Introduction to Decision Support based on Cake Cutting problem

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What is Decision Support?

The goal of Decision Support

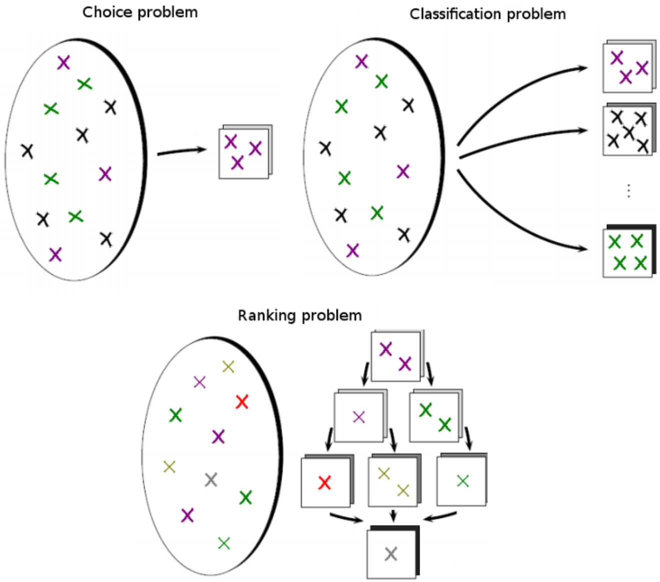
The main aim of decision support is proposing algorithms that simplify a process of making decisions i.e. choosing a new car, camera, etc. In other words, we look for a method which solves a specific decision problem and let us achieve a goal.

Decision problem

A situation where there is a necessity to choose one of at least two possible variants of actions. A decision maker has to answer one of the following questions:

- How to choose the best variant? (Choice problem)
- How to classify variants into decision classes? (Classification problem)
- How to order variants from the best to the worst? (Ordering problem)

What is Decision Support?



Specific areas of Decision Support

- What is a number of decision makers?
- What is a number of criteria?
- Is a consequence of action deterministic or uncertain?

	Theory of Social Choice	Multi-Criteria Decision Making	Decision under Risk and Uncertainty
DM	many	one	one
Criteria	one	many	one
RU	no	no	yes

What do we need to construct decision support algorithm?

Preference information

The information that is given by decision maker in order support solving a problem.

Preference model

Preference model allows to aggregate evaluations on each criterion of specific variant. It is built by preference information given by decision maker. We usually distinguish three types of preference model:

- function,
- relational system,
- set of decision rules.

Criterion

Criterion is a real-valued function reflecting a worth of variants from a particular point of view. Family of criteria should be consistent.

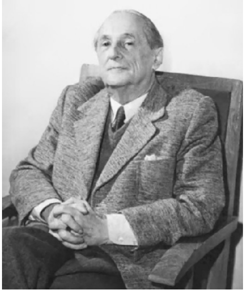
Cake cutting

Cake-cutting is a metaphor for a wide range of real-world problems that involve dividing some continuous object, whether its cake or, say, a tract of land, among people who value its features differently. The ideal method, which solves the problem, should:

- work for any number of players,
- make a division proportional,
- make a division envy-free,
- make a division equitable.

Historical background

The problem was introduced by Hugo Steinhaus - a Polish mathematician and educator who was one of the creators of Lwow School of Mathematics. Cake cutting was strongly studied after The Second World War (1940s).



The model

Let us denote a set of players by $N = 1, \ldots, n$ and our devisible good - the cake - by the interval $[0, 1]$. Moreover we assume that each player is endowed with a valuation function V_i (information preference), which maps a given subinterval $I \subseteq [0, 1]$ to it by player i, $V_i(I)$.
We are certainly interested in allocations $A = (A_1, \ldots, A_n)$, where each A_i is the piece of cake allocated to agent i. Now we can express our criteria in a more formal way:

- Proportionality: for all $i \epsilon N$, $V_i(A_i) \geq \frac{1}{n}$,
- Envy-freeness: for all $i, j \epsilon N$, $V_i(A_i) \geq V_i(A_j)$,
- Equitability: for all $i, j \epsilon N$, $V_i(A_i) = V_j(A_j)$

Proportionality for n = 2: Cut and Choose

1. Player 1 cuts the cake into two equally-valued pieces X_1 and X_2, such that $V_1(X_1) = V_1(X_2) = \frac{1}{2}$
2. Player 2 chooses its preferred piece and player 1 receives the remaining piece.

Proportionality for any n: Banach-Knaster

1. Agent 1 cuts off a piece X such that $V_1(X) = \frac{1}{n}$
2. This piece is then passed around players. Each player i has two options: lets it pass because it is not valuable for him ($V_i(X) < \frac{1}{n}$) or trims it down futher to get X' such that $V_i(X') = \frac{1}{n}$ and then pass it.
3. When the piece makes a full round, the last player that cut something off is obliged to take it.
4. The rest of cake (with cut pieces) is divided in the same way between $n - 1$ players.

Proportionality for any n: Dubins-Spanier

1. In the first step each player $i \epsilon N$ makes a mark at the point x_i such that $V_i([0, x_i]) = \frac{1}{n}$. The player j that made leftmost mark exits with the piece $A_j = [0, x_j]$.
2. If there is only one player left, it receives uclaimed piece of cake else go to step 1.

Proportionality for any n: Even-Paz

1. In the first step each player from a subset $1, \ldots, k$ makes a mark at the point x_i such that $V_i([y, x_i]) = \frac{V_i([y,z])}{2}$.
2. Let $x_1^*, \ldots, x_k^*$ be the marks sorted from left to right. Call the algorithm recursively with players $i_1, \ldots, i_{k/2}$ and the piece $[y, x_{k/2}^*]$, and with players $i_{k/2+1}, \ldots, i_k$ and the piece $[x_{k/2+1}^*, z]$.
3. If there are only one player i and and interval I, assign $A_i = I$

Envy-freeness for n = 3: Selfridge-Conway

1. Agent 1 divides the cake into three eequally-valued pieces $X_1, X_2, X_3 : V_1(X_1) = V_1(X_2) = V_1(X_3) = \frac{1}{3}$
2. Agent 2 trims the most valuable piece according to V_2 to create a tie for most valuable. For example, if $V_2(X_1) > V_2(X_2) \geq V_2(X_3)$, agent 2 removes $X^{'} \subseteq X_1$ such that $V_2(X_1 - X^{'}) = V_2(X_2)$. Let us call $X^{'}$ cake 2 and the rest of pieces cake 1.
3. Agent 3 chooses one of the three pieces of cake 1.
4. If agent 3 chose the trimmes piece, agent 2 chooses between the two other piece of cake 1. Otherwise, agent 2 receives the trimmed piece. Let us denote the agent $i \epsilon 2, 3$ that received the trimmed piece by T_1, and the other agent by T_2.
5. Agent 1 receives the remaining piece of cake 1.
6. Agent T_2 divides cake 2 into three equally-valued pieces.
7. Agents $T_1, 1, T_2$ select a piece of cake 2 each, in that order.

How about an env-free algorithm for any number of players?

- In 1995 Steven Brams and Alan Taylor created procedure which works for more than 3 players and guarantees to produce an envy-free division. However, this algorithm is unbounded which means that it might need to run very large number of steps (cuts) depending on the players' cake preferences.
- Over the past 50 years, many mathematicians and computer scientists, had convinced themselves that there was probably no bounded, envy-free algorithm for dividing cake among n players.
- In 2016 two computer scientists described an algorithm which only depends on a number of palyers. Unfortunetely this is very complex - it requires nn̂n̂n̂n̂n̂. Even for just a handful of players, this number is greater than the number of atoms in the universe.

References

Ariel D. Procaccia (2002)
Cake Cutting Algorithms
http://www.cs.cmu.edu/~arielpro/15896s15/docs/paper10.pdf

Fair cake-cutting
https://en.wikipedia.org/wiki/Fair_cake-cutting

Erica Klarreich (2016)
How to Cut Cake Fairly and Finally Eat It Too
https://www.quantamagazine.org/20161006-new-algorithm-solves-cake-cutting-problem/

Roman Sowiski
Preference Modelling and Decision Support

Ulle Endriss
Computational Social Choice: Autumn 2013
https://staff.science.uva.nl/u.endriss/teaching/comsoc/2013/slides/comsoc-cakes.pdf

The End