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ASSIGNMENT 2

> "*Regn ut flateintegralet*
> $$\iint_T x dS$$
> *der T er delen av sfæren $x^2+y^2+z^2=a^2$ som ligger i første oktant $x \geq 0, y \geq 0, z \geq 0$.*"

$$\iint_T f dS = \iint_A f(\vec{r}(\theta,\varphi))|(\frac{\partial \vec{r}}{\partial \theta} \times \frac{\partial \vec{r}}{\partial \varphi})(\theta,\varphi)| d\theta d\varphi$$

$$\vec{r}(u,v) = a \cdot \sin(\varphi)\cos(\theta)\hat{\imath} + a \cdot \sin(\varphi)\sin(\theta)\hat{\jmath} + a \cdot \cos(\varphi)\hat{\mathrm{k}}$$

$$\therefore \iint_T x dS = \iint_A (a \cdot \sin(\varphi)\cos(\theta))|(\frac{\partial \vec{r}(\theta,\varphi)}{\partial \theta} \times \frac{\partial \vec{r}(\theta,\varphi)}{\partial \varphi})| d\theta d\varphi$$

$$= \iint_A (a \cdot \sin(\varphi)\cos(\theta))| \begin{pmatrix} -a\sin(\varphi)\sin(\theta) \\ a\sin(\varphi)\cos(\theta) \\ 0 \end{pmatrix} \times \begin{pmatrix} a\cos(\varphi)\cos(\theta) \\ a\cos(\varphi)\sin(\theta) \\ -a\sin(\varphi) \end{pmatrix} | d\theta d\varphi$$

$$= \iint_A (a \cdot \sin(\varphi)\cos(\theta))| \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{\mathrm{k}} \\ -a\sin(\varphi)\sin(\theta) & a\sin(\varphi)\cos(\theta) & 0 \\ a\cos(\varphi)\cos(\theta) & a\cos(\varphi)\sin(\theta) & -a\sin(\varphi) \end{vmatrix} | d\theta d\varphi$$

$$= \iint_A (a \cdot \sin(\varphi)\cos(\theta))|\hat{\mathrm{k}} \begin{vmatrix} -a\sin(\varphi)\sin(\theta) & a\sin(\varphi)\cos(\theta) \\ a\cos(\varphi)\cos(\theta) & a\cos(\varphi)\sin(\theta) \end{vmatrix} - a\sin(\varphi) \begin{vmatrix} \hat{\imath} & \hat{\jmath} \\ -a\sin(\varphi)\sin(\theta) & a\sin(\varphi)\cos(\theta) \end{vmatrix} | d\theta d\varphi$$

$$= \iint_A (a \cdot \sin(\varphi)\cos(\theta))|\hat{\mathrm{k}}(-a^2\sin\varphi\cos\varphi\sin^2\theta - a^2\sin\varphi\cos^2\theta\cos\varphi)$$
$$- a\sin\varphi(\hat{\imath}a\sin\varphi\cos\theta + \hat{\jmath}a\sin\varphi\sin\theta)| d\theta d\varphi$$

$$= \iint_A (a \cdot \sin(\varphi)\cos(\theta))| \begin{pmatrix} -a^2\sin^2\varphi\cos\theta \\ -a^2\sin^2\varphi\sin\theta \\ -a^2\sin\varphi\cos\varphi \end{pmatrix} | d\theta d\varphi$$

$$= \iint_A (a \cdot \sin(\varphi)\cos(\theta))\sqrt{(-a^2 \sin^2 \varphi \cos\theta)^2 + (-a^2 \sin^2 \varphi \sin\theta)^2 + (-a^2 \sin\varphi \cos\varphi)^2} d\theta d\varphi$$

$$= \iint_A (a \cdot \sin(\varphi)\cos(\theta))\sqrt{a^4 \sin^4 \varphi \cos^2\theta + a^4 \sin^4 \varphi \sin^2\theta + a^4 \sin^2 \varphi \cos^2 \varphi} d\theta d\varphi$$

$$\begin{aligned}
&= \iint_A (a \cdot \sin(\varphi)\cos(\theta))\sqrt{a^4 \sin^2 \varphi(\sin^2\varphi + \cos^2\varphi)} \\
&= \iint_A (a \cdot \sin(\varphi)\cos(\theta))\sqrt{a^4 \sin^2(\varphi)} d\theta d\varphi \\
&= \iint_A a^3 \cdot \sin^2(\varphi)\cos(\theta) d\theta d\varphi \\
&= \int_0^{\tau/4} \left[\int_0^{\tau/4} a^3 \cdot \sin^2(\varphi)\cos(\theta) d\varphi\right] d\theta \\
&= a^3 \int_0^{\tau/4} \cos(\theta) \left[\int_0^{\tau/4} \sin^2(\varphi) d\varphi\right] d\theta \\
&= a^3 \int_0^{\tau/4} \cos(\theta) \left[\frac{\varphi - \sin(\varphi)\cos(\varphi)}{2}\right]_0^{\tau/4} d\theta \\
&= a^3 \int_0^{\tau/4} \cos(\theta) \frac{\tau/4 - \sin(\tau/4)\cos(\tau/4)}{2} d\theta \\
&= a^3 \int_0^{\tau/4} \cos(\theta) \frac{\tau}{8} d\theta = \frac{\tau a^3}{8} \left[\sin(\theta)\right]_0^{\tau/4}
\end{aligned}$$

$$= \frac{\tau a^3}{8}$$

Overflatearealet til planet i kule-domenet er altså $a^3\tau/8$, der a er radiusen til kula vår.

□

“
En dobbelt deriverbar funksjon $f : \mathbb{R}^2 \to \mathbb{R}$ kalles harmonisk dersom den tilfredstiller Laplace-likningen

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$

.

- *Vis at funskjonene $x^4 - 6x^2y^2 + y^4$ og $e^x sin(y)$ er harmoniske.*
- *Vis at dersom f er harmonisk, så er*

$$\int_C \left(\frac{\partial f}{\partial y}dx - \frac{\partial f}{\partial x}dy\right) = 0$$

for alle enkle, lukkede, glatte kurver C i planet.

”

$$\begin{aligned}
\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} &= 0 \\
\frac{\partial^2 (x^4 - 6x^2y^2 + y^4)}{\partial x^2} + \frac{\partial^2 (x^4 - 6x^2y^2 + y^4)}{\partial y^2} &= 0 \\
\frac{\partial (4x^3 - 12xy^2)}{\partial x} + \frac{\partial (-12x^2y + 4y^3)}{\partial y} &= 0 \\
(12x^2 - 12y^2) + (-12x^2 + 12y^2) &= 0 \\
(x^2 - y^2) + (-x^2 + y^2) &= 0 \\
0 &= 0
\end{aligned}$$

□

$$\begin{aligned}
\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} &= 0 \\
\frac{\partial^2 (e^x sin(y))}{\partial x^2} + \frac{\partial^2 (e^x sin(y))}{\partial y^2} &= 0 \\
\frac{\partial (e^x sin(y))}{\partial x} + \frac{\partial (e^x cos(y))}{\partial y} &= 0 \\
(e^x sin(y)) + (-e^x sin(y)) &= 0 \\
0 &= 0
\end{aligned}$$

Begge funskjonene er dermed harmoniske.

□

Green's teorem sier at $\int_C \boldsymbol{Pdx+Qdy} = \iint_R (\frac{\partial \boldsymbol{Q}}{\partial \boldsymbol{x}} - \frac{\partial \boldsymbol{P}}{\partial \boldsymbol{y}})\boldsymbol{dxdy}$ er gyldig for alle enkle, lukkede og glatte kurver C.

Ved å erstatte $\boldsymbol{P} = \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{y}}$ og $\boldsymbol{Q} = \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}$, og ved å bruke definisjonen til harmonisk funskjon har vi:

$$\int_C \boldsymbol{Pdx + Qdy} = \iint_R (\frac{\partial \boldsymbol{Q}}{\partial \boldsymbol{x}} - \frac{\partial \boldsymbol{P}}{\partial \boldsymbol{y}})\boldsymbol{dxdy}$$

$$\int_C \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{y}}\boldsymbol{dx} + \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}\boldsymbol{dy} = \iint_R (\frac{\partial \boldsymbol{Q}}{\partial \boldsymbol{x}} - \frac{\partial \boldsymbol{P}}{\partial \boldsymbol{y}})\boldsymbol{dxdy}$$

$$\int_C \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{y}}\boldsymbol{dx} + \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}\boldsymbol{dy} = \iint_R (\frac{\partial \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}}{\partial \boldsymbol{x}} - \frac{\partial \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{y}}}{\partial \boldsymbol{y}})\boldsymbol{dxdy}$$

$$\int_C \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{y}}\boldsymbol{dx} + \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}\boldsymbol{dy} = \iint_R (\frac{\partial^2 \boldsymbol{f}}{\partial \boldsymbol{x}^2} - \frac{\partial^2 \boldsymbol{f}}{\partial \boldsymbol{y}^2})\boldsymbol{dxdy}$$

$$\int_C \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{y}}\boldsymbol{dx} + \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}\boldsymbol{dy} = \iint_R \boldsymbol{0dxdy}$$
$$\int_C \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{y}}\boldsymbol{dx} + \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}\boldsymbol{dy} = \boldsymbol{0}$$

Siden vi integrerer på et domene vet vi at svarer blir null uten noen konstanter, og har dermed bevist hypotesen.

$\square$

“ *La $A = [-1,1] \times [-1,1]$ og la $\vec{F} : A \to \mathbb{R}^2$ være gitt ved*

$$\vec{F}(x,y) = \frac{1}{5}(xy + y^2 - 1,\, x^3 - y^2 + 3).$$

- *Vis at $\vec{F}$ definerer en funskjon $\vec{F} : A \to A$ (altså at verdiene til $\vec{F}$ ligger i A).*
- *Vis at $\vec{F} : A \to A$ definerer en kontraksjon. (Hint: Setning 5.5.8 i boka (gjengitt nedenfor) kan bli nyttig her).*
- *Vis at følgenende likningsystem har en unik løsning for $-1 \leq x, y \leq 1$:*

$$\begin{aligned} 5x &= xy + y^2 - 1 \\ 5y &= x^3 - y^2 + 3. \end{aligned}$$

- *Lag et MATLAB/Python script som regner ut en approksimasjon til løsningen i c) ved hjelp av en iterasjon $x_{n+1} = \vec{F}(x_n)$. Programmet skal ta startpunkt x_0 og antall iterasjoner som input. Legg ved et plott av følgende du får med startpunkt i $(0,0)$ og $(-1,1)$.*

”

For at $\vec{F} : A \to A$ skal være sant må $f_x(x,y) = \frac{1}{5}(xy + y^2), \{x \in [-1,1] \wedge y \in [-1,1]\}$ gi et tall mellom -1 og 1. Det samme gjelder for $f_y(x,y) = \frac{1}{5}(x^3 - y^2 + 3)$.

Det er trivielt at de høyeste verdiene av $|F_x|$ må være en del av x, y-parene $(1,-1), (1,1), (-1,-1), (-1,1)$, som gir $F_x = \frac{1}{5}\{-1,1,1,-1\}$. Disse verdiene ligger under domenet A_x. Vi har dermed bevist at $F_x : A_x \to A_x$.

På samme måte ser vi at $|F_y|$ har samme patent. Med samme verdiene får vi $F_y = \frac{1}{5}\{3,3,1,1\}$. Siden vi nå vet at både F_x og F_y er en del av A, vet vi at $F : A \to A$ er gyldig for alle verdiene i F.

$\square$

$$\vec{F}(x,y) = \frac{1}{5}\begin{pmatrix} xy + y^2 - 1 \\ x^3 - y^2 + 3 \end{pmatrix} \tag{3-1}$$

$$\sqrt{|\nabla F_x(\vec{c_x})|^2 + |\nabla F_y(\vec{c_y})|^2} \leq C \tag{3-2}$$

$$J = \begin{bmatrix} \frac{\partial F_x}{\partial x} & \frac{\partial F_x}{\partial y} \\ \frac{\partial F_y}{\partial x} & \frac{\partial F_y}{\partial y} \end{bmatrix} = \begin{bmatrix} \frac{y}{5} & \frac{x+2y}{5} \\ \frac{3x^2}{5} & \frac{-2y}{5} \end{bmatrix} \tag{3-3}$$

$$\sqrt{(\frac{y}{5})^2 + (\frac{x+2y}{5})^2 + (\frac{3x^2}{5})^2 + (\frac{-2y}{5})^2} \leq C \tag{3-4}$$

$$\sqrt{\frac{y^2 + (x+2y)^2 + 3x^4 + 4y^2}{25}} \leq C \tag{3-5}$$

$$\frac{1}{5}\sqrt{y^2 + (x+2y)^2 + 3x^4 + 4y^2} \leq C \tag{3-6}$$

$$\tag{3-7}$$

Siden både $|x|$ og $|y|$ er mindre eller lik 1, vet vi den høyeste mulige verdien til kvadratroten:

$$\frac{1}{5}\sqrt{y^2 + (x+2y)^2 + 3x^4 + 4y^2} \leq C$$
$$\frac{1}{5}\sqrt{1^2 + (1+2)^2 + 3(1)^4 + 4(1)^2} \leq C$$
$$\frac{1}{5}\sqrt{1+9+3+4} \leq C$$
$$\frac{\sqrt{17}}{5} \leq C$$

Siden $\sqrt{17}/5$ er mindre enn $\mathbf{1}$ er funskjonen en kontraksjon.

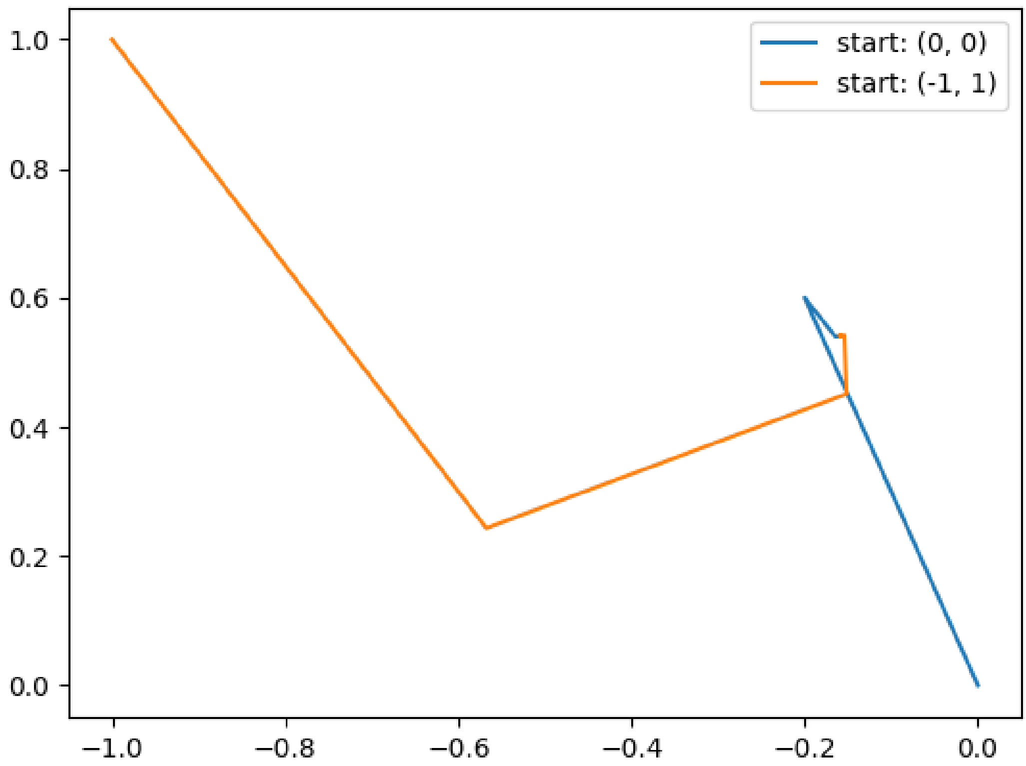
$\square$

$$\vec{F}(x,y) = \begin{Bmatrix} 5x = xy + y^2 - 1 \\ 5y = x^3 - y^2 + 3 \end{Bmatrix} = \begin{pmatrix} 5x - xy - y^2 + 1 \\ 5y - x^3 + y^2 - 3 \end{pmatrix}$$

Siden funskjonen har en $C < 1$ og funskjonen er en avbilding av seg selv ($A \to A$) vet vi at funskjonen konvergerer mot et viss punkt i $[-1, 1]$. På grunn av dette vet vi at det eksisterer et unikt punkt (x, y) som er gyldig for likningsystemet.

$\square$

Som vi ser har det ikke noe å si hvor vi starter, siden funskjonen konvergerer mot $(-0.1586434, 0.54072429)$ i begge iterasjonene.



$\square$

Submitted by Rolf Vidar Mazunki Hoksaas *on April 16, 2020.*