Analysis of Markovian Population Models

Dissertation Defense

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Motivation

- Example
- list other applications: queueing, metabolic networks, switches etc.

Markovian Population Models

Semantics

- counting agents / population size
- continuous time
- exponential jump times / CTMC dynamics

Markovian Population Models

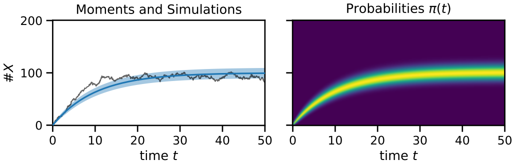
Stationary Distribution – Foster-Lyapunov Functions

- ergodic chains converge to unique distribution
- how does this distribution look like for infinite state-spaces?
- use Foster-Lyapunov function to bound sets
- locally augment functions for tighter sets / bounds

Markovian Population Models

Moment Dynamics

- alternative approach: look at moments instead of states
- expected values, e.g. $E(X), E(X^2)$



- Moment formula

$$\frac{d}{dt}E(f(X_t)) = \sum_{j=1}^{n_R} E\left(\left(f(X_t+v_j)-f(X_t)\right)\alpha_j(X_t)\right)$$

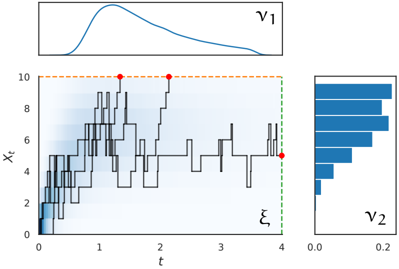
- ODE system not closed

- Analytic integration and resulting martingale process

$$
\begin{aligned}
Z_T := w(T)f(X_T) - w(0)f(X_0) - \int_0^T \frac{dw(t)}{dt} f(X_t)\,dt \\
- \sum_{j=1}^{n_R} \int_0^T w(t)(f(X_t + v_j) - f(X_t))\alpha_j(X_t)\,dt\,.
\end{aligned}
$$

- Crucially, $E\left(Z_T\right) = 0,\ \forall T \geq 0$.

- expected occupation time and exit measures (in relation to expectation of the martingale)
- linear constraints connecting 3 measures (integrate moms and figure)



- semi-definite moment constraints (positive variance as example)
- hint at localizing matrices

Bounding Mean First-Passage Times

Results and Practical Issues

- moment stiffness, re-scaling issue
- some examples

Bounding Mean First-Passage Times

Hausdorff Constraints and Linear Programs

- linear constraints possible if domains (time and space) are finite
- 1D visualization of Hausdorff constraints

Linear Control Variates

Using Correlated RVs with Known Expected Value

- segue: use the same martingale constraints to enhance MC estimation
- use correlations between target RV and martingales (linear regression, i.e. control variates)

Linear Control Variates

Finding Efficient Sets of Control Variates

- time-weighting has a large influence on the correlation
- Infinitely many possibilities (cost needs to be controlled though)
- variates can be highly redundant (correlated) and incur an additional cost
- Alg. 1: Tighten an initial proposal set
- Alg. 2: Re-sample promising candidates

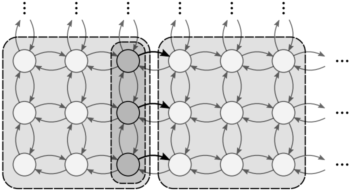
Linear Control Variates

Results

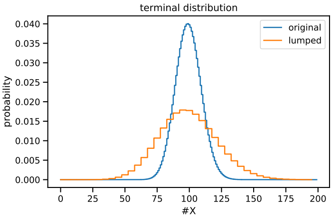
- best example?

State-Space Aggregation

Treating Hyper-Cubes of States as One



- hyper-cube macro-states
- *assumption*: uniform dist. within
- closed-form transition rates

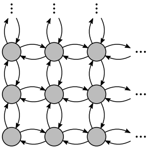


- resulting distribution more “flat”
- locate main probability mass

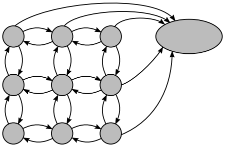
Stationary Distribution

Finite-Space Projection

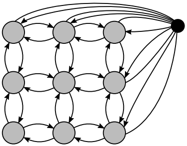
original



sink state



redirect



Stationary Distribution

Iterative Refinement Algorithm

Bridging Problem

Dynamical Analysis Under Initial *and* Terminal Constraints

Importance Sampling

Conclusions and Future Directions

Bibliography

Foster-Lyapunov Functions

Local Augmentation of Foster-Lyapunov Functions

Control Variates in General

Control Variates Selection Algorithm 1

Control Variates Selection Algorithm 2

Semi-definite programming