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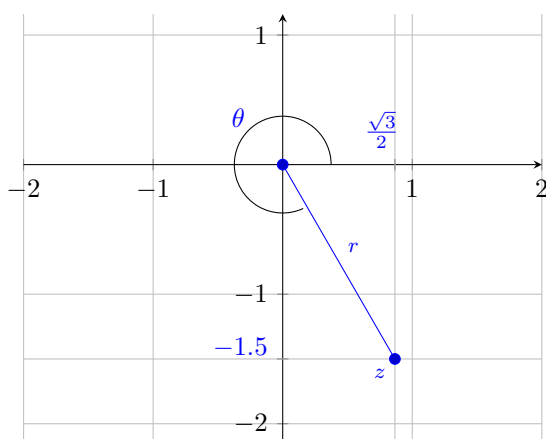
September 13, 2019

OBLIG 1 — Obligatorisk oppgave 1 av 2

> *Skriv det komplekse tallet $z = \frac{6}{\sqrt{3}+3i}$ først på formen $a + ib$ også på polarformen $re^{i\theta}$.*

$$z = \frac{6}{\sqrt{3} + 3i}$$

$$z = \frac{2}{\sqrt{\frac{1}{3}} + i}$$

$$z = \frac{2 \times (\sqrt{\frac{1}{3}} - i)}{(\sqrt{\frac{1}{3}} + i) \times (\sqrt{\frac{1}{3}} - i)}$$

$$z = \frac{2 \times (\sqrt{\frac{1}{3}} - i)}{\frac{1}{3} - i^2}$$

$$z = \frac{2 \times (\sqrt{\frac{1}{3}} - i)}{\frac{4}{3}}$$

$$z = \frac{3 \times (\sqrt{\frac{1}{3}} - i)}{2}$$

$$z = \frac{\sqrt{3}}{2} - \frac{3}{2}i$$

$\square$

With Pythagora's Theorem

$$r = \sqrt{(\frac{\sqrt{3}}{2})^2 + (\frac{3}{2})^2}$$

$$r = \sqrt{\frac{3}{4} + \frac{9}{4}}$$

$$r = \sqrt{3}$$

∎

By the laws of trigonometry

$$\theta' := 2\pi - \theta$$

$$\sin(\theta') = \frac{-1.5}{\sqrt{3}}$$

$$\theta' = \arcsin(\frac{-1.5\sqrt{3}}{3})$$

$$\theta' = \arcsin(\frac{-\sqrt{3}}{2})$$

$$\theta' = -\frac{\pi}{3} \equiv \frac{5\pi}{3}$$

$$\therefore \theta = \frac{\pi}{3}$$

Using $z = \frac{\sqrt{3}}{2} - \frac{3}{2}i$:

∎

And since z is in the fourth quadrant, as $Re(z) > 0 \wedge Im(z) < 0$ we know to use θ':

$$r : \sqrt{3} \wedge \theta : \frac{5\pi}{3} \therefore z = \sqrt{3}e^{\frac{-i\pi}{3}} \equiv \sqrt{3}e^{\frac{5i\pi}{3}}$$

$\square$

> *Finn de to løsningene til likningen $w^2 - w + 1 = 0$, og bruk disse til å finne alle komplekse løsninger til likningen $z^4 - z^2 + 1 = 0$. Gi en faktorisering av $z^4 - z^2 + 1$, først i komplekse førstegradspolynomer og så i reelle andregradspolynomer.*

Assuming $cis(x) \equiv cos(x) + i \cdot sin(x)$, and $i^2 = -1$.

$$a = 1, b = -1, c = 1 \tag{2-1}$$
$$w = \frac{1 \pm \sqrt{1-4}}{2} = \frac{1}{2} \pm \frac{\sqrt{3}}{2}i \tag{2-2}$$
$$\tag{2-3}$$

With $t = z^2$:

$$z^4 - z^2 + 1 = t^2 - t^2 + 1 = 0 \tag{2-4}$$
$$t = \frac{1}{2} \pm \frac{\sqrt{3}}{2}i \tag{2-5}$$
$$\tag{2-6}$$

With $t_0 = \frac{1}{2} + \frac{\sqrt{3}}{2}i = cis(\frac{\pi}{3})$:

$$\sqrt{cis(\tfrac{\pi}{3})}^4 - \sqrt{cis(\tfrac{\pi}{3})}^2 + 1 = 0 \tag{2-7}$$
$$cis^2(\tfrac{\pi}{3}) - cis(\tfrac{\pi}{3}) + 1 = 0 \tag{2-8}$$
$$\tag{2-9}$$

With De Moivres formula:

$$cis(\tfrac{2\pi}{3}) - cis(\tfrac{\pi}{3}) + 1 = 0 \tag{2-10}$$
$$cos(\tfrac{2\pi}{3}) + i \cdot sin(\tfrac{2\pi}{3}) - cos(\tfrac{\pi}{3}) - i \cdot sin(\tfrac{\pi}{3}) + 1 = 0 \tag{2-11}$$
$$-\tfrac{1}{2} + \tfrac{\sqrt{3}}{2}i - \tfrac{1}{2} - \tfrac{\sqrt{3}}{2}i + 1 = 0 \tag{2-12}$$
$$-\tfrac{1}{2} - \tfrac{1}{2} + 1 = 0 \tag{2-13}$$
$$-1 + 1 = 0 \tag{2-14}$$
$$\tag{2-15}$$

Therefore, $z = t_0^{\frac{1}{2}} = cis^{\frac{1}{2}}(\frac{\pi}{3}) = cis(\frac{\pi}{6})$ is a root z_0.

By the complex conjugate root theorem, $z = t_1^{\frac{1}{2}} = cis^{\frac{1}{2}}(-\frac{\pi}{3}) = cis(-\frac{\pi}{6})$ is a root z_0'.

■

By dividing by the original polynom by $(z - z_0)(z - z_0')$:

$$\frac{z^4 - z^2 + 1}{(z - \frac{\sqrt{3}}{2} - \frac{1}{2}i) \cdot (z - \frac{\sqrt{3}}{2} + \frac{1}{2}i)} = 0 \tag{2-16}$$

$$\frac{z^4 - z^2 + 1}{(z - \frac{\sqrt{3}}{2})^2 - (\frac{1}{2}i)^2} = 0 \tag{2-17}$$

$$\frac{z^4 - z^2 + 1}{z^2 - \sqrt{3}z + \frac{3}{4} + \frac{1}{4}} = 0 \tag{2-18}$$

$$\frac{z^4 - z^2 + 1}{z^2 - \sqrt{3}z + 1} = 0 \tag{2-19}$$

By polynomial division: $\qquad z^2 + \sqrt{3}z + 1 = 0 \tag{2-20}$

Thus the original equation is equivalent to the following:

$$(z^2 + \sqrt{3}z + 1) \cdot (z - cis(\frac{\pi}{6})) \cdot (z - cis(\frac{-\pi}{6})) = 0 \tag{2-21}$$

Solving for $(z^2 + \sqrt{3}z + 1) = 0$, viz. $(z - z_1)(z - z_1')$:

$$a = 1, b = \sqrt{3}, c = 1 \tag{2-22}$$

$$z = \frac{-\sqrt{3} \pm \sqrt{3 - 4}}{2} \tag{2-23}$$

$$z = \frac{-\sqrt{3}}{2} \pm \frac{i}{2} = cis(\pi \pm \frac{\pi}{6}) \tag{2-24}$$

$$z_1 = cis(\frac{5\pi}{6}), \qquad z_1' = cis(-\frac{5\pi}{6}) \tag{2-25}$$

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By having z_0, z_0', z_1, and z_1', we can factorize as $(z - z_0) \cdot (z - z_0') \cdot (z - z_1) \cdot (z - z_1') = 0$,

so $z^4 - z^2 + 1 = 0$ is equivalent in complex roots to

$(z - cis(\frac{\pi}{6})) \cdot (z - cis(-\frac{\pi}{6})) \cdot (z - cis(\frac{5\pi}{6})) \cdot (z - cis(-\frac{5\pi}{6})) = 0.$ □

By the denominator of 2-19 we know $(z - z_0)(z - z_0') = z^2 + \sqrt{3}z + 1$,

so $z^4 - z^2 + 1 = 0$ expressed with second degree polynoms is $(z^2 - \sqrt{3}z + 1)(z^2 + \sqrt{3}z + 1)$.

□

" $\qquad$ "

Finn grensene $\lim_{n\to\infty} \frac{3n+2}{\sqrt{4n^2-1}}$ *og* $\lim_{n\to\infty}(\sqrt{n^2-5n}-n)$.

$$\lim_{n\to\infty} \frac{3n+2}{\sqrt{4n^2-1}} =$$

$$\lim_{n\to\infty} \frac{(3n+2)\frac{1}{n}}{\sqrt{4n^2-1}\frac{1}{n}} =$$

$$\lim_{n\to\infty} \frac{3+\frac{2}{n}}{\sqrt{\frac{4n^2-1}{n^2}}} =$$

$$\lim_{n\to\infty} \frac{3+\frac{2}{n}}{\sqrt{4-\frac{1}{n^2}}} =$$

$$\frac{3+\cancel{0}}{\sqrt{4-\cancel{0}}} =$$

$$\frac{3}{2}$$

$$\lim_{n\to\infty}(\sqrt{n^2-5n}-n)$$

$$\lim_{n\to\infty} \frac{(\sqrt{n^2-5n}-n)(\sqrt{n^2-5n}+n)}{\sqrt{n^2-5n}+n}$$

$$\lim_{n\to\infty} \frac{n^2-5n-n^2}{\sqrt{n^2-5n}+n}$$

$$\lim_{n\to\infty} \frac{-5n}{\sqrt{n^2-5n}+n}$$

$$\lim_{n\to\infty} \frac{-5}{\sqrt{1-\frac{5}{n}}+1}$$

$$\frac{-5}{\sqrt{1-\cancel{0}}+1}$$

$$\frac{-5}{2}$$

$\square$ $\qquad\qquad\qquad\qquad$ $\square$

> *Finn de komplekse tallene z som oppfyller likningen $2|z-1| = |z-4|$ og skisser løsningsmengden i det komplekse planet.(Hint: Sett inn $z = x+iy$ og finn en polynomlikning ix og y forløsningsmengden.)*

$$2|z - 1| = |z - 4| \equiv$$

$$4|a + ib - 1| = |a + ib - 4|$$

$$4((a - 1)^2 + (b)^2) = (a - 4)^2 + (b)^2$$

$$4(a - 1)^2 + 4b^2 = a^2 - 8a + 16 + b^2$$
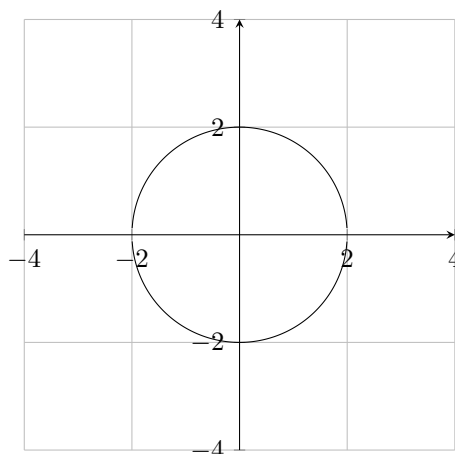
$$4(a^2 - 2a + 1) + 4b^2 = a^2 - 8a + 16 + b^2$$

$$4a^2 - 8a + 4 + 4b^2 = a^2 - 8a + 16 + b^2$$

$$3a^2z + 3b^2 = 12$$

$$b^2 = 4 - a^2$$

$$b = \pm\sqrt{4 - a^2}$$

$$\therefore Im(z) = \pm\sqrt{4 - Re(z)^2}, \exists z \forall \{a \in \mathbb{R} | -2 < a < 2\}$$

Since a circle's point is defined by all the points with radius r from its center, and we can recognize the modulus of the radius to be defined by Pythagora's theorem, as $b^2 + a^2$, we know that the equation defines a circle of radius 2. It's position is the Origin, as there is no transition on any of the direction vectors. □

> *En følge $\{a_n\}$ er definert ved $a_1 = 3, a_{n+1} = 3\sqrt{a_n}$ for $n \geq 1$. Vis at $a_n < 9$ og at $a_{n+1} > a_n$*

for alle n. Forklar hvorfor følgen konvergerer og finn $\lim_{n \to \infty} a_n$

$$
\begin{aligned}
a_1 &:= 3, & (5\text{-}1) \\
a_{n+1} &:= 3\sqrt{a_n} & (5\text{-}2)
\end{aligned}
$$

To be proven by induction:

$$
\begin{aligned}
P_n &:= a_n < 9 & (5\text{-}3) \\
Q_n &:= a_{n+1} > a_n & (5\text{-}4)
\end{aligned}
$$

Prove for n=1:

$$
\begin{aligned}
a_1 &= 3 \therefore a_2 = 3\sqrt{3} & (5\text{-}5) \\
P_1 &:= a_1 < 9 \equiv 3 < 9 \therefore P_1 & (5\text{-}6) \\
Q_1 &:= a_2 > a_1 \equiv 3\sqrt{3} > 3 \therefore Q_1 & (5\text{-}7)
\end{aligned}
$$

Assume Q_k, and prove ($Q_k \implies Q_{k+1}$):

$$
\begin{aligned}
Q_k &:= a_{k+1} > a_k & (5\text{-}8) \\
Q_{k+1} &:= a_{k+2} > 3\sqrt{a_k} & (5\text{-}9) \\
Q_{k+1} &:= 3\sqrt{a_{k+1}} > 3\sqrt{a_k} & (5\text{-}10) \\
Q_{k+1} &:= \sqrt{a_{k+1}} > \sqrt{a_k} & (5\text{-}11) \\
Q_{k+1} &:= a_{k+1} > a_k & (5\text{-}12)
\end{aligned}
$$

$\square$

Assume P_{k+1}, and prove ($P_k \implies P_{k+1}$)

$$
\begin{aligned}
P_k &:= a_k < 9 & (5\text{-}13) \\
P_{k+1} &:= 3\sqrt{a_k} < 9 & (5\text{-}14) \\
P_{k+1} &:= \sqrt{a_k} < 3 & (5\text{-}15) \\
P_{k+1} &:= a_k < 9 & (5\text{-}16)
\end{aligned}
$$

$\square$

The sequence converges since all values increment as n increments, but is bounded by **9**. We also know by this that the sequence's infinum is 3, as $a_1 = 3$ □

Since the sequence is convergent, we know its limit exsits:

$$L = \lim_{n \to \infty} a_n \tag{5-17}$$

$$L = \lim_{n \to \infty} 3\sqrt{a_{n-1}} \tag{5-18}$$

Since all $a_n > 1$:

$$L = 3\sqrt{\lim_{n \to \infty} a_{n-1}} \tag{5-19}$$

By the monotone convergence theorem:

$$L = 3\sqrt{\lim_{n \to \infty} a_n} \tag{5-20}$$

$$L = 3\sqrt{L} \tag{5-21}$$

$$L^2 = 9L \tag{5-22}$$

$$L = 9 \tag{5-23}$$

Thus we know the supremum of the sequence is **9**. □.

Submitted by Rolf Vidar Hoksaas *on September 13, 2019.*