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Answer 1

Let a_n denote the number of edges in Q_n. We can find the number of edges in Q_n with the following equation:

$$a_n = n.2^{n-1}$$

Now, a_{n+1} will be the number of edges in Q_{n+1}, which is

$$a_{n+1} = (n+1).2^{(n+1)-1} = (n+1)2^n$$

Q_{n+1} can also be thought of as two Q_n graphs (each with a_n edges) connected with 2^n edges (since there are 2^n vertices in Q_n).

So, we can write a_{n+1} as $2.a_n + 2^n$.
Therefore, the recurrence relation for the number of edges in a cube graph is:

$$a_{n+1} = 2.a_n + 2^n \text{ for } n \geq 1$$

The base case is $a_0 = 0$ (since Q_0 has no edges). Also, we know $a_1 = 1$, $a_2 = 4$, $a_3 = 12$ and the recurrence relation can be verified with these known cases.

$$a_1 = a_{0+1} = 2.a_0 + 2^0 = 1$$
$$a_2 = a_{1+1} = 2.a_1 + 2^1 = 4$$
$$a_3 = a_{2+1} = 2.a_2 + 2^2 = 12$$

Answer 2

This is an arithmetic sequence since all the consecutive pairs differ with 3.

$$a_n = 1 + 3(n-1) = 3n - 2$$

The generating function for sequence a_n is:

$$G(x) = \sum_{n=0}^{\infty} a_n x^n$$

Substitute a_n into the equation:

$$G(x) = \sum_{n=0}^{\infty}(3n-2)x^n$$

$$G(x) = 3.\sum_{n=0}^{\infty} nx^n - 2.\sum_{n=0}^{\infty} x^n$$

$$G(x) = 3x.\sum_{n=0}^{\infty} nx^{n-1} - 2.\sum_{n=0}^{\infty} x^n$$

$$G(x) = 3x.\frac{d}{dx}\sum_{n=0}^{\infty} x^n - 2.\sum_{n=0}^{\infty} x^n$$

From the Table 1 (at the page 542) in the textbook we know that:

$$\textstyle\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \text{ for } |x| < 1$$

So we have:

$$G(x) = 3x.\frac{d}{dx}.(\frac{1}{1-x}) - 2.\left(\frac{1}{1-x}\right)$$

Since the sum $\sum_{n=0}^{\infty} nx^{n-1}$ is the derivative of the sum $\sum_{n=0}^{\infty} x^n$.

$$G(x) = 3x.\frac{1}{(1-x)^2} - 2.\frac{1}{1-x}$$

So, the generating function for the sequence $< 1, 4, 7, 10, 13, ... >$ is:

$$G(x) = \tfrac{3x}{(1-x)^2} - \tfrac{2}{1-x} \text{ for } |x| < 1$$

Answer 3

Let:

$$G(x) = \sum_{n=0}^{\infty} a_n x^n$$

We know that:

$$xG(x) = \sum_{n=0}^{\infty} a_n x^{n+1} = \sum_{n=1}^{\infty} a_{n-1} x^n$$

From the Table 1 (at the page 542) in the textbook:

$$\textstyle\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \text{ for } |x| < 1$$

The recurrence relation can be rewritten in terms of $G(x)$ as follows:

$$G(x) = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=1}^{\infty} (a_{n-1} + 2^n) x^n$$

$$G(x) = \sum_{n=1}^{\infty} a_{n-1} x^n + \sum_{n=0}^{\infty} 2^n x^n$$

$$G(x) = xG(x) + \sum_{n=0}^{\infty} (2x)^n$$

This simplifies to:

$$G(x) = xG(x) + \frac{1}{1-2x}$$

$$(1-x)G(x) = \frac{1}{1-2x}$$

Solving for $G(x)$ gives:

$$G(x) = \frac{1}{(1-x)(1-2x)} = \frac{-1}{1-x} + \frac{2}{1-2x}$$

Now, we can expand $G(x)$ into a power series to find the a_n:

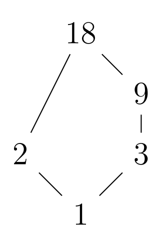
$$G(x) = -1. \sum_{n=0}^{\infty} x^n + 2. \sum_{n=0}^{\infty} (2x)^n$$

So, the solution to the recurrence relation is $a_n = -1 + 2.2^n = -1 + 2^{n+1}$ for $n \geq 0$.

$$a_n = 2^{n+1} - 1$$

Answer 4

(a)



(b)

$$R = \{(1,1), (1,2), (1,3), (1,9), (1,18), (2,2), (2,18), (3,3), (3,9), (3,18), (9,9), (9,18)\}$$

$$R = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

(c) It is a lattice. because for every pair of objects that exists there is a unique greatest lower bound and least upper bound.

(d) R_s is the symmetric closure of R. So we should add symmetric pairs which is not in the R.

All pairs (x, y) where $(x, y) \in R_s \wedge (x, y) \notin R$

$$(2, 1), (3, 1), (9, 1), (18, 1), (18, 2), (9, 3), (18, 3), (18, 9)$$

$$R_s = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

(e) The elements a and b of a poset (S, R) are called comparable if either aRb or bRa. When a and b are elements of S such that neither aRb nor bRa, a and b are called incomparable.

Thus; 2 and 9 are incomparable since neither (2,9) nor (9,2) is not included in R. However, 3 and 18 are comparable since (3,18) is included in R.

Answer 5

(a) A binary relation on a set A is a collection of ordered pairs of elements of A. In other words, it is a subset of the Cartesian product $A^2 = A \times A$ Hence, the number of possible order pairs is n^2.

A binary relation on a set A is reflexive if it relates every element of A to itself. So we need to subtract the diagonal elements:

$$n^2 - n$$

Then, we need to divide it by 2 since a_{ij} must be equal to a_{ji} by symmetric relation property

$$\frac{n^2 - n}{2}$$

The total number of possible relations is

$$2^{\frac{n^2-n}{2}}$$

(b) For anti-symmetric relations, for each pair (a, b) and (b, a) where $a \neq b$ we have three options since we can put either pair in the relation, or none of them, but not both. Also, we are free to include or exclude each diagonal element. So the total number of possible relations is:

$$3^{\frac{n^2-n}{2}}.2^n$$

However, if we want the relation to be reflexive, we must include all the diagonal pairs so we need to delete 2^n term.

The total number of possible relations is

$$3^{\frac{n^2-n}{2}}$$

Answer 6

Let $A = \{1, 2, 3, 4\}$ and

$$R = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

R is an anti-symmetric relation. The transitive closure $R^+ = R + R^2 + ... + R^n$:

$$R^2 = RoR = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

$$R^3 = RoR^2 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$R^4 = RoR^3 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$R+ = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

Thus, the transitive closure of an anti-symmetric relation is NOT always anti-symmetric.