Stellar and Planetary Astrophysics

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Determine the equation of state of a mixture of classical ideal gas and radiation (pressure, internal energy, specific heats, and adiabatic exponents).

Some Definitions

Let us begin this problem by introducing the adiabatic exponents Γ_1, Γ_2, and Γ_3. For any adiabatic process, that is, one that occurs at constant entropy and thus $dQ = TdS = 0$, we have:

$$\Gamma_1 \equiv -\left(\frac{\partial \ln P}{\partial \ln V}\right)_s \quad (1)$$

$$\frac{\Gamma_2}{\Gamma_2 - 1} \equiv \left(\frac{\partial \ln P}{\partial \ln T}\right)_s \quad (2)$$

$$1 - \Gamma_3 = \left(\frac{\partial \ln T}{\partial \ln V}\right)_s \quad (3)$$

Here the letter "s" indicates adiabatic change. From these definitions, note that there are actually only **2 independent** coefficients [2]:

$$\Gamma_1(\Gamma_2 - 1) = \Gamma_2(\Gamma_3 - 1) \quad (4)$$

To calculate these parameters, we will calculate the total pressure P and internal energy U of the mixture. Given that this mixture is made of gas and radiation, the total pressure will consist of both a gas pressure term and a blackbody radiation pressure term. These two components are given by Eq. 5 and Eq. 6 respectively:

$$P_{gas} = \frac{\rho k_B T}{\mu m} = \frac{k_B T}{\mu m} \cdot \frac{1}{V_{spec}} \quad (5)$$

$$P_{rad} = \frac{1}{3} a T^4 \quad (6)$$

where k_B is the Boltzmann constant, a is the radiation constant,[1] T is the temperature of the mixture, μ is the mean molecular weight, $m \equiv m_u = 1/N_A$ is a unity of atomic gass, ρ is the gas density, and

[1] $a = \frac{8\pi^5 k^4}{15 c^3 h^3} = \frac{4\sigma}{c} = 7.565 \cdot 10^{-15}$ erg cm$^{-3} K^{-4}$..

$V \equiv V_{specific} = 1/\rho$ is the specific volume. Adding the two terms yields a total pressure of:

$$\boxed{P = P_{gas} + P_{rad} = \frac{\rho k_B T}{\mu m} + \frac{1}{3}aT^4 = \frac{k_B T}{\mu m V_{spec}} + \frac{1}{3}aT^4} \tag{7}$$

With regard to the mixture's total specific internal energy, we can express the internal energy of the gas and radiation as:

$$U_{rad} = \varepsilon_r V = aT^4 V \tag{8}$$

$$U_{gas} = \frac{3}{2}\frac{k_B T}{\mu m} \tag{9}$$

Therefore,

$$\boxed{U = U_{gas} + U_{rad} = \frac{3}{2}\frac{k_B T}{\mu m} + aT^4 V} \tag{10}$$

As we can see, the total internal energy U depends on volume and temperature, i.e. $U(T, V)$.

The First Law of Thermodynamics

A quasistatic process happens infinitely slowly, so any system that undergoes such thermodynamic process remains close to equilibrium at all times. Any reversible process (i.e. a process whose direction can be reversed with no net increase in entropy) is also quasistatic. The 1st Law of Thermodynamics for quasistatic changes can be written as:

$$dU = dQ + dW = dQ - PdV \rightarrow \boxed{dQ = dU + PdV,} \tag{11}$$

where dQ is the amount of heat supplied to the system. We can rewrite the above by considering an infinitessimal change in the internal energy,

$$dU = \left(\frac{\partial U}{\partial T}\right)_V dT + \left(\frac{\partial U}{\partial V}\right)_T dV \tag{12}$$

Substituting Eq. 12 into Eq. 11, we obtain:

$$dQ = \left[\left(\frac{\partial U}{\partial T}\right)_V dT + \left(\frac{\partial U}{\partial V}\right)_T dV\right] + PdV = 0. \tag{13}$$

Note that for adiabatic conditions, $dQ = 0$. We can now compute the terms that involve the internal energy dU. In particular,

$$\left(\frac{\partial U}{\partial T}\right)_V = \frac{3}{2}\frac{k_B}{\mu m} + 4aT^3 V \tag{14}$$

$$\left(\frac{\partial U}{\partial V}\right)_T = aT^4 \tag{15}$$

Plug Eq. 14 and Eq. 15 into Eq. 13:

$$dQ = \left[\left(\frac{3}{2}\frac{k_B}{\mu m} + 4aT^3 V\right) dT + \left(aT^4\right) dV\right] + \left[\frac{\rho k_B T}{\mu m} + \frac{1}{3}aT^4\right] dV = 0 \tag{16}$$

$$= \left(\frac{3}{2}\frac{k_B}{\mu m} + 4aT^3 V\right) dT + \left(\frac{\rho k_B T}{\mu m} + \frac{4}{3}aT^4\right) dV = 0 \tag{17}$$

Multiply the first term by V/T:

$$= \left(\frac{3}{2}\frac{k_B}{\mu m}\frac{T}{V} + 4aT^4 V\frac{T}{V}\right) \frac{V}{T} dT + \left(\frac{\rho k_B T}{\mu m} + \frac{4}{3}aT^4\right) dV = 0 \tag{18}$$

$$= \left(\frac{3}{2}\frac{k_B}{\mu m}\frac{T}{V} + 4aT^4\right) \frac{V}{T} dT + \left(\frac{\rho k_B T}{\mu m} + \frac{4}{3}aT^4\right) dV = 0 \tag{19}$$

Hence,

$$\boxed{dQ = \left(\frac{3}{2}P_{gas} + 12P_{rad}\right)\frac{dT}{T} + (P_{gas} + 4P_{rad})\frac{dV}{V} = 0.} \tag{20}$$

In addition to dQ, we will also need the term dP later on to calculate the various adiabatic exponents. From Eq. 7, we know that:

$$P(V,T) = P_{gas} + P_{rad} = \frac{k_B T}{\mu m V_{spec}} + \frac{1}{3}aT^4$$

Then, a small increase in pressure will correspond to:

$$dP = \left(\frac{\partial P}{\partial T}\right)_V dT + \left(\frac{\partial P}{\partial V}\right)_T dV, \tag{21}$$

where

$$\left(\frac{\partial P}{\partial T}\right)_V = \frac{k_B}{\mu m V_{spec}} + \frac{4}{3}aT^3 \tag{22}$$

$$\left(\frac{\partial P}{\partial V}\right)_T = -\frac{k_B T}{\mu m V_{spec}^2} \tag{23}$$

Plugging Eq. 22 and Eq. 23 into Eq. 21, we obtain:

$$dP = \left(\frac{k_B}{\mu m V_{spec}} + \frac{4}{3}aT^3\right) dT + \left(-\frac{k_B T}{\mu m V_{spec}^2}\right) dV \tag{24}$$

Divide the dT term by T and readjust first parenthesis as necessary. As for the 2nd term, take out a power of V outside the parenthesis. Then,

$$dP = \left(\frac{k_B}{\mu m}\frac{T}{V_{spec}} + \frac{4}{3}aT^4\right)\frac{dT}{T} + \left(-\frac{k_B}{\mu m}\frac{T}{V_{spec}}\right)\frac{dV}{V} = 0 \tag{25}$$

or simply

$$\boxed{dP = (P_{gas} + 4P_{rad})\frac{dT}{T} - P_{gas}\frac{dV}{V}} \tag{26}$$

Bringing back the definition of Γ_1 and Γ_2, namely

$$\Gamma_1 \equiv -\left(\frac{\partial \ln P}{\partial \ln V}\right)_s, \qquad \frac{\Gamma_2}{\Gamma_2 - 1} \equiv \left(\frac{\partial \ln P}{\partial \ln T}\right)_s$$

we can set dP equal to $\partial P/\partial T$ or $\partial P/\partial V$ and relate the resulting expressions to the adiabatic exponents. In particular,

$$\text{For T:} \quad dP = \left(\frac{\partial P}{\partial T}\right)_{ad} dT = \frac{\frac{P}{P}}{\frac{T}{T}} \left(\frac{\partial P}{\partial T}\right)_{ad} dT = \frac{P}{T} \left(\frac{\frac{\partial P}{P}}{\frac{\partial T}{T}}\right)_{ad} dT = \frac{P}{T} \left(\frac{\partial \ln P}{\partial \ln T}\right)_{ad} dT \tag{27}$$

$$\boxed{dP = \frac{P}{T} \left(\frac{\Gamma_2}{\Gamma_2 - 1}\right) dT} \tag{28}$$

$$\text{For V:} \quad dP = \left(\frac{\partial P}{\partial V}\right)_{ad} dV = \frac{\frac{P}{P}}{\frac{V}{V}} \left(\frac{\partial P}{\partial V}\right)_{ad} dV = \frac{P}{V} \left(\frac{\frac{\partial P}{P}}{\frac{\partial V}{V}}\right)_{ad} dV = \frac{P}{V} \left(\frac{\partial \ln P}{\partial \ln V}\right)_{ad} dV \tag{29}$$

$$\boxed{dP = -\frac{P}{V} (\Gamma_1)\, dV} \tag{30}$$

To summarize, we can express dP in various ways:

$$\boxed{\begin{array}{rcl} dP & = & \left(\frac{\partial P}{\partial T}\right)_V dT + \left(\frac{\partial P}{\partial V}\right)_T dV \\ & = & \left(\frac{\partial P}{\partial T}\right)_{ad} dT = \frac{P}{T} \left(\frac{\partial \ln P}{\partial \ln T}\right)_{ad} dT \\ & = & \frac{P}{T} \left(\frac{\Gamma_2}{\Gamma_2 - 1}\right) dT = -\frac{P}{V} (\Gamma_1)\, dV \end{array}} \tag{31}$$

In the next sections, we will use the equalities involving Γ_2 and Γ_1 to solve for Γ_2 and Γ_1 respectively.

Calculating Γ_2

To find Γ_2, we will use the left-hand side equality in Eq. 31:

$$dP = \left(\frac{\partial P}{\partial T}\right)_V dT + \left(\frac{\partial P}{\partial V}\right)_T dV = \frac{P}{T} \left(\frac{\Gamma_2}{\Gamma_2 - 1}\right) dT \tag{32}$$

Plugging Eq. 22 and Eq. 23, the above becomes:

$$dP = \left(\frac{k_B}{\mu m V_{spec}} + \frac{4}{3} a T^3\right) dT - \left(\frac{k_B T}{\mu m V_{spec}^2}\right) dV = \frac{P}{T} \left(\frac{\Gamma_2}{\Gamma_2 - 1}\right) dT \tag{33}$$

$$(P_{gas} + 4P_{rad}) \frac{dT}{T} - P_{gas} \frac{dV}{V} = P \left(\frac{\Gamma_2}{\Gamma_2 - 1}\right) \frac{dT}{T} = 0 \tag{34}$$

Grouping in terms of the same multipliers:

$$\left[P_{gas} + 4P_{rad} - P \left(\frac{\Gamma_2}{\Gamma_2 - 1}\right)\right] \frac{dT}{T} - P_{gas} \frac{dV}{V} = 0 \tag{35}$$

Given that $P = P_{gas} + P_{rad}$, Eq. 35 becomes:

$$\boxed{\left[P_{gas} + 4P_{rad} - \left(\frac{\Gamma_2}{\Gamma_2 - 1}\right) (P_{gas} + P_{rad})\right] \frac{dT}{T} - P_{gas} \frac{dV}{V} = 0} \tag{36}$$

With Eq. 20 and Eq. 36, we are now ready to write the system of equations that we will use to calculate the Γ_2 adiabatic coefficient:

$$\begin{cases} dQ = \left(\frac{3}{2}P_{gas} + 12P_{rad}\right)\frac{dT}{T} + \left(P_{gas} + 4P_{rad}\right)\frac{dV}{V} & = 0 \\ \left[P_{gas} + 4P_{rad} - \left(\frac{\Gamma_2}{\Gamma_2 - 1}\right)\left(P_{gas} + P_{rad}\right)\right]\frac{dT}{T} - P_{gas}\frac{dV}{V} & = 0 \end{cases} \tag{37}$$

Any system of the form

$$\begin{cases} Ax + By & = 0 \\ Cx + Dy & = 0 \end{cases}$$

can be solved with the simpler expression: $\frac{A}{C} = \frac{B}{D}$. Consequently,

$$\begin{array}{rcc} A & = & \frac{3}{2}P_{gas} + 12P_{rad} \\ B & = & P_{gas} + 4P_{rad} \\ C & = & P_{gas} + 4P_{rad} - \left(\frac{\Gamma_2}{\Gamma_2 - 1}\right)\left(P_{gas} + P_{rad}\right) \\ D & = & -P_{gas} \end{array} \tag{38}$$

Then,

$$\frac{A}{C} = \frac{B}{D} \rightarrow \frac{\frac{3}{2}P_{gas} + 12P_{rad}}{P_{gas} + 4P_{rad} - \left(\frac{\Gamma_2}{\Gamma_2 1}\right)\left(P_{gas} + P_{rad}\right)} = \frac{P_{gas} + 4P_{rad}}{-P_{gas}} \tag{39}$$

Taking

$$\boxed{\begin{array}{rcc} P_{gas} & = & \beta P \\ P_{rad} & = & (1-\beta)\,P \end{array},} \tag{40}$$

Substituting the above into Eq. 39 and eliminating the pressure terms, we obtain:

$$\frac{\frac{3}{2}\beta\cancel{P} + 12(1-\beta)\cancel{P}}{\beta\cancel{P} + 4(1-\beta)\cancel{P} - \left(\frac{\Gamma_2}{\Gamma_2 - 1}\right)\cancel{P}} = \frac{\beta\cancel{P} + 4(1-\beta)\cancel{P}}{-\beta\cancel{P}} \tag{41}$$

$$\frac{\frac{3}{2}\beta + 12 - 12\beta}{\beta + 4 - 4\beta - \left(\frac{\Gamma_2}{\Gamma_2 - 1}\right)} = \frac{\beta + 4 - 4\beta}{-\beta} \tag{42}$$

$$\frac{\left(12 - \frac{21}{2}\beta\right)}{-3\beta + 4 - \left(\frac{\Gamma_2}{\Gamma_2 - 1}\right)} = \frac{-3\beta + 4}{-\beta} \tag{43}$$

$$\left(12 - \frac{21}{2}\right)(-\beta) = (-3\beta + 4)\left(-3\beta + 4 - \left(\frac{\Gamma_2}{\Gamma_2 - 1}\right)\right) = \tag{44}$$

$$= 9\beta^2 - 12\beta + 3\beta\left(\frac{\Gamma_2}{\Gamma_2 - 1}\right) - 12\beta + 16 - 4\left(\frac{\Gamma_2}{\Gamma_2 - 1}\right)$$

$$-12\beta + \frac{21}{2}\beta^2 = 9\beta^2 - 24\beta + 16 + 3\beta\left(\frac{\Gamma_2}{\Gamma_2 - 1}\right) - 4\left(\frac{\Gamma_2}{\Gamma_2 - 1}\right) \tag{45}$$

$$-12\beta + 24\beta + \frac{21}{2}\beta^2 - 9\beta^2 - 16 = (3\beta - 4)\left(\frac{\Gamma_2}{\Gamma_2 - 1}\right)$$

$$12\beta + \left(\frac{21}{2} - \frac{18}{2}\right)\beta^2 - 16 = 12\beta + \frac{3}{2}\beta^2 - 16 = (3\beta - 4)\left(\frac{\Gamma_2}{\Gamma_2 - 1}\right) \tag{46}$$

Hence, the Γ_2 term is equal to:

$$\frac{\Gamma_2}{\Gamma_2 - 1} = \frac{12\beta + \frac{3}{2}\beta^2 - 16}{3\beta - 4} \rightarrow \frac{\Gamma_2 - 1}{\Gamma_2} = \frac{3\beta - 4}{12\beta + \frac{3}{2}\beta^2 - 16} \tag{47}$$

and

$$1 - \frac{1}{\Gamma_2} = \frac{3\beta - 4}{12\beta + \frac{3}{2}\beta^2 - 16} \tag{48}$$

$$\frac{1}{\Gamma_2} = 1 - \frac{3\beta - 4}{12\beta + \frac{3}{2}\beta^2 - 16} = \frac{\left(12\beta + \frac{3}{2}\beta^2 - 16\right) - (3\beta - 4)}{12\beta + \frac{3}{2}\beta^2 - 16} = \frac{9\beta + \frac{3}{2}\beta^2 - 12}{12\beta + \frac{3}{2}\beta^2 - 16} \tag{49}$$

Finally, we find

$$\Gamma_2 = \frac{12\beta + \frac{3}{2}\beta^2 - 16}{9\beta + \frac{3}{2}\beta^2 - 12} \quad \overset{\cdot 2}{=} \quad \boxed{\Gamma_2 = \frac{32 - 24\beta - 3\beta^2}{24 - 18\beta - 3\beta^2}.} \tag{50}$$

Calculating Γ_1

The adiabatic coefficient Γ_1 can be found using the right-hand side equality in Eq. 31 which relates it to dP, that is:

$$dP = \left(\frac{\partial P}{\partial T}\right)_V dT + \left(\frac{\partial P}{\partial V}\right)_T dV \rightarrow \boxed{dP = -\frac{P}{V}(\Gamma_1)\, dV.} \tag{51}$$

Substituting the dP term by Eq. 26, we obtain:

$$(P_{gas} + 4P_{rad})\frac{dT}{T} - P_{gas}\frac{dV}{V} = -\frac{P}{V}(\Gamma_1)\, dV \tag{52}$$

$$(P_{gas} + 4P_{rad})\frac{dT}{T} + [P\Gamma_1 - P_{gas}]\frac{dV}{V} = 0 \tag{53}$$

Our new system of equations will incorporate Eq. 53 (for pressure) and Eq. 20 (for Q):

$$\begin{cases} dQ = \left(\frac{3}{2}P_{gas} + 12P_{rad}\right)\frac{dT}{T} + (P_{gas} + 4P_{rad})\frac{dV}{V} & = 0 \\ (P_{gas} + 4P_{rad})\frac{dT}{T} + [P\Gamma_1 - P_{gas}]\frac{dV}{V} & = 0 \end{cases} \tag{54}$$

We can expand Eq. 53 as:

$$0 = (P_{gas} + 4P_{rad})\frac{dT}{T} + [P\Gamma_1 - P_{gas}]\frac{dV}{V} \tag{55}$$

$$= (P_{gas} + 4P_{rad})\frac{dT}{T} + [(P_{gas} + P_{rad})\Gamma_1 - P_{gas}]\frac{dV}{V}$$

Thus,

$$0 = (P_{gas} + 4P_{rad})\frac{dT}{T} + [P\Gamma_1 - P_{gas}]\frac{dV}{V} = (P_{gas} + 4P_{rad})\frac{dT}{T} + [P_{gas}\Gamma_1 + P_{rad}\Gamma_1 - P_{gas}]\frac{dV}{V}. \tag{56}$$

Once again, solve the system of equations by means of the expression: $\frac{A}{C} = \frac{B}{D}$. In this case,

$$\begin{aligned} A &= \frac{3}{2}P_{gas} + 12P_{rad} \\ B &= P_{gas} + 4P_{rad} \\ C &= P_{gas} + 4P_{rad} \\ D &= P_{gas}\Gamma_1 + P_{rad}\Gamma_1 - P_{gas} \end{aligned}$$

so

$$\frac{\frac{3}{2}P_{gas} + 12P_{rad}}{P_{gas} + 4P_{rad}} = \frac{P_{gas} + 4P_{rad}}{P_{gas}\Gamma_1 + P_{rad}\Gamma_1 - P_{gas}} \tag{57}$$

With the two β expressions shown in Eq. 40, we can rewrite the above as:

$$\frac{\frac{3}{2}\left(\beta \cancel{P}\right) + 12\left(1-\beta\right)\cancel{P}}{\beta\cancel{P} + 4(1-\beta)\cancel{P}} = \frac{\beta\cancel{P} + 4(1-\beta)\cancel{P}}{\beta\cancel{P}\Gamma_1 + (1-\beta)\Gamma_1\cancel{P} - \beta\cancel{P}} \tag{58}$$

Eliminating the pressure terms as earlier, we obtain:

$$\frac{\frac{3}{2}\beta + 12(1-\beta)}{\beta + 4 - 4\beta} = \frac{\beta + 4 - 4\beta}{\beta\Gamma_1 + (1-\beta)\Gamma_1 - \beta} \rightarrow \frac{\frac{3}{2}\beta + 12(1-\beta)}{4 - 3\beta} = \frac{4 - 3\beta}{\beta\Gamma_1 + (1-\beta)\Gamma_1 - \beta}, \tag{59}$$

$$\frac{3}{2}\beta + 12 - 12\beta = \frac{(4-3\beta)^2}{\beta\Gamma_1 + \Gamma_1 - \beta\Gamma_1 - \beta} \rightarrow \Gamma_1 - \beta = \frac{16 - 24\beta + 9\beta^2}{\frac{3}{2}\beta + 12 - 12\beta}, \tag{60}$$

Isolate Γ_1:

$$\Gamma_1 = \beta + \frac{9\beta^2 - 24\beta + 16}{\frac{3}{2}\beta + 12 - \frac{24}{2}\beta} = \beta + \frac{9\beta^2 - 24\beta + 16}{-\frac{21}{2}\beta + 12} = \tag{61}$$

$$= \frac{\beta\left(-\frac{21}{2}\beta + 12\right) + 9\beta^2 - 24\beta + 16}{-\frac{21}{2}\beta + 12} = \frac{-\frac{21}{2}\beta^2 + 12\beta + 9\beta^2 - 24\beta + 16}{\frac{-21}{2}\beta + 12},$$

$$= \frac{-21\beta^2 + 24\beta + 18\beta^2 - 48\beta + 32}{-21\beta + 24},$$

and finally,

$$\boxed{\Gamma_1 = \frac{-3\beta^2 - 24\beta + 32}{24 - 21\beta}.} \tag{62}$$

Calculate Γ_3

To conclude, we will compute the Γ_3 coefficient. To this end, we will use Eq. 4, that is: $\Gamma_1(\Gamma_2 - 1) = \Gamma_2(\Gamma_3 - 1)$. Rearranging the terms, we find that

$$\Gamma_3 - 1 = \frac{\Gamma_1\left(\Gamma_2 - 1\right)}{\Gamma_2} = \Gamma_1 - \frac{\Gamma_1}{\Gamma_2}. \tag{63}$$

First, calculate the ratio Γ_1/Γ_2 with Eq. 50 (Γ_2) and Eq. 62 (Γ_1):

$$\frac{\Gamma_1}{\Gamma_2} = \frac{\frac{-3\beta^2 - 24\beta + 32}{24 - 21\beta}}{\frac{32 - 24\beta - 3\beta^2}{24 - 18\beta - 3\beta^2}} = \frac{-3\beta^2 - 24\beta + 32}{24 - 21\beta} \tag{64}$$

Now, substract the above from Γ_1:

$$\Gamma_1 - \frac{\Gamma_1}{\Gamma_2} = \frac{-3\beta^2 - 24\beta + 32}{24 - 21\beta} - \frac{-3\beta^2 - 24\beta + 32}{24 - 21\beta} = \frac{-\cancel{3\beta^2} - 24\beta + 32 + \cancel{3\beta^2} + 18\beta - 24}{24 - 21\beta}, \tag{65}$$

and simplifying, we get:

$$\Gamma_1 - \frac{\Gamma_1}{\Gamma_2} = \frac{-6\beta + 8}{24 - 21\beta} \tag{66}$$

Calculate Γ_3 using the above result:

$$\Gamma_3 = 1 + \frac{-6\beta + 8}{24 - 21\beta} = \frac{24 - 21\beta - 6\beta + 8}{24 - 21\beta} = \frac{-27\beta + 32}{24 - 21\beta} \to \boxed{\Gamma_3 = \frac{32 - 27\beta}{24 - 21\beta}.} \tag{67}$$

Summary of Results

Putting together Eq. 62, Eq. 50 and Eq. 67, we have found the following expressions for the adiabatic coefficients:

$$\boxed{\begin{array}{lcc} \Gamma_1 & = & \frac{-3\beta^2 - 24\beta + 32}{24 - 21\beta} \\ \Gamma_2 & = & \frac{32 - 24\beta - 3\beta^2}{24 - 18\beta - 3\beta^2} \\ \Gamma_3 & = & \frac{32 - 27\beta}{24 - 21\beta} \end{array}} \tag{68}$$

For a gas, $\beta \to 1$ (perfect gas case), so:

$$\begin{array}{lcc} \Gamma_{1_g} & = & \frac{-3-24+32}{24-21} = \frac{5}{3} \\ \Gamma_{2_g} & = & \frac{32-24-3}{24-18-3} = \frac{5}{3} \\ \Gamma_{3_g} & = & \frac{32-27\beta}{24-21\beta} = \frac{5}{3} \end{array} \to \Gamma_{1_g} = \Gamma_{2_g} = \Gamma_{3_g} \tag{69}$$

and $\Gamma_1 = \Gamma_2 = \Gamma_3 = 5/3 = \gamma$. However, for $\beta \to 0$ (sample filled with black body radiation), we have:

$$\begin{array}{lcc} \Gamma_{1_r} & = & \frac{32}{24} = \frac{4}{3} \\ \Gamma_{2_r} & = & \frac{32}{24} = \frac{4}{3} \\ \Gamma_{3_r} & = & \frac{32}{24} = \frac{4}{3} \end{array} \to \Gamma_{1_r} = \Gamma_{2_r} = \Gamma_{3_r} \tag{70}$$

and $\Gamma_{1_r} = \Gamma_{2_r} = \Gamma_{3_r} = 4/3$. These two evidences can be clearly seen if we play with the coefficients and β (Python notebook[2]):

[2]https://github.com/mbadenas/Stellar-Astrophysics/blob/master/EoS%20Gas%20and%20Radiation.ipynb

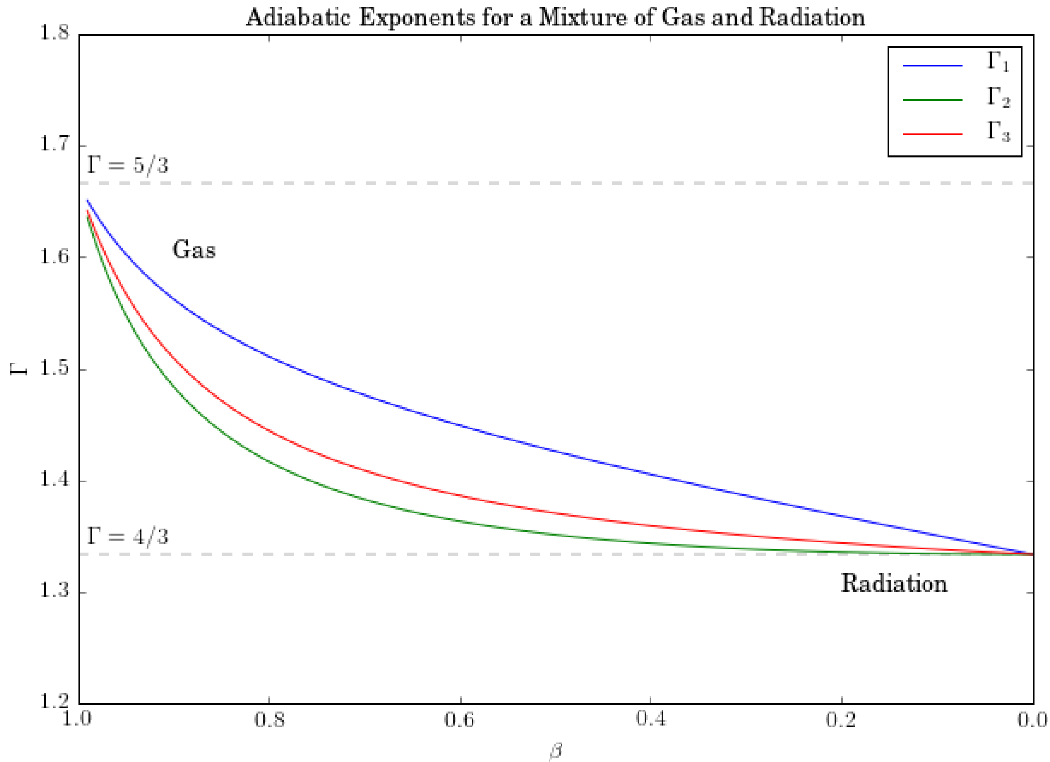

Figure 1: Adiabatic coefficients for a mixture of gas and radiation.

Calculating the Specific Heat Coefficients

The specific heat of a substance, or heat capacity, is the amount of energy needed to change the temperature of 1 kg of the substance by 1°C. For an adiabatic process (i.e. $dQ = 0$), the specific heat at constant volume, denoted by c_V, is defined as

$$c_V = \left(\frac{\partial Q}{\partial T}\right)_V = \left(\frac{\partial U}{\partial T}\right)_V . \tag{71}$$

Using the expression of the total specific energy given by Eq. 10, we obtain:

$$\boxed{c_V = \left(\frac{\partial U}{\partial T}\right)_V = \frac{4}{3}\frac{k_B}{\mu m} + 4aT^3 V} \tag{72}$$

Now, define the quantity

$$\boxed{c_V^0 = \frac{3}{2}\frac{k_B}{\mu m},} \tag{73}$$

which is the specific heat at constant volume for an ideal gas. Then,

$$c_V = \frac{4}{3}\frac{k_B}{\mu m} + 4aT^3V = \frac{3}{2}c_V^0\left(\frac{8}{3}aT^3V\frac{\mu m}{k_B+1}\right) \tag{74}$$

Introducing the definition of β, Eq. 74 becomes:

$$c_V = c_V^0\frac{1}{\beta}\left(\beta\cdot\frac{8}{3}aT^3V\frac{\mu m}{k_B} + \beta\right) = \frac{c_V^0}{\beta}\left(\frac{k_BT}{k_BT+\frac{1}{3}aT^4\mu mV}\cdot\frac{8}{3}aT^3V\frac{\mu m}{k_B} + \beta\right) = \tag{75}$$

$$= \frac{c_V^0}{\beta}\left(\frac{\frac{8}{3}k_BaT^4V\frac{\mu m}{k_B} + k_BT + 7k_BT - 7k_BT}{k_BT + \frac{1}{3}aT`4\mu mV}\right) =$$

$$= \frac{c_V^0}{\beta}\left(\frac{8\left(\frac{1}{3}aT^4V\mu m + k_BT\right) - 7k_BT}{k_BT + \frac{1}{3}aT^4\mu mV}\right),$$

which can be simplified to:

$$\boxed{c_V = \frac{c_V^0}{\beta}\left(8 - 7\beta\right).} \tag{76}$$

We can now find the specific heat at constant pressure from the following relationship:

$$c_P - c_V \;=\; -T\left[\frac{\left(\frac{\partial V}{\partial T}\right)_P^2}{\left(\frac{\partial V}{\partial P}\right)_T}\right] \;=\; -T\left[\frac{\left(\frac{\partial P}{\partial T}\right)_V^2}{\left(\frac{\partial P}{\partial V}\right)_T}\right]. \tag{77}$$

The right-hand side partial derivatives can be calculated using Eq. 73, Eq. 22, and Eq. 23, which are written again below:

$$\left(\frac{\partial P}{\partial T}\right)_V = \frac{k_B}{\mu mV} + \frac{4}{3}aT^3,$$

$$\left(\frac{\partial P}{\partial V}\right)_T = -\frac{k_BT}{\mu mV^2}$$

$$c_V^0 = \frac{3}{2}\frac{k_B}{\mu m}$$

Rewrite the partials of P as a function of c_V^0:

$$\left(\frac{\partial P}{\partial T}\right)_V = \frac{3}{2}\frac{c_V^0}{V} + \frac{4}{3}aT^3 = \frac{1}{6}\left(9\frac{c_v^0}{V} + 8aT^3\right) \tag{78}$$

$$\left(\frac{\partial P}{\partial V}\right)_T = -\frac{3}{2}c_V^0\frac{T}{V^2} \tag{79}$$

Then, Eq. 77 becomes:

$$c_P - c_V = \cancel{-}T\frac{\left[\frac{c_V^0}{\cancel{V}}\left(\frac{8-6\beta}{\beta}\right)\right]^2}{\cancel{-}c_V^0\,\frac{2}{3}\,\frac{\cancel{T}}{\cancel{V^2}}} = \frac{3}{2}c_V^0\frac{1}{9\beta^2}\left(64 + 36\beta^2 - 96\beta\right) = \frac{c_V^0}{\beta^2}\left(\frac{64}{6} + 6\beta^2 - 16\beta\right) \tag{80}$$

Substituing the coefficient c_V by the expression found in Eq. 76 and moving it to the right-hand side yields:

$$
\begin{aligned}
c_P &= \frac{c_V^0}{\beta^2}\left(\frac{64}{6} + 6\beta^2 - 16\beta\right) + \frac{c_V^0}{\beta}\,(8 - 7\beta) = \\
&= \frac{c_V^0}{\beta^2}\left(\frac{32}{3} + 6\beta^2 - 16\beta + 8\beta - 7\beta^2\right)
\end{aligned} \tag{81}
$$

which can be rewritten as:

$$
\boxed{c_P = \frac{c_V^0}{\beta^2}\left(\frac{32}{3} - 8\beta - \beta^2\right).} \tag{82}
$$

As a summary, any adiabatic process filled with a mixture of gas and radiation has the following specific heats at constant pressure and volume:

$$
\boxed{\begin{array}{rcl}
c_V &=& \frac{c_V^0}{\beta}\,(8 - 7\beta) \\
c_P &=& \frac{c_V^0}{\beta^2}\left(\frac{32}{3} - 8\beta - \beta^2\right)
\end{array}} \tag{83}
$$

Once again, we have two limiting cases. If $\beta \to 1$ (perfect gas case), the coefficients are

$$
\begin{array}{rcl}
c_{V_g} &=& c_V^0\,(8 - 7) = c_V^0 = \frac{3}{2}\frac{k_B}{\mu m} \\
c_{P_g} &=& c_V^0\left(\frac{32}{3} - 8 - 1\right) = \frac{5}{3}c_V^0 = \frac{5}{3}\cdot\frac{3}{2}\frac{k_B}{\mu m} = \frac{5}{2}\frac{k_B}{\mu m}
\end{array}, \tag{84}
$$

as expected. If we take the extreme case of $\beta = 0$, c_V and c_P tends to infinity due to the β term in the denominator.

References

[1] Horedt, G. P., "*Polytropes: Applications in Astrophysics and Related Fields,*" Astrophysics and Space Science Library, Volume 306.

[2] Jackiewicz, J., *"Stellar Structure and Evolution,"* Department of Astronomy, New Mexico State University, Fall 2017.