Stellar and Planetary Astrophysics

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Determine the equation of state of a mixture of classical ideal gas and radiation (pressure, internal energy, specific heats, and adiabatic exponents).

Some Definitions

Let us begin this problem by introducing the adiabatic exponents Γ_1, Γ_2, and Γ_3. For any adiabatic process, that is, one that occurs at constant entropy and thus $dQ = TdS = 0$, we have:

$$\Gamma_1 \equiv -\left(\frac{\partial \ln P}{\partial \ln V}\right)_s \tag{1}$$

$$\frac{\Gamma_2}{\Gamma_2 - 1} \equiv \left(\frac{\partial \ln P}{\partial \ln T}\right)_s \tag{2}$$

$$1 - \Gamma_3 = \left(\frac{\partial \ln T}{\partial \ln V}\right)_s \tag{3}$$

Here the letter "s" indicates adiabatic change. From these definitions, note that there are actually only **2 independent** coefficients:

$$\Gamma_1(\Gamma_2 - 1) = \Gamma_2(\Gamma_3 - 1) \tag{4}$$

To calculate these parameters, we will calculate the total pressure P and internal energy U of the mixture. Given that this mixture is made of gas and radiation, the total pressure will consist of both a gas pressure term and a blackbody radiation pressure term. These two components are given by Eq. 5 and Eq. 6 respectively:

$$P_{gas} = \frac{\rho k_B T}{\mu m} = \frac{k_B T}{\mu m} \cdot \frac{1}{V_{spec}} \tag{5}$$

$$P_{rad} = \frac{1}{3} a T^4 \tag{6}$$

where k_B is the Boltzmann constant, a is the radiation constant,[1] T is the temperature of the mixture, μ is the mean molecular weight, $m \equiv m_u = 1/N_A$ is a unity of atomic gass, ρ is the gas density, and $V \equiv V_{specific} = 1/\rho$ is the specific volume. Adding the two terms yields a total pressure of:

$$\boxed{P = P_{gas} + P_{rad} = \frac{\rho k_B T}{\mu m} + \frac{1}{3} a T^4 = \frac{k_B T}{\mu m V_{spec}} + \frac{1}{3} a T^4} \tag{7}$$

[1] $a = \frac{8\pi^5 k^4}{15 c^3 h^3} = \frac{4\sigma}{c} = 7.565 \cdot 10^{-15}$ erg cm$^{-3} K^{-4}$..

(!) While gas pressure is most important in low mass stars, radiation pressure is the term that contributes the most in high mass stars.[2] With regard to the mixture's total specific internal energy, we can express the internal energy of the gas and radiation as:

$$U_{rad} = \varepsilon_r V = aT^4 V \tag{8}$$

$$U_{gas} = \frac{3}{2}\frac{k_B T}{\mu m} \tag{9}$$

Therefore,

$$\boxed{U = U_{gas} + U_{rad} = \frac{3}{2}\frac{k_B T}{\mu m} + aT^4 V} \tag{10}$$

(!) As we can see, the total internal energy U depends on volume and temperature, i.e. $U(T, V)$.

The First Law of Thermodynamics

(!) A quasistatic process happens infinitely slowly, so any system that undergoes such thermodynamic process remains close to equilibrium at all times. Any reversible process (i.e. a process whose direction can be reversed with no net increase in entropy) is also quasistatic. The 1st Law of Thermodynamics for quasistatic changes can be written as:

$$dU = dQ + dW = dQ - PdV \rightarrow \boxed{dQ = dU + PdV,} \tag{11}$$

where dQ is the amount of heat supplied to the system. We can rewrite the above by considering an infinitessimal change in the internal energy,

$$dU = \left(\frac{\partial U}{\partial T}\right)_V dT + \left(\frac{\partial U}{\partial V}\right)_T dV \tag{12}$$

Substituting Eq. 12 into Eq. 11, we obtain:

$$dQ = \left[\left(\frac{\partial U}{\partial T}\right)_V dT + \left(\frac{\partial U}{\partial V}\right)_T dV\right] + PdV = 0. \tag{13}$$

(!) Note that for adiabatic conditions, $dQ = 0$. We can now compute the terms that involve the internal energy dU. In particular,

$$\left(\frac{\partial U}{\partial T}\right)_V = \frac{3}{2}\frac{k_B}{\mu m} + 4aT^3 V \tag{14}$$

$$\left(\frac{\partial U}{\partial V}\right)_T = aT^4 \tag{15}$$

Plug Eq. 14 and Eq. 15 into Eq. 13:

$$dQ = \left[\left(\frac{3}{2}\frac{k_B}{\mu m} + 4aT^3 V\right) dT + \left(aT^4\right) dV\right] + \left[\frac{\rho k_B T}{\mu m} + \frac{1}{3}aT^4\right] dV = 0 \tag{16}$$

[2] http://jila.colorado.edu/~pja/astr3730/lecture16.pdf, Date of Access: 12-12-2017.

$$= \left(\frac{3}{2} \frac{k_B}{\mu m} + 4aT^3 V \right) dT + \left(\frac{\rho k_B T}{\mu m} + \frac{4}{3} aT^4 \right) dV = 0 \tag{17}$$

Multiply the first term by V/T:

$$= \left(\frac{3}{2} \frac{k_B}{\mu m} \frac{T}{V} + 4aT^4 V \frac{T}{V} \right) \frac{V}{T} dT + \left(\frac{\rho k_B T}{\mu m} + \frac{4}{3} aT^4 \right) dV = 0 \tag{18}$$

$$= \left(\frac{3}{2} \frac{k_B}{\mu m} \frac{T}{V} + 4aT^4 \right) \frac{V}{T} dT + \left(\frac{\rho k_B T}{\mu m} + \frac{4}{3} aT^4 \right) dV = 0 \tag{19}$$

Hence,

$$\boxed{dQ = \left(\frac{3}{2} P_{gas} + 12 P_{rad} \right) \frac{dT}{T} + \left(P_{gas} + 4P_{rad} \right) \frac{dV}{V} = 0.} \tag{20}$$

In addition to dQ, we will also need the term dP later on to calculate the various adiabatic exponents. From Eq. 7, we know that:

$$P(V,T) = P_{gas} + P_{rad} = \frac{k_B T}{\mu m V_{spec}} + \frac{1}{3} aT^4$$

Then, a small increase in pressure will correspond to:

$$dP = \left(\frac{\partial P}{\partial T} \right)_V dT + \left(\frac{\partial P}{\partial V} \right)_T dV, \tag{21}$$

where

$$\left(\frac{\partial P}{\partial T} \right)_V = \frac{k_B}{\mu m V_{spec}} + \frac{4}{3} aT^3 \tag{22}$$

$$\left(\frac{\partial P}{\partial V} \right)_T = -\frac{k_B T}{\mu m V_{spec}^2} \tag{23}$$

Plugging Eq. 22 and Eq. 23 into Eq. 21, we obtain:

$$dP = \left(\frac{k_B}{\mu m V_{spec}} + \frac{4}{3} aT^3 \right) dT + \left(-\frac{k_B T}{\mu m V_{spec}^2} \right) dV \tag{24}$$

Divide the dT term by T and readjust first parenthesis as necessary. As for the 2nd term, take out a power of V outside the parenthesis. Then,

$$dP = \left(\frac{k_B}{\mu m} \frac{T}{V_{spec}} + \frac{4}{3} aT^4 \right) \frac{dT}{T} + \left(-\frac{k_B}{\mu m} \frac{T}{V_{spec}} \right) \frac{dV}{V} = 0 \tag{25}$$

or simply

$$\boxed{dP = \left(P_{gas} + 4P_{rad} \right) \frac{dT}{T} - P_{gas} \frac{dV}{V}} \tag{26}$$

(!) In the adiabatic case, pressure depends only on temperature ($PV^\gamma = \text{const}$, $P^{1-\gamma}T^\gamma = \text{const}$, i.e. $P(T)$). Bringing back the definition of Γ_1 and Γ_2, namely

$$\Gamma_1 \equiv -\left(\frac{\partial \ln P}{\partial \ln V}\right)_s , \qquad \frac{\Gamma_2}{\Gamma_2 - 1} \equiv \left(\frac{\partial \ln P}{\partial \ln T}\right)_s$$

we can set dP equal to $\partial P/\partial T$ or $\partial P/\partial V$ and relate the resulting expressions to the adiabatic exponents. In particular,

$$\text{For T:} \quad dP = \left(\frac{\partial P}{\partial T}\right)_{ad} dT = \frac{\frac{P}{P}}{\frac{T}{T}}\left(\frac{\partial P}{\partial T}\right)_{ad} dT = \frac{P}{T}\left(\frac{\frac{\partial P}{P}}{\frac{\partial T}{T}}\right)_{ad} dT = \frac{P}{T}\left(\frac{\partial \ln P}{\partial \ln T}\right)_{ad} dT \tag{27}$$

$$\boxed{dP = \frac{P}{T}\left(\frac{\Gamma_2}{\Gamma_2 - 1}\right) dT} \tag{28}$$

$$\text{For V:} \quad dP = \left(\frac{\partial P}{\partial V}\right)_{ad} dV = \frac{\frac{P}{P}}{\frac{V}{V}}\left(\frac{\partial P}{\partial V}\right)_{ad} dV = \frac{P}{V}\left(\frac{\frac{\partial P}{P}}{\frac{\partial V}{V}}\right)_{ad} dV = \frac{P}{V}\left(\frac{\partial \ln P}{\partial \ln V}\right)_{ad} dV \tag{29}$$

$$\boxed{dP = -\frac{P}{V}\left(\Gamma_1\right) dV} \tag{30}$$

To summarize, we can express dP in various ways:

$$\boxed{\begin{array}{rcl} dP & = & \left(\frac{\partial P}{\partial T}\right)_V dT + \left(\frac{\partial P}{\partial V}\right)_T dV \\ & = & \left(\frac{\partial P}{\partial T}\right)_{ad} dT = \frac{P}{T}\left(\frac{\partial \ln P}{\partial \ln T}\right)_{ad} dT \\ & = & \frac{P}{T}\left(\frac{\Gamma_2}{\Gamma_2 - 1}\right) dT = -\frac{P}{V}\left(\Gamma_1\right) dV \end{array}} \tag{31}$$

In the next sections, we will use the equalities involving Γ_2 and Γ_1 to solve for Γ_2 and Γ_1 respectively.

Calculating Γ_2

To find Γ_2, we will use the left-hand side equality in Eq. 31:

$$dP = \left(\frac{\partial P}{\partial T}\right)_V dT + \left(\frac{\partial P}{\partial V}\right)_T dV = \frac{P}{T}\left(\frac{\Gamma_2}{\Gamma_2 - 1}\right) dT \tag{32}$$

Plugging Eq. 22 and Eq. 23, the above becomes:

$$dP = \left(\frac{k_B}{\mu m V_{spec}} + \frac{4}{3} a T^3\right) dT - \left(\frac{k_B T}{\mu m V_{spec}^2}\right) dV = \frac{P}{T}\left(\frac{\Gamma_2}{\Gamma_2 - 1}\right) dT \tag{33}$$

$$\left(P_{gas} + 4P_{rad}\right)\frac{dT}{T} - P_{gas}\frac{dV}{V} = P\left(\frac{\Gamma_2}{\Gamma_2 - 1}\right)\frac{dT}{T} = 0 \tag{34}$$

Grouping in terms of the same multipliers:

$$\left[P_{gas} + 4P_{rad} - P\left(\frac{\Gamma_2}{\Gamma_2 - 1}\right)\right]\frac{dT}{T} - P_{gas}\frac{dV}{V} = 0 \tag{35}$$

Given that $P = P_{gas} + P_{rad}$, Eq. 35 becomes:

$$\boxed{\left[P_{gas} + 4P_{rad} - \left(\frac{\Gamma_2}{\Gamma_2 - 1}\right)(P_{gas} + P_{rad})\right]\frac{dT}{T} - P_{gas}\frac{dV}{V} = 0} \tag{36}$$

With Eq. 20 and Eq. 36, we are now ready to write the system of equations that we will use to calculate the Γ_2 adiabatic coefficient:

$$\begin{cases} dQ = \left(\frac{3}{2}P_{gas} + 12P_{rad}\right)\frac{dT}{T} + (P_{gas} + 4P_{rad})\frac{dV}{V} & = 0 \\ \left[P_{gas} + 4P_{rad} - \left(\frac{\Gamma_2}{\Gamma_2 - 1}\right)(P_{gas} + P_{rad})\right]\frac{dT}{T} - P_{gas}\frac{dV}{V} & = 0 \end{cases} \tag{37}$$

Any system of the form

$$\begin{cases} Ax + By & = 0 \\ Cx + Dy & = 0 \end{cases}$$

can be solved with the simpler expression: $\frac{A}{C} = \frac{B}{D}$. Consequently,

$$\begin{array}{rcc} A & = & \frac{3}{2}P_{gas} + 12P_{rad} \\ B & = & P_{gas} + 4P_{rad} \\ C & = & P_{gas} + 4P_{rad} - \left(\frac{\Gamma_2}{\Gamma_2 - 1}\right)(P_{gas} + P_{rad}) \\ D & = & -P_{gas} \end{array} \tag{38}$$

Then,

$$\frac{A}{C} = \frac{B}{D} \rightarrow \frac{\frac{3}{2}P_{gas} + 12P_{rad}}{P_{gas} + 4P_{rad} - \left(\frac{\Gamma_2}{\Gamma_2 1}\right)(P_{gas} + P_{rad})} = \frac{P_{gas} + 4P_{rad}}{-P_{gas}} \tag{39}$$

Taking

$$\boxed{\begin{array}{rcc} P_{gas} & = & \beta P \\ P_{rad} & = & (1 - \beta)P \end{array},} \tag{40}$$

Substituting the above into Eq. 39 and eliminating the pressure terms, we obtain:

$$\frac{\frac{3}{2}\beta\cancel{P} + 12(1-\beta)\cancel{P}}{\beta\cancel{P} + 4(1-\beta)\cancel{P} - \left(\frac{\Gamma_2}{\Gamma_2 - 1}\right)\cancel{P}} = \frac{\beta\cancel{P} + 4(1-\beta)\cancel{P}}{-\beta\cancel{P}} \tag{41}$$

$$\frac{\frac{3}{2}\beta + 12 - 12\beta}{\beta + 4 - 4\beta - \left(\frac{\Gamma_2}{\Gamma_2 - 1}\right)} = \frac{\beta + 4 - 4\beta}{-\beta} \tag{42}$$

$$\frac{\left(12 - \frac{21}{2}\beta\right)}{-3\beta + 4 - \left(\frac{\Gamma_2}{\Gamma_2 - 1}\right)} = \frac{-3\beta + 4}{-\beta} \tag{43}$$

$$\left(12 - \frac{21}{2}\right)(-\beta) = (-3\beta + 4)\left(-3\beta + 4 - \left(\frac{\Gamma_2}{\Gamma_2 - 1}\right)\right) = \tag{44}$$

$$= 9\beta^2 - 12\beta + 3\beta\left(\frac{\Gamma_2}{\Gamma_2 - 1}\right) - 12\beta + 16 - 4\left(\frac{\Gamma_2}{\Gamma_2 - 1}\right)$$

$$-12\beta + \frac{21}{2}\beta^2 = 9\beta^2 - 24\beta + 16 + 3\beta\left(\frac{\Gamma_2}{\Gamma_2 - 1}\right) - 4\left(\frac{\Gamma_2}{\Gamma_2 - 1}\right) \tag{45}$$

$$-12\beta + 24\beta + \frac{21}{2}\beta^2 - 9\beta^2 - 16 = (3\beta - 4)\left(\frac{\Gamma_2}{\Gamma_2 - 1}\right)$$

$$12\beta + \left(\frac{21}{2} - \frac{18}{2}\right)\beta^2 - 16 = 12\beta + \frac{3}{2}\beta^2 - 16 = (3\beta - 4)\left(\frac{\Gamma_2}{\Gamma_2 - 1}\right) \tag{46}$$

Hence, the Γ_2 term is equal to:

$$\frac{\Gamma_2}{\Gamma_2 - 1} = \frac{12\beta + \frac{3}{2}\beta^2 - 16}{3\beta - 4} \rightarrow \frac{\Gamma_2 - 1}{\Gamma_2} = \frac{3\beta - 4}{12\beta + \frac{3}{2}\beta^2 - 16} \tag{47}$$

and

$$1 - \frac{1}{\Gamma_2} = \frac{3\beta - 4}{12\beta + \frac{3}{2}\beta^2 - 16} \tag{48}$$

$$\frac{1}{\Gamma_2} = 1 - \frac{3\beta - 4}{12\beta + \frac{3}{2}\beta^2 - 16} = \frac{\left(12\beta + \frac{3}{2}\beta^2 - 16\right) - (3\beta - 4)}{12\beta + \frac{3}{2}\beta^2 - 16} = \frac{9\beta + \frac{3}{2}\beta^2 - 12}{12\beta + \frac{3}{2}\beta^2 - 16} \tag{49}$$

Finally, we find

$$\Gamma_2 = \frac{12\beta + \frac{3}{2}\beta^2 - 16}{9\beta + \frac{3}{2}\beta^2 - 12} \quad \overset{\cdot 2}{=} \quad \boxed{\Gamma_2 = \frac{32 - 24\beta - 3\beta^2}{24 - 18\beta - 3\beta^2}.} \tag{50}$$

Calculating Γ_1

The adiabatic coefficient Γ_1 can be found using the right-hand side equality in Eq. 31 which relates it to dP, that is:

$$dP = \left(\frac{\partial P}{\partial T}\right)_V dT + \left(\frac{\partial P}{\partial V}\right)_T dV \rightarrow \boxed{dP = -\frac{P}{V}\left(\Gamma_1\right) dV.} \tag{51}$$

Substituting the dP term by Eq. 26, we obtain:

$$\left(P_{gas} + 4P_{rad}\right)\frac{dT}{T} - P_{gas}\frac{dV}{V} = -\frac{P}{V}\left(\Gamma_1\right) dV \tag{52}$$

$$\left(P_{gas} + 4P_{rad}\right)\frac{dT}{T} + \left[P\Gamma_1 - P_{gas}\right]\frac{dV}{V} = 0 \tag{53}$$

Our new system of equations will incorporate Eq. 53 (for pressure) and Eq. 20 (for Q):

$$\begin{cases} dQ = \left(\frac{3}{2}P_{gas} + 12P_{rad}\right)\frac{dT}{T} + \left(P_{gas} + 4P_{rad}\right)\frac{dV}{V} & = 0 \\ \left(P_{gas} + 4P_{rad}\right)\frac{dT}{T} + \left[P\Gamma_1 - P_{gas}\right]\frac{dV}{V} & = 0 \end{cases} \tag{54}$$

We can expand Eq. 53 as:

$$0 = \left(P_{gas} + 4P_{rad}\right)\frac{dT}{T} + \left[P\Gamma_1 - P_{gas}\right]\frac{dV}{V} \tag{55}$$

$$= \left(P_{gas} + 4P_{rad}\right)\frac{dT}{T} + \left[\left(P_{gas} + P_{rad}\right)\Gamma_1 - P_{gas}\right]\frac{dV}{V}$$

Thus,

$$0 = (P_{gas} + 4P_{rad})\,\frac{dT}{T} + [P\Gamma_1 - P_{gas}]\,\frac{dV}{V} = (P_{gas} + 4P_{rad})\frac{dT}{T} + [P_{gas}\Gamma_1 + P_{rad}\Gamma_1 - P_{gas}]\,\frac{dV}{V}. \quad (56)$$

Once again, solve the system of equations by means of the expression: $\frac{A}{C} = \frac{B}{D}$. In this case,

$$\begin{aligned} A &= \tfrac{3}{2}P_{gas} + 12P_{rad} \\ B &= P_{gas} + 4P_{rad} \\ C &= P_{gas} + 4P_{rad} \\ D &= P_{gas}\Gamma_1 + P_{rad}\Gamma_1 - P_{gas} \end{aligned}$$

so

$$\frac{\frac{3}{2}P_{gas} + 12P_{rad}}{P_{gas} + 4P_{rad}} = \frac{P_{gas} + 4P_{rad}}{P_{gas}\Gamma_1 + P_{rad}\Gamma_1 - P_{gas}} \quad (57)$$

With the two β expressions shown in Eq. 40, we can rewrite the above as:

$$\frac{\frac{3}{2}\left(\beta \cancel{P}\right) + 12\,(1-\beta)\,\cancel{P}}{\beta\cancel{P} + 4(1-\beta)\cancel{P}} = \frac{\beta\cancel{P} + 4(1-\beta)\cancel{P}}{\beta\cancel{P}\Gamma_1 + (1-\beta)\Gamma_1\cancel{P} - \beta\cancel{P}} \quad (58)$$

Eliminating the pressure terms as earlier, we obtain:

$$\frac{\frac{3}{2}\beta + 12(1-\beta)}{\beta + 4 - 4\beta} = \frac{\beta + 4 - 4\beta}{\beta\Gamma_1 + (1-\beta)\Gamma_1 - \beta} \rightarrow \frac{\frac{3}{2}\beta + 12(1-\beta)}{4 - 3\beta} = \frac{4 - 3\beta}{\beta\Gamma_1 + (1-\beta)\Gamma_1 - \beta}, \quad (59)$$

$$\frac{3}{2}\beta + 12 - 12\beta = \frac{(4-3\beta)^2}{\beta\Gamma_1 + \Gamma_1 - \beta\Gamma_1 - \beta} \rightarrow \Gamma_1 - \beta = \frac{16 - 24\beta + 9\beta^2}{\frac{3}{2}\beta + 12 - 12\beta}, \quad (60)$$

Isolate Γ_1:

$$\Gamma_1 = \beta + \frac{9\beta^2 - 24\beta + 16}{\frac{3}{2}\beta + 12 - \frac{24}{2}\beta} = \beta + \frac{9\beta^2 - 24\beta + 16}{-\frac{21}{2}\beta + 12} = \quad (61)$$

$$= \frac{\beta\left(-\frac{21}{2}\beta + 12\right) + 9\beta^2 - 24\beta + 16}{-\frac{21}{2}\beta + 12} = \frac{-\frac{21}{2}\beta^2 + 12\beta + 9\beta^2 - 24\beta + 16}{\frac{-21}{2}\beta + 12},$$

$$= \frac{-21\beta^2 + 24\beta + 18\beta^2 - 48\beta + 32}{-21\beta + 24},$$

and finally,

$$\boxed{\Gamma_1 = \frac{-3\beta^2 - 24\beta + 32}{24 - 21\beta}.} \quad (62)$$

Calculate Γ_3

To conclude, we will compute the Γ_3 coefficient. To this end, we will use Eq. 4, that is: $\Gamma_1(\Gamma_2 - 1) = \Gamma_2(\Gamma_3 - 1)$. Rearranging the terms, we find that

$$\Gamma_3 - 1 = \frac{\Gamma_1\,(\Gamma_2 - 1)}{\Gamma_2} = \Gamma_1 - \frac{\Gamma_1}{\Gamma_2}. \quad (63)$$

First, calculate the ratio Γ_1/Γ_2 with Eq. 50 (Γ_2) and Eq. 62 (Γ_1):

$$\frac{\Gamma_1}{\Gamma_2} = \frac{\frac{-3\beta^2-24\beta+32}{24-21\beta}}{\frac{32-24\beta-3\beta^2}{24-18\beta-3\beta^2}} = \frac{-3\beta^2-24\beta+32}{24-21\beta} \tag{64}$$

Now, substract the above from Γ_1:

$$\Gamma_1 - \frac{\Gamma_1}{\Gamma_2} = \frac{-3\beta^2-24\beta+32}{24-21\beta} - \frac{-3\beta^2-24\beta+32}{24-21\beta} = \frac{-\cancel{3\beta^2}-24\beta+32+\cancel{3\beta^2}+18\beta-24}{24-21\beta}, \tag{65}$$

and simplifying, we get:

$$\Gamma_1 - \frac{\Gamma_1}{\Gamma_2} = \frac{-6\beta+8}{24-21\beta} \tag{66}$$

Calculate Γ_3 using the above result:

$$\Gamma_3 = 1 + \frac{-6\beta+8}{24-21\beta} = \frac{24-21\beta-6\beta+8}{24-21\beta} = \frac{-27\beta+32}{24-21\beta} \rightarrow \boxed{\Gamma_3 = \frac{32-27\beta}{24-21\beta}.} \tag{67}$$

Summary of Results

Putting together Eq. 62, Eq. 50 and Eq. 67, we have found the following expressions for the adiabatic coefficients:

$$\boxed{\begin{array}{lcc} \Gamma_1 & = & \frac{-3\beta^2-24\beta+32}{24-21\beta} \\ \Gamma_2 & = & \frac{32-24\beta-3\beta^2}{24-18\beta-3\beta^2} \\ \Gamma_3 & = & \frac{32-27\beta}{24-21\beta} \end{array}} \tag{68}$$

For a gas, $\beta \rightarrow 1$ (perfect gas case), so:

$$\begin{array}{lcl} \Gamma_{1_g} & = & \frac{-3-24+32}{24-21} = \frac{5}{3} \\ \Gamma_{2_g} & = & \frac{32-24-3}{24-18-3} = \frac{5}{3} \\ \Gamma_{3_g} & = & \frac{32-27\beta}{24-21\beta} = \frac{5}{3} \end{array} \rightarrow \Gamma_{1_g} = \Gamma_{2_g} = \Gamma_{3_g} \tag{69}$$

and $\Gamma_1 = \Gamma_2 = \Gamma_3 = 5/3 = \gamma$. However, for $\beta \rightarrow 0$ (sample filled with black body radiation), we have:

$$\begin{array}{lcl} \Gamma_{1_r} & = & \frac{32}{24} = \frac{4}{3} \\ \Gamma_{2_r} & = & \frac{32}{24} = \frac{4}{3} \\ \Gamma_{3_r} & = & \frac{32}{24} = \frac{4}{3} \end{array} \rightarrow \Gamma_{1_r} = \Gamma_{2_r} = \Gamma_{3_r} \tag{70}$$

and $\Gamma_{1_r} = \Gamma_{2_r} = \Gamma_{3_r} = 4/3$. These two evidences can be clearly seen if we play with the coefficients and β (Python notebook):

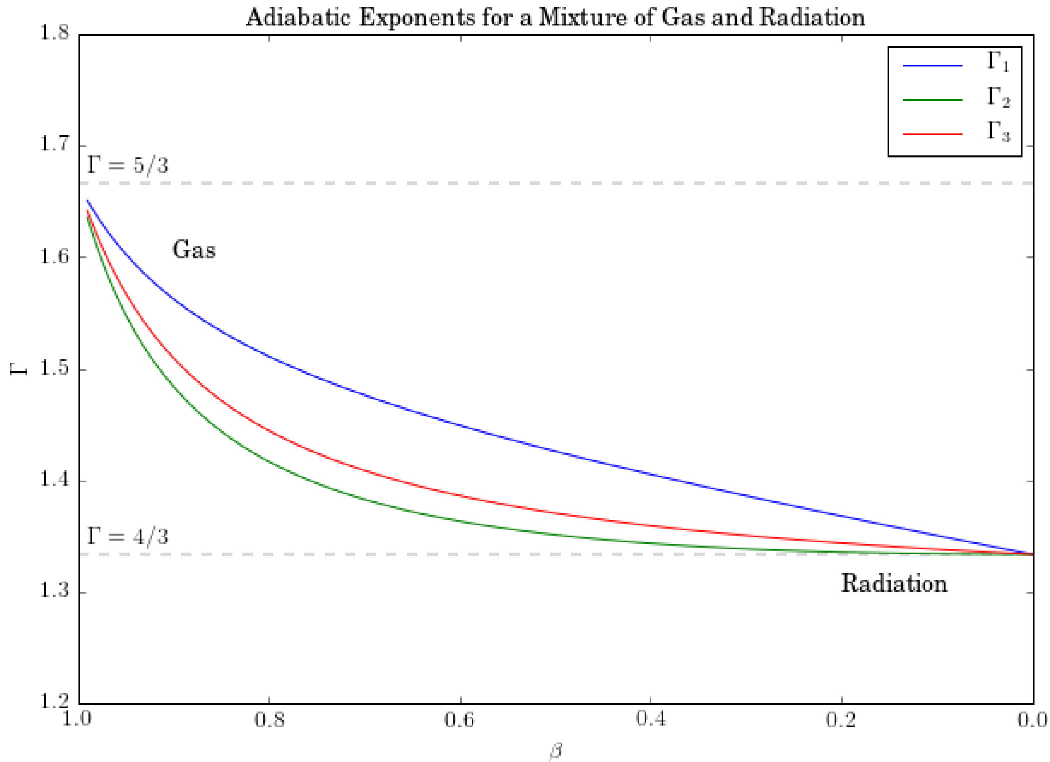

Figure 1: Adiabatic coefficients for a mixture of gas and radiation.

(!) At a low temperature, gas pressure dominates, while at a high temperature radiation pressure dominates.[3] In terms of mass, we have already mentioned how gas pressure is most dominant in low mass stars, whereas radiation pressure is most important in high mass stars. The two contributions are equal when we have P_r.

Calculating the Specific Heat Coefficients

(!) The specific heat of a substance, or heat capacity, is the amount of energy needed to change the temperature of 1 kg of the substance by 1°C. For an adiabatic process (i.e. $dQ = 0$), the specific heat at constant volume, denoted by c_V, is defined as

$$c_V = \left(\frac{\partial Q}{\partial T}\right)_V = \left(\frac{\partial U}{\partial T}\right)_V . \tag{71}$$

Using the expression of the total specific energy given by Eq. 10, we obtain:

$$\boxed{c_V = \left(\frac{\partial U}{\partial T}\right)_V = \frac{4}{3}\frac{k_B}{\mu m} + 4aT^3V} \tag{72}$$

[3]http://www.astro.princeton.edu/~gk/A403/state.pdf, Date of Access: 13-12-2017.

(!) Now, define the quantity

$$\boxed{c_V^0 = \frac{3}{2}\frac{k_B}{\mu m},} \tag{73}$$

(!) which is the specific heat at constant volume for an ideal gas. Then,

$$c_V = \frac{4}{3}\frac{k_B}{\mu m} + 4aT^3V = \frac{3}{2}c_V^0\left(\frac{8}{3}aT^3V\frac{\mu m}{k_B+1}\right) \tag{74}$$

Introducing the definition of β, Eq. 74 becomes:

$$c_V = c_V^0\frac{1}{\beta}\left(\beta\cdot\frac{8}{3}aT^3V\frac{\mu m}{k_B}+\beta\right) = \frac{c_V^0}{\beta}\left(\frac{k_BT}{k_BT+\frac{1}{3}aT^4\mu mV}\cdot\frac{8}{3}aT^3V\frac{\mu m}{k_B}+\beta\right) = \tag{75}$$

$$= \frac{c_V^0}{\beta}\left(\frac{\frac{8}{3}k_BaT^4V\frac{\mu m}{k_B}+k_BT+7k_BT-7k_BT}{k_BT+\frac{1}{3}aT`4\mu mV}\right) =$$

$$= \frac{c_V^0}{\beta}\left(\frac{8\left(\frac{1}{3}aT^4V\mu m+k_BT\right)-7k_BT}{k_BT+\frac{1}{3}aT^4\mu mV}\right),$$

which can be simplified to:

$$\boxed{c_V = \frac{c_V^0}{\beta}\left(8-7\beta\right).} \tag{76}$$

We can now find the specific heat at constant pressure from the following relationship:

$$c_P - c_V = -T\left[\frac{\left(\frac{\partial V}{\partial T}\right)_P^2}{\left(\frac{\partial V}{\partial P}\right)_T}\right] = -T\left[\frac{\left(\frac{\partial P}{\partial T}\right)_V^2}{\left(\frac{\partial P}{\partial V}\right)_T}\right]. \tag{77}$$

The right-hand side partial derivatives can be calculated using Eq. 73, Eq. 22, and Eq. 23, which are written again below:

$$\left(\frac{\partial P}{\partial T}\right)_V = \frac{k_B}{\mu mV} + \frac{4}{3}aT^3,$$

$$\left(\frac{\partial P}{\partial V}\right)_T = -\frac{k_BT}{\mu mV^2}$$

$$c_V^0 = \frac{3}{2}\frac{k_B}{\mu m}$$

Rewrite the partials of P as a function of c_V^0:

$$\left(\frac{\partial P}{\partial T}\right)_V = \frac{3}{2}\frac{c_V^0}{V} + \frac{4}{3}aT^3 = \frac{1}{6}\left(9\frac{c_v^0}{V}+8aT^3\right) \tag{78}$$

$$\left(\frac{\partial P}{\partial V}\right)_T = -\frac{3}{2}c_V^0\frac{T}{V^2} \tag{79}$$

Then, Eq. 77 becomes:

$$c_P - c_V = \cancel{\rho} T \frac{\left[\frac{c_V^0}{\cancel{V}} \left(\frac{8-6\beta}{\beta} \right) \right]^2}{\cancel{\rho} c_V^0 \frac{2}{3} \frac{\cancel{T}}{\cancel{V^2}}} = \frac{3}{2} c_V^0 \frac{1}{9\beta^2} \left(64 + 36\beta^2 - 96\beta \right) = \frac{c_V^0}{\beta^2} \left(\frac{64}{6} + 6\beta^2 - 16\beta \right) \tag{80}$$

Substituing the coefficient c_V by the expression found in Eq. 76 and moving it to the right-hand side yields:

$$\begin{aligned} c_P &= \frac{c_V^0}{\beta^2} \left(\frac{64}{6} + 6\beta^2 - 16\beta \right) + \frac{c_V^0}{\beta} (8 - 7\beta) = \\ &= \frac{c_V^0}{\beta^2} \left(\frac{32}{3} + 6\beta^2 - 16\beta + 8\beta - 7\beta^2 \right) \end{aligned} \tag{81}$$

which can be rewritten as:

$$\boxed{c_P = \frac{c_V^0}{\beta^2} \left(\frac{32}{3} - 8\beta - \beta^2 \right).} \tag{82}$$

(!) As a summary, any adiabatic process filled with a mixture of gas and radiation has the following specific heats at constant pressure and volume:

$$\boxed{\begin{array}{lcc} c_V &=& \frac{c_V^0}{\beta} (8 - 7\beta) \\ c_P &=& \frac{c_V^0}{\beta^2} \left(\frac{32}{3} - 8\beta - \beta^2 \right) \end{array}} \tag{83}$$

(!) Once again, we have two limiting cases. If $\beta \to 1$ (perfect gas case), the coefficients are

$$\begin{array}{lcc} c_{V_g} &=& c_V^0 (8 - 7) = c_V^0 = \frac{3}{2} \frac{k_B}{\mu m} \\ c_{P_g} &=& c_V^0 \left(\frac{32}{3} - 8 - 1 \right) = \frac{5}{3} c_V^0 = \frac{5}{3} \cdot \frac{3}{2} \frac{k_B}{\mu m} = \frac{5}{2} \frac{k_B}{\mu m} \end{array} , \tag{84}$$

(!) as expected. If we take the extreme case of $\beta = 0$, the β terms in the denominator make the expressions of c_V and c_P indefinite.