Homework I

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Problem 1.1.16

Some number of particles, a leaves $x = 0$ and heads toward $x = 1$. If r is the fraction of particles reflected back toward $x = 0$ at $x = 1$ (iteration $n = 1$) then ar particles are returned and $a(1-r)$ particles escape. At iteration $n = 2$ (back at $x = 0$) we have $ar \times r$ particles or ar^2 particles reflected back and, consequently, $ar \times (1-r)$ particles escaping. We write the sequence as

$$ar^0(1-r), ar^1(1-r), ar^2(1-r), ar^3(1-r), \ldots$$

and we can generalize this as

$$ar^{n-1}(1-r) \tag{1}$$

We further notice that odd values of n occur at $x = 1$ and even values of n occur at $x = 0$, so we can write the sequence for each as

$$x = 0 : ar^1(1-r), ar^3(1-r), ar^5(1-r), ar^7(1-r), \ldots \frac{a(1-r)r}{(1-r^2)} \tag{2}$$

$$x = 1 : ar^0(1-r), ar^2(1-r), ar^4(1-r), ar^6(1-r), \ldots \frac{a(1-r)}{(1-r^2)} \tag{3}$$

We recall

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r} \qquad \forall x : |x| < 1$$

So to sum the series at $x = 0$,

$$\begin{aligned}
\sum_{n=0}^{\infty} a(1-r)r^{2n+1} &= \sum_{n=0}^{\infty} [a(1-r)r]r^{2^n} \\
&= \frac{a(1-r)r}{1-r^2} \\
&= \frac{a(1-r)r}{(1-r)(1+r)} \\
&= \frac{ar}{1+r}
\end{aligned}$$

Summing the series at $x = 1$,

$$\begin{aligned}
\sum_{n=0}^{\infty} a(1-r)r^{2n} &= \sum_{n=0}^{\infty}[a(1-r)]r^{2^n} \\
&= \frac{a(1-r)}{1-r^2} \\
&= \frac{a(1-r)}{(1-r)(1+r)} \\
&= \frac{a}{1+r}
\end{aligned}$$

Since the particles begin at $x = 0$ and head toward $x = 1$ first, the largest fraction of particles which can escape at $x = 0$ ($n = 2$) is $\frac{1}{2}$.

Problem 1.6.27

We apply the ratio test to $\sum\limits_{n=0}^{\infty} \frac{100^n}{n^{200}}$ as follows and determine that our series diverges since $\rho > 1$.

$$\begin{aligned}
\rho_n &= \left| \frac{\frac{100^{n+1}}{(n+1)^{200}}}{\frac{100^n}{n^{200}}} \right| \\
\rho &= \lim_{n\to\infty} \left| \frac{\frac{100^{n+1}}{(n+1)^{200}}}{\frac{100^n}{n^{200}}} \right| \\
&= \lim_{n\to\infty} \left| \frac{100n^{200}}{(1+n)^{200}} \right| \\
&= 100
\end{aligned}$$

Problem 1.10.2

Problem 2.4.12

Problem 2.5.5

Problem 2.5.41

Problem 2.5.59