14.7.4 Find $I = \int_0^{2\pi} \frac{\sin^2\theta}{5+3\cos\theta}\,d\theta$

Let $z = e^{i\theta}$ $\quad dz = ie^{i\theta}\,d\theta$

$$= iz\,d\theta$$

$$d\theta = \frac{1}{iz}\,dz$$

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$$

$$= \frac{z+\frac{1}{z}}{2}$$

$$\sin^2\theta = 1-\cos^2\theta$$

$$= 1-\left(\frac{z+\frac{1}{z}}{2}\right)^2$$

Substitute into I $\quad I = \oint_C \frac{\left[1-\left(\frac{z+1/z}{2}\right)^2\right]\cdot\frac{1}{iz}\,dz}{5+3\left(\frac{z+1/z}{2}\right)}$

$$= \frac{1}{i}\oint_C \frac{\frac{-(z-1)^2}{2z^2}}{\frac{(3z+1)(z+3)}{2z}}\,dz$$

$$= \frac{1}{i}\oint_C -\frac{(z-1)^2}{z(3z+1)(z+3)}\,dz$$

Poles $= -\frac{1}{3}, -3, 0$

$$\lim_{z \to -1/3} \left(z + \frac{1}{3}\right) \left[- \frac{(z-1)^2}{z(3z+1)(z+3)} \right] = \frac{2}{3}$$

$$I = \frac{1}{i} 2\pi i \, R\left(-\frac{1}{3}\right)$$

$$= 2\pi \left(\frac{2}{3}\right)$$

$$= \frac{4\pi}{3} \neq \frac{2\pi}{9}$$