Tensor Analysis

Mathematical Methods in the Physical Sciences

Steve Mazza

Naval Postgraduate School
Monterey, CA

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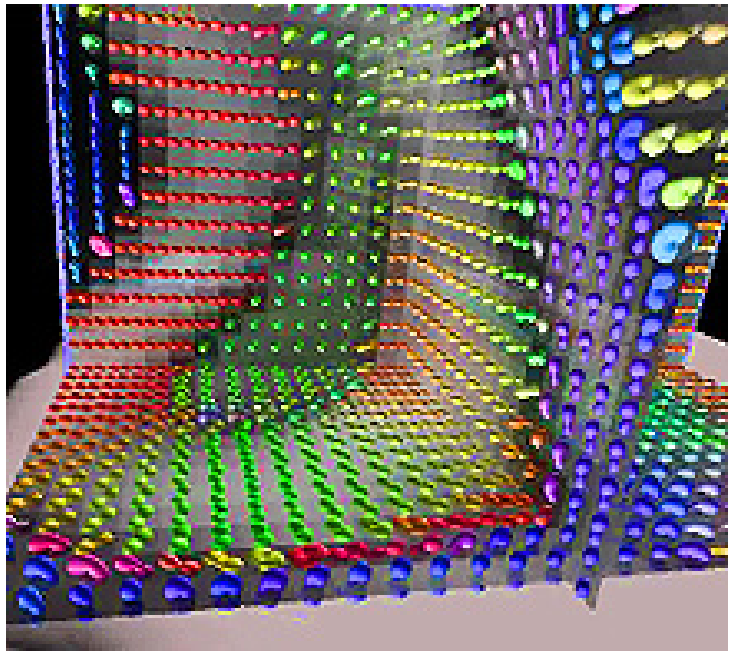
Introduction

- Tensors are designated by their size and *order*.
- Tensors of order 0 are scalars
- Tensors of order 1 are vectors
- A second order tensor has $3^2 = 9$ components



Cartesian Tensors

Tensor Notation and Operations

- For simplicity, we drop the summation sign and assume summation over any index which appears twice in one term.
- Contraction
  - Obtained by setting unlike indices equal and summing
  - Reduces the order by 2
- First and second order tensors can be displayed as matrices.
- Symmetry
  - Symmetric if $T_{ij} = T_{ji}$.
  - Antisymmetric if $T_{ij} = -T_{ji}$.
  - Any second order tensor can be written as a sum of a symmetric and antisymmetric tensor.
- Combination
  - The linear combination of two tensors of order n is a tensor of order n.
  - Addition is not defined for tensors of different order.
- Quotient Rule is useful for identifying components of a tensor.

Inertia Tensor

For a rigid body rotating about a fixed axis, we know that the velocity, ω, and momentum, L, are related by the equation $L = I\omega$ where I is the moment of inertia. But if the rotation axis is not fixed, then I must be replaced by a second order tensor with components I_{jk}.

Kronecker Delta and Levi-Civita Symbol

Kronecker Delta

$$\delta_{ij} = 1 \text{ if } i = j, 0 \text{ otherwise}$$

Levi-Civita Symbol

$$\begin{aligned} \epsilon_{ijk} &= 1 \text{ for an even permutation,} \\ &-1 \text{ for an odd permutation, and} \\ &0 \text{ if any indices are repeated.} \end{aligned}$$

Vector Identities

3-by-3 determinant

$$\det A = a_{1i} a_{2j} c_{3k} \epsilon_{ijk}$$

Dot Product

$$A \cdot B = A_i B_i$$

Cross Product

$$(A \times B)_i = \epsilon_{ijk} B_j C_k$$

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$$(\nabla \times V)_i = \epsilon_{ijk} \frac{\partial}{\partial x_j} V_k$$

Pseudovectors and Pseudotensors

More About Applications

Curvilinear Coordinates

Vector Operations in Orthogonal Curvilinear Coordinates

Non-Cartesian Tensors

Miscellaneous Problems

Questions?