

# Chapter 1

## Example problem: SUPG-stabilised solution of the 2D advection diffusion equation

In this example we discuss the SUPG-stabilised solution of the 2D advection-diffusion problem

### Two-dimensional advection-diffusion problem in a rectangular domain

Solve

$$\text{Pe} \sum_{i=1}^2 w_i(x_1, x_2) \frac{\partial u}{\partial x_i} = \sum_{i=1}^2 \frac{\partial^2 u}{\partial x_i^2} + f(x_1, x_2), \quad (1)$$

in the rectangular domain  $D = \{(x_1, x_2) \in [0, 1] \times [0, 2]\}$ , with Dirichlet boundary conditions

$$u|_{\partial D} = u_0, \quad (2)$$

where the *Peclet number*, Pe the boundary values,  $u_0$ , the source function  $f(x_1, x_2)$ , and the components of the "wind"  $w_i(x_1, x_2)$  ( $i = 1, 2$ ) are given.

We set  $f(x_1, x_2) = 0$  and assign the boundary conditions such that

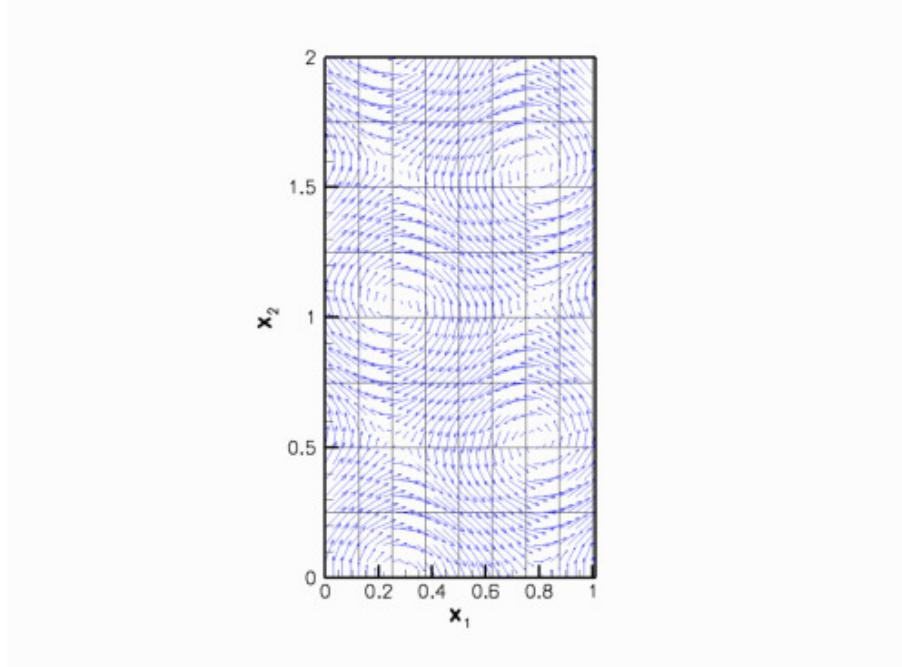
$$u_0(x_1, x_2) = \tanh(1 - \alpha(x_1 \tan \Phi - x_2)), \quad (3)$$

For large values of  $\alpha$ , this boundary data approaches a step, oriented at an angle  $\Phi$  against the  $x_1$ -axis.

In the computations we will impose the "wind"

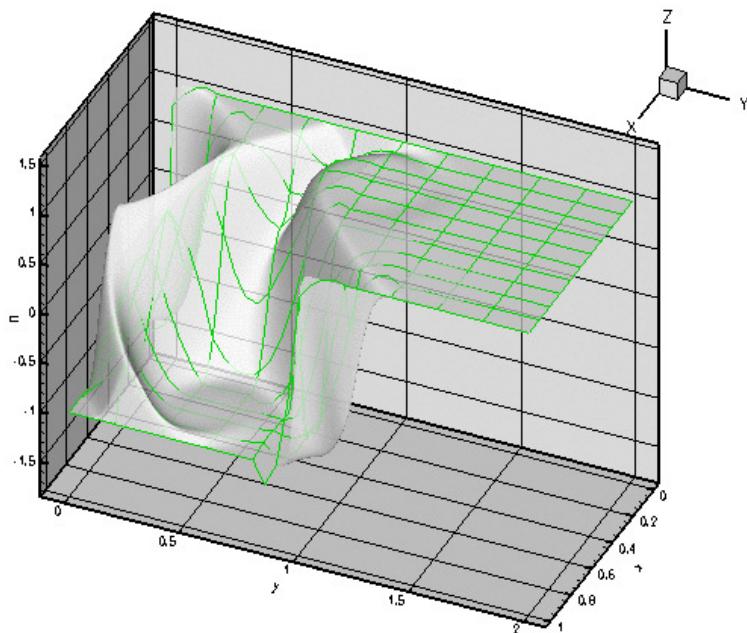
$$\mathbf{w}(x_1, x_2) = \begin{pmatrix} \sin(6x_2) \\ \cos(6x_1) \end{pmatrix}, \quad (4)$$

illustrated in this vector plot:

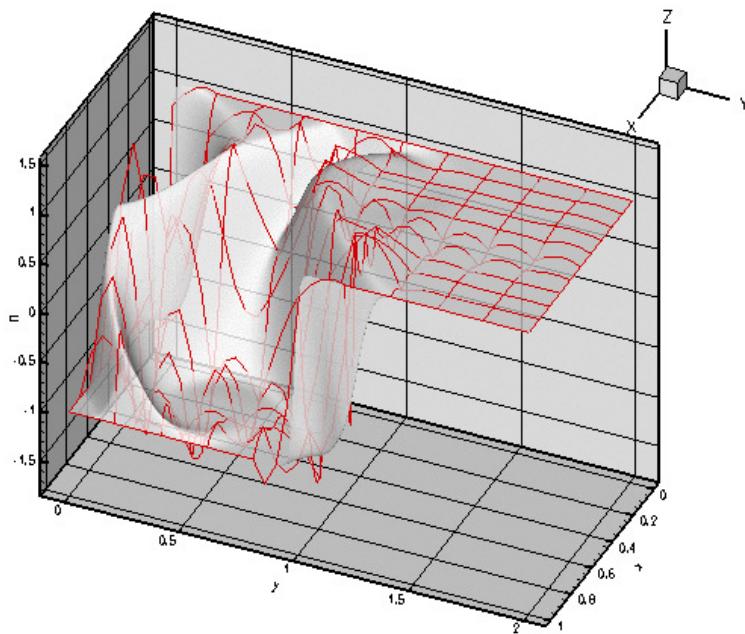


**Figure 1.1 Plot of the wind.**

The figures below show plots of the solution for  $\Phi = 45^\circ$ ,  $\alpha = 50$  and a Peclet number of  $Pe = 200$ , with and without SUPG stabilisation. The wire-mesh plot shows the solution computed on a  $10 \times 10$  mesh, the shaded surface represents the solution obtained from an unstabilised solution on a  $150 \times 150$  mesh. Note how SUPG stabilisation "suppresses the wiggles" on the relatively coarse mesh.



**Figure 1.2 Plot of the SUPG-stabilised solution at different levels of mesh refinement.**



**Figure 1.3 Plot of the unstabilised solution at different levels of mesh refinement.**

## 1.1 The driver code

Overall, the structure of the driver code is very similar to that used for the [problem without stabilisation](#).

### 1.1.1 To be written:

- Discuss SUPG theory.
- Implementation and the role of basis, shape and test functions (our equations are isoparametric)

Until we get around to completing this example, here's the driver code. Fairly self-explanatory, isn't it?

```
//LIC// =====
//LIC// This file forms part of oomph-lib, the object-oriented,
//LIC// multi-physics finite-element library, available
//LIC// at http://www.oomph-lib.org.
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//LIC// Foundation, Inc., 51 Franklin Street, Fifth Floor, Boston, MA
//LIC// 02110-1301 USA.
//LIC//
//LIC// The authors may be contacted at oomph-lib@maths.man.ac.uk.
//LIC// =====
//Driver for a simple 2D adv diff problem with SUPG stabilisation

//Generic routines
#include "generic.h"

// The Poisson equations
#include "advection_diffusion.h"

// The mesh
```

```

#include "meshes/rectangular_quadmesh.h"

using namespace std;
using namespace oomph;

//=====start_of_namespace=====
/// Namespace for global parameters: Unforced problem with
/// boundary values corresponding to a steep tanh step profile
/// oriented at 45 degrees across the domain.
//=====
namespace GlobalPhysicalParameters
{
    /// Peclet number
    double Peclet=200.0;

    /// Parameter for steepness of step in boundary values
    double Alpha=50.0;

    /// Parameter for angle of step in boundary values: 45 degrees
    double TanPhi=1.0;

    /// Some "solution" for assignment of boundary values
    void get_boundary_values(const Vector<double>& x, Vector<double>& u)
    {
        u[0]=tanh(1.0-Alpha*(TanPhi*x[0]-x[1]));
    }

    /// Zero source function
    void source_function(const Vector<double>& x_vect, double& source)
    {
        source=0.0;
    }

    /// Wind
    void wind_function(const Vector<double>& x, Vector<double>& wind)
    {
        wind[0]=sin(6.0*x[1]);
        wind[1]=cos(6.0*x[0]);
    }
} // end of namespace

//===== start_of_problem_class=====
/// 2D AdvectionDiffusion problem on rectangular domain, discretised
/// with refineable 2D QAdvectionDiffusion elements. The specific type
/// of element is specified via the template parameter.
//=====
template<class ELEMENT>
class SUPGAdvectionDiffusionProblem : public Problem
{
public:
    /// Constructor: Pass pointer to source and wind functions, and
    /// flag to indicate if stabilisation is to be used.
    SUPGAdvectionDiffusionProblem(
        AdvectionDiffusionEquations<2>::AdvectionDiffusionSourceFctPt source_fct_pt,
        AdvectionDiffusionEquations<2>::AdvectionDiffusionWindFctPt wind_fct_pt,
        const bool& use_stabilisation);

    /// Destructor. Empty
    ~SUPGAdvectionDiffusionProblem() {}

    /// Update the problem specs before solve: Reset boundary conditions
    /// to the values from the tanh solution and compute stabilisation
    /// parameter.
    void actions_before_newton_solve();

    /// Update the problem after solve (empty)
    void actions_after_newton_solve() {}

    /// Doc the solution.
    void doc_solution();

    /// Overloaded version of the problem's access function to
    /// the mesh. Recasts the pointer to the base Mesh object to
    /// the actual mesh type.
    RectangularQuadMesh<ELEMENT>* mesh_pt()
    {
}

```

```

    return dynamic_cast<RectangularQuadMesh<ELEMENT>*>(
        Problem::mesh_pt());
}

private:
    /// DocInfo object
    DocInfo Doc_info;

    /// Pointer to source function
    AdvectionDiffusionEquations<2>::AdvectionDiffusionSourceFctPt Source_fct_pt;

    /// Pointer to wind function
    AdvectionDiffusionEquations<2>::AdvectionDiffusionWindFctPt Wind_fct_pt;

    /// Flag to indicate if stabilisation is to be used
    bool Use_stabilisation;
};

// end of problem class

=====start_of_constructor=====
/// Constructor for AdvectionDiffusion problem: Pass pointer to
/// source function and wind functions and flag to indicate
/// if stabilisation is to be used.
=====

template<class ELEMENT>
SUPGAdvectionDiffusionProblem<ELEMENT>::SUPGAdvectionDiffusionProblem(
    AdvectionDiffusionEquations<2>::AdvectionDiffusionSourceFctPt source_fct_pt,
    AdvectionDiffusionEquations<2>::AdvectionDiffusionWindFctPt wind_fct_pt,
    const bool& use_stabilisation)
: Source_fct_pt(source_fct_pt), Wind_fct_pt(wind_fct_pt),
  Use_stabilisation(use_stabilisation)
{
    // Set output directory
    if (use_stabilisation)
    {
        Doc_info.set_directory("RESLT_stabilised");
    }
    else
    {
        Doc_info.set_directory("RESLT_unstabilised");
    }

    // Setup mesh
    // # of elements in x-direction
    unsigned n_x=40;

    // # of elements in y-direction
    unsigned n_y=40;

    // Domain length in x-direction
    double l_x=1.0;

    // Domain length in y-direction
    double l_y=2.0;

    // Build and assign mesh
    Problem::mesh_pt() =
        new RectangularQuadMesh<ELEMENT>(n_x,n_y,l_x,l_y);

    // Set the boundary conditions for this problem: All nodes are
    // free by default -- only need to pin the ones that have Dirichlet
    // conditions here
    unsigned num_bound = mesh_pt()->nboundary();
    for(unsigned ibound=0; ibound<num_bound; ibound++)
    {
        unsigned num_nod= mesh_pt()->nboundary_node(ibound);
        for (unsigned inod=0; inod<num_nod; inod++)
        {
            mesh_pt()->boundary_node_pt(ibound,inod)->pin(0);
        }
    } // end loop over boundaries
    // Complete the build of all elements so they are fully functional

    // Loop over the elements to set up element-specific
    // things that cannot be handled by the (argument-free!) ELEMENT
    // constructor: Pass pointer to source function
    unsigned n_element = mesh_pt()->nelement();
    for(unsigned i=0;i<n_element;i++)
    {
        // Upcast from GeneralisedElement to the present element

```

```

ELEMENT *el_pt = dynamic_cast<ELEMENT*>(mesh_pt () ->element_pt(i));

//Set the source function pointer
el_pt->source_fct_pt() = Source_fct_pt;

//Set the wind function pointer
el_pt->wind_fct_pt() = Wind_fct_pt;

// Set the Peclet number
el_pt->pe_pt() = &GlobalPhysicalParameters::Peclet;
}

// Setup equation numbering scheme
cout << "Number of equations: " << assign_eqn_numbers() << std::endl;

} // end of constructor

//=====================================================================
// Update the problem specs before solve: (Re-)set boundary conditions
//=====================================================================
template<class ELEMENT>
void SUPGAdvectionDiffusionProblem<ELEMENT>::actions_before_newton_solve()
{
    // How many boundaries are there?
    unsigned num_bound = mesh_pt ()->nboundary();
    //Loop over the boundaries
    for(unsigned ibound=0;ibound<num_bound;ibound++)
    {
        // How many nodes are there on this boundary?
        unsigned num_nod=mesh_pt ()->nboundary_node(ibound);

        // Loop over the nodes on boundary
        for (unsigned inod=0;inod<num_nod;inod++)
        {
            // Get pointer to node
            Node* nod_pt=mesh_pt ()->boundary_node_pt(ibound,inod);

            // Extract nodal coordinates from node:
            Vector<double> x(2);
            x[0]=nod_pt->x(0);
            x[1]=nod_pt->x(1);

            // Get boundary value
            Vector<double> u(1);
            GlobalPhysicalParameters::get_boundary_values(x,u);

            // Assign the value to the one (and only) nodal value at this node
            nod_pt->set_value(0,u[0]);
        }
    }

    // Now loop over all elements and set the stabilisation parameter
    unsigned n_element = mesh_pt ()->nelement();
    for(unsigned i=0;i<n_element;i++)
    {
        // Upcast from GeneralisedElement to the present element
        ELEMENT *el_pt = dynamic_cast<ELEMENT*>(mesh_pt ()->element_pt(i));

        // Use stabilisation?
        if (Use_stabilisation)
        {
            //Compute stabilisation parameter
            el_pt->compute_stabilisation_parameter();
        }
        else
        {
            //Compute stabilisation parameter
            el_pt->switch_off_stabilisation();
        }
    }
}

} // end of actions before solve

//=====================================================================
// Doc the solution
//=====================================================================
template<class ELEMENT>
void SUPGAdvectionDiffusionProblem<ELEMENT>::doc_solution()
{
    ofstream some_file;

```

```
char filename[100];

// Number of plot points: npts x npts
unsigned npts=5;

// Output solution
//
snprintf(filename, sizeof(filename), "%s/soln%i.dat", Doc_info.directory().c_str(),
Doc_info.number());
some_file.open(filename);
mesh_pt()->output(some_file,npts);
some_file.close();

} // end of doc

===== start_of_main=====
/// Driver code for 2D AdvectionDiffusion problem
=====
int main()
{
    //Set up the problem with stabilisation
    {
        bool use_stabilisation=true;

        // Create the problem with 2D nine-node elements from the
        // QAdvectionDiffusionElement family. Pass pointer to
        // source and wind function.
        SUPGAdvectionDiffusionProblem<QSUPGAdvectionDiffusionElement<2,3> >
        problem(&GlobalPhysicalParameters::source_function,
                &GlobalPhysicalParameters::wind_function,
                use_stabilisation);

        // Solve the problem
        problem.newton_solve();

        //Output the solution
        problem.doc_solution();
    }

    //Set up the problem without stabilisation
    {
        bool use_stabilisation=false;

        // Create the problem with 2D nine-node elements from the
        // QAdvectionDiffusionElement family. Pass pointer to
        // source and wind function.
        SUPGAdvectionDiffusionProblem<QSUPGAdvectionDiffusionElement<2,3> >
        problem(&GlobalPhysicalParameters::source_function,
                &GlobalPhysicalParameters::wind_function,
                use_stabilisation);

        // Solve the problem
        problem.newton_solve();

        //Output the solution
        problem.doc_solution();
    }
}

} // end of main
```

---

## 1.2 Source files for this tutorial

- The source files for this tutorial are located in the directory:

`demo_drivers/advection_diffusion/two_d_adv_diff_SUPG/`

- The driver code is:

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```
demo_drivers/advection_diffusion/two_d_adv_diff_SUPG/two_d_adv_diff_←  
SUPG.cc
```

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### 1.3 PDF file

A [pdf version](#) of this document is available. \