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Answer 1

(a) In order a graph to have an Eulerian circuit: it has to be connected and every vertex in it has to have 0 nodes with odd degrees.

Thus, Yes G has an Eulerian curcuit since the G is connected and every vertex of it has 0 nodes with odd degrees.

An example Eulerian curcuit would be: a-c-b-g-h-l-k-g-c-d-i-l-m-j-i-h-d-e-f-j-e-a

(b) No. Because all nodes have an even number of degrees.

A graph cannot be both an Euler circuit and a path at the same time.

If there were two edges with the odd number of edges, then there would be an Eulerian path which is not a circuit. Because only then we could start and end at the different nodes while building a path.

(c) No, there is not a Hamilton circuit in G.

(d) Yes there is a Hamilton Path that is not a circuit in G. An example Hamilton path: a-c-b-g-h-i-d-e-f-j-m-l-k

(e) The chromatic number of graph is 3.

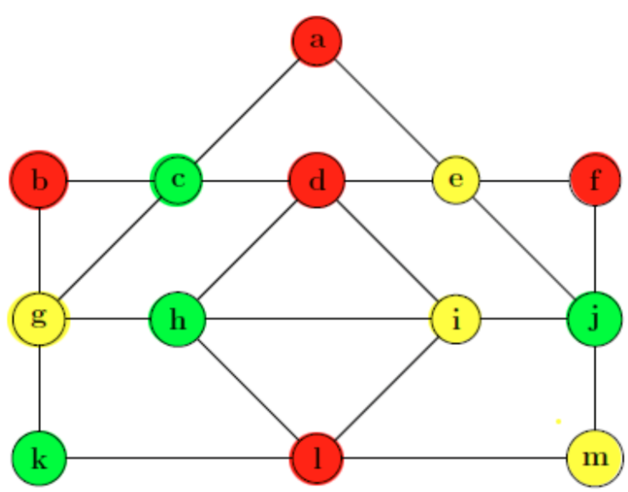

Figure 1: Coloring of graph G

(f) No. Since its chromatic number is three, it is not bipartite. The chromatic number of a bipartite graph is two.

We need to break cycles with odd number of vertices. To do so; we need to delete minimum three edges: (b,c), (e,f) and (h,i). A possible solution is as follows:

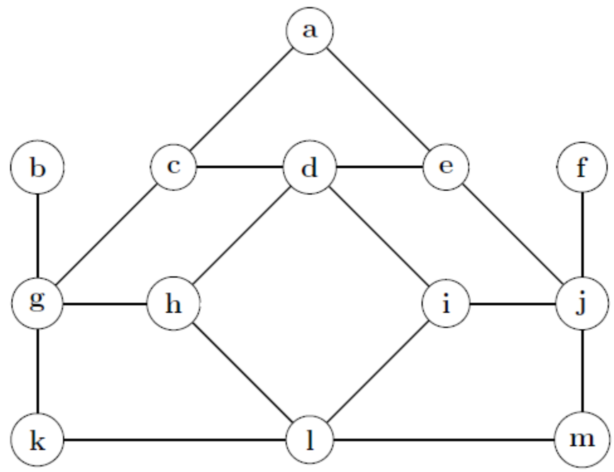

Figure 2: Bipartite version of G

(g) No it doesn't have a complete sub-graph. We can obtain an four edge complete graph selecting d-h-i-l nodes and adding (d,l) edge as shown in the following figure.

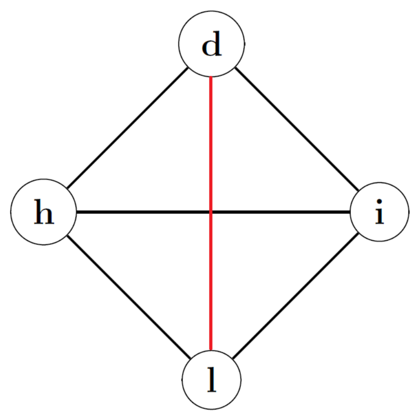

Figure 3: Enter Caption

Answer 2

If two graphs are isomorphic, their following properties must be the same:

- number of vertices

- number of edges
- degrees for corresponding vertices
- number of connected components
- number of loops
- number of parallel edges

Notice that all of these properties are the same. So we need to try couple of times to find if they are isomorphic or not. Thus, yes they are isomorphic because we can find a one-to-one and onto function f from G to H such that:

$$f(a) = a'$$
$$f(b) = c'$$
$$f(c) = e'$$
$$f(d) = g'$$
$$f(e) = b'$$
$$f(f) = h'$$
$$f(g) = d'$$
$$f(h) = f'$$

Answer 3

(a) The chromatic number χ of the cycle graph C_n is 2 if n is even, and 3 if n is odd.

Proof:

1. We set $C_n = P + v_{n-1}v_0$ with $P = v_0v_1v_2 \cdots v_{n-1}$ being a path.

2. For a simple graph with at least one edge, χ is at least 2. Since a path is a non-empty graph, whereby all its vertices are distinct and linked by edges, we can find a valid coloring for P by alternating two colors, say 1 and 2.

3. Starting with v_0, we color vertices with an even index with 1 and vertices with an odd index with 2.

4. For v_{n-1} we have two options. If n is even, $n-1$ is odd, hence v_{n-1} is colored with 2. If n is odd, $n-1$ is even, hence v_{n-1} is colored with 1.

5. But in C_n, v_{n-1} is adjacent to v_0, which is also colored with 1. Hence, the coloring is not valid. Therefore, if n is odd, we need 3 colors.

Thus, this shows that the chromatic number of a cycle graph is 2 if the number of vertices is even, and 3 if the number of vertices is odd.

(b) For a cube graph the chromatic number is 2. This is because a cube graph is a bipartite graph. We can prove it using mathematical induction.

Proof:

Base case: For n=1, the graph is a line with two vertices. It can be colored with 2 colors

Inductive Step: Assume that a n-dimensional cube can be colored with 2 colors. We need to prove that a (n+1)-dimensional cube can also be colored with 2 colors.

A (n+1)-dimensional cube can be formed by taking two n-dimensional cubes and adding edges between corresponding vertices.

Since each n-dimensional cube can be colored with 2 colors by our inductive hypothesis, we can color one cube with 2 colors, and color the other cube with the same 2 colors but reversed. This way, each added edge connects vertices of different colors.

Therefore, a (n+1)-dimensional cube can also be colored with 2 colors.

Answer 4

(a) Let's choose Prim's algorithm. We begin by choosing an initial arbitrary vertex and proceed the tree by adding minimum weight edges occurring to vertices that do not form circuits.

Weight of the MST is 12 and the order of edges added to the tree is as follows (choose vertex "a" as the inital vertex):

Choice	Edge	Cost
1	$\{a, b\}$	1
2	$\{a, d\}$	3
3	$\{b, c\}$	4
4	$\{c, e\}$	2
5	$\{c, f\}$	2

(b) Minimum spanning tree is shown in the following figure:

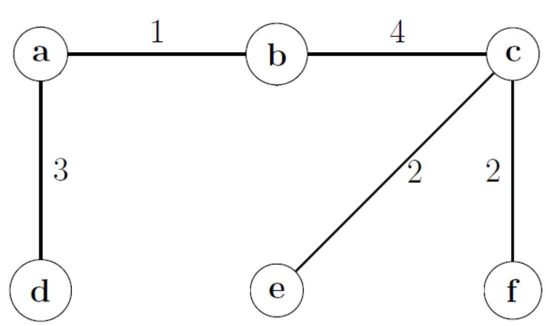

Figure 4: Minimum spanning tree for Q4

(c) The minimum spanning tree of a graph is not necessarily unique. It depends on the edge weights of the graph. If all the edge weights are distinct, then there is only unique MST. However, if some edge weights are equal, then there may be more than one MST with the same total weight.

In our case the MST is not unique. The following figure also shows a MST with weight 12:

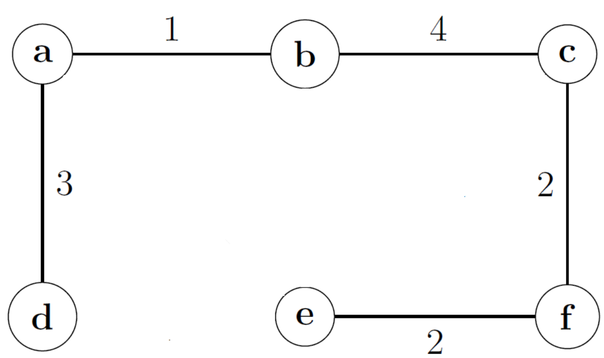

Figure 5: Minimum spanning tree for Q4

Answer 5

(a) In the full binary tree, each internal node has exactly two children and the leaf node has no children. We can use mathematical induction to show that a full binary tree with n vertices has $\frac{n+1}{2}$ leaf vertices, on the number of internal nodes. Assume that I is the number of internal nodes.

Base case: If $I = 0$, then the tree consists of only one node, which is a leaf. So, the number of leaves is $\frac{0+1}{2} = 1$, which is true.

Induction hypothesis: Assume that any full binary tree with I internal nodes has $\frac{I+1}{2}$ leaves.

Induction step: Let T be a full binary tree with $I + 1$ internal nodes. Select an internal node v that has two leaf children, u and w. Remove u and w from T, making v a leaf node. Call the new tree T'. T' has I internal nodes, so by the induction hypothesis, it has $\frac{I+1}{2}$ leaves. Now, restore u and w to T. T has $I+1$ internal nodes and $\frac{I+1}{2}+2$ leaves. Simplifying, we get $\frac{I+1}{2} + 2 = \frac{I+5}{2} = \frac{(I+1)+1}{2}$, which is the desired result.

Therefore, by mathematical induction, a full binary tree with n vertices has $\frac{n+1}{2}$ leaf vertices.

(b) A tree is a special type of connected graph that has **no cycles**. Thus; every tree is a bipartite graph, which means it can be divided into two sets of vertices such that no two vertices in the same set are adjacent. Therefore, the chromatic number of a tree is always 2, regardless of the number of vertices or edges.

(c) Full m-ary tree means that every node has either 0 or m children. If we have n nodes and want to maximize the height of the tree; we can put only one node with m children in one level and make others have zero children. An example illasturation is shown in the following figure:

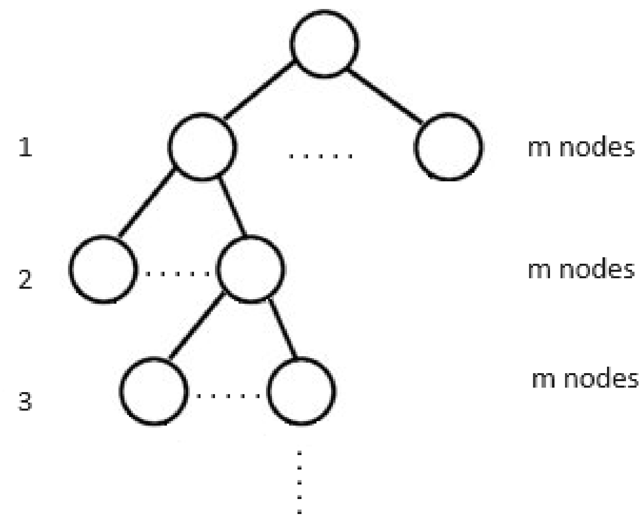

Figure 6: Enter Caption

So; if we say h to the maximum height of the tree(the upper bound of the full m-ary tree).

$$h = \frac{n-1}{m}$$

We need to subtract 1 due to the root node and divide it with m since we will have minimum m nodes for every level to maximize the height.