Solving Equations

Mathematical Methods in the Physical Sciences

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Introduction

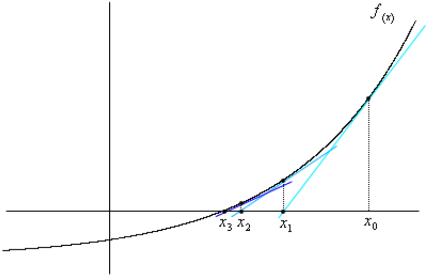
Solving an Equation in One Variable

Given an equation of the form $f(x) = 0$, we want to find a solution to within the accuracy of our computations. We explore four methods

- Newton's Method, which uses a series of linear approximations by taking the derivative
- Poor Man's Newton, which uses a series of linear approximations without taking the derivative (numerical approximation).
- Another linear method that uses bracketing
- Divide and Conquer, which looks a little like a binary search

Newton-Raphson

$$x_{n+1} = x_n - \frac{f(n)}{f'(n)}$$



Geometrically, $(x_{n+1}, 0)$ is the intersection with the x-axis of the tangent to the graph of f at $(x_n, f(x_n))$.

Newton's Method Example

We calculate $\sqrt{5}$ by finding an approximate solution to the equation $f(x) = x^2 - 5 = 0$. We choose a first approximation of $x_0 = 2$, so

$$\begin{aligned}
f(x) &= x^2 - 5 \\
f'(x) &= 2x \\
fLx_0(x) &= f(x_0) + f'(x_0)(x - x_0) \\
&= -1 + 4(x - 2) \\
&= 4x - 9 = \frac{9}{4} = 2.25 \\
fLx_1(x) &= f(x_1) + f'(x_1)(x - x_1) \\
&= \left(\frac{81}{16} - 5\right) + \frac{9}{2}\left(x - \frac{9}{4}\right) \\
&= \frac{9}{2}x - \frac{161}{16} = 2.236111
\end{aligned}$$

Newton's Method Caveats

- Many functions have more than one zero
- Some functions will never converge
- Unfortunate initial guesses can be very misleading
- If f is implicitly defined, we may find values for x_n for which f is undefined.

However, if f goes from negative to positive at the true solution x, and f' is increasing between x and your guess x_0, which is greater than x, then the method will always converge.

Poor Man's Newton

Instead of calculating the derivative at each successive step we use the following approximation

Approximation to the Derivative

$$f'(x) \approx \frac{f(x_i + d) - f(x_i)}{d}$$

Then in general,

Approximation to Newton

$$x_{n+1} = x_n - d\frac{f(x_n)}{f(x_n + d) - f(x_n)}$$

The trick is selecting an appropriate value for d.

Another Linear Method

Divide and Conquer

Solving Two General Equations in Two Variables

Questions?