

# Groups & Vector Spaces

## Mathematical Methods in the Physical Sciences

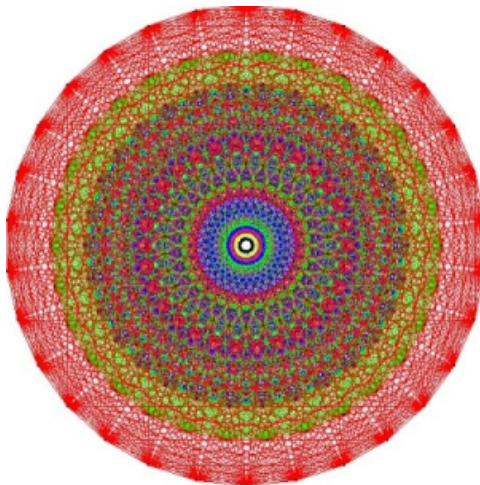
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# Groups



# Definition of Groups

A group is a set of elements,  $G$ , together with a set operation,  $\cdot$ , that satisfies the following conditions:

## Group Conditions

Closure:  $\forall a, b \in G, a \cdot b \in G$

Association:  $\forall a, b, c \in G, (a \cdot b) \cdot c = a \cdot (b \cdot c)$

Identity:  $\exists$  exactly 1 element,  $i \in G \mid \forall a \in G, i \cdot a = a \cdot i = a$

Inversion:  $\forall a \in G \exists b \mid a \cdot b = b \cdot a = i$ , where  $i$  is the identity element.

# Operation Table

## Product

The term *product* is used in the generalized sense.

It is handy to write out an operation table for the group.

## Operation Table on $\pm 1, \pm i$

	1	i	-1	-i
1	1	i	-1	-i
i	i	-1	-i	1
-1	-1	-i	1	i
-i	-i	1	i	-1

# Group Isomorphism

## Definition

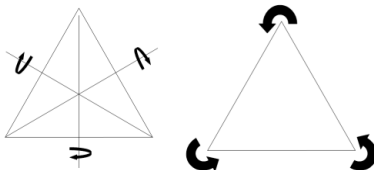
$$f : (G, \cdot) \rightarrow (H, \times) \mid \forall u, v \in G, f(u \cdot v) = f(u) \times f(v)$$

Two groups are considered *isomorphic* if an isomorphism exists between them. We write  $G \cong H$ . Isomorphic groups are considered indistinguishable.

# Group Symmetry

## Definition

The symmetry group is the group of all isometries under which the elements are invariant with regard to the group operation.



## Equilateral Triangle

We consider the example presented on Boas, page 174, where the equilateral triangle is symmetric on three reflections and three rotations.

# Conjugate Elements, Class, Character

## Conjugate Elements

$a, b, \in G$  are called conjugate elements if  $\exists c \in G \mid c^{-1} \cdot a \cdot c = b$ .

All elements of a class describe the same mapping.

## Class

The set of all elements conjugate to a given element form a *class*.

Class is a subset of a group, but not a usually subgroup. All matrices of a class have the same trace (sum of diagonal elements).

## Character

The trace of a matrix is called its character.

All matrices of a class have the same character.

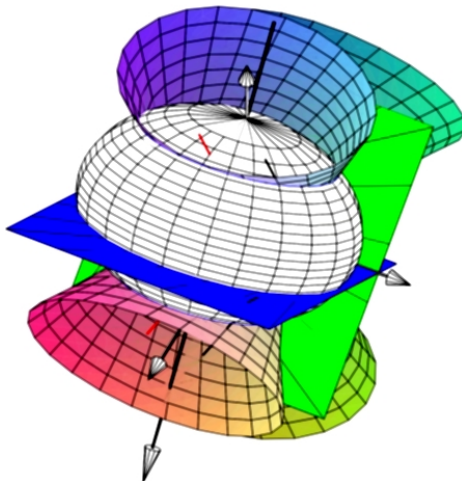
# Irreducible Representations

## Definition

If a group,  $G$ , only has two subrepresentations,  $\emptyset$  and  $G$ , then  $G$  is said to be irreducible.



# Vector Spaces



# Definition of Vector Spaces

A vector space over field  $F$  is a set  $V$  together with two binary operations satisfying following conditions:

## Group Conditions

Closure:  $\forall \vec{u}, \vec{v} \in V, \vec{u} + \vec{v} \in V$

Vector Addition:

Commutation:  $\forall \vec{u}, \vec{v} \in V, \vec{u} + \vec{v} = \vec{v} + \vec{u}$

Association:  $\forall \vec{u}, \vec{v}, \vec{w} \in V, (\vec{u} + \vec{w}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$

Additive Identity:  $\exists \vec{0} \in V \mid \forall \vec{v} \in V, \vec{0} + \vec{v} = \vec{v} + \vec{0} = \vec{v}$

Additive Inverse:  $\forall \vec{v} \in V \exists -\vec{v} \mid \vec{v} + (-\vec{v}) = \vec{0}$

Multiplication:

Distribution 1:  $\forall \vec{u}, \vec{v} \in V, k(\vec{u} + \vec{v}) = k\vec{u} + k\vec{v}$

Distribution 2:  $\forall \vec{v} \in V, \vec{v}(k_1 + k_2) = k_1\vec{v} + k_2\vec{v}$

Association:  $\forall \vec{v} \in V, \vec{v}(k_1 \cdot k_2) = (\vec{v} \cdot k_1)k_2$

Identity:  $\forall \vec{v} \in V, 1 \cdot \vec{v} = \vec{v}$

Zero:  $\forall \vec{v} \in V, 0 \cdot \vec{v} = \vec{0}$

# Inner Product, Norm, Orthogonality

For  $A(x)$  and  $B(x)$  on  $a \leq x \leq b$ , we define the following.

## Inner Product

$$\langle A(x), B(x) \rangle = \int_a^b \overline{A(x)} B(x) dx$$

## Norm

$$\|A(x)\| = \sqrt{\int_a^b \overline{A(x)} A(x) dx}$$

## Orthogonality

$$A(x) \perp B(x) \text{ on } (a, b) \text{ if } \int_a^b \overline{A(x)} B(x) dx = 0$$

# Schwartz's Inequality

Let  $\vec{u}, \vec{v}$  be elements of a vector space. Recall previously that for  $n$ -dimensional space, Schwartz's Inequality states

## Schwartz's Inequality for $n$ -dimensional Space

$$|\vec{u} \cdot \vec{v}| \leq \vec{u} \vec{v}$$

For an inner product space, we modify the definition as follows.

## Schwartz's Inequality for Inner Product Space

$$|\langle \vec{u} | \vec{v} \rangle|^2 \leq \langle \vec{u} | \vec{u} \rangle \langle \vec{v} | \vec{v} \rangle$$

Boas develops a proof of this on page 182.

# Orthonormal Basis

Two functions are *orthonormal* if they

- satisfy the property of orthogonality, and
- have a norm  $= 1$

# Infinite Dimensional Spaces

An *infinite dimensional vector space* is one which does not have a finite basis.

We recall that the set of all linearly independent vectors  $\vec{b}_i$  which, expressed as some finite sum, are capable of describing any vector,  $\vec{v}$ , form the *basis* for our vector space.

As an example, consider  $\mathbb{R}[x]$ , the set of polynomials in  $x$  with real coefficients for which  $x : n \in \mathbb{N}$  forms the basis.

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