

Partial Differentiation

Mathematical Methods in the Physical Sciences

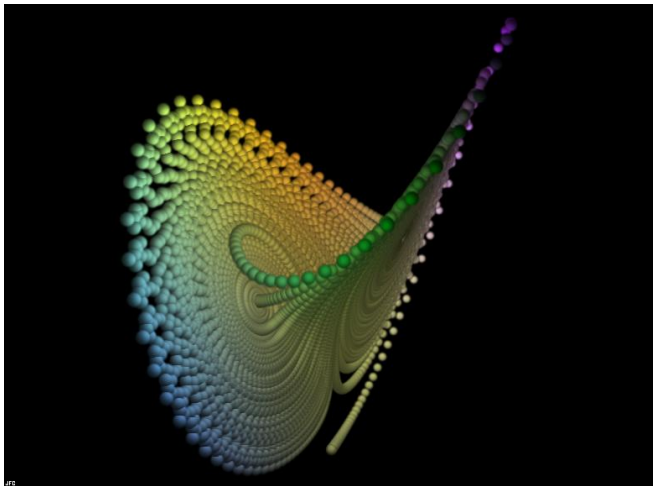
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Definition

Derivatives of a multi-variable function where all variables are held fixed during differentiation except the variable of interest.

To denote this we generically write $\frac{\partial}{\partial r}$, which means the partial derivative with respect to r . We more frequently see $\frac{\partial z}{\partial r}$, which means the partial derivative of z with respect to r . In equations of more than two variables we may see $\left(\frac{\partial z}{\partial r}\right)_x$, which denotes the partial derivative of z with respect to r , holding x constant.

Example

4.1.12: Find $\partial z / \partial y$, holding θ constant

$$\text{given: } z = x^2 + 2y^2, x = r \cos\theta, y = r \sin\theta$$

$$\text{solve for } r: y = r \sin\theta \implies r = \frac{y}{\sin\theta}$$

$$\text{substitute for } r: x = r \cos\theta \implies \frac{y \cos\theta}{\sin\theta}$$

$$\text{substitute for } x: z = \left(\frac{y \cos\theta}{\sin\theta} \right)^2 + 2y^2$$

$$\text{rewrite: } = 2y^2 + y^2 \cot^2\theta$$

$$\begin{aligned} \text{differentiate: } \left(\frac{\partial z}{\partial y} \right)_{\theta} &= 2 \cdot 2y + 2y \cot^2\theta \\ &= 4y + 2y \cot^2\theta \end{aligned}$$

Power Series in Two Variables

Our standard power series expansions can be re-written in terms of partial differential equations.

Definition

$$f(x, y) = \sum_{n=0}^{\infty} \frac{1}{n!} \left(h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right)^n f(a, b)$$

- A power series about a given point for a function of 2 variables is unique.
- Any methods from Chapter 1 may be used.

Power Series Example

We can arrive at the 2-variable expansion by finding the Maclaurin series expansions for sin and cos in the table on page 26 of Boas.

Example 1, Boas, p. 191

$$\begin{aligned}f(x, y) &= \sin x \cos y \\&= \left(x - \frac{x^3}{3!} + \cdots \right) \cdot \left(1 - \frac{y^2}{2!} + \cdots \right) \\&= x - \frac{x^3}{3!} - \frac{xy^2}{2!} + \cdots\end{aligned}$$

Total Differentiation in 2 Variables

$$dz = \frac{\partial}{\partial x} dx + \frac{\partial}{\partial y} dy$$

Approximation: For sufficiently small values of Δx and Δy ,

- $\Delta z = \Delta f = f_x(x, y)\Delta x + f_y(x, y)\Delta y$, and
- $f(x + \Delta x, y + \Delta y) \equiv f(x, y) + f_x(x, y)\Delta x + f_y(x, y)\Delta y$.

Approximations Using Differentials

Example 4, Boas, p. 197

The relative error in length measurement is $\pm 5\%$ and the relative error in radius measurement is $\pm 10\%$. We want to find the largest value that $|dR/R|$ can have.

$$R = \frac{kl}{r^2}$$

$$\ln R = \ln k + \ln l - 2 \ln r$$

$$\begin{aligned}\frac{dR}{R} &= \left| \frac{dl}{l} \right| - 2 \left| \frac{dr}{r} \right| \\ &= 0.05 + 2(0.10) \\ &= 0.25\end{aligned}$$

Chain Rule

The chain rule is a formula for computing the derivative of the composition of two or more functions.

In General

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dx}$$

Find dy/dx if $y = \ln \sin 2x$

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{\sin 2x} \cdot \frac{d}{dx}(\sin 2x) \\ &= \frac{1}{\sin 2x} \cdot \cos 2x \cdot \frac{d}{dx}(2x) \\ &= 2 \cot 2x\end{aligned}$$

Implicit Differentiation

We can differentiate with respect to a given term in some cases where we cannot solve for that term with respect to another.

Given $x + e^x = t$, find dx/dt

We realize that x is a function of t even though we cannot solve x for t directly.

$$\begin{aligned}x + e^x &= t \\ \frac{dx}{dt} + e^x \frac{dx}{dt} &= 1 \\ \frac{dx}{dt} &= \frac{1}{1 + e^x}\end{aligned}$$

This example can be found in Boas, p 202.

Chain Rule (Redux)

We can extend our earlier discussion of the Chain Rule where $z = f(x, y)$ and x and y were functions of some variable t by considering the case where x and y are functions of two variables, s and t . z is a function of both s and t and we want to be able to find $\partial z / \partial s$ and $\partial z / \partial t$.

Chain Rule (Redux) Example

Boas, 4.7.3

Given: $z = xe^{-y}$, $x = \cosh t$, $y = \cos s$

Find: $\frac{\partial z}{\partial s} xe^{-y} = \cosh(t)e^{-\cos(s)}$

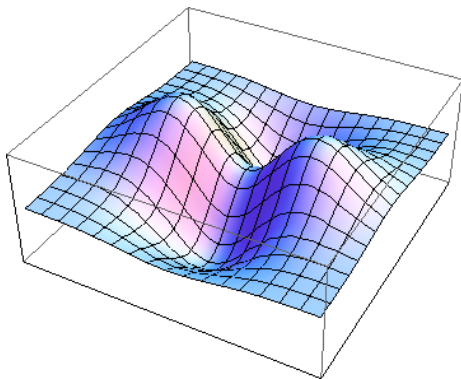
Chain Rule: $\frac{d}{ds} = \frac{de^u}{du} \frac{du}{ds}$, $u = -\cos(s)$

$$\begin{aligned}\text{Constants: } &= \cosh(t)e^{-\cos(s)} \left(-\frac{d}{ds} \cos(s) \right) \\ &= \cosh(t)e^{-\cos(s)} (-(-\sin(s))) \\ &= \cosh(t)e^{-\cos(s)} \sin(s)\end{aligned}$$

Applications

Helps us locate

- Hills
- Valleys
- Saddle Points



FIN