CENG 223

Discrete Computational Structures

Fall 2022-2023
Take Home Exam 1

Solution

Question 1 (25 pts)

a) Show that whether the following statement is a tautology or a contradiction by using a **truth table**.

$$(p \to q) \oplus (p \land \neg q)$$

(5/25 pts)

p	q	$\neg q$	$p \to q$	$p \land \neg q$	$(p \to q) \oplus (p \land \neg q)$
T	T	F	T	F	T
T	F	T	F	T	T
F	T	F	T	F	T
F	F	T	T	F	T

It is a tautology.

b) Show that $p \to ((q \lor \neg p) \to r)$ and $(p \land q) \to r$ are logically equivalent. Use tables 6, 7 and 8 given under the section "*Propositional Equivalences*" in the course textbook and give the reference to the table and the law in each step.

(10/25 pts)

$$
\begin{array}{rcll}
p \to ((q \vee \neg p) \to r) & \equiv & p \to (\neg(q \vee \neg p) \vee r) & \textit{Table 7, line 1} \\
& \equiv & \neg p \vee (\neg(q \vee \neg p) \vee r) & \textit{Table 7, line 1} \\
& \equiv & \neg p \vee (\neg q \wedge \neg\neg p) \vee r & \textit{De Morgan's laws} \\
& \equiv & \neg p \vee (\neg q \wedge p) \vee r & \textit{Double negation law} \\
& \equiv & (\neg p \vee (\neg q \wedge p)) \vee r & \textit{Associative laws} \\
& \equiv & ((\neg p \vee \neg q) \wedge (\neg p \vee p)) \vee r & \textit{Distributive laws} \\
& \equiv & ((\neg p \vee \neg q) \wedge T) \vee r & \textit{Negation laws} \\
& \equiv & (\neg(p \wedge q) \wedge T) \vee r & \textit{De Morgan's laws} \\
& \equiv & (\neg(p \wedge q)) \vee r & \textit{Identity laws} \\
& \equiv & \neg\neg(p \wedge q) \to r & \textit{Table 7, line 1} \\
& \equiv & (p \wedge q) \to r & \textit{Double negation law}
\end{array}
$$

c) (10/25 pts)

- F
- F
- F
- T
- T

Question 2 (30 pts)

Assume the following:

$P(x, y)$: Player x plays on team y.
$T(x, y)$: Team x plays in league y.
$N(x, y)$: Player x is of nationality y.
$W(x, y)$: Team x has won against team y.
$R(x, y)$: Team x is a rival of team y. Note that $R(x, y) \equiv R(y, x)$.

Translate the following sentences into predicate logic using $\vee, \wedge, \rightarrow, \neg, \forall, \exists$. You are not allowed to use any other predicate symbols. They are 5 points each. (Note: You can use constants to denote individual players or specific teams like Can or P).

a) Player Can plays in a team that plays in league L.

b) Every team in league S has at least one Turkish player.

c) Every team in league S has exactly one rival, that is also in league S.

d) Team M has never won against a team that has at least one English player.

e) Exactly two Turkish players play on team G.

f) There are some teams that play in more than one league.

a)

$$\exists a(P(Can, a) \wedge (T(a, L)))$$

b)

$$\forall x(T(x, S) \rightarrow \exists y(N(y, Turkish) \wedge P(y, x)))$$

c)

$$\forall x \forall y \exists z(T(x, S) \rightarrow (R(x, z) \wedge T(z, S)) \wedge ((T(y, S) \wedge R(x, y)) \rightarrow (y \neq z \wedge y \neq x)))$$

d)

$$\neg \exists x \exists y(W(M, x) \wedge P(y, x) \wedge N(y, English))$$

e)

$$\exists x \exists y((P(x, G) \wedge N(x, Turkish) \wedge P(y, G) \wedge N(y, Turkish) \wedge (x \neq y))$$
$$\wedge \forall z((P(z, G) \wedge N(z, Turkish)) \rightarrow ((z \neq x) \wedge (z \neq y))))$$

f)

$$\exists x \exists y \exists z(T(x, y) \wedge T(x, z) \wedge (y \neq z))$$

Question 3 (20 pts)

Prove the following claims by natural deduction. Use only the natural deduction rules $\vee$, $\wedge$, $\rightarrow$, $\neg$, introduction and elimination. If you attempt to make use of a lemma or equivalence, you need to prove it by natural deduction too. Try to keep your answer under 25 lines.

$$p \rightarrow q, (r \wedge s) \rightarrow p, (r \wedge \neg q) \vdash \neg s$$

1.	$p \rightarrow q$	*premise*
2.	$(r \wedge s) \rightarrow p$	*premise*
3.	$r \wedge \neg q$	*premise*
4.	$\neg q$	$\wedge e_2$ 3
5.	p	*assumption*
6.	q	$\rightarrow e$ 1, 5
7.	$\perp$	$\neg e$ 4, 6
8.	$\neg p$	$\neg i$ 5, 7
9.	r	$\wedge e_1$ 3
10.	s	*assumption*
11.	$r \wedge s$	$\wedge i$ 9, 10
12.	p	$\rightarrow e$ 2, 11
13.	$\perp$	$\neg e$ 8, 12
14.	$\neg s$	$\neg i$ 10, 13

Question 4 (25 pts)

Assume the following:

$S(x)$: Student x studies for the exam.
$P(x)$: Student x passes the exam.

The following premises are given;

- Some students need to study for the exam in order to pass.
- Every student passed the exam.

With these premises, it can be claimed that "There is at least one student that studied for the exam".

a) Write these two premises and the claim in predicate logic.

(10/25 pts)

$$\exists x(\neg S(x) \rightarrow \neg P(x)), \forall x P(x) \vdash \exists x S(x)$$

b) Prove the above claim by natural deduction. Use only the natural deduction rules $\vee, \wedge, \rightarrow, \neg, \forall, \exists$ introduction and elimination. If you attempt to make use of a lemma or equivalence, you need to prove it by natural deduction too.

(15/25 pts)

1.	$\exists x(\neg S(x) \rightarrow \neg P(x))$	*premise*
2.	$\forall x P(x)$	*premise*
3.	$\exists x(\neg S(x))$	*assumption*
4.	$\exists x(\neg P(x))$	$\rightarrow e$ 1, 3
5.	$\perp$	$\neg e$ 2, 4
6.	$\exists x(S(x))$	$\neg i$ 3, 5