Vector Analysis

Mathematical Methods in the Physical Sciences

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Introduction

We will extend our discussion of vectors from chapters 3, 4, & 5 with the following broad overview of topics

- Vector products
- Differentiation
- Integration

The end of the chapter contains discussions of vector theorems, which we will save for another lecture.

Applications of Vector Multiplication

We can apply the dot and cross products introduced in Chapter 3, Section 4, to calculate

$$\begin{aligned} \text{Work:} \quad & Fd\ \cos\theta = \vec{F} \cdot \vec{d} \\ \text{Torque:} \quad & rF\ \sin\theta = \vec{r} \times \vec{F} \\ \angle\ \text{Velocity:} \quad & \omega r\ \sin\theta = |\vec{\omega} \times \vec{r}| \end{aligned}$$

Angular velocity is solved by considering that the linear velocity $\vec{v}$ of some point P is equal to $\vec{\omega} \times \vec{r}$, and that the magnitude of $\vec{v} = |\vec{\omega} \times \vec{r}|$, which is also equal to $\omega r\ \sin\theta$.

Triple Products

Triple Scalar Product

$$\vec{A} \cdot \left(\vec{B} \times \vec{C}\right)$$

- Can be thought of as the volume of a parallelepiped.
- Is the determinant of a 3×3 matrix.
- The product is invariant under a circular shift:

Circular Shift Invariant

$$\vec{A} \cdot \left(\vec{B} \times \vec{C}\right) = \vec{B} \cdot \left(\vec{C} \times \vec{A}\right) = \vec{C} \cdot \left(\vec{A} \times \vec{B}\right)$$

- Satisfies an equality under cross product negation:

Negative Cross Product

$$\vec{A} \cdot \left(\vec{B} \times \vec{C}\right) = -\vec{A} \cdot \left(\vec{C} \times \vec{B}\right)$$

Triple Products (continued)

Triple Vector Product

$$\vec{A} \times \left(\vec{B} \times \vec{C}\right) = \left(\vec{A} \cdot \vec{C}\right) \vec{B} - \left(\vec{A} \cdot \vec{B}\right) \vec{C}$$

- Known as the "BAC-CAB" product.
- Useful for simplifying some calculations in physics.
- Is anticommutative:

Anticommutative

$$\left(\vec{A} \times \vec{B}\right) \times \vec{C} = -\vec{C} \times \left(\vec{A} \times \vec{B}\right)$$

- Satisfies Lagrange:

Jacobi Identity

$$\vec{A} \times \left(\vec{B} \times \vec{C}\right) + \vec{B} \times \left(\vec{C} \times \vec{A}\right) + \vec{C} \times \left(\vec{A} \times \vec{B}\right) = 0$$

6.3.2

$$\begin{aligned}
\text{given:}\quad \vec{B} &= 2\hat{i} - \hat{j} + 3\hat{k} && \text{(force)} \\
\vec{C} &= \hat{j} - 5\hat{k} && \text{(displacement)} \\
\text{apply:}\quad W &= F \cdot d \\
\text{find:}\quad \vec{B} \cdot \vec{C} &= B_i C_i + B_j C_j + B_k C_k \\
&= 2 \cdot 0 + (-1) \cdot 1 + 3 \cdot (-5) \\
&= 0 + (-1) + (-15) \\
&= 0 - 1 - 15 \\
&= -16
\end{aligned}$$

Differentiation of Vectors

$\vec{A}'$ is the vector whose components are the derivatives of the components of $\vec{A}$.

Definition

$$\frac{d\vec{A}}{dt} = \frac{dA_x}{dt}\hat{i} + \frac{dA_y}{dt}\hat{j} + \frac{dA_z}{dt}\hat{k}$$

Fields

Fields can be either *scalar* or vector.
A vector field, $\vec{f}(x, y)$ on $\mathbb{R}^2$ can be defined as some scalar function, $P(x, y)$ for the $\hat{i}$ component and some other scalar function, $Q(x, y)$ for the $\hat{j}$ component such that for any given (x, y) we can calculate

$$\vec{f}(x, y) = P(x, y)\hat{i}, Q(x, y)\hat{j}$$

This associates a vector with every point on the *xy* plane.
The term *field* refers to both the region and the physical value of the region.

Consider the points, (x_0, y_0, z_0) and (x, y, z), both located within some scalar field described by $\phi(x, y, z)$. The separation of these points can be described by the distance, s, in the direction $\vec{u}$. From

$$(x, y, z) - (x_0, y_0, z_0) = \hat{u}s = (a\hat{i} + b\hat{j} + c\hat{k})s$$

we derive the parametric equations,

$$x = x_0 + as, y = y_0 + bs, z = z_0 cs$$

Substituting the equations back in $\phi(x, y, z)$, we arrive at ϕ as a functino of s and find that $d\phi/ds = \nabla\phi \cdot \hat{u}$, so

Definition

$$\nabla\phi = \frac{\partial\phi}{\partial x}\vec{i} + \frac{\partial\phi}{\partial y}\vec{j} + \frac{\partial\phi}{\partial z}\vec{k}$$

Expressions Involving ∇

Vector Operator

$$\nabla = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$$

Div

$$\nabla \cdot \vec{V} = \operatorname{div} \vec{V} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$$

Curl

$$\begin{aligned} \nabla \times \vec{V} &= \operatorname{curl} \vec{V} \\ &= \left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right)\hat{i} + \left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x}\right)\hat{j} + \left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)\hat{k} \end{aligned}$$

Expressions Involving ∇ (continued)

The Laplacian

$$\begin{aligned}
\nabla^2\phi &= \nabla\cdot\nabla\phi \\
&= \text{div grad } \phi \\
&= \frac{\partial}{\partial x}\frac{\partial\phi}{\partial x} + \frac{\partial}{\partial y}\frac{\partial\phi}{\partial y} + \frac{\partial}{\partial z}\frac{\partial\phi}{\partial z} +
\end{aligned}$$

6.7.5 (Div)

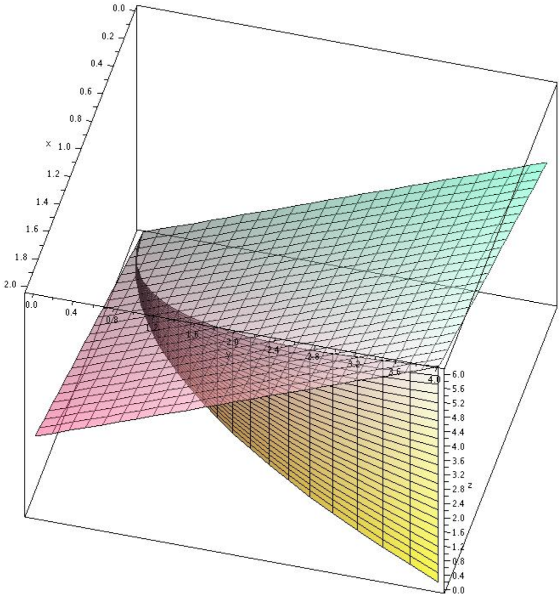
$$\text{given: } \vec{V} = x^2\hat{i} + y^2\hat{j} + z^2\hat{k}$$

$$\begin{aligned}\text{find: } \nabla \cdot \vec{V} &= \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z} \\ &= 2x + 2y + 2z \\ &= 2(x + y + z)\end{aligned}$$

6.7.5 (Curl)

$$\text{given: } \vec{V} = x^2\hat{i} + y^2\hat{j} + z^2\hat{k}$$

$$\begin{aligned}
\text{find: } \nabla \times \vec{V} &= \left(\frac{\partial V_z}{\partial y} - \frac{\partial V_x}{\partial z}\right)\hat{i} \\
&+ \left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x}\right)\hat{j} \\
&+ \left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)\hat{k} \\
&= (0-0)\hat{i} + (0-0)\hat{j} + (0-0)\hat{k} \\
&= 0\hat{i} + 0\hat{j} + 0\hat{k} \\
&= 0
\end{aligned}$$


x
0.0
0.2
0.4
0.6
0.8
1.0
1.2
1.4
1.6
1.8
2.0
y
0.0
0.4
0.8
1.2
1.6
2.0
2.4
2.8
3.2
3.6
4.0
z
6.0
5.6
5.2
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Questions?