Fourier Series Fundamentals

Mathematical Methods in the Physical Sciences

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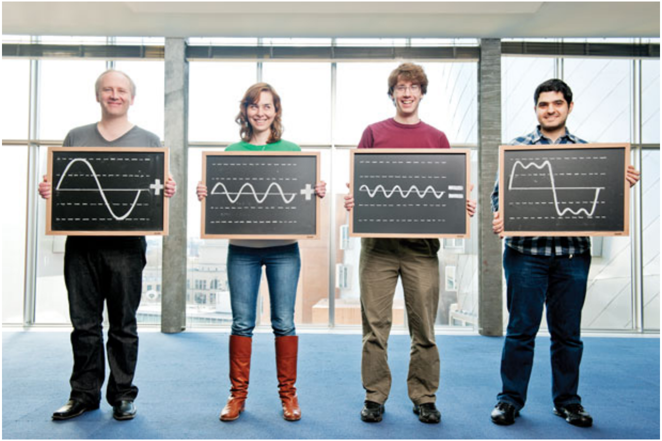
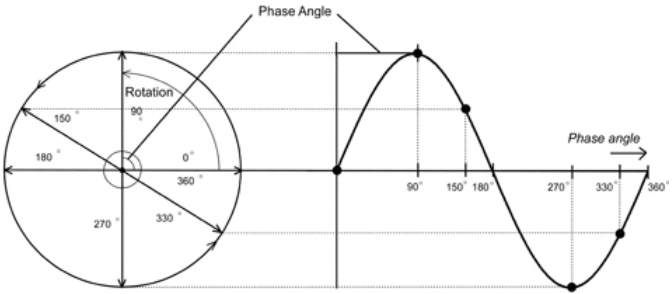
Introduction

Fourier series are like power series but are only used to represent periodic functions.



Simple Harmonic Motion

An object executing simple harmonic motion if its displacement from equilibrium can be written as $A \sin \omega t$ or $A \cos \omega t$ or $A \sin\ (\omega t + \phi)$.



The x and y components are $(A\cos\omega t, A\sin\omega t)$. In the complex plane this could be rewritten as

$$\begin{aligned} z &= x + iy \\ &= A\left(\cos\omega t + i\sin\omega t\right) \\ &= Ae^{i\omega t} \end{aligned}$$

The *amplitude* is the maximum displacement from equilibrium and the *period* is the time of one complete oscillation.

Applications of Fourier Series

In application,

- Fourier series do not tend to converge as rapidly as power series.
- Fourier series can represent discontinuous functions.

Often applied to problems involving,

- Sound
- Light
- Radio waves

Applications of Fourier Series (continued)

Average Value of a Function

Definition

$$\text{average of } f(x) \text{ on } (a,b) = \frac{\int_a^b f(x)dx}{b-a}$$

When the average of a function over a period of time is 0 then the average of the square of the function is often of interest. The average value over 1 period of $\sin^2 nx$ and $\cos^2 nx$ are the same:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \sin^2 nx\ dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos^2 nx\ dx = \frac{\pi}{2\pi} = \frac{1}{2}$$

Fourier Coefficients

We begin by considering only periodic functions of period 2π in terms of $\sin nx$ and $\cos nx$.

Coefficient Form

$$f(x) = \frac{1}{2}a_0 + a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x + \cdots$$
$$+ b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + \cdots$$

The integrals on page 351 are used to find coefficients a_n and b_n.

Coefficient Formulae

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \; dx$$
$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \; dx$$

Fourier Coefficients Example

We want to find the coefficients of a 2π Periodic Square Wave.

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\int_{-\pi}^{\pi} dx = 1$$

All coefficients $a_n = 0$ since $\sin(0) = 0$ and $\sin(\pi) = 0$.

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx = \frac{1}{\pi}\left(-\frac{\cos nx}{n}\right)\Big|_0^{\pi} = \frac{1-\cos n\pi}{n\pi}$$

But since $\cos n\pi = (-a)^n$, $b_n = \dfrac{1-(-1)^n}{n\pi}$, so...

$$f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{1-(-1)^n}{n\pi}\sin nx$$

Dirichlet Conditions

Our series converges to $f(x)$ if it satisfies the following conditions:

- Is periodic of period 2π
- Is single-valued between $-\pi$ and π
- Has finite number of max and min values
- $\int_{-\pi}^{\pi} |f(x)|\, dx$ is finite

We often do not need to evaluate the integral if we can show that $f(x)$ is bounded.

Complex Form of Fourier Series

We can use what we know about the complex representation of sines and cosines to rewrite the Fourrier series.

Complex Representation of Fourrier Series

$$\begin{aligned} f(x) &= c_0 + c_1 e^{ix} + c_{-1} e^{-ix} + c_2 e^{2ix} + c_{-2}^{-2ix} + \cdots \\ &= \sum_{n=-\infty}^{\infty} c_n d^{inx} \end{aligned}$$

We calculate the coefficients, c_n, as follows,

Complex Representation of Fourrier Series

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx$$

Other Intervals

Questions?