The Average Projected Area of Non-Convex Surfaces: A Generalization of Cauchy's Surface Area Formula

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Abstract

We provide a rigorous yet accessible derivation proving that for any closed surface K with surface area S, the average orthographic shadow area over all viewing directions equals

$$\overline{A} = \frac{S \cdot C_M}{4},$$

where $C_M = 1 - \langle AO \rangle$ is the Moeini Convexity Measure and $\langle AO \rangle$ is the area-weighted mean cosine-weighted ambient occlusion. This generalizes Cauchy's surface area formula ($\overline{A} = S/4$) to non-convex shapes by accounting for self-occlusion. This proof shall be named **Maryam′s** theorem in dedication to my mother and the late Maryam Mirzakhani.

1 Introduction and Main Result

Consider a closed 3D shape $K \subset \mathbb{R}^3$ with boundary surface ∂K. When illuminated from different directions, the shape casts shadows of varying sizes. A natural question arises: what is the *average* shadow area over all possible viewing directions?

Theorem 1 (Main Result)**.** *Let K be a compact, piecewise-smooth closed surface with total surface area S. Then the average orthographic projection area over uniformly random orientations is*

$$\boxed{\overline{A} = \frac{S}{4}\left(1 - \langle AO \rangle\right) = \frac{S \cdot C_M}{4}}$$

where $\langle AO \rangle$ is the area-weighted mean ambient occlusion, defined precisely in Section 3.3.

For convex shapes, $\langle AO \rangle = 0$, and this reduces to $\overline{A} = S/4$ (Cauchy's formula). For non-convex shapes, self-occlusion reduces the average shadow area proportionally.

2 Mathematical Setup

2.1 Notation and Definitions

Definition 1 (Surface and Normal)**.** *Let ∂K denote the boundary surface of K, with surface area element $dA(\boldsymbol{x})$. At each point $\boldsymbol{x} \in \partial K$, let $\boldsymbol{n}(\boldsymbol{x})$ denote the outward-pointing unit normal vector.*

Definition 2 (Visibility Function)**.** *For a viewing direction $\boldsymbol{u} \in S^2$ (the unit sphere), define the visibility function*

$$V(\boldsymbol{x}, \boldsymbol{u}) = \begin{cases} 1 & \text{if } \boldsymbol{x} \text{ is visible from direction } \boldsymbol{u} \\ 0 & \text{if } \boldsymbol{x} \text{ is occluded by other parts of } K \end{cases}$$

Precisely, $V(\boldsymbol{x}, \boldsymbol{u}) = 1$ if and only if $\boldsymbol{x}$ is the first surface point encountered when tracing a ray from infinity in direction $-\boldsymbol{u}$.

Definition 3 (Orthographic Projection Area)**.** *For a given viewing direction $\boldsymbol{u} \in S^2$, the orthographic shadow area is*

$$A_{proj}(\boldsymbol{u}) = \int_{\partial K} \underbrace{(\boldsymbol{n}(\boldsymbol{x}) \cdot \boldsymbol{u})_+}_{\textit{cosine factor}} \cdot \underbrace{V(\boldsymbol{x}, \boldsymbol{u})}_{\textit{visibility}} \, dA(\boldsymbol{x}),$$

where $(a)_+ := \max(a, 0)$ ensures only forward-facing surfaces contribute.

Remark 1. *The term $(\boldsymbol{n} \cdot \boldsymbol{u})_+$ represents the effective projected area of an infinitesimal surface patch: when the surface is perpendicular to $\boldsymbol{u}$, the full area contributes; when parallel, it contributes zero (the cosine law).*

2.2 Averaging Over All Directions

The average shadow area over all possible orientations is obtained by integrating over the unit sphere S^2 with the uniform measure:

Definition 4 (Average Shadow Area)**.**

$$\overline{A} := \frac{1}{4\pi} \int_{S^2} A_{proj}(\boldsymbol{u}) \, d\sigma(\boldsymbol{u}),$$

where $d\sigma$ is the standard surface measure on S^2.

3 The Derivation

3.1 Step 1: Interchanging Integration Order

Substituting the definition of $A_{\mathrm{proj}}(\boldsymbol{u})$ into $\overline{A}$:

$$\begin{aligned} \overline{A} &= \frac{1}{4\pi} \int_{S^2} \left[\int_{\partial K} (\boldsymbol{n}(\boldsymbol{x}) \cdot \boldsymbol{u})_+ V(\boldsymbol{x}, \boldsymbol{u}) \, dA(\boldsymbol{x}) \right] d\sigma(\boldsymbol{u}) \\ &= \int_{\partial K} \left[\frac{1}{4\pi} \int_{S^2} (\boldsymbol{n}(\boldsymbol{x}) \cdot \boldsymbol{u})_+ V(\boldsymbol{x}, \boldsymbol{u}) \, d\sigma(\boldsymbol{u}) \right] dA(\boldsymbol{x}), \end{aligned} \tag{1}$$

where we applied Fubini's theorem (valid since the integrand is non-negative and the domain is bounded).

3.2 Step 2: Restricting to the Hemisphere

For a fixed $\boldsymbol{x} \in \partial K$, the term $(\boldsymbol{n}(\boldsymbol{x}) \cdot \boldsymbol{u})_+$ is non-zero only when $\boldsymbol{u}$ lies in the outward-pointing hemisphere:

$$H(\boldsymbol{x}) := \{\boldsymbol{\omega} \in S^2 : \boldsymbol{n}(\boldsymbol{x}) \cdot \boldsymbol{\omega} \geq 0\}.$$

The inner integral from (1) becomes:

$$\begin{aligned} \frac{1}{4\pi} \int_{S^2} (\boldsymbol{n} \cdot \boldsymbol{u})_+ V(\boldsymbol{x}, \boldsymbol{u}) \, d\sigma(\boldsymbol{u}) &= \frac{1}{4\pi} \int_{H(\boldsymbol{x})} (\boldsymbol{n} \cdot \boldsymbol{\omega}) V(\boldsymbol{x}, \boldsymbol{\omega}) \, d\sigma(\boldsymbol{\omega}) \\ &= \frac{1}{4} \cdot \frac{1}{\pi} \int_{H(\boldsymbol{x})} (\boldsymbol{n} \cdot \boldsymbol{\omega}) V(\boldsymbol{x}, \boldsymbol{\omega}) \, d\sigma(\boldsymbol{\omega}). \end{aligned} \tag{2}$$

3.3 Step 3: Introducing Ambient Occlusion

Definition 5 (Cosine-Weighted Ambient Occlusion)**.** *At each point $\boldsymbol{x} \in \partial K$, define the ambient occlusion factor as*

$$AO(\boldsymbol{x}) := \frac{1}{\pi} \int_{H(\boldsymbol{x})} \underbrace{[1 - V(\boldsymbol{x}, \boldsymbol{\omega})]}_{\textit{occlusion indicator}} \cdot \underbrace{(\boldsymbol{n}(\boldsymbol{x}) \cdot \boldsymbol{\omega})}_{\textit{cosine weight}} \, d\sigma(\boldsymbol{\omega}).$$

This measures the fraction of the outward hemisphere that is occluded by the shape itself, weighted by the cosine of the angle from the normal.

Lemma 1 (Hemispherical Integral Normalization)**.** *For any unit normal* $\boldsymbol{n}$*,*

$$\frac{1}{\pi}\int_{H(\boldsymbol{x})}(\boldsymbol{n}\cdot\boldsymbol{\omega})\,d\sigma(\boldsymbol{\omega}) = 1.$$

Proof. Using spherical coordinates with $\boldsymbol{n}$ as the polar axis, where $\boldsymbol{\omega}\cdot\boldsymbol{n} = \cos\theta$:

$$\begin{aligned}\frac{1}{\pi}\int_{H(\boldsymbol{x})}(\boldsymbol{n}\cdot\boldsymbol{\omega})\,d\sigma(\boldsymbol{\omega}) &= \frac{1}{\pi}\int_0^{2\pi}\int_0^{\pi/2}\cos\theta\sin\theta\,d\theta\,d\phi\\ &= \frac{2\pi}{\pi}\int_0^{\pi/2}\cos\theta\sin\theta\,d\theta\\ &= 2\cdot\frac{1}{2}\sin^2\theta\Big|_0^{\pi/2} = 1.\end{aligned}$$

□

Lemma 2 (Visibility and Occlusion Relationship (Conservation of Energy))**.**

$$\frac{1}{\pi}\int_{H(\boldsymbol{x})}(\boldsymbol{n}\cdot\boldsymbol{\omega})V(\boldsymbol{x},\boldsymbol{\omega})\,d\sigma(\boldsymbol{\omega}) = 1 - AO(\boldsymbol{x}).$$

Proof. By Lemma 1 and the definition of $AO(\boldsymbol{x})$:

$$\begin{aligned}1 &= \frac{1}{\pi}\int_{H(\boldsymbol{x})}(\boldsymbol{n}\cdot\boldsymbol{\omega})\,d\sigma(\boldsymbol{\omega})\\ &= \frac{1}{\pi}\int_{H(\boldsymbol{x})}(\boldsymbol{n}\cdot\boldsymbol{\omega})\cdot 1\,d\sigma(\boldsymbol{\omega})\\ &= \frac{1}{\pi}\int_{H(\boldsymbol{x})}(\boldsymbol{n}\cdot\boldsymbol{\omega})[V(\boldsymbol{x},\boldsymbol{\omega}) + (1 - V(\boldsymbol{x},\boldsymbol{\omega}))]\,d\sigma(\boldsymbol{\omega})\\ &= \frac{1}{\pi}\int_{H(\boldsymbol{x})}(\boldsymbol{n}\cdot\boldsymbol{\omega})V(\boldsymbol{x},\boldsymbol{\omega})d\sigma(\boldsymbol{\omega}) + \frac{1}{\pi}\int_{H(\boldsymbol{x})}(\boldsymbol{n}\cdot\boldsymbol{\omega})(1 - V(\boldsymbol{x},\boldsymbol{\omega}))\,d\sigma(\boldsymbol{\omega})\\ &= \frac{1}{\pi}\int_{H(\boldsymbol{x})}(\boldsymbol{n}\cdot\boldsymbol{\omega})V(\boldsymbol{x},\boldsymbol{\omega})\,d\sigma(\boldsymbol{\omega}) + AO(\boldsymbol{x}).\end{aligned}$$

□

3.4 Step 4: Final Derivation

Combining equations (1), (2), and Lemma 2:

$$\begin{aligned}\overline{A} &= \int_{\partial K}\frac{1}{4}[1 - AO(\boldsymbol{x})]\,dA(\boldsymbol{x})\\ &= \frac{1}{4}\int_{\partial K}[1 - AO(\boldsymbol{x})]\,dA(\boldsymbol{x})\\ &= \frac{1}{4}\left[\int_{\partial K}dA(\boldsymbol{x}) - \int_{\partial K}AO(\boldsymbol{x})\,dA(\boldsymbol{x})\right]\\ &= \frac{1}{4}\left[S - \int_{\partial K}AO(\boldsymbol{x})\,dA(\boldsymbol{x})\right].\end{aligned} \tag{3}$$

We define the area-weighted mean ambient occlusion as:

$$\langle AO\rangle := \frac{1}{S}\int_{\partial K}AO(\boldsymbol{x})\,dA(\boldsymbol{x}).$$

Substituting this definition into (3) gives:

$$\begin{aligned}\overline{A} &= \frac{1}{4}[S - S\cdot\langle AO\rangle]\\ &= \frac{S}{4}[1 - \langle AO\rangle]\\ &= \frac{S\cdot C_M}{4}.\end{aligned}$$

This completes the proof of Theorem 1.

4 Special Cases and Physical Applications

4.1 Convex Shapes (Cauchy's Formula)

Theorem 2 (Cauchy's Projection Formula)**.** *For any convex shape with surface area S,*

$$\overline{A} = \frac{S}{4}.$$

Proof. For convex shapes, no point occludes any other, so $V(\boldsymbol{x}, \boldsymbol{\omega}) \equiv 1$ for all $\boldsymbol{x}$ and $\boldsymbol{\omega}$. Thus, from Definition 5, the integrand $[1 - V]$ is always zero:

$$AO(\boldsymbol{x}) = 0 \quad \Rightarrow \quad \langle AO \rangle = 0 \quad \Rightarrow \quad C_M = 1.$$

Therefore, our main result becomes:

$$\overline{A} = \frac{S \cdot C_M}{4} = \frac{S}{4}. \qquad \square$$

4.2 Physical Interpretation: Radiative Heat Transfer

For thermal radiation exchange, the Moeini Convexity Measure directly determines effective heat transfer. A body at temperature T exchanging radiation with an environment at T_{env} has net heat flux proportional to $S_{\text{eff}} = S \cdot C_M$. Points deep in cavities have $\text{AO}(x) \to 1$, exchanging radiation primarily with nearby surfaces at similar temperature, yielding negligible net heat transfer. As $\langle \text{AO} \rangle \to 1$, we find $S_{\text{eff}} \to 0$: highly self-occluded surfaces cannot efficiently exchange thermal radiation with the external environment.

5 Validity and Assumptions

Remark 2 (Regularity Conditions)**.** *The derivation requires:*

1. *∂K is compact and piecewise-C^1 (smooth except on a set of measure zero)*
2. *Polyhedra satisfy this condition almost everywhere*
3. *The visibility function V is measurable*
4. *Fubini's theorem applies (always satisfied for bounded domains with non-negative integrands)*

Remark 3 (Measure-Zero Exceptions)**.** *Grazing rays (where $\boldsymbol{\omega}$ is tangent to the surface) and edge cases occur on sets of measure zero and do not affect the integral values.*

6 Conclusion

We have rigorously derived the relationship

$$\boxed{\overline{A} = \frac{S}{4}(1 - \langle AO \rangle) = \frac{S \cdot C_M}{4}}$$

connecting the average orthographic shadow area to the mean, cosine-weighted ambient occlusion. This formula:

- Generalizes Cauchy's surface area formula to non-convex shapes
- Quantifies precisely how self-occlusion reduces average shadow area
- Provides a direct link between a geometric property (S) and a rendering property ($\langle AO \rangle$)
- Holds for any closed, piecewise-smooth surface

The result confirms that the orthographic shadow area is directly proportional to the Moeini Convexity Measure of the surface.