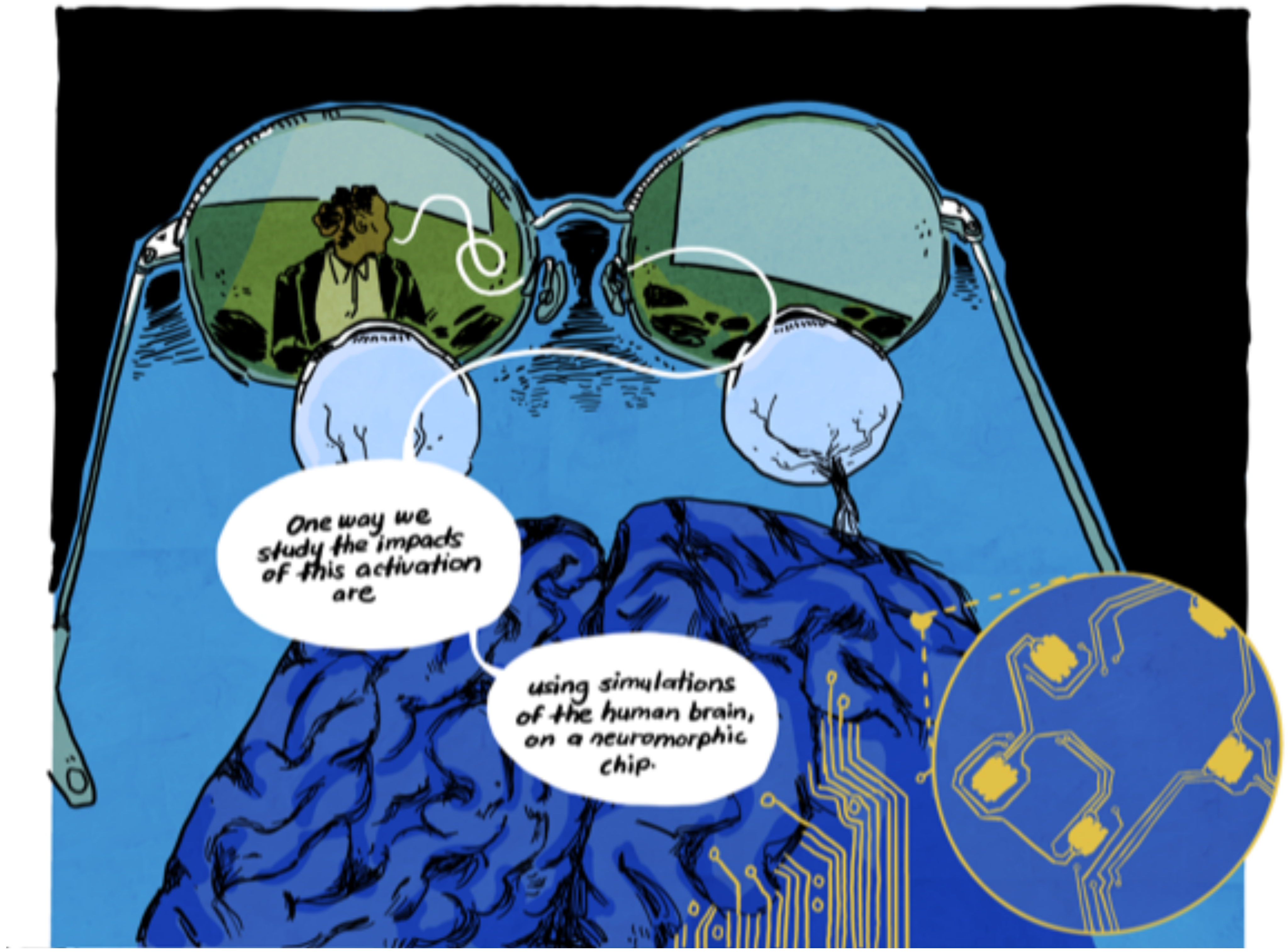
Dynamical systems for behavioral and neural data

Luca Mazzucato

Depts of Biology, Mathematics & Physics
Institute of Neuroscience
University of Oregon

CSHA Computational and Cognitive Neuroscience Summer School



Artwork by Audra McNamee

Dynamical systems for behavioral and neural data

- TODAY: hidden Markov models for **behavior**:
  - Slides
    - Spontaneous movements - HMM with Autoregressive observations
    - Naturalistic foraging - HMM with Linear Model emissions
    - HMM best practices
  - Live coding session
    - Mystery dataset
    - Fit Linear Model, Mixture of Linear Model
    - Sample and fit data w/ LM-HMM
    - Model selection for # of hidden states

Behavior is a complex dynamical system

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words / **actions** ~100 ms

Whisking, sniffing, moving a paw, …

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sentences / **sequences** ~1 s

Walking, rolling the walnut, hiding

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Must get walnut!

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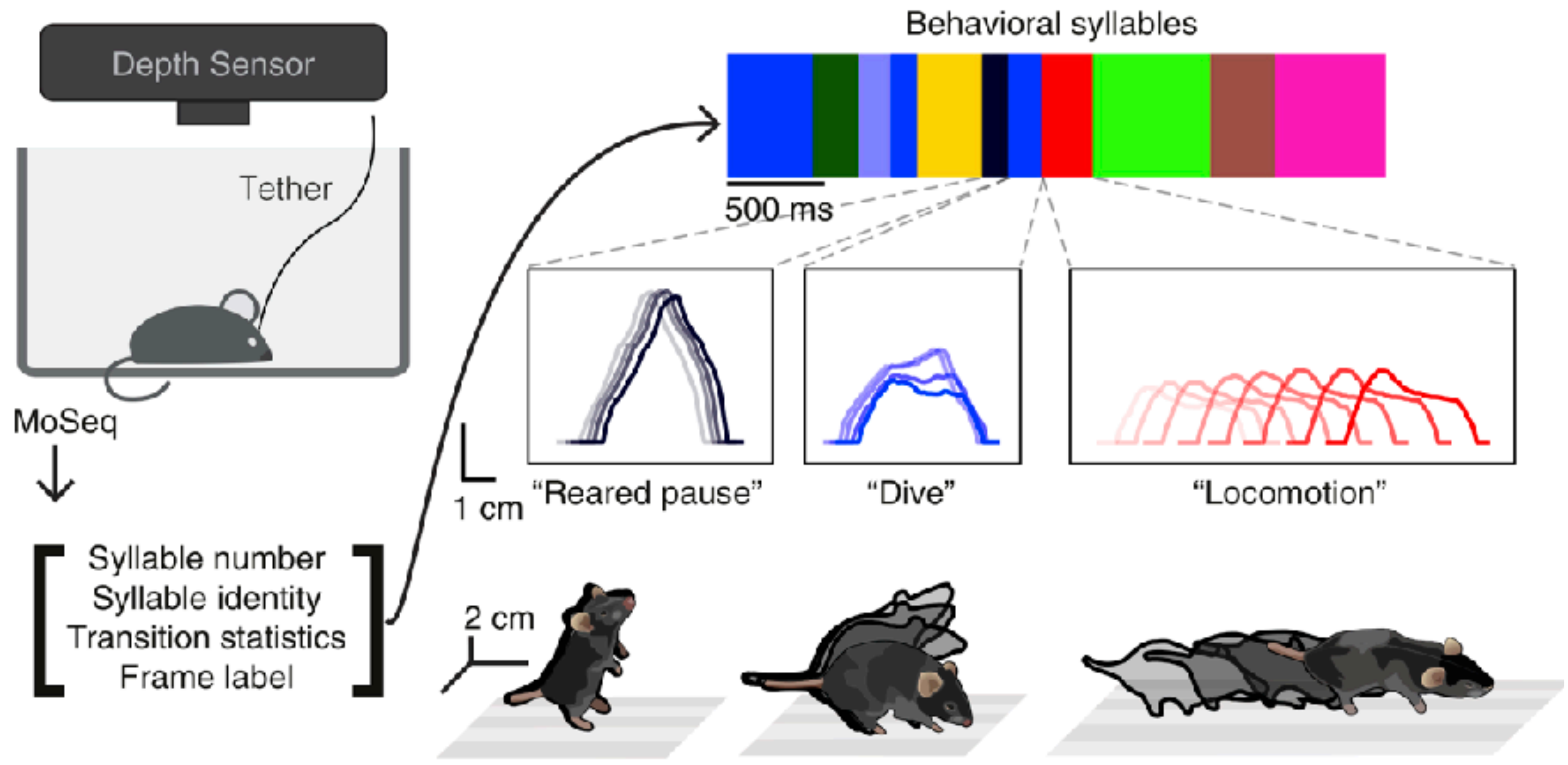
Complex temporal structure in behavior



[Wiltschko et al, 2015]
[Markovits et al, 2018]
[Findley et al, 2021]

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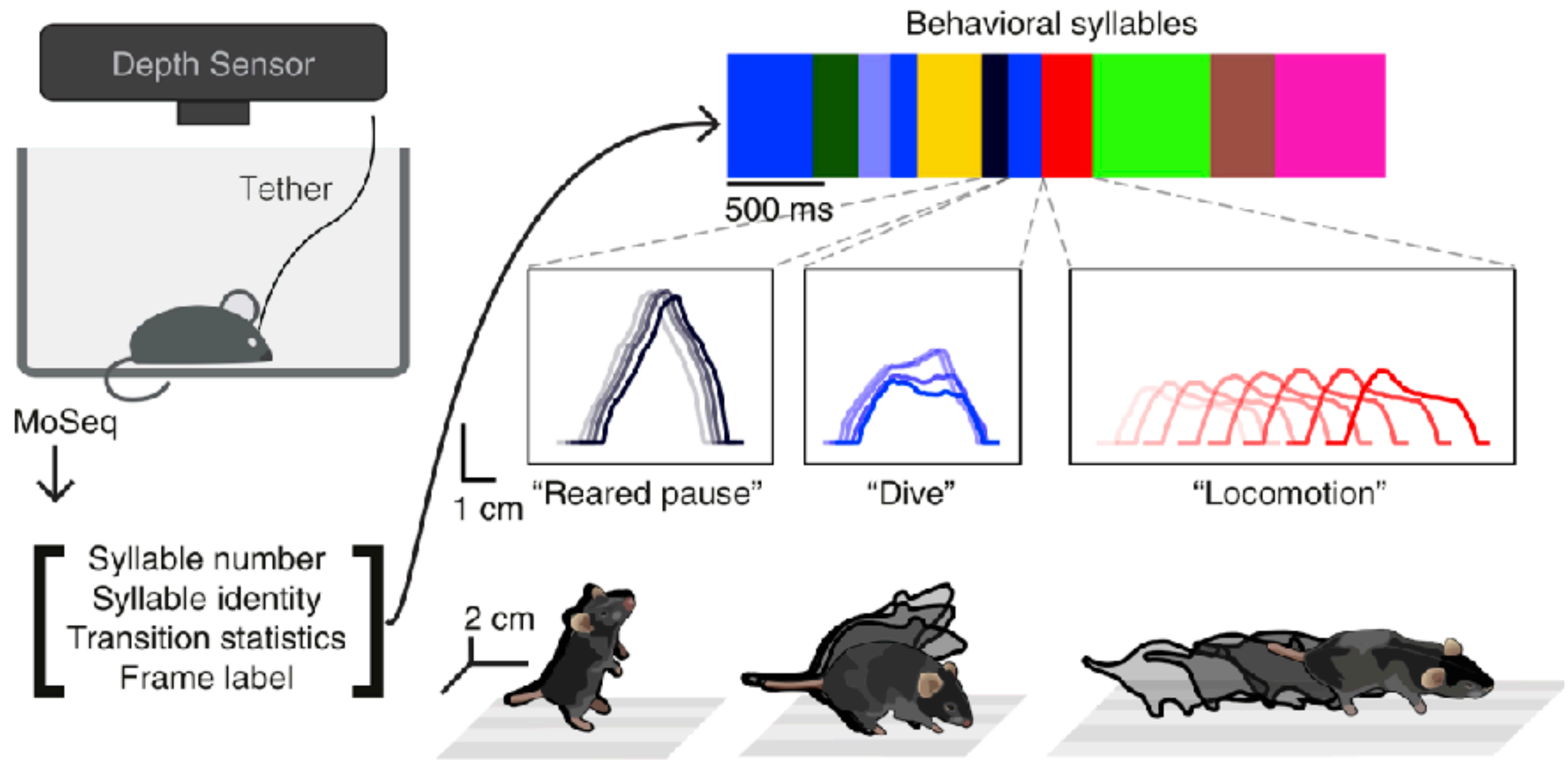

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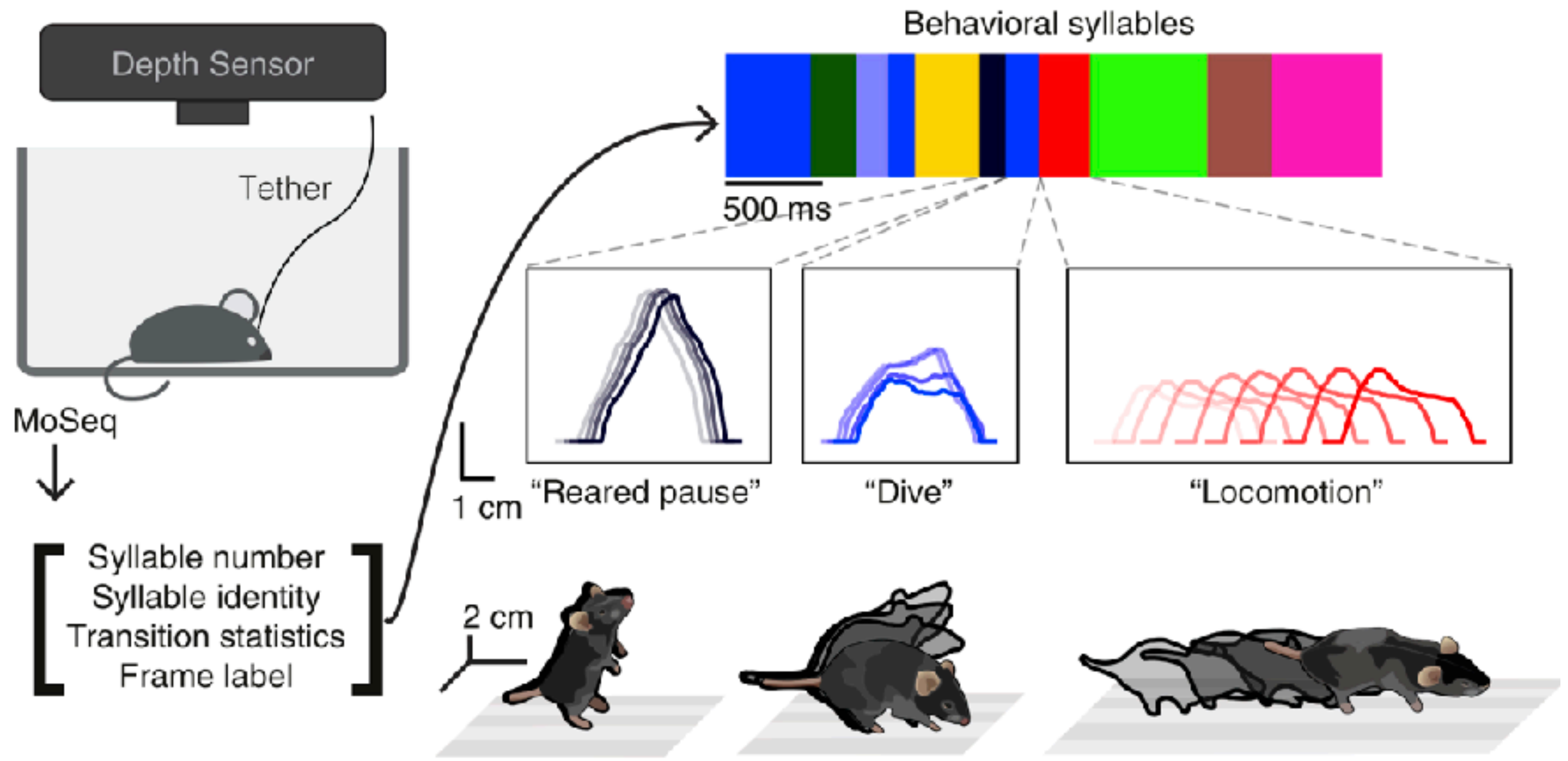

Variability sources:

1 - many syllables / actions

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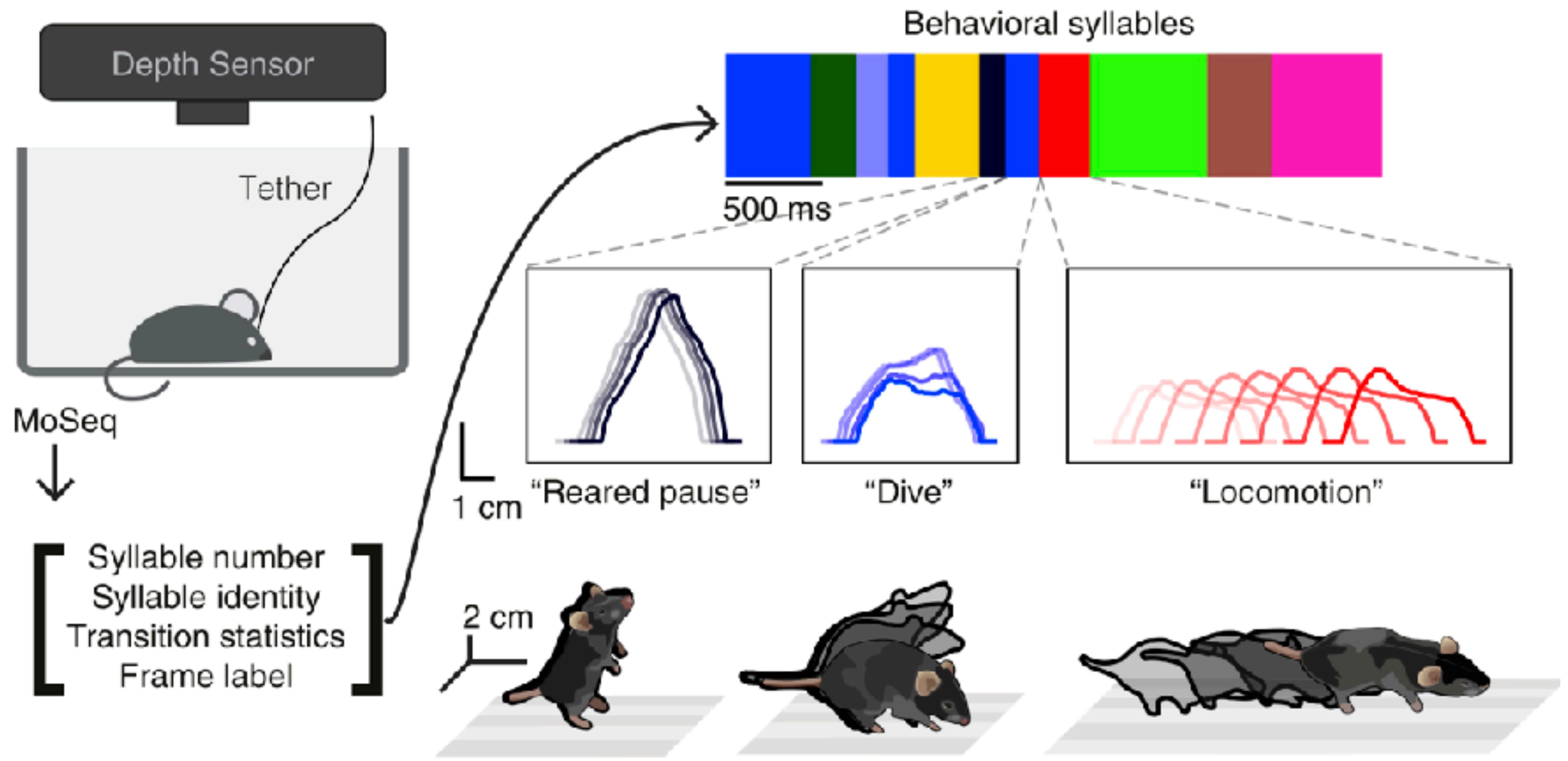

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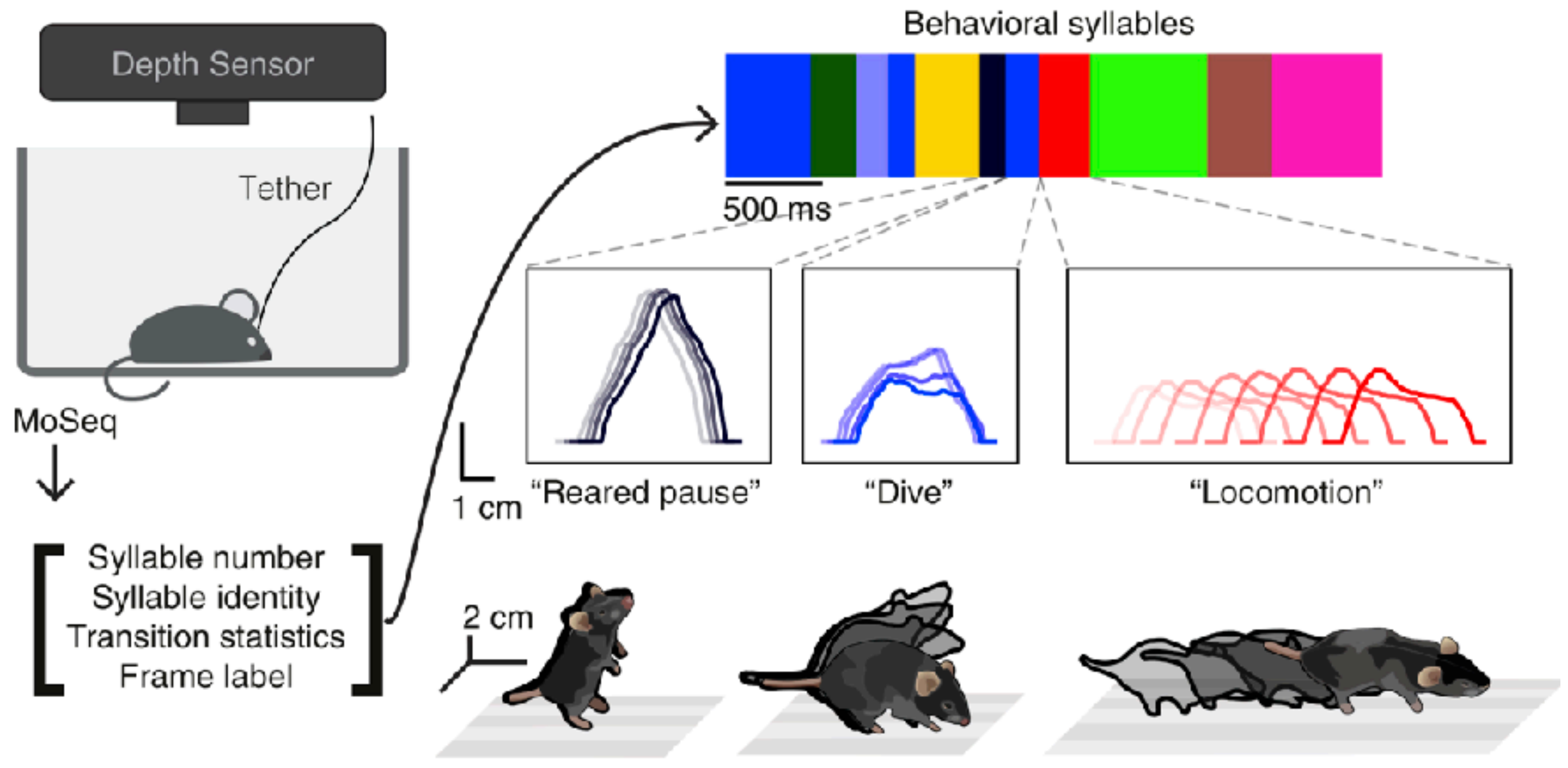

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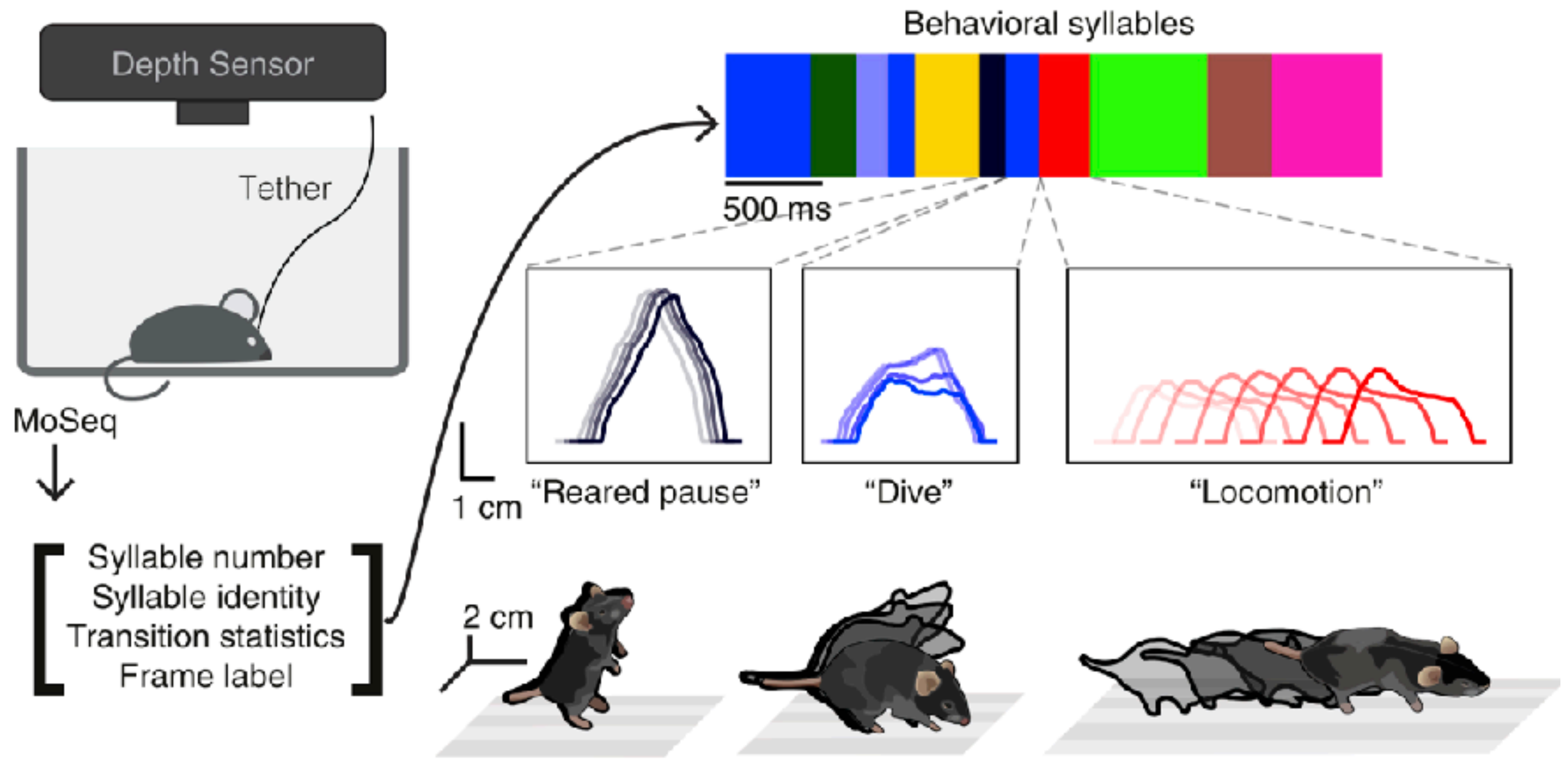

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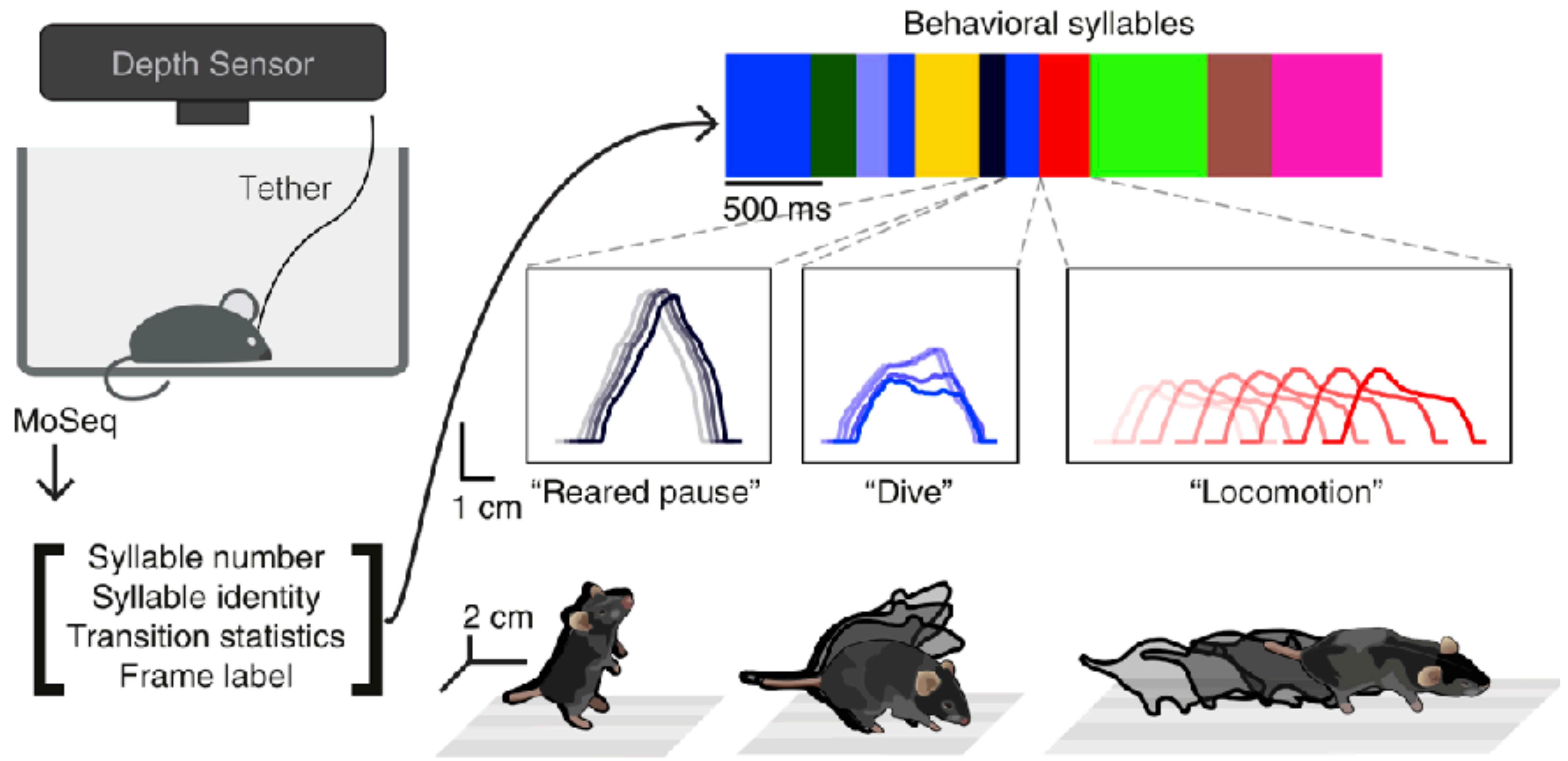

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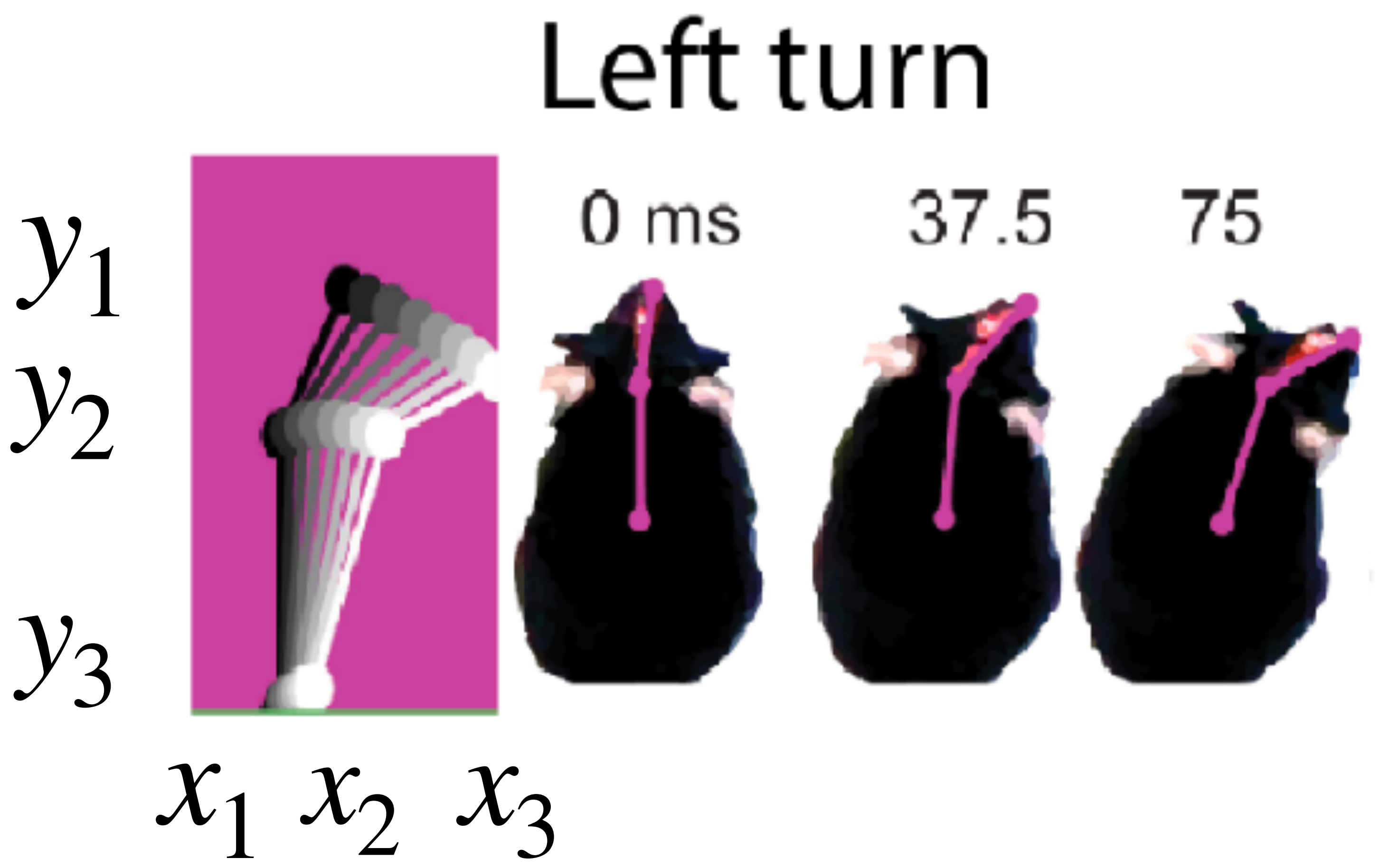
Autoregressive dynamics



$$\vec{r} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ y_3 \end{pmatrix}$$

Autoregressive dynamics

$$\vec{r}_{t+1} = A\vec{r}_t + \vec{b} + \vec{\epsilon}_t$$

A matrix = rotation

b vector = translation

[Wiltschko et al, 2015]
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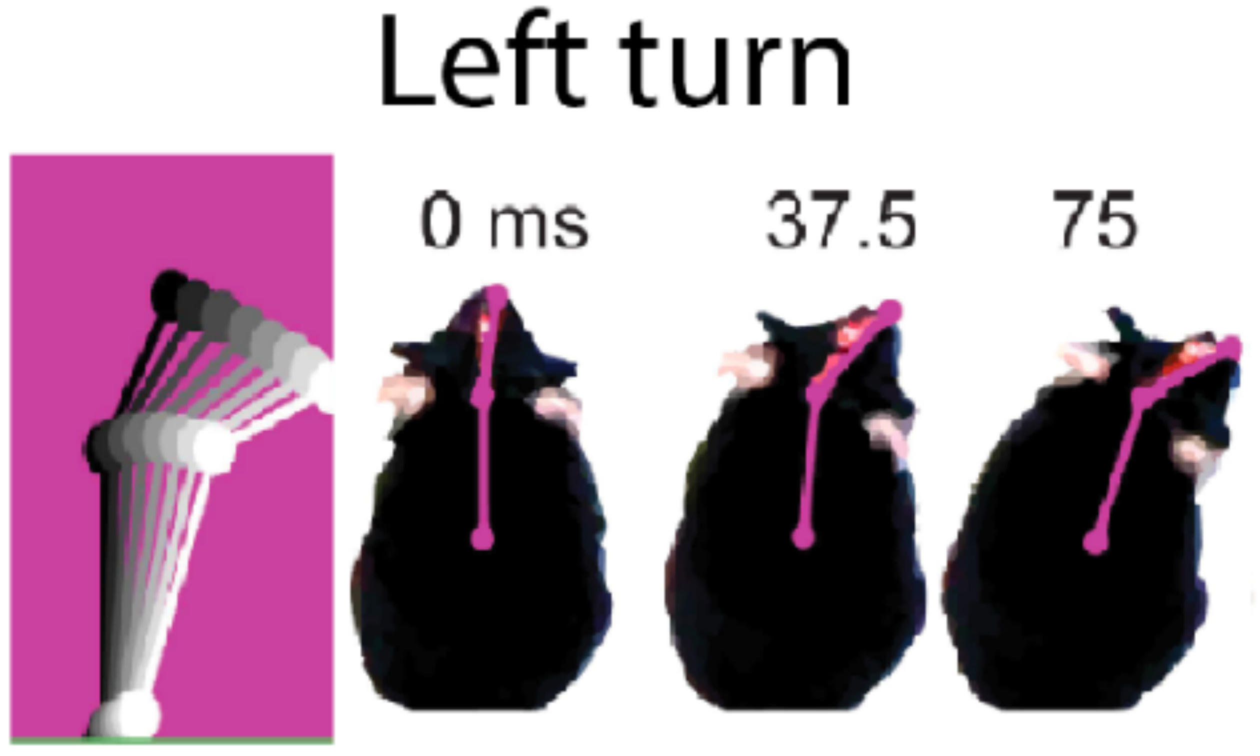

y_1 y_2 y_3

x_1 x_2 x_3

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noise

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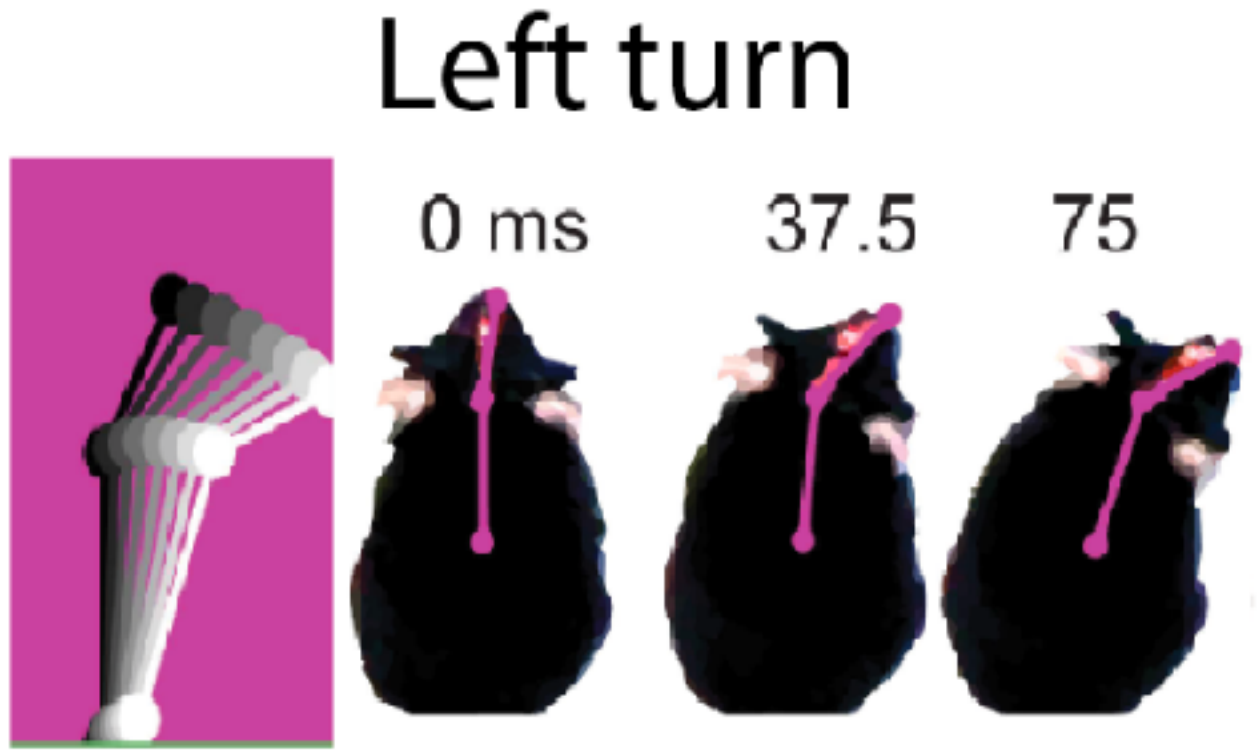
Autoregressive dynamics

y_1

y_2

y_3



$x_1 \; x_2 \; x_3$

$\Downarrow$

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Autoregressive dynamics

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$$\to r_{t-1} \to r_t \to r_{t+1} \to$$

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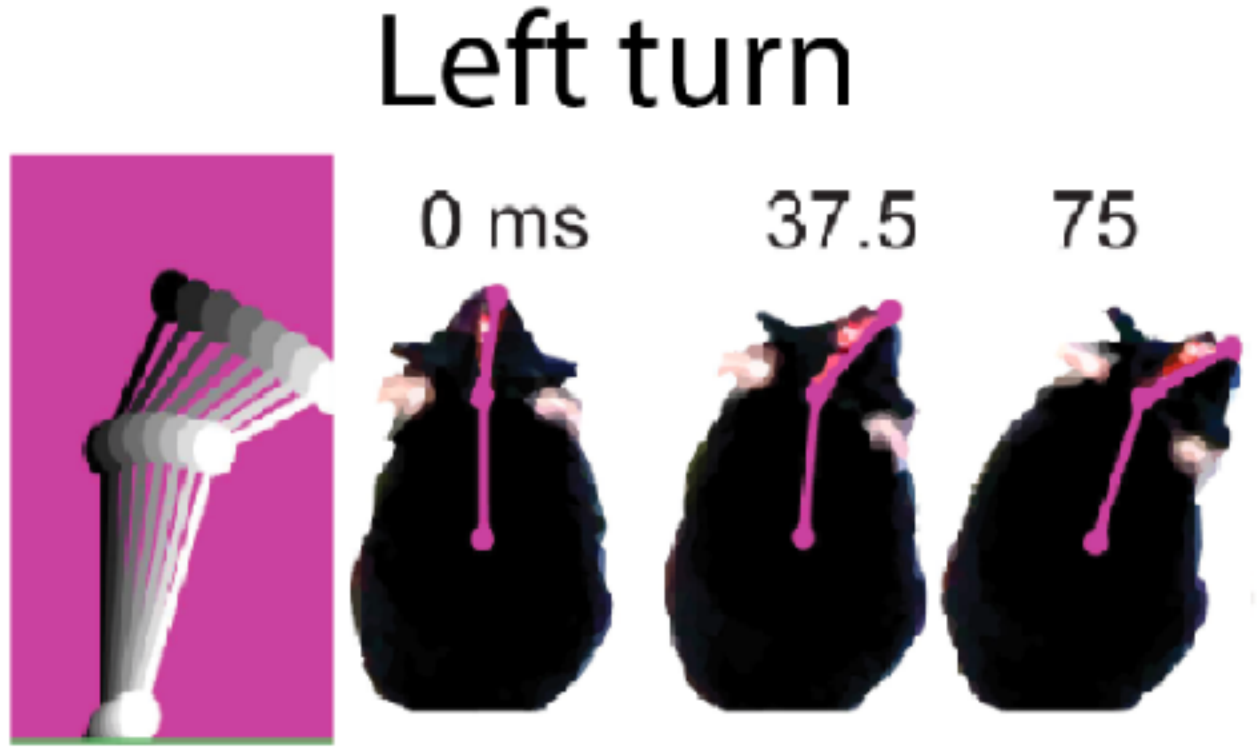
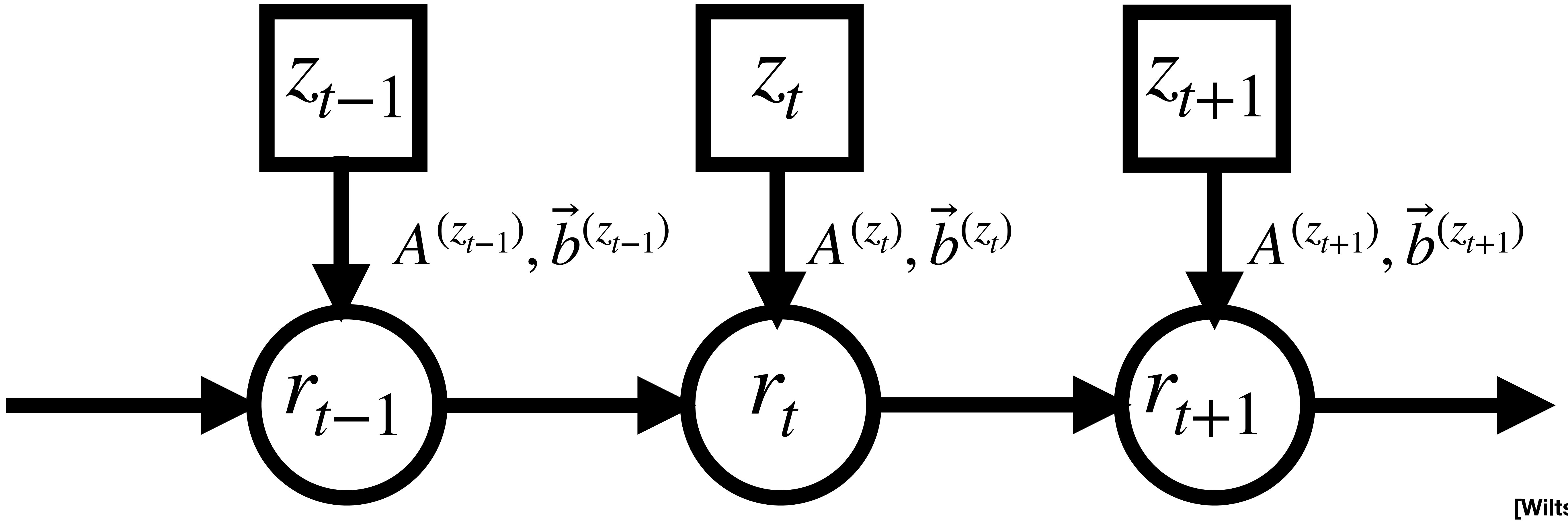


Autoregressive dynamics

$$\vec{r}_{t+1} = A\vec{r}_t + \vec{b} + \vec{\epsilon}_t$$

$z_t \in 1,\ldots,K$ Hidden states (motifs)



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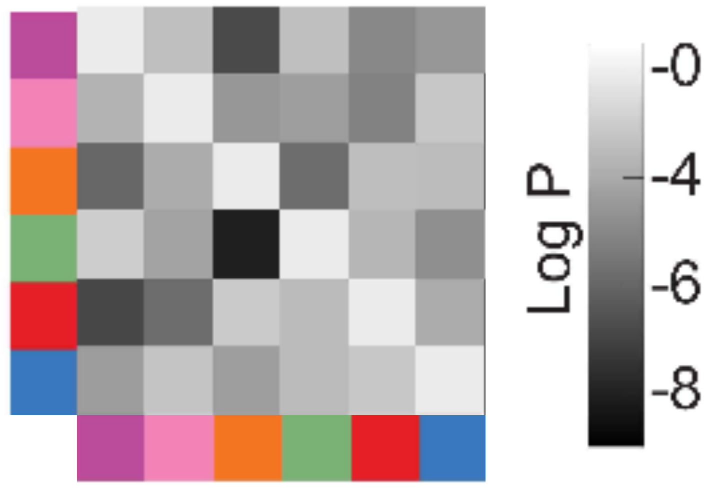
AR Hidden Markov model

Transition probabilities

$$p(z_t = k \,|\, z_{t-1} = l)$$

z_{t-1}



z_t

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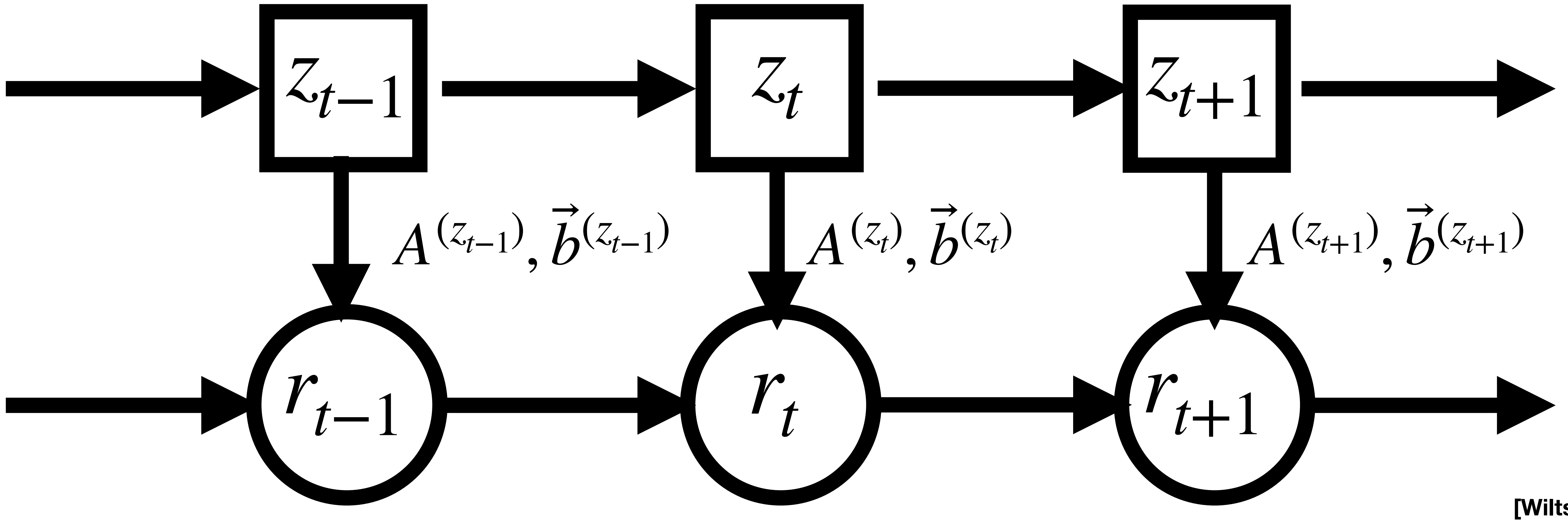

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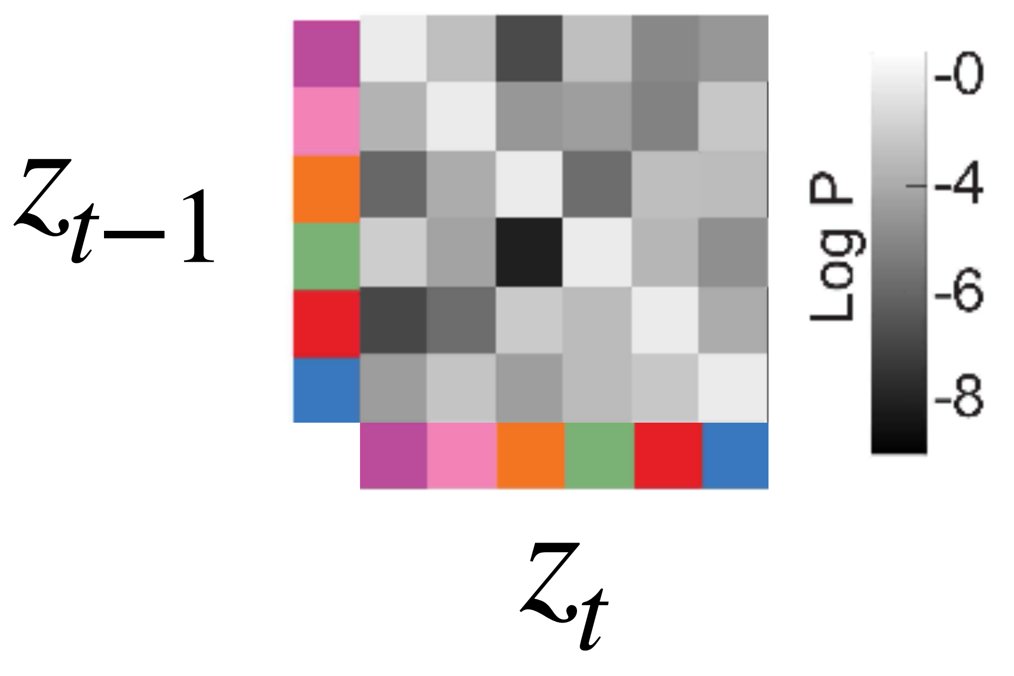

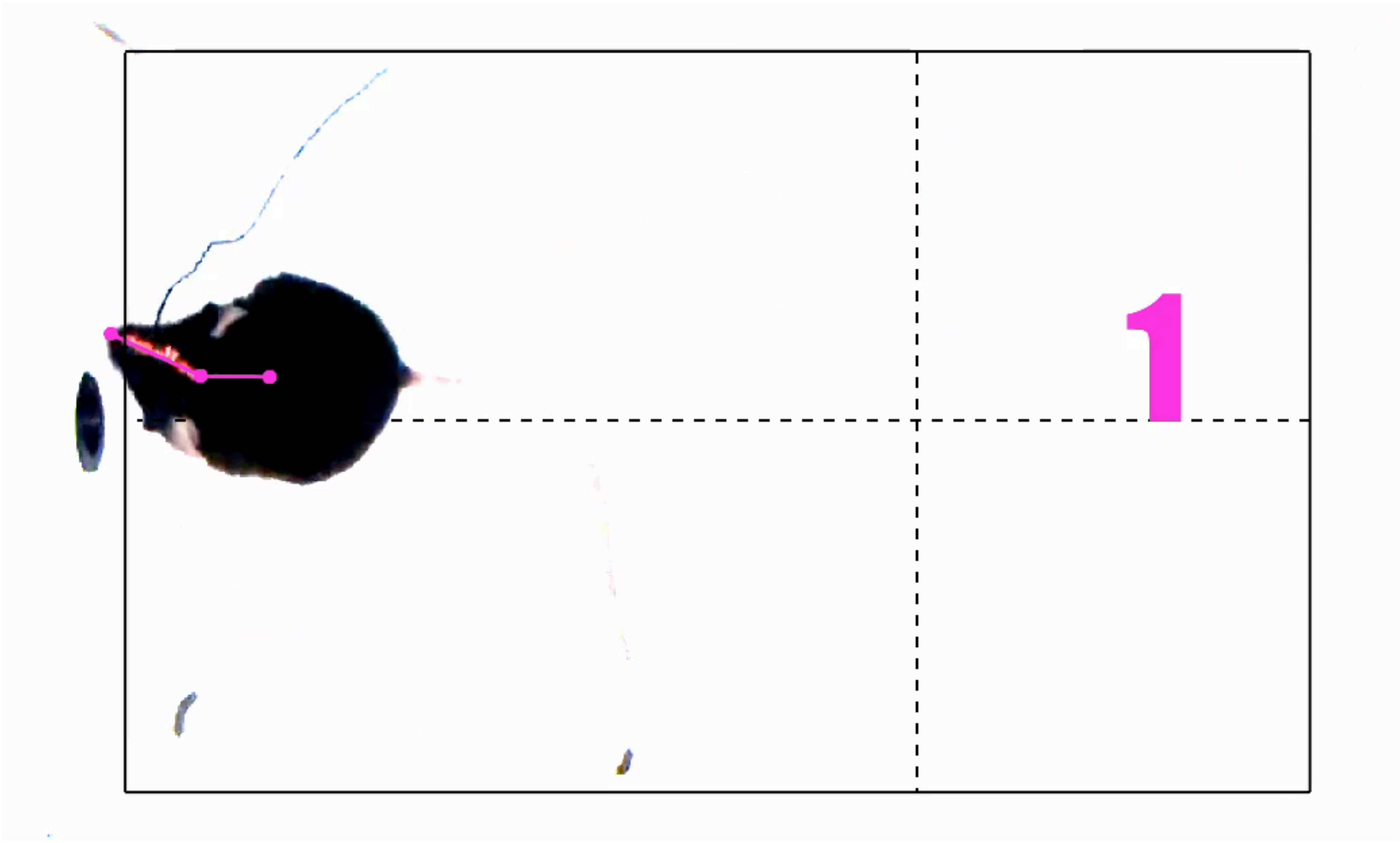

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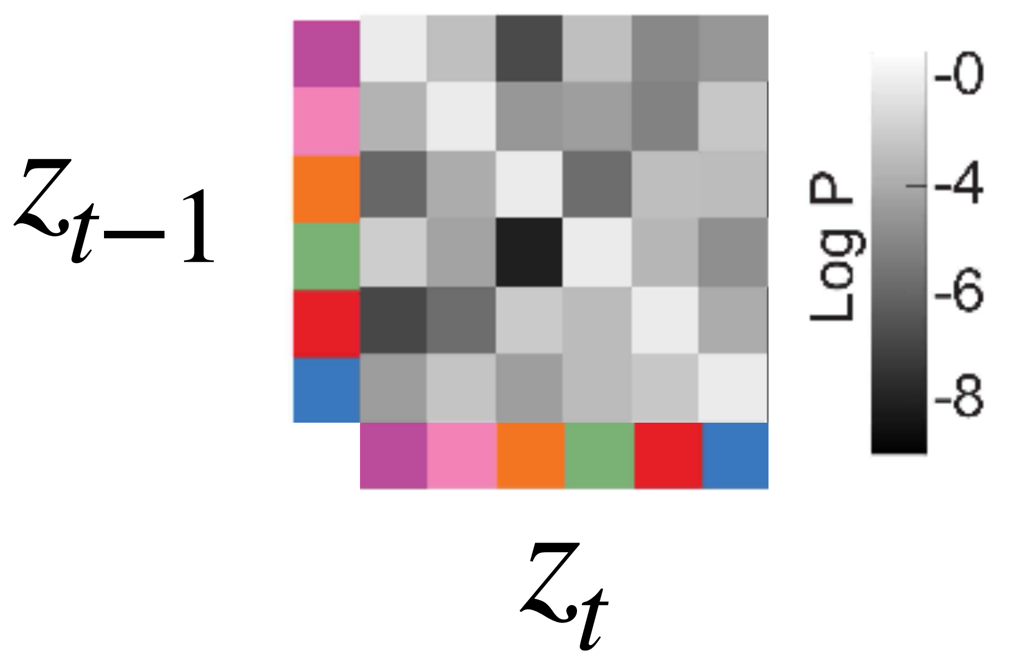

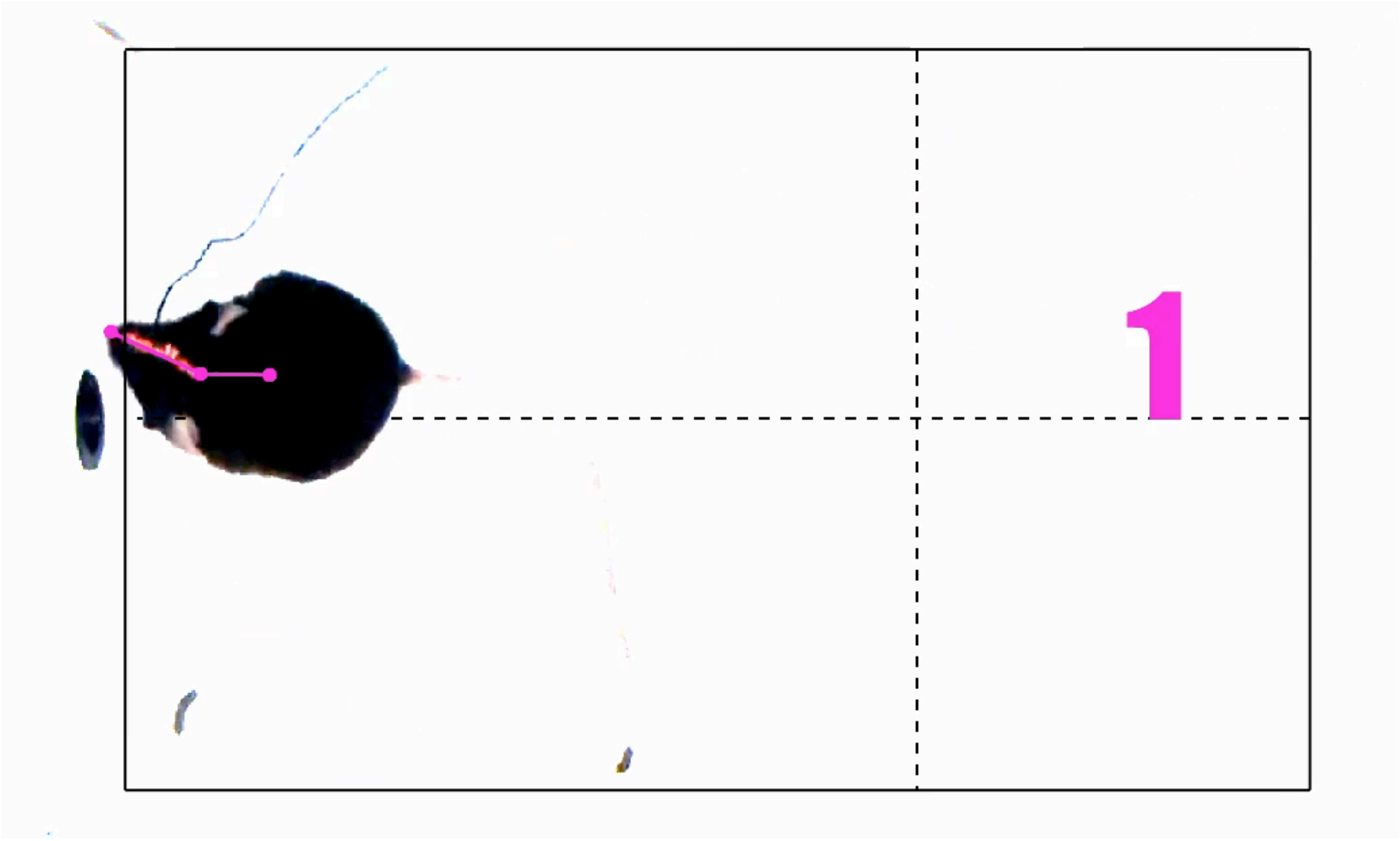

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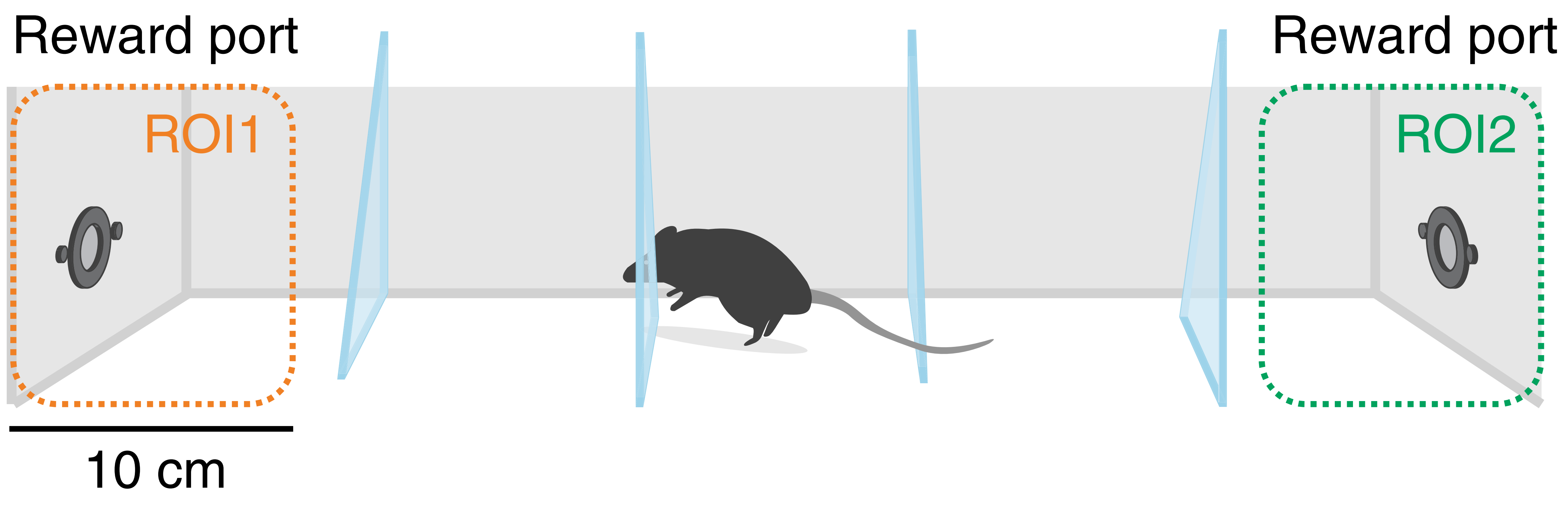
References

HMM: Theory

Applications to freely moving behavior

- AR-HMM:
  - Moseq *[Wiltschko et al, 2015, Markovits et al, 2018][Findley et al, 2021]*
- Switching Linear Dynamical Systems (SLDS)
  - Keypoint Moseq *[Weinreb et al, 2023]*

Naturalistic foraging



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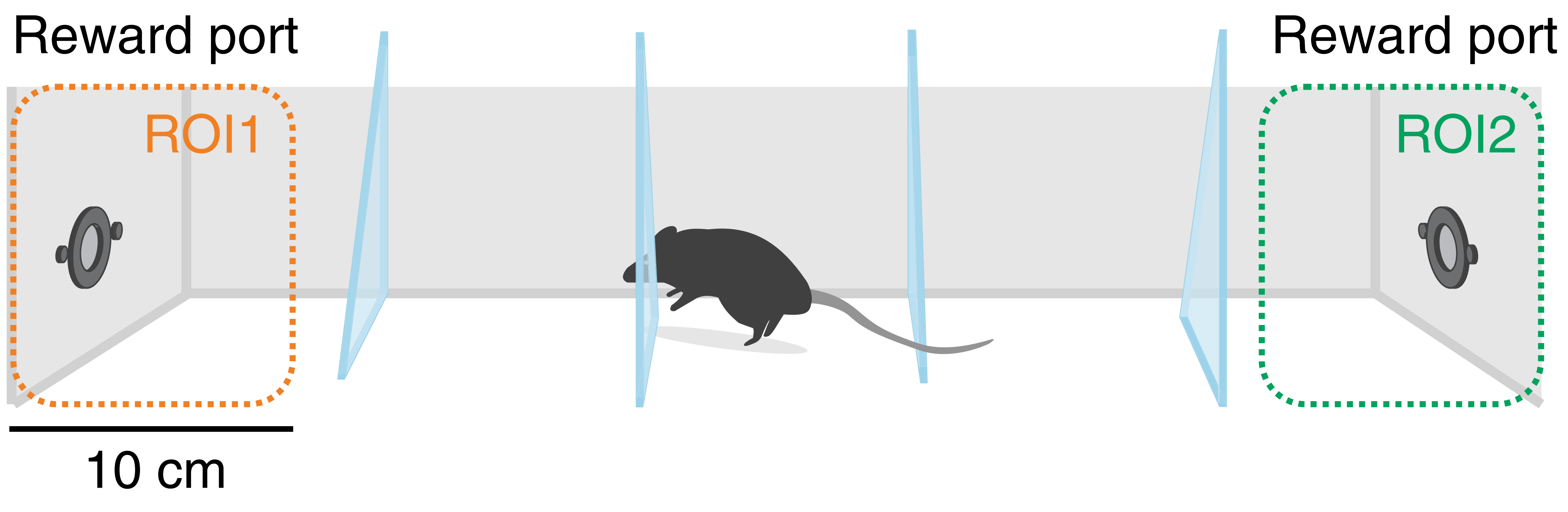
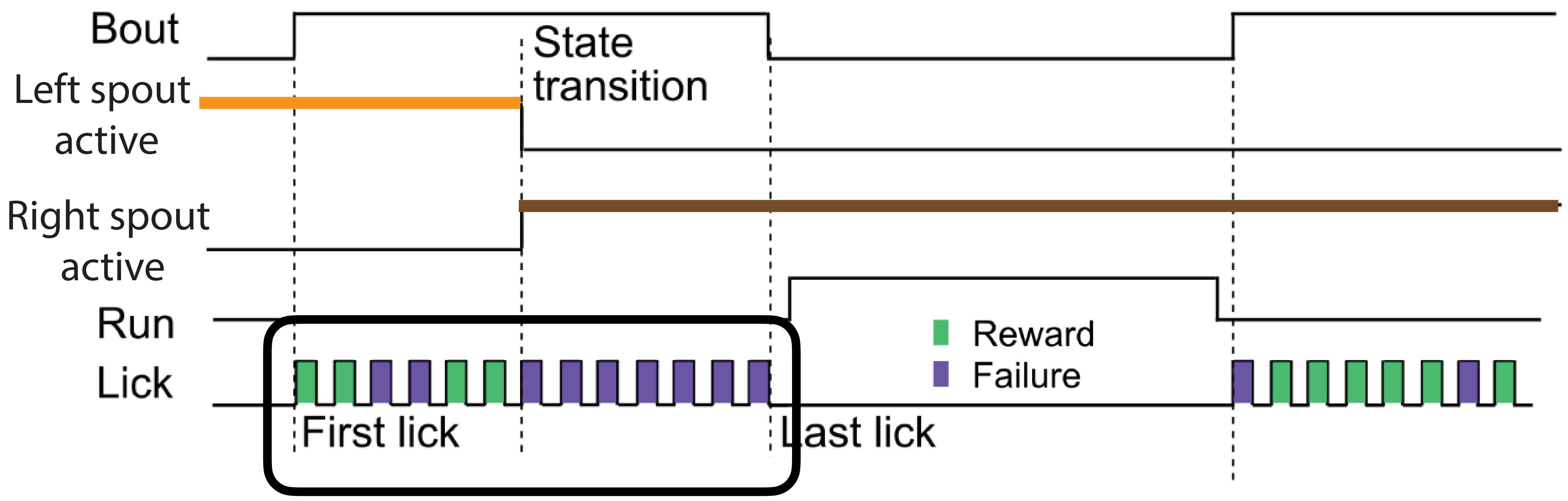


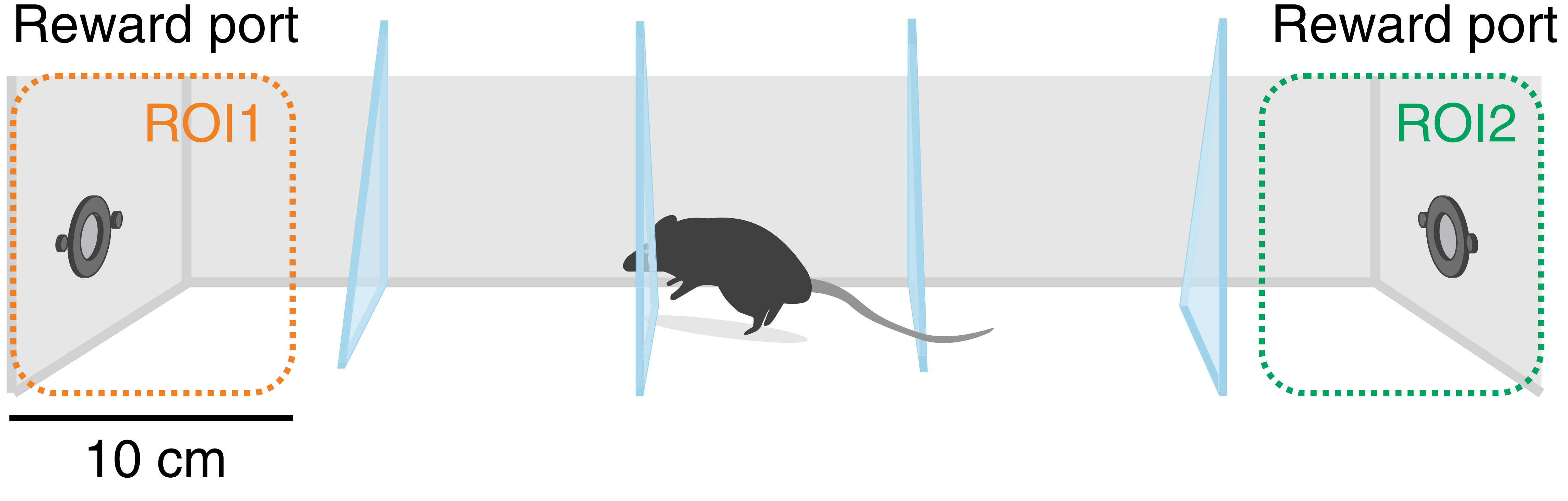


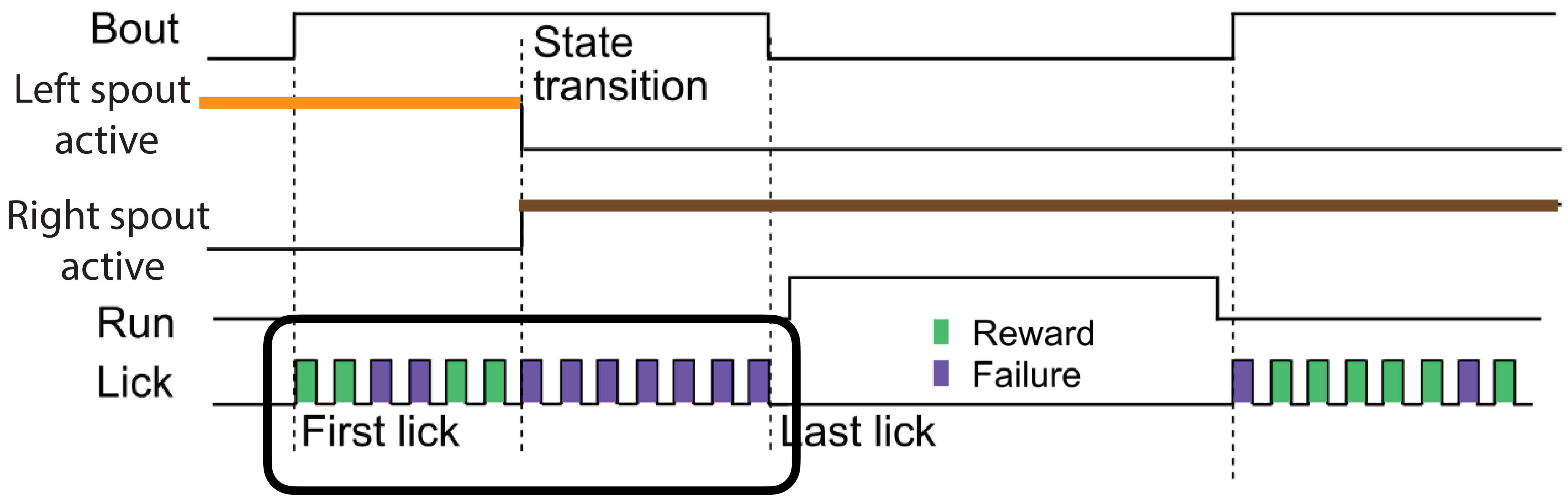
Rewarded lick —> port is active —> stay

Failures. —> prob release?

port inactive?

Naturalistic foraging





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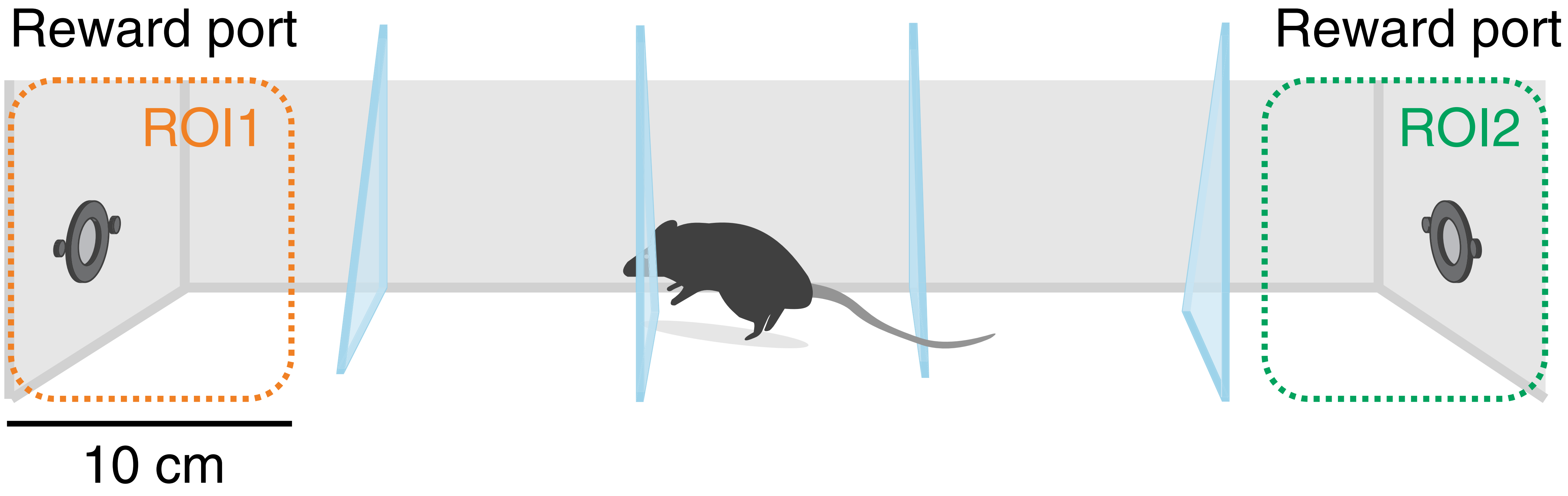
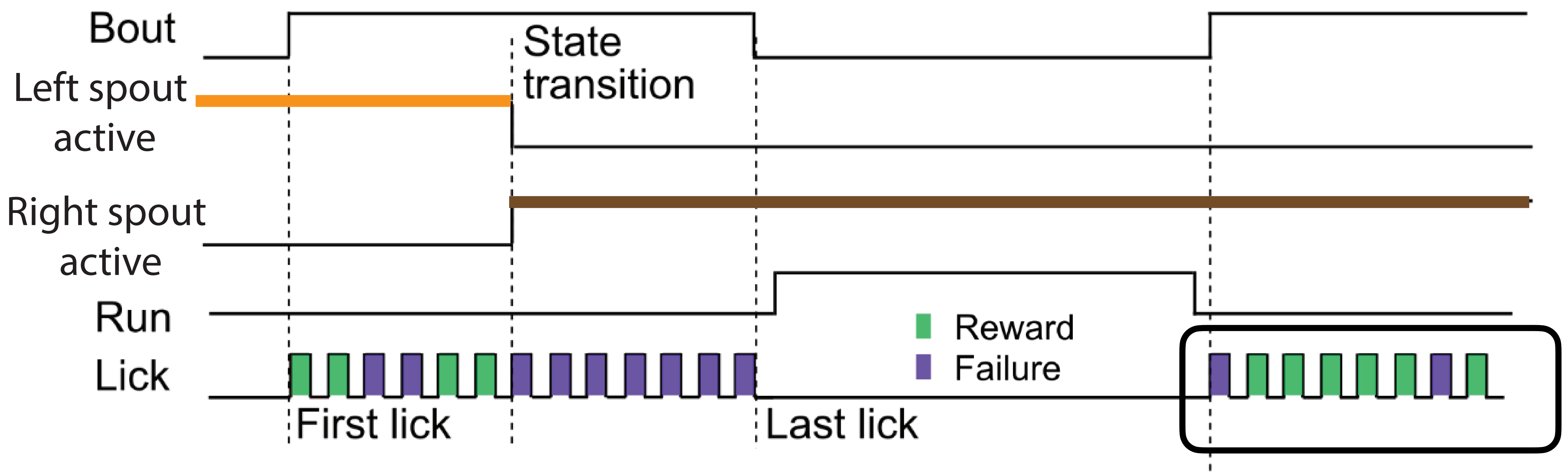
Integrate Rew & Fail, but how?

Naturalistic foraging





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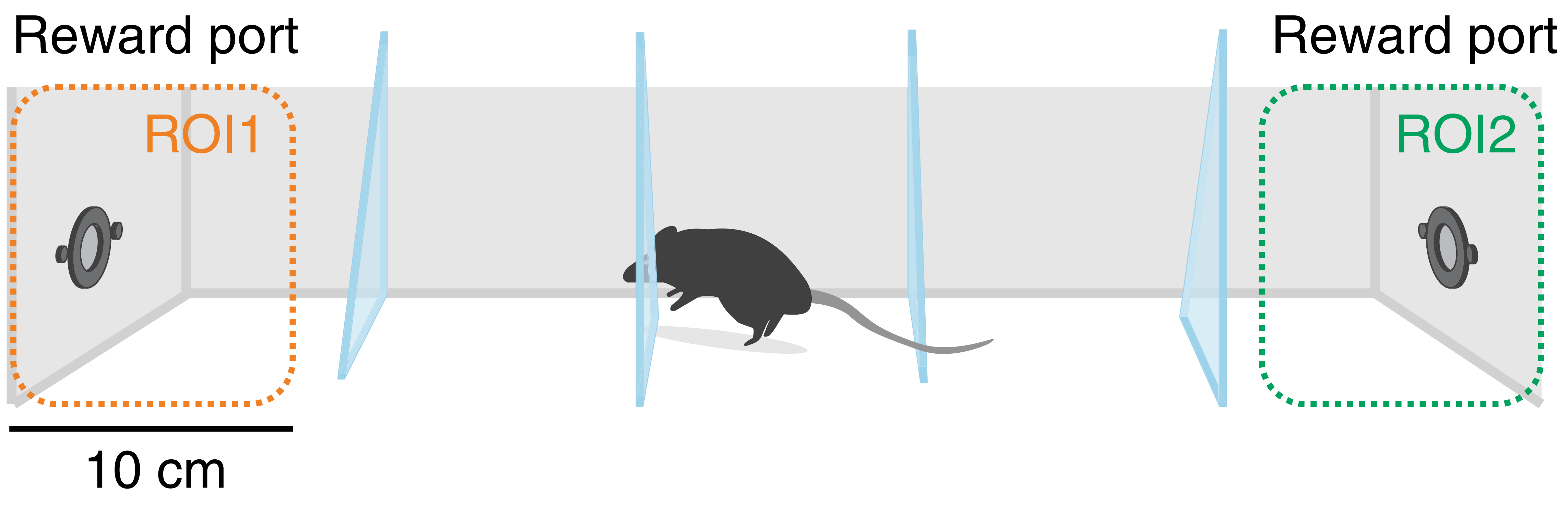

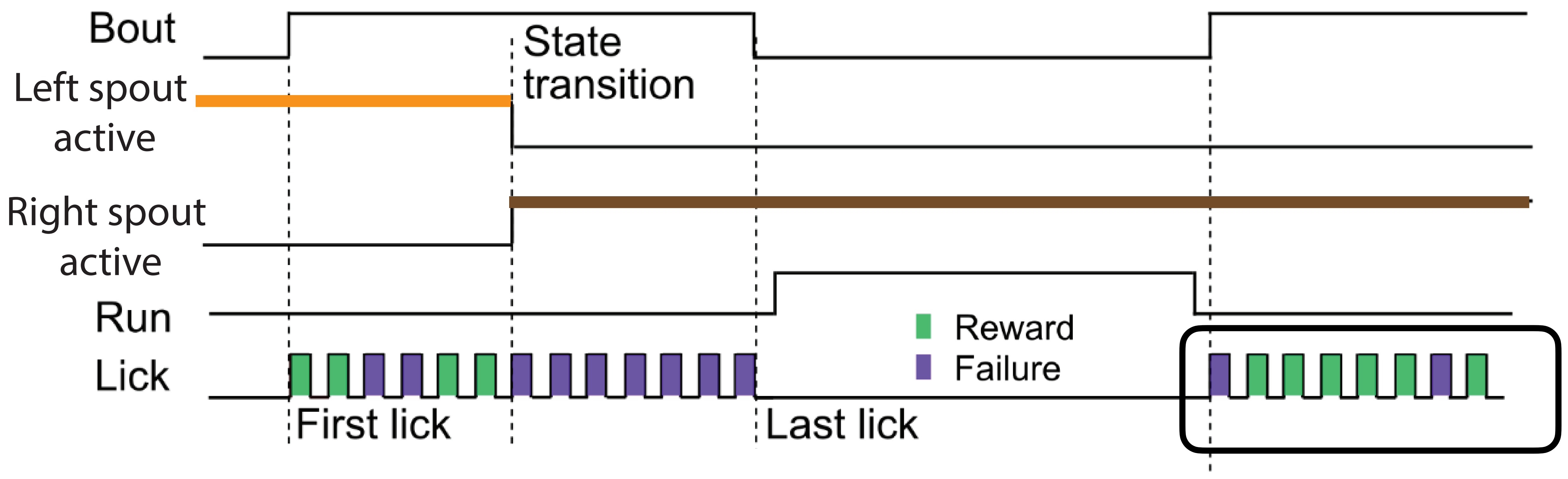

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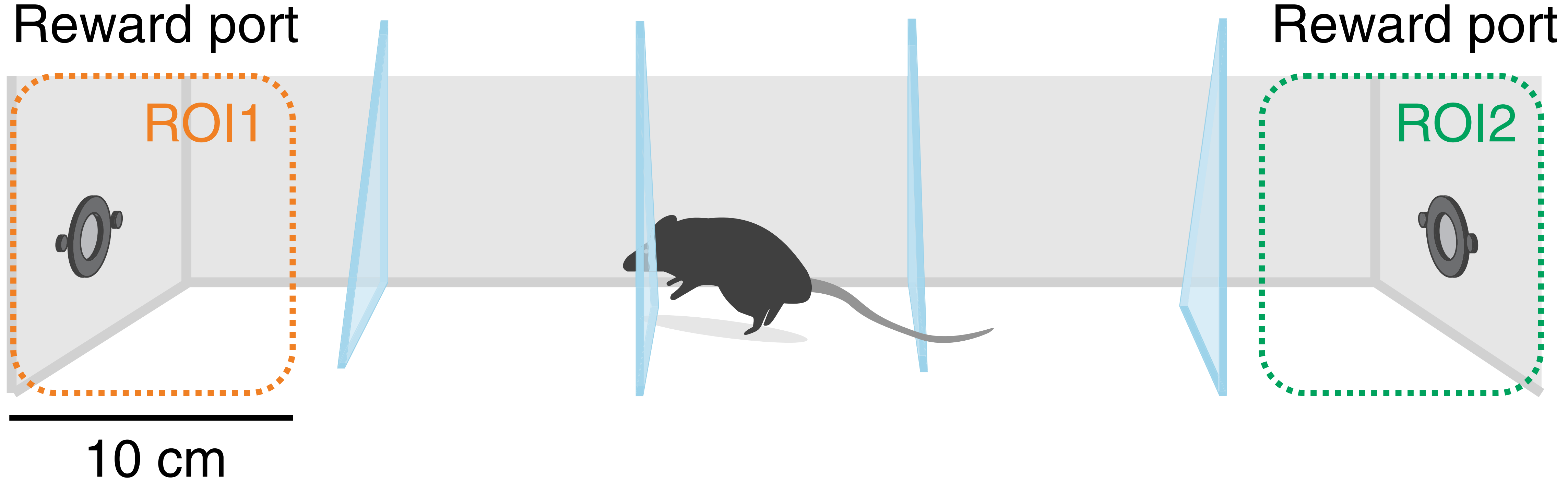
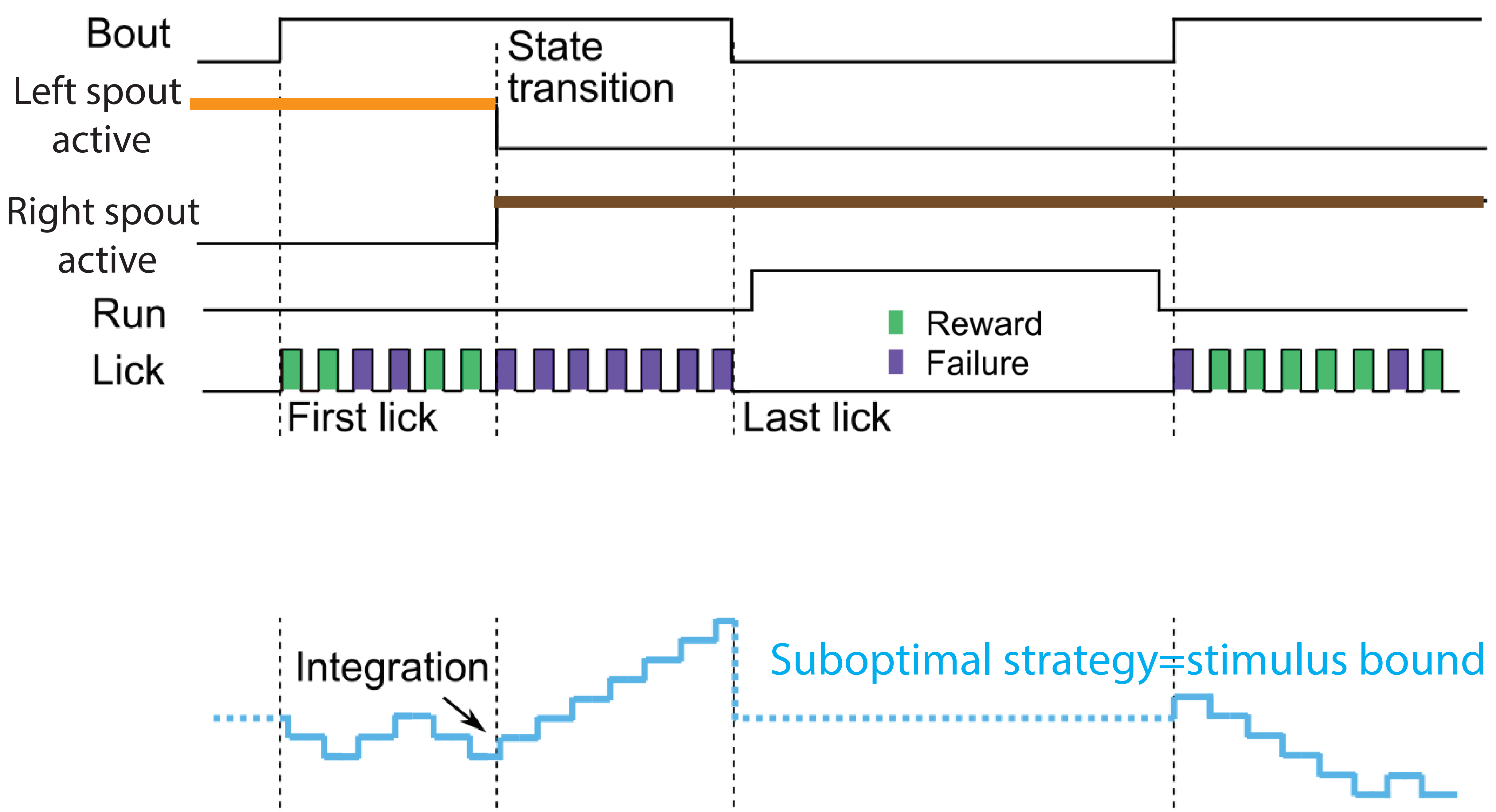
…Alternative strategies for evidence accumulation

Naturalistic foraging





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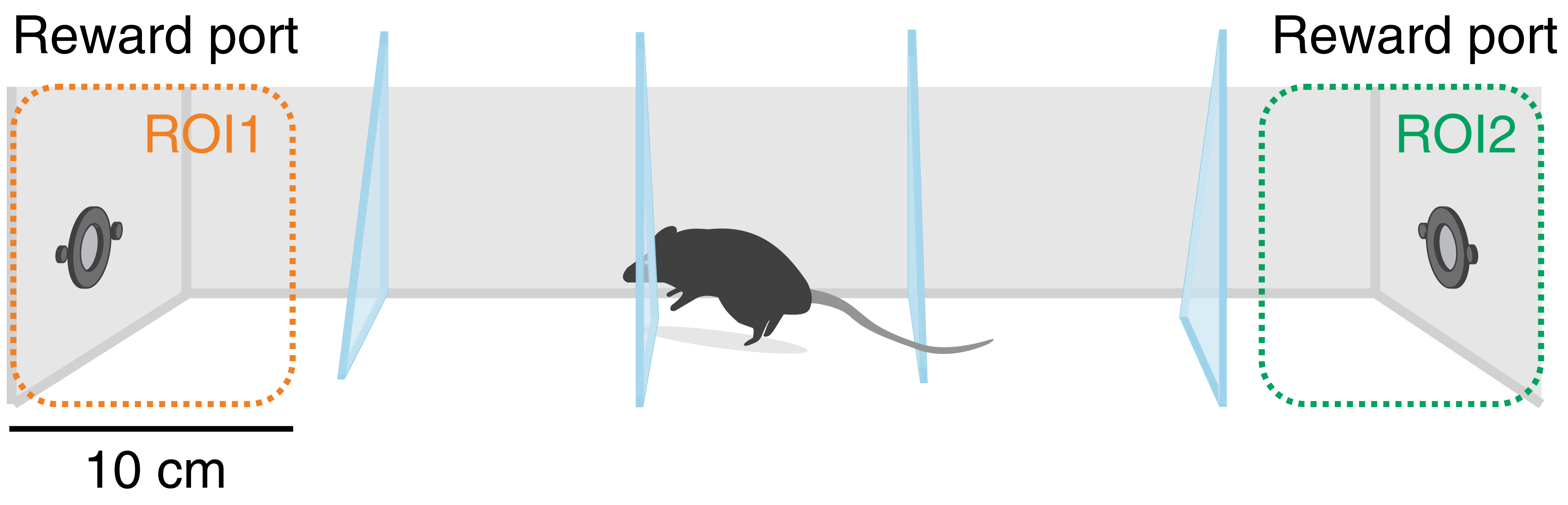

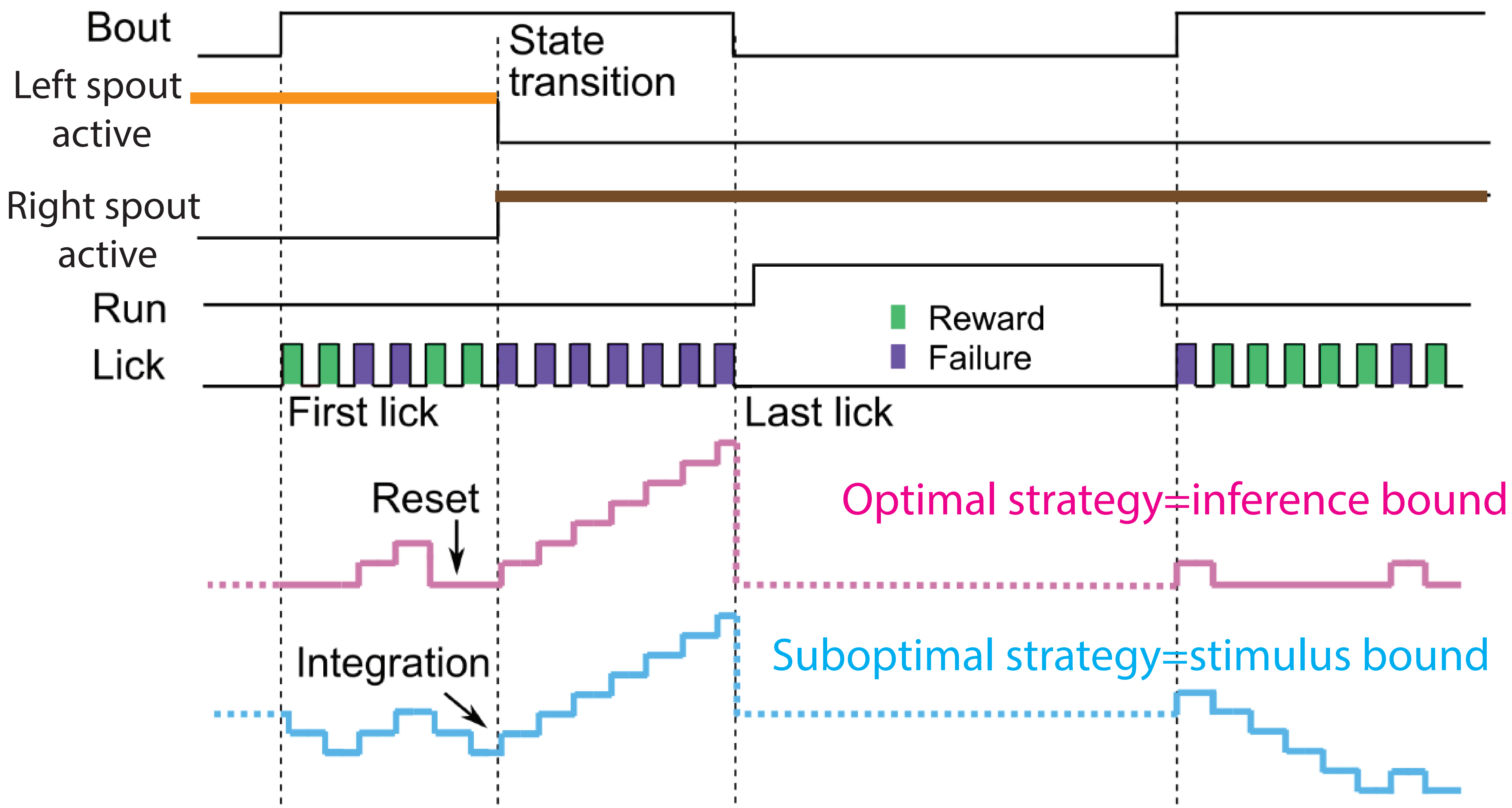

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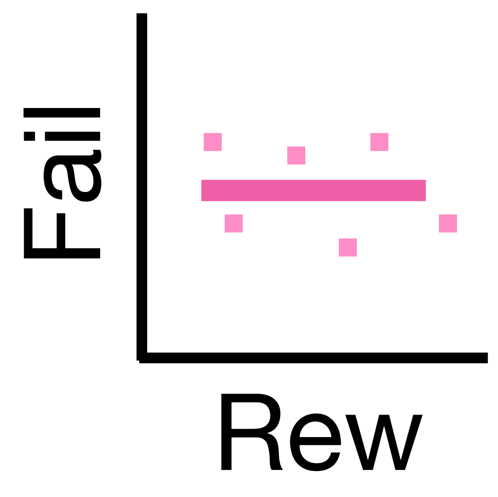
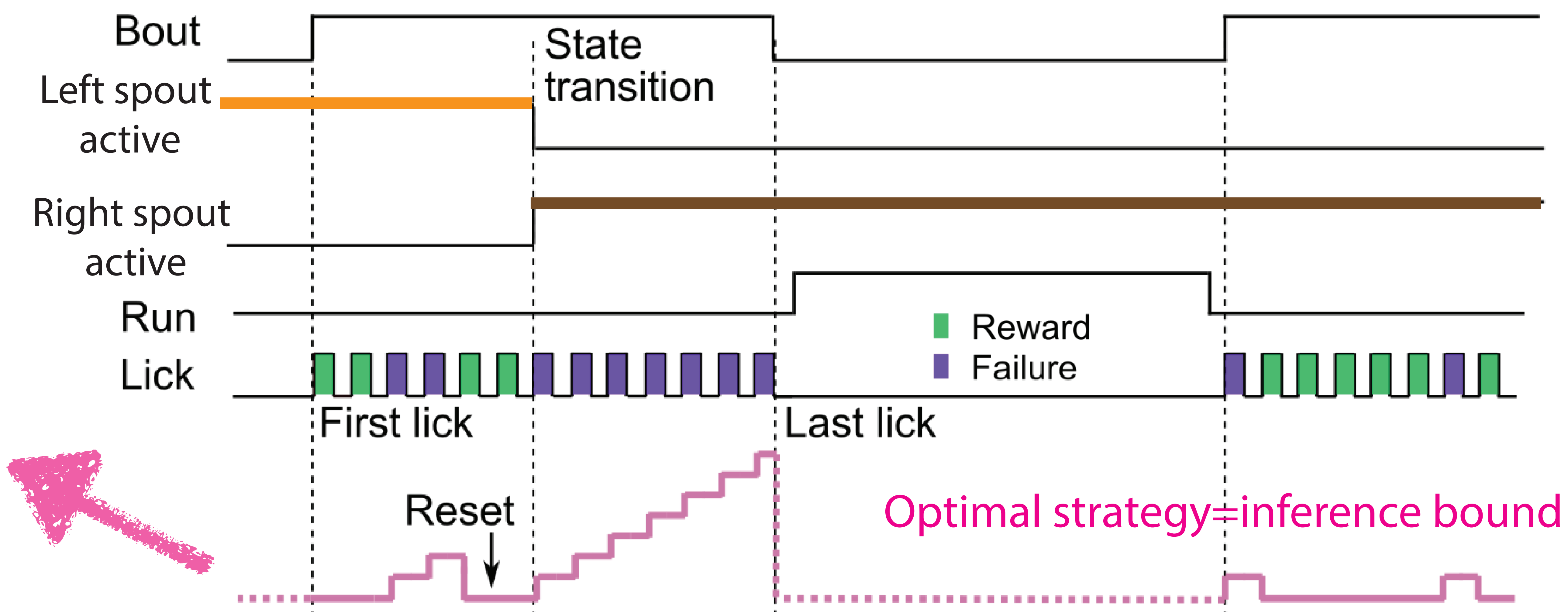




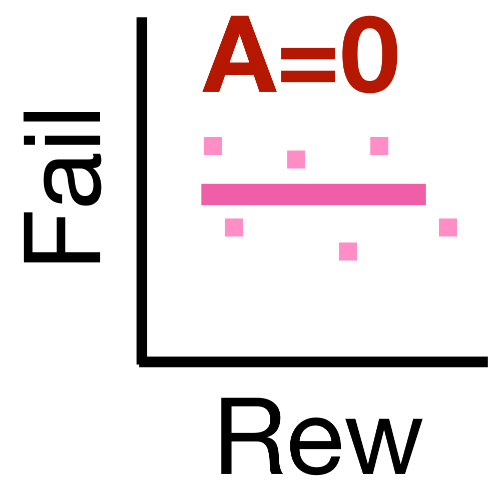
Naturalistic foraging

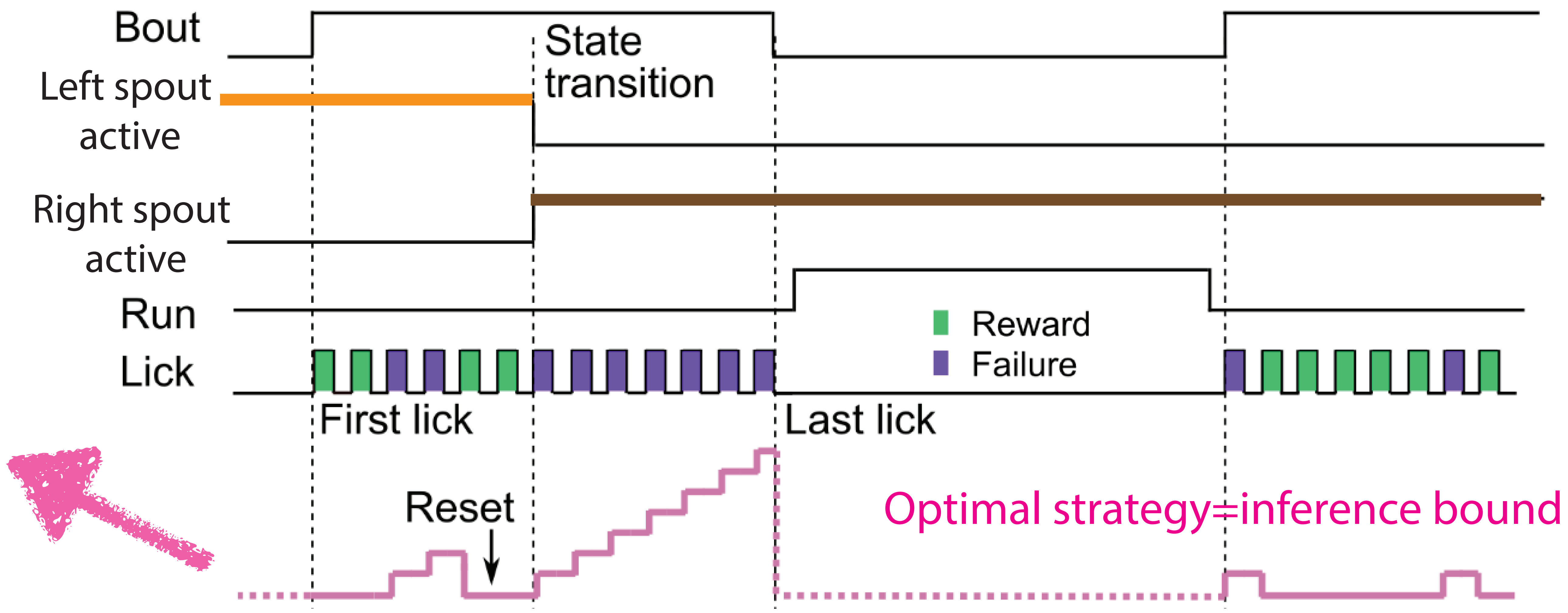
Strategies are **Linear Models**:

Fail = **A** x Rew + b

A = how much they weigh Rew

b = how many Fail they tolerate





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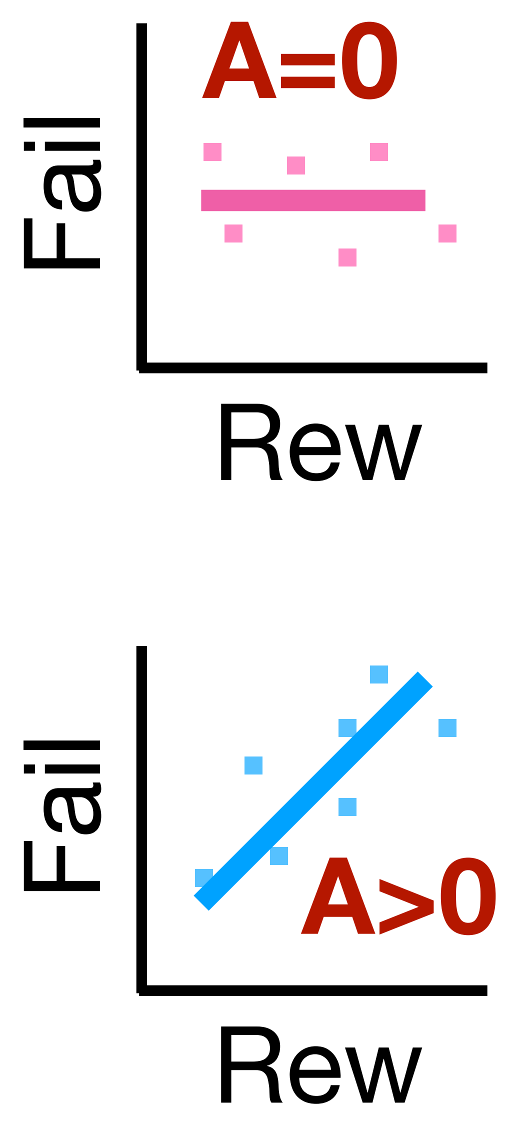

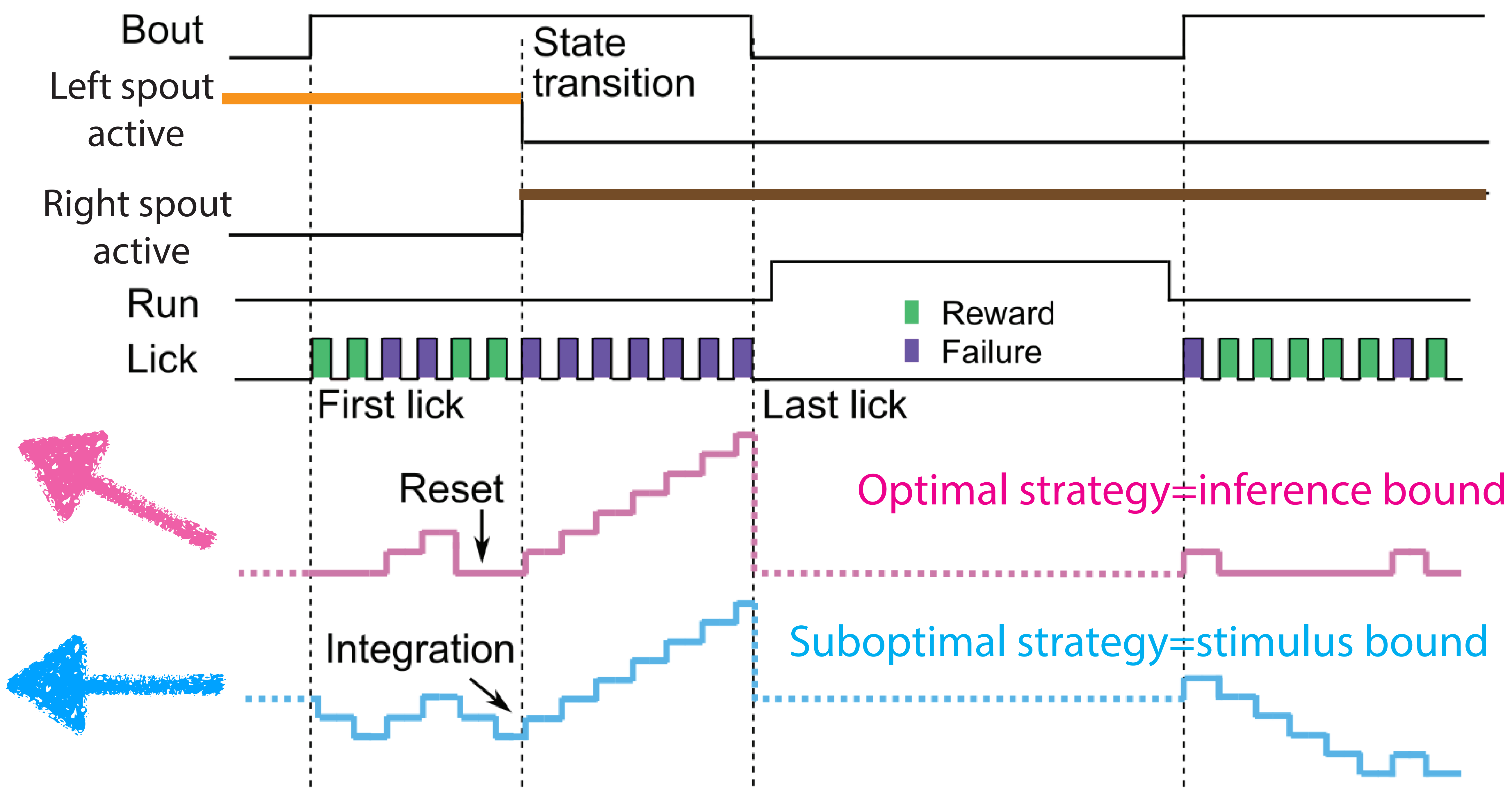

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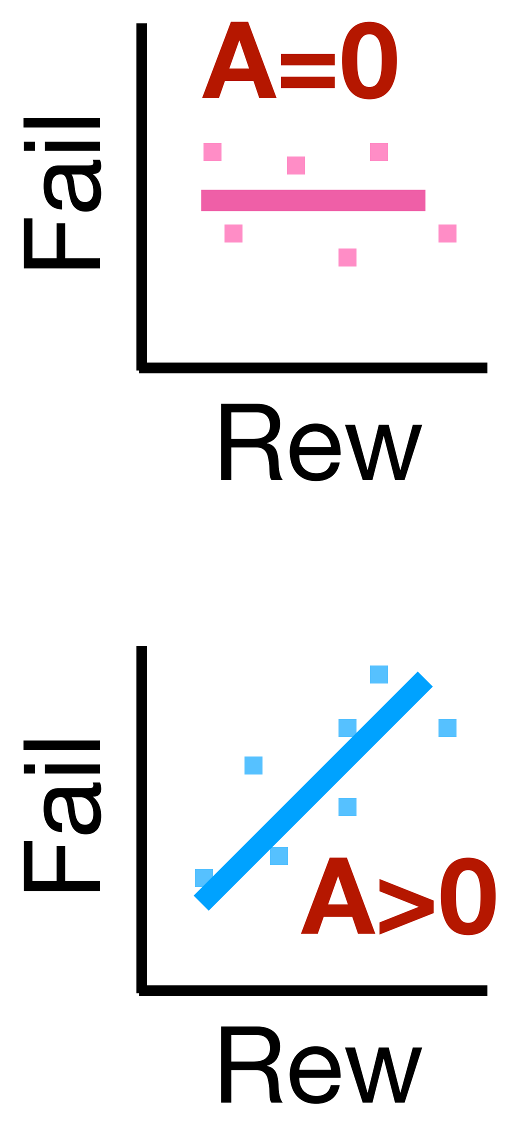
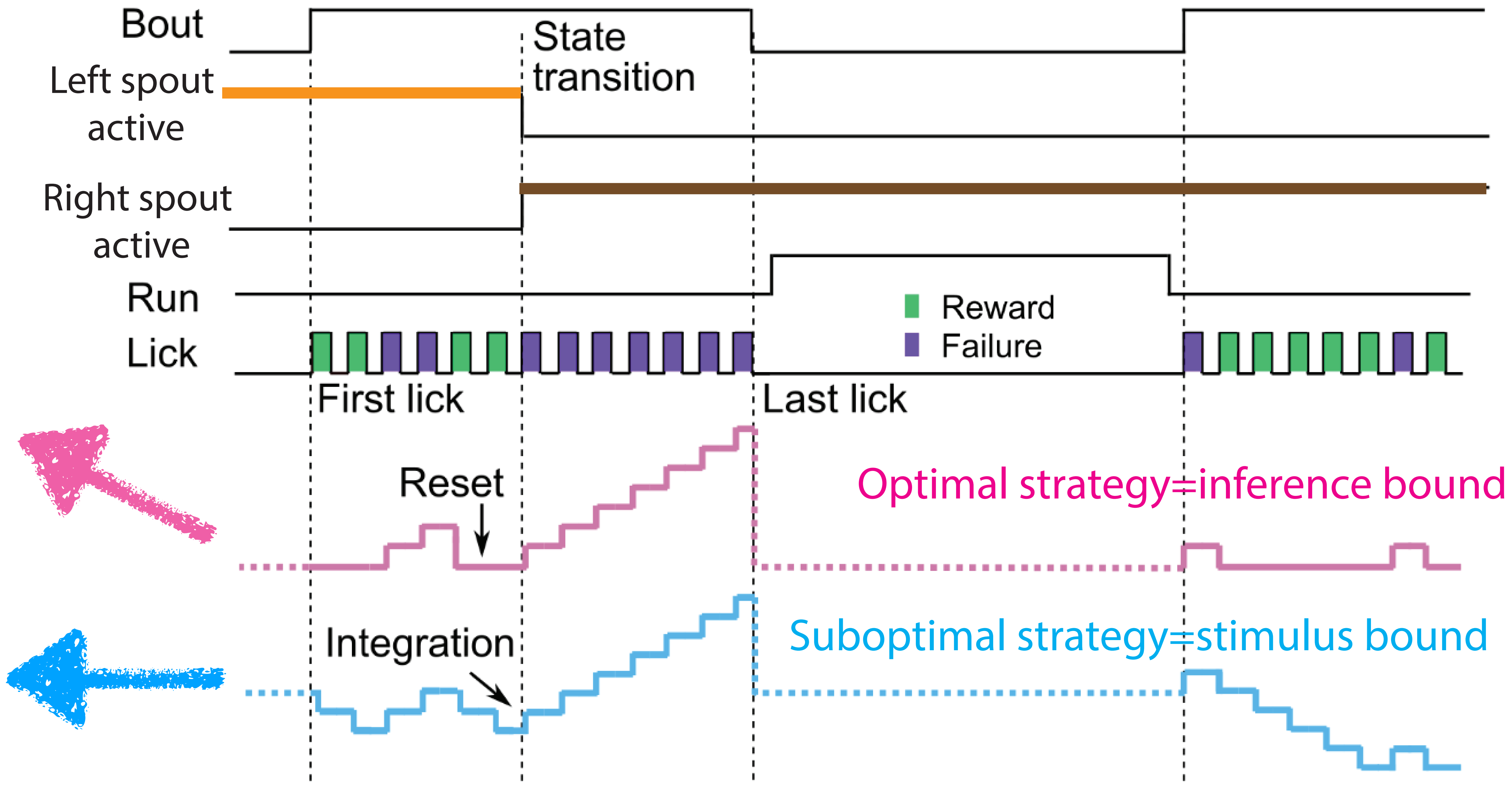
Switching between strategies

Switching between linear models





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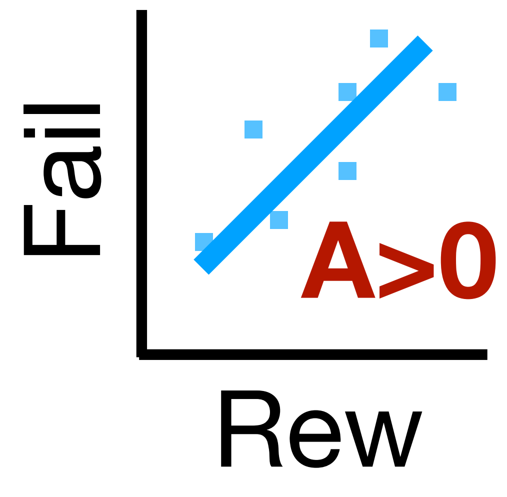

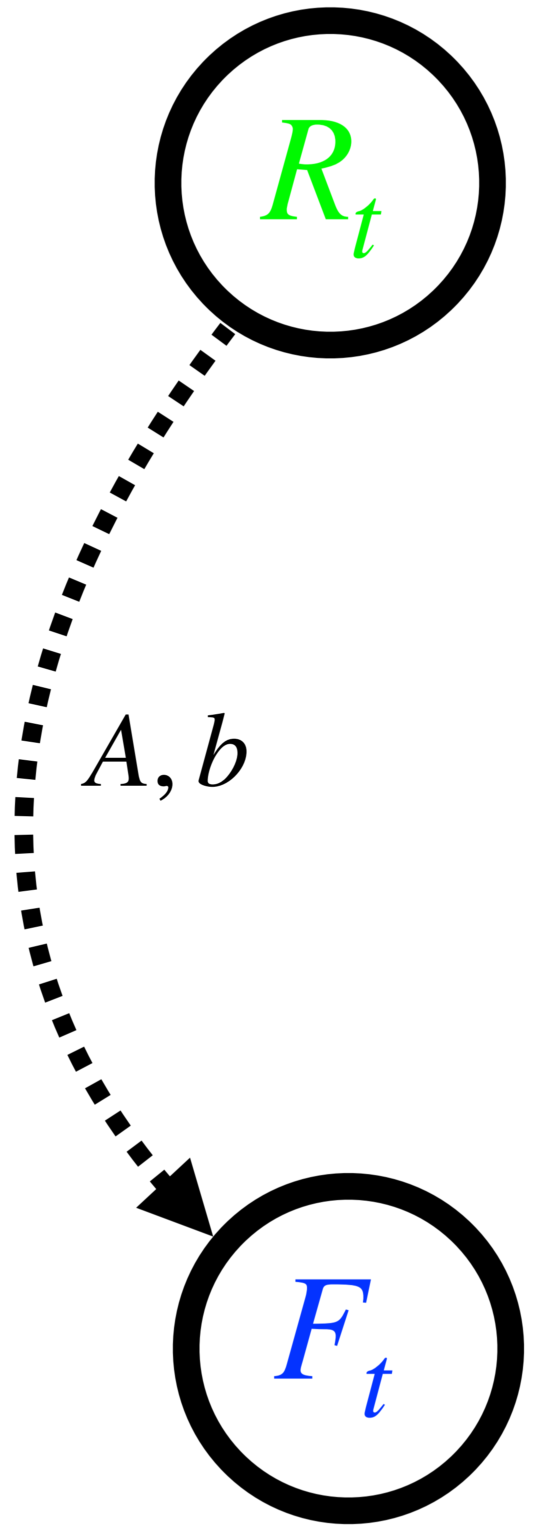

$$p(F_t) = N(AR_t + b, \sigma)$$

Naturalistic foraging

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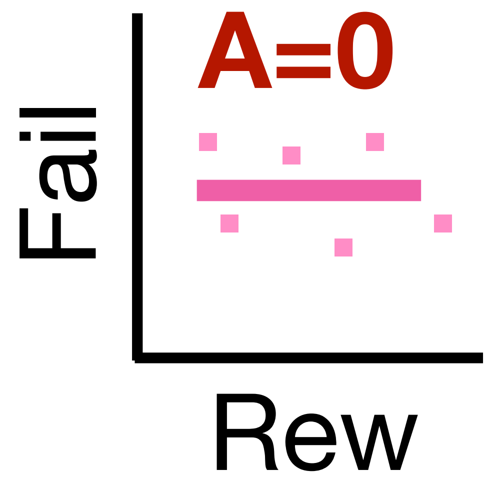
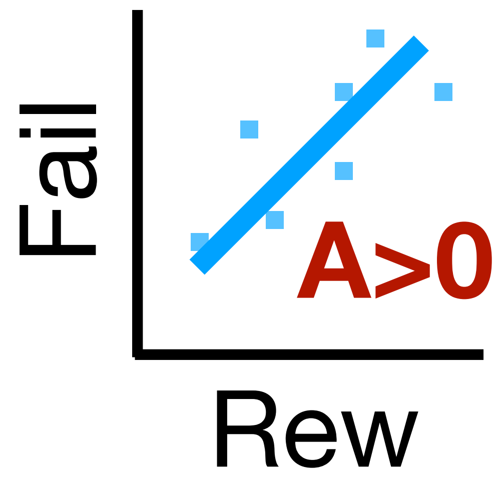
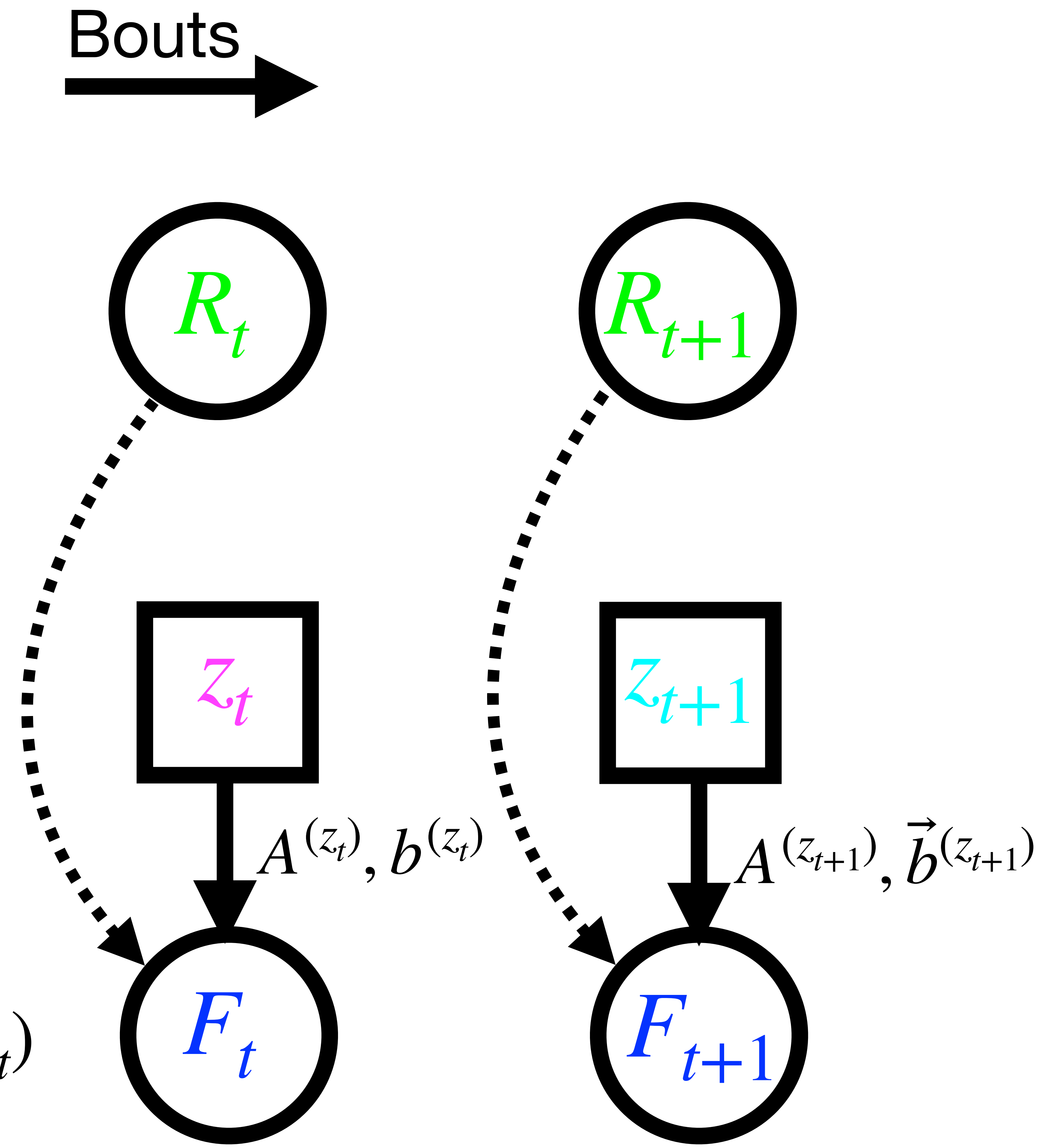
Linear Model parameters A,b depend on hidden state z_t







$$p(F_t) = N(A^{(z_t)} R_t + b^{(z_t)}, \sigma^{(z_t)} \mid z_t)$$

Naturalistic foraging

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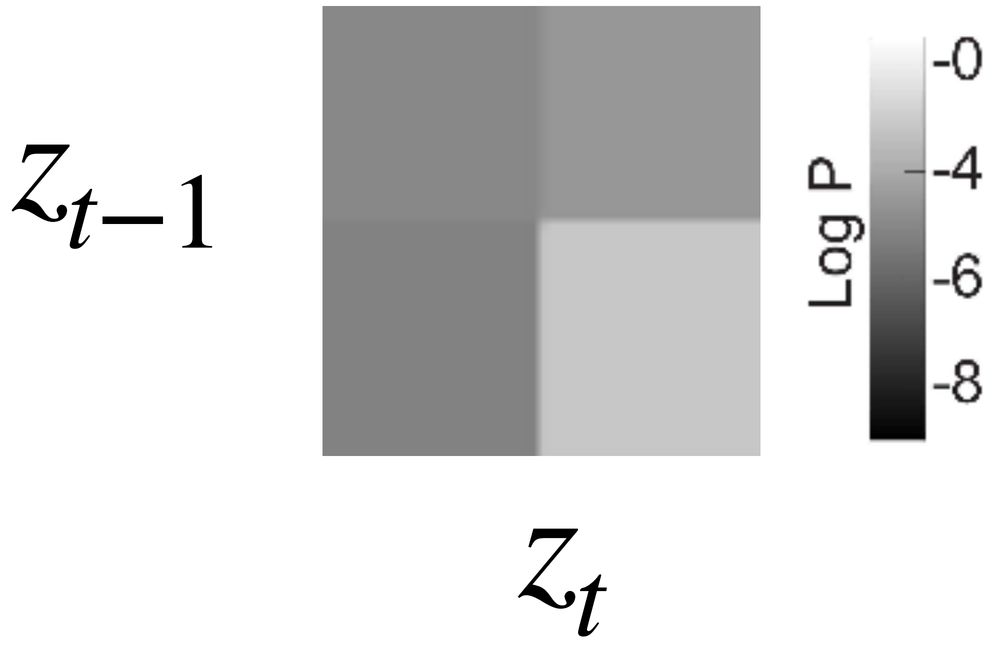
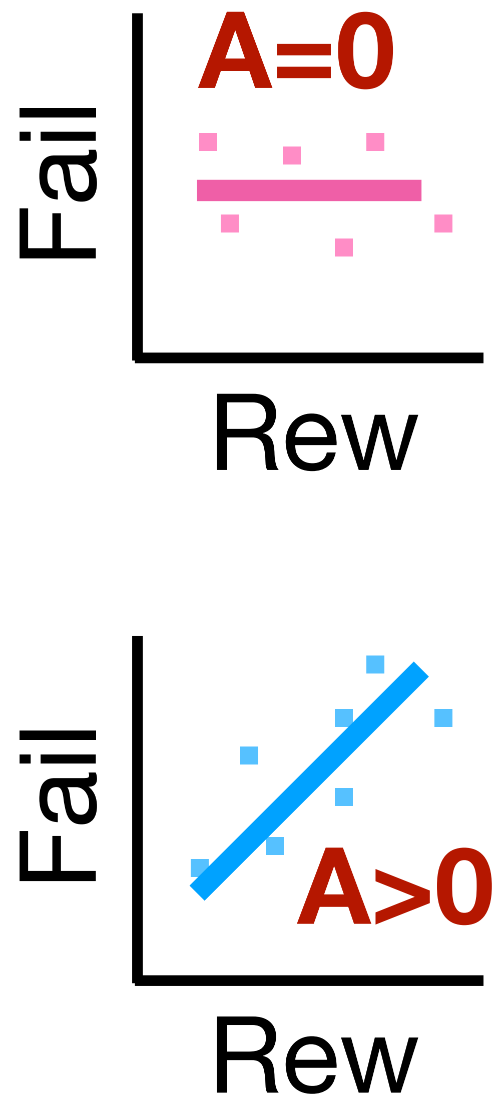
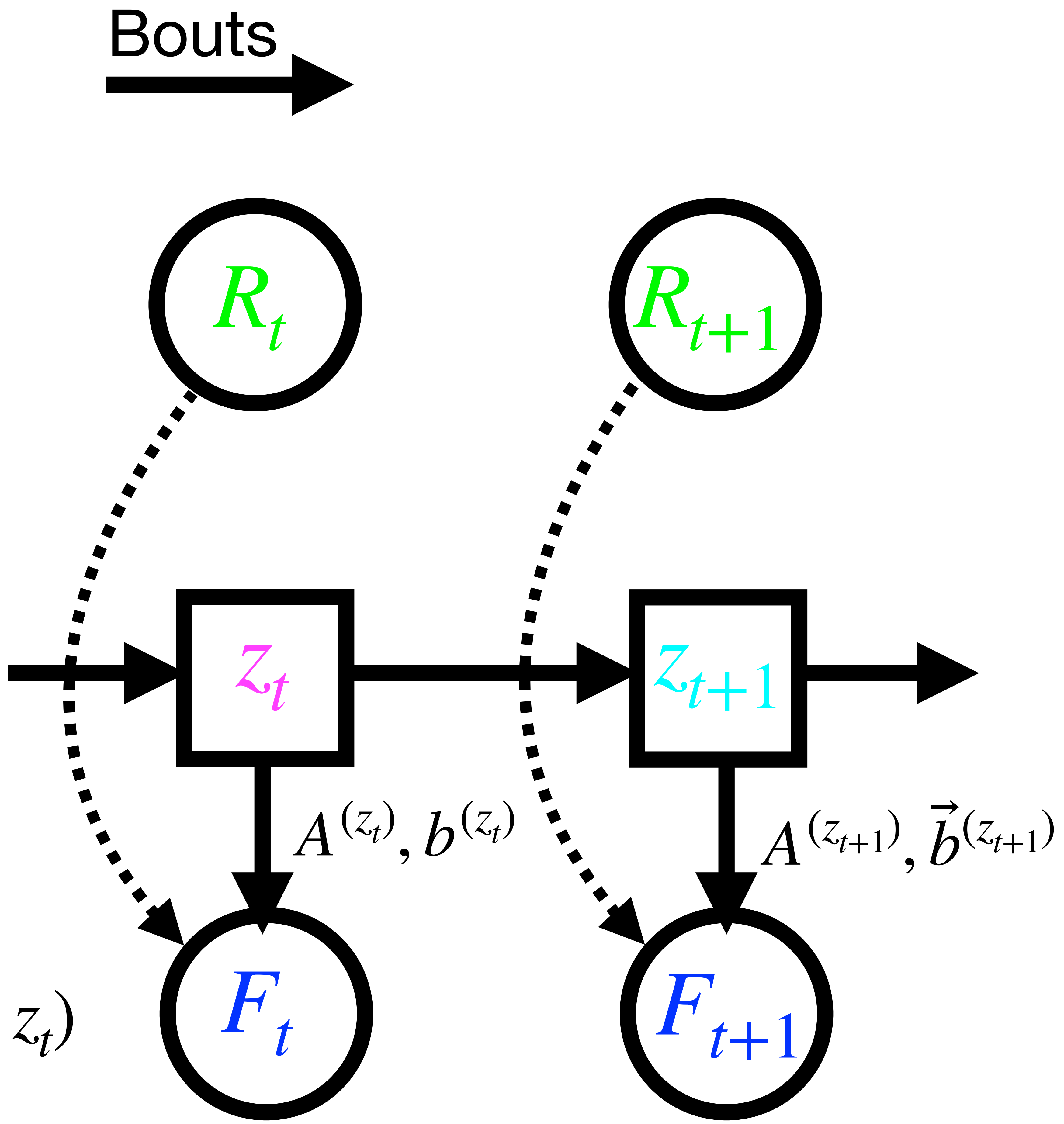
$$p(z_t = k \,|\, z_{t-1} = l)$$



Hidden Markov model with Linear Model emissions





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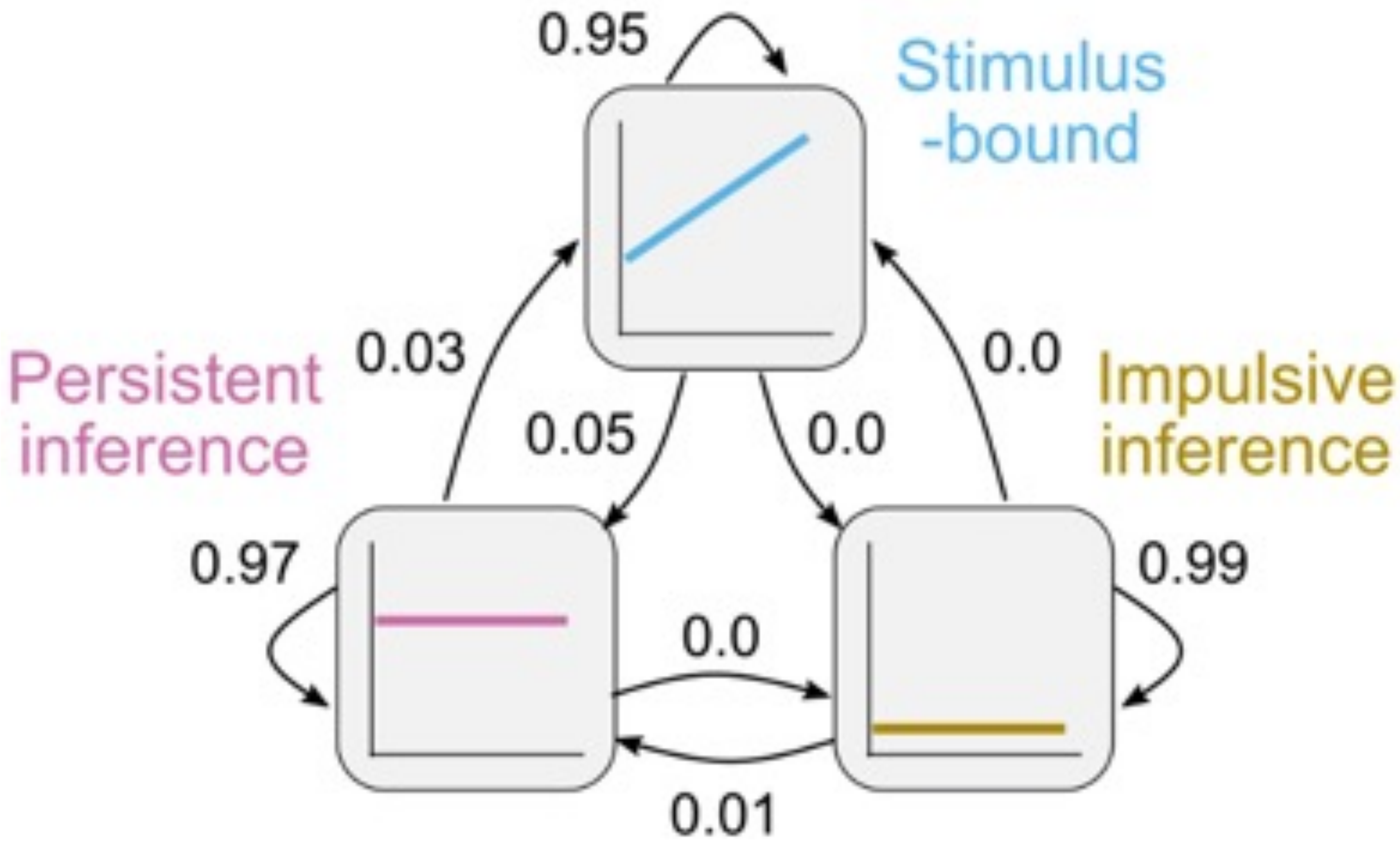

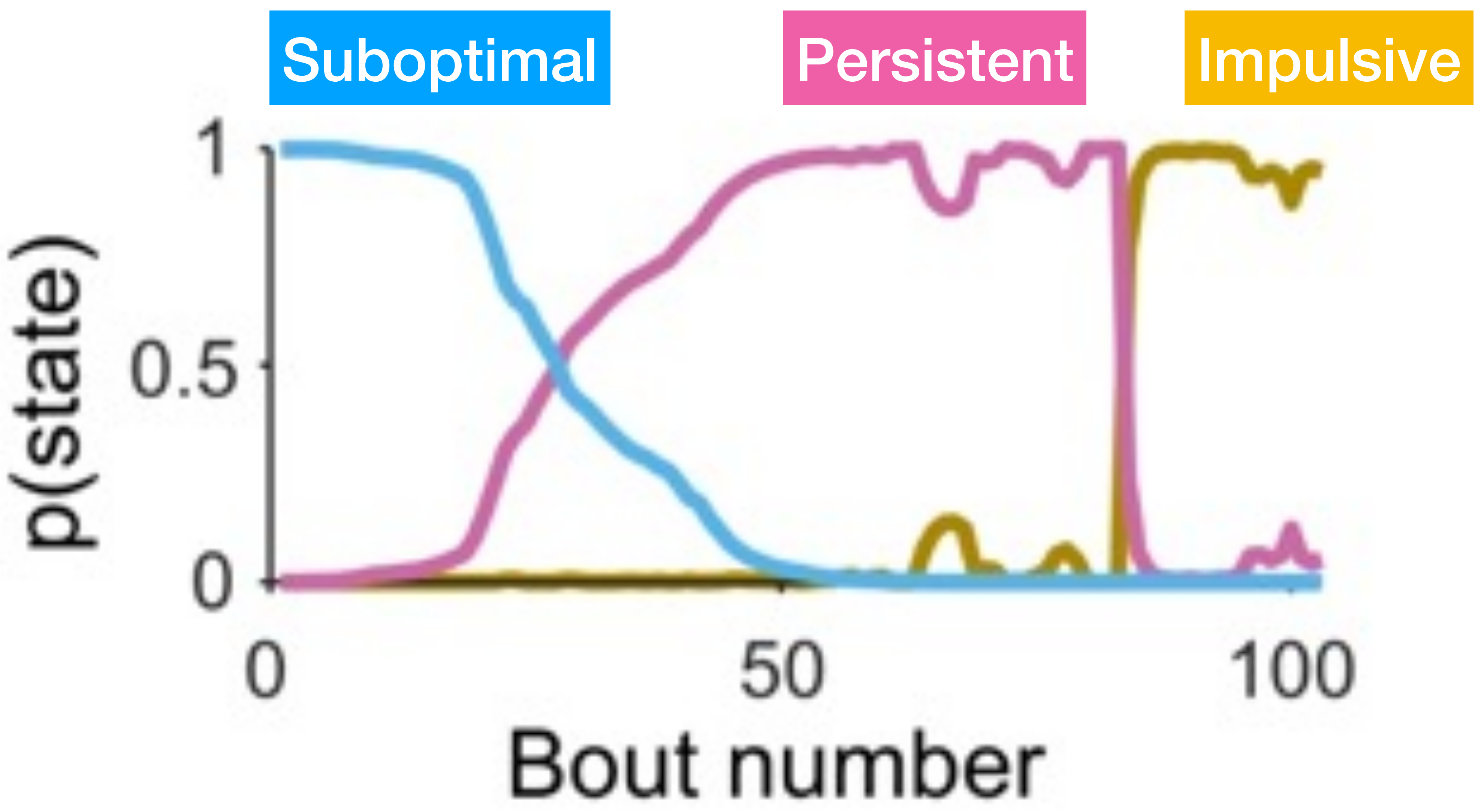

[Cazettes et al 2023]

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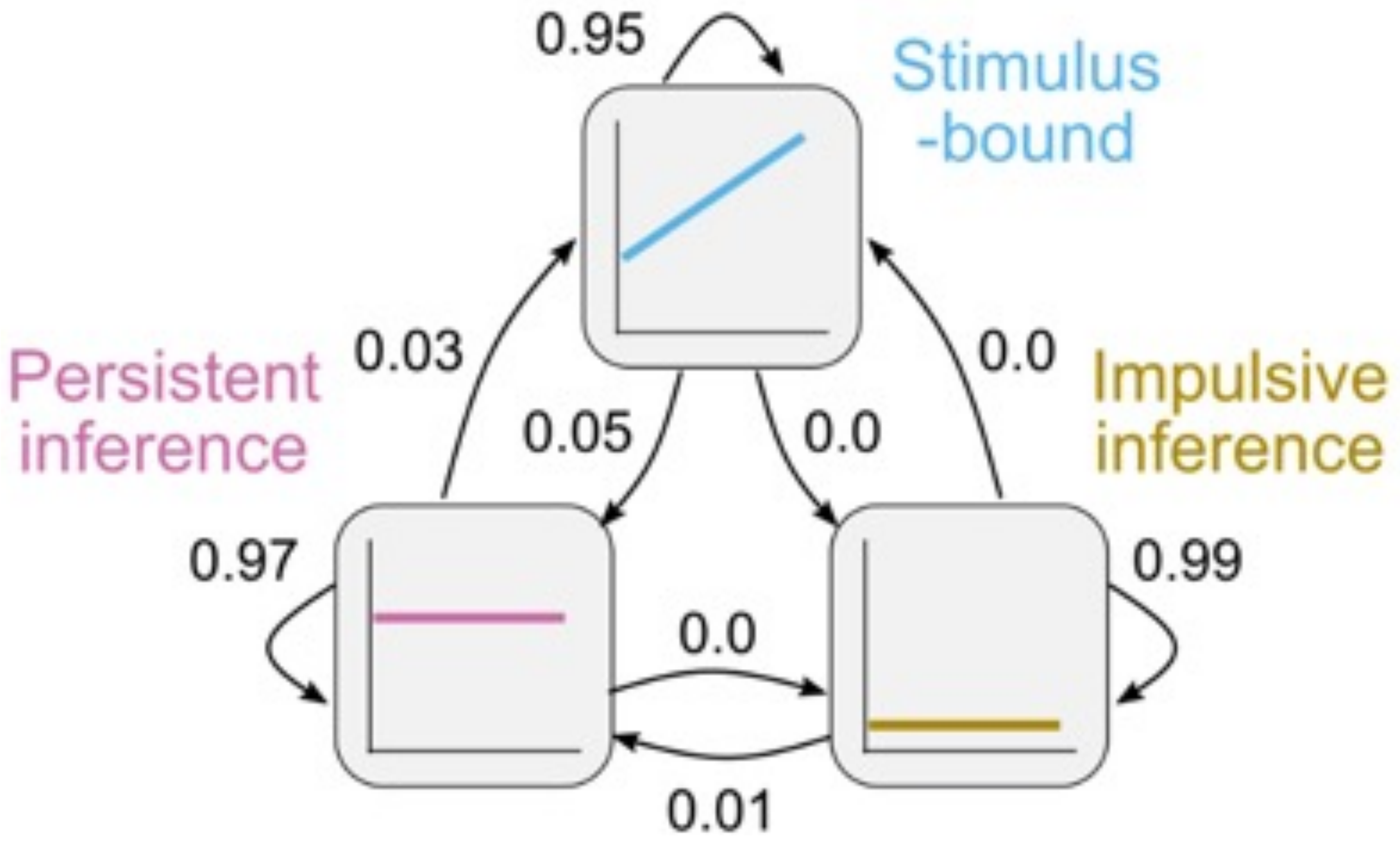
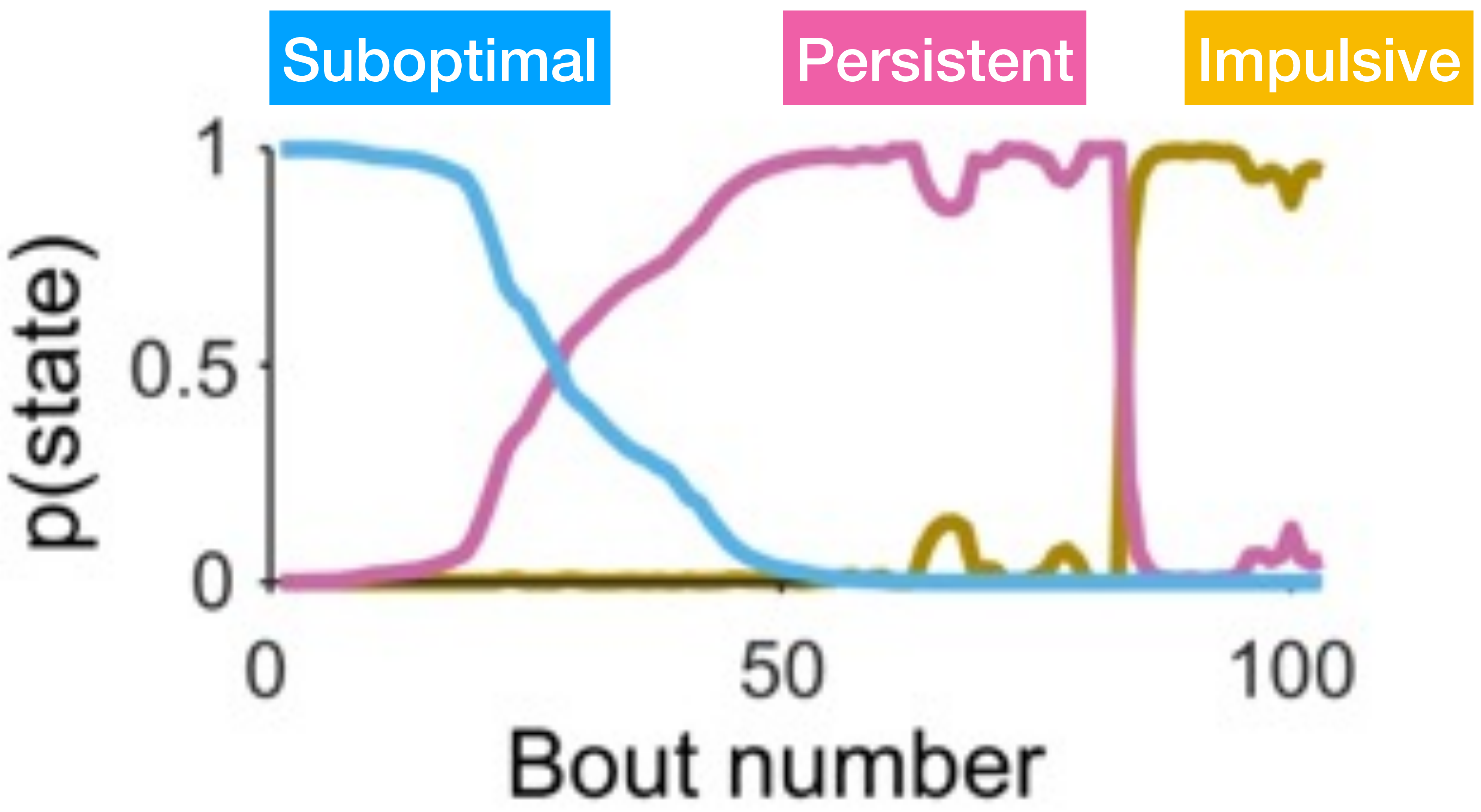


Two optimal strategies, one suboptimal strategy.

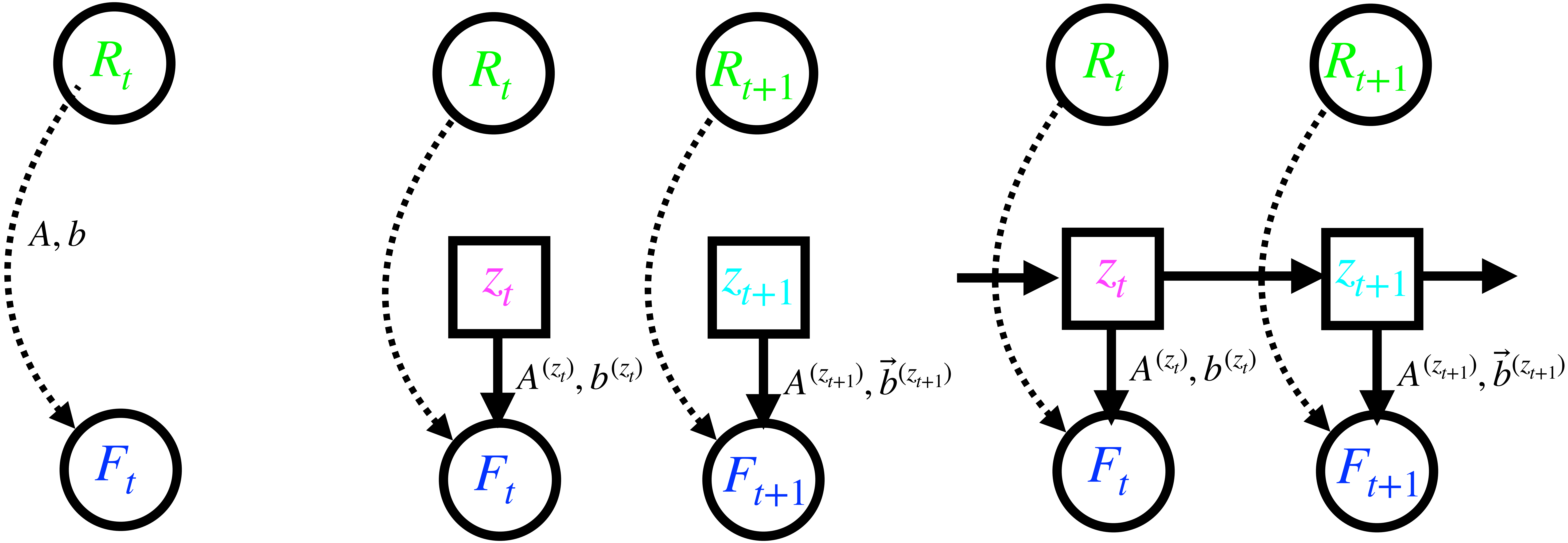


[Cazettes et al 2023]

Coding tutorial

- Load mystery dataset
- Fit models
- Select # hidden states

Fit Linear Model | **Fit mixture of LMs** | **Fit LM-HMM**



HMM with Linear Model emissions

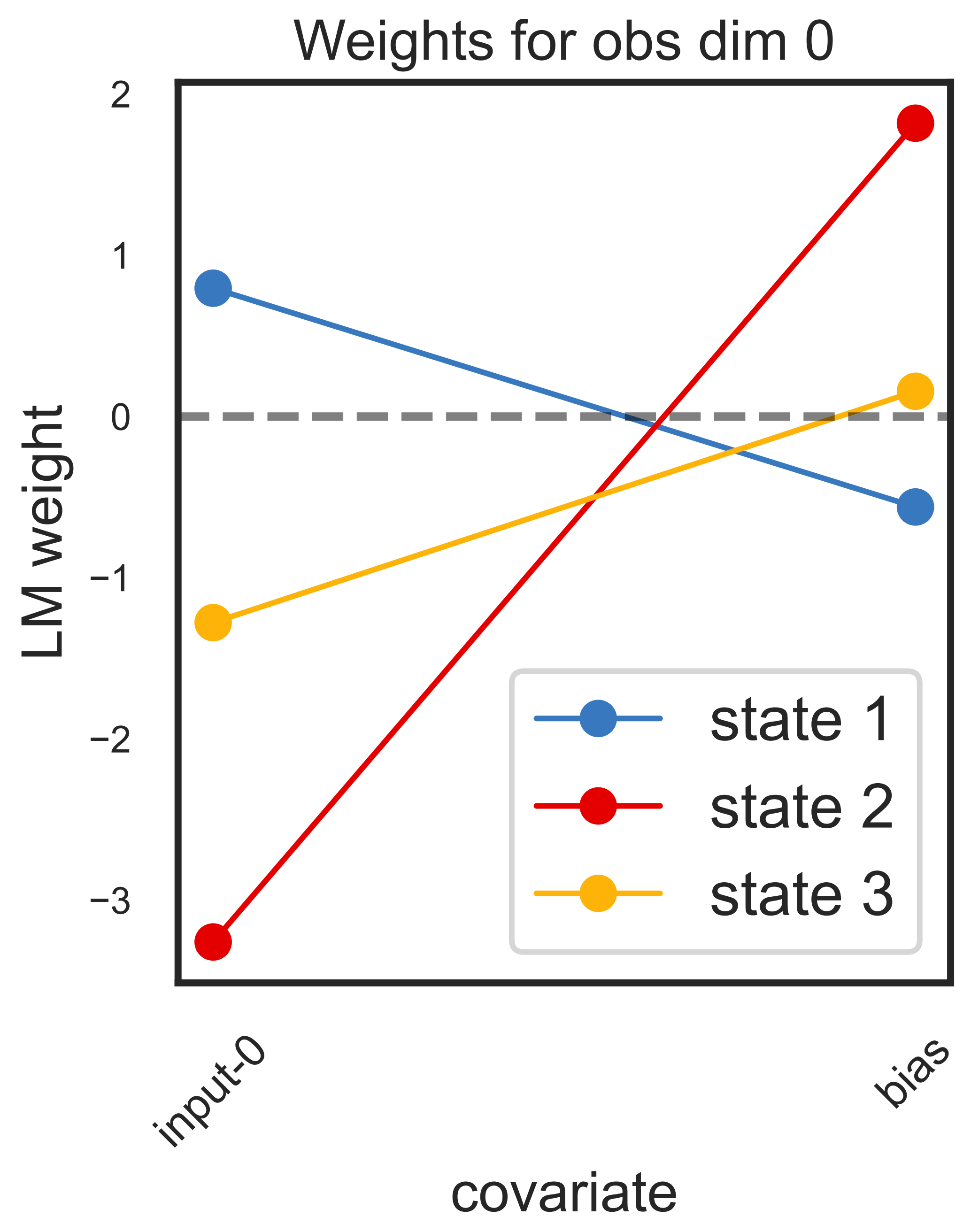
Observations n-dim: $\overrightarrow{F}_t$

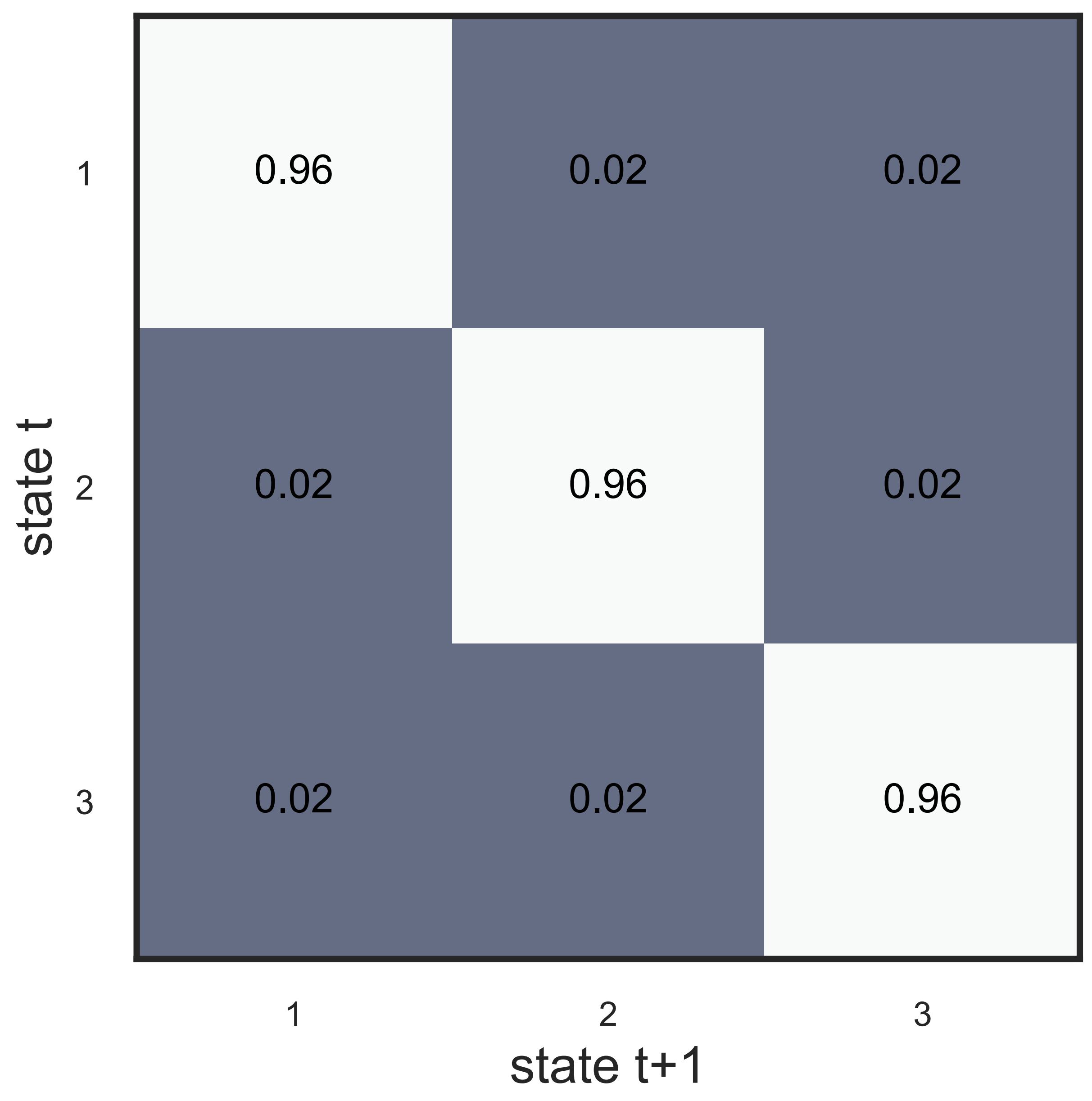
Inputs m-dim: $\overrightarrow{R}_t$

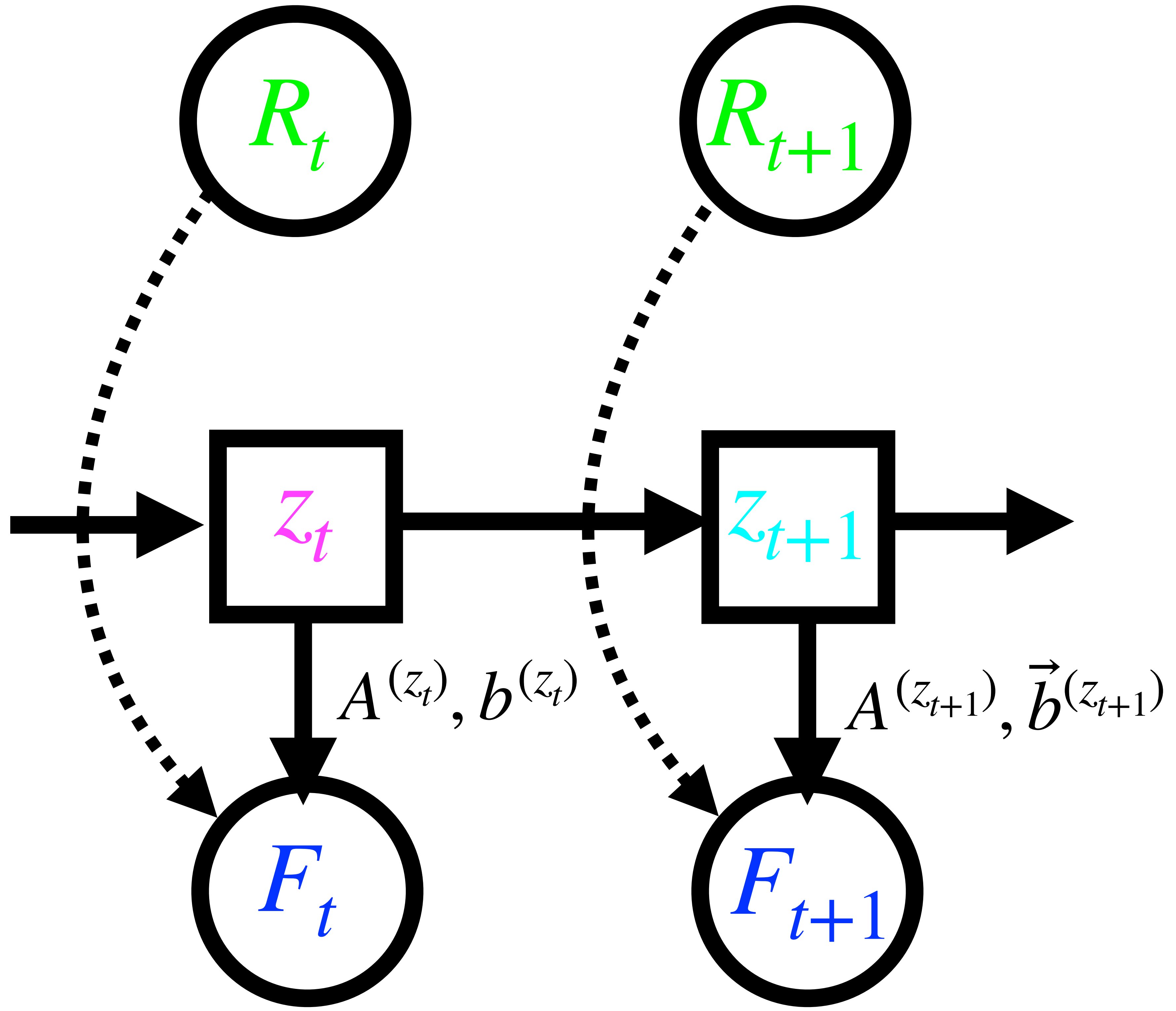
Model: $\overrightarrow{F}_t = A^{(z_t)}\ \overrightarrow{R}_t + b^{(z_t)} + \vec{\epsilon}_t$

Parameters with K hidden states
observation weights $A^{(k)}$, biases $b^{(k)}$,
noise cov $\Sigma^{(k)}$

Transition probabilities $T_{kl} = p(z_t = l \,|\, z_{t-1} = k)$
Initial probabilities: $\pi_k = p(z_0 = k)$







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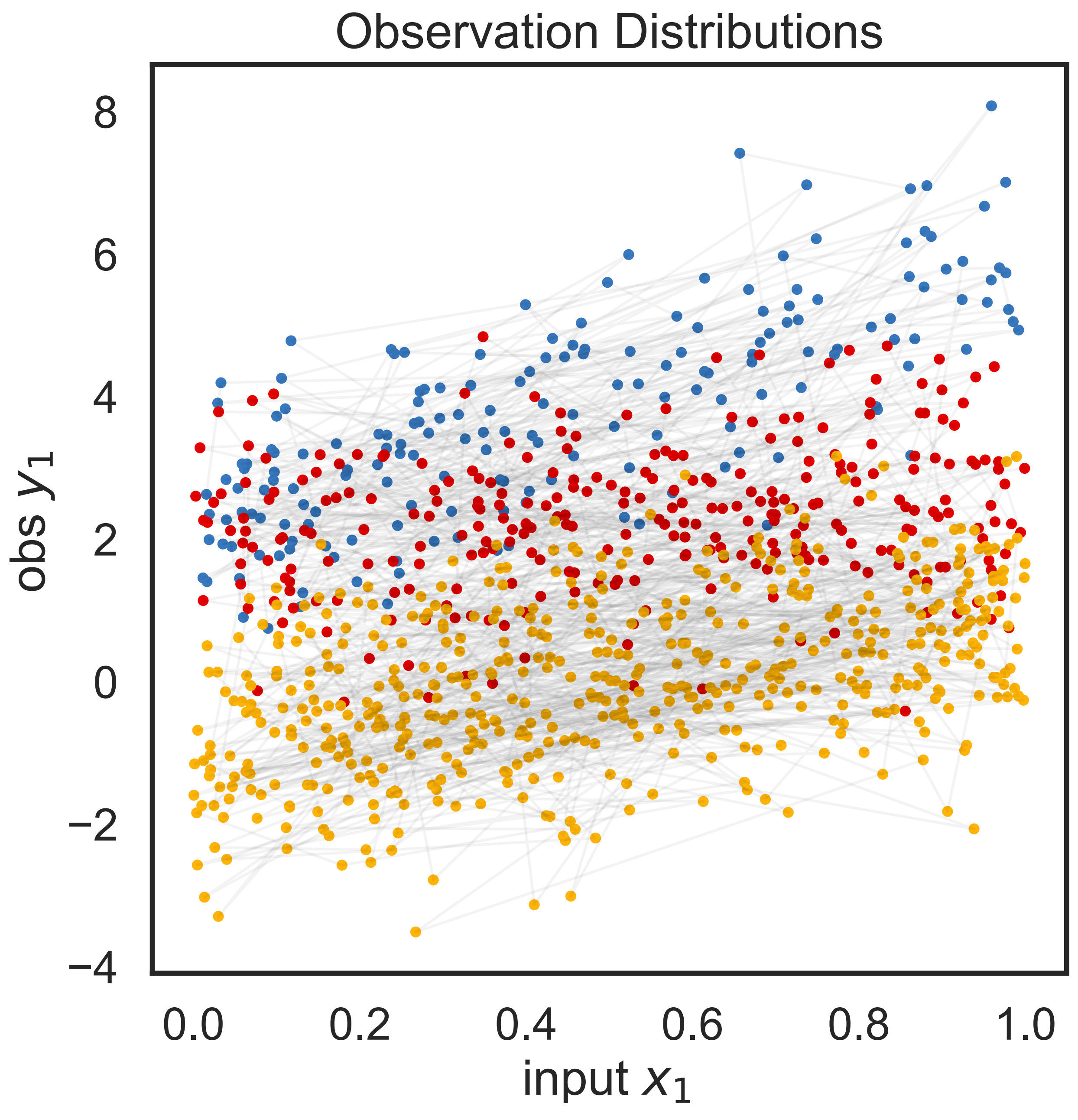
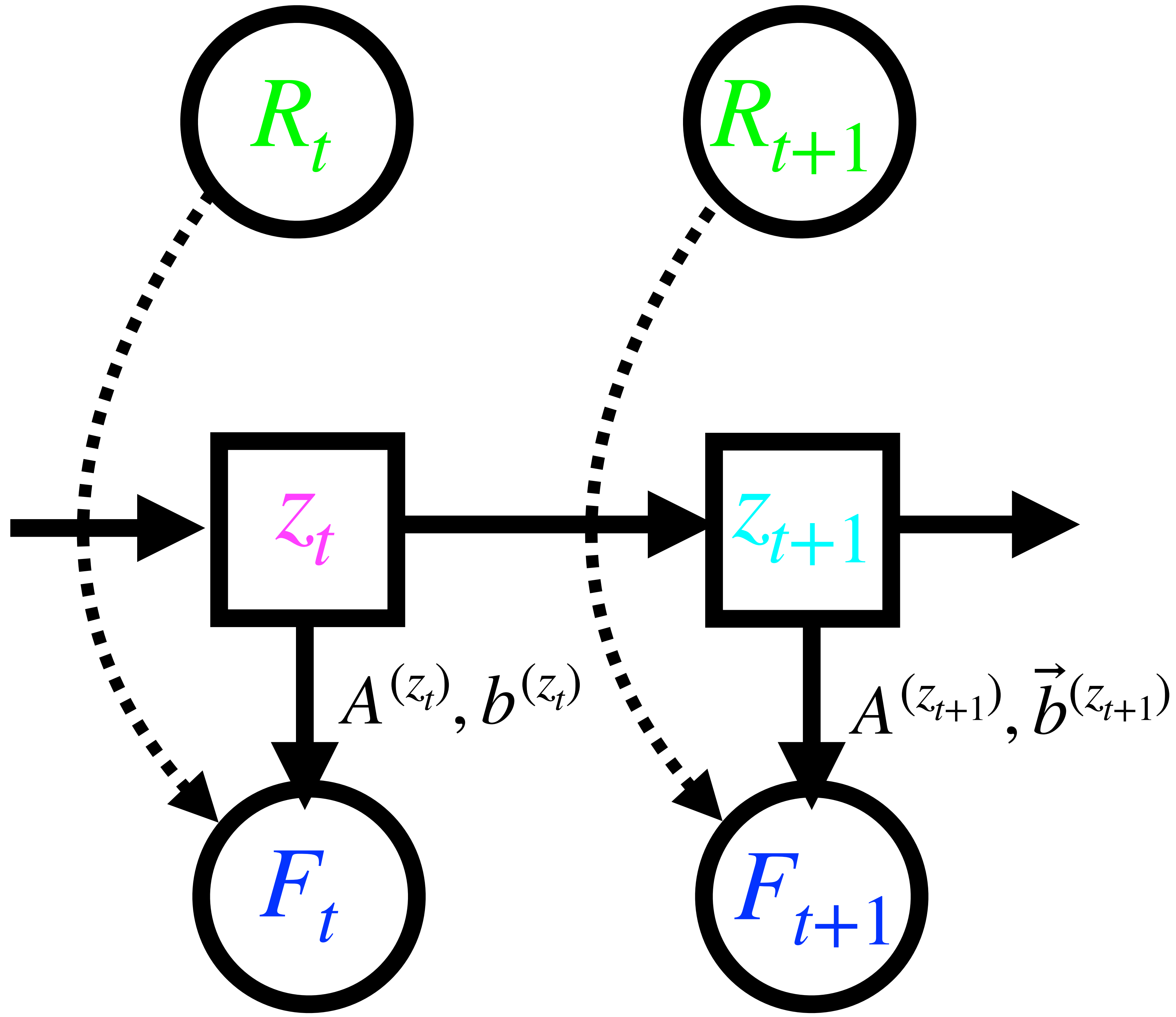
$\overrightarrow{F}_t$



$\overrightarrow{R}_t$



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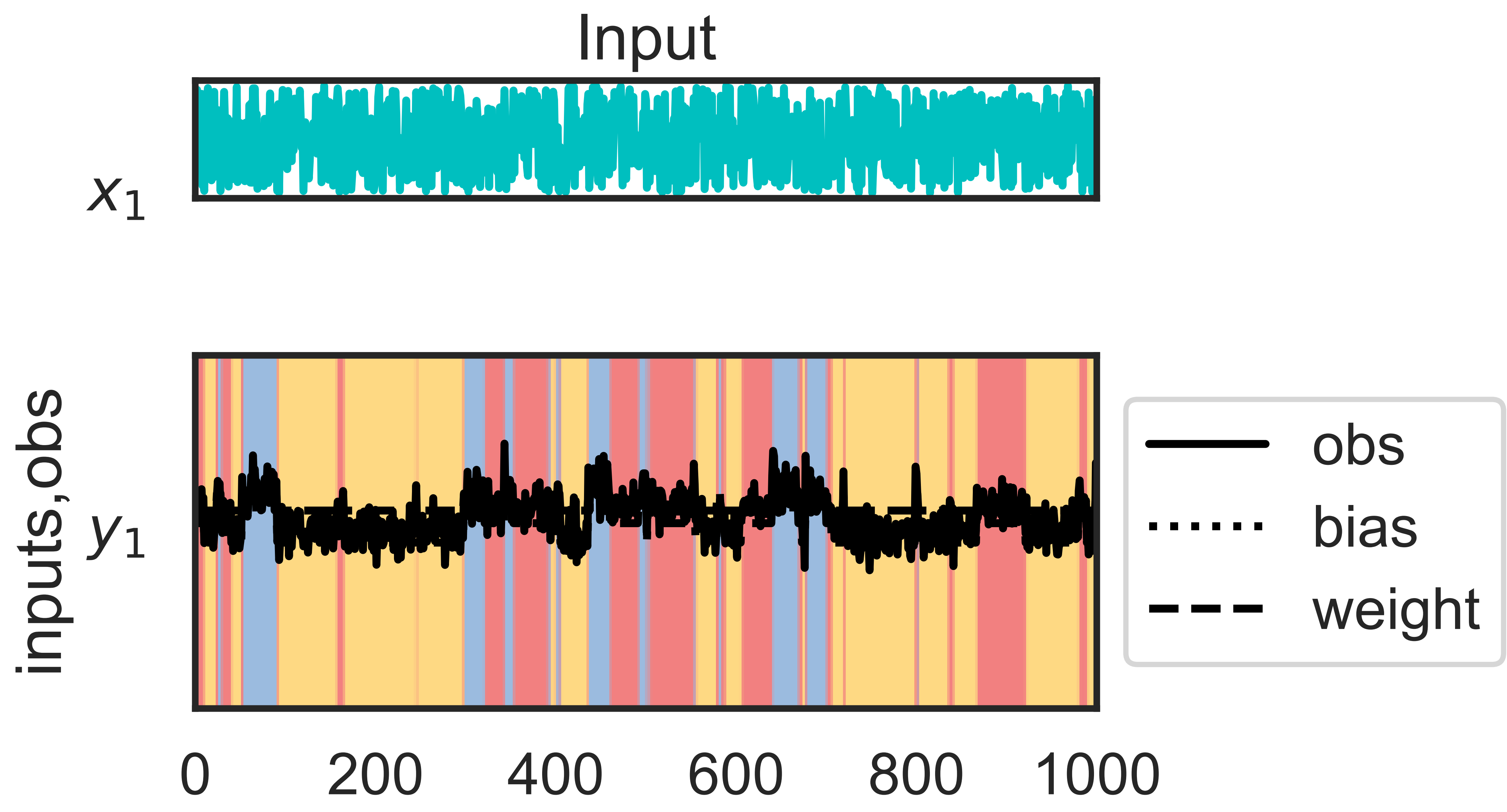
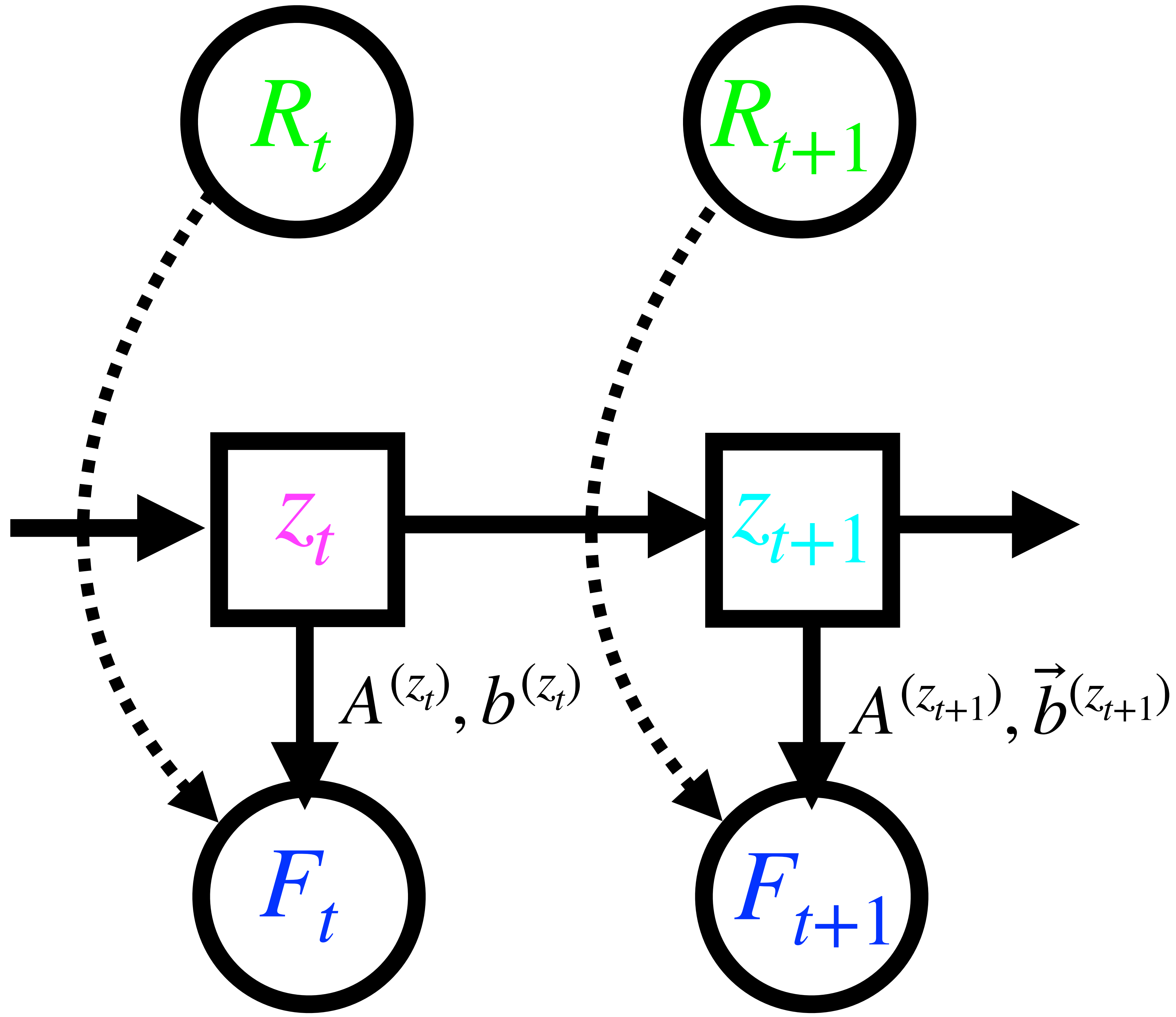
$\vec{R}_t$

$\vec{F}_t$





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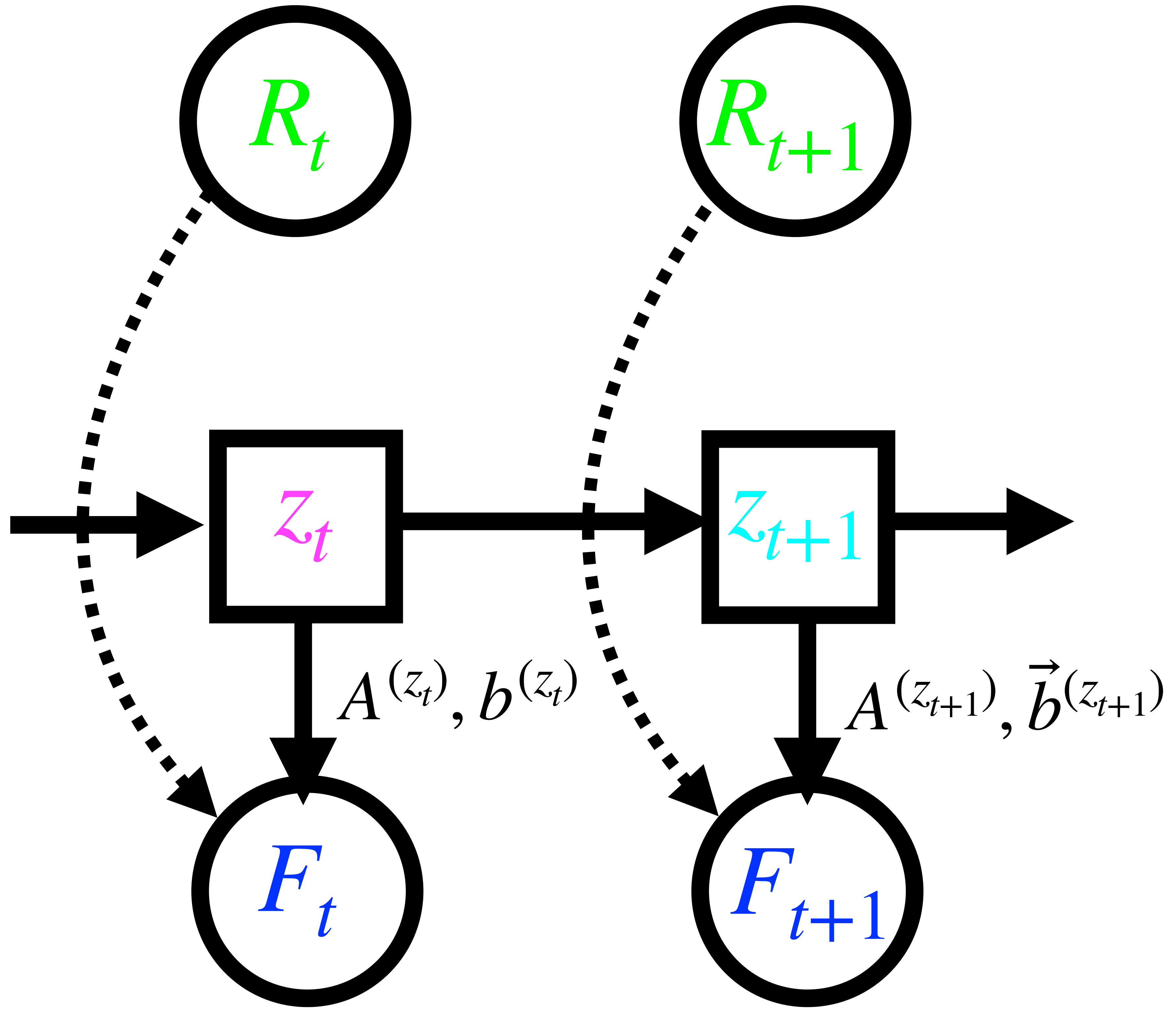
Train model parameters
using Expectation Maximization

Alternate E and M steps
E step: estimate probs with fixed params
M step: estimate params with fixed probs



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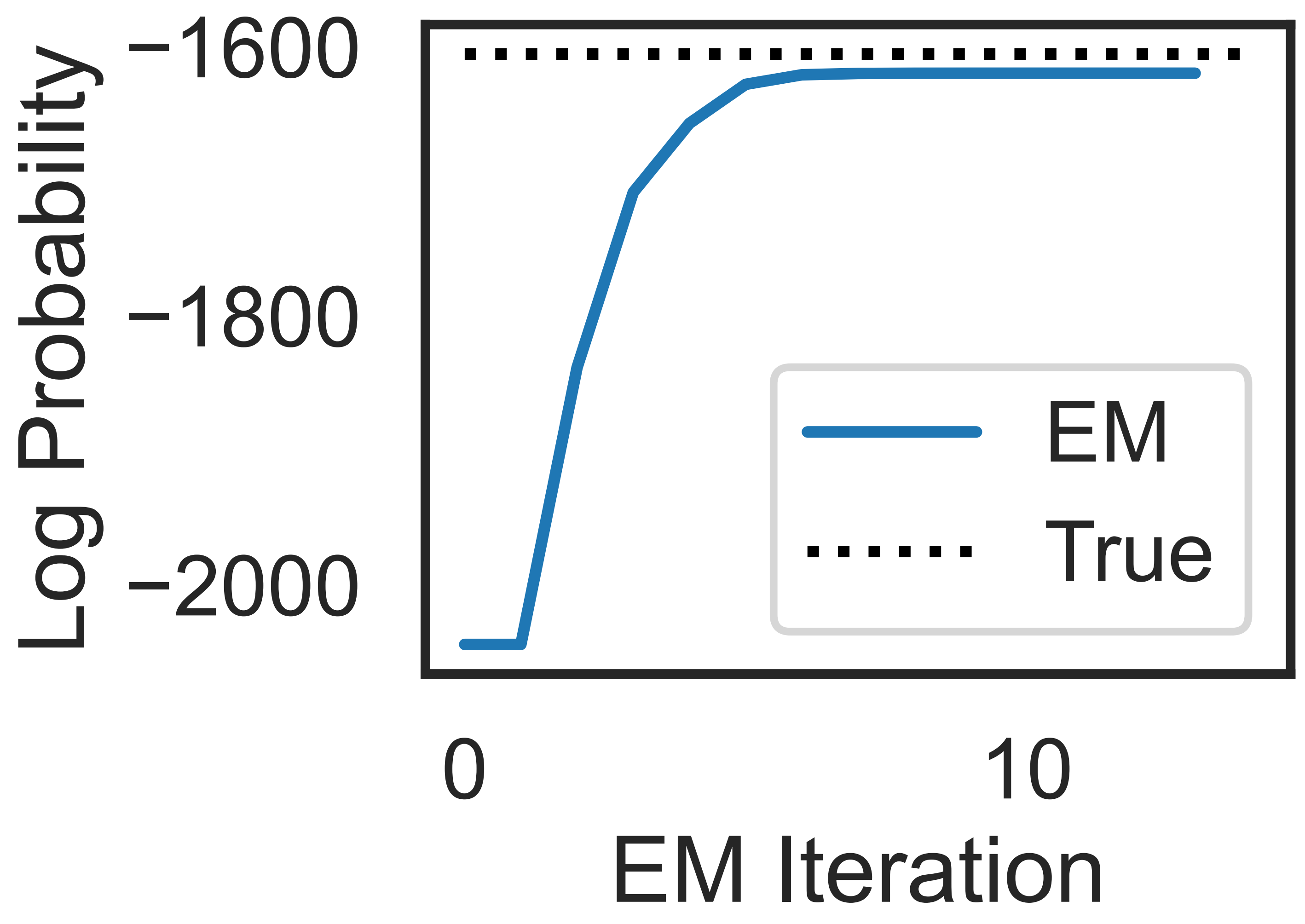

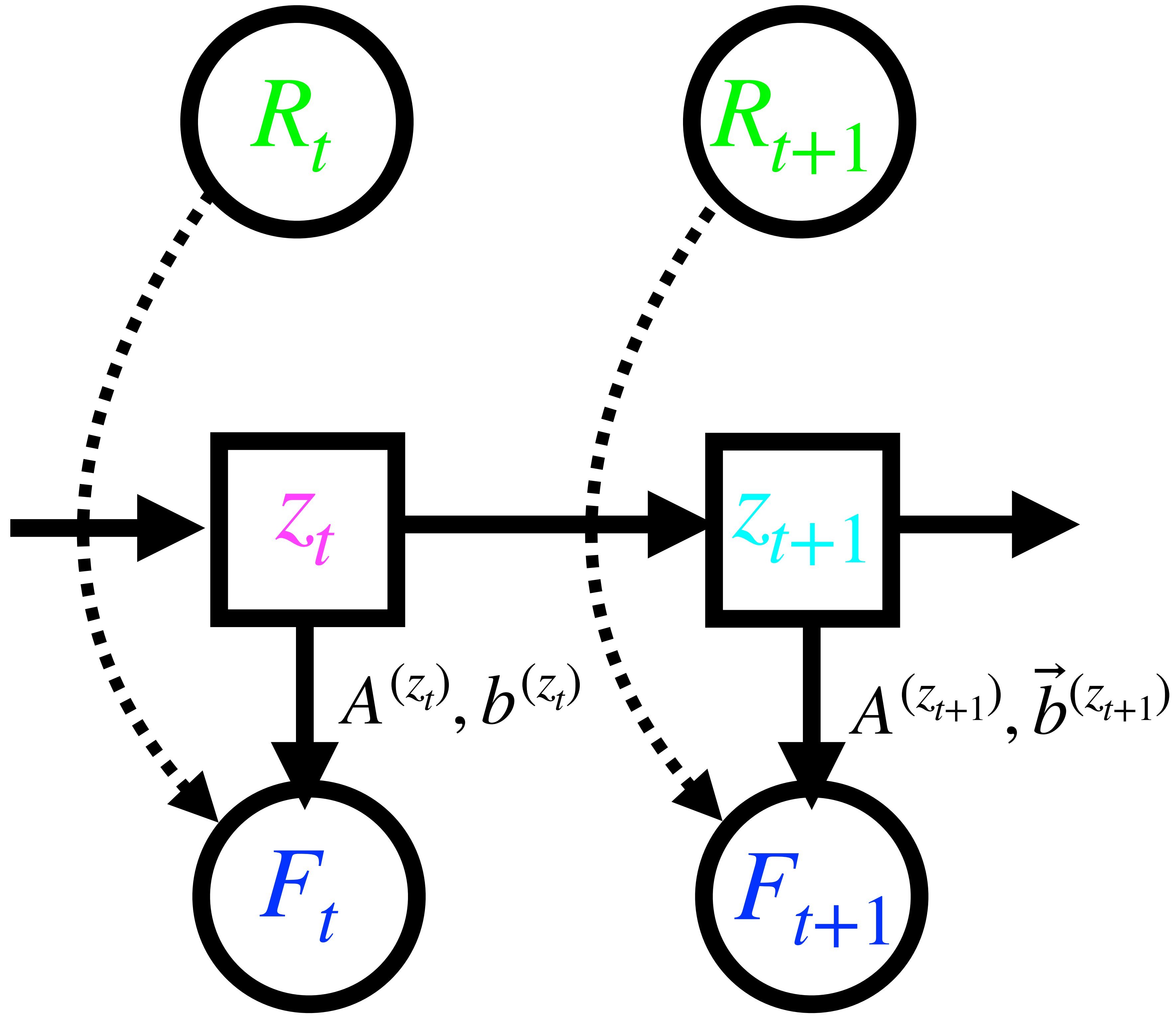

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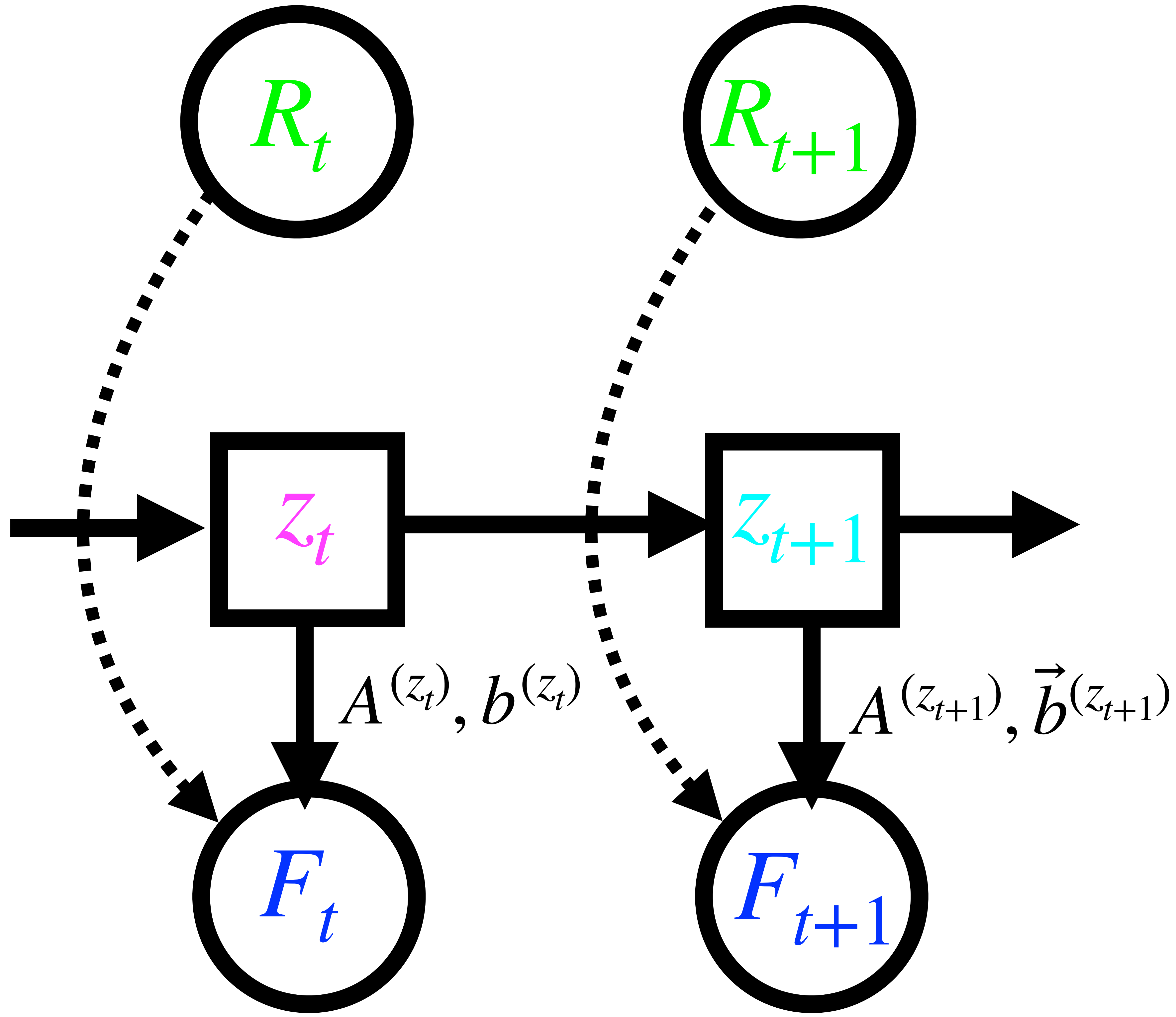
EM is not convex
—> train many models with diverse
params initialization, pick best



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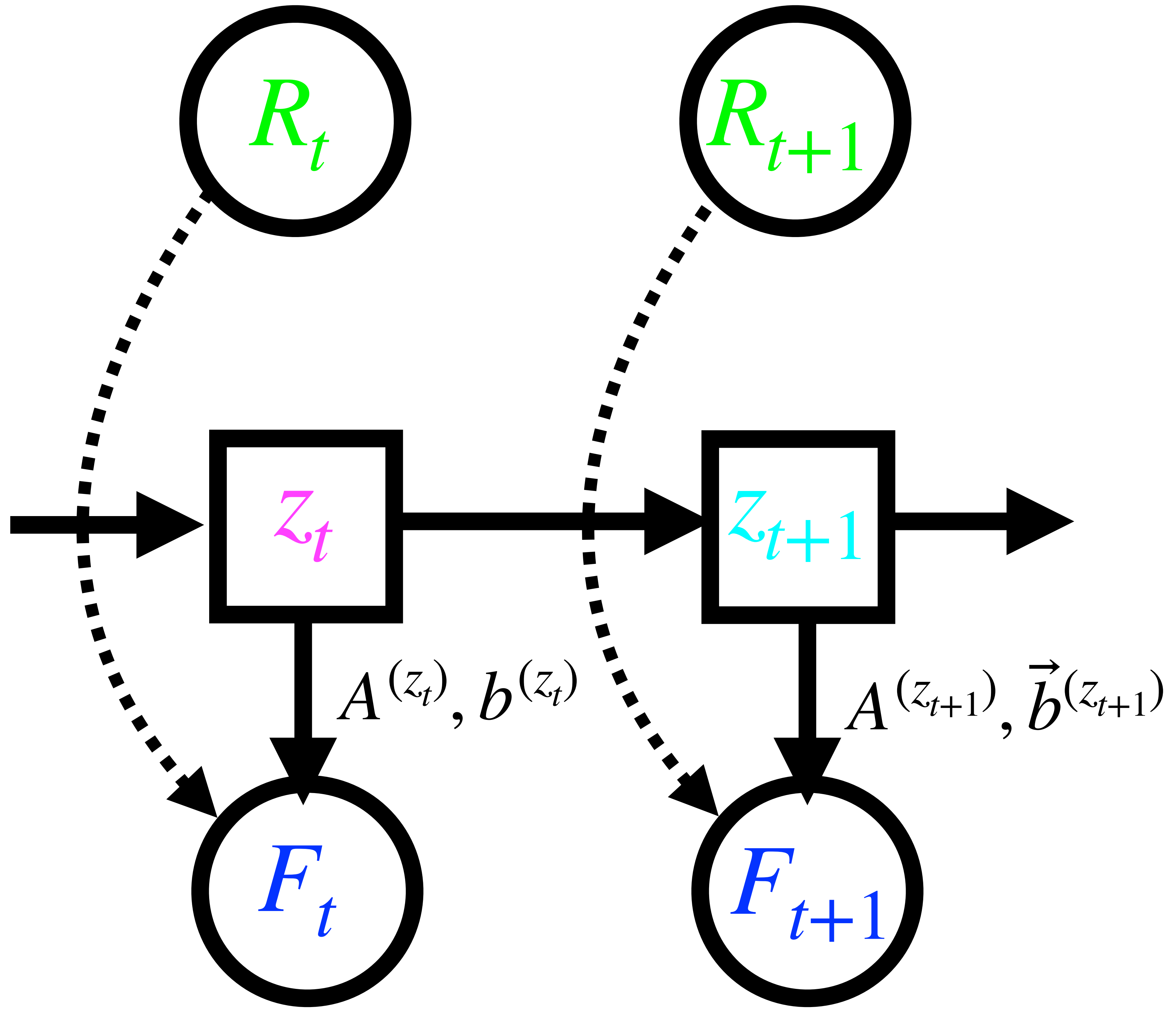

Model selection: Pick # states

Observations n-dim: $\overrightarrow{F}_t$

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How do we choose the
number of states?

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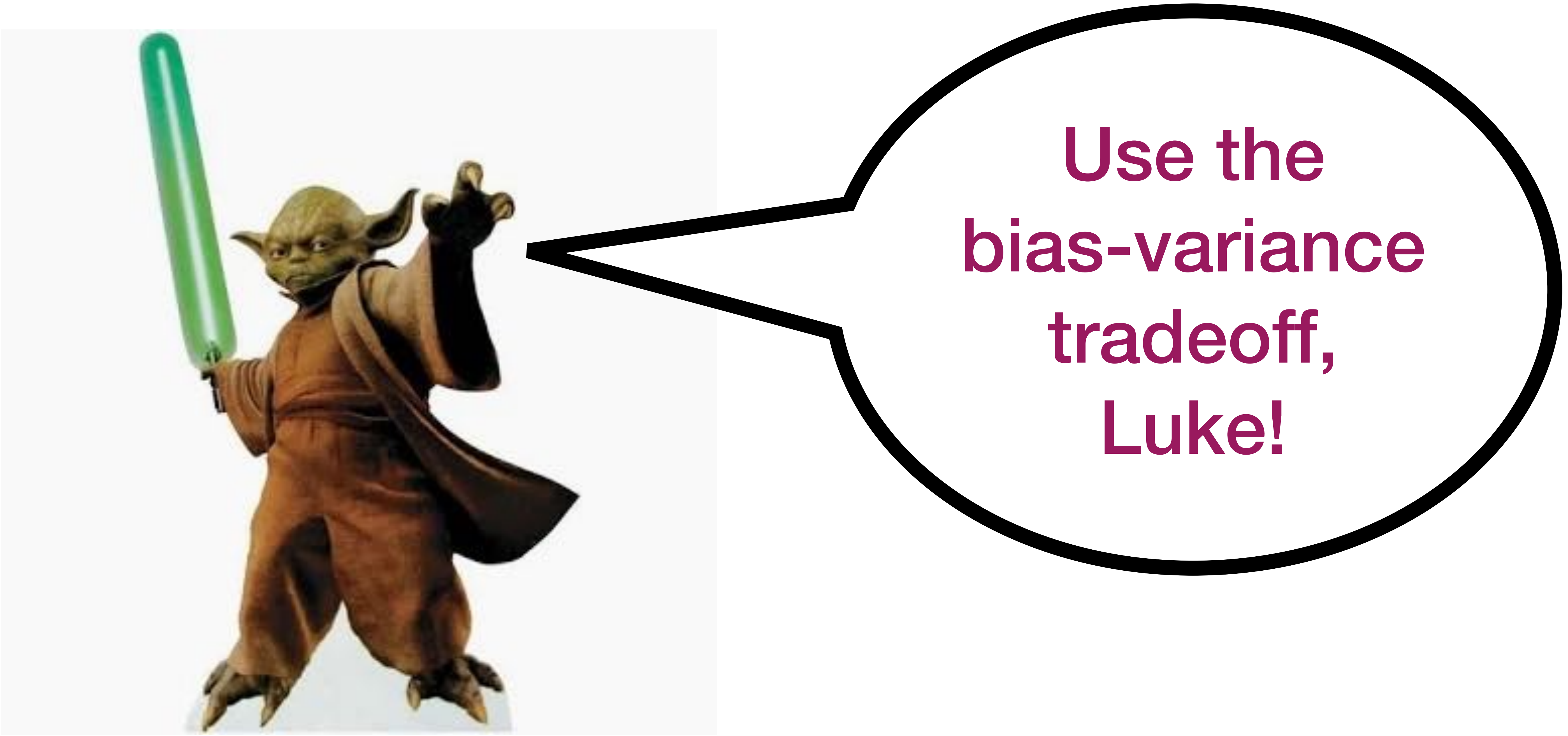

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Initial probabilities: $\pi_k = p(z_0 = k)$

Too few states —> underfit

Too many states —> overfit

Optimal # states —> best fit

Use the bias-variance tradeoff, Luke!

Model selection: xval

Observations n-dim: $\vec{F}_t$

Inputs m-dim: $\vec{R}_t$

Model: $\vec{F}_t = A^{(z_t)}\,\vec{R}_t + b^{(z_t)} + \vec{\epsilon}_t$

Parameters with K hidden states
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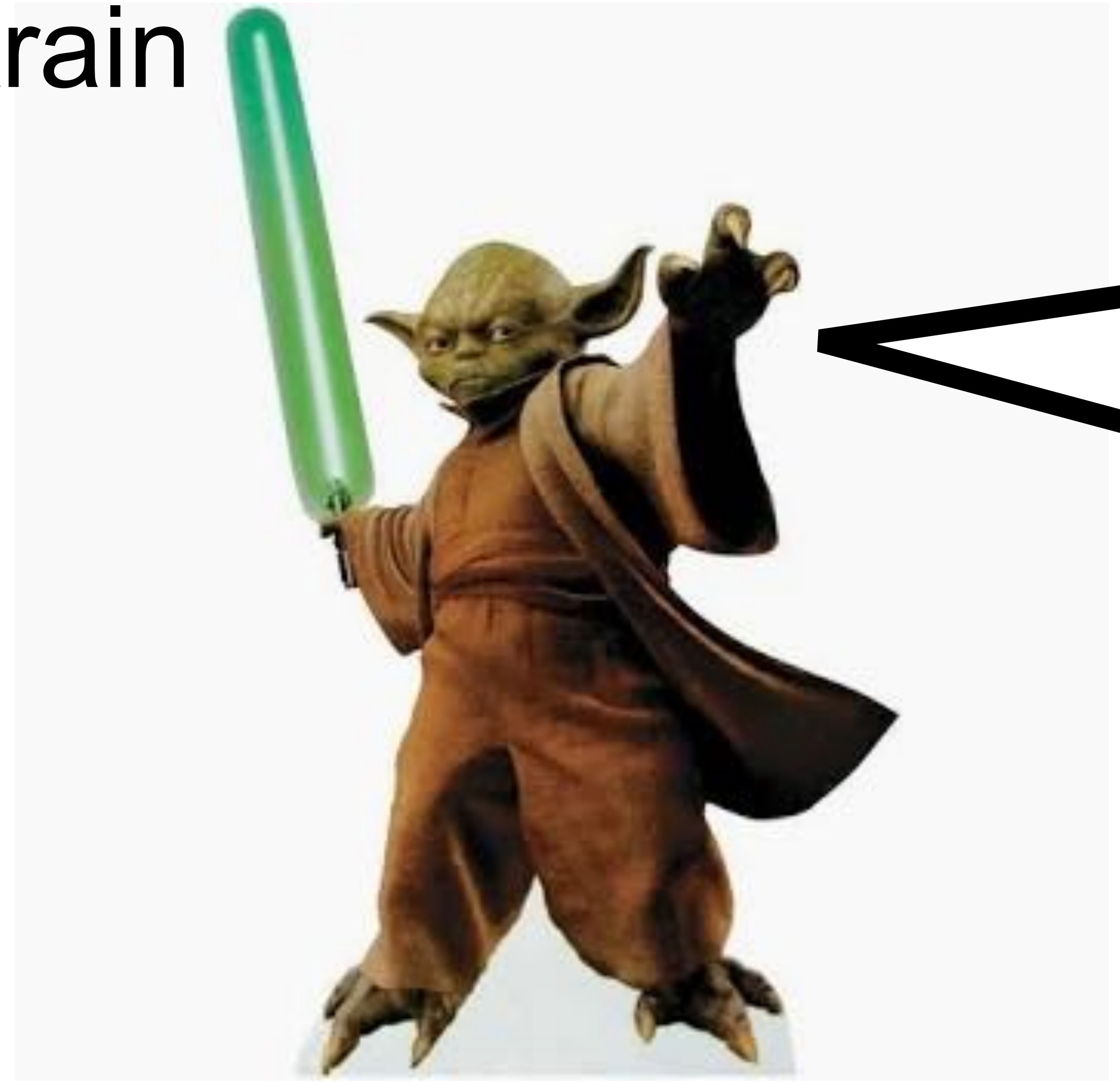
noise cov $\Sigma^{(k)}$

Transition probabilities $T_{kl} = p(z_t = l \,|\, z_{t-1} = k)$
Initial probabilities: $\pi_k = p(z_0 = k)$

1. Split data into training set S_train and test set S_test

2. Fit HMM with K states to S_train, estimate likelihood of test set LL(S_test|S_train)

3. Increase K and find best, or at least elbow



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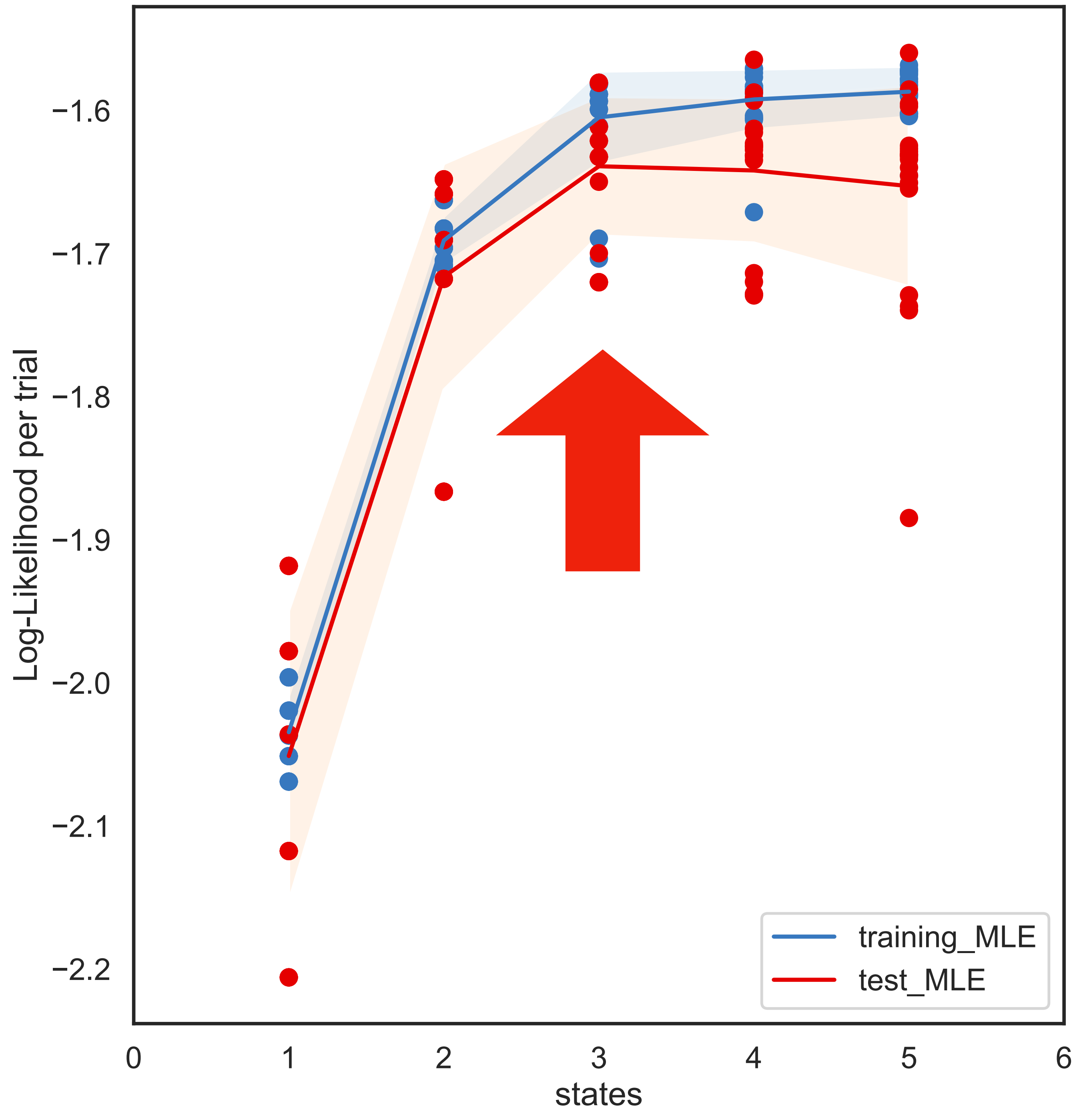


Model selection: BIC/AIC

Observations n-dim: $\overrightarrow{F}_t$

Inputs m-dim: $\overrightarrow{R}_t$

Model: $\overrightarrow{F}_t = A^{(z_t)} \overrightarrow{R}_t + b^{(z_t)} + \vec{\epsilon}_t$

Parameters with K hidden states observation weights $A^{(k)}$, biases $b^{(k)}$, noise cov $\Sigma^{(k)}$

Transition probabilities $T_{kl} = p(z_t = l \mid z_{t-1} = k)$

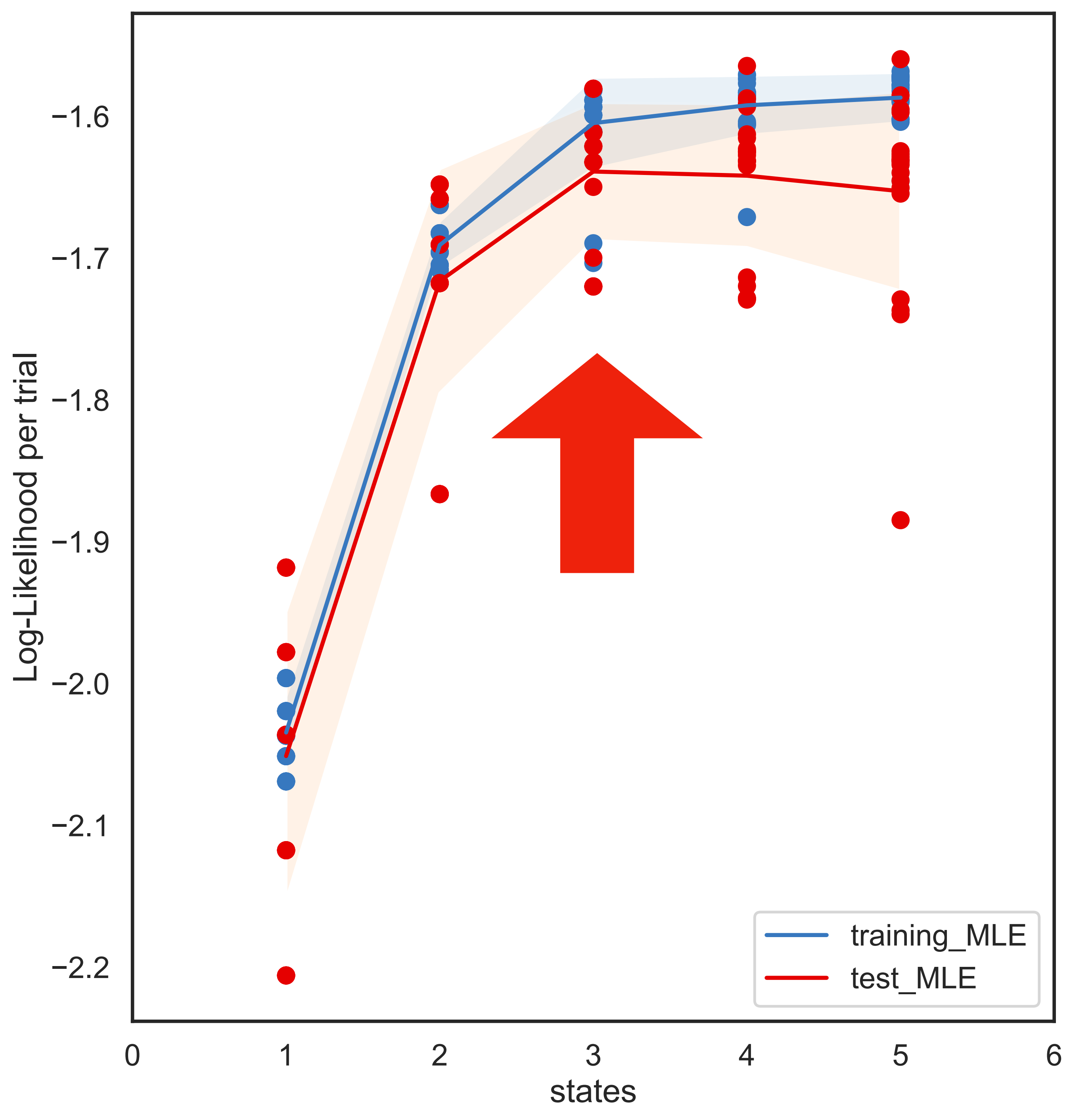
Initial probabilities: $\pi_k = p(z_0 = k)$

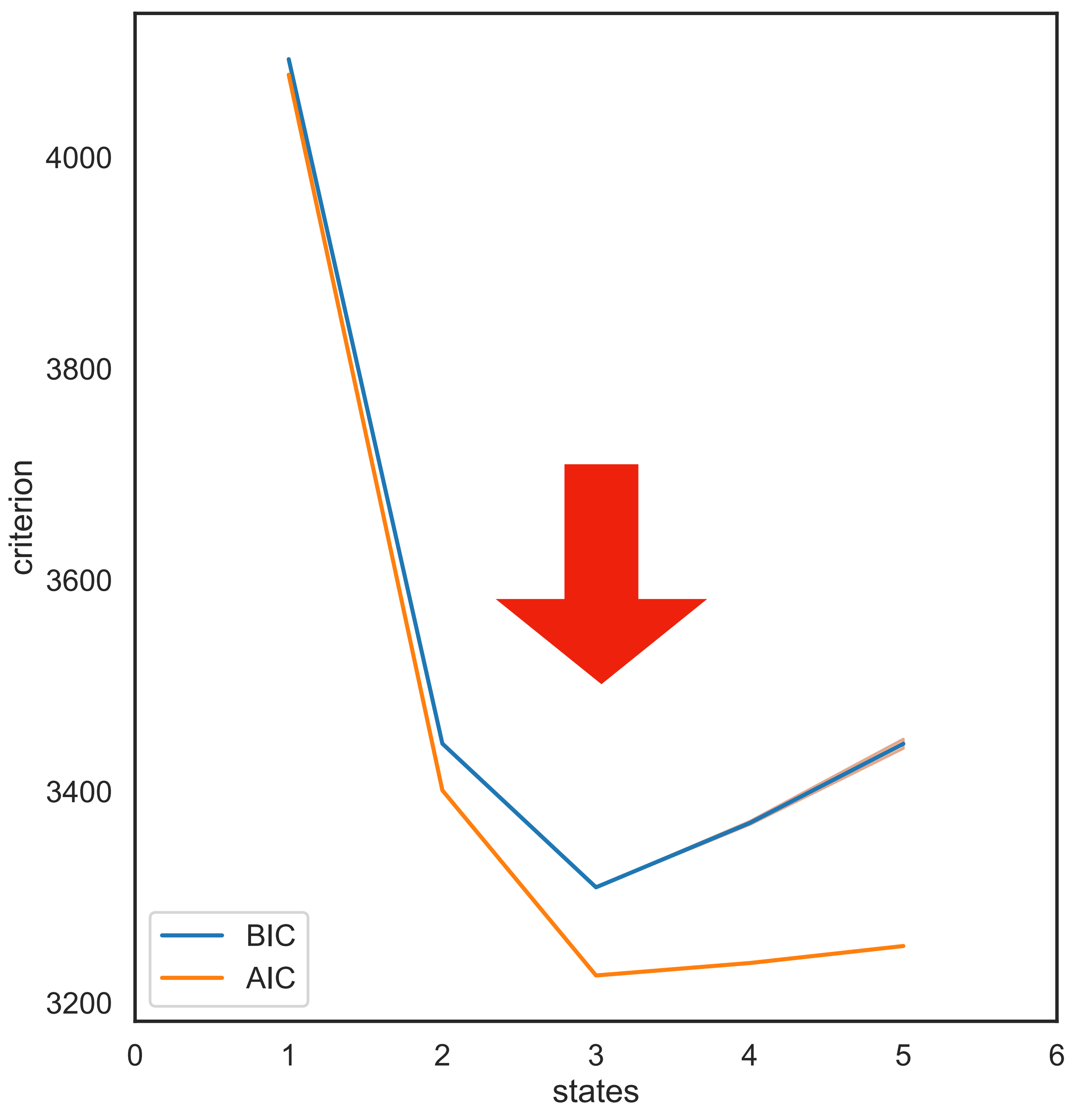
Penalize fit by # of parameters P:

BIC = - 2*LL + P*log(T)

AIC = - 2(LL + 2*P

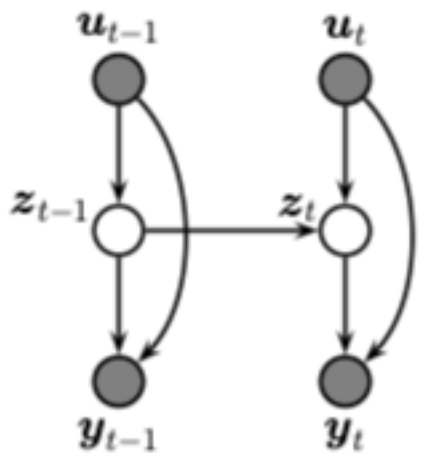
P = K(K-1) + K-1 + K*n*(m+1) + K*n





References

- State space model packages
  - https://github.com/lindermanlab/ssm
    - https://github.com/mazzulab/ssm
  - https://github.com/probml/dynamax



- Freely moving behavior
  - Wiltschko, Alexander B., et al. "Mapping sub-second structure in mouse behavior." (2015).
  - Findley, Teresa M., et al. "Sniff-synchronized, gradient-guided olfactory search by freely moving mice." (2021).

- Decision Making
  - Ashwood, Zoe C., et al. "Mice alternate between discrete strategies during perceptual decision-making." (2022).
  - Cazettes, Fanny, et al. "A reservoir of foraging decision variables in the mouse brain." (2023).