Analysis of Markovian Population Models

Dissertation Defense

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Motivation

- Example
- list other applications: queueing, metabolic networks, switches etc.

Markovian Population Models

Semantics

- counting agents / population size
- continuous time
- exponential jump times / CTMC dynamics
- example: birth-death process

Markovian Population Models

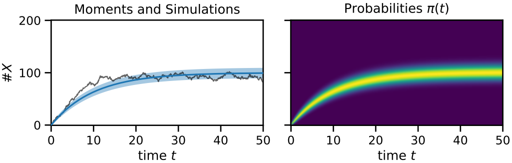
Stationary Distribution – Foster-Lyapunov Functions

- ergodic chains converge to unique distribution
- how does this distribution look like for infinite state-spaces?
- use Foster-Lyapunov function to bound sets
- locally augment functions for tighter sets / bounds

Markovian Population Models

Moment Dynamics

- alternative approach: look at moments instead of states
- expected values, e.g. $E\left(X\right), E\left(X^2\right)$



- Moment formula

$$\frac{d}{dt}E\left(f\left(X_t\right)\right) = \sum_{j=1}^{n_R} E\left(\left(f\left(X_t + v_j\right) - f\left(X_t\right)\right)\alpha_j\left(X_t\right)\right)$$

- ODE system not closed

Markovian Population Models

Martingale Process

- analytic integration and resulting martingale process

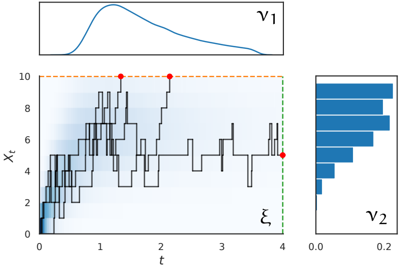
$$Z_T \coloneqq w(T)f(X_T) - w(0)f(X_0) - \int_0^T \frac{dw(t)}{dt} f(X_t)\,dt \\ - \sum_{j=1}^{n_R} \int_0^T w(t)(f(X_t + v_j) - f(X_t))\alpha_j(X_t)\,dt$$

- time-weighting: $w(t) = t^k$, $k \in \mathbb{N}$ or $w(t) = \exp(\lambda t)$
- known expectation: $E\left(Z_T\right) = 0$, $\forall T \geqslant 0$

Bounding Mean First-Passage Times

Martingale Process and Linear Moment Constraints

- expected occupation time and exit measures (in relation to expectation of the martingale)



- $0 = E\left(Z_T\right)$ is a linear moment constraint constraint on ν_1, ν_2, and ξ ($w(t) = t^k$) (TODO: integrate moms and figure)

- semi-definite moment constraints (positive variance as example)
- hint at localizing matrices

Bounding Mean First-Passage Times

Results and Practical Issues

- moment stiffness, re-scaling issue
- some examples

Bounding Mean First-Passage Times

Hausdorff Constraints and Linear Programs

- linear constraints possible if domains (time and space) are finite
- 1D visualization of Hausdorff constraints

Linear Control Variates

Using Correlated RVs with Known Expected Value

- segue: use the same martingale constraints to enhance MC estimation
- use correlations between target RV and martingales (linear regression, i.e. control variates)

Linear Control Variates

Finding Efficient Sets of Control Variates

- time-weighting has a large influence on the correlation
- Infinitely many possibilities (cost needs to be controlled though)
- variates can be highly redundant (correlated) and incur an additional cost
- Alg. 1: Tighten an initial proposal set
- Alg. 2: Re-sample promising candidates

Linear Control Variates

Selection Algorithms

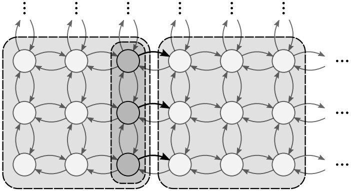
Linear Control Variates

Results

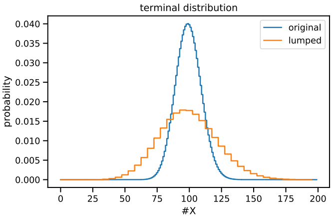
- best example?

State-Space Aggregation

Treating Hyper-Cubes of States as One (Backenköhler et al. 2021a,b)



- hyper-cube macro-states
- *assumption*: uniform dist. within
- closed-form transition rates

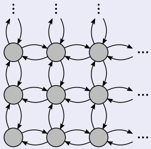


- resulting distribution more "flat"
- locate main probability mass

Stationary Distribution

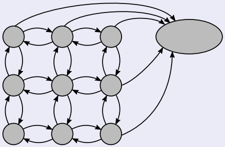
Finite-Space Projection

Original



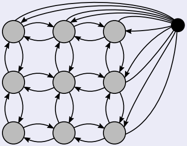
- infinite

Sink state



- transient analysis
- keep track of approx. error

Redirection



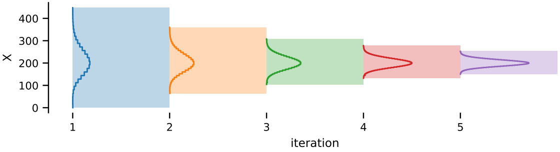
- stationary dist.
- dependent on redirection

Stationary Distribution

Iterative Refinement Algorithm

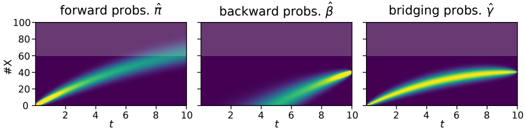
A simple refinement based on approximate solutions:

1. start with macro-states of size 2^k
2. compute approximate distribution
3. remove states with low probability
4. split the remaining states
5. go to step 2



Bridging Problem

Dynamical Analysis Under Initial *and* Terminal Constraints



Forward Probabilities

How the process evolves with time: $\Pr(X_t = x \mid X_0 = 0)$

Backward Probabilities

Probability of ending up in a given state: $\Pr(X_T = 40 \mid X_t = x)$

Bridging Probabilities

In between: $\Pr(X_t = x \mid X_0 = 0, X_T = 40)$

Bridging Problem
Refinement

Importance Sampling

Conclusions and Future Directions

References I

Backenköhler, Michael et al. (2021a). "Analysis of Markov Jump Processes under Terminal Constraints". In: *27th International Conference on Tools and Algorithms for the Construction and Analysis of Systems*. Vol. 12651. Lecture Notes in Computer Science. Springer, pp. 210–229.

— (2021b). "Abstraction-Guided Truncations for Stationary Distributions of Markov Population Models". In: *18th International Conference on Quantitative Evaluation of SysTems*. Vol. 12846. Lecture Notes in Computer Science. Springer, pp. 351–371.