

Groups & Vector Spaces

Mathematical Methods in the Physical Sciences

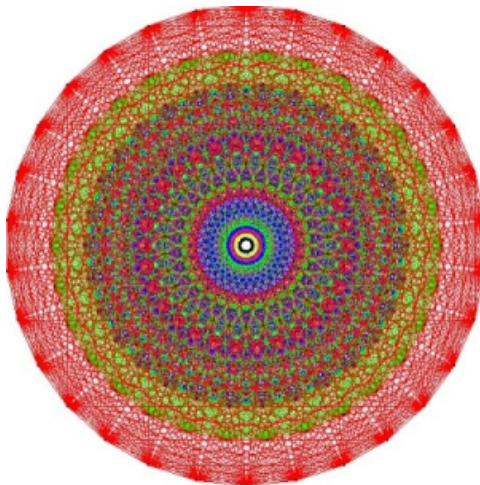
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Groups



Definition of Groups

A group is a set of elements, G , together with a set operation, \cdot , that satisfies the following conditions:

Group Conditions

Closure: $\forall a, b \in G, a \cdot b \in G$

Association: $\forall a, b, c \in G, (a \cdot b) \cdot c = a \cdot (b \cdot c)$

Identity: \exists exactly 1 element, $i \in G \mid \forall a \in G, i \cdot a = a \cdot i = a$

Inversion: $\forall a \in G \exists b \mid a \cdot b = b \cdot a = i$, where i is the identity element.

Operation Table

Product

The term *product* is used in the generalized sense.

It is handy to write out an operation table for the group.

Operation Table on $\pm 1, \pm i$

	1	i	-1	-i
1	1	i	-1	-i
i	i	-1	-i	1
-1	-1	-i	1	i
-i	-i	1	i	-1

Definition

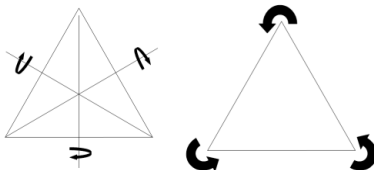
$$f : (G, \cdot) \rightarrow (H, \times) \mid \forall u, v \in G, f(u \cdot v) = f(u) \times f(v)$$

Two groups are considered *isomorphic* if an isomorphism exists between them. We write $G \cong H$. Isomorphic groups are considered indistinguishable.

Group Symmetry

Definition

The symmetry group is the group of all isometries under which the elements are invariant with regard to the group operation.



Equilateral Triangle

We consider the example presented on Boas, page 174, where the equilateral triangle is symmetric on three reflections and three rotations.

Conjugate Elements, Class, Character

Conjugate Elements

$a, b, \in G$ are called conjugate elements if $\exists c \in G \mid c^{-1} \cdot a \cdot c = b$.

All elements of a class describe the same mapping.

Class

The set of all elements conjugate to a given element form a *class*.

Class is a subset of a group, but not a usually subgroup. All matrices of a class have the same trace (sum of diagonal elements).

Character

The trace of a matrix is called its character.

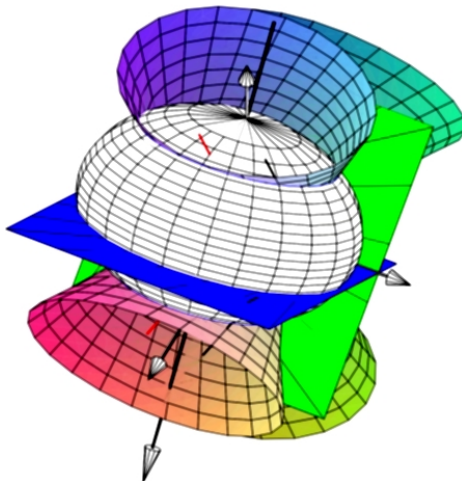
All matrices of a class have the same character.

Irreducible Representations

Definition

If a group, G , only has two subrepresentations, \emptyset and G , then G is said to be irreducible.

Vector Spaces



Definition of Vector Spaces

A vector space over field F is a set V together with two binary operations satisfying following conditions:

Group Conditions

Closure: $\forall \vec{u}, \vec{v} \in V, \vec{u} + \vec{v} \in V$

Vector Addition:

Commutation: $\forall \vec{u}, \vec{v} \in V, \vec{u} + \vec{v} = \vec{v} + \vec{u}$

Association: $\forall \vec{u}, \vec{v}, \vec{w} \in V, (\vec{u} + \vec{w}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$

Additive Identity: $\exists \vec{0} \in V \mid \forall \vec{v} \in V, \vec{0} + \vec{v} = \vec{v} + \vec{0} = \vec{v}$

Additive Inverse: $\forall \vec{v} \in V \exists -\vec{v} \mid \vec{v} + (-\vec{v}) = \vec{0}$

Multiplication:

Distribution 1: $\forall \vec{u}, \vec{v} \in V, k(\vec{u} + \vec{v}) = k\vec{u} + k\vec{v}$

Distribution 2: $\forall \vec{v} \in V, \vec{v}(k_1 + k_2) = k_1\vec{v} + k_2\vec{v}$

Association: $\forall \vec{v} \in V, \vec{v}(k_1 \cdot k_2) = (\vec{v} \cdot k_1)k_2$

Identity: $\forall \vec{v} \in V, 1 \cdot \vec{v} = \vec{v}$

Zero: $\forall \vec{v} \in V, 0 \cdot \vec{v} = \vec{0}$

Inner Product, Norm, Orthogonality

Schwart's Inequality

Orthonormal Basis

Infinite Dimensional Spaces

FIN