Notes - For Thesis

Yaniv Mazor

June 12, 2024

0.1 2D BCRB

Another issue we would like to discuss is the impact of different noise deviations, so far we have only examined the case, where the quantized and analog signal-to-noise ratio (SNR), $\sigma_a \sigma_q$, are equal. However, exploring the different values of this SNR will offer us a new perspective on the impact of each type of data.

Our simulations have validated our previous findings. The Bayesian CRB (BCRB) consistently decreases as the SNR (whether analog or quantized) increases, regardless of factors like dithering. The BCRB remains unaffected by the mixed resolution algorithm; it simply decreases with higher SNR, as can be seen in Fig. (2).

In contrast, linear minimum-mean-squared-error (LMMSE) is more intriguing, as it is evident from Fig (1). When σ_q is held constant, the mean squared error (MSE) generally decreases monotonically. However, when σ_a is constant, the behavior becomes convex. An important distinction, not easily discernible in the two-dimensional graph, is that even though the LMMSE exhibits convex behavior for constant σ_a, the function gradually becomes flatter as σ_a decreases. This makes sense since we know that at high SNR, analog observations become more dominant.

0.2 Quantize LMMSE-Threshold

We would like to explore quantize estimation which is examine different statistical properties. For making such research, we will try the develop $\mathbf{C}_{\mathbf{x}_q,\boldsymbol{\theta}}$ & $\mathbf{C}_{\mathbf{x}_q}$. using the idea of Bussgang therom, and by comparing to the arcsin law, we will try to develop cross-correlation function for two dimensional gaussian variables which passs through quantization.

we would like to find the cross corvariance function of x & Q(y):

$$\begin{aligned}
&\mathbf{C}_{x,Q(y)} = \\
&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xQ(y)\frac{1}{2\pi\sqrt{\sigma_x^2\sigma_y^2-\rho^2}} \\
&e^{\frac{-(\sigma_y^2(x-\mu_x)^2-2\rho(x-\mu_x)(y-\mu_y)+\sigma_x^2(y-\mu_y)^2)}{2(\sigma_x^2\sigma_y^2-\rho^2)}}\,dx\,dy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xQ(y)\frac{1}{2\pi\sqrt{\sigma_x^2\sigma_y^2-\rho^2}} \\
&e^{\frac{-(\sigma_y^2(x-\mu_x)^2-2\rho(x-\mu_x)(y-\mu_y)+\sigma_x^2(y-\mu_y)^2)}{2(\sigma_x^2\sigma_y^2-\rho^2)}}\,dx\,dy \\
&\int_{-\infty}^{\infty} xe^{\frac{-(\sigma_y^2(x-\mu_x)^2-2\rho(x-\mu_x)(y-\mu_y))}{2(\sigma_x^2\sigma_y^2-\rho^2)}}\,dx
\end{aligned} \tag{1}$$

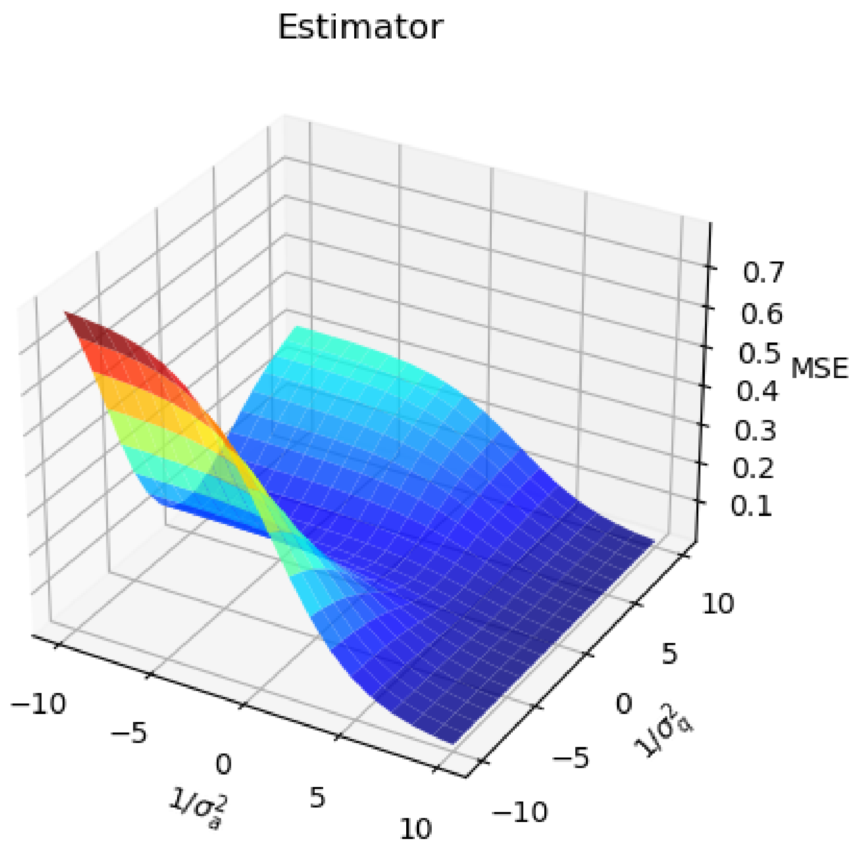


Figure 1: The mean-squared-error (MSE) of the LMMSE as function of quantize & analog SNR

if we define:

$$\frac{1}{2\pi\sqrt{\sigma_x^2\sigma_y^2-\rho^2}}\int_{-\infty}^{\infty} Q(y)e^{\frac{-(\sigma_x^2(y-\mu_y)^2)}{2(\sigma_x^2\sigma_y^2-\rho^2)}}\,dy \triangleq A$$

$$\frac{\rho(y-\mu_y)}{\sigma_y} \triangleq C \tag{2}$$

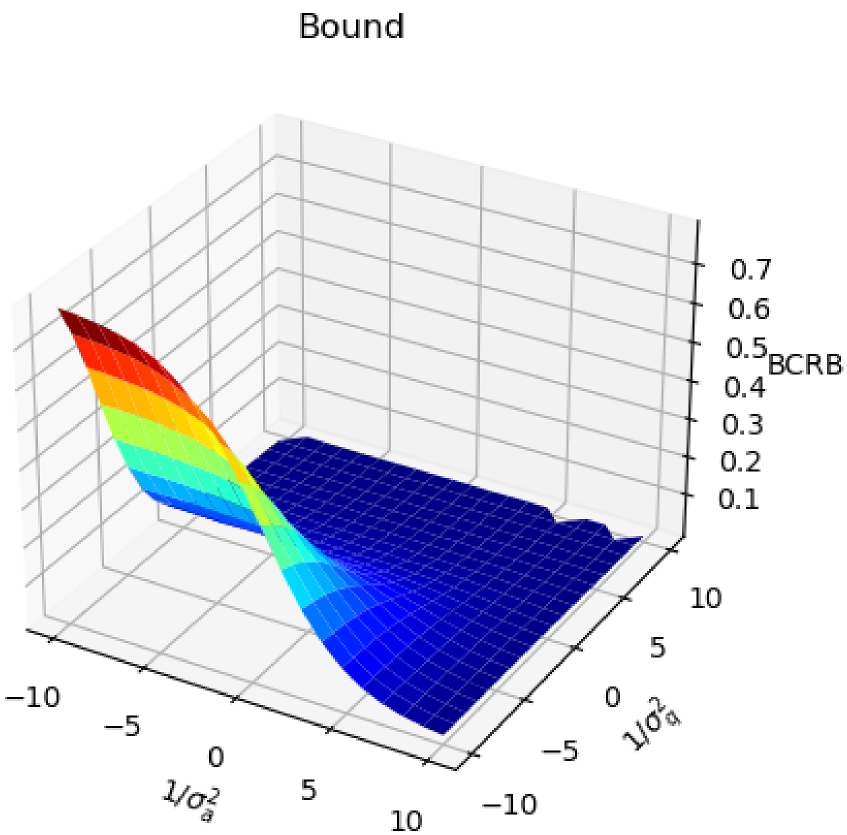


Figure 2: The BCRB as function of quantize & analog SNR

then we will get:

$$... = Ae^{\frac{C^2}{2(\sigma_x^2\sigma_y^2-\rho^2)}} \int_{-\infty}^{\infty} xe^{\frac{-(\sigma_y(x-\mu_x)-C)^2}{2(\sigma_x^2\sigma_y^2-\rho^2)}}\,dx$$

$$= Ae^{\frac{C^2}{2(\sigma_x^2\sigma_y^2-\rho^2)}} \int_{-\infty}^{\infty} xe^{\frac{-\left((x-\mu_x)-\frac{C}{\sigma_y}\right)^2}{2\frac{(\sigma_x^2\sigma_y^2-\rho^2)}{\sigma_y^2}}}\,dx$$

$$= Ae^{\frac{C^2}{2(\sigma_x^2\sigma_y^2-\rho^2)}} \int_{-\infty}^{\infty} xe^{\frac{-\left((x-\mu_x)-\frac{C}{\sigma_y}\right)^2}{2\frac{(\sigma_x^2\sigma_y^2-\rho^2)}{\sigma_y^2}}} \frac{1}{\sqrt{2\pi\frac{(\sigma_x^2\sigma_y^2-\rho^2)}{\sigma_y^2}}}\sqrt{2\pi\frac{(\sigma_x^2\sigma_y^2-\rho^2)}{\sigma_y^2}}\,dx$$

$$= A\sqrt{2\pi\frac{(\sigma_x^2\sigma_y^2-\rho^2)}{\sigma_y^2}}e^{\frac{C^2}{2(\sigma_x^2\sigma_y^2-\rho^2)}} \int_{\substack{-\infty\\3}}^{\infty} \frac{1}{\sqrt{2\pi\frac{(\sigma_x^2\sigma_y^2-\rho^2)}{\sigma_y^2}}}xe^{\frac{-\left(x-(\mu_x+\frac{C}{\sigma_y})\right)^2}{2\frac{(\sigma_x^2\sigma_y^2-\rho^2)}{\sigma_y^2}}}\,dx$$

$$= A\sqrt{2\pi\frac{(\sigma_x^2\sigma_y^2-\rho^2)}{\sigma_y^2}}e^{\frac{C^2}{2(\sigma_x^2\sigma_y^2-\rho^2)}}(\mu_x+\frac{C}{\sigma_y})$$

$$= \frac{1}{2\pi\sqrt{\sigma_x^2\sigma_y^2-\rho^2}}\sqrt{2\pi\frac{(\sigma_x^2\sigma_y^2-\rho^2)}{\sigma_y^2}} \int_{-\infty}^{\infty} Q(y)e^{\frac{-(\sigma_x^2(y-\mu_y)^2)}{2(\sigma_x^2\sigma_y^2-\rho^2)}}\,e^{\frac{C^2}{2(\sigma_x^2\sigma_y^2-\rho^2)}}$$

We can see that for $\sigma_x = \sigma_y = 1$ & $\mu_x = \mu_y = 0$ and $Q(X) = sign(X)$, We will get the usual Bussgang Theorem. In our case:

$$\mathbf{C}_{x,Q(y)} = -\frac{1}{\sqrt{2\pi\sigma_y^2}} \int_{-\infty}^{\tau} e^{\frac{-(\sigma_x^2 - \frac{\rho^2}{\sigma_y^2})(y-\mu_y)^2}{2(\sigma_x^2\sigma_y^2-\rho^2)}} (\frac{\rho}{\sigma_y^2} y + \mu_x - \frac{\rho\mu_y}{\sigma_y^2})\, dy$$
$$+ \frac{1}{\sqrt{2\pi\sigma_y^2}} \int_{\tau}^{\infty} e^{\frac{-(\sigma_x^2 - \frac{\rho^2}{\sigma_y^2})(y-\mu_y)^2}{2(\sigma_x^2\sigma_y^2-\rho^2)}} (\frac{\rho}{\sigma_y^2} y + \mu_x - \frac{\rho\mu_y}{\sigma_y^2})\, dy \tag{4}$$

0.3 bound on the BCRB

0.3.1 Lower bound on the BCRB

We find that the minimum mean-squared-error (MMSE) is lower bounded by the MSE of a 'genie-aided' estimator. This imaginary, non-implementable estimator knows the value of $\mathbf{G}\boldsymbol{\theta} + \mathbf{w}_q$ (i.e., quantized measurements before the quantization). Therefore, it has more information than the MMSE estimator, which results in a lower MSE. Thus, the performance of the 'genie-aided' estimator can be used as a (loose) bound.

It can be shown by using the extension of the data processing inequality for Fisher information [?], that the vector before quantization, $\mathbf{y}$, from (??) contains more information than the output of the quantizer, $\mathbf{x}_q$. That is,

$$\mathbf{J}_q \preceq \mathbf{J}_y, \tag{5}$$

where $\mathbf{J}_q$ is defined in (??) and

$$\mathbf{J}_y \triangleq -\mathrm{E}_{\boldsymbol{\theta}} \left[\mathrm{E}_{\mathrm{y}}|\boldsymbol{\theta} \left[\frac{\partial}{\partial \boldsymbol{\theta}} \left[\frac{\partial}{\partial \boldsymbol{\theta}} l(\mathbf{y}|\boldsymbol{\theta}) \right]^H \right] |\boldsymbol{\theta} \right]. \tag{6}$$

This result has been shown for non-Bayesian estimation with b-quantized data as well [?]. Thus, we can obtain a tractable lower bound on the BCRB by using $\mathbf{J}_y$ instead of $\mathbf{J}_q$. Yet, it is not tight, and we cannot use this bound to optimize the resource allocation problem.

0.3.2 Upper bound on the BCRB

Similarly, by using the extension of the data processing inequality for Fisher information [?] the vector $\mathbf{x}_a$ contains less information than its quantized version, i.e. contains more information. That is

$$\mathbf{J}_{Q(\mathbf{x}_a)} \preceq \mathbf{J}_a, \tag{7}$$

where $\mathbf{J}_a$ is defined in (??) and

$$\mathbf{J}_{Q(\mathbf{x}_a)} \triangleq -\mathrm{E}_{\boldsymbol{\theta}} \left[\mathrm{E}_{\mathrm{Q(x_a)}|\boldsymbol{\theta}} \left[\frac{\partial}{\partial \boldsymbol{\theta}} \left[\frac{\partial}{\partial \boldsymbol{\theta}} l(Q(\mathbf{x}_a)|\boldsymbol{\theta}) \right]^H \right] |\boldsymbol{\theta} \right]. \tag{8}$$

Thus, we can obtain an upper bound on the BCRB by using $\mathbf{J}_{Q(\mathbf{x}_a)}$ instead of $\mathbf{J}_a$. While this upper bound is not tight, it can be useful for analyzing the quantized BCRB, which relies solely on quantized observations.

0.3.3 Stein Bound

A naive bound could be found by extending the data processing inequality for Fisher information matrix (FIM) [**?**] and implementing it on the value of $\mathbf{y} = \mathbf{G}\boldsymbol{\theta} + \mathbf{w_q}$ (i.e., quantized measurements before the quantization), where the fact that finding bound on one of the Bayesian Fisher information matrix (BFIM) in (**??**) and replacing the original BFIM by the new bound yield BCRB's bound. Another option is based on the fact that $\mathbf{J}_{Q(y)} \leq \frac{2}{\pi}\mathbf{J}_y$, as has been explaned in **??** and [**?**]. However, we prefer not to use any 'genie-aided' bound that depends on the value of the pre-quantized variable $\mathbf{y}$.

In [**?**], it has been shown that for the non-Bayesian case, the BFIM of the quantized observations satisfies

$$\mathbf{J}_q(\boldsymbol{\theta}) \geq 2Re\left[\left(\frac{\partial\mu(\boldsymbol{\theta})}{\partial\boldsymbol{\theta}}\right)^H \Sigma^{-1}(\boldsymbol{\theta})\left(\frac{\partial\mu(\boldsymbol{\theta})}{\partial\boldsymbol{\theta}}\right)\right]. \tag{9}$$

Using the monotonicity of the expectation operator for measurable functions:

$$f(\boldsymbol{\theta}) \geq g(\boldsymbol{\theta}) \to E_{\boldsymbol{\theta}}(f(\boldsymbol{\theta})) \geq E_{\boldsymbol{\theta}}(g(\boldsymbol{\theta})), \tag{10}$$

and by applying it on (9), we obtain for our Bayesian scenario:

$$\mathbf{J}_q(\boldsymbol{\theta}) \geq 2E_{\boldsymbol{\theta}}\left(Re\left[\left(\frac{\partial\mu(\boldsymbol{\theta})}{\partial\boldsymbol{\theta}}\right)^H \Sigma^{-1}(\boldsymbol{\theta})\left(\frac{\partial\mu(\boldsymbol{\theta})}{\partial\boldsymbol{\theta}}\right)\right]\right) \tag{11}$$

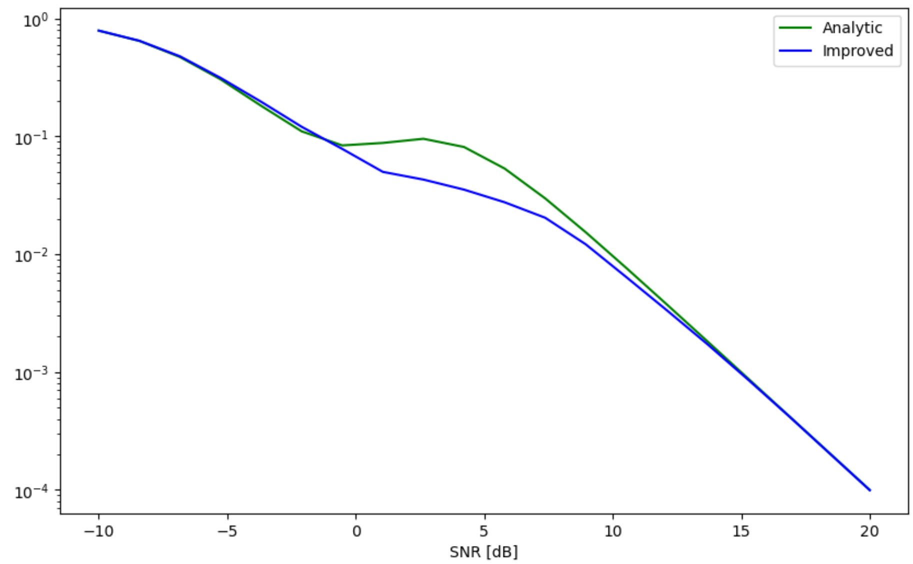
Where $\mu(\boldsymbol{\theta})$ & $\Sigma(\boldsymbol{\theta})$ are the first and second moments of the quantized observations $\mathbf{x_q}$. The computation of the first moment is detailed in the supplemented material, and the second moment could be found at [p.10 [**?**]] for the case where $\tau = \mathbf{0}$, otherwise, its possible to use sample covariacne matrix, and invert it by the algorithm describe in **??**.

Note that (11) is valid for $\tau = 0$, since in this case $\frac{\partial\Sigma}{\partial\boldsymbol{\theta}} = 0$. By inputting the right-hand side of equation (11) into (**??**), instead of the ordinary FIM (**??**), an upper bound on the BCRB can be obtained.

0.4 ALTERNATIVE BOUNDS

We are willing to find other bounds which will be used as an alternative to the BCRB. We would like to see whether any bound can be useful for the resource allocation problem or help in predicting the dithering phenomena area.

One of the alternatives to the BCRB is the Bayesian Weiss-Weinstein Bound (WWB) which has been proposed in the literature [**?**, p. 45]. For mathematical simplicity, we will discuss the scalar case, where: Where $L(x;\theta_1,\theta_2) = \frac{P(x,\theta_1)}{P(x,\theta_2)}$



A common technique is to fix s to 0.5 and then we can re-write the bound as:

$$\text{WWB} \triangleq \max_{h} \quad \frac{h^2 \mathrm{e}^{2\eta(0.5,h)}}{2\left(1 - \mathrm{e}^{\eta(0.5,2h)}\right)} \tag{12}$$

Where

$$\eta(s,h) = \ln \int p^s(\theta + h) p^{1-s}(\theta) \mathrm{e}^{\mu(s,\theta+h,\theta)} d\theta \tag{13}$$

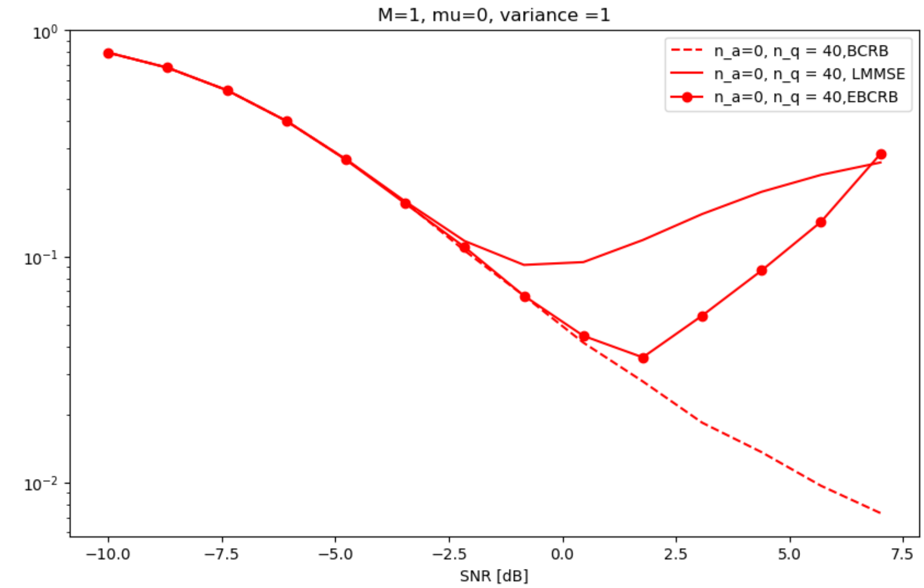
$$\mu(s,\theta_1,\theta_2) = \ln \int p(x|\theta_1)^s p(x|\theta_2)^{1-s} dx \tag{14}$$

Note that for $h \rightarrow 0$, the WWB is converged to the BCRB [**?**].

The computation of the expression provided in (14), which is the semi-invariant moment generating function, poses difficulties when dealing with quantized observations. As a result, the effectiveness of the WWB is reduced for such cases, particularly in scenarios involving channel estimation.

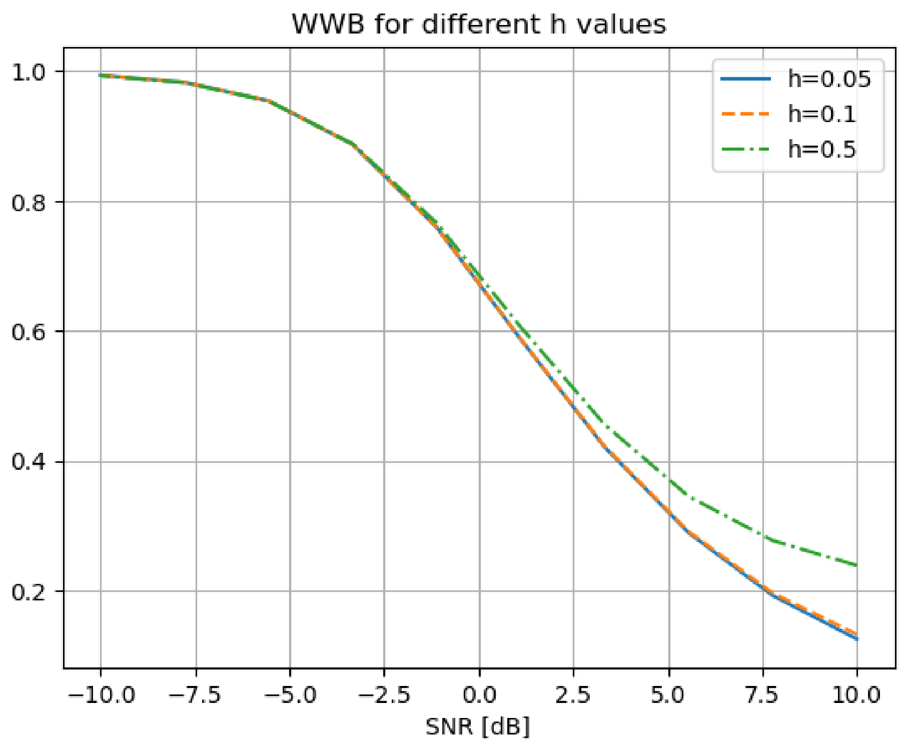
However, we conducted simulations specifically for the case of real-valued scalars, where we only had one quantized observation. The purpose was to gain insight into the new pattern of bounds and determine whether quantization impacted the bound. We aimed to examine if the presence of dithering (resulting in non-monotonic behavior) affected the bounds. It should be noted that dithering leads to variations in the ordering of different LMMSE estimates, rendering our BCRB irrelevant to the resource allocation problem.

Regrettably, we have observed that the behavior of WWB closely resembles that of BCRB. Calculating this bound is challenging and does not provide any advantages in addressing quantization phenomena.

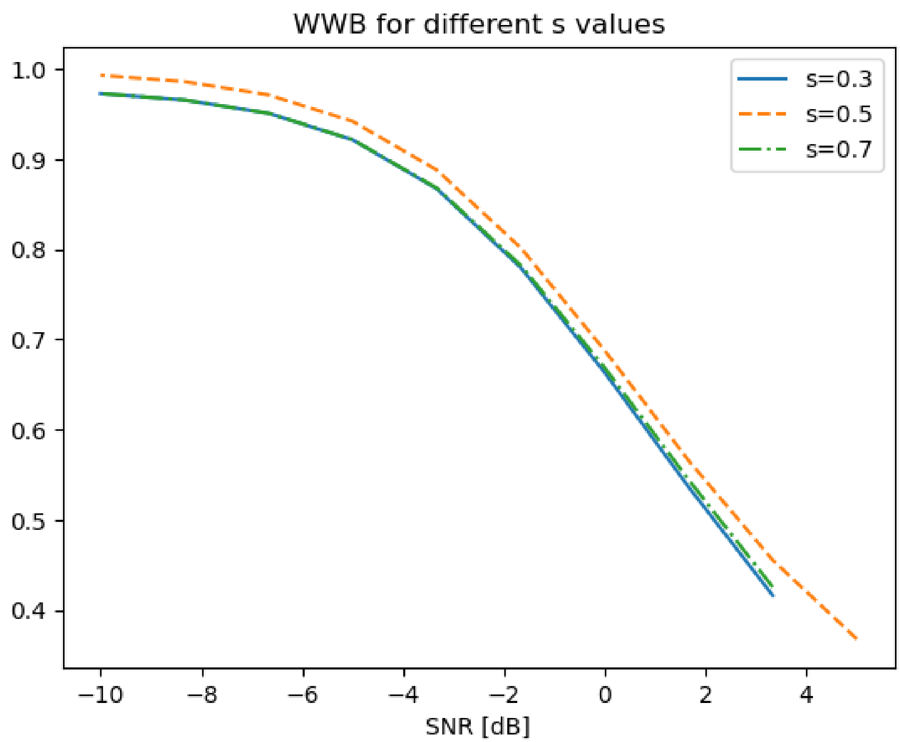


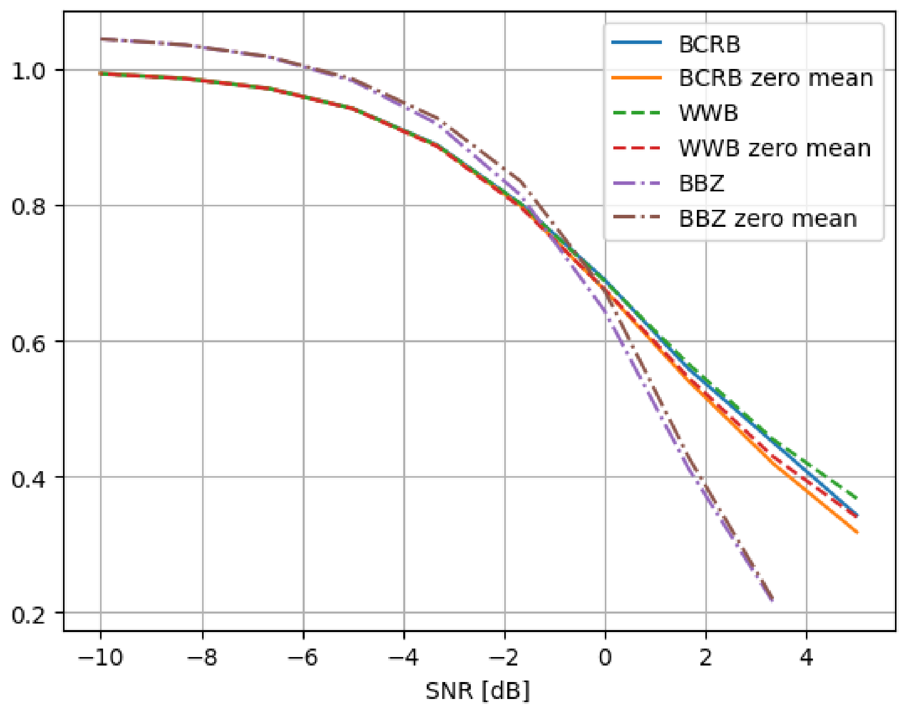
Another potential bound for addressing that issue is known as the Zvi-Zakai Bound [**?**, p. 54].

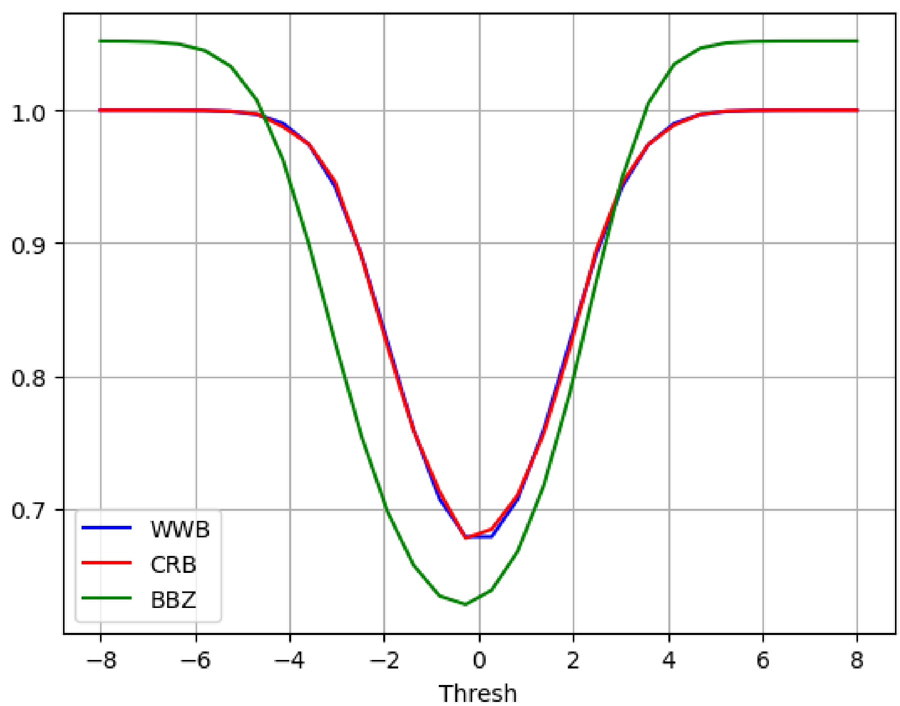
Unlike BCRB or WWB, this bound is not calculated using the covariance inequality. However, the second derivative of (14) is needed in order to compute the bound, which unfortunately makes it unfeasible for the quantized scenario.



WWB for different h values
h=0.05
h=0.1
h=0.5
1.0
0.8
0.6
0.4
0.2
−10.0
−7.5
−5.0
−2.5
0.0
2.5
5.0
7.5
10.0
SNR [dB]


WWB for different s values
s=0.3
s=0.5
s=0.7
1.0
0.9
0.8
0.7
0.6
0.5
0.4
−10
−8
−6
−4
−2
0
2
4
SNR [dB]


BCRB
BCRB zero mean
WWB
WWB zero mean
BBZ
BBZ zero mean
1.0
0.8
0.6
0.4
0.2
−10
−8
−6
−4
−2
0
2
4
SNR [dB]



1.0
0.9
0.8
0.7
WWB
CRB
BBZ
−8
−6
−4
−2
0
2
4
6
8
Thresh