Groups & Vector Spaces

Mathematical Methods in the Physical Sciences

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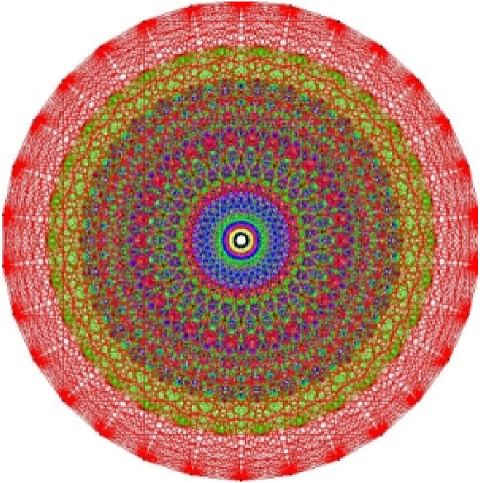
Groups



Definition of Groups

A group is a set of elements, G, together with a set operation, $\cdot$, that satisfies the following conditions:

Group Conditions

Closure: $\forall\ a, b \in G, a \cdot b \in G$

Association: $\forall\ a, b, c \in G, (a \cdot b) \cdot c = a \cdot (b \cdot c)$

Identity: $\exists$ exactly 1 element, $i \in G \mid \forall\ a \in G, i \cdot a = a \cdot i = a$

Inversion: $\forall\ a \in G\ \exists\ b \mid a \cdot b = b \cdot a = i$, where i is the identity element.

Operation Table

Product

The term *product* is used in the generalized sense.

It is handy to write out an operation table for the group.

Operation Table on $\pm 1, \pm i$

	1	i	-1	-i
1	1	i	-1	-i
i	i	-1	-i	1
-1	-1	-i	1	i
-i	-i	1	i	-1

Group Isomorphism

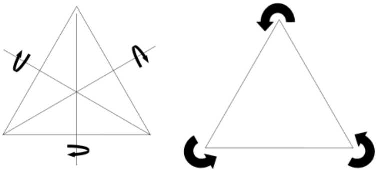
Definition

$f : (G, \cdot) \to (H, \times) \mid \forall\, u, v \in\ G, f(u \cdot v) = f(u) \times f(v)$

Two groups are considered *isomorphic* if an isomorphism exists between them. We write $G \cong H$. Isomorphic groups are considered indistinguishable.

Definition

The symmetry group is the group of all isometries under which the elements are invariant with regard to the group operation.



Equilateral Triangle

We consider the example presented on Boas, page 174, where the equilateral triangle is symmetric on three reflections and three rotations.

Conjugate Elements, Class, Character

Conjugate Elements

$a, b, \in G$ are called conjugate elements if $\exists\ c \in G \mid c^{-1} \cdot a \cdot c = b$.

All elements of a class describe the same mapping.

Class

The set of all elements conjugate to a given element form a *class*.

Class is a subset of a group, but not a usually subgroup. All matrices of a class have the same trace (sum of diagonal elements).

Character

The trace of a matrix is called its character.

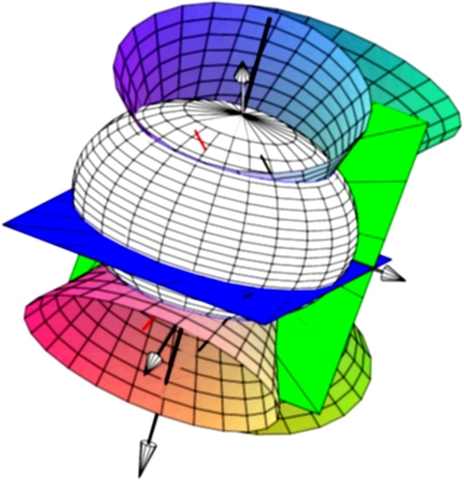
All matrices of a class have the same character.

Irreducible Representations

Definition

If a group, G, only has two subrepresentations, $\emptyset$ and G, then G is said to be irreducible.

Vector Spaces



Definition of Vector Spaces

A vector space over field F is a set V together with two binary operations satisfying following conditions:

Group Conditions

Closure: $\forall\ \vec{u}, \vec{v} \in V, \vec{u} + \vec{v} \in V$

Vector Addition:

$$\begin{aligned} \text{Commutation:} &\quad \forall\ \vec{u}, \vec{v} \in V, \vec{u} + \vec{v} = \vec{v} + \vec{u} \\ \text{Association:} &\quad \forall\ \vec{u}, \vec{v}, \vec{w} \in V, (\vec{u} + \vec{w}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w}) \end{aligned}$$

Additive Identity: $\exists\ \vec{0} \in V \mid \forall\ \vec{v} \in V, \vec{0} + \vec{v} = \vec{v} + \vec{0} = \vec{v}$

Additive Inverse: $\forall\ \vec{v} \in V\ \exists\ -\vec{v} \mid \vec{v} + (-\vec{v}) = 0$

Multiplication:

$$\begin{aligned} \text{Distribution 1:} &\quad \forall\ \vec{u}, \vec{v} \in V, k(\vec{u} + \vec{v}) = k\vec{u} + k\vec{v} \\ \text{Distribution 2:} &\quad \forall\ \vec{v} \in V, \vec{v}(k_1 + k_2) = k_1\vec{v} + k_2\vec{v} \\ \text{Association:} &\quad \forall \vec{v} \in V, \vec{v}(k_1 \cdot k_2) = (\vec{v} \cdot k_1)k_2 \\ \text{Identity:} &\quad \forall\ \vec{v} \in V, 1 \cdot \vec{v} = \vec{v} \\ \text{Zero:} &\quad \forall\ \vec{v} \in V, 0 \cdot \vec{v} = \vec{0} \end{aligned}$$

Inner Product, Norm, Orthogonality

For $A(x)$ and $B(x)$ on $a \leq x \leq b$, we define the following.

Inner Product

$\langle A(x), B(x) \rangle = \int_a^b \overline{Ax} B(x) dx$

Norm

$\|A(x)\| = \sqrt{\int_a^b \overline{Ax} A(x) dx}$

Orthogonality

$A(x) \perp B(x)$ on (a, b) if $\int_a^b \overline{Ax} B(x) dx = 0$

Schwartz's Inequality

Let $\vec{u}, \vec{v}$ be elements of a vector space. Recall previously that for n-dimensional space, Schwartz's Inequality states

Schwartz's Inequality for *n*-dimensional Space

$|\vec{u} \cdot \vec{v}| \leq \vec{u}\vec{v}$

For an inner product space, we modify the definition as follows.

Schwartz's Inequality for Inner Product Space

$|\langle \vec{u} \mid \vec{v} \rangle|^2 \leq \langle \vec{u} \mid \vec{u} \rangle \langle \vec{v} \mid \vec{v} \rangle$

Boas develops a proof of this on page 182.

Orthonormal Basis

Two functions are *orthonormal* if they

- satisfy the property of orthogonality, and
- have a norm $= 1$

Infinite Dimensional Spaces

An *infinite dimensional vector space* is one which does not have a finite basis.
We recall that the set of all linearly independent vectors $\vec{b_i}$ which, expressed as some finite sum, are capable of describing any vector, $\vec{v}$, form the *basis* for our vector space.
As an example, consider $\mathbb{R}[x]$, the set of polynomials in x with real coefficients for which $x : n \in \mathbb{N}$ forms the basis.

Questions?