The Laplace Transform

Mathematical Methods in the Physical Sciences

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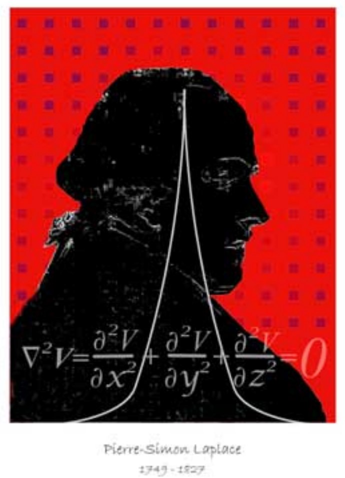
Introduction

Definition

$$L(f) = \int_0^\infty f(t)e^{-pt}dt = F(p)$$

Integral transformation

- Begin with a function $f(t)$
- Multiply by a function of t and p
- Find a definite integral with respect to t



Derivation of L1

Substitute $F(t) = 1$

L1

$$\begin{aligned} F(p) &= \int_0^\infty 1 \cdot d^{-pt} dt \\ &= -\frac{1}{p} e^{-pt} \Big|_0^\infty \\ &= \frac{1}{p} \end{aligned}$$

The real part of p must be > 0.

A table of Laplace transforms can be very handy. One such table is available in Boas, pps 469 - 471. A brief web search will turn up more, as well. The tables contain transforms for both $f(t)$ and $F(p)$ and operate very much like the table for power series in Chapter 1 on page 26.

Laplace transforms can be used to reduce ODEs to simpler algebraic equations. We take the Laplace transform of each term in the differential equation.

$L(y')$

$$\begin{aligned} L(y') &= \int_0^\infty y'(t)e^{-pt}\,dt \\ &= e^{-pt}y(t)\Big|_0^\infty - (-p)\int_0^\infty y(t)e^{-pt}\,dt \\ &= pL(y) - y(0) \\ &= pY - y_0 \end{aligned}$$

Solutions of Differential Equations (continued)

We continue for second order by re-writing y'' as $(y')'$

$L(y'')$

$$\begin{aligned} L(y'') &= pL(y') - y'(0) \\ &= p^2 L(y) - py(0) - y'(0) \\ &= p^2 Y - py_0 - y'_0 \end{aligned}$$

By using the table to look up the Laplace transforms we can find solutions to differential equations quite directly. Using $L15$ and values $a = 3$ and $p = 2$ we directly calculate,

Example 5, page 442

$$\begin{aligned} \int_0^\infty e^{-2t}\left(1 - \cos 3t\right) dt &= \frac{3^2}{2\left(2^2 + 3^2\right)} \\ &= \frac{9}{26} \end{aligned}$$

We see that this is very powerful and convenient.

Questions?