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Answer 1

(a) The statement is true. Let's consider m = 4 points in the set C. Let's denote the 4 points as x1, x2, x3, x4 and their corresponding weights as λ1, λ2, λ3, λ4 such that λ1 + λ2 + λ3 + λ4 = 1 and λi ≥ 0.

Linear combination of 4 points:

λ1x1 + λ2x2 + λ3x3 + λ4x4

Which is equal to:

λ1x1 + λ2x2 + λ3x3 + λ4x4 = λ1x1 + λ2x2 + (λ3 + λ4)(λ3/(λ3 + λ4)x3 + λ4/(λ3 + λ4)x4)

The term in the bracket is a convex combination of x3 and x4 because λ3/(λ3 + λ4) + λ4/(λ3 + λ4) = 1 and both coefficients are non-negative. Since C is a convex set, this combination is in C.

This technique can be performed again until a convex combination of two points is obtained. This combination is also in C because C is a convex set. As a result, each linear combination of four points in set C is likewise a part of set C. This argument can be generalized to any number of points. Thus, the statement is true.

(b) The statement is false. Let's consider two convex functions $f(x) = -x$ and $g(x) = x^2$. Both functions are convex. However, the composition of these two functions $(fog)(x) = f(g(x)) = -(x^2)$ is not convex. Actually it is concave.

Therefore, the composition of two convex functions is not always a convex function.

(c) Let x, v ∈ R^n be such that x + tv ∈ S for all t ∈ R, and let $g(t) = f(x + tv)$. Then, for any t_1, t_2 ∈ R and any λ ∈ [0, 1], we have

$g(st_1 + (1 - \lambda)t_2) =$

$f(x + (\lambda t_1 + (1 - \lambda)t_2)v) =$

$f(\lambda(x + t_1v) + (1 - \lambda)(x + t_2v)) \leq \lambda f(x + t_1v) + (1 - \lambda)f(x + t_2v) = \lambda g(t_1) + (1 - \lambda)g(t_2).$

This shows that g(t) is a convex function on R.

Suppose that S is a convex set and $g(t) = f(x + tv)$ is a convex function on R for all x, v ∈ Rn such that $x + tv$ ∈ S for all t ∈ R. Then, for any x, y ∈ S and any t ∈ [0, 1], we have

$f(tx + (1 - t)y) = f(x + t(y - x)) = g(t),$

where $g(t) = f(x + t(y - x))$. Since $g(t)$ is convex, we have

$g(t) \leq tg(1) + (1-t)g(0) = tf(x + (y-x)) + (1-t)f(x) = tf(y) + (1-t)f(x).$

This shows that f($\cdot$) is a convex function. So; the statement is true.

Answer 2

(a) This set is **not** a σ-algebra on X.

- The first property is satisfied. This is because X is in the set, since X - X is $\emptyset$, which is finite.
- The second property is also satisfied. If U is in the set, then X - U is either finite or is $\emptyset$. So, X - (X - U) is in the set, and the set is closed under complementation.
- The third property is **not** satisfied, because there exist some subsets of X that are in the set, but their union is not in the set. Let X be the set of natural numbers, and U_n be the set n for each n $\in X$. Then U_n is in the set, since X - U_n is finite. However, the union of all U_n is X and is not in the set, since X - X is not finite which is $\emptyset$. Therefore, the set is not closed under countable unions.

(b) This set is a σ-algebra on X.

- The first property is satisfied. X is in the set, because X - X is all of X.
- The second property is satisfied. If U is in the set, then X - U is either countable or is all of X. Therefore, the complement of U, which is X - (X - U), is either countable or is $\emptyset$. In either case, it is in the set. Hence, the set is closed under complementation.
- The third property is satisfied, since if A1, A2, ... are in the set, then X - A1, X - A2, ... are either countable or are all of X. Therefore, A (the union of A1, A2, ..., which is) is the complement of X - A (the intersection of X - A1, X - A2, ...). We know that X - A is either countable or is all of X since the intersection of countable sets is countable and the intersection of all of X with any set is that set. Hence, A is in the set. Therefore, this set is closed under countable unions.

Since this set satisfies all three properties of σ-algebra, it is a σ-algebra on X.

(c) This set is **not** a σ-algebra on X.

- The first property is satisfied, since X is in the set, because X - X is all of X.
- The second property is **not** satisfied. Let X be the set of natural numbers, and let U be the set of even numbers. Since X - U is infinite, U is in the set. However, X - U is not in the set, since U - X is $\emptyset$. It is not infinite. Therefore, the set is not closed under complementation.
- The third property is **not** satisfied. Let X be the set of natural numbers, and let U_n be the set of multiples of n for each n $\in X$. Then U_n is in the set since X - U_n is infinite. However, the union of all U_n is not in the set, since X - ($U_1 \cup U_2 \cup$ ...) is $\emptyset$, which is not infinite. Therefore, the set is not closed under countable unions.

Answer 3

(a) Assume that there is no solution for x in the equation ax $\equiv$ b (mod p), that is b is not divisible by gcd(a,p).

If there is no solution for x in the equation ax $\equiv$ b (mod p), then there is no integer k such that ax - b = kp. This implies that b is not a multiple of p, and hence b mod p is not zero.

Now, by a result called Bezout's identity, there are integers s and t such that as + pt = gcd(a, p). If we take both sides modulo p, $\equiv$ gcd(a, p) (mod p). Since b mod p is not zero, we cannot have as $\equiv$ b (mod p) for any s. Therefore, gcd(a, p) b, as required.

On the other hand, if we assume that b is not divisible by gcd(a, p), then we can use the same reasoning in reverse to show that there is no solution for x in the equation ax $\equiv$ b (mod p). Therefore, the statement is true by contraposition.

(b) Assume that gcd(p_1, p_2) = 1, and show that this implies that the pair of congruences has a solution for x.

If gcd(p_1, p_2) = 1, then there exist integers s and t such that p_1s + p_2t = 1 by Bezout's identity. Multiply both sides by x, $p_1$sx + $p_2$tx = x. Take both sides modulo p_1, $p_2$tx $\equiv$ x (mod p_1). Similarly, take both sides modulo p_2, $p_1$sx $\equiv$ x (mod p_2).

Therefore, if we can find an x that satisfies both $p_2$tx $\equiv$ x (mod p_1) and $p_1$sx $\equiv$ x (mod p_2), then we have a solution for the original pair of congruences. So, the statement is true.

(c) We can use the induction method. Assume that the statement is true for k = 1 and k = 2 and show that this leads to the statement being true for k + 1. If k = 1, then the congruences reduce to a single congruence

a_1x $\equiv$ b_1 (mod p_1)

which has a solution for x of the form x $\equiv$ c (mod p_1), where c is any integer that makes the equation true.

If k = 2, which is already shown to have a solution for x of the form x $\equiv$ c (mod p_1p_2), where c is determined by the Chinese Remainder Theorem.

Let's say the statement is true for k, and consider the system of congruences for k + 1. Then;

a_1x $\equiv$ b_1 (mod p_1) ... a_kx $\equiv$ b_k (mod p_k) a_k+1x $\equiv$ b_k+1 (mod p_k+1)

By the induction, the first k congruences have a solution for x of the form x $\equiv$ c (mod Π), where Π = p_1p_2 ... pk and c is determined by the Chinese Remainder Theorem. The new congruence:

x $\equiv$ c (mod Π)

and combine it with the last congruence

x $\equiv$ b_k+1 (mod pk+1)

to form a new system of two congruences. If Π and p_k+1 are coprime, then we can apply the Chinese Remainder Theorem to find a solution for x of the form x $\equiv$ m (mod Πp_k+1), where m is determined by the theorem. This completes the induction step.

Therefore, the statement is true for any k by induction.

Answer 4

(a) $\Pi i \in Z^+$ X means the Cartesian product of X with itself infinitely many times, which results in a set of all possible endless strings of letters from X, such as (a, b, c, ...), (k, h, n, ...), (t, r, s, ...), etc.

The cartesian product of countably infinite, countable sets is countable. It can be shown by the diagonal argument.

Let the set X = {a, b, ..., z} that contains the letters of the Turkish alphabet. This set has 29 elements, so it is finite and countable. We can list the elements of X in any order. For example: a, b, c, ..., z. We can also assign a number to each letter, for example: 1 - a, 2 - b, 3 - c, ..., 29 - z. This function is also bijective because no two different letters have the same number (injective) and every number from 1 to 29 is assigned to some letter (surjective).

Next, let's consider the Cartesian product of X with itself, denoted by (X)x(X). This set contains all ordered pairs of letters, for example: (a, a), (a, b), (a, c), ..., (z, z). This set has 841 elements, so it is also finite and countable. We can list the elements of (X)x(X) in a systematic way, for example: (a, a), (a, b), (a, c), ..., (a, z), (b, a), (b, b), (b, c), ..., (b, z), ..., (z, a), (z, b), (z, c), ..., (z, z). We can also assign a natural number to each pair of letters, for example; 1 - (a, a), 2 - (a, b), 3 - (a, c), ..., $29^2 = 841$ - (z, z). This function is also bijective.

We can generalize this process to any finite number of Cartesian products of X with itself. For example, the set (X)x(X)x(X) contains all ordered triples of letters, for example: (a, a, a), (a, a, b), (a, a, c), ..., (z, z, z). This set has $29^3 = 24389$ elements, so it is also finite and countable. We can list the elements of (X)x(X)x(X) in a systematic way, for example:

(a, a, a), (a, a, b), (a, a, c), ..., (a, a, z), (a, b, a), (a, b, b), (a, b, c), ..., (a, b, z), ..., (z, z, a), (z, z, b), (z, z, c), ..., (z, z, z).

We can also assign a natural number to each triple of letters, for example; 1 - (a, a, a), 2 - (a, a, b), 3 - (a, a, c), ..., 24389 - (z, z, z). This function is also bijective.

Finally, let's consider the infinite Cartesian product of X with itself, denoted by $\Pi i \in Z^+$ X. The set contains all infinite sequences of letters, for example: (a, a, a, a, ...), (a, a, a, b, ...), (a, a, a, c, ...), ..., (z, z, z, z, ...). This set is infinite and countable at the same time. We can list the elements of $\Pi i \in Z^+$ X in a diagonal way, as shown in the following table:

1	2	3	4	...
a	a	a	a	...
b	a	a	a	...
c	a	a	a	...
...	...	...	...	...
a	b	a	a	...
b	b	a	a	...
c	b	a	a	...
...	...	...	...	...
a	c	a	a	...
b	c	a	a	...
c	c	a	a	...
...	...	...	...	...

We can also assign a natural number to each infinite sequence of letters, for example: 1 - (a, a, a, a, ...), 2 - (b, a, a, a, ...), ..., 30 - (a, b, a, a, ...), ..., 60 - (b, c, a, a, ...), ...

Therefore, the set $\Pi i \in Z^+$ X is countable, since the set can be listed and numbered in a one-to-one correspondence with the natural numbers.

(b) Since each Y_i is countably infinite, there exists a surjective function f_i: $Z^+ \to Y_i$. Assume that each f_i is injective because we can always choose an injective function from $Z^+ \to Y_i$ by removing duplicates.

Consider the function f: $Z^+ x Z^+ \to U_{i=1}^{\infty} Y_i$ defined by f(n, m) = f_n(m) , where f_n is the function associated with the set Y_n. In other words, f takes an ordered pair (n, m) and maps it to the m-th element in the n-th set.

Now, we claim that f is a surjective function. Given any element y in the union $U_{i=1}^{\infty} Y_i$, there exists an n such that y belongs to Y_n. Since f_n is surjective, there exists an m such that f_n(m) = y. Therefore, f(n, m) = f_n(m) = y, and f covers every element in the union.

Since we have constructed a surjective function from $Z^+ x Z^+$ to $U_{i=1}^{\infty} Y_i$, and $Z^+ x Z^+$ is countable, we can conclude that $U_{i=1}^{\infty} Y_i$ is countable.