Supplemental Material for "Bayesian Cramér-Rao Bound for Parameter Estimation Based on Mixed-Resolution Data"

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This document contains supplemental material for the paper [1]. The notation in this document is adapted from [1].

S.I. Regularity conditions for Theorem 1

In the following, we demonstrate that all three regularity conditions of the BCRB (see, e.g. p. 35 in [2]) are satisfied for the considered model. First, we develop the Bayesian log-likelihood function for the considered model, which is

$$\begin{aligned} l(\mathbf{x}_a,\mathbf{x}_q,\boldsymbol{\theta}) &\triangleq \log p(\mathbf{x}_a|\mathbf{x}_q,\boldsymbol{\theta})+\log p(\mathbf{x}_q|\boldsymbol{\theta})+\log p(\boldsymbol{\theta}) \\ &= \log p(\mathbf{x}_a|\boldsymbol{\theta})+\log p(\mathbf{x}_q|\boldsymbol{\theta})+\log p(\boldsymbol{\theta}), \end{aligned} \quad \text{(S-1)}$$

where the last equality is since, given $\boldsymbol{\theta}$, the measurement vectors $\mathbf{x}_a$ and $\mathbf{x}_q$ are independent. Moreover, given $\boldsymbol{\theta}$, the analog measurements, $\mathbf{x}_a$, are of complex Gaussian distribution, $\mathbf{x}_a|\boldsymbol{\theta}\sim\mathcal{CN}(\mathbf{H}\boldsymbol{\theta},\sigma_a^2\mathbf{I}_{N_a})$. Similarly, the vector

$$\mathbf{y}\triangleq\mathbf{G}\boldsymbol{\theta}+\mathbf{w}_q \quad \text{(S-2)}$$

satisfies $\mathbf{y}|\boldsymbol{\theta}\sim\mathcal{CN}(\mathbf{G}\boldsymbol{\theta},\sigma_q^2\mathbf{I}_{N_q})$. Using the properties of the complex Gaussian distribution, the real and imaginary parts of $\mathbf{y}$ are Gaussian and satisfy $\mathrm{Re}\{\mathbf{y}|\boldsymbol{\theta}\}\sim\mathcal{N}(\mathrm{Re}\{\mathbf{G}\boldsymbol{\theta}\},\frac{\sigma_q^2}{2}\mathbf{I}_{N_q})$ and $\mathrm{Im}\{\mathbf{y}|\boldsymbol{\theta}\}\sim\mathcal{N}(\mathrm{Im}\{\mathbf{G}\boldsymbol{\theta}\},\frac{\sigma_q^2}{2}\mathbf{I}_{N_q})$. As a result, the 1-bit quantized log-likelihood function is equal to

$$\begin{aligned} \log p(\mathbf{x}_q|\boldsymbol{\theta}) = \sum_{n=1}^{N_q}&\left(\frac{1}{2}+\frac{\mathrm{Re}\{\mathbf{x}_{q_n}\}}{\sqrt{2}}\right)\log\Phi(\zeta_n^R) \\ &+\left(\frac{1}{2}-\frac{\mathrm{Re}\{\mathbf{x}_{q_n}\}}{\sqrt{2}}\right)\log\left(1-\Phi(\zeta_n^R)\right) \\ &+\left(\frac{1}{2}+\frac{\mathrm{Im}\{\mathbf{x}_{q_n}\}}{\sqrt{2}}\right)\log\Phi(\zeta_n^I) \\ &+\left(\frac{1}{2}-\frac{\mathrm{Im}\{\mathbf{x}_{q_n}\}}{\sqrt{2}}\right)\log\left(1-\Phi(\zeta_n^I)\right), \end{aligned} \quad \text{(S-3)}$$

where ζ_n^R and ζ_n^I are given in (7).

Thus, since $\boldsymbol{\theta}$ is Gaussian and $\mathbf{x}_a|\boldsymbol{\theta}\sim\mathcal{CN}(\mathbf{H}\boldsymbol{\theta},\sigma_a^2\mathbf{I}_{N_a})$, and using (S-3), it can be verified that the log-likelihood function in (S-1) is twice differentiable w.r.t. $\boldsymbol{\theta}\in\mathbb{C}^M$ and absolutely integrable (first and second regularity conditions). Furthermore, it can be verified that the third regularity condition is satisfied:

$$\boldsymbol{\theta}\cdot p(\boldsymbol{\theta}|\mathbf{x}) \xrightarrow{\boldsymbol{\theta}\to\pm\infty} \mathbf{0},\ \forall\mathbf{x}\in\mathcal{C}, \quad \text{(S-4)}$$

since $\lim_{\boldsymbol{\theta}\to\infty}\boldsymbol{\theta}\cdot p(\boldsymbol{\theta}) = \lim_{\boldsymbol{\theta}\to\infty}\int\boldsymbol{\theta}\cdot p(\boldsymbol{\theta}|\mathbf{x})\cdot p(\mathbf{x})d\mathbf{x}=\mathbf{0}$.

S.II. Derivation of (15)

It can be verified that the complex-valued gradient of the quantized measurements log-likelihood from (S-3) is [3]

$$\begin{aligned} &\nabla_{\boldsymbol{\theta}}^T\log p(\mathbf{x}_q|\boldsymbol{\theta}) = \frac{1}{\sqrt{2}\sigma_q}\sum_{n=1}^{N_q}\mathbf{g}_n \\ &\times\left[\frac{\phi(\zeta_n^R)}{\Phi(\zeta_n^R)(\Phi(\zeta_n^R)-1)}\left(\Phi(\zeta_n^R)-\frac{1}{2}-\frac{1}{\sqrt{2}}\mathrm{Re}\{\mathbf{x}_{q_n}\}\right)\right. \\ &\left.-j\frac{\phi(\zeta_n^I)}{\Phi(\zeta_n^I)(\Phi(\zeta_n^I)-1)}\left(\Phi(\zeta_n^I)-\frac{1}{2}-\frac{1}{\sqrt{2}}\mathrm{Im}\{\mathbf{x}_{q_n}\}\right)\right], \end{aligned} \quad \text{(S-5)}$$

where ζ_n^R and ζ_n^I are defined in (7). Furthermore, the mean of the real and imaginary parts of $\mathbf{x}_{q_n}$ given $\boldsymbol{\theta}$ are given by

$$\sqrt{2}\mathrm{E}[\mathrm{Re}\{\mathbf{x}_{q_n}\}|\boldsymbol{\theta}] = 2\Phi(\zeta_n^R)-1, \quad \text{(S-6a)}$$

$$\sqrt{2}\mathrm{E}\left[\mathrm{Im}\{\mathbf{x}_{q_n}\}|\boldsymbol{\theta}\right] = 2\Phi(\zeta_n^I)-1, \quad \text{(S-6b)}$$

respectively. The second moment for both parts is equal to

$$\mathrm{E}[\mathrm{Re}^2\{\mathbf{x}_{q_n}\}|\boldsymbol{\theta}] = \mathrm{E}[\mathrm{Im}^2\{\mathbf{x}_{q_n}\}|\boldsymbol{\theta}] = 0.5. \quad \text{(S-7)}$$

We note that the smoothness assumption is valid for quantized measurements since the expected value of the gradient is zero, as shown using (S-6a), (S-6b), and (S-7).

By substituting (S-5) in (13b), using (S-6a)-(S-7), and because the entries of $\mathbf{x}_q|\boldsymbol{\theta}$ are independent, and the real and imaginary parts of each entry are also independent, we obtain:

$$\begin{aligned} \mathbf{J}_{\mathbf{x}_q|\boldsymbol{\theta}} &= \mathrm{E}_{\mathbf{x}_q|\boldsymbol{\theta}}\left[\nabla_{\boldsymbol{\theta}}^H\log p(\mathbf{x}_q|\boldsymbol{\theta})\nabla_{\boldsymbol{\theta}}\log p(\mathbf{x}_q|\boldsymbol{\theta})\right] \\ &= \frac{1}{2\sigma_q^2}\sum_{n=1}^{N_q}d_n(\boldsymbol{\theta})g_{n_i}^*g_{n_j} = \frac{1}{2\sigma_q^2}\mathbf{G}^H\mathbf{D}(\boldsymbol{\theta})\mathbf{G}, \end{aligned} \quad \text{(S-8)}$$

where $\mathbf{D}(\boldsymbol{\theta}) = \mathrm{diag}\left([d_1(\boldsymbol{\theta}),\cdots,d_M(\boldsymbol{\theta})]\right)$, and $d_n(\boldsymbol{\theta})$ is defined in (6). Thus, we obtain in (S-8) the result in (15).

S.III. Expected value of $\mathbf{x}_q$

The expected value of $\mathbf{x}_{q_n}$ is $\frac{1}{\sqrt{2}}\left(\left(1-2\overline{P_1}\right)+j\left(1-2\overline{P_2}\right)\right)$, where

$$\overline{P_1} = \Pr(\mathrm{Re}(\mathbf{g}_n^T\boldsymbol{\theta})+\mathrm{Re}(w_n)-\tau_n^R<0), \quad \text{(S-9)}$$

$$\overline{P_2} = \Pr(\mathrm{Im}(\mathbf{g}_n^T\boldsymbol{\theta})+\mathrm{Im}(w_n)-\tau_n^I<0). \quad \text{(S-10)}$$

Then, using the Gaussianity of the different terms, it can be verified that $\overline{P_1} = \Phi(\frac{\tau_n^R-\tilde{\mu_R}}{\tilde{\sigma_R}})$, where $\tilde{\mu}_R$ and $\tilde{\sigma}_R$ are the mean and the standard deviation of $\mathrm{Re}(\mathbf{g}_n^T\boldsymbol{\theta})+\mathrm{Re}(u_n)$, respectively. Similarly, we can compute $\overline{P_2}$ via the distribution of $\mathrm{Im}(\mathbf{g}_n^T\boldsymbol{\theta})+\mathrm{Im}(u_n)$. Based on our model from Section II, we get

$$\tilde{\mu}_R = \tilde{\mu}_{Im} = 0, \quad \tilde{\sigma}_R^2 = \tilde{\sigma}_I^2 = \frac{1}{2}(\sigma_q^2+\rho_q). \quad \text{(S-11)}$$

Eventually, taking (S-11) into account, we get

$$\begin{aligned} \sqrt{2}\mathrm{E}[x_{q_n}] = &\left(1-2\Phi\left(\frac{\tau_n^R}{\sqrt{\frac{1}{2}(\sigma_q^2+\rho_q)}}\right)\right) \\ &+j\left(1-2\Phi\left(\frac{\tau_n^{Im}}{\sqrt{\frac{1}{2}(\sigma_q^2+\rho_q)}}\right)\right). \end{aligned} \quad \text{(S-12)}$$

Note that for a zero threshold, $\tau=0$, we obtain $\mathrm{E}[x_{q_n}]=0$.

References

[1] Y. Mazor, I. E. Berman, and T. Routtenberg, "Bayesian Cramér-Rao bound for parameter estimation based on a mixed-resolution data," 2023.

[2] H. L. Van Trees and K. L. Bell, *"Bayesian Bounds for Parameter Estimation and Nonlinear Filtering/Tracking"*. Wiley-IEEE Press, 2007.

[3] Peter J. Schreler, Louis L. Scharf, *"statistical signal processing of complex-valued data - the theory of improper and noncircular signals"*. Cambridge University Press, 2010.