Student Information

Full Name : Mustafa Barış Emektar
Id Number : 2304533

Answer 1

(a) For base case $n = 1$ it holds.

$$2^3 - 3 \quad = \quad 8 - 3 \quad = \quad 5$$

Since 5 is divisible by 5.

Inductive step:

Assume it holds for $n = k$. So, $2^{3k} - 3^k$ is divisible by 5. (inductive hypothesis)

Try to find out if it holds for n = k+1 ;

$$\begin{aligned}
2^{3(k+1)} - 3^{(k+1)} &= 2^3.2^{3k} - 3.3^k \\
&= 8.8^k - 3.3^k \\
&= 8.8^k - 3.3^k - 5.3^k + 5.3^k \\
&= 8.8^k - 8.3^k + 5.3^k \\
&= 8.(8^k - 3^k) + 5.3^k \\
&= 2^3.(2^{3k} - 3^k) + 5.3^k
\end{aligned}$$

$(2^{3k} - 3^k)$ is divisible by assumption. 5.3^k is obviously divisible by 5.

So, the $2^{3n} - 3^n$ is divisible by 5 for all integers $n \geq 1$ by mathematical induction.

(b) For base case $n = 2$ it holds.

$$4^2 - 7.2 - 1 > 0$$

$$16 - 14 - 1 > 0$$

$$1 > 0$$

Inductive step:

Assume it holds for $n = k$ (inductive hypothesis) So;

$$4^k - 7.k - 1 > 0$$

Try to find out if it holds for n = k+1 ;

$$
\begin{aligned}
4^{(k+1)} - 7(k+1) - 1 &= 4.4^k - 7(k+1) - 1 \\
&= 4.4^k - 7k - 8 \\
&= 4.4^k - 7k - 3.7k + 3.7k - 4 - 4 \\
&= 4.4^k - 28k - 4 + 3.7k - 4 \\
&= 4(4^k - 7k - 1) + 3.7k - 4
\end{aligned}
$$

$(4^k - 7k - 1)$ is positive by assumption. $3.7k - 4$ is also positive since $n \geq 2$ and $k \geq 1$ since n = k+1.

So, the $4^n - 7n - 1 > 0$ holds for all integers $n \geq 2$ by mathematical induction.

Answer 2

(a) We must use the combination.

$$
\begin{gathered}
\binom{10}{7} = \frac{10!}{7!(10-7)!} = \frac{10\times9\times8}{3\times2\times1} = 120 \\
\binom{10}{8} = \frac{10!}{8!(10-8)!} = \frac{10\times9}{2\times1} = 45 \\
\binom{10}{9} = \frac{10!}{9!(10-9)!} = \frac{10}{1} = 10 \\
\binom{10}{10} = \frac{10!}{10!(10-10)!} = \frac{10!}{10!\times0!} = \frac{10!}{10!\times1} = 1 \\
\binom{10}{7} + \binom{10}{8} + \binom{10}{9} + \binom{10}{10} = 120 + 45 + 10 + 1 = 176
\end{gathered}
$$

(b) Calculate it by combination.

All combinations:

$$\binom{9}{4} = \frac{9!}{4!(9-4)!} = \frac{9\times8\times7\times6}{4\times3\times2\times1} = 126$$

Only Statistical Methods textbooks combinations:

$$\binom{5}{4} = 5$$

Only Discrete Mathematics textbooks combinations:

$$\binom{4}{4} = 1$$

All combinations with at least one Statistical Methods and one Discrete Mathematics textbooks : $\binom{9}{4} - \binom{5}{4} - \binom{4}{4} = 126 - 5 - 1 = 120$

(c) The number of onto functions from a set with m elements to a set n elements is

$$n^m - \binom{n}{1}(n-1)^m + \binom{n}{2}(n-2)^m - \cdots + (-1)^{n-1}\binom{n}{n-1}1^m$$

To find the number of onto functions from A to B, we can use the formula:

So;

$$3^5 - \binom{3}{1}.2^5 + \binom{3}{2}.1^5$$
$$= 243 - 3.32 + 3.1$$
$$= 243 - 96 + 3$$
$$= 150$$

Answer 3

(a) The question is actually about the pigeonhole principle. We can say the kids are pigeons and we need to create pigeonholes. To do that, the given triangle can be subdivided into 4 equilateral triangles. So, these small triangles are pigeonholes. Now we must distribute the 5 kids into these four triangles. We know that at least two of them must appear in one of these triangles. Also, we know that the maximal distance in one of these small triangles is $500/2 = 250$. So the by pigeonhole principle no matter how much they wander away from each other, as long as they stay in the triangle-shaped circus, there are two of them within 250 meters of each other.

Answer 4

This is a first-order linear recurrence relation with constant coefficients
$a_n = a_n^h + a_n^p$

(a) For a_n^h;

assume a solution of the form $a_n = r^n$

$$r^n = 3.r^{(n-1)}$$

by dividing both sides with $r^{(n-1)}$

$$r = 3$$

which is the characteristic equation. And the root is 3.

So the homogeneous solution is in the form of:

$$a_n^h = C.3^n$$

(b) For a_n^p;

Assume $a_n^p = B.5^n$ where B is a constant.

Substitute the assumed solution;

$$B.5^n = 3.(B.5^{n-1}) + 5^{n-1}$$

Divide both sides with 5^{n-1}

$$5.B = 3.B + 1$$
$$5.B - 3.B = 1$$
$$2.B = 1$$
$$B = \frac{1}{2}$$

Thus, the particular solution is;

$$a_n^p = \frac{1}{2}.5^n$$

By $a_n = a_n^h + a_n^p$

$$a_n = C.3^n + \frac{1}{2}.5^n$$

Use base case to find out C

$$a_1 = C.3^1 + \frac{1}{2}.5^1 = 4$$

$$C = \frac{1}{2}$$

Thus;

$$a_n = \frac{1}{2}.3^n + \frac{1}{2}.5^n = \frac{1}{2}.(3^n + 5^n)$$

(c) For base case $n = 1$ it holds;

$$a_1 = \frac{1}{2}.(3^1 + 5^1) = 4$$

For the inductive step, assume that $n = k$ holds (inductive hypothesis)

$$a_k = \frac{1}{2}.(3^k + 5^k)$$

Try for $n = k + 1$, use the recurrence relation;

$$a_{k+1} = 3.a_k + 5^k$$
$$a_{k+1} = 3.(\frac{1}{2}.(3^k + 5^k)) + 5^k$$
$$a_{k+1} = \frac{1}{2}.(3.3^k + 3.5^k) + 5^k$$
$$a_{k+1} = \frac{1}{2}.(3.3^k + 3.5^k + 2.5^k)$$
$$a_{k+1} = \frac{1}{2}.(3.3^k + 5.5^k)$$
$$a_{k+1} = \frac{1}{2}.(3^{k+1} + 5^{k+1})$$

Thus, the expression found in previous parts is correct and proved by induction.