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OBLIG 1 — Obligatorisk oppgave 1 av 2

> *"Skriv det komplekse tallet $z = \frac{6}{\sqrt{3}+3i}$ først på formen $a + ib$ også på polarformen $re^{i\theta}$."*

$$z = \frac{6}{\sqrt{3}+3i}$$

$$z = \frac{2}{\sqrt{\frac{1}{3}}+i}$$

$$z = \frac{2 \times (\sqrt{\frac{1}{3}} - i)}{(\sqrt{\frac{1}{3}}+i) \times (\sqrt{\frac{1}{3}} - i)}$$

$$z = \frac{2 \times (\sqrt{\frac{1}{3}} - i)}{\frac{1}{3} - i^2}$$

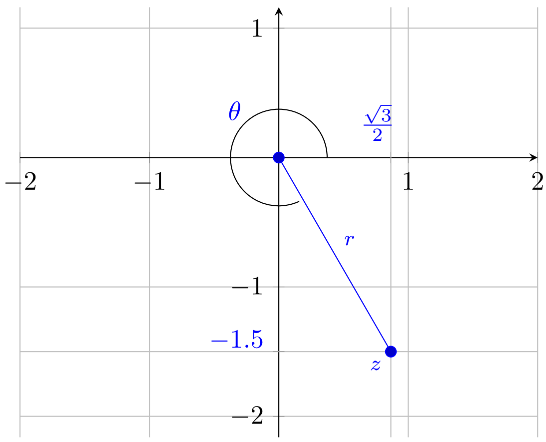
$$z = \frac{2 \times (\sqrt{\frac{1}{3}} - i)}{\frac{4}{3}}$$

$$z = \frac{3 \times (\sqrt{\frac{1}{3}} - i)}{2}$$

$$z = \frac{\sqrt{3}}{2} - \frac{3}{2}i$$

□

Using $z = \frac{\sqrt{3}}{2} - \frac{3}{2}i$:



With Pythagora's Theorem

$$r = \sqrt{(\frac{\sqrt{3}}{2})^2 + (\frac{3}{2})^2}$$

$$r = \sqrt{\frac{3}{4} + \frac{9}{4}}$$

$$r = \sqrt{3}$$

■

By the laws of trigonometry

$$\theta' := 2\pi - \theta$$

$$\sin(\theta') = \frac{-1.5}{\sqrt{3}}$$

$$\theta' = \arcsin(\frac{-1.5\sqrt{3}}{3})$$

$$\theta' = \arcsin(\frac{-\sqrt{3}}{2})$$

$$\theta' = -\frac{\pi}{3} \equiv \frac{5\pi}{3}$$

$$\therefore \theta = \frac{\pi}{3}$$

■

And since z is in the fourth quadrant, as $Re(z) > 0 \wedge Im(z) < 0$ we know to use θ':

$$r : \sqrt{3} \wedge \theta : \frac{5\pi}{3} \therefore z = \sqrt{3}e^{\frac{-i\pi}{3}} \equiv \sqrt{3}e^{\frac{5i\pi}{3}}$$

□

“ *Finn de to løsningene til likningen $w^2 - w + 1 = 0$, og bruk disse til å finne alle komplekse løsninger til likningen $z^4 - z^2 + 1 = 0$. Gi en faktorisering av $z^4 - z^2 + 1$, først i komplekse førstegradspolynomer og så i reelle andregradspolynomer.* ”

Assuming $cis(x) \equiv cos(x) + i \cdot sin(x)$, and $i^2 = -1$.

$$\boldsymbol{a = 1, b = -1, c = 1} \tag{2-1}$$
$$\boldsymbol{w = \frac{1 \pm \sqrt{1-4}}{2} = \frac{1}{2} \pm \frac{\sqrt{3}}{2} i} \tag{2-2}$$
$$\tag{2-3}$$

With $\boldsymbol{t = z^2}$:

$$\boldsymbol{z^4 - z^2 + 1 = t^2 - t^2 + 1 = 0} \tag{2-4}$$
$$\boldsymbol{t = \frac{1}{2} \pm \frac{\sqrt{3}}{2} i} \tag{2-5}$$
$$\tag{2-6}$$

With $\boldsymbol{t_0 = \frac{1}{2} + \frac{\sqrt{3}}{2} i = cis(\frac{\pi}{3})}$:

$$\boldsymbol{\sqrt{cis(\frac{\pi}{3})}^4 - \sqrt{cis(\frac{\pi}{3})}^2 + 1 = 0} \tag{2-7}$$
$$\boldsymbol{cis^2(\frac{\pi}{3}) - cis(\frac{\pi}{3}) + 1 = 0} \tag{2-8}$$
$$\tag{2-9}$$

With De Moivres formula:

$$\boldsymbol{cis(\frac{2\pi}{3}) - cis(\frac{\pi}{3}) + 1 = 0} \tag{2-10}$$
$$\boldsymbol{cos(\frac{2\pi}{3}) + i \cdot sin(\frac{2\pi}{3}) - cos(\frac{\pi}{3}) - i \cdot sin(\frac{\pi}{3}) + 1 = 0} \tag{2-11}$$
$$\boldsymbol{-\frac{1}{2} + \frac{\sqrt{3}}{2} i - \frac{1}{2} - \frac{\sqrt{3}}{2} i + 1 = 0} \tag{2-12}$$
$$\boldsymbol{-\frac{1}{2} - \frac{1}{2} + 1 = 0} \tag{2-13}$$
$$\boldsymbol{-1 + 1 = 0} \tag{2-14}$$
$$\tag{2-15}$$

Therefore, $\boldsymbol{z = t_0^{\frac{1}{2}} = cis^{\frac{1}{2}}(\frac{\pi}{3}) = cis(\frac{\pi}{6})}$ is a root $\boldsymbol{z_0}$.
By the complex conjugate root theorem, $\boldsymbol{z = t_1^{\frac{1}{2}} = cis^{\frac{1}{2}}(-\frac{\pi}{3}) = cis(-\frac{\pi}{6})}$ is a root $\boldsymbol{z_0'}$.

□

“ *Finn grensene* $\lim_{n\to\infty} \frac{3n+2}{\sqrt{4n^2-1}}$ *og* $\lim_{n\to\infty}(\sqrt{n^2-5n}-n)$. ”

$$\lim_{n\to\infty} \frac{3n+2}{\sqrt{4n^2-1}} =$$

$$\lim_{n\to\infty} \frac{(3n+2)\frac{1}{n}}{\sqrt{4n^2-1}\frac{1}{n}} =$$

$$\lim_{n\to\infty} \frac{3+\frac{2}{n}}{\sqrt{\frac{4n^2-1}{n^2}}} =$$

$$\lim_{n\to\infty} \frac{3+\frac{2}{n}}{\sqrt{4-\frac{1}{n^2}}} =$$

$$\frac{3+\cancel{0}}{\sqrt{4-\cancel{0}}} =$$

$$\frac{3}{2}$$

$\square$

$$\lim_{n\to\infty} (\sqrt{n^2-5n}-n)$$

$$\lim_{n\to\infty} \frac{(\sqrt{n^2-5n}-n)(\sqrt{n^2-5n}+n)}{\sqrt{n^2-5n}+n}$$

$$\lim_{n\to\infty} \frac{n^2-5n-n^2}{\sqrt{n^2-5n}+n}$$

$$\lim_{n\to\infty} \frac{-5n}{\sqrt{n^2-5n}+n}$$

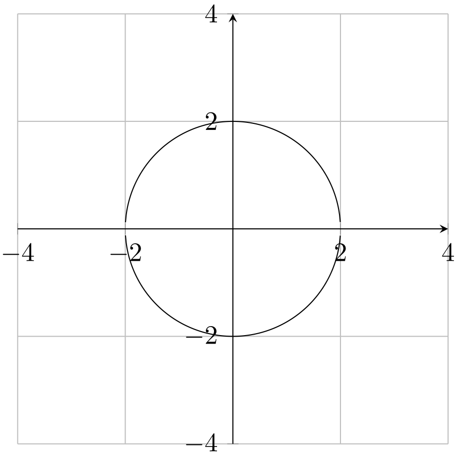
$$\lim_{n\to\infty} \frac{-5}{\sqrt{1-\frac{5}{n}}+1}$$

$$\frac{-5}{\sqrt{1-\cancel{0}}+1}$$

$$\frac{-5}{2}$$

$\square$

“Finn de komplekse tallene z som oppfyller likningen $2|z-1| = |z-4|$ og skisser løsningsmengden i det komplekse planet.(Hint: Sett inn $z = x+iy$ og finn en polynomlikning ix og y forløsningsmengden.)”



$$\begin{aligned}
\mathbf{2}|z-\mathbf{1}| &= |z-\mathbf{4}| \equiv \\
\mathbf{4}|x+iy-\mathbf{1}| &= |x+iy-\mathbf{4}| \\
\mathbf{4}((x-\mathbf{1})^{\mathbf{2}}+(y)^{\mathbf{2}}) &= (x-\mathbf{4})^{\mathbf{2}}+(y)^{\mathbf{2}} \\
\mathbf{4}(x-\mathbf{1})^{\mathbf{2}}+\mathbf{4}y^{\mathbf{2}} &= x^{\mathbf{2}}-\mathbf{8}x+\mathbf{16}+y^{\mathbf{2}} \\
\mathbf{4}(x^{\mathbf{2}}-\mathbf{2}x+\mathbf{1})+\mathbf{4}y^{\mathbf{2}} &= x^{\mathbf{2}}-\mathbf{8}x+\mathbf{16}+y^{\mathbf{2}} \\
\mathbf{4}x^{\mathbf{2}}-\mathbf{8}x+\mathbf{4}+\mathbf{4}y^{\mathbf{2}} &= x^{\mathbf{2}}-\mathbf{8}x+\mathbf{16}+y^{\mathbf{2}} \\
\mathbf{3}x^{\mathbf{2}}+\mathbf{3}y^{\mathbf{2}} &= \mathbf{12} \\
y^{\mathbf{2}} &= \mathbf{4}-x^{\mathbf{2}} \\
y &= \sqrt{\mathbf{4}-x^{\mathbf{2}}}
\end{aligned}$$

$$\therefore Im(z) = \pm\sqrt{\mathbf{4}-Re(z)^{\mathbf{2}}}, \exists z \forall \{x \in Re(z) | -\mathbf{2} < x < \mathbf{2}\}$$

$\square$

> *En følge $\{a_n\}$ er definert ved $a_1 = 3, a_{n+1} = 3\sqrt{a_n}$ for $n \geq 1$. Vis at $a_n < 9$ og at $a_{n+1} > a_n$ for alle n. Forklar hvorfor følgen konvergerer og finn $\lim_{n\to\infty} a_n$*

$$\begin{aligned} \boldsymbol{a_1} &:= \mathbf{3}, && \text{(5-1)} \\ \boldsymbol{a_{n+1}} &:= \mathbf{3}\sqrt{\boldsymbol{a_n}} && \text{(5-2)} \end{aligned}$$

To be proven by induction:

$$\begin{aligned} \boldsymbol{P_n} &:= \boldsymbol{a_n < 9} && \text{(5-3)} \\ \boldsymbol{Q_n} &:= \boldsymbol{a_{n+1} > a_n} && \text{(5-4)} \end{aligned}$$

Prove for n=1:

$$\begin{aligned} \boldsymbol{a_1} &= \boldsymbol{3 \therefore a_2 = 3\sqrt{3}} && \text{(5-5)} \\ \boldsymbol{P_1} &:= \boldsymbol{a_1 < 9 \equiv 3 < 9 \therefore P_1} && \text{(5-6)} \\ \boldsymbol{Q_1} &:= \boldsymbol{a_2 > a_1 \equiv 3\sqrt{3} > 3 \therefore Q_1} && \text{(5-7)} \end{aligned}$$

Assume $\boldsymbol{Q_k}$, and prove $(\boldsymbol{Q_k \implies Q_{k+1}})$:

$$\begin{aligned} \boldsymbol{Q_k} &:= \boldsymbol{a_{k+1} > a_k} && \text{(5-8)} \\ \boldsymbol{Q_{k+1}} &:= \boldsymbol{a_{k+2} > 3\sqrt{a_k}} && \text{(5-9)} \\ \boldsymbol{Q_{k+1}} &:= \boldsymbol{3\sqrt{a_{k+1}} > 3\sqrt{a_k}} && \text{(5-10)} \\ \boldsymbol{Q_{k+1}} &:= \boldsymbol{\sqrt{a_{k+1}} > \sqrt{a_k}} && \text{(5-11)} \\ \boldsymbol{Q_{k+1}} &:= \boldsymbol{a_{k+1} > a_k} && \text{(5-12)} \end{aligned}$$

$\square$

Assume $\boldsymbol{P_{k+1}}$, and prove $(\boldsymbol{P_k \implies P_{k+1}})$

$$\begin{aligned} \boldsymbol{P_k} &:= \boldsymbol{a_k < 9} && \text{(5-13)} \\ \boldsymbol{P_{k+1}} &:= \boldsymbol{3\sqrt{a_k} < 9} && \text{(5-14)} \\ \boldsymbol{P_{k+1}} &:= \boldsymbol{\sqrt{a_k} < 3} && \text{(5-15)} \\ \boldsymbol{P_{k+1}} &:= \boldsymbol{a_k < 9} && \text{(5-16)} \end{aligned}$$

$\square$

The sequence converges since all values increment as $\boldsymbol{n}$ increments, but is bounded by $\mathbf{9}$. We also know by this that the sequence's infinum is 3, as $\boldsymbol{a_1 = 3}$ $\square$

Since the sequence is convergent, we know its limit exsits:

$$\boldsymbol{L} = \lim_{n\to\infty} \boldsymbol{a_n} \tag{5-17}$$

$$\boldsymbol{L} = \lim_{n\to\infty} 3\sqrt{\boldsymbol{a_{n-1}}} \tag{5-18}$$

Since all $\boldsymbol{a_n > 1}$:

$$\boldsymbol{L} = 3\sqrt{\lim_{n\to\infty} \boldsymbol{a_{n-1}}} \tag{5-19}$$

By the monotone convergence theorem:

$$\boldsymbol{L} = 3\sqrt{\lim_{n\to\infty} \boldsymbol{a_n}} \tag{5-20}$$

$$\boldsymbol{L} = 3\sqrt{\boldsymbol{L}} \tag{5-21}$$

$$\boldsymbol{L^2} = 9\boldsymbol{L} \tag{5-22}$$

$$\boldsymbol{L} = 9 \tag{5-23}$$

Thus we know the supremum of the sequence is $\mathbf{9}$. $\square$.

Submitted by Rolf Vidar Hoksaas *on September 13, 2019.*