Final Exam

Steve Mazza

June 15, 2014

Question 1: Given the differential equation $\frac{d^4y}{dx^4} + \frac{d^2y}{dx^2} = 0 \equiv y'''' + Ay''$ derive the equivalent system of first-order ordinary differential equations. This is a fourth order differential equation. What order is the system of equations? Is the system linear or nonlinear? What does such a system of first-order ordinary differential equations represent?

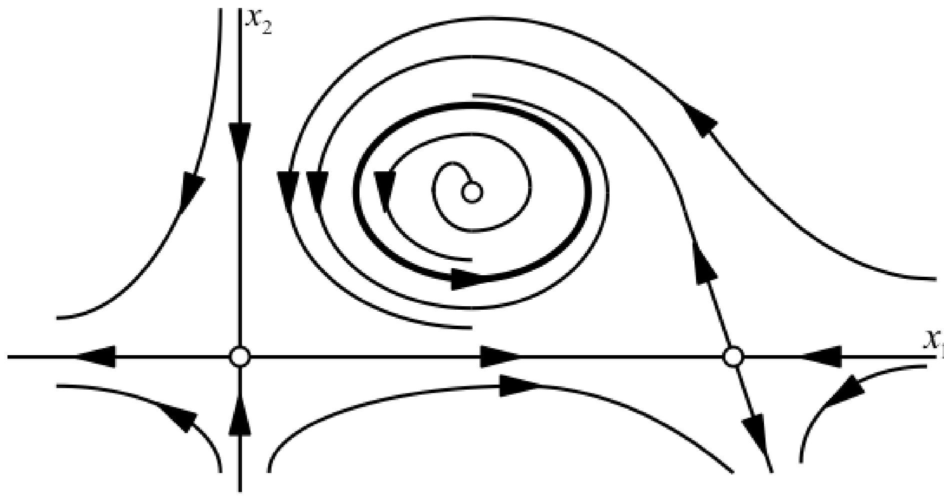
Question 2: The Maxwell-Bloch equations are a sophisticated model for a laser and describe the dynamics of the electric field E, the mean polarization of the atoms P, and the population inversion D:

$$\begin{aligned}\dot{E} &= (P - E)\\ \dot{P} &= \gamma_1(ED - P)\\ \dot{D} &= \gamma_2(\lambda + 1 - D - EP)\end{aligned}$$

where γ_1 and γ_2 are decay rates of the atomic polarization and population inversion, respectively, and λ is a pumping energy parameter. The parameter λ may be positive, negative, or zero; all other parameters are positive. In the simplest case, P and D relax rapidly to steady values, and hence may be eliminated as follows.

1. Assuming $\dot{D} \approx 0$ $\dot{P} \approx 0$, express P and D in terms of E, and thereby derive a first-order equation for the evolution of E.
2. Find all the fixed points of E.
3. Draw the bifurcation diagram of E^* versus λ. Distinguish between stable and unstable branches.

Question 3: What is this an example of? What features are represented?



This is a phase portrait.

From our class notes where this image was taken, "A phase portrait is a qualitative picture of the behavior of a system in phase space. Typically it shows fixed points, the stability of fixed points, closed orbits, and trajectories near the fixed points and closed orbits. The phase portrait above is illustrative only and does not represent a specific system."

Fixed points on this diagram are represented by the hollow dots. There are three; one appears at the intersection of the axes, one appears at the zero on the $x-$axis, and the third is located at the center of the sync of the closed orbit. The closed orbit, located near the center of the diagram, is represented by the heavier closed line with the counter-clockwise directional arrow.

There are trajectories located near the fixed points which indicate stability or instability. Trajectories moving toward fixed points indicate stability. Trajectories moving away from fixed points indicate instability. As seen in our diagram, a fixed point may be stable in one direction and instable in another. A geometric representation of this would be a saddle point. The unstable fixed point inside the closed orbit could be geometrically represented as a hill of bump in the landcape.

Question 4: For the Lorenz equations

$$\begin{aligned}\dot{x} &= \sigma(y-x)\\ \dot{y} &= rx - y - xz\\ \dot{z} &= xy - bz\end{aligned}$$

with $\sigma = 10, r = 28$, and $b = 2.66666$, and initial condition $x = 1.0 + \delta, y = 1.0$, and $z = 10$, determine how long it takes the absolute error between the "true x solution" ($\delta = 0$) to grow from δ to 0.1. Calculate for δ values of 0.01, $10^{-4}, 10^{-6}, 10^{-8}$, and 10^{-10}. What does this tell you about the predictability versus measurement error? Can you estimate the Liapunov exponent? (Strogatz, 366)

Question 5: Consider the iterated map given by

$$x_{n+1} = \begin{cases} rx_n & 0 \le x_n \le 0.5 \\ f(1-x_n) & 0.5 \le x_n \le 1 \end{cases}$$

where $0 < r < 2$. What properties do you expect to see in the orbit diagram? Is there any condition that might cause different behavior? The Liapunov exponent is $\lambda = \ln r$. (Strogatz, 366) What does this tell you about the behavior?

Question 6: In your own words and using no more than one paragraph, describe the difference between complex and complicated systems. That is, in your own opinion what distinguishes the two?

The difference between complexity and complicated systems hinges on the interactions of the parts of the system and on how difficult those interactions make predicting the future state of the system. Emergent properties, a hallmark of complex systems, come from these highly affective interactions. Note that systems with many moving parts and even possibly many interactions but whose outcomes are highly deterministic are not complex, but complicated. The classic example of this is a watch. Lastly, complex systems exhibit a robustness that merely complicated systems do not have. That is, they will continue to function despite the loss of a (small) subset of its parts.

Question 7: How are fractals and complexity related?

According to Mitchell (page 103, "Complexity as Fractal Dimension"), the fractal dimension of an object is a measure of its complexity. A fractal dimension is often a ratio (non-integer) that is a statistical index of complexity indicating how detail changes with scale. This is also referred to as the Hausdorf dimension and quantifies the number of self-similar copies at each level of magnification.

Question 8: Define what an adaptive agent-based model is and briefly describe its characteristics.

Agent based models.

- Ideas drive the development of tools (quarks drive accelerators) or tools may drive the development of ideas (microscopes drive microbiology).
- Agent-based models are computer models that permit the exploration of complex systems in more detail.
- Agent-based models consist of entities of various types (the agents) endowed with limited memory and cognitive ability that are embedded in a network (connected to other agents) and whose behaviors are interdependent (usually locally).
- Agents follow rules. The rules may be simple and fixed or complicated and adaptive.
- Adaptive agents follow metarules, which are rules about how rules can be changed.
- Agents often take discrete actions such as changing location, switching strategy (cooperation vs. defection), or joining or exiting a particular activity.
- Rules are often threshold-based, that is, rules in which the agents behavior remains the same until some threshold is met.
- Threshold effects can cause positive (more of the same) or negative (less of the same) feedbacks.
- Agent-based systems often exhibit complex, emergent behavior.

Question 9: In an engineering system consisting of various parts and mechanisms, what kinds of diversity are most applicable to determining complexity? How might that diversity be measured?

Complexity (from slide 62)

- Variation within a type seldom results in complexity

- Differences between types is often associated with complexity
- Diversity in composition is often associated with complexity

Five kinds of diversity measures (from slide 59)

- Variation measures
- Entropy measures
- Distance measures
- Attribute measures
- Disjoint population measures

Also see slides 60 & 61

Question 10: What approaches are likely to [be] part of any attempt to harness complexity in an inherently complex system?

Traditional approaches to control (command and control) fail for complex systems. Complex systems, however, can be harnessed by considering the primary characteristics of interdependence, connectedness, diversity, and adaptivity. One way to harness complexity is to adjust diversity, which determines the level of exploitation or exploration. Increasing diversity moves the system away from exploitation toward exploration. To a point, increasing diversity helps prevent errors (Linus' Law).

Watch for outliers in a population. It does not take many vocal people to sway the conditions of a complex system. Be wary of spending too much effort chasing small gains in efficiency and always leave room in the system so that failure does not cascade (as in recent bank failures).

Adjusting the interactions to remove unnecessary connections in favor of synergistic ones is another strategy. Incentives may not support your desired outcomes. In hierarchical structures, severing connections to parent organizations increases diversity and exploration.

Additionally, in "Harnessing Complexity," Axelrod & Cohen present a framework for harnessing complexity, which the refer to as the Complex Adaptive Systems approach. They describe various techniques of variation, interaction, and selection that the user of a system can leverage to affect or sway the outcome of a complex adaptive system.

The following ideas are summarized from the section titled, "What a User of the Framework Can Do," in the *Conclusion* of their book.

Variation

- Arrange organizational routines to generate a good balance between exploration and exploitation.
- Link processes that generate extreme variation to processes that select with few mistakes in the attribution of credit.

Interaction

- Build networks of reciprocal interaction that foster trust and cooperation.
- Assess strategies in light of how their consequences can spread.

- Promote effective neighborhoods.
- Do not sow large failures when reaping small efficiencies.

Selection

- Use social activity to promote the growth and spread of valued criteria.
- Look for shorter-term, finer-grained measures of success that can usefully stand in for longer-run, broader goals.

Detailed explanations of these approaches can be found on pages 155 – 158.