Tensor Analysis

Mathematical Methods in the Physical Sciences

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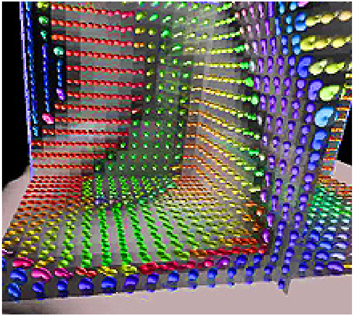
Introduction

- Tensors are designated by their size and *order*.
- Tensors of order 0 are scalars
- Tensors of order 1 are vectors
- A second order tensor has $3^2 = 9$ components
- In general, an n^{th} ranked tensor can be described by 3^n coefficients.



Under *passive rotation* the vectors are fixed and the axes are rotated. We want to know how the components of a displacement vector in one coordinate system are related to its components in a rotated system. A vector $\vec{r}$ has components x, y, z or x', y', z' relative to the two coordinate systems.

	x	y	z
x'	l_1	m_1	n_1
y'	l_2	m_2	n_2
z'	l_3	m_3	n_3

The table lists the cosines of the nine angles between the (x, y, z) and the (x', y', z') axes.

Cartesian Tensors (continued)

Let $\vec{i}, \vec{j}, \vec{k}$ be unit vectors along (x, y, z) axes and $\vec{i'}, \vec{j'}, \vec{k'}$ be unit vectors along (x', y', z'). Then we can represent $\vec{r}$ as follows.

$$\begin{aligned}
r &= \vec{i}x + \vec{j}y + \vec{k}z = \vec{i'}x' + \vec{j'}y' + \vec{k'}z' \\
\vec{r} \cdot \vec{i} &= \vec{i} \cdot \vec{i'}x + \vec{j} \cdot \vec{i'}y + \vec{k} \cdot \vec{i'}z = x' \\
\text{since } \vec{i'} \cdot \vec{i'} &= 1, \text{ and } \vec{i'} \cdot \vec{j'} = \vec{i'} \cdot \vec{k'} = 0 \\
\text{and } \vec{i} \cdot \vec{i'} &= l_1, \vec{j} \cdot \vec{i'} = m_1, \vec{k} \cdot \vec{i'} = n_1 \\
x' &= l_1 x + m_1 y + n_1 z \\
y' &= l_2 x + m_2 y + n_2 z \\
z' &= l_3 x + m_3 y + n_3 z
\end{aligned}$$

These are the transformation equations from (x, y, z) to (x', y', z').

Tensor Notation and Operations

- For simplicity, we drop the summation sign and assume summation over any index which appears twice in one term.
- Contraction
  - Obtained by setting unlike indices equal and summing
  - Reduces the order by 2
- First and second order tensors can be displayed as matrices.
- Symmetry
  - Symmetric if $T_{ij} = T_{ji}$.
  - Antisymmetric if $T_{ij} = -T_{ji}$.
  - Any second order tensor can be written as a sum of a symmetric and antisymmetric tensor.
- Combination
  - The linear combination of two tensors of order n is a tensor of order n.
  - Addition is not defined for tensors of different order.
- Quotient Rule is useful for identifying components of a tensor.

Inertia Tensor

For a rigid body rotating about a fixed axis, we know that the velocity, ω, and momentum, L, are related by the equation $L = I\omega$ where I is the moment of inertia. But if the rotation axis is not fixed, then I must be replaced by a second order tensor with components I_{jk}.

Kronecker Delta and Levi-Civita Symbol

Kronecker Delta

$$\delta_{ij} = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ otherwise} \end{cases}$$

Levi-Civita Symbol

$$\epsilon_{ijk} = \begin{cases} 1 & \text{for an even permutation} \\ -1 & \text{for an odd permutation} \\ 0 & \text{if any indices are repeated} \end{cases}$$

Vector Identities

3-by-3 determinant

$$\det A = a_{1i} a_{2j} c_{3k} \epsilon_{ijk}$$

Dot Product

$$A \cdot B = A_i B_i$$

Cross Product

$$(A \times B)_i = \epsilon_{ijk} B_j C_k$$

Curl

$$(\nabla \times V)_i = \epsilon_{ijk} \frac{\partial}{\partial x_j} V_k$$

Pseudovectors and Pseudotensors

The general case of orthogonal transformations includes reflections.

- If $\det A = 1$ (rotation), it is called a *polar* or true vector.
- If $\det A = -1$ (reflection), it is called an *axial* or pseudovector.

Consider the current density in an anisotropic material like graphite. In general the current density, $\vec{j}$, will be parallel to the applied electric field, $\vec{E}$ according to Ohm's law ($\vec{j} = \sigma \vec{E}$). However, due to the crystalline structure, each component of the density vector depend on all other components of the electric field.[1]

$$
\begin{aligned}
j_1 &= {}_{11}E_1 + {}_{12}E_2 + {}_{13}E_3 \\
j_2 &= {}_{21}E_1 + {}_{22}E_2 + {}_{23}E_3 \\
j_3 &= {}_{31}E_1 + {}_{32}E_2 + {}_{33}E_3
\end{aligned}
$$

Other second rank tensors include thermal conductivity, stress, and strain (see Boas , p. 519). Higher order tensors describe properties related to more than one 2$^{\text{nd}}$ rank tensor (e.g., stiffness = stress + strain).

[1] http://www.doitpoms.ac.uk/tlplib/tensors/what_is_tensor.php

Equivalence between rectangular and cylindrical coordinates. As usual,

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= z \\ \text{so...} \quad dx &= \cos\theta dr - r\sin\theta d\theta \\ dy &= \sin\theta dr + r\cos\theta d\theta \\ dz &= dz \end{aligned}$$

Squaring and reducing, we obtain,

$$ds^2 = dx^2 + dy^2 + dz^2 = dr^2 + r^2 d\theta^2 + dz^2$$

For orthogonal coordinate systems, all cross products will cancel nicely.

Vector Operations in Orthogonal Curvilinear Coordinates

Gradient

$$\nabla u = \sum_{i=1}^{3} e_i \frac{1}{h_i} \frac{\partial u}{\partial x_i}$$

Div

$$\nabla \cdot \vec{V} = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial x_1} (h_2 h_3 V_1) + \frac{\partial}{\partial x_2} (h_1 h_3 V_2) + \frac{\partial}{\partial x_3} (h_1 h_2 V_3) \right]$$

Laplacian

$$\nabla^2 u = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial x_1} \left(\frac{h_2 h_3}{h_1} \frac{\partial u}{\partial x_1} \right) + \frac{\partial}{\partial x_2} \left(\frac{h_1 h_3}{h_2} \frac{\partial u}{\partial x_2} \right) + \frac{\partial}{\partial x_3} \left(\frac{h_1 h_2}{h_3} \frac{\partial u}{\partial x_3} \right) \right]$$

Non-Cartesian Tensors

In spherical coordinates r, θ, ϕ,

$$x = r \sin\theta \cos\phi$$
$$y = r \sin\theta \sin\phi$$
$$z = r \cos\theta$$

We must write relations between the differentials of the variables (Boas, pg. 529).

Covariant Vector

$$V_i' = \frac{\partial x_j}{\partial x_i'} V_j$$

Contravariant Vector

$$V'^i = \frac{\partial x_i'}{\partial x_j} V^j$$

Questions?