

OPEN DISCRETE MATHEMATICS

Discrete Maths Problem Set with Solutions
An Open Source Problem Set for Beginners

Curated by: Mazeyar Moeini Feizabadi

The Contributors

A specail thanks to the following people

Muhammad Usaid

Marina Shehzad

Mudasir Hanif Shaikh

Shehryar Mughal

Rayyan ul Haq

1 Problems

1.1 Sets and Functions

1. Let A be the set of all the students taking Discrete Maths with Muhammad Imtiaz, B be the set of all the students taking Discrete Maths with Abdul Samad and C be the set of all the students taking Mechanics. Universal set will be set of all the students in Habib University. Express each of the following sets as a combination of A, B and C, using **both** Venn diagrams and set operators:
 - (a) The set of all the students taking DM and Mechanics.
 - (b) The set of all the students not taking DM with Muhammad Imtiaz or taking Mechanics.
 - (c) The set of all the students taking Mechanics but not DM.
 - (d) The set of all the students taking Mechanics or DM but not both.
2. Using formal definitions, prove or disprove the following (when disproving, also provide instance/s when statement holds true):

Given arbitrary sets A, B, C :

- (a) $A \times B = B \times A$
 - (b) $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$
 - (c) $A = B$ if A and B are two sets with the same power set
 - (d) $(A \cap B) \cup (A \cap \overline{B}) = A$
3. Suppose A, B and C are sets and $C \neq \emptyset$. Show that if $A \times C = B \times C$, then $A = B$
 4. Show that $\sum_{j=1}^n (a_j - a_{j-1}) = a_n - a_0$, where a_0, a_1, \dots, a_n is a sequence of real numbers.
 5. Find the domain, co-domain and range of $h(x)$ where

$$h(x) = f(g(x))$$

$g(x)$ and $f(x)$ are defined as

$$f(x) = \sin(\pi x)$$

$$f : \mathbb{Q} \rightarrow \mathbb{R}$$

$$g(x) = \frac{2x}{3}$$

$$g : \mathbb{Z} \rightarrow \mathbb{Q}$$

6. Let f be function from \mathbb{Z} to \mathbb{Z} defined by

$$f(x) = x^3 + 20$$

Prove or disprove the following:

- (a) f is not injective
- (b) f is surjective
- (c) f is bijective

1.2 Counting and Sequences

1. Prove the following:

- (a) $\sum_{i=1}^n (2i-1)^2 = \frac{1}{3}n(2n-1)(2n+1)$
- (b) $\sum_{i=1}^n i(i+2) = \frac{1}{6}n(n+1)(2n+7)$
- (c) $\sum_{i=1}^n \frac{1}{i(i+1)} = \frac{n}{n+1}$
- (d) $\sum_{i=1}^n i(2^i) = 2^{n+1}(n-1) + 2$
- (e) $\sum_{i=1}^n i(i!) = (n+1)! - 1$

2. Suppose that your 5 friends are roommates over a winter trip to Kaghan. Unfortunately, this means that they also share a sock drawer. In this drawer your friends have 10 pairs of socks.

Each pair of socks is a different color (Habib aspires to be a diverse community), but is folded together so that no roommate is ever wearing two different colors of socks.

- (a) How many different roommate-sock combinations are there?
 - (b) Each day, the roommates record the combination of sock colors that they are wearing in a special diary. How many such combinations are possible?
 - (c) Assuming that the roommates randomly choose their socks each morning, what is the probability that the socks colored red, teal, orange, pink and magenta are all worn by the roommates on a specific day?
 - (d) Suppose that one roommate decides he/she likes the pink, red and teal socks more than all of the rest and decides to hoard them so that no one else can wear them. Now, how many distinct roommate-sock combinations are there?
 - (e) The other roommates grow frustrated with the hoarding roommate and tell him/her that if the pink, red and teal socks are not returned then he/she is not allowed to wear the other colors of socks. The roommate does not relent. Now, how many distinct roommate-sock combinations are there?
 - (f) How many color combinations of socks are now possible in the diary?
3. (a) Let there be m positive integers represented by b_1, b_2, \dots, b_m such that they satisfy the following condition $b_1 + b_2 + b_3 + \dots + b_m = n$. Show that number of ways we can choose the values of b_1, b_2, \dots, b_m is

$$\binom{n-1}{m-1}$$

For example if $m = 3$ and $n = 4$ then the ways we can choose the values for b_i will be

$$b_1 + b_2 + b_3 = n$$

$$1 + 1 + 2 = 4$$

$$1 + 2 + 1 = 4$$

$$2 + 1 + 1 = 4$$

$$\text{which also equal to } \binom{4-1}{3-1} = 3$$

(Note: positive integers does not include zero)

- (b) Figure out another way to count the above scenario and use the principle of double counting to prove the equation below - the principle of double counting is a combinatorial trick to count the same thing in two different ways using two different formulas in order to establish that the formulas are equal.

$$\binom{n-1}{m-1} = \sum_{k=1}^{n-m+1} \binom{n-k-1}{m-2}$$

- (c) Show that if there is a set of positive integers $S = \{b_1, b_2, b_3, \dots, b_m\}$ such that the sum of S is n and the following condition holds $2^m - 1 > n$ then we can form two non empty disjoint subsets of S such that both the subsets will have the same sum

4. Prove the following:

(a) $\binom{n}{k} = \binom{n}{n-k}$

(b) $\sum_{i=1}^n i = \binom{n+1}{2}$

(c) $\sum_{k=0}^n \binom{n}{k} = 2^n$

5. Take a look at pascals triangle it has many different patterns and sequences.

							1							
						1		1						
					1		2		1					
				1		3		3		1				
			1		4		6		4		1			
		1		5		10		10		5		1		
1		6		15		20		15		6		1		

(1)

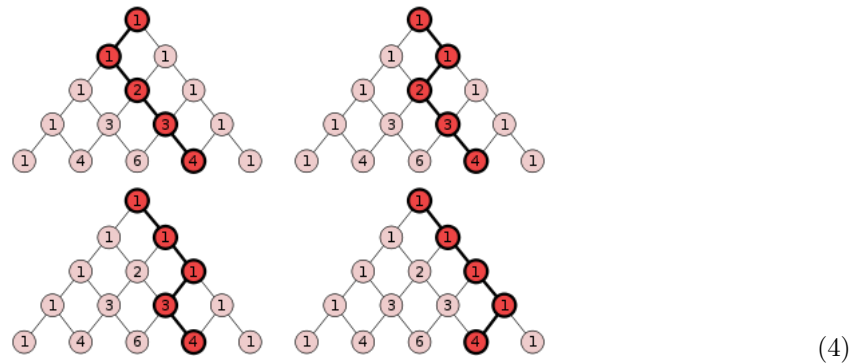
(a) You can also write pascals triangle as a table, using this prove pascals identity in figure 3.

1																			
1	1																		
1	2	1																	
1	3	3	1																
1	4	6	4	1															
1	5	10	10	5	1														
1	6	15	20	15	6	1													

(2)

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k} \quad (3)$$

(b) We can also look at pascals triangle in the form of a graph. We can observe that the highlighted paths end at the 4 node because there are 4 different paths to reach it.



Now let's look the following expansion.

$$(x + y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4 \quad (5)$$

Hopefully by now you should have seen the similarities between the expansion and pascals triangle. This is formally known as Binomial Theorem.

$$(x + y)^n = \sum_{k=0}^n \binom{n}{k} x^{n-k} y^k \quad (6)$$

Give a mathematical proof or explain why the Binomial Theorem is true.

6. Asadullah Khan Ghalab has recently discovered the [Fibonacci series](#) and the mathematical beauty inherent to it. The Fibonacci Series is an infinite sequence of integers, starting with 1 and 2 and defined recursively after that, for the n th term in the array, as $F(n) = F(n-1) + F(n-2)$. Through research on the Fibonacci Series, Ghalib also discovered an interesting set derived from the Fibonacci recurrence.

The [Wythoff array](#) is an infinite 2D-array of integers where the n th row is formed from the Fibonacci recurrence using starting numbers n and $\lfloor \phi \cdot (n+1) \rfloor$ where $n \in \mathbb{N}$ and ϕ is the [golden ratio](#) 1.618 (3 sf).

Ghalib recently studied sets in his discrete math class and needs your help.

1	2	3	5	8	13	21	...
4	7	11	18	29	47	76	...
6	10	16	26	42	68	110	...
9	15	24	39	63	102	165	...
12	20	32	52	84	136	220	...
14	23	37	60	97	157	254	...
17	28	45	73	118	191	309	...
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

- (a) Ghalib is curious whether the Fibonacci series is countable. Prove that it is.
- (b) Ghalib wants you to consider the Modified Wythoff as any array derived from the original, where each entry of the leading diagonal (marked in blue) of the original 2D-Array is replaced with an integer that does not occur in that row. Prove to Ghalib that the Modified Wythoff Array is countable.
7. After being disappointed with the “impressive” performance of his students, the discrete mathematics professor at the GradesDontDefineYou University cancels his weekly lecture and decides to talk to the

students about their recent performance instead. Each student responds differently citing different reasons for their supposedly bad performance. There are over 100 students taking this course, of those only 76 responses are deemed **not invalid** by the professor. The Professor saves those responses in a sheet, assigning each of these 76 responses a unique index between 0 and 149, inclusive. The invalid responses are simply discarded. Show that **at least two responses** have indices that are consecutive integers.

2 Solutions

2.1 Sets and Functions

1. (a) $(A \cup B) \cap C$
 (b) $\bar{A} \cup C$
 (c) $\overline{(A \cup B)} \cap C$
 (d) $(A \cup B \cup C) - ((A \cup B) \cap C)$
2. (a) $A \times B$ is defined as $\{(a, b) | a \in A \wedge b \in B\}$. $B \times A$ is defined as $\{(b, a) | b \in B \wedge a \in A\}$. But we know that $(a, b) \neq (b, a)$ unless $a = b$. Therefore $A \times B \neq B \times A$ unless $A = B$.
 (b) To Prove equality, we will prove $(A \cup B) \cap C \subseteq (A \cap C) \cup (B \cap C)$ and $(A \cap C) \cup (B \cap C) \subseteq (A \cup B) \cap C$
 Considering Left Hand Side

$$x \in (A \cup B) \cap C$$

$$x \in (A \cup B) \wedge x \in C$$

$$(x \in A \vee x \in B) \wedge x \in C$$

$$(x \in A \wedge x \in C) \vee (x \in B \wedge x \in C)$$

$$x \in (A \cap C) \wedge x \in (B \cap C)$$

$$x \in ((A \cap C) \cup (B \cap C))$$

Considering Right Hand Side

$$x \in ((A \cap C) \cup (B \cap C))$$

$$x \in (A \cap C) \vee x \in (B \cap C)$$

$$(x \in A \wedge x \in C) \vee (x \in B \wedge x \in C)$$

$$(x \in A \vee x \in B) \wedge (x \in C)$$

$$(x \in A \cup B) \wedge (x \in C)$$

$$x \in ((A \cup B) \cap C)$$

Q.E.D

- (c) i. By definition, $\mathcal{P}(A)$ is the set of all subsets that can be generated from A, if A and B generate the exact same collection of valid subsets, then it must be that A and B contain the same elements and are therefore equal
 ii. This part can also be proved by proving it's contrapositive i.e $A \neq B \rightarrow \mathcal{P}(A) \neq \mathcal{P}(B)$. $\mathcal{P}(A)$ is defined as the set of all the possible subsets of A. If A and B are not equal, their sets of all the possible subsets will also not be equal.

Q.E.D

(d) Considering Right Hand Side

$$x \in ((A \cap B) \cup (A \cap \bar{B}))$$

$$x \in (A \cap B) \vee x \in (A \cap \bar{B})$$

$$(x \in A \wedge x \in B) \vee (x \in A \wedge x \in \bar{B})$$

$$x \in A \wedge (x \in B \vee x \in \bar{B})$$

$$x \in A \wedge (x \in B \vee x \notin B)$$

$$x \in A$$

Considering Left Hand Side

We have to proof that $A \subseteq (A \cap B) \cup (A \cap \bar{B})$ We can proof this by proving it's contrapositive. i.e

$$\begin{aligned}
 & \forall x((x \notin ((A \cap B) \cup (A \cap \bar{B})) \rightarrow x \notin A) \\
 & \quad x \notin ((A \cap B) \cup (A \cap \bar{B})) \\
 & \quad \neg(x \in ((A \cap B) \cup (A \cap \bar{B}))) \\
 & \quad \neg(x \in ((A \cap B) \vee x \in (A \cap \bar{B}))) \\
 & \quad \neg(x \in (A \cap B)) \wedge \neg(x \in (A \cap \bar{B})) \\
 & \quad \neg(x \in A \wedge x \in B)) \wedge \neg(x \in A \wedge x \in \bar{B})) \\
 & \quad (\neg x \in A \vee \neg x \in B)) \wedge (\neg x \in A \vee \neg(x \in \bar{B})) \\
 & \quad (x \notin A \vee x \notin B) \wedge (x \notin A \vee (x \in B)) \\
 & \quad (x \notin A) \vee (x \notin B \wedge x \in B) \\
 & \quad x \notin A
 \end{aligned}$$

Hence Proved that both sides are subsets of each other and are equal.

3. Possible solutions:

- i. Consider the contrapositive: $A \neq B \rightarrow A \times C \neq B \times C$.

Consider sets A, B such that $A \neq B$. Then there exists some $x \in A \cap \bar{B}$ or its mirror $y \in \bar{A} \cap B$. Without loss of generality, suppose that x exists. Then $(x, -) \in (A \cap \bar{B}) \times C$ exists. If $(x, -) \in (A \cap \bar{B}) \times C$, then $(x, -) \in A \times C$ and $(x, -) \notin B \times C$. Since $A \times C$ and $B \times C$ differ over x , $A \times C \neq B \times C$. Q.E.D.

- ii. We set up a correspondence between sets $A \times C$ and $B \times C$ such that $\forall p \in A \times C, q \in B \times C$ (p maps to q iff $p = q$). We know this correspondence is injective because there is no p which does not correspond to a twin member q . This result follows directly from the assumption that $A \times C = B \times C$. Furthermore, this correspondence is surjective because it is injective and $|A \times C| = |B \times C|$.

The one-to-one correspondence $\forall p, q$ (p maps to q iff $p = q$) could be elaborated as $\forall a, b$ ($(a, -)$ maps to $(b, -)$ iff $(a, -) = (b, -)$), where $a \in A, b \in B$. Since $(a, -) = (b, -)$ for all elements in the respective sets (as per the definition of the Cartesian Product), $a = b$ for all a and b . Therefore $A = B$. Q.E.D.

4. If we just write out what the sum means, we see that parts of successive terms cancel, leaving only two terms:

$$\sum_{j=1}^n (a_j - a_{j-1}) = a_1 - a_0 + a_2 - a_1 + a_3 - a_2 + \dots + a_{n-1} - a_{n-2} + a_n - a_{n-1} = a_n - a_0$$

5. Here, $g(x)$ takes in something from \mathbb{Z} and then outputs something in \mathbb{Q} . Then, $f(g(x))$ takes in $g(x)$, which is from \mathbb{Q} , and outputs something in \mathbb{R} . Therefore, for $h(x)$:

Domain: \mathbb{Z}

Co-domain: \mathbb{R}

However, to figure out the range we need to combine the two functions:

$$h(x) = \sin\left(\frac{2x}{3}\right)$$

Range: $[-1, 1]$

6. (a) f is **injective**:

Proof:

Injunction can formally be written as,

$$\forall a, b (f(a) = f(b) \implies a = b)$$

Using our function we can write this as:

$$\forall a, b (a^3 + 20 = b^3 + 20 \implies a = b)$$

$$\equiv \forall a, b (a^3 = b^3 \implies a = b)$$

$$\equiv \forall a, b (a = b \implies a = b)$$

$$\equiv \forall a, b (\mathbf{T})$$

$$\equiv \mathbf{T} \quad \blacksquare$$

- (b) **f is not surjective:** The answer to this should be easy to see. We just need to find one out of many counter examples to prove that f is not surjective.

Surjection is formally written as:

$$\forall y \exists x (f(x) = y)$$

In order to disprove this we need to find a particular $y \in \mathbb{Z}$ such that $f(x) \neq y$ for all $x \in \mathbb{Z}$. 1 is one such value, that is,

$$f(x) \neq 1 \quad \forall x \in \mathbb{Z}$$

Therefore f is not surjective. \blacksquare

Refer to page 145 of your course book for more on how to prove/disprove injunction and/or surjection.

- (c) **f is not bijective:**

A necessary condition for f to be bijective is that it has to be surjective. Since f is not surjective therefore it is not bijective. \blacksquare

2.2 Counting and Sequences

1. (a) Base Case:- For $n = 1$,

$$(2(1) - 1)^2 = \frac{1}{3}(1)(2(1) - 1)(2(1) + 1)$$

$$(1)^2 = \frac{1}{3}(1)(1)(3)$$

$$1 = 1$$

Inductive Step:- We assume that

$$\sum_{i=1}^k (2i - 1)^2 = \frac{1}{3}k(2k - 1)(2k + 1)$$

To Prove:-

$$\sum_{i=1}^{k+1} (2i - 1)^2 = \frac{1}{3}(k + 1)(2(k + 1) - 1)(2(k + 1) + 1) = \frac{1}{3}(k + 1)(2k + 1)(2k + 3)$$

$$\begin{aligned}
& \sum_{i=1}^{k+1} (2i-1)^2 \\
& \sum_{i=1}^k (2i-1)^2 + (2(k+1)-1)^2 \\
& \frac{1}{3}k(2k-1)(2k+1) + (2k+1)^2 \\
& \frac{1}{3}(2k+1)(k(2k-1) + 3(2k+1)) \\
& \frac{1}{3}(2k+1)(2k^2 + 5k + 3) \\
& \frac{1}{3}(2k+1)(k+1)(2k+3) \\
& Q.E.D.
\end{aligned}$$

(b) Base Case:- For $n = 1$,

$$\begin{aligned}
1(1+2) &= \frac{1}{6}(1)(1+1)(2(1)+7) \\
3 &= \frac{1}{6}(2)(9) \\
3 &= 3
\end{aligned}$$

Inductive Step:- We assume that

$$\sum_{i=1}^k i(i+2) = \frac{1}{6}k(k+1)(2k+7)$$

To Prove:-

$$\begin{aligned}
\sum_{i=1}^{k+1} i(i+2) &= \frac{1}{6}(k+1)(k+2)(2k+9) \\
& \sum_{i=1}^{k+1} i(i+2) \\
& \sum_{i=1}^k i(i+2) + (k+1)(k+3) \\
& \frac{1}{6}k(k+1)(2k+7) + (k+1)(k+3) \\
& \frac{1}{6}(k+1)(k(2k+7) + 6(k+3)) \\
& \frac{1}{6}(k+1)(2k^2 + 15k + 18) \\
& \frac{1}{6}(k+1)(k+2)(2k+9) \\
& Q.E.D.
\end{aligned}$$

(c) Base Case:- For $n = 1$,

$$\frac{1}{1(1+1)} = \frac{1}{1+1}$$

$$\frac{1}{2} = \frac{1}{2}$$

Inductive Step:- We assume that

$$\sum_{i=1}^k \frac{1}{i(i+1)} = \frac{k}{k+1}$$

To Prove:-

$$\sum_{i=1}^{k+1} \frac{1}{i(i+1)} = \frac{k+1}{k+2}$$

$$\sum_{i=1}^{k+1} \frac{1}{i(i+1)}$$

$$\sum_{i=1}^k \frac{1}{i(i+1)} + \frac{1}{(k+1)(k+2)}$$

$$\frac{k}{k+1} + \frac{1}{(k+1)(k+2)}$$

$$\frac{k(k+2)+1}{(k+1)(k+2)}$$

$$\frac{k^2+2k+1}{(k+1)(k+2)}$$

$$\frac{(k+1)^2}{(k+1)(k+2)}$$

$$\frac{(k+1)}{(k+2)}$$

Q.E.D.

(d) Base Case:- For $n = 1$,

$$1(2^1) = 2^{1+1}(1-1) + 2$$

$$2 = 2^2(0) + 2$$

$$2 = 2$$

Inductive Step:- We assume that

$$\sum_{i=1}^k i(2^i) = 2^{k+1}(k-1) + 2$$

To Prove:-

$$\sum_{i=1}^{k+1} i(2^i) = 2^{k+2}(k) + 2$$

$$\sum_{i=1}^{k+1} i(2^i)$$

$$\begin{aligned}
& \sum_{i=1}^k i(2^i) + (k+1)(2^{k+1}) \\
& 2^{k+1}(k-1) + 2 + (k+1)(2^{k+1}) \\
& 2^{k+1}((k-1) + (k+1)) + 2 \\
& 2^{k+1}(2k) + 2 \\
& 2^{k+2}(k) + 2 \\
& Q.E.D.
\end{aligned}$$

(e) Base Case:- For $n = 1$,

$$\begin{aligned}
1(1!) &= (1+1)! - 1 \\
1 &= 2 - 1 \\
1 &= 1
\end{aligned}$$

Inductive Step:- We assume that

$$\sum_{i=1}^k i(i!) = (k+1)! - 1$$

To Prove:-

$$\begin{aligned}
& \sum_{i=1}^{k+1} i(i!) = (k+2)! - 1 \\
& \sum_{i=1}^{k+1} i(i!) \\
& \sum_{i=1}^k i(i!) + (k+1)((k+1)!) \\
& (k+1)! - 1 + (k+1)((k+1)!) \\
& (k+1)!(1 + (k+1)) - 1 \\
& (k+1)!(k+2) - 1 \\
& (k+2)(k+1)! - 1 \\
& (k+2)! - 1 \\
& Q.E.D.
\end{aligned}$$

2. (a) Since the arrangement has to be taken in consideration for roommate-sock combinations, the possibilities would be: ${}^{10}P_5 = 30,240$.
- (b) Since the colour is disregarded over here and it does not matter if the same colour combination is repeated by different individuals, the possibilities would be: ${}^{10}C_5 = 252$
- (c) The colours mentioned are one specific combination (red, teal, orange, pink and magenta) out of a total of ${}^{10}C_5$ so the probability of this combination occurring is reduced to: $\frac{{}^5C_5}{{}^{10}C_5} = \frac{1}{252}$
- (d) Since one roommate hoards 3 pairs of socks, there are 7 remaining sock pairs to share from for the other 4 friends. Now he has two distinct possibilities: Either to choose from his own 3 pairs (one's he has hoarded) or one from the other 7 pairs that his friends are sharing. This would make the final permutation: ${}^3P_1 \times {}^7P_4 + {}^7P_5 = 5040$
- (e) Now since there is a restriction introduced - the hoarder is not allowed to wear the other colors of socks. This eliminates his choice and he can only wear from the ones he has hoarded. The new permutation is: ${}^3P_1 \times {}^7P_4 = 2520$

- (f) Again, in this situation only colour combinations are considered. For this, the arrangement can be overlooked. The complete combination of the colours the hoarder could possibly wear from his collection of 3 pairs and the other friends from their 7 pairs is: ${}^3C_1 \times {}^7C_4 = 105$
3. (a) We model this situation as distributing n apples among m children, where we add an extra condition that all of them must receive at least one apple. Disregarding the condition, the question is a simple combination with repetition problem so how can we incorporate the condition. Instead of trying to distribute n apples among m children we say we will give one apple to each child then find the number of way of distribute the remaining apples among the children. There fore the number of apples left handing m apples, one to each child, will be $n - m$ and the number of ways to distribute these among m children, applying the following formula of combination with repetition

$$\binom{(n) + (r) - 1}{(r) - 1}$$

we have

$$\binom{(n - m) + (m) - 1}{(m) - 1}$$

$$\binom{n - 1}{m - 1}$$

- (b) To count the above in a different way we say that we will hand the first child k apples where k cannot be less than 1 and cannot be greater than $n - m - 1$ because at least one apple needs be to handed to other $m - 1$ children. For each value of k there will be a certain number of ways to distribute the $n - k$ remaining apples among the $m - 1$ children. Following the same strategy as before and handing one apple to each of the $m - 1$ children we are left with $n - k - m + 1$ apples to be distributed among $m - 1$ children, applying combination with repetition

$$\binom{(n - k - m + 1) + (m - 1) - 1}{(m - 1) - 1}$$

$$\binom{n - k - 1}{m - 2}$$

From here we can say that if we sum the number of ways to distribute the remaining apples over all possible values of k it will be the same as the number ways of distributing n apples among m children where each gets at least one and hence,

$$\binom{n - 1}{m - 1} = \sum_{k=1}^{n-m+1} \binom{n - k - 1}{m - 2}$$

- (c) The first thing to note is that all b_i will be different since they belong to a set. Secondly none of them will have a value less than one that is the least value any element can possibly have is one. The number of possible subsets that S can have is equal to 2^m since it has m elements and excluding the empty subset it is $2^m - 1$. Now the largest subset will be equal to S itself and have sum n that is the maximum sum any subset can have will be n and the minimum sum a subset can have is 1. Thus the total possible values of the sum of a subset can range from 1 to n that is the sum of a subset can take n possible values and we have $2^m - 1$ non-empty subsets. From the question we know $2^m - 1 > n$ thus by the pigeon hole principle there must be at least two subsets which have the same sum. When we have two such sets we can just remove the common elements in them and we will have two non empty disjoint subsets that have the same sum.
4. Prove the following

(a) Simple substitution.

$$\frac{n!}{k!(n-k)!} = \frac{n!}{(n-k)!(n-(n-k))!}$$

$$\frac{n!}{k!(n-k)!} = \frac{n!}{(n-k)!k!}$$

NOTE: This also shows symmetry in each row of Pascal's triangle.

(b)

$$\sum_{i=1}^n i = \binom{n+1}{2}$$

$$\frac{n^2+n}{2} = \binom{n+1}{2}$$

$$\frac{n^2+n}{2} = \frac{(n+1)!}{2!(n+1-2)!}$$

$$\frac{n^2+n}{2} = \frac{(n+1)!}{2!(n+1-2)!}$$

$$\frac{n^2+n}{2} = \frac{(n+1)n!}{2!(n-1)!}$$

Using: $(n-1)! = \frac{n!}{n}$

$$\frac{n^2+n}{2} = \frac{(n+1)n!}{2! \frac{n!}{n}}$$

$$\frac{n^2+n}{2} = \frac{(n+1)n}{2!}$$

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(c) First we can expand the sum.

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n$$

We can interperate this a little differently.

We think of each choose function being summed as the number of ways to choose k thing from a set of cardinality n .

For example from $S = \{1, 2, 3\}$

Ways to can choose 0 thing = 1. $\{\}$

Ways to can choose 1 thing = 3. $\{\{1\}, \{2\}, \{3\}\}$

Ways to can choose 2 thing = 3. $\{\{1, 2\}, \{2, 3\}, \{3, 1\}\}$

Ways to can choose 3 thing = 1. $\{\{1, 2, 3\}\}$

Numerically this looks like.

$$\binom{3}{0} + \binom{3}{1} + \binom{3}{2} + \binom{3}{3} = 2^3$$

$$1 + 3 + 3 + 1 = 8$$

By counting number of ways to choose $0 - n$ things from a set of size n , essentially we are counting the cardinality of the powerset.

5. (a) We can say the columns are the k value and the rows are the n value. From looking at the table we can realize the recursive relation. That the value of the current node is the sum the one above and to the left $\binom{n-1}{k-1}$ and one above $\binom{n-1}{k}$. This is just a formal representation of how most people derive Pascal's triangle by hand.
- (b) If we expand a binomial we write it as.

$$(x + y)^4 = (x + y)(x + y)(x + y)(x + y)$$

Conventionally we would expand this algebraically, however we can view the expansion with combinatorics. If we were to multiply this out(FOIL), it would be the same as asking how many different ways can you choose $x^n y^k$ where $n + k = 4$ for n 's and k 's. NOTE: the 4 in $n + k = 4$ corresponds to the fourth row of Pascal's Triangle.

We can see if $n = 4, k = 0$, there is only one way to choose this and that is where from each parenthesis the x is chosen a multiplied.

$$(x + y)^4 = (x_1 + y_1)(x_2 + y_2)(x_3 + y_3)(x_4 + y_4)$$

$$x^4 = x_1 x_2 x_3 x_4$$

Another example is where $n = 1, k = 3$, there is 4 ways to choose this through.

$$(x + y)^4 = (x_1 + y_1)(x_2 + y_2)(x_3 + y_3)(x_4 + y_4)$$

$$xy^3 = x_1 y_2 y_3 y_4$$

$$xy^3 = y_1 x_2 y_3 y_4$$

$$xy^3 = y_1 y_2 x_3 y_4$$

$$xy^3 = y_1 y_2 y_3 x_4$$

Now we can connect this Pascal's Triangle in graph form. We can think that moving left corresponds with choosing a x and moving right corresponds with a y . From the graph we can see the four different ways to 4 node in the fourth row.

left, right, right, right

right, left, right, right

right, right, left, right

right, right, right, left

6. (a) If we create a function $f(x)$ where $x \in \mathbb{Z}^+$,

$$f(x) = \begin{cases} 1 & \text{if } x=1 \\ 2 & \text{if } x=2 \\ f(x-1) + f(x-2) & x>2 \end{cases}$$

Through this function, we can see that all positive integers are being mapped to the Fibonacci series. Since each integer maps to a unique element in the Fibonacci, the function is injective. Since each element in the Fibonacci is being mapped from a unique integer, the function is surjective therefore the function is bijective and since we have proved a bijection from the set of positive integers to the Fibonacci series, we have proved that the Fibonacci series is countable.

- (b) Every row in the Wythoff array is a Fibonacci sequence. As we have proved in part (a) that the Fibonacci series is countable, every row of the Wythoff is countable, as every row is a modified Fibonacci series. If we replace each leading diagonal in each row, we will be replacing each with a number not present in the row, meaning every element in the row will be a unique number. Because of this, each modified row will still remain countable and bijective with \mathbb{Z}^+

If we take the cartesian product of \mathbb{Z}^+ , $\mathbb{Z}^+ \times \mathbb{Z}^+$, we can see each tuple in $\mathbb{Z}^+ \times \mathbb{Z}^+$ as (a,b). a can be seen as the row number and b can be seen as the column number. With this we can establish a bijection between $\mathbb{Z}^+ \times \mathbb{Z}^+$ and the Wythoff array similar to how we did it in part (a). As each element of the Wythoff array has a different column number and row number, there is a surjection between $\mathbb{Z}^+ \times \mathbb{Z}^+$ and the Wythoff. As we have already established that each row will have no repeating elements, if an element does repeat in the Wythoff, it will be in a different column and hence will be related to a different tuple of $\mathbb{Z}^+ \times \mathbb{Z}^+$, thus establishing an injection. As we have established both an injection and surjection, we can say that the Wythoff array is bijective with $\mathbb{Z}^+ \times \mathbb{Z}^+$. As \mathbb{Z}^+ is countable, $\mathbb{Z}^+ \times \mathbb{Z}^+$ is also countable and since we have established a bijection between the Wythoff and $\mathbb{Z}^+ \times \mathbb{Z}^+$, we can safely say that the Wythoff is countable.