# The Average Projected Area of Non-Convex Surfaces: A Generalization of Cauchy's Formula

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### Abstract

We provide a rigorous yet accessible derivation proving that for any closed surface $K$ with surface area $S$, the average orthographic shadow area over all viewing directions equals

$$\overline{A} = \frac{S \cdot C_M}{4},$$

where $C_M = 1 - \langle AO \rangle$ is the Moeini Convexity Measure and $\langle AO \rangle$ is the area-weighted mean cosine-weighted ambient occlusion. This generalizes Cauchy's surface area formula ($\overline{A} = S/4$) to non-convex shapes by accounting for self-occlusion. This proof shall be named **Maryam's** theorem in dedication to my mother and the late Maryam Mirzakhani.

## 1  Introduction and Main Result

Consider a closed 3D shape $K \subset \mathbb{R}^3$ with boundary surface $\partial K$. When illuminated from different directions, the shape casts shadows of varying sizes. A natural question arises: what is the *average* shadow area over all possible viewing directions?

**Theorem 1** (Main Result). *Let $K$ be a compact, piecewise-smooth closed surface with total surface area $S$. Then the average orthographic projection area over uniformly random orientations is*

$$\boxed{\overline{A} = \frac{S}{4}\left(1 - \langle AO \rangle\right) = \frac{S \cdot C_M}{4}}$$

*where $\langle AO \rangle$ is the area-weighted mean ambient occlusion, defined precisely in Section 3.3.*

For convex shapes, $\langle AO \rangle = 0$, and this reduces to $\overline{A} = S/4$ (Cauchy's formula). For non-convex shapes, self-occlusion reduces the average shadow area proportionally.

## 2  Mathematical Setup

### 2.1  Notation and Definitions

**Definition 1** (Surface and Normal). *Let $\partial K$ denote the boundary surface of $K$, with surface area element $dA(\boldsymbol{x})$. At each point $\boldsymbol{x} \in \partial K$, let $\boldsymbol{n}(\boldsymbol{x})$ denote the outward-pointing unit normal vector.*

**Definition 2** (Visibility Function). *For a viewing direction $\boldsymbol{u} \in S^2$ (the unit sphere), define the visibility function*

$$V(\boldsymbol{x}, \boldsymbol{u}) = \begin{cases} 1 & \text{if } \boldsymbol{x} \text{ is visible from direction } \boldsymbol{u} \\ 0 & \text{if } \boldsymbol{x} \text{ is occluded by other parts of } K \end{cases}$$

*Precisely, $V(\boldsymbol{x}, \boldsymbol{u}) = 1$ if and only if $\boldsymbol{x}$ is the first surface point encountered when tracing a ray from infinity in direction $-\boldsymbol{u}$.*

**Definition 3** (Orthographic Projection Area). *For a given viewing direction $\boldsymbol{u} \in S^2$, the orthographic shadow area is*

$$A_{proj}(\boldsymbol{u}) = \int_{\partial K} \underbrace{(\boldsymbol{n}(\boldsymbol{x}) \cdot \boldsymbol{u})_+}_{cosine\ factor} \cdot \underbrace{V(\boldsymbol{x}, \boldsymbol{u})}_{visibility}\, dA(\boldsymbol{x}),$$

*where $(a)_+ := \max(a, 0)$ ensures only forward-facing surfaces contribute.*

**Remark 1.** *The term $(\boldsymbol{n} \cdot \boldsymbol{u})_+$ represents the effective projected area of an infinitesimal surface patch: when the surface is perpendicular to $\boldsymbol{u}$, the full area contributes; when parallel, it contributes zero (the cosine law).*

## 2.2 Averaging Over All Directions

The average shadow area over all possible orientations is obtained by integrating over the unit sphere $S^2$ with the uniform measure:

**Definition 4** (Average Shadow Area).

$$\overline{A} := \frac{1}{4\pi} \int_{S^2} A_{proj}(\boldsymbol{u})\, d\sigma(\boldsymbol{u}),$$

*where $d\sigma$ is the standard surface measure on $S^2$.*

# 3 The Derivation

## 3.1 Step 1: Interchanging Integration Order

Substituting the definition of $A_{\mathrm{proj}}(\boldsymbol{u})$ into $\overline{A}$:

$$\begin{aligned}
\overline{A} &= \frac{1}{4\pi} \int_{S^2} \left[ \int_{\partial K} (\boldsymbol{n}(\boldsymbol{x}) \cdot \boldsymbol{u})_+ V(\boldsymbol{x}, \boldsymbol{u})\, dA(\boldsymbol{x}) \right] d\sigma(\boldsymbol{u}) \\
&= \int_{\partial K} \left[ \frac{1}{4\pi} \int_{S^2} (\boldsymbol{n}(\boldsymbol{x}) \cdot \boldsymbol{u})_+ V(\boldsymbol{x}, \boldsymbol{u})\, d\sigma(\boldsymbol{u}) \right] dA(\boldsymbol{x}),
\end{aligned} \tag{1}$$

where we applied Fubini's theorem (valid since the integrand is non-negative and the domain is bounded).

## 3.2 Step 2: Restricting to the Hemisphere

For a fixed $\boldsymbol{x} \in \partial K$, the term $(\boldsymbol{n}(\boldsymbol{x}) \cdot \boldsymbol{u})_+$ is non-zero only when $\boldsymbol{u}$ lies in the outward-pointing hemisphere:

$$H(\boldsymbol{x}) := \{ \boldsymbol{\omega} \in S^2 : \boldsymbol{n}(\boldsymbol{x}) \cdot \boldsymbol{\omega} \geq 0 \}.$$

The inner integral from (1) becomes:

$$\begin{aligned}
\frac{1}{4\pi} \int_{S^2} (\boldsymbol{n} \cdot \boldsymbol{u})_+ V(\boldsymbol{x}, \boldsymbol{u})\, d\sigma(\boldsymbol{u}) &= \frac{1}{4\pi} \int_{H(\boldsymbol{x})} (\boldsymbol{n} \cdot \boldsymbol{\omega}) V(\boldsymbol{x}, \boldsymbol{\omega})\, d\sigma(\boldsymbol{\omega}) \\
&= \frac{1}{4} \cdot \frac{1}{\pi} \int_{H(\boldsymbol{x})} (\boldsymbol{n} \cdot \boldsymbol{\omega}) V(\boldsymbol{x}, \boldsymbol{\omega})\, d\sigma(\boldsymbol{\omega}).
\end{aligned} \tag{2}$$

## 3.3 Step 3: Introducing Ambient Occlusion

**Definition 5** (Cosine-Weighted Ambient Occlusion). *At each point $\boldsymbol{x} \in \partial K$, define the ambient occlusion factor as*

$$AO(\boldsymbol{x}) := \frac{1}{\pi} \int_{H(\boldsymbol{x})} \underbrace{[1 - V(\boldsymbol{x}, \boldsymbol{\omega})]}_{occlusion\ indicator} \cdot \underbrace{(\boldsymbol{n}(\boldsymbol{x}) \cdot \boldsymbol{\omega})}_{cosine\ weight}\, d\sigma(\boldsymbol{\omega}).$$

*This measures the fraction of the outward hemisphere that is occluded by the shape itself, weighted by the cosine of the angle from the normal.*

**Lemma 1** (Hemispherical Integral Normalization). *For any unit normal $\boldsymbol{n}$,*

$$\frac{1}{\pi} \int_{H(\boldsymbol{x})} (\boldsymbol{n} \cdot \boldsymbol{\omega}) \, d\sigma(\boldsymbol{\omega}) = 1.$$

*Proof.* Using spherical coordinates with $\boldsymbol{n}$ as the polar axis, where $\boldsymbol{\omega} \cdot \boldsymbol{n} = \cos\theta$:

$$\frac{1}{\pi} \int_{H(\boldsymbol{x})} (\boldsymbol{n} \cdot \boldsymbol{\omega}) \, d\sigma(\boldsymbol{\omega}) = \frac{1}{\pi} \int_0^{2\pi} \int_0^{\pi/2} \cos\theta \sin\theta \, d\theta \, d\phi$$

$$= \frac{2\pi}{\pi} \int_0^{\pi/2} \cos\theta \sin\theta \, d\theta$$

$$= 2 \cdot \frac{1}{2} \sin^2\theta \Big|_0^{\pi/2} = 1. \qquad \square$$

**Lemma 2** (Visibility and Occlusion Relationship (Conservation of Energy)).

$$\frac{1}{\pi} \int_{H(\boldsymbol{x})} (\boldsymbol{n} \cdot \boldsymbol{\omega}) V(\boldsymbol{x}, \boldsymbol{\omega}) \, d\sigma(\boldsymbol{\omega}) = 1 - AO(\boldsymbol{x}).$$

*Proof.* By Lemma 1 and the definition of $AO(\boldsymbol{x})$:

$$1 = \frac{1}{\pi} \int_{H(\boldsymbol{x})} (\boldsymbol{n} \cdot \boldsymbol{\omega}) \, d\sigma(\boldsymbol{\omega})$$

$$= \frac{1}{\pi} \int_{H(\boldsymbol{x})} (\boldsymbol{n} \cdot \boldsymbol{\omega}) \cdot 1 \, d\sigma(\boldsymbol{\omega})$$

$$= \frac{1}{\pi} \int_{H(\boldsymbol{x})} (\boldsymbol{n} \cdot \boldsymbol{\omega}) [V(\boldsymbol{x}, \boldsymbol{\omega}) + (1 - V(\boldsymbol{x}, \boldsymbol{\omega}))] \, d\sigma(\boldsymbol{\omega})$$

$$= \frac{1}{\pi} \int_{H(\boldsymbol{x})} (\boldsymbol{n} \cdot \boldsymbol{\omega}) V(\boldsymbol{x}, \boldsymbol{\omega}) d\sigma(\boldsymbol{\omega}) + \frac{1}{\pi} \int_{H(\boldsymbol{x})} (\boldsymbol{n} \cdot \boldsymbol{\omega})(1 - V(\boldsymbol{x}, \boldsymbol{\omega})) \, d\sigma(\boldsymbol{\omega})$$

$$= \frac{1}{\pi} \int_{H(\boldsymbol{x})} (\boldsymbol{n} \cdot \boldsymbol{\omega}) V(\boldsymbol{x}, \boldsymbol{\omega}) \, d\sigma(\boldsymbol{\omega}) + AO(\boldsymbol{x}). \qquad \square$$

## 3.4 Step 4: Final Derivation

Combining equations (1), (2), and Lemma 2:

$$\overline{A} = \int_{\partial K} \frac{1}{4} [1 - AO(\boldsymbol{x})] \, dA(\boldsymbol{x})$$

$$= \frac{1}{4} \int_{\partial K} [1 - AO(\boldsymbol{x})] \, dA(\boldsymbol{x})$$

$$= \frac{1}{4} \left[ \int_{\partial K} dA(\boldsymbol{x}) - \int_{\partial K} AO(\boldsymbol{x}) \, dA(\boldsymbol{x}) \right]$$

$$= \frac{1}{4} \left[ S - \int_{\partial K} AO(\boldsymbol{x}) \, dA(\boldsymbol{x}) \right]. \tag{3}$$

We define the area-weighted mean ambient occlusion as:

$$\langle AO \rangle := \frac{1}{S} \int_{\partial K} AO(\boldsymbol{x}) \, dA(\boldsymbol{x}).$$

Substituting this definition into (3) gives:

$$\overline{A} = \frac{1}{4} [S - S \cdot \langle AO \rangle]$$

$$= \frac{S}{4} [1 - \langle AO \rangle]$$

$$= \frac{S \cdot C_M}{4}.$$

This completes the proof of Theorem 1.

# 4 Special Cases and Physical Applications

## 4.1 Convex Shapes (Cauchy's Formula)

**Theorem 2** (Cauchy's Projection Formula). *For any convex shape with surface area $S$,*

$$\overline{A} = \frac{S}{4}.$$

*Proof.* For convex shapes, no point occludes any other, so $V(\boldsymbol{x}, \boldsymbol{\omega}) \equiv 1$ for all $\boldsymbol{x}$ and $\boldsymbol{\omega}$. Thus, from Definition 5, the integrand $[1 - V]$ is always zero:

$$AO(\boldsymbol{x}) = 0 \quad \Rightarrow \quad \langle AO \rangle = 0 \quad \Rightarrow \quad C_M = 1.$$

Therefore, our main result becomes:

$$\overline{A} = \frac{S \cdot C_M}{4} = \frac{S}{4}. \qquad \square$$

## 4.2 Physical Interpretation: Radiative Heat Transfer

For thermal radiation exchange, the Moeini Convexity Measure directly determines effective heat transfer. A body at temperature $T$ exchanging radiation with an environment at $T_{\text{env}}$ has net heat flux proportional to $S_{\text{eff}} = S \cdot C_M$. Points deep in cavities have $AO(x) \to 1$, exchanging radiation primarily with nearby surfaces at similar temperature, yielding negligible net heat transfer. As $\langle AO \rangle \to 1$, we find $S_{\text{eff}} \to 0$: highly self-occluded surfaces cannot efficiently exchange thermal radiation with the external environment.

# 5 Validity and Assumptions

**Remark 2** (Regularity Conditions). *The derivation requires:*

1. *$\partial K$ is compact and piecewise-$C^1$ (smooth except on a set of measure zero)*

2. *Polyhedra satisfy this condition almost everywhere*

3. *The visibility function $V$ is measurable*

4. *Fubini's theorem applies (always satisfied for bounded domains with non-negative integrands)*

**Remark 3** (Measure-Zero Exceptions). *Grazing rays (where $\boldsymbol{\omega}$ is tangent to the surface) and edge cases occur on sets of measure zero and do not affect the integral values.*

# 6 Conclusion

We have rigorously derived the relationship

$$\boxed{\overline{A} = \frac{S}{4}(1 - \langle AO \rangle) = \frac{S \cdot C_M}{4}}$$

connecting the average orthographic shadow area to the mean, cosine-weighted ambient occlusion. This formula:

- Generalizes Cauchy's surface area formula to non-convex shapes

- Quantifies precisely how self-occlusion reduces average shadow area

- Provides a direct link between a geometric property ($S$) and a rendering property ($\langle AO \rangle$)

- Holds for any closed, piecewise-smooth surface

The result confirms that the orthographic shadow area is directly proportional to the Moeini Convexity Measure of the surface.