Groups & Vector Spaces

Mathematical Methods in the Physical Sciences

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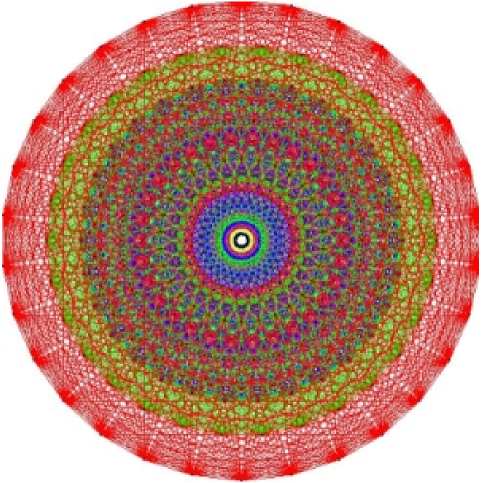
Groups



Definition of Groups

A group is a set of elements, G, together with a set operation, $\cdot$, that satisfies the following conditions:

Group Conditions

Closure: $\forall\ a, b \in G, a \cdot b \in G$

Association: $\forall\ a, b, c \in G, (a \cdot b) \cdot c = a \cdot (b \cdot c)$

Identity: $\exists$ exactly 1 element, $i \in G \mid \forall\ a \in G, i \cdot a = a \cdot i = a$

Inversion: $\forall\ a \in G\ \exists\ b \mid a \cdot b = b \cdot a = i$, where i is the identity element.

Product

The term *product* is used in the generalized sense.

It is handy to write out an operation table for the group.

$\pm 1, \pm i$

	1	i	-1	-i
1	1	i	-1	-i
i	i	-1	-i	1
-1	-1	-i	1	i
-i	-i	1	i	-1

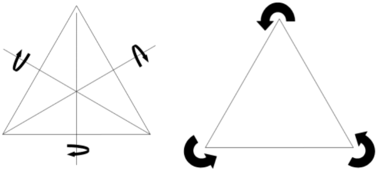
Definition

$f : (G, \cdot) \to (H, \times) \mid \forall\ u, v \in\ G, f(u \cdot v) = f(u) \times f(v)$

Two groups are considered *isomorphic* if an isomorphism exists between them. We write $G \cong H$. Isomorphic groups are considered indistinguishable.

Definition

The symmetry group is the group of all isometries under which the elements are invariant with regard to the group operation.



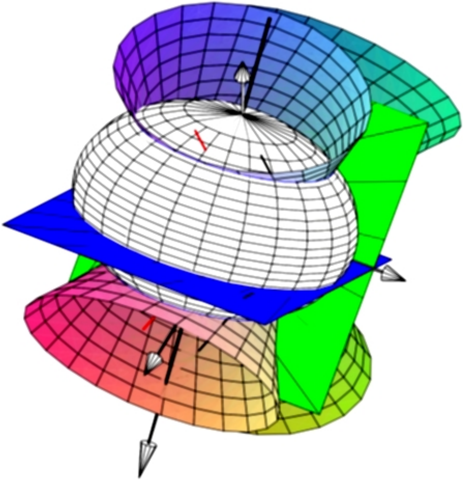
Equilateral Triangle

We consider the example presented on Boas, page 174, where the equilateral triangle is symmetric on three reflections and three rotations.

Conjugate Elements, Class, Character

Irreducible Representations

Infinite Groups



Definition of Vector Spaces

A vector space over field F is a set V together with two binary operations satisfying following conditions:

Group Conditions

Closure: $\forall\ \vec{u}, \vec{v} \in V, \vec{u} + \vec{v} \in V$

Vector Addition:

Commutation: $\forall\ \vec{u}, \vec{v} \in V, \vec{u} + \vec{v} = \vec{v} + \vec{u}$

Association: $\forall\ \vec{u}, \vec{v}, \vec{w} \in V, (\vec{u} + \vec{w}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$

Additive Identity: $\exists\ \vec{0} \in V \mid \forall\ \vec{v} \in V, \vec{0} + \vec{v} = \vec{v} + \vec{0} = \vec{v}$

Additive Inverse: $\forall\ \vec{v} \in V\ \exists\ -\vec{v} \mid \vec{v} + (-\vec{v}) = 0$

Multiplication:

Distribution 1: $\forall\ \vec{u}, \vec{v} \in V, k(\vec{u} + \vec{v}) = k\vec{u} + k\vec{v}$

Distribution 2: $\forall\ \vec{v} \in V, \vec{v}(k_1 + k_2) = k_1\vec{v} + k_2\vec{v}$

Association: $\forall \vec{v} \in V, \vec{v}(k_1 \cdot k_2) = (\vec{v} \cdot k_1)k_2$

Identity: $\forall\ \vec{v} \in V, 1 \cdot \vec{v} = \vec{v}$

Zero: $\forall\ \vec{v} \in V, 0 \cdot \vec{v} = \vec{0}$

Inner Product, Norm, Orthogonality

Schwart's Inequality

Orthonormal Basis

Infinite Dimensional Spaces

Questions?