# Power Series

## Mathematical Methods in the Physical Sciences

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Power series are series where the $n^{\text{th}}$ term is a constant times $x^n$ or a constant times $(x-a)^n$ where $a$ is also constant.

**Definition: Boas, p. 20**

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

$$\sum_{n=0}^{\infty} a_n (x-a)^n = a_0 + a_1(x-a) + a_2(x-a)^2 + a_3(x-a)^3 + \cdots$$

# Power Series Examples

The following are two examples.

**Example #1: Boas, p. 20**

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + \frac{(-1)^{x+1}x^n}{n} + \cdots$$

**Example #2: Boas, p. 20**

$$1 + \frac{(x+2)}{\sqrt{2}} + \frac{(x+2)^2}{\sqrt{3}} + \cdots + \frac{(x+2)^n}{\sqrt{(n+1)}} + \cdots$$

# Interval of Convergence

Convergence of the power series depends on the value of $x$. We can use the ratio test to find values of $x$ that cause the series to converge.

**Example: Boas, p. 20**

$$\rho_n = \left| \frac{(-x)^{n+1}}{2^{n+1}} \div \frac{(-x)^n}{2^n} \right|$$
$$\rho = \left| \frac{x}{2} \right|$$

Since the series will converge for values $\rho < 1$, we can see that it holds for all values $|x| < 2$ and diverges for all values $|x| > 2$.

# Theorems About Power Series

There are four theorems about power series that we want to introduce.

**Theorem #1: Boas, p. 23**

A power series can be integrated and differentiated.

It is convenient that the resulting power series converges to the derivative or integral of the original power series's functional equivalent and, furthermore, it does so within the same interval.

# Theorems About Power Series (continued)

**Theorem #2: Boas, p. 23**

Power series may be added, subtracted, multiplied, or divided.

The resulting series will converge at least in the common interval of convergence.

**No Division by Zero!**

Division of power series holds only so long as the denominator series is not zero at $x = 0$.

# Theorems About Power Series (continued)

**Theorem #3: Boas, p. 23**

One series may be substituted in another.

This holds so long as the values of the substituted series are in the interval of convergence or the other.

# Theorems About Power Series (continued)

**Theorem #4: Boas, p. 23**

The power series of a function is unique.

For any function there is exactly one converging power series of the form $\sum_{n=0}^{\infty} a_n x^n$ .

# Expanding Functions in Power Series

We wish to derive a power series that represents a given function. Following along with the example from Boas, p. 23, we begin this by assuming that the power series exists.

**Step #1**

$\sin x = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$

We then select coefficients such that the function's value is preserved at $x = 0$.

**Step #2**

$a_0 = 0$ and $\sin 0 = 0$

Next we differentiate term by term.

**Step #3**

$\cos x = a_1 + 2a_2 x + 3a_3 x^2 + \cdots$

# Expanding Functions in Power Series (continued)

We continue setting $x = 0$ and differentiating as follows

## Continuing...

$$
\begin{aligned}
-\sin x &= 2a_2 + 3 \cdot 2a_3 x + 4 \cdot 3a_4 x^2 + \cdots \\
-\cos x &= 3 \cdot 2a_3 + 4 \cdot 3 \cdot 2a_4 x + \cdots \\
\sin x &= 4 \cdot 3 \cdot 2a_4 + 5 \cdot 4 \cdot 3 \cdot 2a_5 x + \cdots \\
\cos x &= 5 \cdot 4 \cdot 3 \cdot 2a_5 + \cdots
\end{aligned}
$$

We can see that this results in the values $a_2 = 0$, $a_3 = -\frac{1}{3!}$, $a_4 = 0$, $a_5 = \frac{1}{5!}$, etc. Lastly, we perform substitution back into the original equation.

## Substitution

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

# Techniques for Obtaining Power Series Expansions

We outlay and demonstrate several additional methods of obtaining power series expansions to provide alternatives to the differentiation-substitution method.

- Multiplying a series by polynomial or by another series
- Division of two series or of a series by a polynomial
- Binomial series
- Substitution of a polynomial or series or the variable in another series
- Combination of methods
- Taylor series using the basic Maclaurin series
- Using a computer

The textbook strongly recommends memorizing the following:

**Save for Reference: Boas, p. 26**

converge

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1a)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \qquad \forall x$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \qquad \forall x$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x^2}{2!} + \frac{x^3}{3!} - \frac{x^4}{4!} + \cdots \qquad \forall x$$

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \qquad \forall x : \{-1 < x \le 1\}$$

$$(1+x)^p = \sum_{n=0}^{\infty} \binom{p}{n} x^n = 1 + px + \frac{p(p-1)}{2!}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 + \cdots \qquad \forall x : \{|x| < 1\}$$

This is very straight forward and is carried out by simply performing the multiplication and collecting the terms.

### Multiplication by a Polynomial: Boas, p. 26

$$\begin{aligned}(x+1)\sin x &= (x+1)\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right) \\ &= x + x^2 - \frac{x^3}{3!} - \frac{x^4}{3!} + \frac{x^5}{5!} - \frac{x^6}{5!} \cdots\end{aligned}$$

### Multiplication by Another Series: Boas, p. 27

$$e^x \cos x = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right)\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right)$$

Division works exactly like multiplication with the caveat that we must prevent division by zero.

**Division by a Polynomial: Boas, p. 27**

$$\frac{1}{x}\ln(1+x) = 1 - \frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + \cdots$$

**Division by Another Series**

Division by another series is a somewhat lengthier process for which there is an excellent example in the textbook on page 28.

If we expand the binomial series $(a+b)^n$ for values $a = 1$, $b = x$, and $n = p$ then we will get the last of the formula presented on the slide at the beginning of this section.

**Example: Boas, p. 28**

$$\begin{aligned}
\frac{1}{1+x} &= (a+x)^{-1} \\
&= 1 - x + \frac{(-1)(-2)}{2!}x^2 + \frac{(-1)(-2)(-3)}{}x^3 + \cdots \\
&= 1 - x + x^2 - x^3 + \cdots \\
&= \sum_{n=0}^{\infty} (-x)^n
\end{aligned}$$

In order to obtain the series for $e^{-x^2}$ we use what we know about the series for $e^x$ and replace $x$ with $-x^2$ as follows.

**Example: Boas, p. 29**

$$\begin{aligned} e^{-x^2} &= 1 - x^2 + \frac{(-x^2)^2}{2!} + \frac{(-x^2)^3}{3!} + \cdots \\ &= 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots \end{aligned}$$

We show how we use the binomial and integral methods together.

**Example: Boas, p. 30**

$$\int_0^x \frac{dt}{1+t^2} = \operatorname{atan} t\Big|_0^x = \operatorname{atan} x$$

$$(1+t^2)^{-1} = 1 - t^2 + t^4 - t^6 + \cdots$$

$$\int_0^x \frac{dt}{1+t^2} = t - \frac{t^3}{3} + \frac{t^5}{5} - \frac{t^7}{7} + \cdots$$

The Maclaurin series provides us with an alternative method to the formulas for obtaining a Taylor series.

**Maclaurin Series**

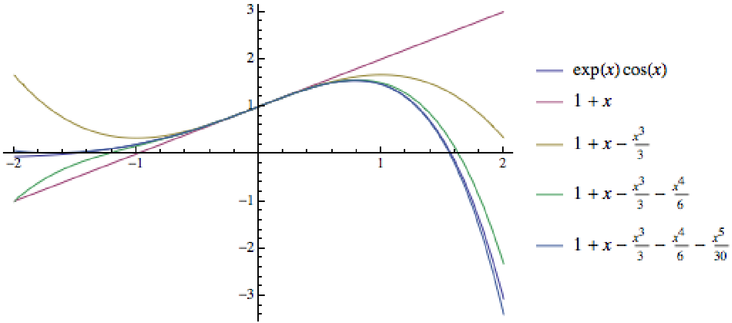
$$\ln x = \ln\left[1 + (x-1)\right]$$

Then we replace $x$ with $(x-1)$

**Substitution**

$$\begin{aligned}\ln x &= \ln\left[1 + (x-1)\right] \\ &= (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \cdots\end{aligned}$$

Approximate $e^x \cos x$ by plotting partial sums.



# Reliability of Series Approximations

**There are functions for which power series do not exist.**

Not all functions have series expansions. Consider $\frac{1}{x}$.

**It is possible to derive an erroneous series expansion of a function for which one does not exist.**

**It is possible to derive a converging series that does not accurately represent the function being expanded.**

Consider the function $e^{-(1/x^2)}$ which expands to $0 + 0 + 0 + \cdots$ but which is non-zero for $x^2 > 0$.

**Alternating Series: Boas, p. 34**

If $S = \sum_{n=1}^{\infty} a_n$ is an alternating series with $|a_n + 1| < |a_n|$, and $\lim_{n\to\infty} a_n = 0$, then $|S - (a_1 + a_2 + \cdots + a_n)| \leq |a_n + 1|$.

Estimating the error in series approximation for alternating converging series can be accomplished by observing the absolute value of the first neglected term.

The previous method provides results that can contain nontrivial errors when applied to non-alternating series. The results are the same order of magnitude but can be off by a significant factor. We can refine our results by using the following formula.

### Non-alternating Series: Boas, p. 35

If $S = \sum_{n=0}^{\infty} a_n x^n$ converges for $|x| < 1$, and if $|a_{n+1}| < |a_n|$ for $N$,

then $\left| S - \sum_{n=0}^{N} a_n x^n \right| < |a_{N+1} x^{N+1}| \div (1 - |x|)$.

# Some Uses of Series

Uses of series include:

- Performing accurate numerical computation
- Summing series
- Approximating integrals
- Evaluating indeterminate forms
- Approximating series