^2 + \sqrt{3}z + 1$

so we could express $z^4 - z^2 + 1 = 0$ the by second degree polynoms as $(z^2 - \sqrt{3}z + 1)(z^2 + \sqrt{3}z + 1)$.

□

> *Finn grensene* $\lim_{n\to\infty} \frac{3n+2}{\sqrt{4n^2-1}}$ *og* $\lim_{n\to\infty}(\sqrt{n^2-5n}-n)$.

$$\lim_{n\to\infty} \frac{3n+2}{\sqrt{4n^2-1}} =$$

$$\lim_{n\to\infty} \frac{(3n+2)\frac{1}{n}}{\sqrt{4n^2-1}\frac{1}{n}} =$$

$$\lim_{n\to\infty} \frac{3+\frac{2}{n}}{\sqrt{\frac{4n^2-1}{n^2}}} =$$

$$\lim_{n\to\infty} \frac{3+\frac{2}{n}}{\sqrt{4-\frac{1}{n^2}}} =$$

$$\frac{3+\cancel{0}}{\sqrt{4-\cancel{0}}} =$$

$$\frac{3}{2}$$

$\square$

$$\lim_{n\to\infty} (\sqrt{n^2-5n}-n)$$

$$\lim_{n\to\infty} \frac{(\sqrt{n^2-5n}-n)(\sqrt{n^2-5n}+n)}{\sqrt{n^2-5n}+n}$$

$$\lim_{n\to\infty} \frac{n^2-5n-n^2}{\sqrt{n^2-5n}+n}$$

$$\lim_{n\to\infty} \frac{-5n}{\sqrt{n^2-5n}+n}$$

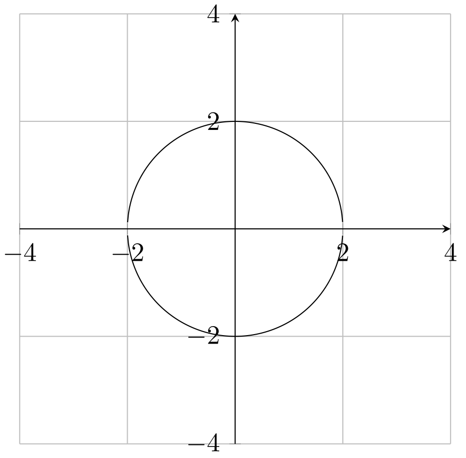
$$\lim_{n\to\infty} \frac{-5}{\sqrt{1-\frac{5}{n}}+1}$$

$$\frac{-5}{\sqrt{1-\cancel{0}}+1}$$

$$\frac{-5}{2}$$

$\square$

“ *Finn de komplekse tallene z som oppfyller likningen $2|z-1| = |z-4|$ og skisser løsningsmengden i det komplekse planet.(Hint: Sett inn $z = x+iy$ og finn en polynomlikning ix og y forløsningsmengden.)* ”



$$2|z-1| = |z-4| \equiv$$

$$4|x+iy-1| = |x+iy-4|$$

$$4((x-1)^2 + (y)^2) = (x-4)^2 + (y)^2$$

$$4(x-1)^2 + 4y^2 = x^2 - 8x + 16 + y^2$$

$$4(x^2 - 2x + 1) + 4y^2 = x^2 - 8x + 16 + y^2$$

$$4x^2 - 8x + 4 + 4y^2 = x^2 - 8x + 16 + y^2$$

$$3x^2 + 3y^2 = 12$$

$$y^2 = 4 - x^2$$

$$y = \pm\sqrt{4 - x^2}$$

$$\therefore Im(z) = \pm\sqrt{4 - Re(z)^2}, \exists z \forall \{x \in Re(z) | -2 < x < 2\}$$

$\square$

“ *En følge $\{a_n\}$ er definert ved $a_1 = 3, a_n{+}1 = 3\sqrt{a_n}$ for $n \geq 1$. Vis at $a_n < 9$ og at $a_n{+}1 > a_n$* ” *for alle n. Forklar hvorfor følgen konvergerer og finn $\lim_{n\to\infty} a_n$*

$\square$.

Submitted by Rolf Vidar Hoksaas *on September 13, 2019.*