Groups & Vector Spaces

Mathematical Methods in the Physical Sciences

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Groups

Definition of Groups

A group is a set of elements, G, together with a set operation, $\cdot$, that satisfies the following conditions:

Group Conditions

Closure: $\forall\ a, b \in G, a \cdot b \in G$

Association: $\forall\ a, b, c \in G, (a \cdot b) \cdot c = a \cdot (b \cdot c)$

Identity: $\exists$ exactly 1 element, $i \in G \mid \forall\ a \in G, i \cdot a = a \cdot i = a$

Inversion: $\forall\ a \in G\ \exists\ b \mid a \cdot b = b \cdot a = i$, where i is the identity element.

Operation Table

Group Symmetry

Conjugate Elements, Class, Character

Irreducible Representations

Infinite Groups

Vector Spaces

Definition of Vector Spaces

A vector space over field F is a set V together with two binary operations satisfying following conditions:

Group Conditions

Closure: $\forall\ \vec{u}, \vec{v} \in V, \vec{u} + \vec{v} \in V$

Vector Addition:

$$\begin{aligned} \text{Commutation:} &\quad \forall\ \vec{u}, \vec{v} \in V, \vec{u} + \vec{v} = \vec{v} + \vec{u} \\ \text{Association:} &\quad \forall\ \vec{u}, \vec{v}, \vec{w} \in V, (\vec{u} + \vec{w}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w}) \end{aligned}$$

Additive Identity: $\exists\ \vec{0} \in V \mid \forall\ \vec{v} \in V, \vec{0} + \vec{v} = \vec{v} + \vec{0} = \vec{v}$

Additive Inverse: $\forall\ \vec{v} \in V\ \exists\ -\vec{v} \mid \vec{v} + (-\vec{v}) = 0$

Multiplication:

$$\begin{aligned} \text{Distribution 1:} &\quad \forall\ \vec{u}, \vec{v} \in V, k(\vec{u} + \vec{v}) = k\vec{u} + k\vec{v} \\ \text{Distribution 2:} &\quad \forall\ \vec{v} \in V, \vec{v}(k_1 + k_2) = k_1\vec{v} + k_2\vec{v} \\ \text{Association:} &\quad \forall \vec{v} \in V, \vec{v}(k_1 \cdot k_2) = (\vec{v} \cdot k_1)k_2 \\ \text{Identity:} &\quad \forall\ \vec{v} \in V, 1 \cdot \vec{v} = \vec{v} \\ \text{Zero:} &\quad \forall\ \vec{v} \in V, 0 \cdot \vec{v} = \vec{0} \end{aligned}$$

Inner Product, Norm, Orthogonality

Schwart's Inequality

Orthonormal Basis

Infinite Dimensional Spaces

Questions?