Partial Differentiation

Mathematical Methods in the Physical Sciences

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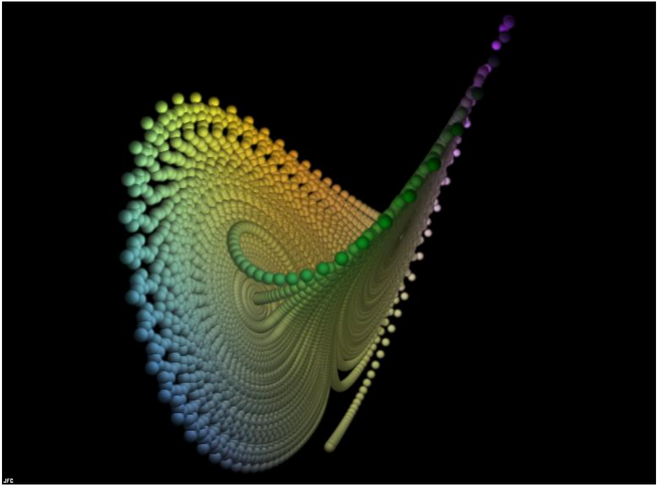
Lorenz Attractor



Introduction

Definition

Derivatives of a multi-variable function where all variables are held fixed during differentiation except the variable of interest.

To denote this we generically write $\dfrac{\partial}{\partial r}$, which means the partial derivative with respect to r. We more frequently see $\dfrac{\partial z}{\partial r}$, which means the partial derivative of z with respect to r. In equations of more than two variables we may see $\left(\dfrac{\partial z}{\partial r}\right)_x$, which denotes the partial derivative of z with respect to r, holding x constant.

Example

4.1.12: Find $\partial z/\partial y$, holding θ constant

$$\begin{aligned}
\text{given: } & z = x^2 + 2y^2, x = r\ \cos\theta, y = r\ \sin\theta \\
\text{solve for } r\text{: } & y = r\ \sin\theta \implies r = \frac{y}{\sin\theta} \\
\text{substitute for } r\text{: } & x = r\ \cos\theta \implies \frac{y\ \cos\theta}{\sin\theta} \\
\text{substitute for } x\text{: } & z = \left(\frac{y\ \cos\theta}{\sin\theta}\right)^2 + 2y^2 \\
\text{rewrite: } & = 2y^2 + (\cot\theta)^2 \\
\text{differentiate: } & = 2 \cdot 2y + 0 \\
& = 4y
\end{aligned}$$

Power Series in Two Variables

Our standard power series expansions can be re-written in terms of partial differential equations.

Definition

$$f(x,y) = \sum_{n=0}^{\infty} \frac{1}{n!} \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y} \right)^n f(a,b)$$

- A power series about a given point for a function of 2 variables is unique.
- Any methods from Chapter 1 may be used.

Power Series Example

We can arrive at the 2-variable expansion by finding the Maclaurin series expansions for sin and cos in the table on page 26 of Boas.

Example 1, Boas, p. 191

$$
\begin{aligned}
f(x,y) &= \sin x \, \cos y \\
&= \left(x - \frac{x^3}{3!} + \cdots \right) \cdot \left(1 - \frac{y^2}{2!} + \cdots \right) \\
&= x - \frac{x^3}{3!} - \frac{xy^2}{2!} + \cdots
\end{aligned}
$$

Total Differentiation in 2 Variables

$$dz = \frac{\partial}{\partial x} dx + \frac{\partial}{\partial y} dy$$

Approximation: For sufficiently small values of Δx and Δy,

- $\Delta z = \Delta f = f_x(x,y)\Delta x + f_y(x,y)\Delta u$, and
- $f(x+\Delta x, y+\Delta y) \equiv f(x,y) + f_x(x,y)\Delta x + f_y(x,y)\Delta y$.

Approximations Using Differentials

Example 4, Boas, p. 197

The relative error in length measurement is $\pm 5\%$ and the relative error in radius measurement is $\pm 10\%$. We want to find the largest value that $|dR/R|$ can have.

$$
\begin{aligned}
R &= \frac{kl}{r^2} \\
\ln R &= \ln k + \ln l - 2\ln r \\
\frac{dR}{R} &= \left|\frac{dl}{l}\right| - 2\left|\frac{dr}{r}\right| \\
&= 0.05 + 2(0.10) \\
&= 0.25
\end{aligned}
$$

Chain Rule

The chain rule is a formula for computing the derivative of the composition of two or more functions.

In General

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dx}$$

Find dy/dx if $y = \ln \sin 2x$

$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{\sin 2x} \cdot \frac{d}{dx}(\sin 2x) \\ &= \frac{1}{\sin 2x} \cdot \cos 2x \cdot \frac{d}{dx}(2x) \\ &= 2 \cot 2x \end{aligned}$$

We can differentiate with respect to a given term in some cases where we cannot solve for that term with respect to another.

Given $x + e^x = t$, find dx/dt

We realize that x is a function of t even though we cannot solve x for t directly.

$$\begin{aligned} x + e^x &= t \\ \frac{dx}{dt} + e^x \frac{dx}{dt} &= 1 \\ \frac{dx}{dt} &= \frac{1}{1 + e^x} \end{aligned}$$

This example can be found in Boas, p 202.

Chain Rule (Redux)

We can extend our earlier discussion of the Chain Rule where $z = f(x, y)$ and x and y were functions of some variable t by considering the case where x and y are functions of two variables, s and t. z is a function of both s and t and we want to be able to find $\partial z/\partial s$ and $\partial z/\partial t$.

Chain Rule (Redux) Example

Boas, 4.7.3

$$\text{Given: } z = xe^{-y}, x = \cosh\ t, y = \cos\ s$$

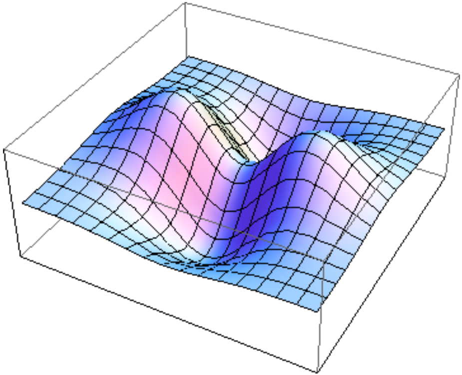
$$\text{Find: } \frac{\partial z}{\partial s} xe^{-y} = \cosh(t)e^{-\cos(s)}$$

$$\text{Chain Rule: } \frac{d}{ds} = \frac{de^u}{du}\frac{du}{ds}, u = -cos(s)$$

$$\begin{aligned}
\text{Constants: } &= cosh(t)e^{-cos(s)}\left(-\frac{d}{ds}cos(s)\right) \\
&= cosh(t)e^{-cos(s)}\left(-\left(-sin(s)\right)\right) \\
&= cosh(t)e^{-cos(s)}sin(s)
\end{aligned}$$

Helps us locate

- Hills
- Valleys
- Saddle Points



Questions?