Real Analysis

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Contents

Let's first talk about why we need analysis in general in the first place. Algebra allows us to define certain algebraic structures, which are essentially sets with operations. These operations are defined to have a finite number of arguments. For example, let's take a look at the negation $x \mapsto -x$ and the addition $x, y \mapsto x + y$ operations in a group G. We can compose these operations up to any finite length n, removing the parentheses due to associativity, but note that the "sum" below is not a single operation. It is a composition of $n - 1$ operations.

$$x_1 + x_2 + \ldots + x_n \in G \tag{1}$$

$$-(-(\ldots(-x))) \in G \tag{2}$$

This is still well defined due to closure, but what if we wanted to do this an infinite number of times?

$$x_1 + x_2 + \ldots =? \tag{3}$$

$$\ldots(-(-x)) =? \tag{4}$$

For someone who has learned about sequences and series in high school, this may not be a big jump in logic, but it is. The objects above are not even well-defined and trying to define them with algebraic tools is equivalent to the famous Zeno's paradox. So we simply need to add more tools in order to define these new mathematical objects, which we call *series*. To define series, we need to first define sequences. Can we do this with algebra? Yes, since we can simply model it as a function.

Definition 0.1 (Sequence)

A sequence is a function $f : \mathbb{N} \to X$. We usually denote a sequence by writing out the first few terms of the sequence, followed by an ellipsis.

$$a_1 = f(1), a_2 = f(2), \ldots \tag{5}$$

or as an indexed set over the naturals $\{a_i\}_{i \in \mathbb{N}}$.

Therefore, we can consider series as a sequence of finite sums, each element which is well-defined.

$$x_1, x_1 + x_2, x_1 + x_2 + x_3, \ldots \tag{6}$$

For any $n \in \mathbb{N}$, we can get the value of $a_n = \sum_{i=1}^{n} a_i$, but can we say something about the limiting behavior of a_n? That is, maybe we can just slap a value x onto this series such that it doesn't "break" any of the rules we have in the finite sense. Unfortunately, it is not possible to define such values for all series, but it is possible for some of them, which we call *convergent series.* To rigorously determine which ones are convergent and which ones are not, we need the tools of topology and analysis. Defining the concept of sequences that model infinitely composed operations is what allows us to define differentiation and integration.

Great, we've motivated the need for analysis, but before jumping straight into real analysis, let's talk about what analysis in general works with. It studies functions of the form $f : X \to Y$, and minimally both X, Y must be *Banach spaces*, i.e. complete normed vector spaces over some field $\mathbb{F}$. Almost all flavors of analysis, including real ($\mathbb{R}$), complex ($\mathbb{C}$), multivariate ($\mathbb{R}^n$), p-adic, and functional (infinite-dimensional Banach spaces) analysis require *at least* a Banach space structure. Why are Banach spaces so great? Well if we were to define convergence in X or Y, then it only makes sense to talk about convergence with respect to a topology. So X, Y must at least be topological spaces. It would also be bad if we were to take a sequence in X and find out that it converges to some element outside of X. Therefore, we want a notion of *completeness* in the sense that all sequences that "get closer," i.e. Cauchy sequences, actually converge in X. Unfortunately, while convergence of sequences is preserved under homeomorphisms (and is thus a topological property), convergence of Cauchy sequences is not.[1] Furthermore, the notion of uniform convergence is a metric space property, not a topological one. Therefore, the concept of distances is crucial to the construction of analysis. As for the norm, I'm still not sure why we need this.[2]

[1]Consider the sequence $a_n = 1/(n + 1)$ in $(0, 1)$ and the map $f(x) = 1/x$ to the set $(1, +\infty)$. a_n is Cauchy but $f(a_n)$ is not.

[2]Aspinwall and Ng told me this, but I'm not sure why. The Frechet derivative seems like it can be purely defined with a metric.

But in college courses such as real and complex analysis, why do we say we work over the *fields* $\mathbb{R}$ and $\mathbb{C}$ rather than the Banach spaces $\mathbb{R}$ and $\mathbb{C}$? This is because of the following theorem.

Theorem 0.1 ()

Every field $\mathbb{F}$ is a 1-dimensional vector space over itself.

Therefore, when we talk about the *field* $\mathbb{R}$, we are really treating it as a vector space $\mathbb{R}$ over the field $\mathbb{R}$.[3] Every other structure beyond this is a "bonus" property that gives us extra tools to prove stronger properties. The most notable is the total ordering on $\mathbb{R}$, which allows us to define upper/lower bounds and other real-analysis specific theorems like the intermediate value theorem or the mean value theorem. Other structures include the inner product or the measure.

Now that we've taken in the big picture, for each type of analysis, we should construct the underlying relevant Banach space. At the very least, we can with the tools of set theory and algebra define the rationals $\mathbb{Q}$ as an ordered field over the quotient space $\mathbb{Z} \times \mathbb{Z}/\sim$. Furthermore, $\mathbb{Q}$ itself is a normed vector space (over $\mathbb{Q}$)[4] and the only thing we need now is completeness.

1. If the norm on $\mathbb{Q}$ is defined as the normal absolute value (Euclidean norm), completing it gives $\mathbb{R}$ as an ordered field which also has a compatible order as that of $\mathbb{Q}$. We study functions mapping to and from $\mathbb{R}$ with *single-variable real analysis*.

2. If we take the *p-adic* norm, then completing it with respect to this gives the *p-adic numbers*, which also forms a field but loses the ordering. We deal with functions over the p-adics with *p-adic analysis*.

3. We can construct $\mathbb{C}$ by taking $\mathbb{R}^2$ and endowing it with a bit more structure. We get *complex analysis*.

4. We can construct $\mathbb{R}^n$ and $\mathbb{C}^n$ by easily defining its vector space structure and then endowing it with a norm, and showing that it is complete with respect to the norm-induced metric. This is known as *multivariate analysis*.

5. With all these defined, we can define Banach function spaces like L^p and perform analysis on operators $f : L^p \to L^q$. This is *functional analysis*.

What we have talked about so far was Cauchy completeness, but there is a different type of completeness called *Dedekind completeness*, also equivalently known as the *least-upper-bound (LUB) property*, defined only on ordered sets (with no other structure). It turns out that in an ordered field, the two forms of completeness are equivalent.[5] Therefore, many real analysis textbooks tend to use Dedekind completeness when constructing the reals, but Cauchy completeness is in a sense more "fundamental." We will go through both independent constructions of $\mathbb{R}$ involving both types of completeness since both are used in future theorems.

1. *Construction from Cauchy Sequences.* We verify that $\mathbb{Q}$ is a field and endow it with the standard Euclidean metric $d(x, y) = |x - y|$. We can then construct a new quotient space S of Cauchy sequences in $\mathbb{Q}$, define all the ordered field operations/relations, and finally show that S satisfies Cauchy completeness. Most would end here and claim that this is $\mathbb{R}$, but we must also prove the Archimedean property with this order. Once done, now we can truly claim $S = \mathbb{R}$.

2. *Construction from Dedekind Cuts.* We verify that $\mathbb{Q}$ is a field, put an order on it, and verify that it is an ordered field. We then construct a new set D of *Dedekind cuts* from $\mathbb{Q}$, define the compatible ordered field operations/relations, and show that this new set D satisfies the least-upper bound property. We claim that $D = \mathbb{R}$.

[3] Thanks to Prof. Lenny Ng for clarifying this.

[4] Note that while we define the norm and metric to usually map to $\mathbb{R}^+$, $\mathbb{R}$ isn't even defined yet and so to avoid circular definitions, we define the norm on the rationals to have codomain $\mathbb{Q}$.

[5] Actually, this is not true. Dedekind completeness is equivalent to Cauchy completeness plus the Archimedean property. An example of a Cauchy-complete non Archimidean field is the field F of rational functions over $\mathbb{R}$, with positive cone consisting of those functions f/g such that the leading coefficients of f, g have the same algebraic sign. The Cauchy completion of this into the equivalence classes of Cauchy sequences in F results in a non-Archimedean field.

1 The Real Numbers

By constructing $\mathbb{Q}$ and its topology in my algebra and topology notes, we can talk about convergence. The first question to ask (if you were the first person inventing the reals) is "how do I know that there exists some other numbers at all?" The first clue is trying to find the side length of a square with area 2. As we see, this number is not rational.

Theorem 1.1 ($\sqrt{2}$ is Not Rational)

There exists no $x \in \mathbb{Q}$ s.t. $x^2 = 2$.

Proof.

We can "imagine" that a square with area 2 certainly exists, but the fact that its side length is undefined is certainly unsettling. I don't know about you, but I would try to "invent" $\sqrt{2}$. We can maybe do this in multiple ways.

1. I write out the decimal expansion one by one, which gives our first exposure to sequences.

$$1, 1.4, 1.41, 1.414, \ldots \tag{7}$$

   It is clear that on $\mathbb{Q}$, this sequence does not converge. Our intuition tells that that if the terms get closer and closer to each other, they must be getting closer and closer to *something*, though that something is not in $\mathbb{Q}$. This motivates the definition for *Cauchy completeness*.

2. I would write out maybe some nested intervals so that $\sqrt{2}$ *must* lie within each interval.

$$[1, 2] \supset [1.4, 1.5] \supset [1.41, 1.42] \supset \ldots \tag{8}$$

   This motivates the definition of *nested-interval completeness*.

3. I would define the set of all rationals such that $x^2 < 2$, and try to define $\sqrt{2}$ as the max or supremum of this set. We will quickly find that neither the max nor the supremum exists in $\mathbb{Q}$, and this motivates the definition for *Dedekind completeness*.

All three of these methods points at the same intuition that there should not be any "gaps" or "missing points" in the set that we will construct to be $\mathbb{R}$. This contrasts with the rational numbers, whose corresponding number line has a "gap" at each irrational value.

1.1 Dedekind Completeness

Definition 1.1 (Dedekind Cut)

A **Dedekind cut** is a partition of the rationals $\mathbb{Q} = A \sqcup A'$ satisfying the three properties.[a]

1. $A \neq \emptyset$ and $A \neq \mathbb{Q}$.[b]
2. $x < y$ for all $x \in A, y \in A'$.
3. The maximum element of A does not exist in $\mathbb{Q}$.

The minimum of A' may exist in $\mathbb{Q}$, and if it does, the cut is said to be **generated** by $\min A'$.

[a] This can really be defined for any totally ordered set.
[b] By relaxing this property, we can actually complete $\mathbb{Q}$ to the extended real number line.

Note that in $\mathbb{Q}$, there will be two types of cuts:

1. ones that are generated by rational numbers, such as

$$A = \{x \in \mathbb{Q} \mid x < 2/3\}, A' = \{x \in \mathbb{Q} \mid x \geq 2/3\} \tag{9}$$

2. and the ones that are not

$$A = \{x \in \mathbb{Q} \mid x^2 < 2\}, A' = \{x \in \mathbb{Q} \mid x^2 \geq 2\} \tag{10}$$

We can intuitively see that the set of all Dedekind cuts (A, A') will "extend" the rationals into a bigger set. We can then define some operations and an order to construct this into an ordered field, and finally it will have the property that we call "completeness."

Definition 1.2 (Dedekind Completeness)

A totally ordered algebraic field $\mathbb{F}$ is **complete** if every Dedekind cut of $\mathbb{F}$ is generated by an element of $\mathbb{F}$.

Theorem 1.2 ()

$\mathbb{Q}$ is not Dedekind-complete.

Proof.

The counter-example is given above for the cut

$$A = \{x \in \mathbb{Q} \mid x^2 < 2\}, A' = \{x \in \mathbb{Q} \mid x^2 \geq 2\} \tag{11}$$

Now we have the tools to define the reals, giving us the beefy theorem.

Theorem 1.3 (Reals as the Dedekind-Completion of Rationals)

Let $\mathbb{R}$ be the set of all Dedekind cuts (A, A') of $\mathbb{Q}$ of $\mathbb{Q}$. For convenience we can uniquely represent (A, A') with just A since $A' = \mathbb{Q} \setminus A$. By doing this we can intuitively think of a real number as being represented by the set of all smaller rational numbers. Let A, B be two Dedekind cuts. Then, we define the following order and operations.

1. *Order.* $A \leq_{\mathbb{R}} B \iff A \subset B$.
2. *Addition.* $A +_{\mathbb{R}} B \coloneqq \{a +_{\mathbb{Q}} b \mid a \in A, b \in B\}$.
3. *Additive Identity.* $0_{\mathbb{R}} \coloneqq \{x \in \mathbb{Q} \mid x < 0\}$.
4. *Additive Inverse.* $-B \coloneqq \{a - b \mid a < 0, b \in (\mathbb{Q} \setminus B)\}$.
5. *Multiplication.* If $A, B \geq 0$, then we define $A \times_{\mathbb{R}} B \coloneqq \{a \times_{\mathbb{Q}} b \mid a \in A, b \in B, a, b \geq 0\} \cup 0_{\mathbb{R}}$. If A or B is negative, then we use the identity $A \times B = -(A \times_{\mathbb{R}} -B) = -(-A \times_{\mathbb{R}} B) = (-A \times_{\mathbb{R}} -B)$ to convert A, B to both positives and apply the previous definition.
6. *Multiplicative Identity.* $1_{\mathbb{R}} = \{x \in \mathbb{Q} \mid x < 1\}$.
7. *Multiplicative Inverse.* If $B > 0$, $B^{-1} \coloneqq \{a \times_{\mathbb{Q}} b^{-1} \mid a \in 1_{\mathbb{R}}, b \in (\mathbb{Q} \setminus B)\}$. If B is negative, then we compute $B^{-1} = -((-B)^{-1})$ by first converting to a positive number and then applying the definition above.

We claim that $(\mathbb{R}, +_{\mathbb{R}}, \times_{\mathbb{R}}, \leq_{\mathbb{R}})$ is a totally ordered field, and the canonical injection $\iota : \mathbb{Q} \to \mathbb{R}$ defined

$$\iota(q) = \{x \in \mathbb{Q} \mid x < q\} \tag{12}$$

is an ordered field isomorphism. Finally, by construction $\mathbb{R}$ is Dedekind-complete.

Definition 1.3 (Least Upper Bound Property)

A totally ordered algebraic field $\mathbb{F}$ (must it be a field?) is complete if every nonempty set of F having an upper bound must have a least upper bound (supremum) in F.

Theorem 1.4 (LUB is Equivalent to GLB)

A set $(X, \leq)$ has the least upper bound property iff it has the greatest lower bound property.[a]

[a] Every set bounded below has a greatest lower bound.

Proof.

We will prove one direction since the other is the same logic. Let $S \subset X$ be a nonempty set that is bounded below by some $l \in X$. Let $L \subset X$ be the set of all lower bounds of S. Since l exists, it is nonempty. Furthermore, L is bounded above by any element of S. Due to LUB property L has a least upper bound, call it $z = \sup L$. We claim that $z = \inf S$.

1. z is a lower bound of S. Assume that it is not. Then there exists $s \in S$ s.t. $s < z$. But by construction s is an upper bound for L and so z s not the *least* upper bound, a contradiction.
2. z is a *greatest* lower bound. Assume that z is not. Then there exists a $z' \in X$ s.t. $z < z' \leq s$ for all $s \in S$. But since z', z are lower bounds, this means $z, z' \in L$ by definition and $z < z'$ contradicts the fact that z is an upper bound of L.

We are done.

Theorem 1.5 (Dedekind Completeness Equals Least-Upper-Bound Property)

Dedekind completeness is equivalent to the least upper bound property.

Proof.

Definition 1.4 (Archimidean Principle)

An ordered ring $(X, +, \cdot, \leq)$ that embeds the naturals $\mathbb{N}$[a] is said to obey the **Archimedean principle** if given any $x, y \in X$ s.t. $x, y > 0$, there exists an $n \in \mathbb{N}$ s.t. $\iota(n) \cdot x > y$. Usually, we don't care about the canonical injection and write $nx > y$.

[a] as in, there exists an ordered ring homomorphism $\iota : \mathbb{N} \to X$

By the canonical injections $\mathbb{N} \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{R}$, we can talk about whether this set has the Archimedean property. In fact Dedekind completeness does imply it.

Theorem 1.6 ()

$\mathbb{R}$ satisfies the Archimedean principle.

Proof.

Assume that this property doesn't hold. Then for any fixed x, $nx < y$ for all $n \in \mathbb{N}$. Consider the set

$$A = \bigcup_{n \in \mathbb{N}} (-\infty, nx), \quad B = \mathbb{R} \setminus A \tag{13}$$

A by definition is nonempty, and B is nonempty since it contains y. Then, we can show that $a \in A, b \in B \implies a < b$ using proof by contradiction. Assume that there exists $a' \in A, b' \in B$ s.t. $a' > b'$. Since $a' \in A$, there exists a $n' \in \mathbb{N}$ s.t. $a' \in (-\infty, n'x) \iff a' < n'x$. But by transitivity of order, this means $b' < n'x \iff b' \in (-\infty, n'x) \implies b' \in A$.

Going back to the main proof, we see that A is upper bounded by y, and so by the least upper bound property it has a supremum $z = \sup A$.

1. If $z \in A$, then by the induction principle[a] $z + x \in A$, contradicting that z is an upper bound.
2. If $z \notin A$, then by the induction principle[b] $z - x \notin A \implies z - x \in B$. Since every element of B upper bounds A and since $x > 0$, this means that $z - x < z$ is a smaller upper bound of A, contradicting that z is a least upper bound.

Therefore, it must be the case that $nx > y$ for some $n \in \mathbb{N}$.

[a] Note that $\mathbb{N}$ is defined recursively as $1 \in \mathbb{N}$ and if $n \in \mathbb{N}$, then $n + 1 \in \mathbb{N}$.
[b] The contrapositive of the recursive definition of $\mathbb{N}$ is: if $n \notin \mathbb{N}$, then $n - 1 \notin \mathbb{N}$.

1.2 Cauchy Completeness

Definition 1.5 (Cauchy Sequence)

A sequence a_n in a metric space (X, d) is a **Cauchy sequence** if for every $\epsilon > 0$, there exists an N s.t.

$$d(a_i, a_j) < \epsilon \tag{14}$$

for every $i, j > N$. We call this **Cauchy convergence**.

Note that it is not sufficient to say that a sequence is Cauchy by claiming that each term becomes arbitrarily close to the preceding term. That is,

$$\lim_{n \to \infty} |x_{n+1} - x_n| = 0 \tag{15}$$

For example, look at the sequence

$$a_n = \sqrt{n} \implies a_{n+1} - a_n = \frac{1}{\sqrt{n+1} + \sqrt{n}} < \frac{1}{2\sqrt{n}} \tag{16}$$

However, it is clear that a_n gets arbitrarily large, meaning that a finite interval can contain at most a finite number of terms in $\{a_n\}$.

It is trivial that convergence implies Cauchy convergence, but the other direction is not true. Therefore, we would like to work in a space where these two are equivalent, and this is called completeness.

Definition 1.6 (Cauchy Completeness)

A metric space (X, d) is complete if every Cauchy sequence in that space converges to an element in X.

Theorem 1.7 ()

$\mathbb{Q}$ is not Cauchy-complete.

Proof.

Let a_n be the largest number x up to the nth decimal expansion such that x^2 does not exceed 2. The first few terms are

$$1.4, 1.41, 1.414, \ldots \tag{17}$$

Therefore, we can construct the reals as equivalence classes over Cauchy sequences. Rather than using the order, we take advantage of the metric.

Theorem 1.8 (Reals as the Cauchy-Completion of the Rationals)

Let $\mathbb{R}$ be the quotient space of all Cauchy sequences (x_n) of rational numbers with the equivalence relation $(x_n) = (y_n)$ iff their difference tends to 0.[a] That is, for every rational $\epsilon > 0$, there exists an integer N s.t. for all naturals $n > N$, $|x_n - y_n| < \epsilon$.

1. *Order.* $(x_n) \leq_{\mathbb{R}} (y_n)$ iff $x = y$ or there exists $N \in \mathbb{N}$ s.t. $x_n \leq_{\mathbb{Q}} y_n$ for all $n > N$.
2. *Addition.* $(x_n) + (y_n) \coloneqq (x_n + y_n)$.
3. *Additive Identity.* $0_{\mathbb{R}} \coloneqq (0_{\mathbb{Q}})$.
4. *Additive Inverse.* $-(x_n) \coloneqq (-x_n)$.
5. *Multiplication.* $(x_n) \times_{\mathbb{R}} (y_n) = (x_n \times_{\mathbb{Q}} y_n)$.
6. *Multiplicative Identity.* $1_{\mathbb{R}} \coloneqq (1)$.
7. *Multiplicative Inverse.* $(x_n)^{-1} \coloneqq (x_n^{-1})$.

We claim that $(\mathbb{R}, +_{\mathbb{R}}, \times_{\mathbb{R}}, \leq_{\mathbb{R}})$ is a totally ordered field, and the canonical injection $\iota : \mathbb{Q} \to \mathbb{R}$ defined

$$\iota(q) = (q) \tag{18}$$

is an ordered field isomorphism. Finally, by construction $\mathbb{R}$ is Cauchy-complete.

[a] This equivalence class reflects that the same real number can be approximated in many different sequences. In fact, this shows *by definition* that $1, 1, \ldots$ and $0.9, 0.99, 0.999, \ldots$ are the same number!

1.3 Nested Intervals Completeness

The final way we prove is using nested-intervals completeness.

Definition 1.7 (Nested Interval Completeness, Cantor's Intersection Theorem)

Let F be a totally ordered algebraic field. Let $I_n = [a_n, b_n]$ ($a_n < b_n$) be a sequence of closed intervals, and suppose that these intervals are nested in the sense that

$$I_1 \supset I_2 \supset I_3 \supset \ldots$$

where

$$\lim_{n \to +\infty} b_n - a_n = 0$$

F is complete if the intersection of all of these intervals I_n contains exactly one point. That is,

$$\bigcup_{n=1}^{\infty} I_n \in F$$

Note that defining nested intervals requires only an ordered field. One may look at this and try to ask if this is a specific instance of the following conjecture: The intersection of a nested sequence of nonempty closed sets in a topological space has exactly 1 point. This claim may not even make sense, actually. If we define nested in terms of proper subsets, then for a finite topological space a sequence cannot exist since we will run out of open sets and so this claim is vacuously true and false. If we allow $S_n = S_{n+1}$ then we can just select $X \supset X \supset \ldots$, which is obviously not true. However, a slightly weaker claim is that every proper nested non-empty closed sets has a non-empty intersection is a consequence of compactness.

Theorem 1.9 ()

$\mathbb{Q}$ is not nested-interval complete.

Proof.

Consider the intervals $[a_i, b_i]$ where a_i is the largest number x up to the nth decimal expansion such that x^2 does not exceed 2, and b_i is the smallest number x up to the nth decimal expansion such that x^2 is not smaller than 2. The first few terms are

$$[1.4, 1.5], [1.41, 1.42], [1.414, 1.415], \ldots \tag{19}$$

Therefore, we can complete $\mathbb{Q}$. It turns out that this is equivalent to the construction using Dedekind cuts, and by definition this new set is nested interval complete. However, like Cauchy completeness, this actually does not imply the Archimedean property.

1.4 Properties of the Real Line

Now that we have completed it, we can define the real numbers.

Definition 1.8 (The Real Numbers)

The **set of real numbers**, denoted $\mathbb{R}$, is a totally ordered complete Archimedean field.

It seems that the real numbers is *any* set that satisfies the definition above. Does this mean that there are multiple real number lines?

Example 1.1 (Multiple Reals?)

For example, let us construct three distinct sets satisfying these axioms:

1. A line $\mathbb{L}$ with $+$ associated with the translation of $\mathbb{L}$ along itself and $\cdot$ associated with the "stretching/compressing" of the line around the additive origin 0.
2. An uncountable list of numbers with possibly infinite decimal points, known as the decimal number system.
$$\ldots, -2.583\ldots, \ldots, 0, \ldots, 1.2343\ldots, \ldots, \sqrt{2}, \ldots \tag{20}$$
3. A circle with a point removed, with addition and multiplication defined similarly as the line.

We will show that there is only one set, up to isomorphism, that satisfies all these properties.

Theorem 1.10 (Uniqueness)

$\mathbb{R}$ is unique up to field isomorphism. That is, if two individuals construct two ordered complete Archimedean fields $\mathbb{R}_A$ and $\mathbb{R}_B$, then
$$\mathbb{R}_A \simeq \mathbb{R}_B \tag{21}$$

Proof.

The proof is actually much longer than I expected, so I draw a general outline.[a] We want to show how to construct an isomorphism $f : \mathbb{R}_A \to \mathbb{R}_B$.

1. Realize that there are unique embeddings of $\mathbb{N}$ in $\mathbb{R}_A$ and $\mathbb{R}_B$ that preserve the inductive principle, the order, closure of addition, and closure of multiplication, the additive identity, and the multiplicative identity. Call these ordered doubly-monoid (since it's a monoid w.r.t. $+$ and $\times$) homomorphisms ι_A, ι_B.
2. Construct an isomorphism $f_1 : \iota_A(\mathbb{N}) \to \iota_B(\mathbb{N})$ that preserves the inductive principle, order, addition, and multiplication. This is easy to do by just constructing $f_1 = \iota_B \circ \iota_A^{-1}$.
3. Extend f_1 to the ordered ring isomorphism f_2 by explicitly defining what it means to map additive inverses, i.e. negative numbers.

4. Extend f_2 to the ordered field isomorphism f_3 by explicitly defining what it means to map multiplicative inverses, i.e. reciprocals.
5. Extend f_3 to the ordered field isomorphism on the entire domain $\mathbb{R}_A$ and codomain $\mathbb{R}_B$. There is no additional operations that we need to support, but we should explicitly show that this is both injective and surjective, which completes our proof.

[a]Followed from here.

Corollary 1.1 (Dedekind and Cauchy Completness are Equivalent for Reals)

Let $\mathbb{R}_D, \mathbb{R}_C$ be the Dedekind and Cauchy completion of $\mathbb{Q}$. Then $\mathbb{R}_D \simeq \mathbb{R}_C$.[a]

[a]Note that this is only true for totally ordered Archimidean fields! The two completeness properties are not equal in general!

The second new property is that the reals are uncountable.

Theorem 1.11 (Cantor's Diagonalization)

The real numbers are uncountable.

Proof.

We proceed by contradiction. Suppose the real numbers are countable. Then there exists a bijection $f : \mathbb{N} \to \mathbb{R}$. This means we can list all real numbers in $[0, 1]$ as an infinite sequence.[a]

$$\begin{aligned} f(1) &= 0.a_{11}a_{12}a_{13}\ldots \\ f(2) &= 0.a_{21}a_{22}a_{23}\ldots \\ f(3) &= 0.a_{31}a_{32}a_{33}\ldots \\ &\vdots \end{aligned}$$

where each a_{ij} is a digit between 0 and 9.
Now construct a new real number $r = 0.r_1r_2r_3\ldots$ where:

$$r_n = \begin{cases} 1 & \text{if } a_{nn} \neq 1 \\ 2 & \text{if } a_{nn} = 1 \end{cases} \tag{22}$$

This number r is different from $f(n)$ for every $n \in \mathbb{N}$, since r differs from $f(n)$ in the nth decimal place. Therefore $r \in [0, 1]$ but $r \notin \text{range}(f)$, contradicting that f is surjective. Thus our assumption that the real numbers are countable must be false.

[a]This must be explicitly proven, but we can take the set of all Cauchy sequences of rationals in their decimal expansion and construct the reals this way.

Provide examples of ordered, Cauchy-complete fields that are not Archimedean.

1.5 Roots, Exponentials, and Logarithms

Now we will focus on some other operations that become well-defined in the reals. We know that x^n for $n \in \mathbb{N}$ denotes repeated multiplication and x^{-1} denotes the multiplicative inverse. We need to build up on this notation. As a general outline, we will show that x^{-n} is well defined, then $x^q, q \in \mathbb{Q}$ is well-defined, and finally $x^r, r \in \mathbb{R}$ is well-defined. For the naturals, we have defined x^n as the repeated multiplication of n. It is trivial that the canonical injection $\iota_0 : \mathbb{N} \to \mathbb{R}$ commutes with the exponential map of naturals. We prove that $\iota_1 : \mathbb{Z} \to \mathbb{R}$ also commutes.

Lemma 1.1 (Integer Exponents)

We have

1. For $x_1, \ldots, x_n \in \mathbb{R}$, $(x_1 \ldots x_n)^{-1} = x_n^{-1} \ldots x_1^{-1}$.
2. For $x \in \mathbb{R}$, $x > 0$, $(x^n)^{-1} = (x^{-1})^n$. This value is denoted x^{-n}.
3. For $x \in \mathbb{R}$ and $w, z \in \mathbb{Z}$, $x^{w+z} = x^w x^z$.
4. For $w, z \in \mathbb{Z}$, $x^{wz} = (x^z)^w = (x^w)^z$.

Proof.

Listed.

1. The proof is trivial, but for $n = 2$ and $x_1 = x, x_2 = y$, we see that by associativity, $(x^{-1}x^{-1})(xy) = y^{-1}(x^{-1}x)y = y^{-1}y = 1$ and we know inverses are unique.
2. Set $x_i = x$ using (1).
3. If $w, z > 0$ this is trivial by the associative property. If either or both are negative, say $w < 0 < z$, then we set $w' = -w > 0$ and using (2) we know that
$$x^w x^z = (x^{-1})^{w'} x^z = x^{-w'+z} = x^{w+z} \tag{23}$$
by associativity in the second last equality.

Therefore, we have successfully defined x^z for all $z \in \mathbb{Z}$, and if z is negative, we're allowed to "swap" the -1 and $|z|$ in the exponents. Now we want to extend this into rational exponents, first by proving the existence and uniqueness of nth roots for any real. The proof is a little involved, but the general idea is that we want to use the LUB property to define the nth root as the supremum of a set.

Theorem 1.12 (Existence of Nth Roots)

For any real $x > 0$ and every $n \in \mathbb{N}$ there is one and only one positive real $y \in \mathbb{R}$ s.t. $y^n = x$. This is denoted $x^{1/n}$.

Proof.

Let E be the set consisting of all reals $t \in \mathbb{R}$ s.t. $t^n < x$. We show that

1. it is nonempty. Consider $t = x/(1+x)$. Then $0 \leq t < 1 \implies t^n \leq t < x$. Thus $t \in E$ and E is nonempty.
2. it is bounded. Consider any number $s = 1 + x$. Then $s^n \geq s > x$, so $s \notin E$, and $s = 1 + x$ is an upper bound of E.

Therefore, E is a nonempty set that is upper bounded, so it has a least upper bound, called $y = \sup E$. We claim that $y^n = x$, proving by contradiction. For both cases, we use the fact that the identity $b^n - a^n = (b-a)(b^{n-1} + b^{n-2}a + \ldots + a^{n-1})$ gives the inequality
$$b^n - a^n < (b-a)nb^{n-1} \text{ for } 0 < a < b \tag{24}$$

1. Assume $y^n < x$. Then we choose a fixed $0 < h < 1$ s.t.
$$h < \frac{x - y^n}{n(y+1)^{n-1}} \tag{25}$$
Then by putting $a = y, b = y + h$, we have
$$(y+h)^n - y^n < hn(y+h)^{n-1} < hn(y+1)^{n-1} < x - y^n \tag{26}$$
and thus $y^n < (y+h)^n < x$. This means that $y + h \in E$, and so y is not an upper bound.

2. Assume $y^n > x$. Then we set a fixed number

$$k = \frac{y^n - x}{ny^{n-1}} \tag{27}$$

Then $0 < k < y$. If we take any $t \in \mathbb{R}$ s.t. $t \geq y - k$, this implies that $t^n \geq (y-k)^n \implies -t^n \geq -(y-k)^n$, and so

$$y^n - t^n \leq y^n - (y-k)^n < kny^{n-1} = y^n - x \tag{28}$$

Thus $t^n > x$ and $t \notin E$. So it must be the case that $t < y - k$, and so $y - k$ is an upper bound of E, contradicting that y is least.

At this point, rooting has been introduced as sort of an independent map from exponentiation. We show that they have the nice property of commuting.

Lemma 1.2 (Rooting and Exponentiation Commute)

For $p \in \mathbb{Z}, q \in \mathbb{N}$ and $x \in \mathbb{R}$ with $x > 0$, we have

$$(x^p)^{1/q} = (x^{1/q})^p \tag{29}$$

Proof.

If $p > 0$, then let $r = (x^p)^{1/q}$. By definition $r^q = x^p$. Let $s = x^{1/q}$ By definition $s^q = x$. Therefore $r^q = (s^q)^p = s^{qp}$ from the lemma on integer exponents. But since roots are well-defined and unique

$$r = (r^q)^{1/q} = (s^{qp})^{1/q} = s^p \implies (x^p)^{1/q} = (x^{1/q})^p \tag{30}$$

If $p = 0$, this is trivially 0, and if $p < 0$ the by the same logic with $p = -p'$ for $p' > 0$ and $y = x^{-1} > 0$. we know

$$(x^p)^{1/q} = \left((y^{-1})^{-p'}\right)^{1/q} = (y^{-(-p')})^{1/q} = (y^{p'})^{1/q} \tag{31}$$

$$= (y^{1/q})^{p'} = ((x^{-1})^{1/q})^{p'} = (x^{1/q})^{-p'} = (x^{1/q})^p \tag{32}$$

Theorem 1.13 (Rational Exponential Function)

Given $m, p \in \mathbb{Z}$ and $n, q \in \mathbb{N}$, prove that

$$(b^m)^{1/n} = (b^p)1/q \tag{33}$$

Hence it makes sens to define $b^r = (b^m)^{1/n}$, since every element of the equivalence class r of each rational number maps to the same value.

Proof.

Since $m/n = p/q \implies mq = np$,

$$b^{mq} = b^{np} \implies (b^m)^q = (b^p)^n \tag{34}$$

$$\implies b^m = ((b^m)^q)^{1/q} = ((b^p)^n)^{1/q} \tag{35}$$

$$\implies b^m = ((b^p)^{1/q})^n \tag{36}$$

$$\implies (b^m)^{1/n} = (b^p)^{1/q} \tag{37}$$

Therefore we can define for any $r \in \mathbb{Q}$

$$x^r = x^{m/n} = (x^m)^{1/n} = (x^{1/n})^m \tag{38}$$

where the final equality holds from the commutativity of rooting and exponentiation.

It turns out that this is a homomorphism.

Corollary 1.2 (Rational Exponential Function is a Homomorphism)

The rational exponential function is a homomorphism. That is, given $r, s \in \mathbb{Q}$ and $x \in \mathbb{R}$,

$$x^{r+s} = x^r \cdot x^s \tag{39}$$

Proof.

Let $r = m/n, s = p/q$. Then

$$\begin{aligned}
x^{r+s} = x^{m/n+p/q} &= x^{\frac{mq+np}{nq}} && \text{(addition on } \mathbb{Q}) \\
&= (x^{mq+np})^{1/nq} && \text{(exp and roots commute)} \\
&= (x^{mq} + x^{np})^{1/nq} && \text{(int exp lemma)} \\
&= (x^{mq})^{1/nq}(x^{np})^{1/nq} && \text{(int exp lemma)} \\
&= x^{mq/nq}x^{np/nq} && \text{(exp and roots commute)} \\
&= x^{m/n}x^{p/q} && \text{(relation from } \mathbb{Q})
\end{aligned}$$

With rational exponents defined, we can use the least upper bound property to define a consistent extension of a real exponent.

Lemma 1.3 ()

If $r \in \mathbb{Q}$ with $r \geq 0$, then for $x \in \mathbb{R}$, $x > 1$, $1 \leq b^r$.

Proof.

Let $r = m/n$. Then $x^r = x^{m/n} = (x^m)^{1/n}$. Since $1 < x$, and $m \geq 0$, we have

$$1 \leq x \leq x^2 \leq \ldots \leq x^m \implies 1 \leq b^m \tag{40}$$

Now set $y = x^{m/n}$ and assume that $y < 1$. Then

$$x^m = y^n < y^{n-1} < \ldots < y < 1 \tag{41}$$

and so $x^m < 1$, which is a contradiction. So it must be the case that $y > 1$.

Lemma 1.4 (Monotonicity of Rational Exponents)

If $x, y \in \mathbb{R}$, then for any rational $r \in \mathbb{Q}$ with $r < x + y$, there exists a $p, q \in \mathbb{Q}$ s.t. $p < x, q < y$ and $p + q = r$. The converse is true as well.

Proof.

$r < x + y \implies r - y < x$. By density of $\mathbb{Q}$ in $\mathbb{R}$, we can choose $r - y < p < x$. Then $-r + y > -p > x \implies r - r + y > r - p > r - x \implies y > r - p > r - x$, and we set $q = r - p$. We are done. The converse is trivial since given $p, q \in \mathbb{Q}$ with $p < x, q < y$, then by the ordered field properties $p + q < x + y$.

Corollary 1.3 (Real Exponential Function)

Given $x \in \mathbb{R}$, we define

$$B(x) := \{x^q \in \mathbb{R} \mid q \in \mathbb{Q},\ q \leq x\} \tag{42}$$

We claim that given $r \in \mathbb{R}$,

$$x^r := \sup B(r) \tag{43}$$

is well-defined and is a homomorphism extension of the rational exponential function. That is,

$$\sup B(x + y) = \sup B(x) \cdot \sup B(y) \tag{44}$$

Proof.

To show that $x^r := \sup B(r)$ where $B(r) = \{x^t \in \mathbb{R} \mid t \in \mathbb{Q}, t \leq r\}$,

1. We show it's an upper bound. Assume it wasn't. Then $x^r < x^t$ for some $t \in \mathbb{Q}$ satisfying $t \leq r$. But $t \leq r \implies 0 \leq r - t$, and by the previous lemma, $1 \leq x^{r-t}$. So $1 \leq x^{r-t} = x^r x^{-t} = x^r (x^t)^{-1} \implies x^t \leq x^r$, which is a contradiction.
2. We show that it is least. Assume that it is not. Then $\exists r' \in \mathbb{Q}$ s.t. $x^t \leq x^{r'}$ and $r' < r$. Now let $s \in \mathbb{Q}$ be an element between r' and r, which is guaranteed to exist due to density of rationals in reals. But $s < r$, so by definition $x^s \in B(r)$, but

$$0 < s - r' \implies 1 < b^{s-r'} \tag{45}$$
$$\implies b^{r'}(b^{r'})^{-1} < b^s(b^{-r'}) \tag{46}$$
$$\implies 1 < b^s(b^{r'})^{-1} \tag{47}$$
$$\implies b^{r'} < b^s \tag{48}$$

and so $b^{r'}$ is not an upper bound for $B(r)$. By contradiction, b^r is least.

Since this is defined, the analogous definition for real numbers is consistent with that of hte rationals, and it is upper bounded by the Archimedean principle, so such a supremum must exist. Note that t is rational. For the second part, from the previous lemma and the homomorphism properties of the rational exponent,

$$B(x + y) = B'(x + y) := \{b^{p+q} \in \mathbb{R} \mid p, q \in \mathbb{Q}, p \leq x, q \leq y\} \tag{49}$$
$$= \{b^p b^q \in \mathbb{R} \mid p, q \in \mathbb{Q}, p \leq x, q \leq y\} \tag{50}$$
$$\tag{51}$$

Therefore we can treat B and B' as the same set.

1. Prove upper bound $\sup B(x + y) \leq \sup B(x) \sup B(y)$. Given $\alpha \in B'(x + y)$, there exists $p_\alpha, q_\alpha \in \mathbb{Q}$ (with $p_\alpha < x$, $q_\alpha < y$) s.t. $b^{p_\alpha} b^{q_\alpha} = \alpha$. But

$$b^{p_\alpha} b^{q_\alpha} \leq \sup_{p_\alpha}\{b^{p_\alpha}\} \cdot \sup_{q_\alpha}\{b^{q_\alpha}\} = \sup B(x) \sup B(y) \tag{52}$$

2. To prove least, assume there exists $K \in \mathbb{R}$ s.t. $\sup B'(x + y) \leq K < \sup B(x) \sup B(y)$. Then, since the image of b^x is always positive, we assume $0 < K$. We bound its factors as so:

$K < \sup B(x) \sup B(y) \implies K/\sup B(x) < \sup B(y)$. By density of the rationals, there exists a $\beta \in \mathbb{Q}$, s.t.

$$\frac{K}{\sup B(x)} < \beta < \sup B(y) \tag{53}$$

This means $K/\beta < \sup B(x)$ and $\beta < \sup B(y)$. But this means that there exists $\phi, \gamma \in B(x), B(y)$ s.t. $K/\beta < \phi, \beta < \gamma \implies K = (K/\beta) \cdot \beta < \phi\gamma \implies \phi\gamma \in B'(x+y)$ by definition. So K is not an upper bound.

Furthermore, this is an isomorphism, and the inverse is defined. Let's define this analytically.

Theorem 1.14 (Logarithm)

For $b > 1$ and $y > 0$, there is a unique real number x s.t. $b^x = y$. We claim

$$x = \sup\{w \in \mathbb{R} b^w < y\} \tag{54}$$

x is called the **logarithm of y to the base b**.

Proof.

We use the inequality $b^n - 1 \leq n(b-1)$ for all $n \in \mathbb{N}$.[a] By substituting $b = b^{1/n}$ (valid since $b > 1 \iff b^{1/n} > 1$) so $b - 1 \geq n(b^{1/n} - 1)$. Now set some $t > 1$, and by Archimidean principle, we can choose some $n \in \mathbb{N}$ s.t. $n > \frac{b-1}{t-1}$. Then $n(t-1) > b-1$, and with the inequality derived we get

$$n(t-1) > b - 1 \geq n(b^{1/n} - 1) \implies t > b^{1/n} \tag{55}$$

This allows us to prove 2 things.

1. If w satisfies $b^w < y$, then $b^{w+(1/n)} < y$ for sufficiently large n. Setting $t = yb^{-w}$ (which is greater than 1 since $b^w < y$) gives $y \cdot b^{-w} > b^{1/n} \implies b^w b^{1/n} < y \implies b^{w+(1/n)} < y$.
2. If w satisfies $b^w > y$, then $b^{w-(1/n)} > y$ for sufficiently large n. Setting $t = b^w y^{-1}$ (which is greater than 1 since $b^w > y$) gives $b^w y^{-1} > b^{1/n} \implies b^{w-(1/n)} > y$.

Now we can prove existence. Let A the set of all w s.t. $b^w < y$. We claim that $x = \sup A$.

1. Assume that $b^x < y$. We know that there exists $n \in \mathbb{N}$ s.t. $b^{x+(1/n)} < y \implies x + (1/n) \in A$, contradicting that x is an upper bound.
2. Assume that $b^x >$. We know that there exists $n \in \mathbb{N}$ s.t. $b^{x-(1/n)} > y \implies x - (1/n)$ is also an upper bound for A, contradicting that x is least. Therefore $b^x = y$.

We now prove uniqueness. Assume that there are two such x's , call them x, x'. By total ordering and $x \neq x'$, WLOG let $x > x' \implies x - x' > 0 \implies b^{x-x'} > 1$. By density of rationals, since we can choose $r \in \mathbb{R}$ s.t. $0 < r < x - x'$, we have $1 < b^r < b^{x-x'}$ and so $B(r) \subset B(x - x')$. Since $1 < b^{x-x'} \implies 1 \cdot b^{x'} < b^{x-x'} \cdot b^{x'} = b^x$, we have $b^{x'} < b^x$ and they cannot both by y. So $x = x'$.

[a] We prove by induction. For $n = 1$ $b^1 - 1 \leq 1(b-1)$. Assume that this holds for some n. Then $b^{n+1} - 1 = b^{n+1} - b + b - 1 = b(b^n - 1) + (b-1) \geq bn(b-1) + (b-1) = (bn+1)(b-1) \geq (n+1)(b-1)$, where the last step follows since $b \geq 1 \implies bn \geq n \implies bn + 1 \geq n + 1$.

1.6 The Extended Reals and Hyperreals

Great! We have officially constructed the reals, and we can finally feel satisfied about defining metrics, norms, and inner products as mappings to the codomain $\mathbb{R}$. Now let's make the concept of infinite numbers a bit more rigorous. In short, what we do is just add the numbers $\pm\infty$ to $\mathbb{R}$, which we call the extended reals, and try and extend the properties from $\mathbb{R}$ to the extended reals. We will see that not all properties can be transferred.

Theorem 1.15 (Extended Real Number Line)

The **extended real number line** is the set $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$. We define the following operations.

1. *Order.* $-\infty \leq x$ and $x \leq +\infty$ for all $x \in \overline{\mathbb{R}}$.
2. *Addition.* $+\infty - \infty = 0$. $x + \infty = +\infty$ and $x - \infty = -\infty$ for all $x \in \mathbb{R}$.
3. *Multiplication.*

$$x \times \infty = \begin{cases} +\infty & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -\infty & \text{if } x < 0 \end{cases} \tag{56}$$

It turns out that this is still Dedekind-complete, which is nice. Unfortunately we lose a lot of structure.

1. this is not even a field since the multiplicative inverse of $\pm\infty$ is not defined.
2. the Archimedean principle does not hold
3. we cannot define a metric nor a norm.
4. we can define the order topology, however.

The loss of the field property is quite bad, and we might want to recover this. Therefore, we can add more elements that serve to be the multiplicative inverse of infinity. We call these inverses *infinitesimals* and the new set the *hyperreal numbers*.

Theorem 1.16 (Hyperreals)

The **hyperreals**

In fact, when Newton first invented calculus, the hyperreals were what he worked with, and you can surprisingly build a good chunk of calculus with this. Even though this is a dead topic at this point, a lot of modern notation is based off of this number system, so it's good to see how it works. For example, when we write the integral

$$\int f(x)\,dx \tag{57}$$

we are saying that we are taking the uncountable sum of the terms $f(x)\,dx$, the multiplication of the real number $f(x)$ and the infinitesimal number dx living in the hyperreals. Unfortunately, we cannot fully construct a rigorous theory of calculus with only infinitesimals. However, in practice (especially physics) people tend to manipulate and do algebra with infinitesimals, so having a good foundation on what you can and cannot do with them is practical. While the focus won't be on *smooth infinitesimal analysis (SIA)*, I will include some alternate constructions later purely with infinitesimals.

1.7 Euclidean Space

Congratulations! We have rigorously constructed both the reals and complex numbers, and this becomes the cornerstone to construct other fundamental sets. Now we consider spaces of the form $\mathbb{R}^n$ or $\mathbb{C}^n$, which we call *Euclidean spaces*, and construct them. This is actually quite easy since we understand linear algebra.

Definition 1.9 (Convex Sets)

A set S is convex if for every point $x, y \in S$, the point

$$z = tx + (1-t)y \in S \tag{58}$$

where $0 \leq t \leq 1$.

2 Euclidean Topology

With the construction of the real line and the real space, the extra properties of completeness, norm, and order (for the real line) allows us to restate these topological properties in terms of these "higher-order" properties. It also proves much more results than for general topological spaces. Therefore, the next few sections will focus on reiterating the topological properties of $\mathbb{R}$ and $\mathbb{R}^n$ (this can be done slightly more generally for metric spaces, but we talk about this in point-set topology). In this section, we will restate the notion of open sets, limit points, compactness, connectedness, and separability. Then we can continue in the next section sequences and their limits, and after that we describe continuity. Once this is done, we can focus constructing the derivative and integral, which are unique to Banach spaces.

2.1 Open Sets

It is well-known that the set of open-balls of a metric space (X, d) is indeed a topology, which we prove in point-set topology. Once we prove this, we have access to a whole suite of theorems on topological spaces that we can just apply to $\mathbb{R}^n$. We will restate many of these topological theorems for completeness but will not prove them. However, if any of these theorems use any other structure, such as order/metrics/norms/-completeness, we will have to prove them.

Definition 2.1 (Topology)

Let X be a set and $\mathscr{T}$ be a family of subsets of X. Then $\mathscr{T}$ is a **topology** on X[a] if it satisfies the following properties.

1. *Contains Empty and Whole Set*:
$$\emptyset, X \in \mathscr{T} \tag{59}$$
2. *Closure Under Union.* If $\{U_\alpha\}_{\alpha \in A}$ is a class of sets in $\mathscr{T}$, then
$$\bigcup_{\alpha \in A} U_\alpha \in \mathscr{T} \tag{60}$$
3. *Closure Under Finite Intersection*: If $U_1, \ldots, U_n$ is a finite class of sets in $\mathscr{T}$, then
$$\bigcap_{i=1}^{n} U_i \in \mathscr{T} \tag{61}$$

A **topological space** is denoted $(X, \mathscr{T})$.

[a] I will use script letters to denote topologies and capital letters to denote sets.

Theorem 2.1 (Euclidean Topology)

Let $\tau_{\mathbb{R}}$ (which we denote as $\mathscr{T}$) be the set of subsets S of $(\mathbb{R}^n, ||\cdot||)$ satisfying the property that if $x \in S$, then there exists an open ϵ-ball $B(x, \epsilon)$ s.t. $B \subset S$. $\mathscr{T}$ is a topology of $\mathbb{R}^n$.

Proof.

We prove the following three properties.
1. $\emptyset, \mathbb{R}^n$ are open.
2. For any collection $\{G_\alpha\}_\alpha$ of open sets, $\cup_\alpha G_\alpha$ is open.
3. For any finite collection $G_1, \ldots, G_n$ of open sets, $\cap_{i=1}^n G_i$ is open.

Listed.
1. Let $x \in \cup_\alpha G_\alpha$. Then, $x \in G_k$ for some k and since G_k is open, there exists a $B_\epsilon(x) \subset G_k \subset \cup_\alpha G_\alpha$, proving that $\cup_\alpha G_\alpha$ is open.

2. Let $x \in \cap_{i=1}^n G_i$. Then, $x \in G_i$ for every i, and so for each G_i, there exists an $\epsilon_i > 0$ s.t. $B_{\epsilon_i}(x) \subset G_i$. Since the set $\{e_i\}$ is finite, we can take

$$\epsilon = \min_i \{\epsilon_i\}$$

and see that $B_\epsilon(x) \subset G_i$ for all i, which implies that $B_\epsilon(x) \subset \cap_{i=1}^n G_i$. Since we have proved the existence of ϵ, $\cap_{i=1}^n G_i$ is open.

Definition 2.2 (Open Set)

An **open set** is an element of $\mathscr{T}$.

1. An **open neighborhood**, or sometimes just the **neighborhood**, of $x \in \mathbb{R}^n$ is an open set U_x containing x.
2. A **punctured neighborhood** is $U_x^\circ = U_x \setminus \{x\}$.

Theorem 2.2 (Equivalence to Open Ball Topology)

$\mathscr{T}$ is equal to the topology $\mathscr{T}'$ generated by the basis $\mathscr{B}$ of open balls

$$B(x, r) := \{y \in \mathbb{R}^n \mid ||x - y|| < r\} \tag{62}$$

Proof.

Let $\mathscr{T}$ be the Euclidean topology and $\mathscr{T}'$ be the open ball topology.

1. We show $\mathscr{T} \subset \mathscr{T}'$.
2. We show $\mathscr{T}' \subset \mathscr{T}$.

By defining the topology, we have automatically defined a bunch of topological objects and properties. For clarification, we will restate them.

Corollary 2.1 ()

An open ball is an open set.

Proof.

Given $x \in B_r(p)$, we can imagine that x will always have some space between it and the boundary. We want to show that there exists some $\epsilon > 0$ s.t. $B_\epsilon(x) \subset B_r(p)$. That is, given any point $y \in B_\epsilon(x)$, we can show that $y \in B_r(p)$. Since $||x - p|| < r$, there exists some space $0 < r - ||x - p||$. There always exists a real number $0 < \epsilon < r - ||x - p||$, so given $y \in B_\epsilon(x)$, we can bound

$$||y - p|| = ||y - x + x - p|| \leq ||y - x|| + ||x - p|| \leq \epsilon + ||x - p|| < r \tag{63}$$

Example 2.1 ()

Here are some examples of sets which are open and not open.

1. $U = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \neq 1\}$ is open since for every point $x \in U$, we just need to find a radius $\epsilon > 0$ that is smaller than its distance to the unit circle.
2. $(a, b) \times (c, d) \subset \mathbb{R}^2$ is open since given a point x, we can take the minimum of its distance between the two sides of the rectangle and construct an open ball.
3. $S = \{(x, y) \in \mathbb{R}^2 : xy \neq 0\}$ is open since given a point $x \in S$, we can take the minimum of the distance between it and the x and y axes.

4. The set of all complex z such that $|z| \leq 1$ is not open since we cannot construct open balls at the boundary points that are fully contained in the set.
5. The set $S = \{1/n\}_{n \in \mathbb{N}}$ is not open since given any point $x = 1/n$, we can construct an open ball with radius $\epsilon < 1/(n+1)$, which contains irrationals that are not in S.

Definition 2.3 (Interior Point)

A point $p \in S$ is an **interior point** if there exists an neighborhood N of p such that $N \subset S$.

An interior point means that we can always contain the point in S with some “breathing room." By definition an open set is a set where all of its points are interior points. A set is then said to be open if every point has this breathing room. This can be useful when defining differentiation at a point within an open set, since we can always find a neighborhood to take limits on.

Now that we have defined the Euclidean topology, we will prove that the features of topological objects can be reduced to features in $\mathbb{R}^n$.

Theorem 2.3 (Convexity)

An open ball is convex in a normed vector space.

Proof.

The normed part is important here, as the properties of the metric is not sufficient. Given $B_r(p)$, $x, y \in B_r(p)$ implies that $||x-p|| < r$ and $||y-p|| < r$. Therefore,

$$\begin{aligned} ||tx + (1-t)y - p|| &= ||tx - tp + (1-t)y - (1-t)p|| && (64) \\ &\leq t||x-p|| + (1-t)||y-p|| && (65) \\ &= tr + (1-t)r = r && (66) \end{aligned}$$

What happens if we weaken it to a metric?

2.2 Limit Points and Closure

Definition 2.4 (Limit Point)

A point $p \in \mathbb{R}^n$ is a **limit point** of $S \subset \mathbb{R}^n$ if every punctured neighborhood of p has a nontrivial intersection with X.[a] The set of all limit points of S is denoted S'.

[a] The definition just means that if we take a point and draw smaller and smaller circles around it, the circle itself should still overlap with S, no matter how small it gets.

Theorem 2.4 ()

Let $A_1, \ldots, A_n$ be a finite collection of sets. Then

$$\bigcup_{i=1}^n A_i' = \left(\bigcup_{i=1}^n A_i \right)'$$

Proof.

Let the LHS be W and the RHS be V. If $x \in W$, $x \in A_i'$ for some i, and so for all $\epsilon > 0$, there exists a $B_\epsilon^\circ(x)$ s.t.

$$B_\epsilon^\circ(x) \cap A_i \neq \emptyset \implies B_\epsilon^\circ(x) \cap \left(\bigcup_{i=1}^n A_i \right) \neq \emptyset$$

which means that $x \in V$. Now assume that $x \in V$. Then for all $\epsilon > 0$, there exists a $B_\epsilon^\circ(x)$ s.t.

$$B_\epsilon^\circ(x) \cap \left(\bigcup_{i=1}^n A_i \right) \neq \emptyset$$

which implies that $B_\epsilon^\circ(x) \cap A_i \neq \emptyset$ for some i, which means that $x \in A_i' \subset W$.

A closed set can be defined in many equivalent ways for arbitrary topological spaces. The more general proof is done in topology, but we still prove it in the context of analysis.

Definition 2.5 (Closed Set)

A **closed set** $S \in \mathbb{R}^n$ is a set that contains all of its limit points.

Theorem 2.5 (Alternative Definition of Closed Set)

A set S is closed iff S^c is open.

Proof.

We prove both ways:

1. ($\rightarrow$) Given that S is closed, then let $x \in S^c$. x is not a limit point of S since if it were, then it would be in S, and so there exists a punctured open neighborhood $B_\epsilon^\circ(x)$ of x s.t. $S \cap B_\epsilon^\circ(x) = \emptyset$. Since $x \notin S$, we also have $S \cap B_\epsilon(x) = \emptyset$, which implies that $B_\epsilon(x) \subset S^c$. Since for every $x \in S^c$, there exists a $B_\epsilon(x) \subset S^c$, S^c is open.
2. ($\leftarrow$) For simplicity, it suffices to prove if S open, then S^c is closed. Given that S is open, we have for every $x \in S$, there exists $B_\epsilon(x) \subset S$, which implies that $B_\epsilon(x) \cap S^c = \emptyset$. Since there exists an $B_\epsilon(x)$ that does not contain points in S^c, x cannot be a limit point of S^c, and so there exists no limit points of S^c in S. Therefore, all limit points of S^c are in S^c, proving that S^c is closed.

Theorem 2.6 ()

We have the following topological properties:

1. For any collection $\{F_\alpha\}_\alpha$ of closed sets, $\cap_\alpha F_\alpha$ is closed.
2. For any finite collection $F_1, \ldots, F_n$ of open sets, $\cup_{i=1}^n F_i$ is closed.

Proof.

Listed.

1. Let x be a limit point of $\cap_\alpha F_\alpha$, and we want to show that $x \in \cap_\alpha F_\alpha$. By definition of limit points, for every $\epsilon > 0$, we have
$$B_\epsilon(x) \cap \left(\bigcap_\alpha F_\alpha \right)$$
which means that $B_\epsilon(x) \cap F_\alpha \neq \emptyset$ for all α. This means that x is a limit point for every F_α,

and since they are all closed, $x \in F_\alpha$ for all α, which implies that $x \in \cap_\alpha F_\alpha$.

We can intuitively see a few properties about this. First, a finite set S of points does not have any limit points, since if we draw small enough circles around a $p \in S$, then at some point the circle will not contain any more points (remember that we're talking about deleted neighborhoods). Following this, we can deduce that a limit point must always have an infinite number of points close to it, as in no matter how small the circle gets, there are always an infinite number of points contained within that circle. This also means that if p is a limit point, then we can construct a sequence of points in S that converges to p, since every open ball with smaller and smaller radii will still have points in S.

Theorem 2.7 ()

If p is a limit point of S, then every neighborhood of p contains infinitely many points of S. The converse is also true trivially.

Proof.

Assume p is a limit point and that there exists a finite number of points within a deleted neighborhood $B^\circ_r(p)$. Then, we can enumerate them $p_1, p_2, \ldots, p_n$ by their distances to p, with

$$d(p_1, p) \leq d(p_2, p) \leq \ldots \leq d(p_n, p) \tag{67}$$

Since $p_1 \neq p$, we have $d(p_1, p) > 0$ and so, we can choose an $0 < \epsilon < d(p_1, p)$ s.t. $B^\circ_\epsilon(p)$ does not contain any of the p_i's. This neighborhood does not contain any elements of S and so p is not a limit point.

Corollary 2.2 ()

A finite set has no limit points.

Proof.

If S is a finite set, then every neighborhood of every point p in $\mathbb{R}^n$ will have at most finite points, which, by the previous theorem, is not a limit point.

We show a very useful result that will make things much more convenient when proving the following theorems and exercises. This is quite intuitive, since it shows that the limit points of a finite union of sets is the same as the finite union of the limit points of each set. This is clearly not true for infinite unions:

1. Look at the countable set $\mathbb{Q} \subset \mathbb{R}$. Each $\{q\}' = \emptyset$, but $\mathbb{Q}' = \mathbb{R}$.
2. Look at the uncountable set $\mathbb{R}$. Each $\{x \in \mathbb{R}\}' = \emptyset$, but $\mathbb{R}' = \mathbb{R}$.

Now, we give two more definitions for convenience of deriving open and closed sets from any arbitrary set.

Definition 2.6 (Closure)

Given a set S, let the set of all limit points of S be denoted S'. The **closure** of S is the set $\overline{S} = S \cup S'$. It is the smallest closed set that contains S.

Definition 2.7 (Interior)

Given a set S, the **interior** of S is denoted S°, the set of all interior points of S. It is the largest open set that is within S.

Theorem 2.8 ()

Let E be a nonempty set of real numbers which is bounded above. Let $y = \sup E$. Then $y \in \overline{E}$. Hence $y \in E$ if E is closed.

Proof.

Assume that y is not a limit point of E. Then, there exists some $\epsilon > 0$ s.t. $(y - \epsilon, y + \epsilon)$ does not intersect with E. This means that $y - \epsilon$ is an upper bound of E, and so y is not the supremum.

Theorem 2.9 ()

If X is a metric space and $E \subset X$, then

1. $\overline{E}$ is closed.
2. $E = \overline{E}$ if and only if E is closed.
3. $\overline{E} \subset F$ for every closed set $F \subset X$ such that $E \subset F$. That is, if $E \subset F$ closed, then "increasing" the size of E to its closure will not make it greater than F.

Proof.

Listed.

1. Let x be a limit point of $\overline{E}$. Then, for every $\epsilon > 0$, we have $B_\epsilon(x) \cap \overline{E} \neq \emptyset$, which means that either $B_\epsilon(x) \cap E \neq \emptyset$ (in which case $x \in E' \implies x \in \overline{E}$ and we are done) or $B_\epsilon(x) \cap E' \neq \emptyset$. We wish to prove that in the latter case, x being a limit point of E' still implies that x is a limit point of E. Since $B_\epsilon(x) \cap E' \neq \emptyset$, there must exist a $y \in B_\epsilon(x) \cap E'$. Since $y \in E'$, we can construct an open ball $B_\delta(y)$ containing elements of E, and since $B_\epsilon(x)$ is open, we can contain $B_\delta(y)$ entirely within $B_\epsilon(x)$. Therefore,
$$B_\delta(y) \cap E \neq \emptyset \implies B_\epsilon(x) \cap E \neq \emptyset$$
therefore, $x \in E' \implies x \in \overline{E}$.
2. If E is closed, then $E' \subset E \implies \overline{E} = E \cup E' = E$. If $E = \overline{E} = E \cup E'$, then $E' \subset E \implies E$ is closed.
3. Since $E \subset F$, it suffices to prove that $E' \subset F$. Consider a limit point x of E. Then every punctured open neighborhood of x satisfies $B_\epsilon^\circ(x) \cap E \neq \emptyset$. But since $E \subset F$, we have
$$B_\epsilon^\circ(x) \cap F \neq \emptyset$$
and so x is also a limit point of F. But since F is closed, $x \in F$. Therefore, $\overline{E} = E \cup E' \subset F$.

The first two statements (1) and (2) imply the following.

Corollary 2.3 ()

The closure of the closure of E is equal to the closure of E.

Proof.

We know that $\overline{\overline{E}} \supset \overline{E}$, so we must prove that $\overline{\overline{E}} \subset \overline{E}$, which is equivalent to proving that $\overline{E}' \subset \overline{E}$. Let $x \in \overline{E}'$, i.e. is a limit point of $\overline{E}$. Then, for every $\epsilon > 0$, we have $B_\epsilon(x) \cap \overline{E} \neq \emptyset$. Pick a point y from this intersection, and since $B_\epsilon(x)$ is open, we can construct an open ball $B_\delta(y)$ fully contained in $B_\epsilon(x)$. Since $y \in \overline{E}$, y is a limit point of E, which implies

$$B_\delta(y) \cap E \neq \emptyset \implies B_\epsilon(x) \cap E \neq \emptyset \tag{68}$$

and therefore x is a limit point of E, $x \in \overline{E}$.

2.3 Compactness

Definition 2.8 (Open Cover)

An **open cover** of a set E in a metric space X is a collection $\{G_\alpha\}$ of open subsets of X such that $E \subset \cup_\alpha G_\alpha$.

Definition 2.9 (Compact Set)

A subset S of a metric space X is said to be **compact** if every open cover of S contains a finite subcover.

While openness behaves differently depending on its embedding space, compactness stays constant. Therefore, we don't have to worry about talking about which space a compact set is embedded in.

Theorem 2.10 (Compactness is Preserved Under Subspace Topology)

Suppose $K \subseteq Y \subseteq X$. Then K is compact relative to X if and only if K is compact relative to Y.

Proof.

We can prove bidirectionally.

1. Suppose that K is compact in X. Then given any open cover $\{U_\alpha\}_\alpha$ of K, there exists a finite subcover $\{U_i\}_i$. Now let there exist an open cover $\{V_\alpha\}$ in Y, but every $V_\alpha = U_\alpha \cap Y$ for some U_α open in X. Therefore, we can take the finite subcover $\{V_i = V_i \cap Y\}_i$.
2. Suppose that K is compact in Y. Then given any open cover $\{V_\alpha\}$ of K, there exists a finite subcover $\{V_i\}_i$. Now let there exist an open cover $\{U_\alpha\}$ in X. Then we set $\{V_\alpha = U_\alpha \cap Y\}_\alpha$, which has a finite subcover $\{V_i = U_i \cap Y\}$, and therefore we can take $\{U_i\}$ as our finite subcover in X.

Theorem 2.11 ()

A finite union of compact sets is compact.

Proof.

It suffices to prove for two sets A, B by induction. Take an arbitrary cover $\mathscr{L}$ of $A \cup B$. Then $\mathscr{L}$ is a cover of A, so it has a finite subcover $\mathscr{F} \subset \mathscr{L}$. It is also a cover of B, so it has a finite subcover $\mathscr{G} \subset \mathscr{L}$. Therefore, $\mathscr{F} \cup \mathscr{G} \subset \mathscr{L}$ is a cover of $A \cup B$, and since it is the union of finite covers, it is finite.

As we will see in the following theorems, compact sets behave well with closed sets. In fact, compactness is in a form a stronger notion than closedness.

Theorem 2.12 ()

Compact subsets of metric spaces are closed.

Proof.

We would like to show that if A is compact in X, then A^c is open. What we would like to do is if we have some $x \in A^c$, then we must prove that there exists some open set $B_\epsilon(x)$ that is disjoint with A. For every point $a \in A$, we can construct an open balls $V_a = B_{d(x,a)/2}(a)$ and $U_a = B_{d(x,a)/2}(x)$. We know that if $y \in B_{d(x,a)/2}(a)$, then assuming $y \in B_{d(x,a)/2}(x)$ will give

$$d(x,a) \leq d(x,y) + d(y,a) < \frac{d(x,a)}{2} + \frac{d(x,a)}{2} = d(x,a) \tag{69}$$

which is absurd. Since $\{V_a\}_{a \in A}$ forms an open covering of A, then by compactness we can take a finite subcover $V_{a_1}, \ldots, V_{a_n}$, along with the respective neighborhoods of x $U_{a_1}, \ldots, U_{a_n}$. Since we have established

$$V_{a_i} \cap U_{a_i} = \emptyset \implies \bigcap_{i=1}^n V_{a_i} \cap \left(\bigcup_{i=1}^n U_{a_i} \right) = \emptyset \tag{70}$$

and since $\cap_{i=1}^n V_{a_i}$ is open (as it is the intersection of open sets) and disjoint from an open cover of A and hence from A, we have proved that A^c is open, and so A is closed.

Theorem 2.13 ()

Closed subsets of compact sets are compact.

Proof.

Let $C \subset K \subset X$ with K compact and C closed. Then let $\{U_\alpha\}$ be an open cover of C. Then C^c is open in X, and so $\{U_\alpha\} \cup \{C^c\}$ is an open cover of K, so it has a finite subcover S.

1. If $C^c \notin S$, then we have a finite subcover of C.
2. If $C^c \in S$, then we can take the element out to get a finite subcover of C.

Therefore we have constructed a way to make a finite subcover. C is compact.

Corollary 2.4 ()

If F is closed and K is compact, then $F \cap K$ is compact.

The general notion of compactness[6] for topological spaces is not needed for analysis. Rather, we make use of the following theorem which allows us to focus on the compactness of subsets in Euclidean spaces $\mathbb{R}^n$.

Theorem 2.14 (Heine-Borel)

Let $E \subset \mathbb{R}^k$. The following are equivalent.

1. E is closed and bounded
2. E is compact.

[6] According to Terry Tao, a compact set is "small," in the sense that it is easy to deal with. While this may sound counterintuitive at first, since $[0, 1]$ is considered compact while $(0, 1)$, a subset of $[0, 1]$, is considered noncompact. More generally, a set that is compact may be large in area and complicated, but the fact that it is compact means we can interact with it in a finite way using open sets, the building blocks of topology. That finite collection of open sets makes it possible to account for all the points in a set in a finite way. This is easily noticed, since functions defined over compact sets have more controlled behavior than those defined over noncompact sets. Similarly, classifying noncompact spaces are more difficult and less satisfying.

3. Every infinite subset of E has a limit point in E.

Example 2.2 ()

An open set in $\mathbb{R}^2$ is not compact. Take the open rectangle $R = (0,1)^2 \subset \mathbb{R}^2$. There exists an infinite cover of R

$$R = \bigcup_{n=0}^{\infty} (0,1) \times \left(0, \frac{2^{n+1}-1}{2^{n+1}}\right)$$

that does not have a finite subcover.

Clearly, the limit point of an open set is its boundary points. Note that a sequence of points can also have a limit point.

Theorem 2.15 (Bolzano-Weierstrass Theorem)

Every bounded sequence in $\mathbb{R}^n$ has a limit point.

Proof.

The fact that the infinite sequence is bounded means that there exists some closed subset $I \in \mathbb{R}^n$ that contains all point of the sequence. By definition I is compact, so by the Heine-Borel theorem, every cover of I has a finite subcover.
Now, assume that there exists an infinite sequence in I that is not convergent, i.e. has no limit point. Then, each point $x_i \in I$ would have a neighborhood $U(x_i)$ containing at most a finite number of points in the sequence. We can define I such that the union of the neighborhoods is a cover of I. That is,

$$I \subset \bigcup_{i=1}^{\infty} U(x_i)$$

However, since every $U(x_i)$ contains at most a finite number of points, we must have an infinite open neighborhoods to cover $I \implies$ we cannot have a finite subcover. This contradicts the fact that I is compact.

In fact, compactness actually implies completeness.

Theorem 2.16 ()

Compact metric spaces are complete.

So far, we've been pretty abstract about compact sets. In general, it's pretty easy to prove that a set is not compact. We just need to find one example of an open cover that does not have a finite subcover. To prove that set *is* compact, we must show that for *every* open cover, we can get a finite subcover. This sounds quite daunting, but here is a special theorem that can start us off, and the theorems above allow us to construct more compact sets. We will need the third interpreation of completeness of the reals: nested intervals completeness.

Theorem 2.17 (Nested Intervals Theorem)

If $\{I_n = [a_n, b_n]\}$ in $\mathbb{R}$ is a sequence of nested closed intervals, then

$$\bigcap_{i=1}^{\infty} I_n \neq \emptyset \tag{71}$$

Proof.

Note that $\{a_n\}$ is bounded above by b_1. Therefore by LUB property it must have a supremum, call it $x = \sup_n\{a_n\}$. Then, we see that $a_n \leq x \leq b_n$ for all n, and so x is in the intersection.

Corollary 2.5 ()

Every closed interval is compact.

Proof.

Let $I = [a, b]$. Then if $x, y \in I$, $|x - y| \leq b - a = \delta$. Now by contradiction, suppose that there exists an open cover $\{U_\alpha\}$ of I which contains no finite subcover of I. Then letting $c = (a + b)/2$, at least one of the two intervals $[a, c], [c, b]$ cannot have a finite subcovering (otherwise their finite union can be covered). WLOG let it be $[a, c]$. We keep subdividing and get the sequence of nested intervals.

$$I \supset I_1 \supset I_2 \supset \ldots \tag{72}$$

We know that I_n is not covered by any finite subcollection of $\{U_\alpha\}$ and if $x, y \in I_n$, then $|x-y| < 2^{-n}\delta$. From the nested intervals theorem, there exists a point z lying in every I_n. There must then be an open neighborhood U_z in the open cover, and by definition of openness there exists a $\epsilon > 0$ s.t. $z \in B_\epsilon(z) \subset U_z$. By the Archimidean property, we can set n so large that $2^{-n}\delta < \epsilon$ and this means that $B_\epsilon(z) \supset I_n$, which contradicts the fact that I_n is not covered by a finite subcollection. Therefore I is compact.

2.4 Connectedness

Definition 2.10 (Separate, Connected Sets)

Two subsets A and B of a metric space X are said to be **separated** if both $A \cap \overline{B}$ and $\overline{A} \cap B$ are empty, i.e. if no point of A lies in the closure of B and no point of B lies in the closure of A.

Example 2.3 ()

It is clear that separate sets imply disjointness. However, this is not true for the other way around.

1. $(0, 1)$ and $[1, 2)$ are disjoint but not separate.
2. The rationals and irrationals are disjoint, but not separate.

Theorem 2.18 ()

A subset E of the real line $\mathbb{R}$ is connected if and only if it has the following property: if $x \in E, y \in E$ and $x < z < y$, then $z \in E$.

Proof.

2.5 Separability

2.6 Perfect Sets

Definition 2.11 (Perfect Sets)

A set P is perfect if it is closed and all of its points are limit points of P. In other words, the limit points of P and P itself coincide.

$$P' = P \tag{73}$$

Theorem 2.19 ()

Let P be a nonempty perfect set in $\mathbb{R}^k$. Then P is uncountable.

3 Sequences in Euclidean Space

We have already defined sequences, but we'll add to our vocabulary to describe and classify sequences and series. To make this chapter a bit more self-contained, we redefine sequences and series.

Definition 3.1 (Sequence)

A sequence on a space X is a function $f : \mathbb{N} \to X$. We usually denote the sequence as (x_n), where $x_n = f(n)$.[a]

[a]Note that (x_n) as a sequence is different from $\{x_n\}$ as a set.

Definition 3.2 (Constant Sequence)

Let X be any set.

1. $\{x_i\}$ is a **constant sequence** if $x_i = x$ for all i
2. $\{x_i\}$ is an **ultimately constant sequence** if $x_i = x$ for all $i > N$ for some $N \in \mathbb{N}$. If $x = 0$ (assuming that an identity exists), then $\{a_i\}$ is **finary**.

3.1 Convergence in Metric Topologies

Now that we have defined sequences, we want to talk about their convergence, and if they do converge, *what* they converge to.

Definition 3.3 (Limit of Sequence in Topological Space)

A number $x \in \mathbb{R}$ is called the **limit of the sequence** $\{x_n\}$, written

$$\lim_{n \to \infty} x_n = x \tag{74}$$

if for every neighborhood U_x there exists an index N such that $x_n \in U_x$ for all $n > N$. If x is the limit of (x_n), then we say that (x_n) **converges** to x. If the limit of (x_n) is not well defined or finite, then we say that (x_n) is **divergent**.

Note that x being the limit of a sequence (x_i) is stronger than the claim that x is a limit point of $\{x_i\}$. If we consider the sequence $0, 1, 0, 1, \ldots$, we can see that both 0 and 1 are limit points, but the limit does not exist. We would like to define some notion of limit points in the language of sequences. We can precisely do this by treating a sequence as a set and talking about subsequential limits.

Definition 3.4 (Subsequences)

A **subsequence** of $\{a_n\}$ is a sequence $\{a_{\gamma_k}\}$, where $\{\gamma_k\}$ is a strictly increasing infinite subset of $\mathbb{N}$.

Definition 3.5 (Partial Limits)

The **partial limit** of a sequence $\{x_n\}$ is the limit of any of its subsequence.

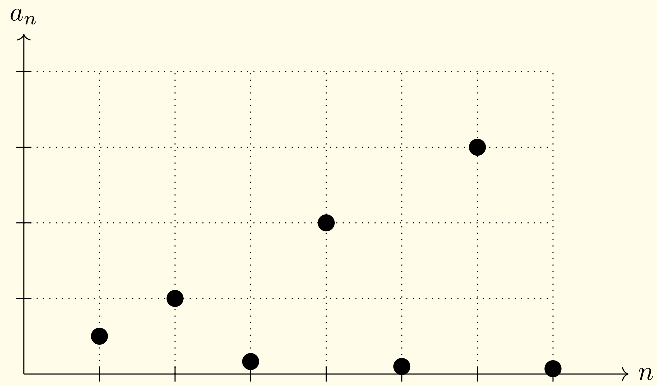


Figure 1: Two partial limits of the sequence $a_n = 1/n$ for n odd and $n/2$ for n even, is $+\infty$ and 0.

Lemma 3.1 (Partial Limits Equivalent to Limit Point)

Given a sequence (x_n), x is a limit point of $\{x_n\}$ iff there exists a subsequence of (x_n) that converges to x.

Immediately from the properties of the metric topology we know the following.

Definition 3.6 (Limit of a Sequence in Metric Space)

x is the limit of (x_n) if for every $\epsilon > 0$, there exists an index $N \in \mathbb{N}$ such that

$$d(x_n, x) < \epsilon \forall n > N \tag{75}$$

The next few theorems help us develop some intuition behind convergence of sequences. This is where the concept of limit points from topology becomes connected to sequences. In here, a limit point of a sequence is a viable candidate for which a sequence converges to.

Theorem 3.1 (Sufficient Conditions for Convergence in Metric Space)

Let $\{x_n\}$ be a sequence in a metric space X.

1. $\{x_n\}$ converges to $x \in X$ if and only if every neighborhood of x contains x_n for all but finitely many n.
2. If $\{x_n\}$ converges to x, then x is unique.
3. If $\{x_n\}$ converges, then $\{x_n\}$ is bounded.
4. If $E \subset X$ and x is a limit point of E, then there exists a sequence $\{x_n\}$ in E that converges to x.

Proof.

Listed.

1. ($\implies$) Let $p_n \to p \in X$. Then, for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ s.t. $d(p, p_n) < \epsilon$ for all $n > N$. Given neighborhood $B_\epsilon(p), x_n \in B_\epsilon(p)$ for all $n > N \implies$ at most N elements are not in $B_\epsilon(p)$. ($\impliedby$) Now for any $\epsilon > 0$, let every $B_\epsilon(p)$ contain all but finitely many p_n. Enumerate them $\{x_{n_k}\}_{k=1}^K$, and let
$$\alpha = \max\{n_k\}$$
This means that there exists an $\alpha \in \mathbb{N}$ s.t. $p_n \in B_\epsilon(p)$ for all $n > \alpha$, which implies that $p_n \to p$.
2. Assume $\{p_n\}$ converges to $p, p' \in X$, with $p \neq p'$. Then for all $\epsilon > 0$ there exists $N_1, N_2 \in \mathbb{N}$

s.t. $d(p, p_n) < \epsilon$ for all $n > N_1$ and $d(p', p_n) < \epsilon$ for all $n > N_2$. This means that there exists a $N = \max\{N_1, N_2\}$ satisfying the above. Since $p \neq p'$, set $\epsilon = d(p, p')/2$. Then,

$$d(p, p_n) < \frac{d(p, p')}{2} \text{ and } d(p', p_n) < \frac{d(p, p')}{2}$$

which implies that by adding both sides and invoking triangle inequality, we have

$$d(p, p') \leq d(p, p_n) + d(p', p_n) < d(p, p')$$

which is absurd.

3. Choose any $\epsilon > 0$. Then from (a), $B_\epsilon(x)$ contains all but finitely many $V = \{x_{n_k}\}_{k=1}^{K}$. Take

$$M = \max\{\epsilon, d(x_{n_1}, x), \ldots, d(x_{n_K}, x)\}$$

and so $d(x, x_n) < M$ for all $n \in \mathbb{N}$.[a]

4. We can explicitly construct one. Let $p \in E'$. Then choose $\epsilon = 1, \frac{1}{2}, \frac{1}{3}, \ldots$ and for every $\epsilon > 0$, $B_\epsilon(p) \cap E \neq \emptyset$. Choose a p_n within this intersection for every $\epsilon = \frac{1}{n}$. Then, we have $\{p_n\}$ contained in E. We want to show that this converges to p. Take any $\epsilon > 0$, then there exists $N \in \mathbb{N}$ s.t. $0 < \frac{1}{N} < \epsilon$, and for every $n > N$, $\frac{1}{n} < \frac{1}{N} < \epsilon$. Therefore, for every $n > N$,

$$p_n \in B_{1/n}(p) \subset B_\epsilon(p)$$

which means that $p_n \in B_\epsilon(p)$ for all $n > N$, implying that $\lim_{n\to\infty} p_n = p$.

[a] This is also a direct result of every metric topology being Hausdorff.

Lemma 3.2 ()

$\{x_n\}$ converges to x if and only if every subsequence of $\{x_n\}$ converges to x.

Proof.

Let $p_n \to p$. Then, take any subsequence $\{p_{n_k}\}$ of $\{p_n\}$. For any $\epsilon > 0$, there exists a $N \in \mathbb{N}$ s.t. $d(p, p_n) < \epsilon$ for all $n > N$. Since N is finite and the n_k's are unbounded, there must exist a $K \in \mathbb{N}$ s.t. $n_k > N$ if $k > K$. Therefore, given any $\epsilon > 0$, we have proved the existence of a $K \in \mathbb{N}$ s.t. $k > K \implies n_k > N$, which implies by convergence of p_n, that

$$d(p_{n_k}, p) < \epsilon \tag{76}$$

which by definition means that p_{n_k} converges to p. Now, for the other direction, given $\{p_n\}$ with every subsequence converging to p, we can take the subsequence $\{p_n\}$ itself ($n_k = k$), which converges to p.

Theorem 3.2 ()

If $\{x_n\}$ is a sequence in a compact metric space X, then some subsequence of $\{x_n\}$ converges to a point of X.

Proof.

Corollary 3.1 (Bolzano-Weierstrass Theorem)

Every bounded sequence in $\mathbb{R}^n$ contains a convergent subsequence.

Proof.

It suffices to prove that there exists a monotonic sequence within a bounded sequence $\{x_n\}$.

Theorem 3.3 (Nested Compact Sets)

Listed.

1. If $\overline{E}$ is the closure of a set E in a metric space X, then
$$\operatorname{diam} \overline{E} = \operatorname{diam} E$$
2. If K_n is a sequence of compact sets in X s.t. $K_n \supset K_{n+1}$ for $n \in \mathbb{N}$ and if
$$\lim_{n\to\infty} \operatorname{diam} K_n = 0$$
then $\cap_{n=1}^{\infty} K_n$ consists of exactly one point.

So far, we have talked about the limits of a sequence, which may or may not exist. But we do know from 3.2.2 that there *always* exists a subsequence that is either convergent or tends to $\pm\infty$. In this section, we focus on these subsequential limits. Here is our first result.

Theorem 3.4 ()

The subsequential limits of a sequence (x_n) in a metric space form a closed subset of X.

Proof.

Let E be the set of subsequential limits and $y \in E'$ be a limit point of E. We must show that $y \in E$.[a] We will construct a subsequence (x_{n_k}) that converges to y. Given any $\epsilon > 0$, we can see that the $B_{\epsilon/2}(y) \cap E \neq \emptyset$, so choose an element z_ϵ. Furthermore, z_ϵ means that it is a limit point of $\{x_n\}$, and so $B_{\epsilon/2}(z) \cap \{x_n\} \neq \emptyset$, call this $x(\epsilon)$. Therefore, by the triangle inequality,

$$|y - x(\epsilon)| \leq |y - z| + |z - x(\epsilon)| = \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \tag{77}$$

and so we can take a point from the sequence (x_n) for every ϵ. Now we do this for $\epsilon = 1/n$, and choose n_k that is greater than its previous by restricting the sequence to that past n_{k-1}. Doing this gives a subsequence which converges to y. Therefore $y \in E$.

[a] Intuitively, we can see that y is infinitesimally close to E, which consists of points infinitesimally close to $\{x_n\}$, and so y should be infinitesimally close to $\{x_n\}$.

3.2 Convergence in Reals

The properties of a general metric space give us some general conditions to determine whether a sequence converges in $\mathbb{R}^n$. To add to our toolbox in determining convergence, we will focus on sufficient conditions for convergence in $\mathbb{R}$, which has the additional properties of being an ordered field. These additional structures unlock a whole new suite of theorems in convergence. Why do we want to focus on just real-valued, i.e. numerical, sequences? First is that since $\mathbb{R}^n$ is constructed as the product topology of $\mathbb{R}$, we can prove a lot about continuity of functions in $\mathbb{R}^n$ by proving limits in $\mathbb{R}$, and letting the construction of the product topology do the rest. Second, the codomain of many natural structures such as inner products, norms, and measures lie in the reals, and we often need to prove convergence of these values.

3.2.1 Completeness

So far, in order to show that a sequence is convergent, we must identify a real number first and then show using the ϵ-δ definition that it converges. This might be overkill in a case where we just want to prove that a sequence converges, but we don't care what it converges to. Unsurprisingly, we use the fact that the sequence lives in the reals. We can determine convergence by using Cauchy-completeness, which gives us the "theorem" (though it is really a fact by construction).

Theorem 3.5 (Cauchy-Convergence Criterion)

A cauchy sequence in $\mathbb{R}$ converges.

The second result is an immediate consequence of Dedekind completeness, which is equivalent to Cauchy completeness in the reals.

Definition 3.7 (Monotonic Sequences)

Let X be an ordered set. $\{x_n\}$ is

1. **increasing** if $x_{n+1} > x_n$ for all n.
2. **decreasing** if $x_{n+1} < x_n$ for all n.
3. **nondecreasing** if $x_{n+1} \geq x_n$ for all n.
4. **nonincreasing** if $x_{n+1} \leq x_n$ for all n.

Sequences of these types are called **monotonic**.

Lemma 3.3 (Convergence Criterion for Monotonic Sequences)

In order for a nondecreasing (nonincreasing) sequence to be convergent, it is necessary and sufficient that it is bounded above (or below).

Proof.

It satisfies to prove the first case, as the second case can be done similarly without much difficulty. Let $x_n \leq x_{n+1}$. Then the set $\{x_n\}$ is bounded above in $\mathbb{R}$, which has the least upper bound property, and so there exists a least upper bound x. We claim that the sequence converges to x. For every $\epsilon > 0$, since it is least, there exists at least one $x_N \in (x - \epsilon, x)$. By monotonicity, this means that $x_n \in (x - \epsilon, x)$ for all $n \geq N$, and so the sequence converges to x.

3.2.2 Properties of Order

We are able to see how both Cauchy and Dedekind completeness of the reals define convergence in $\mathbb{R}$. Now let's squeeze a bit more out of the total ordering to gain some properties of convergence and divergence.

Theorem 3.6 (Preservation of Ordering Between Sequences and Limits)

Given convergent sequences $\{x_n\}$ and $\{y_n\}$, if

$$\lim_{n\to\infty} x_n < \lim_{n\to\infty} y_n \tag{78}$$

then there exists an index $N \in \mathbb{N}$ such that $x_n < y_n$ for all $n > N$.

Proof.

Given $x_n \to x, y_n \to y$ and $x < y (\in \mathbb{R})$, then for every $\epsilon > 0$, there exists $N_1, N_2 \in \mathbb{N}$ s.t. $d(x, x_n) < \epsilon$ for all $n > N_1$ and $d(y, y_n) < \epsilon$ for all $n > N_2$. Setting $N = \max\{N_1, N_2\}$, we can say the same for all $n > N$. We choose $\epsilon = \frac{y-x}{2} > 0$. Then, there exists $N \in \mathbb{N}$ s.t. $x_n \in (x - \epsilon, x + \epsilon)$ and $y_n \in (y - \epsilon, y + \epsilon)$ for all $n > N$. Therefore, if $a \in (x - \epsilon, x + \epsilon)$ and $b \in (y - \epsilon, y + \epsilon)$, then

$$a < \sup B_\epsilon(x) = x + \epsilon = y - \epsilon = \inf B_\epsilon(y) < b$$

which implies that $x_n < y_n$ for all $n > N$.

Theorem 3.7 (Squeeze Theorem for Sequences)

Given sequences $\{x_n\}, \{y_n\}, \{z_n\}$ such that

$$x_n \leq y_n \leq z_n$$

for all $n > N$, if $\{x_n\}$ and $\{z_n\}$ both converge to the same limit, then the sequence $\{y_n\}$ also converges to that limit. That is,

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n = A \implies \lim_{n\to\infty} y_n = A$$

Proof.

We first prove that if there exists a $N \in \mathbb{N}$ s.t. $a_n \leq b_n$ for all $n > N$, then $\lim_{n\to\infty} a_n = a \leq b = \lim_{n\to\infty} b_n$. Assume this weren't true, that $a > b$. Then for $\epsilon = \frac{a-b}{2} > 0$, there must exist $M \in \mathbb{N}$ s.t. $a_n \in (a - \epsilon, a + \epsilon)$ and $b_n \in (b - \epsilon, b + \epsilon)$ for all $n > M$. But

$$b_n < \sup(b - \epsilon, b + \epsilon) = b + \epsilon = a - \epsilon = \inf(a - \epsilon, a + \epsilon) < a_n \tag{79}$$

which contradicts $a_n \leq b_n$. Therefore, $a \leq b$. Therefore, we can use this to get

$$A = \lim_{n\to\infty} x_n \leq \lim_{n\to\infty} y_n \leq \lim_{n\to\infty} z_n = A \implies \lim_{n\to\infty} y_n = A \tag{80}$$

Note that while a convergent sequence can be visualized quite easily by the Cauchy convergence criterion, there are many way in which a sequence can be divergent.

1. Increasing/decreasing indefinitely
2. Oscillating between two constant values
3. Oscillating between a value tending to $+\infty$ and a value tending to $-\infty$
4. Many other classes of divergence

Definition 3.8 (Sequence Tending to Infinity)

The sequence $\{x_n\}$ **tends to positive infinity** if for each number c there exists $N \in \mathbb{N}$ such that $x_n > c$ for all $n > N$. It is denoted

$$x_n \to +\infty \text{ or } \lim_{n\to\infty} x_n = +\infty \tag{81}$$

We define sequences that **tend to negative infinity** similarly. And $\{x_n\}$ **tends to infinity** if for each c there exists $N \in \mathbb{N}$ such that $|x_n| > c$ for all $n > N$, which is written

$$x_n \to \infty \tag{82}$$

Note that

$$x_n \to +\infty \text{ or } x_n \to -\infty \implies x_n \to \infty \tag{83}$$

but the converse is not necessarily true. The simple example is the sequence $x_n = (-1)^n n$. Also, it is important to know that a sequence may be unbounded and yet not tend to $+\infty$, $-\infty$, or ∞.

Example 3.1 (Unbounded Sequence that Doesn't tend to ∞)

The sequence $x_n = n^{(-1)^n}$ is divergent yet does not tend to positive infinity, negative infinity, nor infinity.

However, if a sequence is unbounded we can find a convergent subsequence that does tend to infinity.

Theorem 3.8 ()

If (x_n) is not bounded above then it has a subsequence $x_{n_k} \to +\infty$.

Proof.

We can construct such a subsequence.

Therefore, we can construct a subsequential limit to $\pm\infty$ if (x_n) is not bounded. If it is bounded, then by the Bolzano-Weierstrass theorem it contains a convergent subsequence. Therefore, we have the following.

Corollary 3.2 ()

From each sequence of real numbers there exists either a convergent subsequence or a subsequence tending to infinity.

Example 3.2 ()

We claim that

$$\lim_{n\to\infty} \sqrt[n]{n} = 1 \tag{84}$$

3.3 Arithmetic

Theorem 3.9 (Arithmetic on Limits)

Given that $\{x_n\}, \{y_n\}$ are numerical sequences with $y_n \neq 0$ for all n, and let

$$\lim_{n\to\infty} x_n = A, \qquad \lim_{n\to\infty} y_n = B \neq 0$$

then,

$$\begin{aligned}
&\lim_{n\to\infty} (x_n + y_n) = A + B \\
&\lim_{n\to\infty} (c x_n) = cA \\
&\lim_{n\to\infty} (x_n \cdot y_n) = A \cdot B \\
&\lim_{n\to\infty} \frac{x_n}{y_n} = \frac{A}{B}
\end{aligned}$$

It immediately follows that the set of all convergent sequences in $\mathbb{R}^\mathbb{N}$ is a subspace of $\mathbb{R}^\mathbb{N}$.

Proof.

Assume that

$$\lim_{n\to\infty} x_n = A \text{ and } \lim_{n\to\infty} y_n = B \neq 0$$

This means that for every $\epsilon > 0$, there exists $N_1, N_2 \in \mathbb{N}$ such that

$$|x_n - A| < \epsilon \text{ for all } n > N_1$$
$$|y_n - B| < \epsilon \text{ for all } n > N_2$$

Therefore, for a given ϵ, we wish to prove that there exists a N such that for all $n > N$,

$$1. |(x_n + y_n) - (A + B)| < \epsilon$$
$$2. |cx_n - cA| < \epsilon$$
$$3. |(x_n y_n) - (AB)| < \epsilon$$
$$4. \left|\frac{x_n}{y_n} - \frac{A}{B}\right| < \epsilon$$

1. By the triangle inequality, we can see that

$$|(x_n + y_n) - (A + B)| = |x_n - A| + |y_n - B|$$

   Since we can choose the error between x_n and A for $n > N_1$, and y_n and B for $n > N_2$ as small as we want, we set it to $\epsilon/2$. Then, we have

$$|(x_n + y_n) - (A + B)| = |x_n - A| + |y_n - B| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

   for all $n > N = \max\{N_1, N_2\}$. Therefore, for a given ϵ, there exists an N such that

$$|(x_n + y_n) - (A + B)| < \epsilon \text{ for all } n > N$$

2. This proof is easy. For a given ϵ, we choose the error to be $\frac{\epsilon}{c}$.

$$|x_n - A| < \frac{\epsilon}{c} \text{ for all } n > N_1$$

   Then, there exists natural number N_1 such that

$$|cx_n - cA| < c|x_n - A| = c\frac{\epsilon}{c} = \epsilon \text{ for all } n > N_1$$

3. We first observe that since the limit of $\{y_n\}$ exists, it must be bounded by a value, say B. That is,

$$|y_n| < Y \text{ for all } n \in \mathbb{N}$$

   Then, we see that

$$\begin{aligned} |x_n y_n - AB| &= |(x_n y_n - Ay_n) + (Ay_n - AB)| \\ &< |x_n y_n - Ay_n| + |Ay_n - AB| \\ &= |y_n||x_n - A| + |A||y_n - B| \end{aligned}$$

   Suppose $\epsilon > 0$ is given. Then, we can set the error bounds freely; there exists $N_1, N_2 \in \mathbb{N}$ such that

$$|x_n - A| < \frac{\epsilon}{2Y} \text{ for all } n > N_1$$
$$|y_n - B| < \frac{\epsilon}{2|A|} \text{ for all } n > N_2$$

Then, we can see that

$$|x_n y_n - AB| \leq |y_n||x_n - A| + |A||y_n - B| < Y \cdot \frac{\epsilon}{2Y} + |A|\frac{\epsilon}{2|A|} = \epsilon$$

for all $n > N = \max\{N_1, N_2\}$.

4. We use the estimate

$$\left|\frac{A}{B} - \frac{x_n}{y_n}\right| = \frac{|x_n||y_n - B| + |y_n||x_n - A|}{y_n^2} \cdot \frac{1}{1 - \delta(y_n)}, \quad \delta(y_n) = \frac{|y_n - B|}{|y_n|}$$

For a given $\epsilon > 0$, we find natural numbers N_1, N_2 such that

$$\begin{aligned} |x_n - A| &< \min\left\{1, \frac{\epsilon|B|}{8}\right\} \text{ for all } n > N_1 \\ |y_n - B| &< \min\left\{\frac{|B|}{4}, \frac{\epsilon B^2}{16(|A|+1)}\right\} \text{ for all } n > N_2 \end{aligned}$$

From this we can deduce that

$$|x_n| = |x_n - A + A| < |x_n - A| + |A| < |A| + 1$$

and

$$\begin{aligned} &|B| = |y_n + B - y_n| < |y_n| + |B - y_n| \\ &\Longrightarrow |y_n| > |B| - |y_n - B| > |B| - \frac{|B|}{4} > \frac{|B|}{2} \\ &\Longrightarrow \frac{1}{|y_n|} < \frac{2}{|B|} \\ &\Longrightarrow 0 < \delta(y_n) = \frac{|y_n - B|}{|y_n|} < \frac{|B|/4}{|B|/2} = \frac{1}{2} \\ &\Longrightarrow 1 - \delta(y_n) > \frac{1}{2} \\ &\Longrightarrow 0 < \frac{1}{1 - \delta(y_n)} < 2 \end{aligned}$$

So, we can substitute

$$\begin{aligned} |x_n| \cdot \frac{1}{y_n^2} \cdot |y_n - B| &< (|A| + 1) \cdot \frac{4}{B^2} \cdot \frac{\epsilon \cdot B^2}{16(|A|+1)} = \frac{\epsilon}{4} \\ \left|\frac{1}{y_n}\right| \cdot |x_n - A| &< \frac{2}{|B|} \cdot \frac{\epsilon|B|}{8} = \frac{\epsilon}{4} \end{aligned}$$

into the final equation to get

$$\left|\frac{A}{B} - \frac{x_n}{y_n}\right| < \epsilon \text{ for all } n > N = \max\{N_1, N_2\}$$

Example 3.3 ()

We claim that

$$\lim_{n \to \infty} \frac{n}{q^n} = 0 \text{ if } q > 1$$

Since $x_n = \frac{n}{q^n} \implies x_{n+1} = \frac{n+1}{nq} x_n$ for $n \in \mathbb{N}$. Since

$$\lim_{n\to\infty} \frac{n+1}{nq} = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)\frac{1}{q} = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right) \cdot \lim_{n\to\infty} \frac{1}{q} = 1 \cdot \frac{1}{q} = \frac{1}{q} < 1$$

there exists an index N such that $\frac{n+1}{nq} < 1$ for $n > N$. Thus, we have

$$x_n > x_{n+1} = x_n \cdot \frac{n+1}{nq} \text{ for } n > N$$

which means that the sequence will be monotonically decreasing from index N on. The terms of the sequence

$$x_{N+1} > x_{N+2} > x_{N+3} > \ldots$$

are positive (bounded below) and are monotonically decreasing, so it must have a limit.
Finding the actual limit is easy. Let $x = \lim_{n\to\infty} x_n$. It follows from the relation $x_{n+1} = \frac{n+1}{nq} x_n$ that

$$x = \lim_{n\to\infty} \big(x_{n+1}\big) = \lim_{n\to\infty} \left(\frac{n+1}{nq} x_n\right) = \lim_{n\to\infty} \frac{n+1}{nq} \cdot \lim_{n\to\infty} x_n = \frac{1}{q} x$$

which implies that $\left(1 - \frac{1}{q}\right) = 0 \implies x = 0$.

Definition 3.9 (Cauchy Product)

The Cauchy Product is the direct convolution of two sequences.

Definition 3.10 (Recursive Sequence)

Sometimes, a sequence may be defined **recursively**, where the nth term contains a combination of the $n-1$ terms before it.

3.4 Limsup and Liminf

The superior and inferior limits represent some sort of "bound" on the sequence in the long run. That is, on the long run, the terms of the sequence (x_n) cannot be greater than its superior limit and cannot be less than its inferior limit. With this interpretation, the following definition should be clear.

Definition 3.11 (Inferior, Superior Limits)

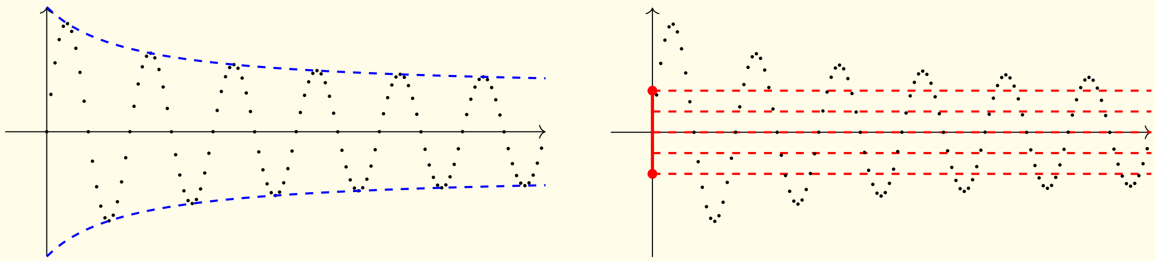
The **superior/inferior limit** of a sequence (x_n) is defined in the equivalent ways.

1. Given that E is the set of all partial limits, the limsup/liminf is the supremum/infimum of E.

$$\limsup_{n\to\infty} x_n := \sup\{E\} \qquad \liminf_{n\to\infty} x_n := \inf\{E\} \tag{85}$$

2. The limsup/liminf is the limit of the sequence of supremums/infimums of the elements up to k.

$$\limsup_{n\to\infty} x_n := \lim_{n\to\infty} \sup_{k\geq n} x_k \qquad \limsup_{n\to\infty} x_n := \lim_{n\to\infty} \inf_{k\geq n} x_k \tag{86}$$



(a) In order to find the limsup, we first look the whole sequence in $\mathbb{N}$ and find the supremum. We now "decrease" our domain from $\mathbb{N}$ to $\{2, \ldots\}$, then $\{3, \ldots\}$, then $\{4, \ldots\}$ and so on, continuing to label the supremum of the sequence. The limit of this sequence of supremums is the limsup.

(b) The 5 red lines marked in the middle (along with infinitely many others) are viable partial limits because one can choose a subsequence such that all of its points after a certain n lie in some ϵ-neighborhood of the limit. Therefore, we claim that the limsup/inf is the supremum of this set E.

Figure 2: Two ways to visualize the superior and inferior limits of the divergent sequence $x_n = \big(\frac{1}{x+1} + 0.5\big)\sin(2\pi x)$. The left is the limit of the supremum, and the right is the supremum of the closed set of subsequential limits.

Example 3.4 (Computing Limsup and Liminf)

We give some basic examples.

1. Let $x_n = (-1)^n$. Then $E = \{-1, +1\}$ and
$$\limsup_{n\to\infty} x_n = 1, \qquad \liminf_{n\to\infty} x_n = -1 \tag{87}$$
2. Let $x_n = (-1)^n/[1 + (1/n)]$. Then
$$\limsup_{n\to\infty} x_n = 1, \qquad \liminf_{n\to\infty} x_n = -1 \tag{88}$$

Let's give two warnings. First, limsup and liminfs do *not* behave like limits under addition and multiplication. That is,
$$\limsup x_n + \limsup y_n \neq \limsup x_n + y_n \tag{89}$$

Example 3.5 (Counterexamples of Arithmetic Consistency of Limit superior)

Consider $(x_n) = (-1)^n$ and $y_n = (-1)^{n+1}$. Then
$$\limsup x_n = \limsup y_n = 1, \qquad \liminf x_n = \liminf y_n = -1 \tag{90}$$
But $(x_n + y_n) = 0$, so
$$\limsup x_n + y_n = \liminf x_n + y_n = 0 \tag{91}$$

Second, note that even though we are talking about subsequential limits, the limsup and liminf are *not* subsequential limits! It is the supremum of subsequential limits E, which may or may not be in E.

Example 3.6 (Limsup that is not attained by any subsequential limit)

This should be a sequence not in $\mathbb{R}$.

However, in $\mathbb{R}$, it turns out that the limsup and liminf are both contained in E, so we are fine.

Lemma 3.4 ()

If (x_n) is a sequence in $\mathbb{R}$, then

1. the limsup is indeed a subsequential limit, i.e. $\limsup x_n \in E$.
2. If $x > \limsup x_n$, then $\exists N \in \mathbb{N}$ s.t. $n \geq N \implies x_n < x$.

Proof.

For the first claim, there are two cases to consider. If (x_n) is unbounded from above, then $\exists (x_{n_k})$ such that $x_{n_k} \to +\infty \implies \infty = \limsup x_n \in E$. If (x_n) is bounded from above, then the subsequential limits of (x_n) are either in (x_n) or they are limit points of x_n. This implies that the set E consists of points either in $\{x_n\}$ or are limit points of the set $\{x_n\} \implies \sup E$ is in E since it's a limit point.
For the second claim, if there are infinitely many terms of the sequence larger than x, then we could find a subsequence (x_{n_k}) with $x_{n_k} > x$ for all k. Therefore (x_n) has a subsequential limit which must be $\geq x$. Every subsequential limit of (x_{n_k}) is also a subsequential limit of (x_n). This contradicts $\limsup x_n = \sup E$.

Theorem 3.10 (Requirements of Partial Limits for Limit to Exist)

Here are two results in which we can use partial limits to determine if a sequence has a limit or not.

1. A sequence has a limit or tends to $\pm\infty$ if and only if its inferior and superior limits are the same.
$$\limsup x_n = \liminf x_n = x \implies \lim_{n \to +\infty} x_n = x \tag{92}$$
2. A sequence converges if and only if every subsequence of it converges.

Proof.

For (1), we pick $x + \epsilon > x$. Then every term past some N_1 must be less than $x + \epsilon$. By the same logic, we have N_2 for $x - \epsilon < x$. So take $N = \max\{N_1, N_2\}$, which is contained in the ϵ-ball around x.

Theorem 3.11 (Ordering on Subsequential Limits)

If $s_n \leq t_n$ for $n \geq N$, where N is fixed, then

$$\liminf_{n \to \infty} s_n \leq \liminf_{n \to \infty} t_n$$
$$\limsup_{n \to \infty} s_n \leq \liminf_{n \to \infty} t_n$$

Example 3.7 ()

We claim
$$\lim_{n \to \infty} n^{1/n} = 1 \tag{93}$$
We can consider $x_n = n^{1/n} - 1$ and want to show that $x_n \to 0$. We have $x_n \geq 0$. If $n > 1$, then $n = (x_n + 1)^n \geq x_n^2 \cdot \frac{n(n-1)}{2}$ from the binomial theorem. This means that
$$x_n^2 \leq \frac{2}{n-1} \implies 0 \leq x_n \leq \sqrt{\frac{2}{n-1}} \to 0 \tag{94}$$
And so by the squeeze theorem, $x_n \to 0$.

Example 3.8 ()

If $x > 1, \alpha \in \mathbb{R}$, then

$$\lim_{n \to +\infty} \frac{n^\alpha}{x^n} = 0 \tag{95}$$

3.5 Convergence Tests for Real Series

Definition 3.12 (Series over $\mathbb{R}$)

Given a sequence of real numbers (x_n), the **series (of partial sums)** is the sequence

$$(s_n) = \sum_{k=1}^{n} x_k \tag{96}$$

The **sum of the series** is the limit of (s_n). Usually we define (s_n) implicitly and use the summation notation.

$$\sum_{n=1}^{\infty} x_n \coloneqq \lim_{n \to \infty} s_n \tag{97}$$

1. If the sequence (s_n) converges to s, the series is **convergent**, written

$$\sum x_n < +\infty \tag{98}$$

2. If the sequence does not converge, it is **divergent**.
3. If the series of partial sums of $(|x_n|)$ converges, then it is said to be **absolutely convergent**.[a]

$$\sum_{n=1}^{\infty} |x_n| \tag{99}$$

[a]Clearly, every absolutely convergent series because $\left|\sum_{n=1}^{\infty} a_n\right| \leq \sum_{n=1}^{\infty} |a_n|$.

We must reiterate a few warnings here. Note that the series $\sum x_n$ is simply notation and should *not* be treated as an "infinite sum." Such a thing does not exist for algebraic structures which have finary operations. More specifically, given a series, we cannot in general split nor combine series, and we cannot reindex nor rearrange (an infinite number of) terms. However, we can manipulate each term for a fixed index.

Example 3.9 (Disasters of Reindexing and Rearranging)

Let us take the series $\sum 0$. We clearly know that the corresponding sequence of partial sums $0, 0, \ldots$

is convergent to 0. But if we do this series of steps.

$$\begin{aligned}
\sum_{n=1}^{\infty} 0 &= \sum_{n=1}^{\infty} n - n && \text{(Can manipulate terms)} \\
&= \sum_{n=1}^{\infty} n - \sum_{n=1}^{\infty} n && \text{(Cannot split series)} \\
&= 1 + \sum_{n=2}^{\infty} n - \sum_{n=1}^{\infty} n && \text{(Can take 1st term out)} \\
&= 1 + \sum_{n=1}^{\infty} (n+1) - \sum_{n=1}^{\infty} n && \text{(Cannot reindexing)} \\
&= 1 + \sum_{n=1}^{\infty} (n+1) - n && \text{(Cannot combine series)} \\
&= 1 + \sum_{n=1}^{\infty} 1 && \text{(Can manipulate terms)} \\
&= 1 + \infty = +\infty && (100)
\end{aligned}$$

The wrong steps show that the series is divergent.

We have seen the consequences of these mistakes that beginners make and are often on popular media. However, note that we can always do splitting, combining, reindexing, and rearranging for *finite sums*, which are algebraically defined. Later on, we will show that some of these operations are allows for series that we know are convergent.

Since the convergence of a series is equivalent to convergence of its sequence of partial sums, applying the Cauchy convergence criterion to the sequence $\{s_n\}$ leads to the following theorem.

Theorem 3.12 (Cauchy Convergence Criterion for Series)

The series $a_1 + \ldots + a_n + \ldots$ converges if and only if for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $m \geq n > N$,

$$|a_n + \ldots + a_m| < \epsilon \tag{101}$$

Corollary 3.3 (nth Term Test)

A necessary (but not sufficient) condition for convergence of the series $a_1 + \ldots a_n + \ldots$ is that the terms tend to 0 as $n \to \infty$. That is, it is necessary that

$$\lim_{n\to\infty} a_n = 0 \tag{102}$$

Proof.

It suffices to set $m = n$ in the Cauchy convergence criterion. This would mean that for every $\epsilon > 0$ there exists a $N \in \mathbb{N}$ such that

$$|a_n| = |a_n - 0| < \epsilon \text{ for all } n > N \tag{103}$$

which, by definition, means that $\{a_n\}$ converges to 0.

Nothing so far is really suprising here. The Cauchy convergence criterion really just follows from the definition of Cauchy completeness, and the nth term test is pretty trivial. The way that we will build up convergence tests is by proving some special cases of convergence and then using the direct comparison test to then classify further series.

Example 3.10 (Telescoping Series)

A **telescoping series** is a series in which the partial sums can cancel out. An example is the series of partial sums of the sequence $(x_n) = \frac{1}{n(n+1)}$. In here, the series term is

$$\begin{aligned} s_n &= \sum_{k=1}^{n} \frac{1}{k(k+1)} && (104) \\ &= \sum_{k=1}^{n} \frac{1}{k} - \frac{1}{k+1} && (105) \\ &= \sum_{k=1}^{n} \frac{1}{k} - \sum_{k=1}^{n} \frac{1}{k+1} && (106) \\ &= \sum_{k=1}^{n} \frac{1}{k} - \sum_{k=2}^{n+1} \frac{1}{k} && (107) \\ &= \frac{1}{1} + \left(\sum_{k=2}^{n} \frac{1}{k} \right) - \left(\sum_{k=2}^{n} \frac{1}{k} \right) - \frac{1}{n+1} && (108) \\ &= 1 + \left(\sum_{k=2}^{n} \frac{1}{k} - \frac{1}{k} \right) - \frac{1}{n+1} && (109) \\ &= 1 - \left(\sum_{k=2}^{n} 0 \right) - \frac{1}{n+1} && (110) \\ &= 1 - \frac{1}{n+1} && (111) \end{aligned}$$

Note that all of the examples that we have done here are for finite sums, so they are all legal.

Example 3.11 (Geometric Series)

The series $\sum_{n=0}^{\infty} q^n$ is called a **geometric series**.

$$1 + q + q^2 + \ldots + q^n + \ldots \tag{112}$$

is called the **geometric series**. We can see that

1. $|q| \geq 1 \iff \sum q^n$ is divergent. $|q| \geq 1 \implies |q|^n \geq 1$, and so the terms q^n does not converge to 0, and the nth term test is not met.
2. $|q| < 1 \iff \sum q^n$ is convergent. We can use the identity

$$s_n = 1 + q + \ldots + q^{n-1} = \frac{1 - q^n}{1 - q} \implies \lim_{n \to \infty} \frac{1 - q^n}{1 - q} = \frac{1}{1 - q} \tag{113}$$

since $\lim_{n \to \infty} q^n = 0$ if $|q| < 1$.

The Cauchy convergence criterion can be used to prove the direct comparison test.

Theorem 3.13 (Direct Comparison Test)

For some fixed N, if

1. If $|x_n| \leq y_n$ for all $n \geq N$ and $\sum y_n$ converges, then $\sum x_n$ converges.
2. If $x_n \geq y_n \geq 0$ for all $n \geq N$ and $\sum y_n$ diverges, then $\sum x_n$ diverges.

Example 3.12 (Comparison with Telescoping Series)

We can prove the special case a geometric series with the direct comparison test. We claim that $\sum_{n=1}^\infty \frac{1}{n^2}$ is finite. We can see that

$$\frac{1}{n^2} \leq \frac{2}{n(n+1)} \tag{114}$$

where the series of the terms in the RHS is telescoping and therefore converges. So by the direct comparison test, $\sum \frac{1}{n^2}$ converges.

Now we prove another corollary of the Cauchy convergence criterion.

Theorem 3.14 (Cauchy Condensation Test)

If $a_1 \geq a_2 \geq \ldots \geq 0$, the series $\sum_{n=1}^\infty a_n$ converges if and only if the series

$$\sum_{k=0}^\infty 2^k a_{2^k} = a_1 + 2a_2 + 4a_4 + 8a_8 + \ldots \tag{115}$$

converges.

Proof.

Letting $A_k = a_1 + a_2 + \ldots + a_k$ and $S_n = a_1 + 2a_2 + \ldots + 2^n a_{2^n}$, it is clear that by adding up the inequalities

$$\begin{aligned}
&a_2 \leq a_2 \leq a_1 \\
&2a_4 \leq a_3 + a_4 \leq 2a_2 \\
&4a_8 \leq a_5 + a_6 + a_7 + a_8 \leq 4a_4 \\
&\ldots \\
&2^n a_{2^{n+1}} \leq a_{2^n+1} + \ldots + a_{2^{n+1}} \leq 2^n a_{2^n},
\end{aligned}$$

we get

$$\frac{1}{2}(S_{n+1} - a_1) \leq A_{2^{n+1}} - a_1 \leq S_n \tag{116}$$

Since the sequences $\{A_k\}$ and $\{S_k\}$ are nondecreasing, and hence from the inequalities we can conclude that they are either both bounded above (which means that they are both convergent since it is a bounded, nondecreasing series) or both unbounded above (which means that they are both divergent since they are nondecreasing and unbounded).

Corollary 3.4 (p-series Test)

The series

$$\sum_{n=1}^\infty \frac{1}{n^p} \tag{117}$$

converges for $p > 1$ and diverges for $p \leq 1$.[a]

[a]This sort of reminds you of u-substitution. For example, look at $\int_1^\infty f(t)\,dt = \int_0^\infty e^u f(e^u)\,du$, where the convergence of LHS $\iff$ convergence of RHS.

Proof.

Suppose $p \geq 0$. By the previous theorem, the series converges or diverges simultaneously with the series

$$\sum_{k=0}^\infty 2^k \frac{1}{(2^k)^p} = \sum_{k=0}^\infty (2^{1-p})^k \tag{118}$$

which is really just a geometric series. A necessary and sufficient condition for the convergence of this series is that $2^{1-p} < 1$, that is, $p > 1$.
Now suppose $p \leq 0$. The series is then clearly divergent since all of the terms are larger than 1.

Example 3.13 (Harmonic Series)

The **harmonic series**

$$1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} + \ldots \tag{119}$$

seems at first glance to be converging since the terms converge to 0. However, it does not pass the Cauchy condensation test since

$$\sum_{n=1}^\infty 2^n x_n = \sum_{n=1}^\infty 2^n \frac{1}{2^n} = \sum_{n=1}^\infty 1 = +\infty \tag{120}$$

As you can see, this increases logarithmically, so in early calculators it was hard to numerically detect divergence (you would have to double the number of series terms to get a linear increase).

3.6 Ratio and Root Tests

Now we introduce the root and ratio tests, which are derived by the comparison test with a geometric series. The ratio test is used more day-to-day, but not as decisive as the root test. Both tests have a similar flavor.

Theorem 3.15 (Ratio Test)

Suppose the limit $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \alpha$ exists for the series $\sum_{n=1}^\infty a_n$. Then,

1. $\alpha < 1 \implies \sum a_n$ converges absolutely.
2. $\alpha > 1 \implies \sum a_n$ diverges.
3. $\alpha = 1 \implies \sum a_n$ is inconclusive.

Alternatively, if

1. $\limsup |a_{n+1}/a_n| = \alpha < 1$, then $\sum a_n$ converges
2. If $\exists N$ s.t. $|a_{n+1}/a_n| \geq 1$ for all $n \geq N$, then $\sum a_n$ diverges.

Proof.

Since $\limsup \left|\frac{a_{n+1}}{a_n}\right| = \alpha < 1$, fix any $\alpha < \beta < 1$. Then $\exists N$ s.t. if $n > N$, $|a_{n+1}/a_n| < \beta$. So $|a_{N+1}| < \beta |a_N| \implies |a_{N+2}| < \beta^2 |a_N|$. So letting $C = |a_N|$, for all $m \geq N$,

$$|a_m| \leq \frac{C}{\beta^N}\beta^m \implies |a_m| \leq \tilde{C}\beta^m \text{ for all } m \geq N \tag{121}$$

So $\sum a_n$ converges by comparison test since $\sum \beta^m < \infty$ when $\beta < 1$.

Theorem 3.16 (Root Test)

Let $\sum_{n=1}^{\infty} a_n$ be a given series and

$$\alpha = \limsup_{n\to\infty} \sqrt[n]{|a_n|} \tag{122}$$

Then,

1. $\alpha < 1 \implies \sum a_n$ converges.
2. $\alpha > 1 \implies \sum a_n$ diverges.
3. $\alpha = 1 \implies \sum a_n$ is inconclusive.

Proof.

Listed.

1. If $\limsup \sqrt[n]{|a_n|} = \alpha < 1$, take any $\alpha < \beta < 1$. Then $\exists N \in \mathbb{N}$ s.t. if $n \geq N$, then $|a_n|^{1/n} < \beta \iff |a_n| < \beta^n$. Since $\beta < 1$, $\sum \beta^n < \infty$, and by comparison test, $\sum a_n$ converges.
2. Suppose $\alpha > 1$. Then $\limsup |a_n|^{1/n} = \alpha > 1$. So there exists a subsequence (a_{n_k}) s.t. $(|a_{n_k}|^{1/n_k}) \to \alpha > 1$. This means $\exists N$ s.t. for $n \geq N$, $|a_{n_k}|^{1/n_k} > 1 \implies |a_{n_k}| > 1$. But this fails the nth term test.
3. We do not claim anything and so there's nothing to prove.

Example 3.14 (Root Test Inconclusive Results)

Consider $\sum \frac{1}{n} = +\infty$, but from the root test

$$\sqrt[n]{\frac{1}{n}} \to 1, \text{ so } \alpha = 1 \tag{123}$$

Consider $\sum \frac{1}{n^2} < +\infty$, but from from the root test

$$\sqrt[n]{\frac{1}{n^2}} = \left(\frac{1}{n^{1/n}}\right)^2 \to 1, \text{ so } \alpha = 1 \tag{124}$$

Example 3.15 ()

The sequence $\sum \frac{c^n}{n!}$ always converges for $c \in \mathbb{R}$.

Theorem 3.17 (Weierstrass M-test for Absolute Convergence)

Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be series. Suppose there exists an index $N \in \mathbb{N}$ such that $|a_n| \leq b_n$ for all $n > N$. Then,

$$\sum_{n=1}^{\infty} b_n \text{ converges } \implies \sum_{n=1}^{\infty} a_n \text{ converges absolutely} \tag{125}$$

We finally conclude by giving a theorem about the convergence of some special sequences.

Theorem 3.18 (Special Sequences)

Some special sequences:

1. If $p > 0$, then $\lim_{n\to\infty} \frac{1}{n^p} = 0$.
2. If $p > 0$, then $\lim_{n\to\infty} \sqrt[n]{p} = 1$.
3. $\lim_{n\to\infty} \sqrt[n]{n} = 1$.

4. If $p > 0$ and α is real, then $\lim_{n\to\infty} \frac{n^\alpha}{(1+p)^n} = 0$.
5. If $|x| < 1$, then $\lim_{n\to\infty} x^n = 0$.

3.7 Euler's Number and Trigonometric Functions

Definition 3.13 (Euler's Number)

We define **Euler's number** as

$$e \coloneqq \sum_{n=0}^{\infty} \frac{1}{n!} \tag{126}$$

The first thing we should do is show that it converges, this is a one-liner.

$$\sum_{n=0}^{\infty} \frac{1}{n!} = 1 + 1 + \sum_{n=2}^{\infty} \frac{1}{n!} \leq 2 + \sum_{n=2}^{\infty} \frac{1}{n(n-1)} \tag{127}$$

Theorem 3.19 (Euler's Number as a Limit)

We have

$$\lim_{n\to+\infty} \left(1 + \frac{1}{n}\right)^n = e \tag{128}$$

Proof.

Let us define the sequence

$$t_n = \sum_{k=0}^{n} \frac{1}{k!}, \qquad s_n = \left(1 + \frac{1}{n}\right)^n \tag{129}$$

We know that $t_n \to e$, and we want to show that $s_n \to e$. We do this with the squeeze theorem.

1. We can see that

$$\begin{aligned}
s_n &= \left(1 + \frac{1}{n}\right)^n && (130)\\
&= \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} \left(\frac{1}{n}\right)^k && (131)\\
&= 1 + 1 + \frac{n(n-1)}{2!}\frac{1}{n^2} + \frac{n(n-1)(n-2)}{3!}\frac{1}{n^3} + \ldots && (132)\\
&= 1 + 1 + \frac{1}{2!}(1)\left(1 - \frac{1}{n}\right) + \frac{1}{3!}(1)\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) + \ldots + \frac{1}{n!}(1)\prod_{k=1}^{n-1}\left(1 - \frac{k}{n}\right) && (133)\\
&\leq \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots + \frac{1}{n!} = t_n && (134)
\end{aligned}$$

and so $s_n < t_n \implies \limsup s_n \leq \limsup t_n = e$.

2. Let $m \leq n$ be fixed. Then,

$$s_n \geq 1 + 1 + \frac{1}{2!}\left(1 - \frac{1}{n}\right) + \ldots + \frac{1}{m!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\ldots\left(1 - \frac{m-1}{n}\right) \tag{135}$$

since we are just taking the first m positive terms of the element. Therefore, letting $n \to +\infty$ and keeping m fixed, we get

$$\liminf_{n\to+\infty} s_n \geq 1 + 1 + \frac{1}{2!} + \ldots + \frac{1}{n!} \text{ for all } m \in \mathbb{N} \tag{136}$$

which implies $\liminf s_n \geq t_m$ for all $m \in \mathbb{N}$, and now letting $m \to +\infty$, we have $\liminf s_n \geq \liminf t_m = e$.

Now we prove the irrationality of e. It is usually extremely difficult to prove that an arbitrary number is irrational, e.g. π^e or π^{e^e}.

Theorem 3.20 (e is Irrational)

e is irrational.

Proof.

Letting $t_n = \sum_{k=0}^n \frac{1}{k!}$, we have

$$e - t_n = \sum_{k=n+1}^{\infty} \frac{1}{k!} \tag{137}$$
$$= \frac{1}{(n+1)!}\left(1 + \frac{1}{n+2} + \frac{1}{(n+3)(n+2)} + \dots\right) \tag{138}$$
$$< \frac{1}{(n+1)!}\underbrace{\left(1 + \frac{1}{n+2} + \frac{1}{(n+2)^2} + \dots\right)}_{\text{geometric}} \tag{139}$$
$$= \frac{1}{(n+1)!}\left(\frac{1}{1-(1/(n+2)!)}\right) \tag{140}$$
$$= \frac{1}{n!n} \cdot \underbrace{\frac{(n+2)n}{(n+1)^2}}_{<1} \tag{141}$$
$$= \frac{1}{n!n} \tag{142}$$

Note that we can combine and split sums since we know that e is convergent. Now suppose that $e = p/q$. Then,

$$0 < q!(e - t_q) < \frac{1}{q} \tag{143}$$

But $q!e$ is an integer and $q!t_q$ is also an integer. So we have $q! \cdot \frac{p}{q}$, an integer, between 0 and 1, which is a contradiction.

Since we have defined some number $e \in \mathbb{R}$, we know that exponential exist, and therefore we the function $x \mapsto e^x$ is well-defined. In fact, it is so important that we have a separate name for it.

Definition 3.14 (Exponential Function)

The **exponential function** is generally referred to as the function $x \mapsto e^x$.

There is a nice series representation.

Theorem 3.21 (Exponential Function as a Series)

We have

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \tag{144}$$

Proof.

Now that this is done, we can define the trigonometric functions formally as such.

Definition 3.15 (Trigonometric Functions)

We have

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \tag{145}$$

$$\tag{146}$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \tag{147}$$

4 Limits and Continuity of Functions

We now extend our analysis to real-valued functions over a metric space. The ones that we will be particularly interested in are *continuous functions*. But before this, let's introduce a new notation. Given a metric space X, we will talk about a variable x approaching a particular value $a \in X$, denoted $x \to a$. But this isn't clear. When we talk about the concept of something approaching another thing, we have two definitions.

1. A *sequence* can approach to its limit, which is a *point*.
2. A *point* can be a limit point of a *set*.

When we write $x \to a$, we are talking about some indeterminate variable x and a point a, it isn't immediately clear what this means. As we will soon define, this will refer to a neighborhood of a or equivalently to *all* sequences converging to a. So we can think of $x \to a$ as notation for all sequences $(x_n) \to a$.

Definition 4.1 (Constant and Ultimately Constant Functions)

Given a real-valued function $f : E \longrightarrow \mathbb{R}$ defined on domain $E \subset \mathbb{R}$,

1. f is a **constant function** if $f(x) = A$ for all $x \in E$
2. f is called **ultimately constant** as $x \to a$ if it is constant in some deleted neighborhood $\mathring{U}(a)$, where a is a limit point of E.

Definition 4.2 (Bounded and Ultimately Bounded Functions)

Given a real-valued function $f : E \longrightarrow \mathbb{R}$ defined on domain $E \subset \mathbb{R}$,

1. f is **bounded**, **bounded above**, or **bounded below** respectively if there is a number $C \in \mathbb{R}$ such that $|f(x)| < C$, $f(x) < C$, or $C < f(x)$ for all $x \in E$.
2. f is **ultimately bounded**, **ultimately bounded above**, or **ultimately bounded below** as $x \to a$ if it is bounded, bounded above, or bounded below in some deleted neighborhood $\mathring{U}_E(a)$.

Example 4.1 (Unbounded but Ultimately Bounded)

The function

$$f(x) = \sin\frac{1}{x} + x\cos\frac{1}{x} \tag{148}$$

for $x \neq 0$ is not bounded on the domain of definition, but it is ultimately bounded as $x \to 0$.

4.1 Limits of Functions

Definition 4.3 (Limit of a Function)

Let $f : X \to Y$ be a map between metric spaces, with $E \subset X$ and $p \in E'$ (note the limit point!). We say $f(x) \to q$ as $x \to p$, i.e.

$$\lim_{x \to p} f(x) = q \tag{149}$$

if it meets the following equivalent conditions.

1. ϵ-δ *Definition.* If $\forall \epsilon > 0$, $\exists \delta > 0$ s.t. $0 < d_X(x, p) < \delta \implies d_Y(f(x), q)) < \epsilon$.[a]

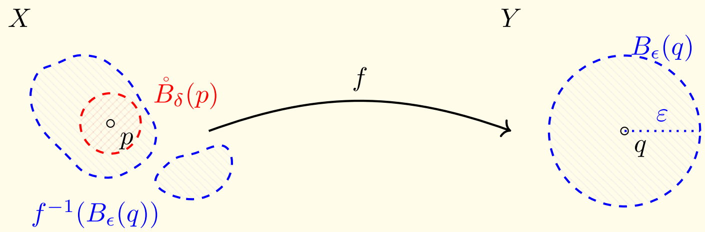


Figure 3: Said in one line, the preimage of any open ball around $y = f(x)$ must contain some open deleted open ball around x.

2. *Sequential Definition.* If for all sequences $(x_n) \to p$, $f(x_n) \to q$.

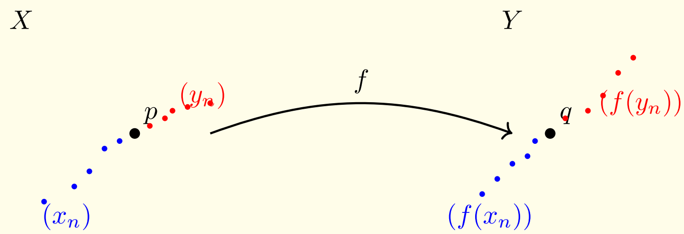


Figure 4: For every sequence that converges to the left, the new sequence mapped through f converges to q. Note that we choose the points x_n to be in the "deleted" neighborhood $E \setminus a$ (neighborhood E with point a removed) to force us to choose a sequence that is not $a, a, \ldots$. That is, it forces us to choose different points for the sequence.

[a]Note that the strictly inequality $0 < d_X(x, p)$ is important to ensure that $x \neq p$, since functions can jump at p.

Proof.

We prove equivalence.

1. $(\to)$. Assume $\lim_{x \to p} f(x) = q$. Let $(x_n) \in E$ s.t. $x_n \to p$ with $x_n \neq p$. We wish to show that $f(x_n) \to q$. Let $\epsilon > 0$. Then $\exists \delta > 0$ s.t. $0 < d_X(x, p) < \delta \implies d_Y(f(x), q) < \epsilon$. Since $\delta > 0$, by definition $\exists N \in \mathbb{N}$ s.t. if $n \geq N$, $d_X(x_n, p) < \delta \implies d_Y(f(x_n), q)$.

Sometimes, the ϵ-δ definition is good, but a lot of the times the sequential definition is good enough and more insightful.

Example 4.2 (Limit of the Signum Function)

The function sgn: $\mathbb{R} \longrightarrow \mathbb{R}$ defined

$$\operatorname{sgn} x = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases} \tag{150}$$

has no limit as $x \to 0$.

First, it is ludicrous that the limit would be any number that is not $\{-1, 0, 1\}$. If we assume that $A \notin \{-1, 0, 1\}$, then we can choose any arbitrarily small ϵ-neighborhood of A that does not include the three numbers. Clearly, there doesn't exist any $\delta > 0$ such that the deleted δ-neighborhood of 0 maps to a set completely contained in the ϵ-neighborhood of A. That is,

$$\operatorname{sgn}\big(\mathring{U}_\delta(0)\big) = \{-1, 1\} \not\subset U_\epsilon(A) \tag{151}$$

It doesn't even intersect the ϵ-neighborhood at all.

1. If $A = 1$, we can construct a ϵ-neighborhood V_A for $\epsilon = \frac{1}{2}$. Clearly, there exists no open neighborhood U_0 of 0 that is entirely mapped to V, since U_0 contains both negative numbers and 0 and hence must be mapped to $0, -1$.
2. Similarly, given the ($\epsilon = \frac{1}{2}$)-neighborhood of $A = -1$, there exists no open neighborhood U_0 of 0 that is entirely mapped to it, since U_0 contains both positive numbers and 0 and hence must be mapped to $0, 1$.
3. Finally, given the ($\epsilon = \frac{1}{2}$)-neighborhood of $A = 0$, there exists no open neighborhood U_0 of 0 that is entirely mapped to it, since U_0 contains both positive and negative numbers and hence must be mapped to ± 1.

Therefore, the limit does not exist.

Example 4.3 (Limit of Absolute Value of Signum Function)

We will show that

$$\lim_{x \to 0} |\operatorname{sgn} x| = 1 \tag{152}$$

We construct a ϵ-neighborhood $U_\epsilon(1)$ around 1. Given this neighborhood, we can imagine choosing the deleted δ-neighborhood $\mathring{U}_\delta(0)$ around 0. Since every element in $\mathring{U}_\delta(0)$ maps to 1, it is clearly in U_ϵ. In fact, for arbitrarily small $\epsilon > 0$, we can choose **any** $\delta > 0$ since everything in $\mathbb{R} \setminus 0$ maps to 1. We can visualize this in $\mathbb{R}^2$ as

Theorem 4.1 (Arithmetic on Limits of Functions)

Given two numerical valued functions $f, g : E \subset \mathbb{R} \longrightarrow \mathbb{R}$ with a common domain where $g(x) \neq 0$ for all $x \in E$, let

$$\lim_{x \to a} f(x) = A, \qquad \lim_{x \to a} g(x) = B \tag{153}$$

then,

$$\begin{aligned}
&\lim_{x \to a} (f + g)(x) = A + B \\
&\lim_{x \to a} (cf)(x) = cA \\
&\lim_{x \to a} (f \cdot g)(x) = A \cdot B \\
&\lim_{x \to a} \left(\frac{f}{g}\right)(x) = \frac{A}{B}
\end{aligned}$$

Proof.

Cauchy sequence criterion for a limit immediately proves this.

We end this with a theorem connecting the relationship between a limit of a function as $x \to a$ and its ultimate behavior as $x \to a$.

Theorem 4.2 ()

Let $f : E \longrightarrow \mathbb{R}$ be a function. Then,

1. f is ultimately the constant A as $x \to a$ implies that $\lim_{x \to a} f(x) = A$.
2. $\lim_{x \to a} f(x)$ implies that f is ultimately bounded as $x \to a$.

Definition 4.4 (Infinitesimal Function)

A function $f : E \subset \mathbb{R} \longrightarrow \mathbb{R}$ is said to be **infinitesimal** as $x \to a$ if

$$\lim_{x \to a} f(x) = 0$$

Lemma 4.1 (Sums, Products of Infinitesimals)

It is clear that if α, β are infinitesimal as $x \to a$, then

1. $\alpha + \beta$ is infinitesimal as $x \to a$
2. $\alpha \cdot \beta$ is infinitesimal as $x \to a$

Furthermore, if α is infinitesimal and β is ultimately bounded as $x \to a$, then the product $\alpha \cdot \beta$ is infinitesimal as $x \to a$.

Proof.

We prove all three statements.

1. Assume that α and β are infinitesimal as $x \to a$. Then, let us fix a small $\epsilon > 0$. This means that for every $\frac{\epsilon}{2}$ there exists an open deleted neighborhood $\mathring{U}'(a)$ such that its image $\alpha\big(\mathring{U}'(a)\big) \subset U'_{\epsilon/2}(0) \subset \mathbb{R}$. Additionally, for every $\frac{\epsilon}{2}$ there exists an open deleted neighborhood $\mathring{U}''(a)$ such that its image $\beta\big(\mathring{U}''(a)\big) \subset U'_{\epsilon/2}(0) \subset \mathbb{R}$. Thus, for the deleted neighborhood
$$\mathring{U}(a) \subset \mathring{U}'(a) \cup \mathring{U}''(a)$$
we can see that for all $x \in \mathring{U}(a)$,
$$|(\alpha + \beta)(x)| = |\alpha(x) + \beta(x)| \leq |\alpha(x)| + |\beta(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$
and hence $(\alpha + \beta)\big(\mathring{U}(a)\big) \subset U_\epsilon(0)$.
2. This case is a special case of assertion 3. That is, every function that has a limit is ultimately bounded.
3. Since $\beta(x)$ is ultimately bounded, this means that there exists a constant M and an open deleted neighborhood $\mathring{U}'(a) \subset E$ such that for all $x \in \mathring{U}'(a)$, its image is bounded: $|\beta(x)| < M$. Let us fix a small $\epsilon > 0$. Then, by definition of the limit, for every $\frac{\epsilon}{M}$ there exists an open deleted neighborhood $\mathring{U}''(a)$ such that its image $\beta\big(\mathring{U}''(a)\big) \subset U_{\epsilon/M}(0) \subset \mathbb{R}$. Therefore, for the deleted neighborhood
$$\mathring{U}(a) \subset \mathring{U}'(a) \cup \mathring{U}''(a)$$
we can see that for all $x \in \mathring{U}(a)$,
$$|(\alpha \cdot \beta)(x)| = |\alpha(x)\beta(x)| = |\alpha(x)||\beta(x)| < \frac{\epsilon}{M} \cdot M = \epsilon$$
Therefore, $(\alpha \cdot \beta)\big(\mathring{U}(a)\big) \subset U_\epsilon(0)$.

Note that in proving these properties of the limits, we have used the following fact about open deleted neighborhoods around a.

1. $\mathring{U}(a)$ is not the empty set.
2. Given open deleted neighborhoods $\mathring{U}'(a)$ and $\mathring{U}''(a)$, there exists an open deleted neighborhood in the intersections of these neighborhoods.
$$\mathring{U}(a) \subset \mathring{U}'(a) \cup \mathring{U}''(a)$$

Theorem 4.3 (Representation of a Convergent Function as a Shift of its Infinitesimal)

Given a function $f : E \subset \mathbb{R} \longrightarrow \mathbb{R}$, its limit exists and

$$\lim_{x \to a} f(x) = A \tag{154}$$

if and only if f can be represented as

$$f(x) = A + \alpha(x) \tag{155}$$

where α is infinitesimal as $x \to a$. We can visualize this theorem by thinking of a function f that results from a "shift" of an infinitesimal.

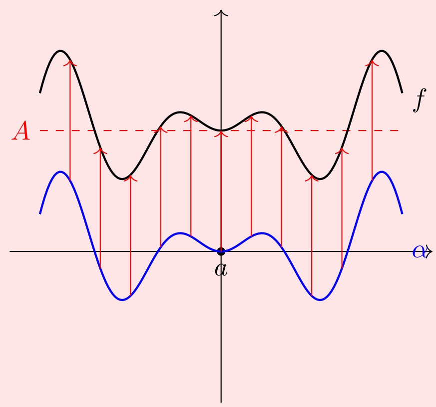


Figure 5: Shift of $f(x) = \frac{1}{2} x \sin(3x) + 2$.

Finally, we reiterate some limit theorems already stated for sequences, but now corresponding to functions. Interpreting the function limit as the Cauchy sequence definition of limits renders the proofs of these theorems trivial.

Theorem 4.4 (Bounds on Limits of Functions)

If the functions $f, g : E \to \mathbb{R}$ are such that

$$\lim_{x \to a} f(x) = A < B = \lim_{x \to a} g(x) \tag{156}$$

then there exists a deleted neighborhood $U_\delta(a)$ in E at each point of which $f(x) < g(x)$.

Theorem 4.5 (Squeeze Theorem for Limits of Functions)

Given the functions $f, g, h : E \subset \mathbb{R} \longrightarrow \mathbb{R}$ such that

$$f(x) \leq g(x) \leq h(x) \text{ for all } x \in E \tag{157}$$

then,

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = C \implies \lim_{x \to a} g(x) = C \tag{158}$$

4.2 Asymptotic Behavior of Functions

Definition 4.5 (Little-O Notation)

The function $f : E \longrightarrow \mathbb{R}$ is said to be **infinitesimal compared with the function** $g : E \longrightarrow \mathbb{R}$ as $x \to a$, written (by abuse of notation) $f = o(g)$ as $x \to a$, if

$$\lim_{x \to a} \frac{f(x)}{g(x)} = 1$$

or in other words, if f/g is an infinitesimal function as $x \to a$. Therefore, $f = o(1)$ as $x \to a$ means that f is infinitesimal as $x \to a$. [a]

[a]Note that writing $f = o(g)$ is again, an abuse of notation. $f = o(g)$ is really a shorthand way of writing that f is in the class of functions that is infinitesimal compared with the function g.

Intuitively, $f = o(g)$ means that the ratio between $f(x)$ and $g(x)$ will tend to infinity as $x \to a$ (this does not mean that f will be infinitely greater than g, however!).

Example 4.4 (Linear vs Quadratic)

For example, looking at the two functions $f(x) = x^2$ and $g(x) = x$, we have

1. $x^2 = o(x)$ as $x \to 0$ (since $\frac{x^2}{x} = x$ is infinitesimal as $x \to 0$)
2. $x = o(x^2)$ as $x \to \infty$ (since $\frac{x}{x^2} = \frac{1}{x}$ is infinitesimal as $x \to \infty$)

We can visualize $g/f(x)$ tending to infinity within a neighborhood of 0 and $f/g(x)$ tending to infinity within a neighborhood of ∞.

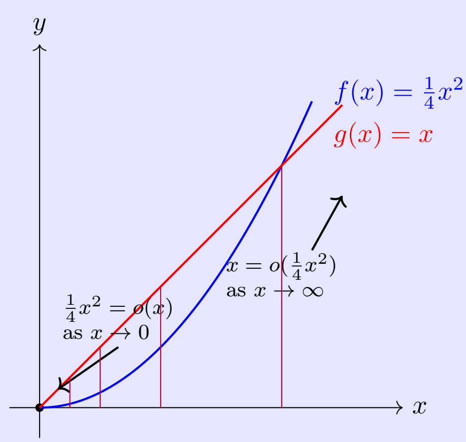


Figure 6

Definition 4.6 (Orders of Infinitesimals, Infinities)

If $f = o(g)$ and g is infinitesimal as $x \to a$, then f is an **infinitesimal of higher order than** g **as** $x \to a$. Furthermore, if f and g are infinite functions as $x \to a$ and $f = o(g)$ as $x \to a$, then g is a **higher order infinity than** f **as** $x \to a$.

Definition 4.7 (Big-O Notation)

By abuse of notation, $f = O(g)$ as $x \to a$ means that

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \infty \tag{159}$$

or in other words, f/g is ultimately bounded as $x \to a$. In particular, $f = O(1)$ as $x \to a$ means that f is bounded within a certain neighborhood $U(a)$ of a.

Definition 4.8 (Functions of Same Order)

The functions f and g are of the same over as $x \to a$, written

$$f \asymp g \text{ as } x \to a \tag{160}$$

if $f = O(g)$ and $g = O(f)$ as $x \to a$. Intuitively, this means that the ratio between f and g within some deleted neighborhood of a is finite.
Note that the condition that f and g be of the same order as $x \to a$ is (by definition of ultimately bounded functions) equivalent to the condition that there exist $c_1, c_2 > 0$ and an open neighborhood $U(a)$ such that the relations

$$c_1|g(x)| \leq |f(x)| \leq c_2|g(x)| \tag{161}$$

is true for $x \in U(a)$.

Definition 4.9 (Asymptotic Equivalence of Functions)

For functions f and g, if

$$\lim_{x \to a} \frac{f(x)}{g(x)} = 1 \tag{162}$$

we say that f **behaves asymptotically like** g **as** $x \to a$, or that f **is equivalent to** g **as** $x \to a$, written

$$f \sim g \text{ as } x \to a \tag{163}$$

Moreover, $\sim$ is an equivalence relation, which means that

1. $f \sim f$ as $x \to a$
2. $f \sim g$ as $x \to a \implies g \sim f$ as $x \to a$
3. $f \sim g$ and $g \sim h$ as $x \to a \implies f \sim h$ as $x \to a$

We list a few examples in order to develop some sort of visual intuition for when two functions are asymptotically equivalent.

Example 4.5 (Both Converges at Finite Value to Nonzero Finite Value)

If $f(a) = g(a) \neq 0$, then $f \sim g$ trivially since the ratio of f and g converges to 1 within a neighborhood of a.

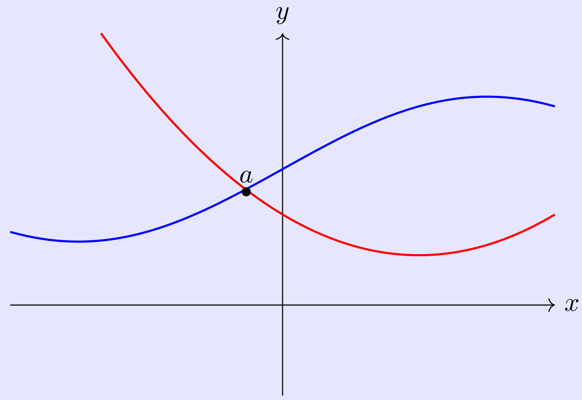
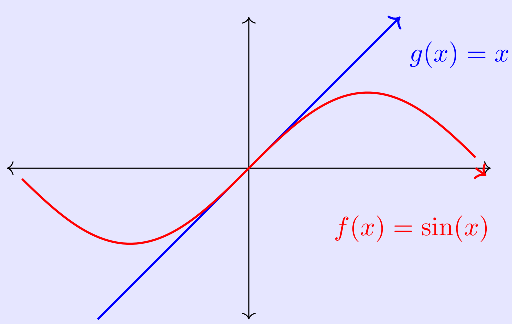


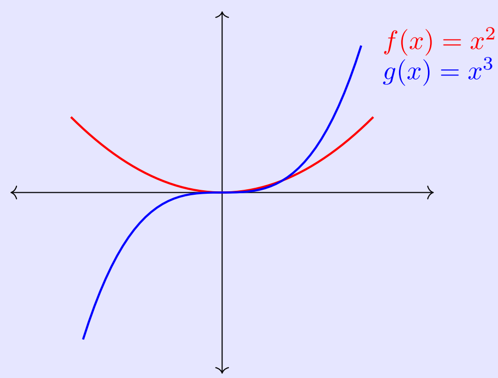
Figure 7

Example 4.6 (Both Converges at Finite Value to 0)

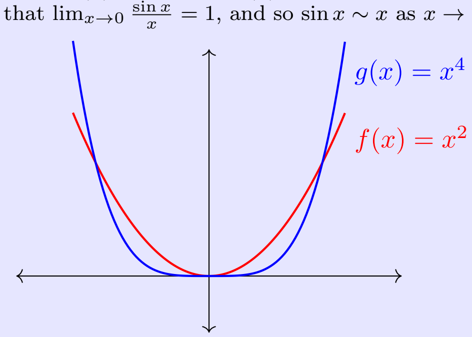
When $f(a) = g(a) = 0$, it may be f may be equivalent to g or one function may be infinitesimally smaller than the other.



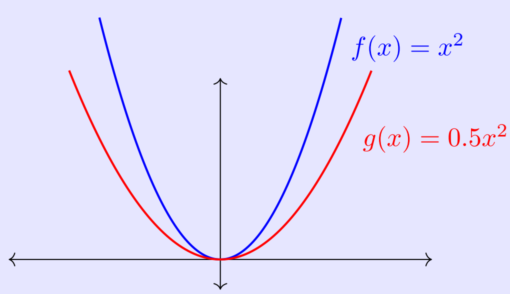
(a) When $f(x) = \sin x$ and $g(x) = x$, then $f \sim g$ since we see that $\lim_{x\to 0} \frac{\sin x}{x} = 1$, and so $\sin x \sim x$ as $x \to 1$.



(b) When $f(x) = x^2$ and $g(x) = x^3$, then $\lim_{x\to 0} \frac{x^3}{x^2} = 0$, and so $x^3 \not\sim x^2$. In fact, $x^3 = o(x^2)$.



(c) When $f(x) = x^2$ and $g(x) = x^4$, then $\lim_{x\to 0} \frac{x^4}{x^2} = 0$, and so $x^4 \not\sim x^2$. In fact, $x^4 = o(x^2)$.

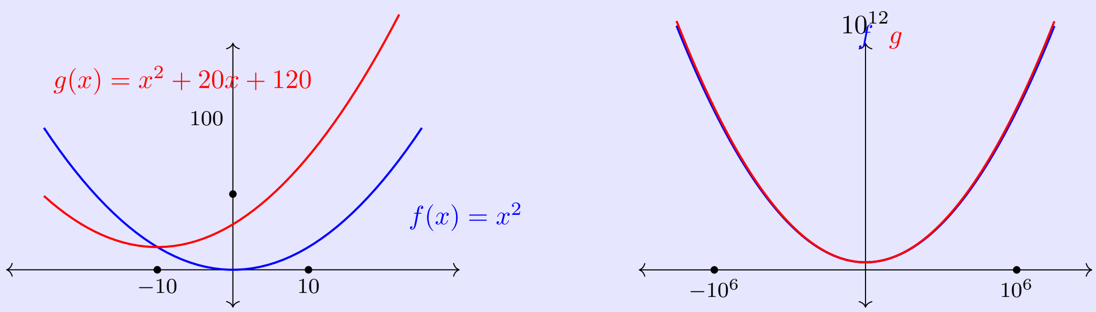


(d) When $f(x), g(x) = x^2, 0.5x^2$, then $\lim_{x\to 0} \frac{0.5x^2}{x^2} = \frac{1}{2}$. So $0.5x^2 \not\sim x^2$.

Figure 8: Examples of different scenarios.

Example 4.7 (Analyzing at Infinity)

When analyzing the behavior of functions as $x \to \infty$, we can picture the two graphs of f and g on the plane and "zoom out" to see if the ratio of the values converge to 1. This would mean that as $x \to \infty$, we should see the graphs overlapping more and more.



(a) Comparison of $f(x) = x^2$ and $g(x) = x^2 + 10x + 100$

(b) Asymptotic behavior with $g/f \approx 1$

Figure 9: taking $f(x) = x^2$ and $g(x) = x^2 + 10x + 100$, we can see that the discrepancy is high around a neighborhood of $x = 0$. But as $x \to +\infty$, we get $\lim_{x\to+\infty} \frac{x^2+10x+100}{x^2} = 1$, and so the graphs look like they are overlapping. Notice that even though the absolute difference $|(x^2 + 10x + 100) - x^2| = |10x + 100|$ tends to infinity, this difference increases infinitesimally compared to f and g.

From this, we can see that if $f \sim g$ as $x \to a$, then their difference

$$f - g = o(g) = o(f) \tag{164}$$

That is, $(f - g)(x)$ is infinitesimal compared to g or f (doesn't matter which one we compare it to). This leads to our next section, where we formalize this concept with absolute and relative errors.

4.2.1 Approximations of Functions

It is useful to note that since the relation $\lim_{x\to a} \gamma(x) = 1$ is equivalent to

$$\gamma(x) = 1 + \alpha(x), \text{ where } \lim_{x\to a} \alpha(x) = 0 \tag{165}$$

the relation $f \sim g$ as $x \to a$ is equivalent to saying that

$$\frac{f(x)}{g(x)} = \gamma(x), \text{ where } \lim_{x\to a} \gamma(x) = 1 \tag{166}$$

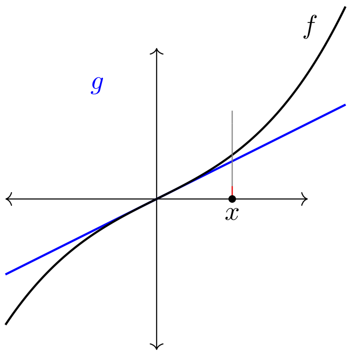
which implies

$$f(x) = g(x) + \alpha(x)g(x) = g(x) + o(g(x)) \text{ as } x \to a \tag{167}$$

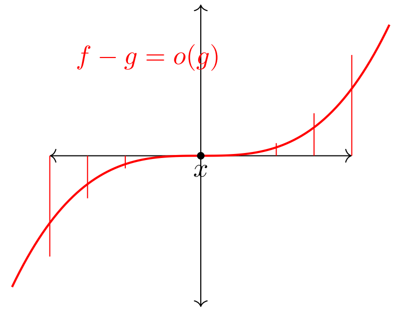
or, symmetrically,

$$g(x) = f(x) + \alpha(x)f(x) = f(x) + o(f(x)) \text{ as } x \to a \tag{168}$$

This means that f can be exactly represented by another function g, plus another (error) function $o(g(x))$ that is infinitesimal compared to g.



(a) Functions f, g, and their difference



(b) Behavior of $f - g$ (little-o of g)

Figure 10: Visualization of asymptotic behavior where $f - g = o(g)$

Note that it is not a sufficient condition that the error function be infinitesimal! The error function $f-g$ must be infinitesimal *compared to* g! This tells us that not only does the error function decrease infinitesimally, but also is infinitesimal compared to the approximation function we already have, which is in general a much stronger claim. This representation of certain types functions will provide the foundation for differential calculus when we talk about "good" approximations for a function.

Definition 4.10 (Relative Error)

Since $f \sim g$ as $x \to a$ means that

$$f(x) = g(x) + \alpha(x) g(x) = g(x) + o(g(x)) \tag{169}$$

we can define the **relative error** of g as an approximation of f to be

$$|\alpha(x)| = \left| \frac{f(x) - g(x)}{g(x)} \right| \tag{170}$$

Clearly, since $f \sim g$, the relative error must be infinitesimal as $x \to a$.

We use the following lemma to check whether two functions are asymptotically equivalent.

Lemma 4.2 ()

$f \sim g$ as $x \to a$ if and only if the relative error of g is infinitesimal as $x \to a$.

Example 4.8 ()

We claim that

$$x^2 + x = \left(1 + \frac{1}{x}\right) x^2 \sim x^2 \text{ as } x \to \infty \tag{171}$$

We see that the absolute error of this approximation $|(x^2 + x) - x^2| = |x|$ tends to infinity, but the relative error $\frac{|x|}{x^2} = \frac{1}{|x|} \to 0$ as $x \to \infty$.

Theorem 4.6 (Prime Number Theorem)

Let $\pi(x)$ be the number of prime numbers strictly less than x. Then $\pi \sim \frac{x}{\ln x}$ as $x \to +\infty$, or more precisely,

$$\pi(x) = \frac{x}{\ln x} + o\left(\frac{x}{\ln x}\right) \text{ as } x \to +\infty \tag{172}$$

Example 4.9 ()

It is a fact that $\lim_{x \to 0} \frac{sin x}{x} = 1$, so we have $\sin x \sim x$ as $x \to 0$. So,

$$\sin x = x + o(x) \text{ as } x \to 0 \tag{173}$$

The following theorem proves useful when computing limits.

Theorem 4.7 ()

If $f \sim \tilde{f}$ as $x \to a$, then

$$\lim_{x \to a} f(x)g(x) = \lim_{x \to a} \tilde{f}(x)g(x) \tag{174}$$

provided one of these limits exist.

Theorem 4.8 (Properties of $o(g)$ and $O(g)$ Functions)

For $x \to a$,

1. $o(f) + o(f) = o(f)$
2. $o(f)$ is also $O(f)$
3. $o(f) + O(f) = O(f)$
4. $O(f) + O(f) = O(f)$
5. If $g(x) \not\equiv 0$, then

$$\frac{o(f(x))}{g(x)} = o\left(\frac{f(x)}{g(x)}\right), \text{ and } \frac{O(f(x))}{g(x)} = O\left(\frac{f(x)}{g(x)}\right) \tag{175}$$

4.3 Continuous Functions

Definition 4.11 (Continuity of a Function)

A function f is **continuous at point** a if for any neighborhood $V\big(f(a)\big)$ of $f(a)$, there is a neighborhood $U(a)$ of a whose image under the mapping f is contained in $V\big(f(a)\big)$.
Generalizing this, we say that a function is **(globally) continuous** if the preimage of every neighborhood in its codomain is an open set in its domain.

Lemma 4.3 (Existence of Limits of Continuous Functions)

$f : E \longrightarrow \mathbb{R}$ is continuous at $a \in E$, where a is a limit point of E if and only if

$$\lim_{x \to a} f(x) = f(a) \tag{176}$$

Proof.

The limit equaling $f(a)$ means that, by definition, for any arbitrarily small deleted neighborhood of $f(a)$, denoted $U_{f(a)} \setminus f(a)$, its preimage will be an open neighborhood of a, which itself will contain an open set.

This also means that we can use the Cauchy limit definition to defined continuity of a function at a point. That is, for any sequence $\{a_n\}$ of point in codomain E which converges to point a, the function f is continuous at a if the corresponding sequence $\{f(a_n)\}$ converges to $f(a)$.

Theorem 4.9 ()

This means that the continuous functions commute with the operation of passing to the limit at a point.

$$\lim_{x \to a} f(x) = f\Big(\lim_{x \to a} x\Big) \tag{177}$$

Lemma 4.4 (Properties of Continuous Functions)

Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m,\ g : \mathbb{R}^m \longrightarrow \mathbb{R}^p$ with $c \in \mathbb{R}$.

1. f continuous at $x_0 \implies cf$ continous at x_0.
2. f, g continuous at $x_0 \implies f + g$ continuous at x_0.
3. Let $m = 1$. f, g continuous at $x_0 \implies fg$ continuous at x_0.
4. f continuous at x_0 and $f(x) \neq 0 \forall x \in \mathbb{R}^n \implies 1/f$ continuous at x_0.
5. If $f(x) = \big(f_1(x), f_2(x), ..., f_n(x)\big)$ coordinate-wise, then
$$f \text{ continuous at } x_0 \iff f_1, f_2, ..., f_m \text{ continuous at } x_0 \tag{178}$$
6. f continuous at x_0 and g continuous at $y_0 = f(x_0) \implies g \circ f$ continuous at x_0.

Proof.

This is an immediate result of the equivalence of a function being continuous at point a and its limit at point a existing.

Theorem 4.10 (Local Properties of Continuous Functions)

Let $f : E \longrightarrow \mathbb{R}$ be a function that is continuous at the point $a \in E$. Then,

1. f is bounded in some neighborhood $U(a)$.
2. If $f(a) \neq 0$, then in some neighborhood $U(a)$ all the values of the function have the same sign as $f(a)$.
3. If the function $g : U(a) \subset E \longrightarrow \mathbb{R}$ is defined in some neighborhood of a and is continuous at a, then the following functions
$$\begin{aligned} &(f+g)(x) \\ &(f \cdot g)(x) \\ &\Big(\frac{f}{g}\Big)(x) \text{ where } g(a) \neq 0 \end{aligned}$$
are also defined in $U(a)$ and continuous at a.
4. If the function $g : Y \longrightarrow \mathbb{R}$ is continuous at a point $b \in Y$ and f is such that $f : E \longrightarrow Y$, $f(a) = b$, and f is continuous at a, then the composite function
$$g \circ f : E \longrightarrow \mathbb{R}$$
is defined on E and continuous at a. This is easy to see because given the open neighborhood of $g(b)$, we know for a fact that $U_\delta(a)$ maps completely into $U_\epsilon(b)$, and that $U_\epsilon(b)$ maps completely into $U_\kappa(g(b))$ and so the composition of these mappings must mean that $U_\delta(a)$ maps completely into $U_\kappa(g(b))$.

Example 4.10 ()

An algebraic polynomial

$$P(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots + a_{n-1} x + a_n \tag{179}$$

is a continuous function on $\mathbb{R}$. Since $f(x) = x$ and $f(x) = c$ are continuous functions, by induction on x, we can multiply them together to find that $f(x) = x^n$ is continuous, which implies that ax^n is continuous, which implies that the sums of these functions are also continuous.

4.3.1 Intermediate and Extreme Value Theorem

Unlike local properties, the global property of a function is a property involving the entire domain of definition of the function.

Theorem 4.11 (Compact Sets to Compact)

If $f : X \to Y$ is continuous and $K \subset X$ is compact, then $f(K)$ is compact in Y.

Proof.

Corollary 4.1 (Extreme Value Theorem)

A continuous real-valued function over a compact set attains its maximum and minimum.

Theorem 4.12 (Intermediate Value Theorem)

If a function f is continuous on an open interval and assumes values $f(a) = A, f(b) = B$, then for any number $C \in (A, B)$, there is a point c between a and b such that $f(c) = C$.

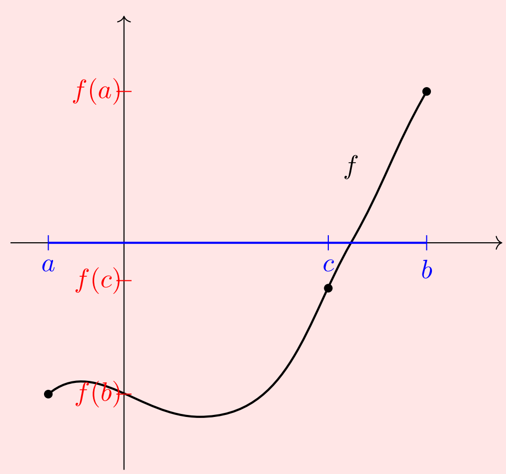


Figure 11: Illustration of a continuous function with a root in the interval $[a, b]$

Proof.

This following proof provides a very simple algorithm for finding the zero of the equation $f(x) = 0$ on an interval whose endpoints has values with opposite signs. Note that the colloquial description of the intermediate value theorem, that is is impossible to pass continuously from positive to negative values without assuming the value 0 along the way), assumes more than they state. That is, this theorem is actually dependent on the domain of definition: that is is a closed interval, or more generally, that it is **connected**.

4.3.2 Inverse Function Theorem

We begin by introducing this intuitive lemma.

Lemma 4.5 ()

A continuous mapping $f : E \longrightarrow \mathbb{R}$ of a closed interval $E = [a, b]$ into $\mathbb{R}$ is injective if and only if the function f is strictly monotonic on $[a, b]$.
Furthermore, every strictly monotonic function $f : X \subset \mathbb{R} \longrightarrow \mathbb{R}$ (for arbitrary X) has an inverse

$$f^{-1} : f(X) \subset \mathbb{R} \longrightarrow \mathbb{R}$$

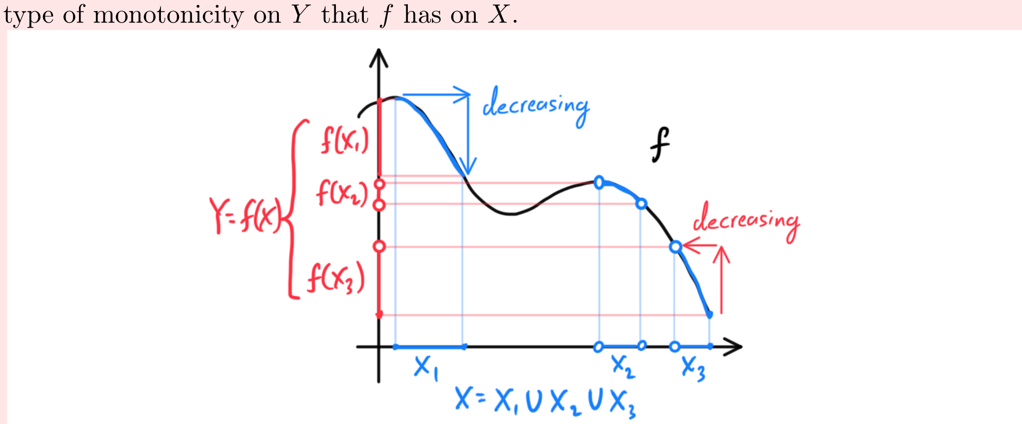
with the same kind of monotonicity on $f(X)$ that f has on X.

Lemma 4.6 (Criterion for Continuity of a Monotonic Function)

A monotonic function $f : E \longrightarrow \mathbb{R}$ defined on a closed interval $E = [a, b]$ is continuous if and only if its set of values $f(E)$ is the closed interval with endpoints $f(a)$ and $f(b)$.
Note that both conditions imply that there are no points of discontinuities in the graph of f.

Theorem 4.13 (Inverse Function Theorem)

A function $f : X \longrightarrow \mathbb{R}$ that is strictly monotonic on a set $X \subset \mathbb{R}$ has an inverse $f^{-1} : Y \longrightarrow \mathbb{R}$ defined on the set $Y = f(X)$ of values of f. The function $f^{-1} : Y \longrightarrow \mathbb{R}$ is monotonic and has the same type of monotonicity on Y that f has on X.



If in addition, X is a closed interval $[a, b]$ and f is continuous on X, then the set $Y = f(X)$ is the closed interval with endpoints $f(a)$ and $f(b)$ and the function $f^{-1} : Y \longrightarrow \mathbb{R}$ is continuous on it.

Example 4.11 (Sin and Arcsin)

The function $f(x) = \sin x$ is increasing and continuous on the closed interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Hence, the restriction to the closed interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ has an inverse $x = f^{-1}(y)$, called the **arcsin**, and denoted by $x = \arcsin y$.

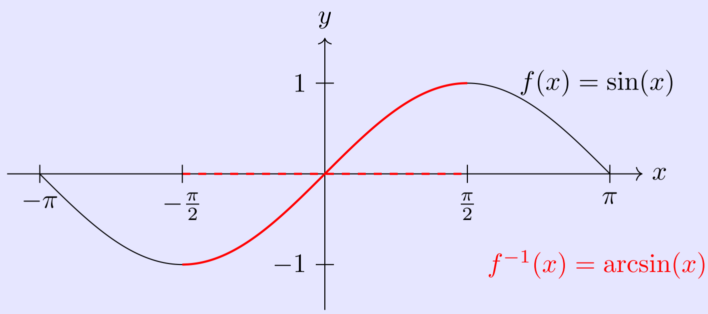


Figure 12: This function is defined on the closed interval $\big[-\sin\big(-\frac{\pi}{2}\big), \sin\big(-\frac{\pi}{2}\big)\big] = [-1, 1]$ and increases continuously from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$.

4.4 Uniform Continuity

Roughly speaking, a function f is uniformly continuous if it is possible to guarantee that $f(x)$ and $f(y)$ be as close to each other as we please by requiring only that x and y be sufficiently close to each other. Intuitively, uniform continuity says that given any two points x, y in the domain where their distance is arbitrarily small (δ apart), we can guarantee that the distance between $f(x), f(y)$ is at maximum some arbitrarily small ϵ.

Definition 4.12 (Uniform Continuity)

A function $f : E \longrightarrow \mathbb{R}$ is **uniformly continuous** on a set $E \subset \mathbb{R}$ if for every $\epsilon > 0$, there exists $\delta > 0$ such that

$$\big|f(x_1) - f(x_2)\big| < \epsilon \tag{180}$$

for all points $x_1, x_2 \in E$ such that $|x_1 - x_2| < \delta$.

Example 4.12 (Uniformly Continuous)

The following visual shows the radical function $f(x) = \sqrt{x}$ defined on $\mathbb{R}^+$. We can see that it satisfies uniform continuity because the graph does not escape the top and/or bottom of the $\epsilon \times \delta$ window, no matter where the box is located on the graph. More strictly speaking, no matter what we set the ϵ (how long the box is), uniform continuity says that we can choose a sufficient δ (width of the box) such that the graph does not escape the top/bottom of the window no matter where the window is.

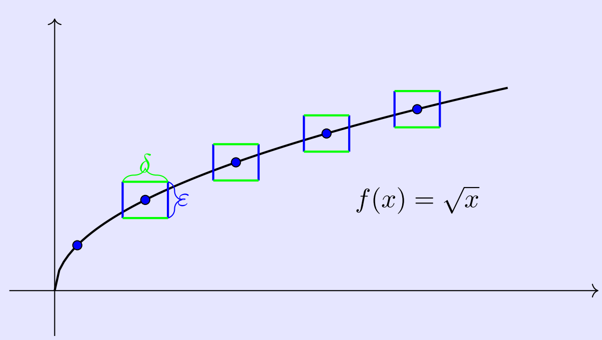


Figure 13: Graph of $f(x) = \sqrt{x}$ with ε-δ rectangles at various points

Example 4.13 (Not Uniformly Continuous)

We can clearly see that the function $f(x) = 1/x$ is not uniformly continuous, since the graph escapes the $\epsilon \times \delta$ window at some point (marked in red). More strictly speaking, given any length ϵ of the window, we cannot create a thin-enough δ box that will contain the graph, since as $x \to 1$, the function becomes unbounded. That is, arbitrarily thin boxes don't help when the slope is arbitrarily steep.

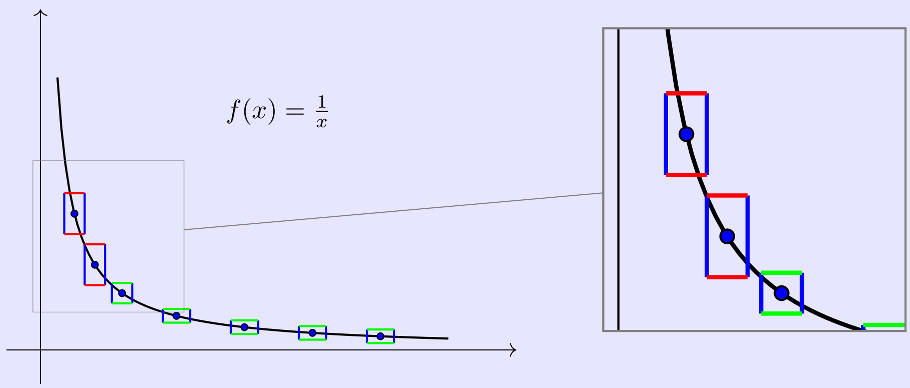
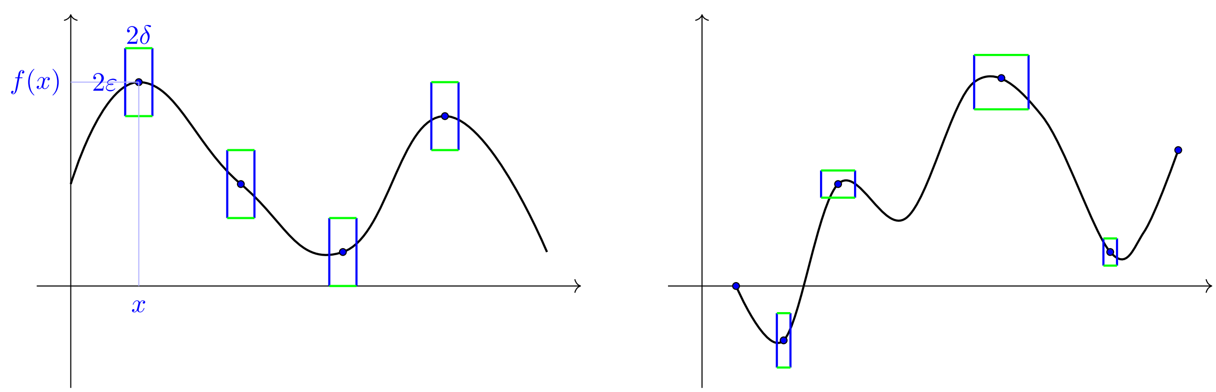


Figure 14: Graph of $f(x) = \frac{1}{x}$ with epsilon-delta boxes and magnified view

To compare uniform continuity with regular continuity, we can adapt this alternate (yet equivalent interpretation): Let there exist function $f : E \longrightarrow \mathbb{R}$. Given any $\epsilon > 0$, we can choose a $\delta > 0$ such that given any point $x \in E$ and $f(x)$, as long as a second point y is δ away from x, then $f(y)$ is ϵ away from $f(x)$. This visualization would lead to there being a $2\epsilon \times 2\delta$ window around point x. Therefore, given a certain $\epsilon > 0$, the way we choose δ is only dependent on ϵ, and so it must be a function of ϵ:

$$\delta = \delta(\epsilon) \tag{181}$$

However, in continuity, there just has to exist *some* δ-neighborhood of x such that its image is contained in the ϵ-neighborhood of $f(x)$.



(a) Uniform continuity means that the box above does not change dimensions no matter where the point is (hence, the name uniform).

(b) In continuity, there are no restrictions on the dimensions of this box. It just has to exist for every point, through a function of ϵ.

Figure 15: Uniform continuity vs continuity.

With this intuition, it is easy to see the result below.

Lemma 4.7 (Uniform Continuity Implies Continuity)

If f is uniformly continuous on the set E, it is continuous at each point of that set. However, the converse is not generally true.

A natural question one might ask is: under what assumptions is the converse true?

Theorem 4.14 (Cantor's Theorem on Uniform Continuity)

A function that is continuous on a closed interval is uniformly continuous on that interval.

Example 4.14 ()

Let $f : \mathbb{R} \longrightarrow \mathbb{R}$, $f(x) = 3x + 7$. Then f is uniformly continuous. Choose $\epsilon > 0$. Let $\delta = \epsilon/3$. Choose $x, y \in \mathbb{R}$ and assume $|x - y| < \delta$. Then,

$$|f(x) - f(y)| = |3x + 7 - 3y - 7| = 3|x - y| < 3\delta = \epsilon \tag{182}$$

Example 4.15 ()

Let $f : (0, 4) \subset \mathbb{R} \longrightarrow \mathbb{R}$, $f(x) = x^2$. Then f is uniformly continuous on $(0, 4)$. Choose $\epsilon > 0$. Let $\delta = \epsilon/8$. Choose $x, y \in (0, 4)$ and assume $|x - y| < \delta$. Then,

$$|f(x) - f(y)| = |x^2 - y^2| = (x + y)|x - y| < (4 + 4)|x - y| = 8\delta = \epsilon \tag{183}$$

4.5 Lipshitz and Holder Continuity

In both examples, the function satisfied an inequality of form

$$|f(x_1) - f(x_2)| \leq M|x_1 - x_2| \tag{184}$$

this is called the *Lipshitz inequality*. Lipshitz continuity is a strong form of uniform continuity for functions. Intuitively, a Lipshitz continuous function is limited in how fast it can change (by the Lipshitz constant).

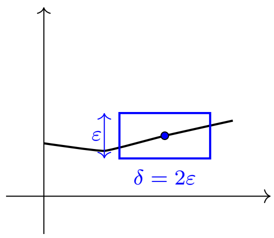
Definition 4.13 (Lipshitz Continuous Function)

Given $f : E \subset \mathbb{R} \longrightarrow \mathbb{R}$, f is **Lipshitz continuous** if there exists a $M > 0$—called the *Lipschitz constant*—such that for all $x, y \in E$,

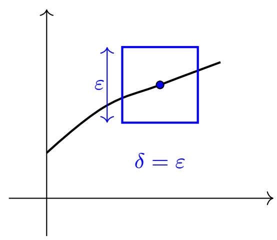
$$|f(x) - f(y)| \leq M|x - y| \tag{185}$$

Note that Lipshitz continuity pops up as a very natural extension of uniform continuity. The inequality above just means that given an ϵ, we can choose a δ such that a linear multiple of δ is always greater than ϵ. This means that Lipshitz continuity is just uniform continuity such that the δ function is linear:

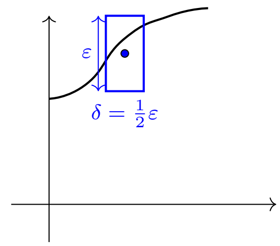
$$\delta = \delta(\epsilon) = \frac{1}{M}\epsilon \tag{186}$$



(a) $M = \frac{1}{2}$



(b) $M = 1$



(c) $M = 2$

Figure 16: Relationship between slope M and the ratio of δ to ε

Definition 4.14 (Bi-Lipshitz Continuity)

A function $f : E \subset \mathbb{R}$ is **Bi-Lipshitz continuous** if there exists constant $M \geq 1$ such that for all real $x, y \in E$,

$$\frac{1}{M}|x - y| \leq |f(x) - f(y)| \leq M|x - y|$$

It immediately follows that for $x \neq y$, $|f(x) - f(y)|$ cannot equal 0, which means that a bilipshitz map is injective. A bilipshitz map is really just Lipshitz map with its inverse also being Lipshitz.

Theorem 4.15 ()

A bilipshitz map f is a homeomorphism onto its image.

Definition 4.15 (Holder Continuity)

Given $f : E \subset \mathbb{R} \to \mathbb{R}$, f is **α-Holder continuous** if there exists a $C > 0$ such that for all $x, y \in E$,

$$|f(x) - f(y)| \leq M|x - y|^\alpha \tag{187}$$

4.6 Discontinuity

If the function $f : E \longrightarrow \mathbb{R}$ is not continuous at a point of E, then this point is called a **point of discontinuity**, or simply a **discontinuity** of f. That is, a is a point of discontinuity of f if for some neighborhood $V(f(a))$ of $f(a)$, there exists no neighborhood of a whose image under the mapping f is contained in $V(f(a))$. There are three types of discontinuities, ranging from least to most extreme.[7]

Definition 4.16 (Removable Discontinuity)

A **removable discontinuity** is characterized by the fact that the limit $\lim_{x \to a} f(x) = A$ exists, but $A \neq f(a)$.

[7]Note that strictly speaking, a removable discontinuity is really a discontinuity of first kind, but in this context we distinguish them.

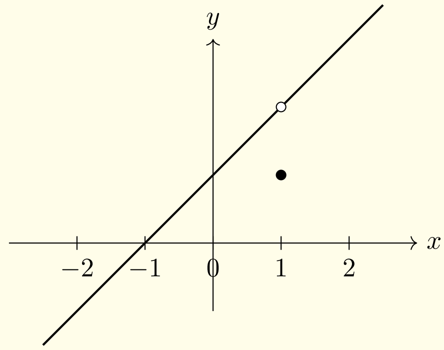


Figure 17: A function with a removable discontinuity at $x = 1$. The function is defined as $f(x) = \frac{x^2-1}{x-1}$ for $x \neq 1$ and $f(1) = 1$. The limit of the function as x approaches 1 is 2 (shown by the open circle), but the function value at $x = 1$ is 1 (shown by the filled circle).

This means that we can modify f and define a new function $\tilde{f} : E \longrightarrow \mathbb{R}$ as

$$\tilde{f}(x) = \begin{cases} f(x), & x \in E \setminus a \\ A, & x = a \end{cases} \tag{188}$$

which would be continuous on E.

Definition 4.17 (Jump/Step Discontinuity, of First Kind)

A **discontinuity of first kind**, also known as a jump/step discontinuity, is characterized by both the left and right-hand limits

$$\lim_{x \to a-0} f(x) \text{ and } \lim_{x \to a+0} f(x) \tag{189}$$

existing, but at least one of them is not equal to the value $f(a)$ that the function assumes at a.

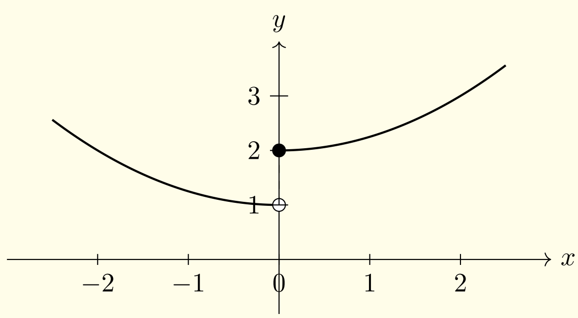


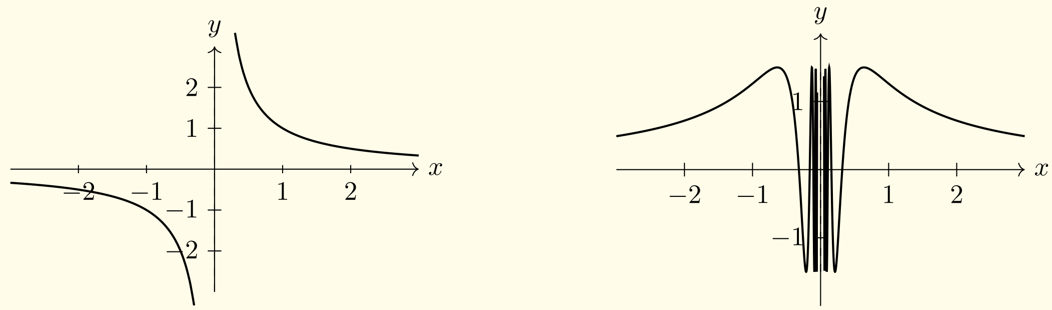
Figure 18: A function with a step discontinuity at $x = 0$. The function is defined as $f(x) = 1 + 0.25x^2$ for $x < 0$ and $f(x) = 2 + 0.25x^2$ for $x \geq 0$. The limit from the left $\lim_{x \to 0^-} f(x) = 1$ is shown by the open circle, while the function value at $x = 0$ is $f(0) = 2$ shown by the filled circle. The dashed line highlights the jump in value.

Definition 4.18 (Essential Discontinuity, of Second Kind)

A **discontinuity of second kind**, also known as an essential discontinuity, is characterized by at least one of the two limits

$$\lim_{x \to a-0} f(x) \text{ and } \lim_{x \to a+0} f(x) \tag{190}$$

not existing.



(a) The function $f(x) = \frac{1}{x}$ has an infinite discontinuity at $x = 0$

(b) The function $f(x) = |1.5 \sin\left(\frac{1}{x}\right)|$ has an oscillatory discontinuity at $x = 0$

Figure 19: Examples of discontinuities of the second kind, where the limit does not exist as x approaches the point of discontinuity

Example 4.16 (Dirichlet Function)

The Dirichlet function, defined

$$\mathcal{D}(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q} \\ 0, & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases} \tag{191}$$

is discontinuous at every point, and obviously all of its discontinuities are of second kind, since in every interval there are both rational and irrational numbers and therefore there exists no limit at any point $a \in \mathbb{R}$.
More specifically, given any point $a \in \mathbb{R}$, assume that a is rational. We can set $\epsilon = 0.1$-neighborhood around the value 1, but no matter how small we let δ, the interval $(a - \delta, a + \delta)$ will contain both rationals and irrationals, meaning that it will map to $\{0, 1\}$ always, which is not fully contained in $(0.9, 1.1)$.

Here is a slightly more interesting example.

Example 4.17 (Riemann Function)

Let the Riemann function $\mathcal{R}$ be defined

$$\mathcal{R}(x) = \begin{cases} \frac{1}{n}, & \text{if } x = \frac{m}{n} \in \mathbb{Q}, \text{ where } \gcd(m,n) = 1 \\ 0, & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases} \tag{192}$$

We first note that for any point $a \in \mathbb{R}$, any bounded neighborhood $U(a)$ of it, and any number $N \in \mathbb{N}$, the neighborhood $U(a)$ contains only a finite number of rational numbers $\gtrdot n$, where $n < N$. By shrinking the neighborhood, we can assume that the denominators of all rational numbers in the neighborhood are larger than N, since rationals with larger denominators have smaller gaps between them. Thus, at any point $x \in U(a) \setminus a$, we have

$$\big|\mathcal{R}(x)\big| < \frac{1}{N} \tag{193}$$

and therefore $\lim_{x \to a} \mathcal{R}(x) = 0$ at any point $a \in \mathbb{R} \setminus \mathbb{Q}$. Hence, the Riemann function is continuous at any irrational number.

5 Differentiation of Single-Variable Functions

5.1 Definition using Limits or Infinitesimals

In general, there are two ways that we define a derivative: as a limit and with infinitesimals. In a standard analysis course we use limits, but in differential equations physicists tend to use the language of infinitesimals. This is where our introduction of the hyperreals in smooth infinitesimal analysis (SIA) will be useful.

Definition 5.1 (Differentiation as a Limit)

Given $f : [a, b] \subset \mathbb{R} \to \mathbb{R}$, for $x \in [a, b]$, define the **difference quotient** as

$$\frac{f(x) - f(y)}{x - y} \tag{194}$$

If the following limit exists, we define it's value—called the **derivative** of f at x—as

$$f'(x) \coloneqq \lim_{y \to x} \frac{f(x) - f(y)}{x - y} \tag{195}$$

and say f is **differentiable** at x. If f is differentiable at x for every $x \in E$, then f is said to be *differentiable over* E.

So if the derivative exists, we can just treat it as a new function $g(x) = f'(x)$. Often, textbooks introduce the limit as

$$f'(x) \coloneqq \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \tag{196}$$

These two are equivalent definitions since the following two different quotients

$$\phi(y) = \frac{f(x) - f(y)}{x - y}, \qquad \gamma(h) = \frac{f(x + h) - f(x)}{h} \tag{197}$$

are related in the sense that $\phi(y) = \gamma(y - x)$. So the following two limits exist simultaneously (or fail to exist simultaneously). It turns out that if they do both exist, then

$$\lim_{y \to x} \phi(y) = \lim_{y \to x} \gamma(y - x) = \lim_{y \to 0} \gamma(y) = \lim_{h \to 0} \gamma(h) \tag{198}$$

where the only nontrivial equality is the second equality, which is true (should be shown). Now we present the second method using infinitesimals.

Definition 5.2 (Diferentiable Function)

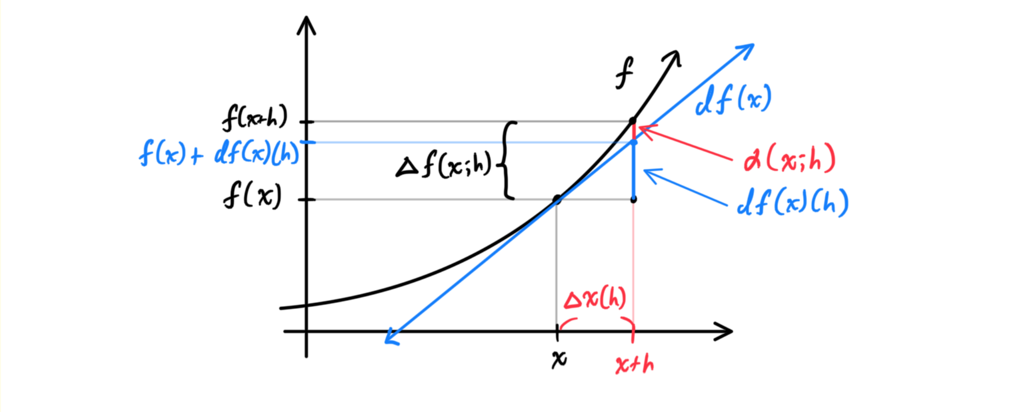
A function $f : E \subset \mathbb{R} \longrightarrow \mathbb{R}$ is **differentiable** at a given point x (that is a limit point of E) if there exists a linear function $h \mapsto df(x)h$ (called the **differential of** f) and an infinitesimal $\alpha(x; h) = o(h)$ as $h \to 0$, such that

$$f(x + h) - f(x) = df(x)(h) + \alpha(x; h)$$

Note that x is fixed; what we are really interested here is the h value. Furthermore,

1. $\Delta x(h) \equiv (x + h) - x = h$ is called the **increment of the argument**
2. $\Delta f(x; h) \equiv f(x + h) - f(x)$ is called the **increment of the function**

They are often denoted (inappropriately) by the symbols Δx and $\Delta f(x)$ representing functions of h. The differential and the infinitesimal can be visualized below.



Definition 5.3 (Derivative)

Given function $f : E \subset \mathbb{R} \longrightarrow \mathbb{R}$, the number

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

is called the **derivative** of the function f at x. This equality can also be written in the equivalent form:

$$\frac{f(x+h) - f(x)}{h} = f'(x) + \alpha(h)$$

where α is infinitesimal as $h \to 0$. This also also equivalent to:

$$f(x+h) - f(x) = f'(x)h + o(h)$$

where the error term $o(h) \to 0$ as $h \to 0$.

Note that we have defined the differentiability of a function at a point and the existence of its derivative at a point completely separately. But it turns out that the existence of this arbitrary number $f'(x)$ we call the "derivative," defined

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

actually has an equivalent form of

$$f(x+h) - f(x) = f'(x)h + o(h)$$

But since $f'(x)$ is in $\mathbb{R}$, the function $h \mapsto f'(x)h$ is linear and $o(h)$ is infinitesimal, so it is in the form

$$f(x+h) - f(x) = df(x)(h) + \alpha(x;h)$$

which, by definition, means that it is differentiable! Therefore, we have determined the equivalence between the differentiability of a function at a point and the existence of its derivative at the same point. Furthermore, this function $h \mapsto f'(x)h$ is precisely the differential of f, meaning that

$$df(x)(h) = f'(x)h$$

Furthermore,

$$\Delta f(x;h) - df(x)(h) = \alpha(x;h)$$

and $\alpha(x;h) = o(h)$ as $h \to 0$, or in other words, the difference between the increment of the function and the value of the function $df(x)$ in h is an infinitesimal of higher order than the first in h. For this reason, we say that the differential is the **principal linear part of the increment of the function**.

In particular, if $f(x) \equiv x$, then we have $f'(x) \equiv 1$ and

$$dx(h) = 1 \cdot h = h$$

Substituting this equality into $df(x)(h) = f'(x)h$, we get

$$df(x)(h) = f'(x)\, dx(h)$$

or without the input parameter h,

$$df(x) = f'(x)\, dx$$

Note that this is an equality between two functions of h. From this, we obtain the familiar **Leibniz notation** of the derivative:

$$\frac{df(x)(h)}{dx(h)} = f'(x) \iff \frac{df(x)}{dx} = f'(x)$$

That is, the function $\frac{df(x)}{dx}$, which is the ratio of the functions $df(x)$ and dx, is constant and equals $f'(x)$.

Let us try to construct successive approximations to an arbitrary function $f : E \longrightarrow \mathbb{R}$ at a given limit point x_0. That is, we find a function g such that

$$f = g + o(g)$$

Depending on what g is, we can construct better approximations of f.

Example 5.1 (Constant Approximation)

The 0th order approximation is when g is a constant. That is, $g \equiv c_0$ for some $c_0 \in \mathbb{R}$. This means

$$f(x) = c_0 + o(c_0) = c_0 + o(1) \text{ as } x \to x_0 \tag{199}$$

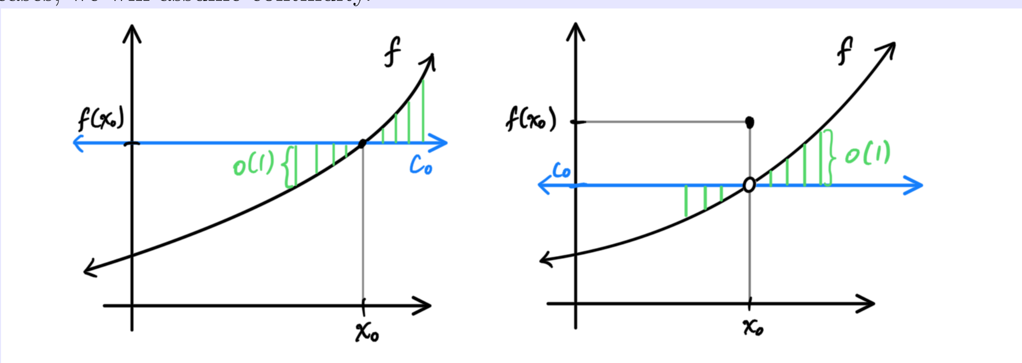
More precisely, we want this difference $f(x) - c_0$ to be $o(1)$ as $x \to x_0$, which means that it is simply infinitesimal. Visualizing this, we can see that given a constant approximation (labeled in blue) to a function at x_0, its error term (labeled in green) is in fact, infinitesimal. All this boils down to the fact that

$$\lim_{x \to x_0} f(x) = c_0$$

If the function is continuous at x_0, then

$$\lim_{x \to x_0} f(x) = f(x_0)$$

and naturally $c_0 = f(x_0)$. Both the continuous (left) and noncontinuous case (right) is shown, but in most cases, we will assume continuity.



Example 5.2 (Linear Approximation)

The 1st order approximation is a linear function that approximates f as

$$f(x) = c_0 + c_1(x - x_0) + o(x - x_0) \text{ as } x \to x_0 \tag{200}$$

Following the previous logic, assuming f continuous means that $c_0 = f(x_0)$. Furthermore, as $x \to x_0$

$$\begin{aligned} f(x) = c_0 + c_1(x - x_0) + o(x - x_0) &\implies c_1 = \frac{f(x) - c_0 - o(x - x_0)}{x - x_0} && (201) \\ &\implies c_1 = \frac{f(x) - c_0}{x - x_0} - \frac{o(x - x_0)}{x - x_0} && (202) \\ &\implies c_1 = \frac{f(x) - c_0}{x - x_0} - o(1) && (203) \\ &\implies c_1 = \lim_{x \to x_0} \frac{f(x) - c_0}{x - x_0} = f'(x_0) && (204) \end{aligned}$$

But this just means that $f'(x_0) = c_1$, Note that before, we have proved the equivalence of the existence of a derivative at x_0 with differentiability at x_0 (which itself means that there exists a linear approximation $df(x)(h)$ that is a function of h). Here, we have created a linear approximation with respect to $x = x_0 + h$, rather than h (shifted the function).
Therefore, the function

$$\alpha(x) = f(x_0) + f'(x_0)(x - x_0)$$

provides the best linear approximation to the function f in a neighborhod of x_0 in the sense that for any other function $\beta(x)$ of the form

$$\beta(x) = c_0 + c_1(x - x_0)$$

we have $f(x) - \beta(x) \neq o(x - x_0)$ as $x \to x_0$. The graph of the function α is the straight line

$$y - f(x_0) = f'(x_0)(x - x_0)$$

This leads to the definition of our familiar tangent line.

Definition 5.4 (Tangent Line)

If a function $f : E \longrightarrow \mathbb{R}$ is differentiable at a point $x_0 \in E$, the line defined by

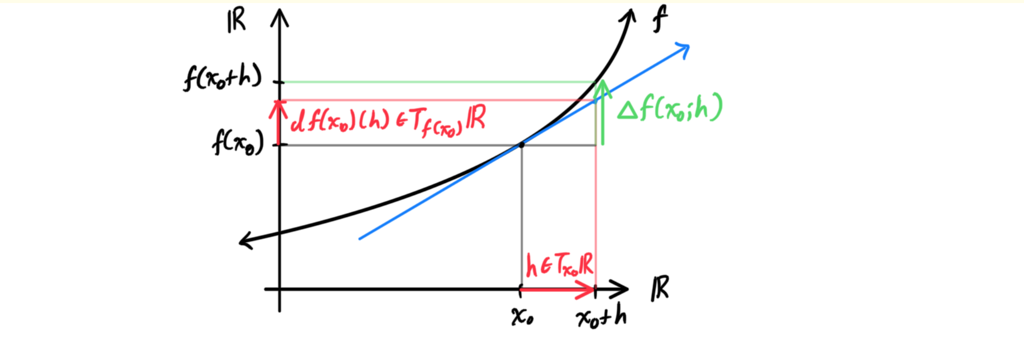
$$y - f(x_0) = f'(x_0)(x - x_0) \tag{205}$$

is called the **tangent** to the graph of f at the point $(x_0, f(x_0))$.

Tangent spaces.

Definition 5.5 (Tangent Space)

Given function $f : E \longrightarrow \mathbb{R}$ and a point $x_0 \in E$, the increment of the argument $h = x - x_0$ can be regarded as a vector attached to the point x_0 and defining the transition from x_0 to $x_0 + h$. h is called a **tangent vector**, and the set of all such vectors as $T_{x_0}\mathbb{R}$. Similarly, we denote $T_{y_0}\mathbb{R}$ the set of all displacement vectors from the point y_0 along the y-axis.



Then, we can see that the differential is a mapping

$$df(x_0) : T_{x_0}\mathbb{R} \longrightarrow T_{f(x_0)}\mathbb{R}$$

Note that that there are two functions to pay attention to here:

1. The true increment of f, defined $h \mapsto f(x_0 + h) - f(x_0) = \Delta f(x_0; h)$ (labeled in green).
2. The differential $h \mapsto f'(x_0)h = df(x_0)(h)$, which gives the increment of the tangent to the graph for increment h in the argument (labeled in red).

Example 5.3 ()

Let $f(x) = \sin x$. Then we will show that $f'(x) = \cos x$.

$$\begin{aligned}\lim_{h\to 0} \frac{\sin(x+h) - \sin(x)}{h} &= \lim_{h\to 0} \frac{2\sin\left(\frac{h}{2}\right)\cos\left(x + \frac{h}{2}\right)}{h} \\ &= \lim_{h\to 0} \cos\left(x + \frac{h}{2}\right) \cdot \lim_{h\to 0} \frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)} = \cos(x)\end{aligned}$$

Here, we have used the theorem on the limit of a product, the continuity of the function $\cos(x)$, the equivalence $\sin t \sim t$ as $t \to 0$, and the theorem on the limit of a composite function.

Example 5.4 ()

We will show that $\cos'(x) = -\sin(x)$.

$$\begin{aligned}\lim_{h\to 0} \frac{\cos(x+h) - \cos(x)}{h} &= \lim_{h\to 0} \frac{-2\sin\left(\frac{h}{2}\right)\sin\left(x + \frac{h}{2}\right)}{h} \\ &= -\lim_{h\to 0} \sin\left(x + \frac{h}{2}\right) \cdot \lim_{h\to 0} \frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)} = -\sin(x)\end{aligned}$$

Lemma 5.1 (Differentiability Implies Continuity)

If f is differentiable at x, it is continuous at x.

Proof.

If f is differentiable at x, then the derivative

$$f'(x) = \lim_{y\to x} \frac{f(x) - f(y)}{x - y} \tag{206}$$

exists. Therefore,

$$0 = f'(x) \cdot 0 = f'(x)\big(\lim_{y \to x} x - y\big) \tag{207}$$
$$= \lim_{y \to x} \frac{f(x) - f(y)}{x - y} \cdot \lim_{y \to x} x - y \tag{208}$$
$$= \lim_{y \to x} \frac{f(x) - f(y)}{x - y}(x - y) \tag{209}$$
$$= \lim_{y \to x} f(x) - f(y) \tag{210}$$

which implies that $f(x) = \lim_{y \to x} f(y)$, and hence f is continuous at x.

Therefore, we can see that the set of differentiable functions is a subset of the set of continuous functions.

5.2 Rules of Differentiation

Lemma 5.2 (Arithmetic)

If f and g are differentiable at x, then

1. $f + g$ is differentiable at x with
$$(f + g)'(x) = f'(x) + g'(x) \tag{211}$$
2. fg is differentiable at x with
$$(fg)' = f'(x)g(x) + f(x)g'(x) \tag{212}$$
3. f/g is differentiable at x with
$$\bigg(\frac{f}{g}\big)\bigg)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2} \tag{213}$$

Proof.

The proof for addition is pretty trivial, so we will prove for multiplication and division. For products, let's not take the quotient just yet.

$$(fg)(x) - (fg)(y) = f(x)g(x) - f(y)g(y) \tag{214}$$

We know something about $f(x) - f(y)$ and $g(x) - g(y)$, so try to put it into this form.

$$\big(f(x) - f(y)\big)g(x) + f(y)\big(g(x) - g(y)\big) \tag{215}$$

Therefore,

$$\frac{(fg)(x) - (fg)(y)}{x - y} = \underbrace{\frac{f(x) - f(y)}{x - y}}_{\text{exists}} g(x) + f(y) \underbrace{\frac{g(x) - g(y)}{x - y}}_{\text{exists}} \tag{216}$$

So by taking limits, f is continuous so $f(y) \to x$ as $y \to x$, and we finally have

$$f'(x)g(x) + f(x)g'(x) \tag{217}$$

For the quotient rule, it suffices to show from the product rule that $(1/g)'(x) = -\frac{g'(x)}{g(x)^2}$.

In proving properties of differentiability, it is useful to observe that

$$\lim_{y \to x} \frac{f(x) - f(y)}{x - y} \iff f(x) - f(y) = (x - y)(f'(x) + E) \tag{218}$$

for some function E where $\lim_{y \to x} E(x) = 0$. This is known as Taylor's formula. This is similar to the decomposition of sequences into a constant plus an infinitesimal sequence.

From this, we can find the derivative of polynomials.

Corollary 5.1 (Polynomial Derivatives)

The following are true.

1. The derivative of a constant function is 0
2. The derivative of the identity function $f(x) = x$ is 1.
3. The derivative of $f(x) = x^n$ is nx^{n-1}.
4. The derivative of a polynomial $f(x) = a_n x^n + \ldots + a_0$ can then be found.

Proof.

Theorem 5.1 (Chain Rule)

Let $f : [a, b] \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function, fix $x \in [a, b]$, and assume differentiable $g : I \to \mathbb{R}$ where $f(x) \in I$. Then, $h = g \circ f$ is differentiable at x with the derivative

$$h'(x) = g'(f(x))f'(x) \tag{219}$$

Proof.

We have

$$g\big(f(x)\big) - g\big(f(y)\big) = \big(f(x) - f(y)\big)\big(g'(f(x)) + E_{f(y) \to f(x)}\big) \tag{220}$$

Now we divide by $x - y$.

$$\frac{g(f(x)) - g(f(y))}{x - y} - \frac{f(x) - f(y)}{x - y} \cdot \big(g'(f(x)) + E_{f(y) \to f(x)}\big) \tag{221}$$

Now if $x \to y$, then f is continuous, which implies $f(y) \to f(x)$ and so $E \to 0$, and so by taking this limit, the above evaluates to

$$f'(x) \cdot g'(f(x)) \tag{222}$$

We could have also done

$$\frac{g(f(x)) - g(f(y))}{x - y} = \frac{g(f(x)) - g(f(y))}{f(x) - f(y)} \cdot \frac{f(x) - f(y)}{x - y} \to g'(f(x)) \cdot f'(x) \tag{223}$$

Theorem 5.2 (Differentiation of Inverse Functions over $\mathbb{R}$)

Let $E_1, E_2 \subset \mathbb{R}$, and $f : E_1 \longrightarrow E_2$ and $f^{-1} : E_2 \longrightarrow E_1$ be mutually inverse and continuous at points $x_0 \in E_1$ and $f(x_0) = y_0 \in E_2$. If f is differentiable at x_0 and $f'(x_0) \neq 0$, then f^{-1} also differentiable at the point y_0, and

$$\big(f^{-1}\big)^{-1}(y_0) = \big(f'(x_0)\big)^{-1} \iff df^{-1}(y_0) = \big(df(x_0)\big)^{-1}$$

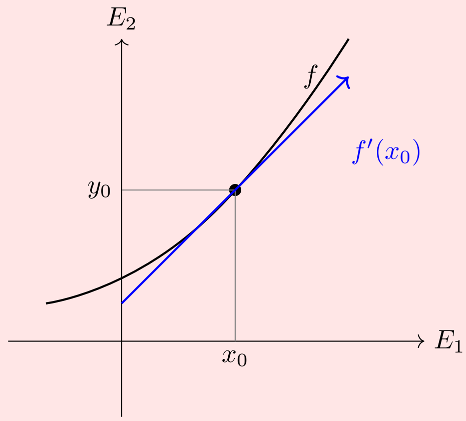


Figure 20: Relationship between a function f and its inverse f^{-1}, showing how their derivatives are related

Note that if we knew in advance that f^{-1} was differentiable at y_0 (which is a stronger hypothesis), we can find immediately by the identity

$$(f^{-1} \circ f)(x) = x$$

and the theorem on the differentiation of a composite function that

$$(f^{-1})'(y_0) \cdot f'(x_0) = 1$$

Note that if the hypothesis was satisfied, but $f'(x_0) = 0$, then f^{-1} would not be differentiable since it would have an undefined differential.

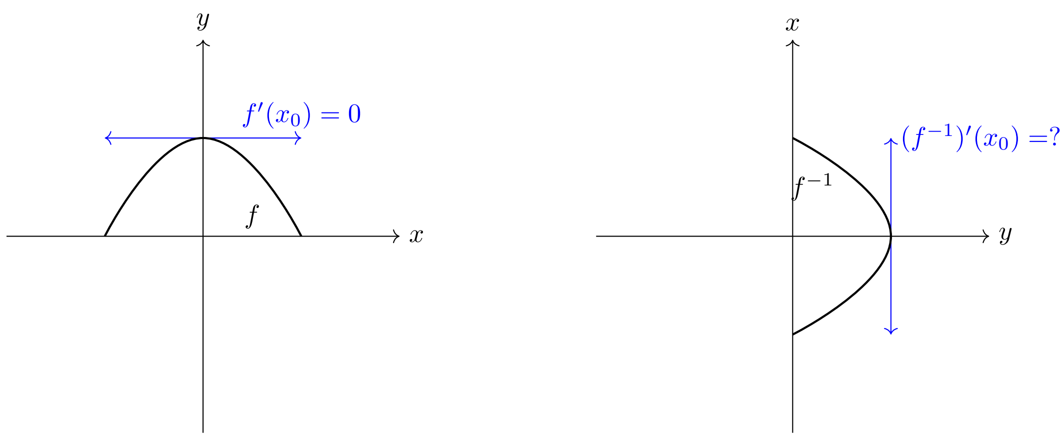


Figure 21: A function through three points and its inverse relation

5.2.1 Basic Properties; Derivatives of Composite, Inverse Functions

Theorem 5.3 (Arithmetic)

If functions $f, g : E \longrightarrow \mathbb{R}$ are differentiable at a point $x \in E$, then

1. their sum is differentiable at x, and

$$d(f+g)(x) = df(x) + dg(x) \iff (f+g)'(x) = (f'+g')(x)$$

2. their product is differentiable at x, and

$$d(f \cdot g)(x) = g(x)df(x) + f(x)dg(x) \iff (f \cdot g)'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x)$$

3. their quotient is differentiable at x if $g(x) \neq 0$, and

$$d\Big(\frac{f}{g}\Big)(x) = \frac{g(x)df(x) - f(x)dg(x)}{g^2(x)} \iff \left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$$

It is clear that $c \cdot df(x) = d(cf)(x)$, it is clear that the derivative is a linear operator from the space of all functions differentiable at x_0 the space of all functions.

Proof.

Since f and g are differentiable at x, there exists the differential $df(x)(h) = f'(x)h$ and $dg(x) = g'(x)h$ where

$$\begin{aligned} f(x+h) &= f(x) + df(x)(h) + o(h) = f(x) + f'(x)h + o(h) \\ g(x+h) &= g(x) + dg(x)(h) + o(h) = g(x) + g'(x)h + o(h) \end{aligned}$$

From this relation, we can clearly see that a certain property of the differential automatically implies the same property of the derivative. (Remember that $f'(x)$ and $g'(x)$ are not functions! They are scalars defined on fixed point x.)

1. Even though this derivation may be a bit long, every step is included to minimize ambiguity.

$$\begin{aligned} (f+g)(x+h) - (f+g)(x) &= \big(f(x+h) + g(x+h)\big) - \big(f(x) + g(x)\big) \\ &= \big(f(x+h) - f(x)\big) + \big(g(x+h) - g(x)\big) \\ &= \big(df(x)(h) + o(h)\big) + \big(dg(x)(h) + o(h)\big) \\ &= \big(f'(x)h + o(h)\big) + \big(g'(x)(h) + o(h)\big) \\ &= \big(f'(x) + g'(x)\big)h + o(h) \\ &= (f' + g')(x)(h) + o(h) \\ &= d(f+g)(x)h + o(h) \end{aligned}$$

2. For the product rule, we have

$$\begin{aligned} (f \cdot g)(x+h) - (f \cdot g)(x) &= f(x+h)g(x+h) - f(x)g(x) \\ &= \big(f(x) + df(x)(h) + o(h)\big)\big(g(x) + dg(x)(h) + o(h)\big) - f(x)g(x) \\ &= \big(f(x) + f'(x)h + o(h)\big)\big(g(x) + g'(x)h + o(h)\big) - f(x)g(x) \end{aligned}$$

Expanding this gives

$$\big(f'(x)g(x) + f(x)g'(x)\big)h + \big(f(x) + g(x)\big)o(h) + \\ f'(x)g'(x)h^2 + \big(f'(x) + g'(x)\big)ho(h) + \big(o(h)\big)^2$$

but note that since $f(x), g(x), f'(x), g'(x)$ are constants, we see that

(a) $\big(f(x) + g(x)\big)o(h) = o(h)$ because

$$\lim_{h \to 0} \frac{\big(f(x) + g(x)\big)o(h)}{h} = \big(f(x) + g(x)\big) \lim_{h \to 0} \frac{o(h)}{h} = 0$$

(b) $f'(x)g'(x)h^2 = o(h)$ since

$$\lim_{h \to 0} \frac{f'(x)g'(x)h^2}{h} = f'(x)g'(x) \lim_{h \to 0} h = 0$$

(c) $\big(f'(x) + g'(x)\big)ho(h) = o(h)$ because

$$\lim_{h \to 0} \frac{\big(f'(x) + g'(x)\big)ho(h)}{h} = \big(f'(x) + g'(x)\big) \lim_{h \to 0} o(h) = 0$$

In fact, this term is of $o(h^2)$.

(d) We can see that $(o(h))^2 = o(h)$ since

$$\lim_{h \to 0} \frac{(o(h))^2}{h} = \lim_{h \to 0} \frac{o(h)}{h} \cdot \lim_{h \to 0} o(h) = 0 \cdot 0 = 0$$

In fact, $(o(h))^2 = o(h^2)$.

Therefore, the above simplifies to

$$(f \cdot g)(x+h) - (f \cdot g)(x) = \big(f'(x)g(x) + f(x)g'(x)\big)h + o(h)$$

But this means that the differential (best approximation) $d(f \cdot g)(x)$ must be

$$(f \cdot g)'(x)(h) = (f \cdot g)'(x)h = \big(f'(x)g(x) + f(x)g'(x)\big)h$$

3. Since the function $g(x) \neq 0$ at point x, then by continuity we can assume that there exists a neighborhood $U(x)$ where the image of that neighborhood does not vanish. That is, we can guarantee that $g(x+h) \neq 0$ for sufficiently small values of h. We assume h is small in the following computations.

$$\begin{aligned}
\left(\frac{f}{g}\right)(x+h) - \left(\frac{f}{g}\right)(x) &= \frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)} \\
&= \frac{1}{g(x)g(x+h)}\big(f(x+h)g(x) - f(x)g(x+h)\big) \\
&= \left(\frac{1}{g^2(x)} + o(1)\right)\Big(\big(f(x) + f'(x)h + o(h)\big)g(x) \\
&\qquad - f(x)\big(g(x) + g'(x)h + o(h)\big)\Big) \\
&= \left(\frac{1}{g^2(x)} + o(1)\right)\Big(\big(f'(x)g(x) - f(x)g'(x)\big)h + o(h)\Big) \\
&= \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}h + o(h)
\end{aligned}$$

Note that here we have used the continuity of g at the point x and the fact that $g(x) \neq 0$ to deduce that

$$\lim_{h \to 0} \frac{1}{g(x)g(x+h)} = \frac{1}{g^2(x)} \iff \frac{1}{g(x) + g(x+h)} = \frac{1}{g^2(x)} + o(1)$$

where $o(1)$ is infinitesimal as $h \to 0$.

Theorem 5.4 (Chain Rule)

Let there be functions $f : E_1 \subset \mathbb{R} \longrightarrow E_2 \subset \mathbb{R}$ is differentiable at a point $x \in E_1$ and the function $g : E_2 \subset \mathbb{R} \longrightarrow \mathbb{R}$ is differentiable at point $y = f(x) \in E_2$, with respective differentials

$$\begin{aligned}
df(x) &: T_x\mathbb{R} \longrightarrow T_y\mathbb{R} \\
dg(y) &: T_y\mathbb{R} \longrightarrow T_{g(y)}\mathbb{R}
\end{aligned}$$

Then the composite function $g \circ f : E_1 \longrightarrow \mathbb{R}$ is differentiable at x, and $d(g \circ f)(x) : T_x\mathbb{R} \longrightarrow T_{g \circ f(x)}\mathbb{R}$ is

$$d(g \circ f)(x) = dg(y) \circ df(x) \iff (g \circ f)'(x) = g'\big(f(x)\big) \circ f'(x)$$

Proof.

We will denote the increment of the argument with the variables h and t. Then, by differentiability of f and g, we have

$$\begin{aligned} f(x+h) - f(x) = f'(x)h + o(h) \text{ as } h \to 0 \\ g(y+t) - g(y) = g'(y)t + o(t) \text{ as } t \to 0 \end{aligned}$$

Since the function $o(t)$ can be represented as $o(t) = \gamma(t)t$, where $\gamma = o(1)$ and hence is infinitesimal as $t \to 0$, meaning that we can assume $\gamma(0) = 0$ (since $o(t)$ is defined for $t = 0$).
We can think of the displacement of x as like a chain reaction: As$x \mapsto x+h$, $f(x) \mapsto f(x+h)$, which we could interpret as $y \mapsto y+t$ and hence means that $g(y) \mapsto g(y+t)$. So, setting $f(x) = y$ and $f(x+h) = y+t$, by differentiability and hence continuity of f at point x, we can conclude that $t \to 0$ as $h \to 0$. So, we have

$$\gamma\big(f(x+h) - f(x)\big) = \gamma\big((y+t) - y\big) = \gamma(t) = \alpha(h) \to 0 \text{ as } h \to 0$$

Thus, we get

$$\begin{aligned} o(t) = \gamma(t)t &= \gamma\big(f(x+h) - f(x)\big)\big(f(x+h) - f(x)\big) \\ &= \alpha(h)\big(f'(x)h + o(h)\big) \\ &= \alpha(h)f'(x)h + \alpha(h)o(h) \\ &= o(h) + o(h) = o(h) \text{ as } h \to 0 \\ (g \circ f)(x+h) - (g \circ f)(x) &= g\big(f(x+h)\big) - g\big(f(x)\big) \\ &= g(y+t) - g(y) \\ &= g'(y)t + o(t) \\ &= g'\Big(f(x)\big)\big(f(x+h) - f(x)\big) + o\big(f(x+h) - f(x)\big) \\ &= g'\big(f(x)\big)\big(f'(x)h + o(h)\big) + o\big(f(x+h)\big) - f(x)\big) \\ &= g'\big(f(x)\big)\big(f'(x)h\big) + g'\big(f(x)\big)\big(o(h)\big) + o\big(f(x+h) - f(x)\big) \end{aligned}$$

Since $g'\big(f(x)\big)\big(o(h)\big)$ is really just a constant multiplied by a function that is $o(h)$, it is $o(h)$. $o\big(f(x+h) - f(x)\big)$. As for $o\big(f(x+h) - f(x)\big)$, we see that since $f(x+h) - f(x) = t$, a function that is $o\big(f(x+h) - f(x)\big)$ becomes infinitesimal compared to t as $t \to 0$. As already stated before, we have

$$o\big(f(x+h) - f(x)\big) = o(h) \text{ as } h \to 0$$

and thus, we proved that

$$\begin{aligned} (g \circ f)(x+h) - (g \circ f)(x) &= g'(y)f'(x)h + o(h) \\ &= \big(dg(y) \circ df(x)\big)(h) + o(h) \end{aligned}$$

5.3 Theorems of Differentiable Functions

Theorem 5.5 (Local Extrema of Differentiable Functions Have Vanishing Derivative)

Let $f : [a, b] \to \mathbb{R}$ and assume f has a local maximum at $c \in (a, b)$ with f differentiable at c. Then $f'(c) = 0$.

Proof.

Let us pick two sequences—a left one and a right one—that converges to x from either side.

1. If $x > c$ (with x sufficiently close to c), then $f(c) \geq f(x)$ and so

$$\frac{f(c) - f(x)}{c - x} \leq 0 \implies f'(c) = \lim_{c-x} \frac{f(c) - f(x)}{c - x} \leq 0 \tag{224}$$

2. If $x < c$, then $f(c) \geq f(x)$ and so

$$\frac{f(c) - f(x)}{c - x} \geq 0 \implies f'(c) = \lim_{c-x} \frac{f(c) - f(x)}{c - x} \geq 0 \tag{225}$$

So $0 \leq f'(c) \leq 0 \implies f'(c) = 0$.

Note that it is generally not true that $f'(c) = 0$ if $c = a$ or $c = b$, i.e. at the endpoints.

Theorem 5.6 (Rolle's Theorem)

Suppose $f : [a, b] \to \mathbb{R}$ is differentiable on (a, b). Then, if $f(a) = f(b)$, then there exists a $c \in (a, b)$ such that $f'(c) = 0$.

Proof.

Since f is continuous on $[a, b]$, it has to attain its global max and min values somewhere in $[a, b]$. If either is in (a, b), then the derivative is 0. If max and min are attained on $\{a, b\}$, then since $f(a) = f(b)$, this implies that $f(x) = f(a)$ for all $x \in [a, b]$, which implies $f'(x) = 0$.

Theorem 5.7 (Mean Value Theorem)

Assume $f : [a, b] \to \mathbb{R}$ is differentiable. Then there exists a $c \in (a, b)$ for which

$$f'(c) = \frac{f(b) - f(a)}{b - a} \iff f'(c)(b - a) = f(b) - f(a) \tag{226}$$

Proof.

Just use Rolle's on

$$g(x) = f(x) - \left[\frac{f(b) - f(a)}{b - a}(x - a) + f(a) \right] \tag{227}$$

which satisfies $g(a) = g(b) = 0$, and so there must exist some $c \in (a, b)$ such that $g'(c) = 0$, i.e.

$$f'(c) - \frac{f(b) - f(a)}{b - a} = 0 \implies f'(c) = \frac{f(b) - f(a)}{b - a} \tag{228}$$

Geometrically, this means that there exists a tangent line somewhere at $\zeta \in (a, b)$ that is parallel the secant line connecting the two points $\big(a, f(a)\big)$ and $\big(b, f(b)\big)$.

Some remarks:

1. Physically, if x is interpreted as time and $f(b) - f(a)$ as the amount of displacement over the time $b - a$ of a particle moving along the line, this theorem says that the velocity $f'(x)$ of the particle at some time $\zeta \in (a, b)$ is such that if the particle had moved with constant velocity $f'(\zeta)$ over the whole time interval, it would have been displaced by the same amount $f(b) - f(a)$. We call $f'(\zeta)$ the **average velocity** over the time interval $[a, b]$.
2. Note that the Mean Value Theorem is important in that it connects the increment of a function over a finite interval with the derivative of the function on that interval. Up to now, we have characterized only the local (infinitesimal) increment of a function in terms of the derivative or differential at a given point. MVT connects the increment of a function over a **finite** interval with the derivative of the function.

The MVT actually leads to multiple useful corollaries.

Theorem 5.8 (Derivative of a Monotonic Function)

Given function $f : [a, b] \longrightarrow \mathbb{R}$ that is differentiable on (a, b),

$$\begin{aligned}
f'(x) > 0 &\implies f \text{ is increasing} \\
f'(x) \geq 0 &\iff f \text{ is nondecreasing} \\
f'(x) \equiv 0 &\iff f \text{ is constant} \\
f'(x) \leq 0 &\iff \text{ is nonincreasing} \\
f'(x) < 0 &\implies f \text{ is decreasing}
\end{aligned}$$

Note the one-sided direction for the strict inequalities.[a] The reverse implication is a bit weaker.

$$\begin{aligned}
f \text{ is increasing} &\implies f'(x) \geq 0 \\
f \text{ is decreasing} &\implies f'(x) \leq 0
\end{aligned}$$

[a]Think of the function $f(x) = x^3$, which is strictly increasing, but has derivative $f'(0) = 0$ at $x = 0$.

Proof.

If $x_1 < x_2$ are two points of the interval, then the MVT

$$f(x_2) - f(x_1) = f'(\zeta)(x_2 - x_1) \tag{229}$$

shows that the sign of the left hand side must equal that of the right.

Corollary 5.2 (Derivative of a Constant Function)

A function that is continuous on a closed interval $[a, b]$ is constant on it if and only if its derivative equals 0 at every point of the interval $[a, b]$ or the open interval (a, b).
Therefore, if the derivatives $f_1'(x)$ and $f_2'(x)$ of two functions $f_1(x)$ and $f_2(x)$ are equal on some interval (that is, $f_1'(x) = f_2'(x)$ on the interval), then the difference

$$(f_1 - f_2)(x) = f_1(x) - f_2(x) \tag{230}$$

is constant.

Proof.

Given constant function f, the MVT equation

$$0 = f(x_2) - f(x_1) = f'(\zeta)(x_2 - x_1) \tag{231}$$

implies that $f'(\zeta) = 0$ for all $x_1, x_2 \in E$. It follows that by the arithmetic properties of the derivative, given two functions f_1, f_2 with the same derivative on an interval, the derivative of their difference $(f_1 - f_2)' = 0$, and therefore must be constant on that interval.

Theorem 5.9 (IVT For Derivatives)

Suppose $f : [a, b] \to \mathbb{R}$ is a real and differentiable function and suppose $f'(a) < \lambda < f'(b)$. Then there exists $x \in (a, b)$ such that $f'(x) = \lambda$.[a]

[a] i.e. f doesn't have to be continuous, but it must have a middle value.

Proof.

Let $g(x) = f(x) - \lambda x$. Then g is differentiable with $g'(x) = f'(x) - \lambda$. But this implies that

1. $g'(a) = f'(a) - \lambda < 0$, which implies that

$$\frac{g(t_1) - g(a)}{t_1 - a} < 0 \implies g(t_1) - g(a) < 0 \implies g(t_1) < g(a) \tag{232}$$

   for some $t_1 > a$ sufficiently close to a.
2. $g'(b) = f'(b) - \lambda > 0$, which implies that

$$\frac{g(t_2) - g(b)}{t_2 - b} > 0 \implies g(t_2) - g(b) > 0 \implies g(t_2) > g(b) \tag{233}$$

   for some $t_2 < b$ sufficiently close to b.

By the mean value theorem there exists $x \in (a, b)$ s.t. $g'(x) = 0 \implies f'(x) = \lambda$.

Corollary 5.3 (Derivatives Cannot Have Jump Discontinuities)

You can't have a jump discontinuity[a] for derivatives.

[a] Also called a discontinuity of the first kind.

That is, the derivative of a differentiable function cannot "jump," so it's like the IVT of derivatives. However, it may as well have discontinuities of the second kind.

Example 5.5 (Derivative Might Jump if Not Differentiable)

A non-example is $f(x) = |x|$. It is not differentiable over $[-1, 1]$, and so we see a jump in the derivative.

The following theorem is a useful generalization of Lagrange's theorem.

Theorem 5.10 (Cauchy's Finite-Increment Theorem)

Let $x = x(t)$ and $y = y(t)$ be functions that are continuous on a closed interval $[\alpha, \beta]$ and differentiable on the open interval (α, β). Then, there exists a point $\tau \in [\alpha, \beta]$ such that

$$x'(\tau)\big(y(\beta) - y(\alpha)\big) = y'(\tau)\big(x(\beta) - x(\alpha)\big) \tag{234}$$

If in addition $x'(t) \neq 0$ for each $t \in (\alpha, \beta)$, then $x(\alpha) \neq x(\beta)$ and we have the equality

$$\frac{y(\beta) - y(\alpha)}{x(\beta) - x(\alpha)} = \frac{y'(\tau)}{x'(\tau)} \tag{235}$$

5.4 Extrema and Concavity

Similarly, we can connect the concepts of extrema and derivatives.

Theorem 5.11 (First Derivative Test)

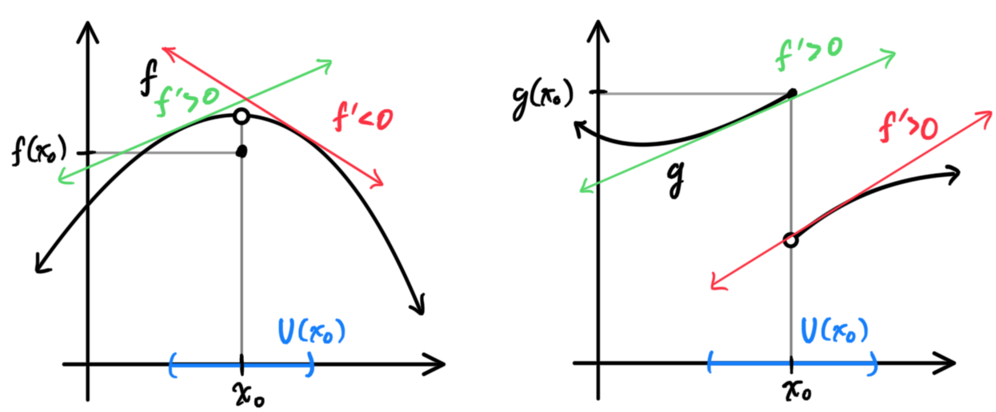
Let function $f : E \longrightarrow \mathbb{R}$ be defined in a neighborhood $U(x_0)$ of point x_0, which is continuous at x_0 and differentiable in $\mathring{U}(x_0)$, a deleted neighborhood of x_0. (Note that this is broader hypothesis than just assuming that f be differentiable at x_0.) Let

$$\mathring{U}^-(x_0) \equiv \{x \in U(x_0) \mid x < x_0\}, \;\; \mathring{U}^+(x_0) \equiv \{x \in U(x_0) \mid x > x_0\}$$

That is, $\mathring{U}^-(x_0)$ is the left portion of $\mathring{U}(x_0)$ and $\mathring{U}^+(x_0)$ is the right portion of $\mathring{U}(x_0)$. Then,

1. $(x_0, f(x_0))$ is strict local minimum if $f'(x) < 0$ in $\mathring{U}^-(x_0)$ and $f'(x) > 0$ in $\mathring{U}^+(x_0)$.
2. $(x_0, f(x_0))$ is strict local maximum if $f'(x) > 0$ in $\mathring{U}^-(x_0)$ and $f'(x) < 0$ in $\mathring{U}^+(x_0)$.
3. $(x_0, f(x_0))$ has no extremum at x_0 if $f'(x) > 0$ in both $\mathring{U}^-(x_0), \mathring{U}^+(x_0)$, or if $f'(x) < 0$ in both $\mathring{U}^-(x_0), \mathring{U}^+(x_0)$.

Note that if there is a discontinuity at a point x_0, then this theorem does not apply. For example, $(x_0, f(x_0))$ in the graph below is a local minimum, even though the derivatives to the left of x_0 are positive and those to the right of x_0 are negative (within neighborhood $U(x_0)$). Similarly, $(x_0, g(x_0))$ is a local maximum, even though the derivative to the left and to the right of x_0 are both positive.



Theorem 5.12 (2nd, nth Derivative Test)

Let function $f : E \longrightarrow \mathbb{R}$ be defined on a neighborhood $U(x_0)$ of x_0 has derivatives of order up to n

inclusive at x_0. If its derivatives up to the $(n-1)$th order vanishes

$$f'(x_0) = f''(x_0) ... = f^{(n-1)}(x_0) = 0$$

but the nth derivative at x_0 does **not** vanish

$$f^{(n)}(x_0) \neq 0$$

then
1. n is odd $\implies$ there is no local extremum at x_0
2. n is even $\implies$ there is a local extremum at x_0
   (a) $f^{(n)}(x_0) > 0 \implies$ it is a strict local minimum
   (b) $f^{(n)}(x_0) < 0 \implies$ it is a strict local maximum

Definition 5.6 (Convex, Concave Functions)

A function $f : (a, b) \longrightarrow \mathbb{R}$ defined on an open interval $(a, b) \subset \mathbb{R}$ is **convex** if the inequality

$$f(\alpha_1 x_1 + \alpha_2 x_2) \leq \alpha_1 f(x_1) + \alpha_2 f(x_2)$$

holds and **concave**, or **convex upward**, if the inequality

$$f(\alpha_1 x_1 + \alpha_2 x_2) \geq \alpha_1 f(x_1) + \alpha_2 f(x_2)$$

holds for all pairs of points $x_1, x_2 \in (a, b)$ and any numbers $\alpha_1, \alpha_2 \geq 0$ such that $\alpha_1 + \alpha_2 = 1$. If this inequality is strict whenever $x_1 \neq x_2$ and $\alpha_1 \alpha_2 \neq 0$, the function is said to be **strictly convex** and **strictly concave**, respectively.

The following is also another equivalent condition for a function to be convex over (a, b).

Theorem 5.13 ()

A function $f : (a, b) \longrightarrow \mathbb{R}$ that is differentiable on the open interval (a, b) is convex on (a, b) if and only if its graph contains no points below any tangent drawn to it.

Theorem 5.14 (2nd Derivatives of Convex Functions)

Given a function $f : (a, b) \longrightarrow \mathbb{R}$ that is differentiable in its domain,
1. f is convex $\iff$ f' is nondecreasing on (a, b) $\iff$ $f'' \geq 0$ on (a, b)
2. f is strictly convex $\iff$ f' is increasing on (a, b) $\iff$ $f'' > 0$ on (a, b)
3. f is concave $\iff$ f' is nonincreasing on (a, b) $\iff$ $f'' \leq 0$ on (a, b)
4. f is strictly concave $\iff$ f' is decreasing on (a, b) $\iff$ $f'' < 0$ on (a, b)

Definition 5.7 (Inflection Point)

Let $f : E \longrightarrow \mathbb{R}$ be a function defined and differentiable on a neighborhood $U(x_0)$. If the function is convex downward (resp. upward) on the set $\mathring{U}^-(x_0) = \{x \in U(x_0) \mid x < x_0\}$ and convex upward (resp. downward) on $\mathring{U}^+(x_0) = \{x \in U(x_0) \mid x > x_0\}$, then the point

$$\big(x_0, f(x_0)\big)$$

is called a **inflection point of the graph**.

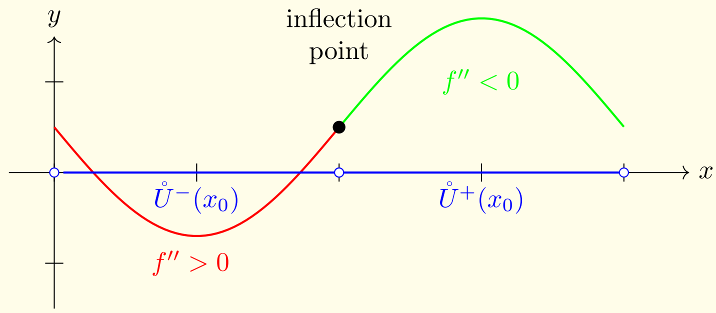


Figure 22: Curve with changing concavity and inflection point at π

5.5 Theorems of Continuously Differentiable Functions

Now continuously differentiable functions are called *smooth functions*, denoted $f \in C^1([a, b])$.

Theorem 5.15 (C^1 Implies Lipschitz)

A continuously differentiable function is Lipschitz continuous.

Proof.

Example 5.6 ()

Suppose f is twice-differentiable on $\mathbb{R}$ and that M_0, M_1, M_2 are the least upper bounds of $|f(x)|$, $|f'(x)|$, and $|f''(x)|$. Then $M_1^2 \leq 4M_0M_2$.

Theorem 5.16 (L'Hopital's Rule)

Suppose f, g are continuously differentiable functions with $f(c) = g(c) = 0$ and $g'(c) \neq 0$. Then,

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)} \tag{236}$$

Proof.

We have

$$\begin{aligned}
\lim_{x \to c} \frac{f(x)}{g(x)} &= \lim_{x \to c} \frac{f(x) - f(c)}{g(x) - g(c)} && (237) \\
&= \lim_{x \to c} \frac{\frac{f(x)-f(c)}{x-c}}{\frac{g(x)-g(c)}{x-c}} && (238) \\
&= \frac{\lim_{x \to c} \frac{f(x)-f(c)}{x-c}}{\lim_{x \to c} \frac{g(x)-g(c)}{x-c}} && (239) \\
&= \lim_{x \to c} \frac{f'(x)}{g'(x)} && (240)
\end{aligned}$$

Example 5.7 ()

Let $f(x) = \sin x$ and $g(x) = -0.5x$. Then, the function

$$h(x) = \frac{f(x)}{g(x)} = \frac{\sin x}{-0.5x} \tag{241}$$

is clearly undefined at $x = 0$. However, we can solve the limit using L'Hopital's rule to get

$$\lim_{x \to 0} \frac{\sin x}{-0.5x} = \lim_{x \to 0} \frac{\cos x}{-0.5} = -2 \tag{242}$$

Therefore, $h : \mathbb{R} \setminus 0 \longrightarrow \mathbb{R}$ can be completed to continuous function on all of $\mathbb{R}$ by defining the extension:

$$H(x) \equiv \begin{cases} h(x), & x \neq 0 \\ -2, & x = 0 \end{cases} \tag{243}$$

5.6 Higher Order Derivatives

Note that the mean value theorem states that given differentiable $f : [a, b] \to \mathbb{R}$, there exists a $c \in (a, b)$ s.t.

$$f(b) = f(a) + f'(c)(b - a) \tag{244}$$

which is like a first order approximation. We would like to attain a second order approximation using the fact that f is twice differentiable. To do this, recall how we proved the MVT. We subtracted a linear function from $f(x)$ to get a new function $g(x)$ satisfying $g(a) = g(b) = 0$. We will do the same here.

Theorem 5.17 (Taylor's Theorem of 2nd Order)

If f is twice differentiable on $[a, b]$, then

$$f(b) = f(a) + f'(b)(b - a) + \frac{f''(c)}{2}(b - a)^2 \tag{245}$$

for some $c \in (a, b)$. This is like a mean value theorem for the second order, where only the final term is dependent on c.

Proof.

Let us define the function

$$g(x) \coloneqq f(x) - f(a) - f'(a)(x - a) - M(x - a)^2 \tag{246}$$

where M was chosen such that $g(b) = 0$. Now notice that

$$g(a) = f(a) - f(a) - 0 - 0 = 0 \tag{247}$$

$$g'(a) = f'(a) - f'(a) - 0 = 0 \tag{248}$$

and so by using Rolle's theorem on g, there exists a $c_1 \in (a, b)$ s.t.

$$g'(c_1)(b - a) = g(b) - g(a) = 0 - 0 = 0 \implies g'(c_1) = 0 \tag{249}$$

therefore, we can use Rolle's theorem again on g' and claim there exists a $c \in (a, c_1)$ s.t.

$$g''(c)(c_1 - a) = g'(c_1) - g'(a) = 0 - 0 = 0 \implies g''(c) = 0 \tag{250}$$

This gives us all we need. By taking the double derivative of g, we get

$$0 = g''(c) = f''(c) - 2M \implies M = \frac{f''(c)}{2} \tag{251}$$

and substituting this in gives

$$0 = g(b) = f(b) - f(a) - f'(a)(b-a) - \frac{f''(c)}{2}(b-a)^2 \tag{252}$$

We can continue this process to get a nth order approximation.

Theorem 5.18 (Taylor's Theorem)

Suppose $f : [a,b] \to \mathbb{R}$, is nth differentiable over $[a,b]$ and $f^{(n+1)}$ exists over (a,b). Then there exists a $c \in (a,b)$ s.t.

$$f(b) = \left(\sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(b-a)^k \right) + \frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1} \tag{253}$$

Proof.

We do the exact same process. Let us define

$$P(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k \tag{254}$$

and set

$$g(x) \coloneqq f(x) - P(x) - M(x-a)^{n+1} \tag{255}$$

where M was chosen such that $g(b) = 0$. Now notice that evaluating on g gives us

$$g(a) = g'(a) = g''(a) = \ldots = g^{(n)}(a) = 0 \tag{256}$$

So by using Rolle's theorem on g, there exists a $c_1 \in (a,b)$ s.t. $g'(c_1) = 0$. Therefore we can use Rolle's theorem on g' to show there exists a $c_2 \in (a, c_1)$. We keep doing this until we show that there exists a $c \in (a,b)$ s.t. $g^{(n)}(c) = 0$. With this, we can directly evaluating the nth derivative of g to find

$$0 = g^{(n)}(c) = f^{(n)}(c) - n!M \implies M = \frac{f^{(n)}(c)}{n!} \tag{257}$$

We can continue this pattern to get a quadratic approximation of f in the form

$$f(x) = c_0 + c_1(x-x_0) + c_2(x-x_0)^2 + o\big((x-x_0)^2\big) \text{ as } x \to x_0 \tag{258}$$

As we have done in the previous subsection, we can derive (assuming continuity of f) $c_0 = f(x_0), c_1 = f'(x_0)$. To derive what c_2 should be, we see that the equation above implies

$$c_2 = \frac{f(x) - c_0 - c_1(x-x_0) - o\big((x-x_0)^2\big)}{(x-x_0)^2} = \frac{f(x) - c_0 - c_1(x-x_0)}{(x-x_0)^2} - o(1) \tag{259}$$

which means

$$c_2 = \lim_{x \to x_0} \frac{f(x) - c_0 - c_1(x-x_0)}{(x-x_0)^2} \tag{260}$$

Extending this, if we are seeking a polynomial $P_n(x_0; x) = c_0 + c_1(x-x_0) + \ldots + c_n(x-x_0)^n$ such that

$$f(x) = c_0 + c_1(x-x_0) + \ldots + c_n(x-x_0)^n + o\big((x-x_0)^n\big) \text{ as } x \to x_0 \tag{261}$$

we would find

$$c_0 = \lim_{x \to x_0} f(x) \tag{262}$$

$$c_1 = \lim_{x \to x_0} \frac{f(x) - c_0}{x - x_0} \tag{263}$$

$$c_2 = \lim_{x \to x_0} \frac{f(x) - c_0 - c_1(x - x_0)}{(x - x_0)^2} \tag{264}$$

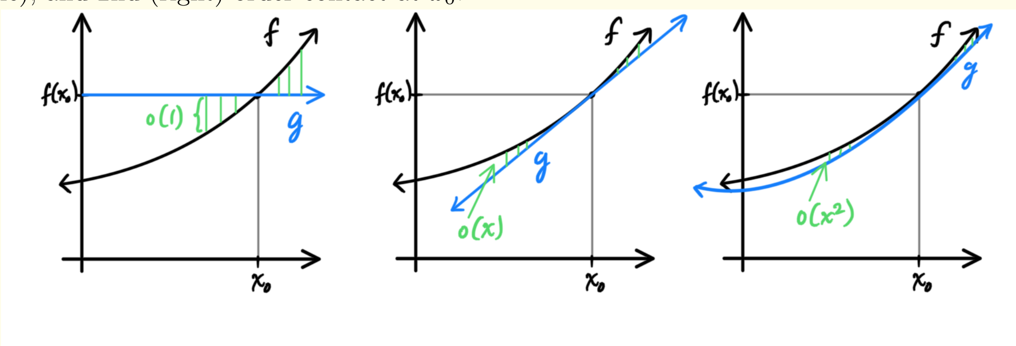
$$\ldots = \ldots \tag{265}$$

$$c_n = \lim_{x \to x_0} \frac{f(x) - (c_0 + \ldots + c_{n-1}(x - x_0)^{n-1})}{(x - x_0)^n} \tag{266}$$

We formalize the order of these approximations by analyzing their error bound.

Definition 5.8 (nth Order Contact)

If $f, g : E \longrightarrow \mathbb{R}$ are continuous at point x_0 and $(f - g)(x) = o\big((x - x_0)^n\big)$ as $x \to x_0$, then we say that f and g have **nth order contact at** x_0, or more precisely, **contact of order at least** n. The following visual shows approximations g of an arbitrary function f that have 0th (left), 1st (middle), and 2nd (right) order contact at x_0.



Lemma 5.3 (Leibniz' Formula)

Let $u(x)$ and $v(x)$ be functions having derivatives up to order n inclusive on a common set E. Then,

$$(uv)^{(n)} = \sum_{m=0}^{n} \binom{n}{m} u^{(n-m)} v^{(m)}$$

This means that given a polynomial $P_n(x) = c_0 + c_1(x - x_0) + \ldots + c_n(x - x_0)^n$, then

$$\begin{aligned} P_n(x_0) &= 0 \\ P_n'(x_0) &= 1! c_1 \\ P_n''(x_0) &= 2! c_2 \\ \ldots &= \ldots \\ P_n^{(n)}(x_0) &= n! c_n \\ P_n^{(k)}(x_0) &= 0 \text{ for } k > n \end{aligned}$$

and thus the polynomial $P_n(x)$ can be written as

$$P_n(x) = P_n^{(0)}(x_0) + \frac{1}{1!} P_n^{(1)}(x_0)(x - x_0) + \frac{1}{2!} P_n^{(2)}(x_0)(x - x_0)^2 + \ldots + \frac{1}{n!} P_n^{(n)}(x_0)(x - x_0)^n$$

5.6.1 Taylor's Formula

From the following results one may deduce that the more derivatives of two functions coincide (including the derivative of the 0th order) at a point, the better these functions approximate each other in a neighborhood of that point. Using Leibniz's rule, approximations up to a certain degree at a point can be expressed as a polynomial

$$P_n(x_0; x) = P_n(x_0) + \frac{P_n'(x_0)}{1!}(x - x_0) + ... + \frac{P_n^{(n)}(x_0)}{n!}(x - x_0)^n$$

where each coefficient of the polynomial

Definition 5.9 (Taylor Polynomial)

If a function $f : E \longrightarrow \mathbb{R}$ has derivatives of all orders $n \in \mathbb{N}$ at a point x_0, the unique series

$$P_n(x_0; x) = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + ... + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

is the **Taylor polynomial of order** n **of** $f(x)$ **at** x_0. We can see that the derivatives of f and P_n coincide up to order n.

Definition 5.10 (Analytic Functions)

We cannot assume that the Taylor series of an infinitely differentiable function converges to the function f within a neighborhood $U(x_0)$, nor can we assume that it converges at all! These types of "nice" functions that have a Taylor approximation within the neighborhood of x_0 are called **analytic functions** and can be written in the form

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + ... + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + r_n(x_0; x)$$

where r is called the **remainder term**.

Example 5.8 (Infinitely Differentiable, Non-Analytic Function)

A example of a non-analytic function is

$$f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases} \tag{267}$$

which looks like the following.

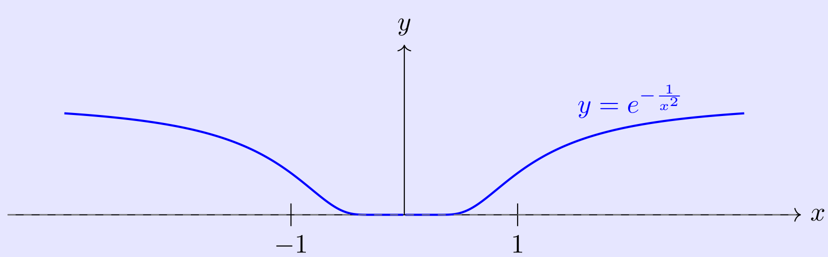


Figure 23: Graph of the function $y = e^{-\frac{1}{x^2}}$. This function equals 0 at $x = 0$ and approaches 1 as $|x|$ approaches infinity. One can verify that the derivative $f^{(k)}(0) = 0$ for all k and hence the Taylor series is identically equal to 0, while $f(x) \neq 0$ if $x \neq 0$.

The relationship between these different conditions is nicely summarized in the figure.

f infinitely differentiable at $x_0 \iff$ Taylor series of f exists at x_0

Taylor series converges at x_0

Taylor series converges to f at $x_0 \iff f$ is analytic

The following lemma proves why Taylor Polynomials are considered a "good" approximations to analytic functions.

Lemma 5.4 (Infinitesimality of Functions with Vanishing Derivative up to Order n)

Given a function $\varphi : E \longrightarrow \mathbb{R}$ defined on a closed interval E with endpoint x_0, let its derivatives vanish up to order n at x_0. That is

$$\varphi(x_0) = \varphi'(x_0) = \ldots = \varphi^{(n)}(x_0) = 0$$

Then, $\varphi = o\big((x - x_0)^n\big)$ as $x \to x_0$.

Proof.

We prove by induction. For $n = 1$, the definition of differentiability states that

$$\varphi(x) = \varphi^(x_0) + \varphi'(x - x_0) + o(x - x_0) \text{ as } x \to x_0$$

and so we have proved that

$$\varphi(x_0) = \varphi'(x_0) = 0 \implies \varphi(x) = o(x - x_0) \text{ as } x \to x_0$$

Now, suppose this assertion has been proved for order $n = k - 1 \geq 1$. That is, we have shown that

$$\varphi(x_0) = \ldots = \varphi^{(k-1)}(x_0) = 0 \implies \varphi = o\big((x - x_0)^{k-1}\big) \text{ as } x \to x_0$$

Then we must show that this is valid for order $n = k \geq 2$. Assume that

$$\varphi(x_0) = \varphi'(x_0) = \ldots = \varphi^{(k)}(x_0) = 0$$

We can see that this is equivalent to

$$(\varphi')'(x_0) = (\varphi')^{(2)}(x_0) = \ldots = (\varphi')^{(k-1)} = 0$$

and therefore by the induction assumption, we have

$$\varphi' = o\big((x - x_0)^{k-1}\big) \text{ as } x \to x_0$$

which means that we can put it in form

$$\varphi(x) = \alpha(x)(x - x_0)^{k-1} \text{ so that } \lim_{x \to x_0} \varphi(x) = \lim_{x \to x_0} \alpha(x) = 0$$

From the mean value theorem and substituting what we have above, we get

$$\begin{aligned}\varphi(x) = \varphi(x) - \varphi(x_0) &= \varphi'(\zeta)(x - x_0) \\ &= \varphi(\zeta)(\zeta - x_0)^{k-1}(x - x_0)\end{aligned}$$

where $\zeta \in (x_0, x)$. However, this implies that $|\zeta - x_0| < |x - x_0|$, and thus, as $x \to x_0$, $\zeta \to x_0$, which then makes $\alpha(\zeta) \to 0$. Since

$$|\varphi(x)| \leq |\alpha(\zeta)||x - x_0|^{k-1}|x - x_0| = |\alpha(\zeta)||x - x_0|^k$$

This means that $\varphi(x)$ is bounded by function $|\alpha(\zeta)||x - x_0|^k$, which is $o\big((x - x_0)^k\big)$, and so

$$\varphi = o\big((x - x_0)^k\big) \text{ as } x \to x_0$$

By induction, this works for all orders n.

Theorem 5.19 (Peano's Form of the Remainder)

Given analytic function $f : E \longrightarrow \mathbb{R}$, a point $x_0 \in E$, and its nth order Taylor polynomial $P_n(x_0; x)$ around x_0, P_n is a "good" approximation of f in the fact that its error term is $o\big((x - x_0)^n\big)$. That is,

$$f(x) = P_n(x_0; x) + o\big((x - x_0)^n\big) \text{ as } x \to x_0$$

This equation where $r_n(x; x_0) = o\big((x - x_0)^n\big)$ is called the **Peano's form of the remainder**.

Proof.

Since the Taylor polynomial $P_n(x_0; x)$ is constructed from the requirement that its derivatives up to order n inclusive must coincide with the corresponding derivatives of f at x_0, it follows that

$$r_n(x_0; x_0) \equiv f^{(k)}(x_0) - P_n^{(k)}(x_0; x_0) = 0 \text{ for } k = 0, 1, \ldots, n$$

Using the previous lemma, a this means that $r_n(x; x_0) = o\big((x - x_0)^n\big)$ as $x \to x_0$.

Theorem 5.20 (Lagrange Form of the Remainder)

If $f : E \longrightarrow \mathbb{R}$ has derivatives of order $n + 1$ on the open interval with endpoints x_0 and x, then

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \ldots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + r_n(x; x_0)$$

where

$$r_n(x; x_0) = \frac{f^{(n+1)}(\zeta)}{(n+1)!}(x - x_0)^{n+1}$$

This form is called **Taylor's formula with the Lagrange form of the remainder**. Furthermore, this form says that if $f^{(n+1)}(x)$ is bounded in a neighborhood of x_0, it also implies the formula

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \ldots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + O\big((x - x_0)^{n+1}\big)$$

Therefore, we can use this boundedness of $f^{(n+1)}$ to find the maximum error bound

$$|r_n(x; x_0)|$$

of $P_n(x; x_0)$.

Proof.

It is a direct result from the lemma. This is actually a generalization of the mean value theorem but for higher orders.

Corollary 5.4 (Table of Asymptotic Formulas for Elementary Functions)

We write the Maclaurin series (Taylor series around $x = 0$) for elementary functions. Note that these error terms are $O(x^{n+1})$ (bounded compared to x^{n+1}) and $o(x^n)$ (infinitesimal compared to x^n).

$$\begin{aligned}
e^x &= 1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \ldots + \frac{1}{n!}x^n + O(x^{n+1}) \\
\cos x &= 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \ldots + \frac{(-1)^n}{(2n)!}x^{2n} + O(x^{2n+2}) \\
\sin x &= x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \ldots + \frac{(-1)^n}{(2n+1)!}x^{2n+1} + O(x^{2n+3}) \\
\cosh x &= 1 + \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \ldots + \frac{1}{(2n)!}x^{2n} + O(x^{2n+2}) \\
\sinh x &= x + \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \ldots + \frac{1}{(2n+1)!}x^{2n+1} + O(x^{2n+3}) \\
\ln(1+x) &= x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \ldots + \frac{(-1)^n}{n}x^n + O(x^{n+1}) \\
(1+x)^\alpha &= 1 + \frac{\alpha}{1!}x + \frac{\alpha(\alpha-1)}{2!}x^2 + \ldots + \frac{\alpha(\alpha-1)\ldots(\alpha-n+1)}{n!}x^n + O(x^{n+1})
\end{aligned}$$

6 Riemann Integration

We would like to define an integral. We do this essentially by defining the Riemann sums for a particular partition, which is a number in $\mathbb{R}$. If we consider the set of all such Riemann sums, somehow bound them in a way. Then we define the upper and lower Riemann sums, and then consider the set of all Riemann sums. By doing so, we can construct two sets that are lower bounded and

Definition 6.1 (Partition)

Let $[a, b]$ be an interval. A **partition** P of $[a, b]$ is a set of $P = \{x_0, \ldots, x_n\}$ (note that this is finite!) s.t.

$$a = x_0 \leq x_1 \leq \ldots \leq x_{n-1} \leq x_n = b \tag{268}$$

with $\Delta x_i = x_i - x_{i-1}$ for $i = 1, \ldots, n$.

In some textbooks, we also define a *partition with distinguished points* which simply is a partition P along with some set of ξ_i's that land in each interval. This allows for extra degrees of freedom for choosing points.

The natural way to define the Riemann integral is as the limit of the finite Riemann sums as partitions gets finer and finer. But we must be careful in saying what "finer" means. It is not simply as the number of partitions $n \to \infty$, since this may lead to multiple subsequential values of convergence by increasing the partition within different subsets of $[a, b]$.

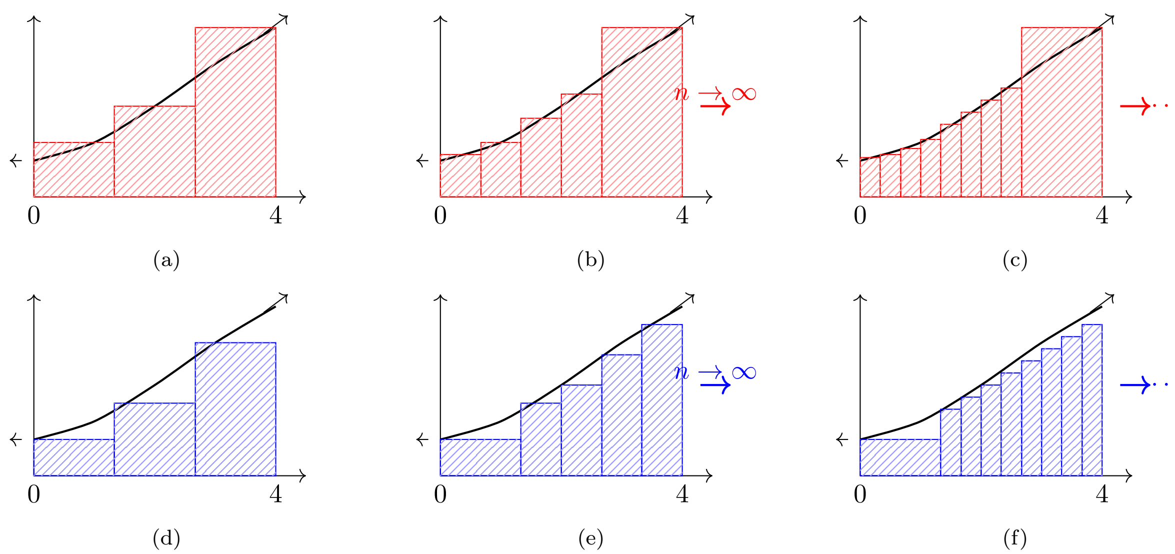
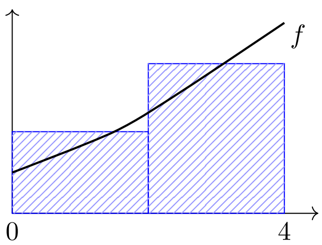
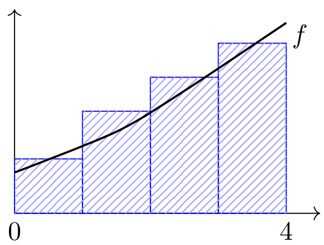


Figure 24: Upper (top row) and lower (bottom row) Riemann sums with refinement of partition. In the upper row, the rightmost rectangle remains fixed while other rectangles become thinner. In the lower row, the leftmost rectangle remains fixed while other rectangles become thinner.

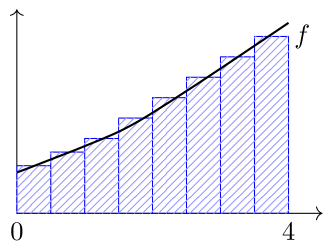
An alternative way is to have the partitions all converge "uniformly" as in the maximum length of an interval in a partition must go to 0.



(a) $\lambda(P) = 2$



(b) $\lambda(P) = 1$



(c) $\lambda(P) = 0.5$

Figure 25: Approximating an integral with increasingly fine partitions

A cleaner way it to simply look at the set of all partitions along with the set of the corresponding upper and lower Riemann sums, and then hope that they behave nicely with each other. This is the approach we will take.

Definition 6.2 (Riemann Sums with Respect to Partition)

Let P be a partition of $[a, b]$ and $f : [a, b] \to \mathbb{R}$ be bounded. Then, the **upper and lower Riemann sum** is defined

$$M_i = \sup_{\Delta x_i} f(x), \qquad m_i = \inf_{\Delta x_i} f(x) \tag{269}$$

Now define

$$U(P, f) = \sum_{i=1}^{n} M_i \Delta x_i, \qquad L(P, f) = \sum_{i=1}^{n} m_i \Delta x_i \tag{270}$$

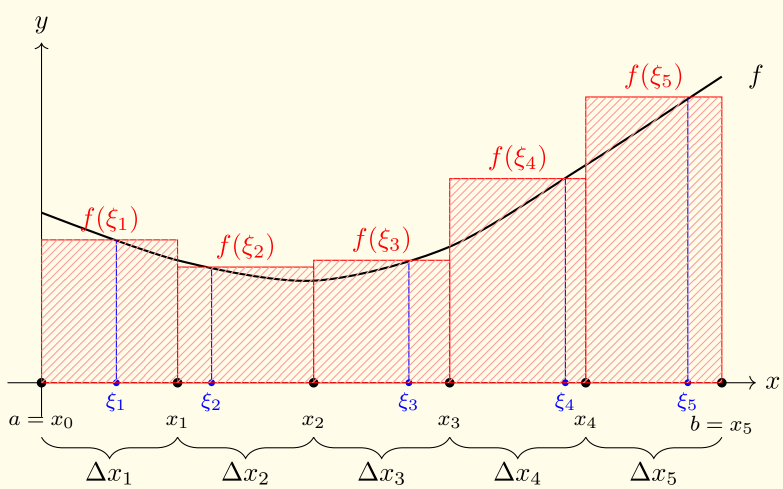


Figure 26: Riemann sum approximation using sample points ξ_i within each subinterval. This is known as a Riemann sum of a partition with distinguished points. The Riemann sum is a mapping that takes in a partition with distinguished points $p = (P, \xi)$ on the closed interval $[a, b]$ and outputs a number representing the total area of the Riemann sums.

Definition 6.3 (Riemann Integral)

Now, given the same assumptions, the **upper and lower Rimemann integrals** of $f(x)$ are defined

$$\int_a^{\bar{b}} f(x)\,dx \coloneqq \inf_P U(P,f), \qquad \int_{\bar{a}}^b f(x)\,dx \coloneqq \sup_P L(P,f) \tag{271}$$

If the upper and lower Riemann integrals are equal, then f is said to be **Riemann integrable** over $[a,b]$, denoted $f \in \mathcal{R}([a,b])$.[a]

[a] Where $\mathcal{R}(X)$ is the set of all Riemann integrable functions over X.

Great, so we've defined Riemann integrable functions, but it's hard to determine whether a function is Riemann integrable—and if so—what the value of the integral is. We will determine the first problem by talking about sufficient conditions for Riemann integrability, and then introduce the fundamental theorem of calculus to address computability.

Definition 6.4 (Refinement)

P^* is a **refinement** of P if $P \subset P^*$. If P_1, P_2 are two partitions, then their **common refinement** $P^* = P_1 \cup P_2$.

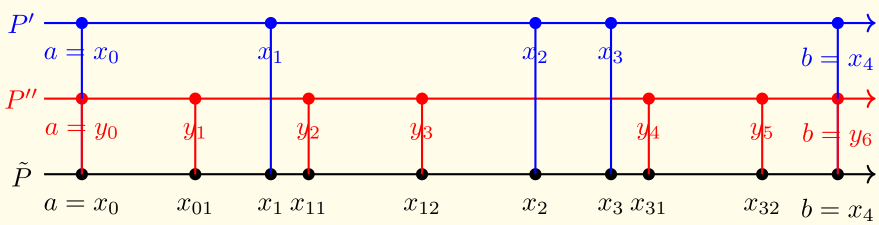


Figure 27: Partitions P' and P'' with their common refinement $\tilde{P}$

Lemma 6.1 (Fundamental Lemma)

If P^* is a refinement of P and $f : [a,b] \to \mathbb{R}$ is bounded, then

$$L(P,f) \leq L(P^*,F) \leq U(P^*,F) \leq U(P,f) \tag{272}$$

Proof.

By induction on the number of points we add to P to get P^*, we might as well assume that $P^* = P \cup \{x_*\}$. So,

$$\begin{align} P &= \{a = x_0, x_1, \ldots, x_{n-1}, x_n\} \tag{273}\\ P^* &= \{a = x_0, x_1, \ldots, x_{i-1}, x_i, x_*, x_{i+1}, \ldots, x_{n-1}, x_n\} \tag{274}\\ & \tag{275} \end{align}$$

Now let's compute $L(f,P^*) - L(f,P)$. Since the only intervals affected are $[x_i, x_{i+1}]$, we have

$$\begin{align} L(f,P^*) - L(f,P) &= \inf_{[x_i,x_*]} f(x)(x_* - x_i) + \inf_{[x_*,x_{i+1}]} f(x)(x_{i+1} - x_*) - \inf_{[x_i,x_{i+1}]} f(x)(x_{i+1} - x_i) \tag{276}\\ &= \Big(\underbrace{\inf_{[x_i,x_*]} f(x) - \inf_{[x_*,x_{i+1}]} f(x)}_{>0}\Big)(x_* - x_i) + \Big(\underbrace{\inf_{[x_*,x_{i+1}]} f(x) - \inf_{[x_i,x_{i+1}]}}_{>0}\Big)(x_{i+1} - x_*) \tag{277} \end{align}$$

which is therefore greater than 0.

Theorem 6.1 (Lower and Upper Integrals as Bounds of Each Other)

We claim

$$\int_{\bar{a}}^{b} f(x)\,dx \leq \int_{a}^{\bar{b}} f(x)\,dx \tag{278}$$

Proof.

Given P_1, P_2 partitions, let $P^* = P_1 \cup P_2$ be their common refinement. Then, from the theorem above,

$$L(P_2, f) \leq L(P^*, f) \leq U(P^*, f) \leq U(P, f) \tag{279}$$

So taking the supremum over all partitions P_2 and fixing P_1 gives

$$\int_{\bar{a}}^{b} f(x)\,dx = \sup_{P_2} L(P_2, f) \leq \sup_{P_2} U(P_1, f) = U(P_1, f) \tag{280}$$

Then taking the infimum over all partitions P_1 gives us

$$\int_{\bar{a}}^{b} f(x)\,dx = \inf_{P_1} \int_{\bar{a}}^{b} f(x)\,dx \leq \inf_{P_1} U(P_1, f) = \int_{a}^{\bar{b}} f(x)\,dx \tag{281}$$

where we note that the infimum does not affect the terms that do not depend on P_1.

6.1 Conditions for Integrability

We have seen some bounds of the upper and lower integrals, and defined the Riemann integral. However, checking Riemann integrability is quite tedious, since we have to take the supremum and infimum over all possible partitions. The following theorem is extremely useful as it only requires us to find *one* partition given some ϵ. In some sense, this is similar to the Cauchy convergence criterion.

Theorem 6.2 (Cauchy)

$f \in \mathcal{R}$ iff $\forall \epsilon > 0$, there exists partition P such that $U(P, f) - L(P, f) < \epsilon$.

Proof.

We prove bidirectionally. The reverse implication is easy, but for the forward direction you must use refinements.

1. $(\leftarrow)$. Pick any partition P. Since

$$L(f, P) \leq \int_{\bar{a}}^{b} f(x)\,dx \leq \int_{a}^{\bar{b}} f(x)\,dx \leq U(f, P) \tag{282}$$

   This implies that

$$0 \leq \int_{a}^{\bar{b}} f(x)\,dx - \int_{\bar{a}}^{b} f(x)\,dx \leq U(f, P) - L(f, P) < \epsilon \tag{283}$$

   and since any nonnegative number less than any positive number must be 0 (since there are no infinitesimals in $\mathbb{R}$), the LHS is 0, and the result is proven.

2. $(\rightarrow)$. f is Riemann integrable, so

$$\int_a^{\bar{b}} f(x)\,dx = \int_{\bar{a}}^b f(x)\,dx \iff \inf_P U(f,P) = \sup_Q L(f,Q) \tag{284}$$

for partitions P, Q. So we can find P that gets really close to the infimum and same for Q close to the supremum, i.e. there exists a P, Q, such that

$$U(f,P) < \int_a^{\bar{b}} f(x)\,dx + \frac{\epsilon}{2}, \qquad L(f,Q) > \int_{\bar{a}}^b f(x)\,dx - \frac{\epsilon}{2} \tag{285}$$

Now take the common refinement $P^* = P \cup Q$, and so by the fundamental lemma,

$$\int_{\bar{a}}^b f(x)\,dx - \frac{\epsilon}{2} < L(f,Q) \leq L(f,P^*) \leq U(f,P^*) \leq U(f,P) < \int_a^{\bar{b}} f(x)\,dx + \frac{\epsilon}{2} \tag{286}$$

which implies that $0 \leq U(f,P^*) - L(f,P^*) < \epsilon$.

Note that a necessary condition of f being Riemann integrable is that f is bounded. In fact it is defined that way. You may know that a sufficient condition of integrability is that it is continuous, but we can prove something slightly weaker.

Definition 6.5 (Oscillation)

Given an interval I, the **oscillation** of f on I is defined

$$\operatorname*{osc}_I(f) \coloneqq \sup_I(f) - \inf_I(f) \tag{287}$$

Intuitively, a function f is Riemann integrable if we can make $U(f,P) - L(f,P)$ as small as we wish. This is the case if we can find a sufficiently refined partition P such that the oscillation on f on each interval is small.

Lemma 6.2 (Functions with Vanishing Osillations are Riemann Integrable)

Let f be a bounded on a closed interval $[a,b]$. If, for every $\epsilon > 0$, there exists a partition P such that

$$\sum_{i=0}^{n-1} \operatorname*{osc}_{[x_i, x_{i+1}]} f < \epsilon \tag{288}$$

then f is Riemann integrable.

Proof.

Given $\epsilon > 0$, choose $\epsilon/(b-a)$. By assumption we can find a partition P in which the total oscillation is bounded above by $\epsilon/(b-a)$. Therefore,

$$U(P,f) - L(P,f) = \sum_{i=0}^{n-1} \sup_{[x_i, x_{i+1}]} f(x)\Delta x_i - \sum_{i=0}^{n-1} \inf_{[x_i, x_{i+1}]} f(x)\Delta x_i \tag{289}$$

$$= \sum_{i=0}^{n-1} \Big(\sup_{[x_i, x_{i+1}]} f(x) - \inf_{[x_i, x_{i+1}]} f(x) \Big) \Delta x_i \tag{290}$$

$$< \sum_{i=0}^{n-1} \operatorname*{osc}_{[x_i, x_{i+1}]} f \Delta x_i \tag{291}$$

$$\leq \sum_{i=0}^{n-1} \frac{\epsilon}{b-a} \Delta x_i \tag{292}$$

$$= \frac{\epsilon}{b-a} \sum_{i=0}^{n-1} \Delta x_i \tag{293}$$

$$= \frac{\epsilon}{b-a} (b-a) = \epsilon \tag{294}$$

Here is a classic example of a non-integrable function.

Example 6.1 (Non-Integrability of the Dirichlet Function)

The Dirichlet function

$$\mathcal{D}(x) \equiv \begin{cases} 1, & \text{for } x \in \mathbb{Q} \\ 0, & \text{for } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases} \tag{295}$$

on the interval $[0,1]$ is not integrable on that interval. For any partition P of $[0,1]$ we can find in each interval Δ_i both a rational point ξ_i' and an irrational point ξ_i''. Then, we can see that the lower and upper Riemann sums do not necessarily converge to each other since

$$\sigma(f; P, \xi') = \sum_{i=1}^{n} 1 \cdot \Delta x_i = 1 \text{ while } \sigma(f; P, \xi'') = \sum_{i=1}^{n} 0 \cdot \Delta x_i = 0 \tag{296}$$

as $\lambda(P) \to 0$.

Example 6.2 ()

Is there a function f that is discontinuous on a dense set of $[0,1]$ but still Riemann integrable?

With this, we can use the uniform continuity of continuous functions over a compact set to place a bound on the oscillation of each subinterval—and thus a bound on the oscillation of the whole interval.

Theorem 6.3 (Continuous Functions are Riemann Integrable)

f continuous on $[a,b]$ $\implies$ f is Riemann integrable on $[a,b]$.

Proof.

If f is continuous, then by EVT it is bounded and uniformly continuous. Therefore we can take the evenly-partitioned intervals of $[a,b]$ and by uniform continuity, the oscillation tends to 0, and we are done.

Perhaps more explicitly, we wish to show that for all $\epsilon > 0$, there exists partition P s.t. $U(P, f) - L(P, f) < \epsilon$. Now let $\epsilon > 0$, and since it's uniformly continuous, take $\delta > 0$ s.t.

$$|x - y| < \delta \implies |f(x) - f(y)| < \frac{\epsilon}{2(b-a)} \tag{297}$$

Let $N \in \mathbb{N}$ be so large that $\frac{b-a}{N} < \delta$. Now consider the partition of $[a, b]$ given by $x_i = a + \frac{b-a}{N} i$ for $0 \leq i < N$. Intuitively, we want these subintervals to be so small that f will not deviate too widely. So it better be the case that $\frac{b-a}{N} < \delta$. So, we have

$$\begin{aligned}
U(P, f) - L(P, f) &= \sum_{i=1}^{n} \sup_{[x_i, x_{i+1}]} f(x) \Delta x_i - \sum_{i=1}^{n} \inf_{[x_i, x_{i+1}]} f(x) \Delta x_i && (298)\\
&= \sum_{i=1}^{n} (\sup_{[x_i, x_{i+1}]} f(x) - \inf_{[x_i, x_{i+1}]} f(x)) \Delta x_i && (299)\\
&< \sum_{i=0}^{N-1} \frac{\epsilon}{2(b-a)} \Delta x_i && (300)\\
&= \frac{\epsilon}{2(b-a)} \cdot (b-a) < \frac{\epsilon}{2} < \epsilon && (301)
\end{aligned}$$

We can actually make a stronger claim.

Corollary 6.1 (Integrability of Discontinuous Functions)

If a bounded function f on a closed interval $[a, b]$ is continuous everywhere except at a finite set of points, then $f \in \mathcal{R}[a, b]$.

Corollary 6.2 (Integrability of Monotonic Functions)

A bounded monotonic function on a closed interval is integrable on that interval.

Theorem 6.4 (Continuous Compositions of Integrable Functions are Integrable)

Let $f \in \mathcal{R}([a, b])$. Assume $\phi : \mathbb{R} \to \mathbb{R}$ is continuous. Then $\phi \circ f \in \mathcal{R}([a, b])$.

Proof.

Since $f \in \mathcal{R}([a, b])$ is bounded, let $|f(x)| \leq M$ for all $x \in [a, b]$ for some $M \geq 0$. Now let $K = \sup_{t \in [-M, M]} \phi(t)$, which exists since $[-M, M]$ is compact and ϕ is continuious. ϕ is also uniformly continiuous on $[-M, M]$.
Now let $\epsilon > 0$. Then there exists a $\delta > 0$ s.t. $|t - s| < \delta \implies |\phi(t) - \phi(s)| < \epsilon$. Consequently,

$$|f(x) - f(y)| < \delta \implies |\phi(f(x)) - \phi(f(y))| < \epsilon \tag{302}$$

Since $f \in \mathcal{R}([a, b])$, we can find a partition P of $[a, b]$ s.t.

$$U(f, P) - L(f, P) < \delta^2 \implies \sum_{i=1}^{n-1} \big(\sup_{[x_i, x_{i+1}]} f - \inf_{[x_i, x_{i+1}]} f \big) \Delta x_i < \delta^2 \tag{303}$$

Let

$$A = \{i \mid \sup_{[x_i, x_{i+1}]} f - \inf_{[x_i, x_{i+1}]} f < \delta\} \tag{304}$$
$$B = \{i \mid \sup_{[x_i, x_{i+1}]} f - \inf_{[x_i, x_{i+1}]} f \geq \delta\} \tag{305}$$

Colloquially, we can think of A as the "good" intervals with small oscillations, and B as the "bad" intervals with larger oscillations. So,

$$\sum_{i \in B} \Delta x_i = \frac{1}{\delta} \sum_{i \in B} \delta \Delta x_i \leq \frac{1}{\delta} \sum_{i \in B} \operatorname*{osc}_{[x_i, x_{i+1}]} \Delta x_i < \frac{1}{\delta} \delta^2 = \delta \tag{306}$$

Now, compute

$$U(\phi(f), P) - L(\phi(f), P) = \sum_i \text{osc}_{[x_i, x_{i+1}]}(\phi(f)) \Delta x_i \tag{307}$$
$$= \sum_{i \in A} \text{osc}_{[x_i, x_{i+1}]}(\phi(f)) \Delta x_i + \sum_{i \in B} \text{osc}_{[x_i, x_{i+1}]}(\phi(f)) \Delta x_i \tag{308}$$

In the good sets, if $f(x)$'s are within δ of each other, the oscillation by uniform continuity implies $\text{osc}(\phi(f)) < \epsilon$. In the bad set, we have $\text{osc}_{[x_i, x_{i+1}]}(\phi(f)) < 2K$, so the above can be bounded by

$$'' \leq \epsilon \sum_{i \in A} \Delta x_i + \sum_{i \in B} 2K \Delta x_i \tag{309}$$
$$\leq \epsilon(b - a) + 2K\delta \tag{310}$$
$$< \epsilon(b - a + 2K) \tag{311}$$

where the penultimate step is due to $\sum_{i \in B} \Delta x_i < \delta$.

However, contrary to intuition, f, g both integrable does not imply that $g \circ g$ is integrable. We present a counterexample.

Example 6.3 (Composition of Integrable Functions May Not be Integrable)

Consider the functions

$$|sgn|(x) \equiv \begin{cases} 1 & x \neq 0 \\ 0 & x = 0 \end{cases}$$

and the Riemann function

$$\mathcal{R}(x) \equiv \begin{cases} \frac{1}{n} & x = \frac{m}{n} \in \mathbb{Q}, \gcd(m, n) = 1 \\ 0 & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

We can see that $\mathcal{R}$ is continuous at all irrational points and discontinuous at all rational points except 0, meaning that it is integrable ($\mathbb{Q}$ has measure zero). Then, the composition of these two functions is precisely the Dirichlet function

$$\mathcal{D}(x) = |sgn| \circ \mathcal{R}$$

which is not integrable.

6.2 Linearity over Functions and Intervals of the Integral

The most important properties of integrable functions is that it is a vector space, and the definite integral is a linear map.

Theorem 6.5 (The Vector Space of Integrable Functions)

The set of Riemann integrable functions $\mathcal{R}[a,b]$ over closed interval $[a,b]$ is a vector space. That is, given $f, g \in \mathcal{R}[a,b]$ and $c \in \mathbb{R}$, then

1. $(f+g) \in \mathcal{R}[a,b]$
2. $(cf) \in \mathcal{R}[a,b]$

which makes $\mathcal{R}([a,b])$ into a $\mathbb{R}$-vector space.

Proof.

We prove the following properties of a vector space.

1. If $c \in \mathbb{R}$ and $f \in \mathcal{R}$, then we wish to show that $cf \in \mathcal{R}$ and $\int cf = c \int f$.
   (a) If $c > 0$, then $U(cf, P) = cU(f, P)$, and $L(cf, P) = cL(f, P)$.
   (b) If $c < 0$, then $U(cf, P) = cL(f, P)$, and $L(cf, P) = cU(f, P)$.

   So, for all $\epsilon > 0$, we can find P s.t.
   $$U(f, P) - L(f, P) < \frac{\epsilon}{c} \implies U(cf, P) - L(cf, P) < \epsilon \tag{312}$$
   and so $cf \in \mathcal{R}$
2. If $f_1, f_2 \in \mathcal{R}$, then
   $$\operatorname{osc}_E(f_1 + f_2) \leq \operatorname{osc}_E(f_1) + \operatorname{osc}_E(f_2) \text{ since } \begin{cases} \sup_E(f_1+f_2) \leq \sup_E(f_1) + \sup_E(f_2) \\ \inf_E(f_1+f_2) \geq \inf_E(f_1) + \inf_E(f_2) \end{cases} \tag{313}$$
   for all $E \subset [a,b]$, which implies that $f_1 + f_2 \in \mathcal{R}$.

Theorem 6.6 (Integral is a Linear Map)

For fixed $a, b \in \mathbb{R}$ with $a < b$, $f \mapsto \int_a^b f$ is a linear map on $\mathcal{R}([a,b])$, i.e. a dual vector.

Proof.

Removing the a, b for convenience, we first show that $\int f_1 + f_2 = \int f_1 + \int f_2$. Let $\epsilon > 0$. Then there exists P_i s.t.
$$U(f_i, P_i) < \int f_i + \epsilon \tag{314}$$
for $i = 1, 2$. Define $P = P_1 \cup P_2$ as the common refinement. Then
$$U(f_i, P) < \int f_i + \epsilon \tag{315}$$
and so
$$\int f_1 + f_2 \leq U(f_1 + f_2, P) \leq U(f_1, P) + U(f_2, P) \tag{316}$$
$$\leq 2\epsilon + \int f_1 + \int f_2 \tag{317}$$

which implies $\int f_1 + f_2 \leq \int f_1 + \int f_2$. To prove the other way, we see that

$$\int (-f_1) + (-f_2) \leq \int (-f_1) + \int (-f_2) \tag{318}$$

and so

$$-\int f_1 + f_2 \leq -\Big(\int f_1 + \int f_2 \Big) \implies \int f_1 + f_2 \geq \int f_1 + \int f_2 \tag{319}$$

For scalar multiplication, we can do similarly.

Theorem 6.7 ()

Given that $f \in \mathcal{R}([a,b])$,

1. $fg \in \mathcal{R}[a,b]$
2. $|f| \in \mathcal{R}[a,b]$
3. $|\int f| \leq \int |f|$.[a]

[a] This will later allow us to define inner products on function spaces.

Proof.

Listed.

1. A nice trick is that
$$fg = \frac{1}{4}\big((f+g)^2 - (f-g)^2\big) \tag{320}$$
which is in $\mathcal{R}([a,b])$ since the sum, difference, and squaring functions are all continuous, and hence the composition $\phi(f,g)$ is Riemann integrable.
2. $\phi(x) = |x|$ is continuous, so $\phi(f) \in \mathcal{R}$.
3. Note that if $f \geq 0$, then $\int_a^b f \geq 0$. Consider $|f| - f$ and $|f| + f$, both ≥ 0. They are integrable as the image of f composed with continuous functions. So we have
$$\int |f| + f \geq 0 \implies \int |f| \geq -\int f \tag{321}$$
$$\int |f| - f \geq 0 \implies \int |f| \geq \int f \tag{322}$$
and so taking the maximum of the right hand side gives $\int |f| \geq |\int f|$.

Example 6.4 ()

Consider the space $X = C([a,b])$. Define $d : X \times X \to \mathbb{R}_0^+$ as

$$d(f,g) \coloneqq \int_a^b |f(x) - g(x)|\, dx \tag{323}$$

Then d is a metric. Note that in $\mathcal{R}([a,b])$, it is *not* a metric since $d(f,g) = 0 \not\Longleftrightarrow f = g$. Consider two functions that are different in 1 point.

Theorem 6.8 (Restrictions of Integrable Functions)

The restriction of f in any $[c,d] \subset [a,b]$, denoted $f\big|_{[c,d]}$, is in $\mathcal{R}[c,d]$

Proof.

Theorem 6.9 (Integral is Additive Over Intervals)

We have $\int_a^c + \int_c^b = \int_a^b$.

Proof.

Let P be a partition of $[a, b]$. If $c \in P$, then we can view $P = P_1 \cup P_2$. If $c \notin P$, consider $P \cup \{c\}$. Then we have

$$U(f, P) = U(f, P_1) + U(f, P_2) \tag{324}$$
$$U(f, P) = L(f, P_1) + L(f, P_2) \tag{325}$$

So $f \in \mathcal{R}([a, c])$, $f \in \mathcal{R}([c, b])$.

6.3 Monotonicity, Mean Value Theorem, and Change of Basis

We now show and prove the method what we call "u-substitution" for definite integration.

Theorem 6.10 (Change of Variable)

If $\varphi : [\alpha, \beta] \longrightarrow [a, b]$ is a continuously differentiable mapping such that $\varphi(\alpha) = a$ and $\varphi(\beta) = b$, then for any continuous function $f(x)$ on $[a, b]$ the function $f\big(\varphi(t)\big)\varphi'(t)$ is continuous on the closed interval $[\alpha, \beta]$ and

$$\int_a^b f(x)\,dx = \int_\alpha^\beta f\big(\varphi(t)\big)\varphi'(t)\,dt$$

Proof.

We prove a slightly weaker form of the theorem with the additional hypothesis that φ is strictly monotonic.

Theorem 6.11 (Change of Variable, U-Substitution)

Let $f \in \mathcal{R}([a, b])$ and $\varphi : [c, d] \to [a, b]$ is a strictly increasing continuous function. Then, $g(y) = (f \circ \varphi)(y) \in \mathcal{R}([c, d])$, and

$$\int_c^d g(y)\,dy = \int_a^b f(x)\,dx \tag{326}$$

Proof.

Lemma 6.3 (Monotonicity of the Integral)

If $a \leq b$, $f_1, f_2 \in \mathcal{R}[a, b]$, and $f_1(x) \leq f_2(x)$ for every $x \in [a, b]$, then

$$\int_a^b f_1(x)\,dx \leq \int_a^b f_2(x)\,dx \tag{327}$$

This immediately implies that given constants m, M such that $m \leq f(x) \leq M$ at each $x \in [a,b]$, we have
$$m \cdot (b-a) \leq \int_a^b f(x)\,dx \leq M \cdot (b-a) \tag{328}$$
In particular, if $0 \leq f(x)$ on $[a,b]$, then
$$0 \leq \int_a^b f(x)\,dx \tag{329}$$

Theorem 6.12 (Mean Value Theorem of the Integral)

Given $f \in \mathcal{R}[a,b]$, with
$$m = \inf_{x \in [a,b]} f(x), \qquad M = \sup_{x \in [a,b]} f(x) \tag{330}$$
Then
1. there exists a number $\mu \in [m, M]$ such that
$$\int_a^b f(x)\,dx = \mu \cdot (b-a) \tag{331}$$
2. Furthermore, if $f \in C[a,b]$, it there exists a point $\xi \in [a,b]$ such that
$$\int_a^b f(x)\,dx = f(\xi)(b-a) \tag{332}$$

Theorem 6.13 (Bonnet's Formula)

If $f, g \in \mathcal{R}[a,b]$ and g is a monotonic function on $[a,b]$, then there exists a point $\xi \in [a,b]$ such that
$$\int_a^b (f \cdot g)(x)\,dx = g(a) \int_a^\xi f(x)\,dx + g(b) \int_\xi^b f(x)\,dx \tag{333}$$

6.4 Fundamental Theorem of Calculus

Let $f \in \mathcal{R}[a,b]$, and let us choose an $x \in [a,b]$ in order to construct the function
$$F(x) \equiv \int_a^x f(t)\,dt \tag{334}$$
which is called an integral with a variable upper limit.

Theorem 6.14 (First Fundamental Theorem of Calculus)

Define $F : [a,b] \to \mathbb{R}$ by
$$F(x) \coloneqq \int_a^x f(t)\,dt \tag{335}$$
Then
1. F is continuous.
2. If F is continuous at x_0, then $F'(x_0) = f(x_0)$.

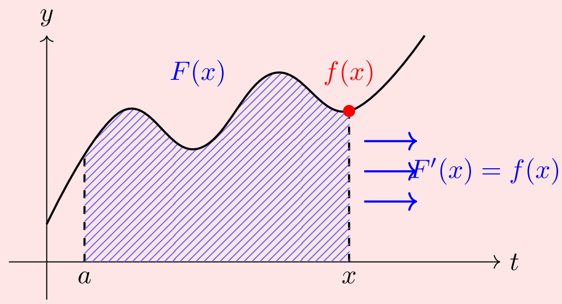


Figure 28: This theorem amazingly tells us that the rate at which the integral F is increasing at x (represented by the increasing area under the curve of f) is equal to the value of f at the point x itself!

Proof.

Listed.

1. Since $f \in \mathbb{R}([a, b])$, let $M = \sup_{x \in [a,b]} |f(x)| < +\infty$. WLOG let $x, y \in [a, b]$ with $x < y$. Then, we can use the "trick" by writing the difference of F as an integral, which follows from linearity of the integral over an interval. So, we have
$$|F(x) - F(y)| = \left| \int_x^y f(t)\, dt \right| \leq \int_x^y |f(t)|\, dt \tag{336}$$
$$\leq \int_x^y M\, dt = M|y - x| \tag{337}$$
So given $\epsilon > 0$, we can take $\delta = \epsilon / M$ and F is continuous.
2. Now let's claim
$$\lim_{h \to 0} \frac{1}{h} \big(F(x_0 + h) - F(x_0) - f(x_0) h \big) = 0 \iff F'(x_0) = f(x_0) \tag{338}$$
since if the limit exists, we can add $f(x_0)$ to both sides. The term in the limit is
$$\frac{1}{h} \left| \int_a^{x_0+h} f(t)\, dt - \int_a^{x_0} f(t)\, dt - f(x_0) h \right| \leq \frac{1}{h} \left| \int_{x_0}^{x_0+h} f(t)\, dt - h f(x_0) \right| \tag{339}$$
Now we do a trick that is simple but powerful. Notice that $hf(x_0) = \int_{x_0}^{x_0+h} f(x_0)\, dt$, so we can join it with the integral.[a] So,
$$'' = \frac{1}{h} \left| \int_{x_0}^{x_0+h} f(t) - f(x_0)\, dt \right| \tag{340}$$
$$\leq \frac{1}{h} \int_{x_0}^{x_0+h} \big| f(t) - f(x_0) \big|\, dt \tag{341}$$
$$\leq \frac{1}{h} \int_{x_0}^{x_0+h} \sup_{t \in [x_0, x_0+h]} \big| f(t) - f(x_0) \big|\, dt \tag{342}$$
Note that the supremum term in the integral is just a number, so evaluating it and taking the limit as $h \to 0$ gives
$$\sup_{t \in [x_0, x_0+h]} |f(t) - f(x_0)| \to 0 \text{ as } h \to 0 \tag{343}$$
since f is continuous at x_0.

[a]Elgindi talked about how simple tricks can go a long way, e.g. the guy who was a master of Cauchy-Schwarz inequality.

Corollary 6.3 ()

Every bounded function $f : [a, b] \longrightarrow \mathbb{R}$ on the closed interval $[a, b]$ and has only a finite number of points of discontinuity has a primitive, and every primitive of f on $[a, b]$ has the form

$$\mathcal{F}(x) := \int_a^x f(t)\, dt + c$$

where c is a constant.

Theorem 6.15 (Second Fundamental Theorem of Calculus)

Let f be a real-valued function on a closed interval $[a, b]$ with $\mathcal{F}$ any primitive of f on $[a, b]$. If f is Riemann-integrable (i.e. f bounded with finite points of Lebesgue measure zero) on $[a, b]$, then

$$\int_a^b f(x)\, dx = \mathcal{F}\Big|_a^b \equiv \mathcal{F}(b) - \mathcal{F}(a) \tag{344}$$

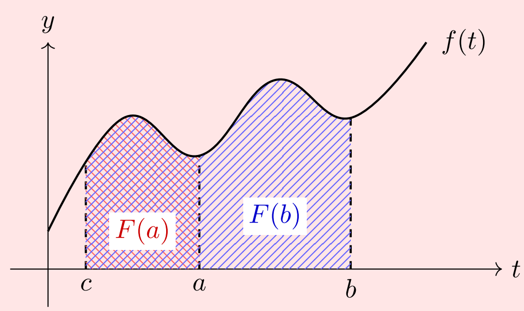


Figure 29: Graphical illustration of the Fundamental Theorem of Calculus, showing how the definite integral equals the difference of antiderivative values.

Proof.

We already know that a bounded function on a closed interval having a finite number of discontinuities is integrable, and by the corollary, we are guaranteed an existence of a primitive $\mathcal{F}(x)$ of the function f on $[a, b]$ with the form

$$\mathcal{F}(x) \equiv \int_a^x f(t)\, dt + c \tag{345}$$

Setting $x = a$, we find that $c = \mathcal{F}(a)$, and so

$$\mathcal{F}(x) \equiv \int_a^x f(t)\, dt + \mathcal{F}(a) \tag{346}$$

Evaluating $\mathcal{F}$ at $x = b$ gives

$$\int_a^b f(t)\, dt = \mathcal{F}(b) - \mathcal{F}(a) \tag{347}$$

Now a direct application of the fundamental theorem of calculus is the integration by parts. By the product rule of differentiation, we have

$$(u \cdot v)'(x) = (u' \cdot v)(x) + (u \cdot v')(x) \tag{348}$$

where by hypothesis, $u' \cdot v, u \cdot v'$ are continuous and hence integrable on $[a, b]$. Using the linearity of the integral and the 2nd fundamental theorem of calculus, we get

$$(u \cdot v)(x)\Big|_a^b = \int_a^b (u' \cdot v)(x)\, dx + \int_a^b (u \cdot v')(x)\, dx \tag{349}$$

Theorem 6.16 (Integration by Parts)

Suppose $F, G : [a,b] \to \mathbb{R}$ are differentiable, with $F' = f, G' = g \in \mathcal{R}([a,b])$. Then

$$\int_a^b F(x)g(x)\, dx = F(x)G(x)\Big|_a^b - \int_a^b f(x)G(x)\, dx \tag{350}$$

Proof.

Theorem 6.17 (Integral Form of the Remainder)

If $f : E \longrightarrow \mathbb{R}$ has continuous derivatives up to order n on the closed interval $[a, x]$, then Taylor's formula holds

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \ldots + \frac{f^{(n-1)}(a)}{(n-1)!}(x-a)^{n-1} + r_{n-1}(a;x) \tag{351}$$

where

$$r_{n-1}(a;x) = \frac{1}{(n-1)!}\int_a^x f^{(n)}(t)(x-t)^{n-1}\, dt \tag{352}$$

This form is called **Taylor's formula with the integral form of the remainder**.

Proof.

Using the 2nd fundamental theorem and the definite integration by parts formula, we can carry out the following chain of transformations, assuming continuity and differentiability when needed.

$$\begin{aligned}
f(x) - f(a) &= \int_a^x f'(t)\, dt \\
&= -\int_a^x f'(t)(x-t)'\, dt \\
&= -f'(t)(x-t)\Big|_a^x + \int_a^x f''(t)(x-t)\, dt \\
&= f'(a)(x-a) - \frac{1}{2}\int_a^x f''(t)\big((x-t)^2\big)'\, dt \\
&= f'(x-a) - \frac{1}{2}f''(t)(x-t)^2\Big|_a^x + \frac{1}{2}\int_a^x f'''(t)(x-t)^2\, dt \\
&= f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2 - \frac{1}{2\cdot 3}\int_a^x f'''(t)\big((x-t)^3\big)'\, dt \\
&= \ldots \\
&= f'(a)(x-a) + \ldots + \frac{1}{(n-1)!}f^{(n-1)}(a)(x-a)^{n-1} + r_{n-1}(a;x)
\end{aligned}$$

where $r_{n-1}(a;x)$ is given by the integral formula mentioned.

6.5 Integration over Paths and Rectifiable Curves

Definition 6.6 (Integration For Vector Valued Functions)

A function $f : [a, b] \to R^d$ is Riemann integrable if $f = (f_1, \ldots, f_d)$ and each component $f_i : [a, b] \to \mathbb{R}$ is in $\mathcal{R}([a, b])$. The integral is defined

$$\int_a^b f(x)\, dx = \left(\int_a^b f_1, \ldots, \int_a^b f_d \right) \tag{353}$$

Now since the codomain is $\mathbb{R}^d$, we can use the Euclidean norm $|v| \coloneqq \left(\sum_i v_i^2 \right)^{1/2}$ on it.

Theorem 6.18 ()

If $f \in \mathcal{R}([a, b], \mathbb{R}^d)$, then $|f| \in \mathcal{R}([a, b], \mathbb{R}^d)$ and

$$\left| \int f \right| \leq \int |f| \tag{354}$$

Proof.

If $f \in \mathcal{R}([a, b], \mathbb{R}^d)$, then $f_i \in \mathcal{R}([a, b])$, and so

$$|f| = \sqrt{f_1^2 + \ldots f_d^2} \in \mathcal{R} \tag{355}$$

since $x \mapsto x^2$ and $x \mapsto \sqrt{x}$ are continuous. Now consider the vector $v = \int_a^b f$. Then

$$|v| = \left| \int_a^b f \right| \implies |v|^2 = \sum_{j=1}^d v_j^2 = \sum_{j=1}^d v_j \int_a^b f_j \tag{356}$$

$$= \int_a^b \sum_{j=1}^d v_j f_j \tag{357}$$

$$= \int_a^b \sum_{j=1}^d v_j f_j \tag{358}$$

$$= \int_a^b \langle v, f(t) \langle \, dt \tag{359}$$

$$\leq \int_a^b |v|\, |f(t)|\, dt \tag{360}$$

and so

$$|v|^2 \leq |v| \cdot \int_a^b |f(t)|\, dt \implies |v| \leq \int_a^b |f(t)|\, dt \tag{361}$$

Definition 6.7 (Curve)

A **curve** is a function $\gamma : [0, 1] \to \mathbb{R}^d$.

1. If $\gamma(0) = \gamma(1)$, then it is a **closed curve**.
2. If γ is injective, then it is called a **simple curve**.

Curves are usually continuous but does not have to be.

Example 6.5 ()

The curve can have different parameterizations and/or image. For example, the two are different curves with the image in $S^1 \subset \mathbb{R}^2$.

$$\gamma(t) = (\cos(2\pi t), \sin(2\pi t)) \tag{362}$$
$$\tilde{\gamma}(t) = (\cos(4\pi t), \sin(4\pi t)) \tag{363}$$

Definition 6.8 (Length of a Curve)

Given a curve $\gamma : [0,1] \to \mathbb{R}^d$ and partition P of $[0,1]$, let

$$\Lambda(\gamma, P) = \sum_{i=1}^N |\gamma(x_i) - \gamma(x_{i-1})| \tag{364}$$

i.e. the sum of the straight line distances between the curves. The **length** of the curve is defined as

$$\Lambda(\gamma) \coloneqq \sup_P \Lambda(\gamma, P) \tag{365}$$

If the length is finite, then we call this a **rectifiable curve**.

Example 6.6 ()

Consider the curve given by

$$\gamma(t) = \left(t, t\sin\frac{1}{t}\right) \tag{366}$$

γ is continuous but $\gamma(t) < +\infty$.

For most continuous curves, this is not finite, but there is a sufficient condition for it to be finite.

Theorem 6.19 (C^1 Curves are Rectifiable)

If $\gamma : [0,1] \to \mathbb{R}^d$ is continuously differentiable, then γ is rectifiable, and

$$\Lambda(\gamma) = \int_0^1 |\gamma'(t)|\, dt \tag{367}$$

Proof.

Since $\gamma'(t)$ is continuous, then $|\gamma'(t)|$ is continuous and $|\gamma'(t)|$ is Riemann integrable. Now is P is any partition of $[0,1]$, then

$$\begin{aligned}
\Lambda(x, P) = \sum_{i=1}^n |\gamma(t_i) - \gamma(t_{i-1})| &= \sum_{i=1}^n \left| \int_{t_{i-1}}^{t_i} \gamma'(s)\, ds \right| && \text{(Fund. Thm. of Calc.)} \\
&\leq \sum_{i=1}^n \int_{t_{i-1}}^{t_i} |\gamma'(s)|\, ds && (368) \\
&= \int_{t_0}^{t_n} |\gamma'(s)|\, ds && (369)
\end{aligned}$$

So we've proved one inequality. Now we prove the other. Let $\epsilon > 0$ be given. Then since $\gamma'(t)$ is continuous on compact $[0,1]$, it must be uniformly continuous on $[0,1]$. So $\exists \delta > 0$ s.t.

$$|s - t| < \delta \implies |\gamma'(s) - \gamma'(t)| < \epsilon \tag{370}$$

Now take a partition P of $[0,1]$ s.t. $|t_i - t_{i-1}| < \delta$ for each $1 \leq i \leq N$. ...

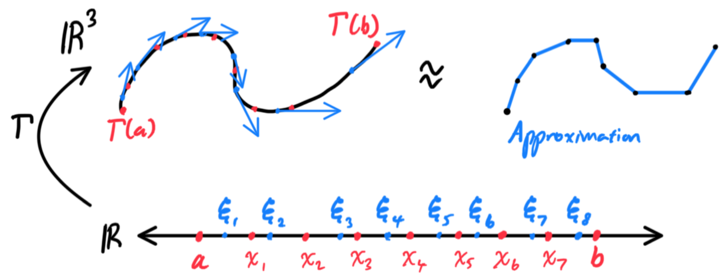


Figure 30: We can visualize this by partitioning the interval $[a,b]$ into the intervals Δ_i, each with point $\xi_i \in \Delta_i$. This would partition the path to $\Gamma(\Delta_i)$, each with points $\Gamma(\xi_i)$, and at each point $\Gamma(\xi_i)$, we can imagine the velocity vector of the curve. By taking the magnitude of this vector $\Gamma'(\xi_i)$, we multiply it by the length of the interval Δx_i to get one rectangle, creating an approximation for one partition of the path.

Corollary 6.4 (Length of the Graph of a C^1 Function)

An immediate result of this formula is the formula for the length of a graph of a function $f : [a,b] \longrightarrow \mathbb{R}$ in $\mathbb{R}^2$, by looking at the paramaterization $t \mapsto (t, f(t))$.

$$\Lambda(\gamma) = \int_a^b \sqrt{1 + (f'(t))^2}\, dt \tag{371}$$

The question on the effect of paramaterization on the integral now arises.

Definition 6.9 (Admissible Change of Parameter)

The path $\tilde{\Gamma} : [\alpha, \beta] \longrightarrow \mathbb{R}^3$ is obtained from $\Gamma : [a,b] \longrightarrow \mathbb{R}^3$ by an **admissible change of parameter** if there exists a smooth mapping

$$T : [\alpha, \beta] \longrightarrow [a,b]$$

such that $T(\alpha) = a, T(\beta) = b$, $T'(\tau) > 0$ (that is, the reparamaterization T is monotonic) on $[\alpha, \beta]$, and

$$\tilde{\Gamma} = \Gamma \circ T$$

The series of mappings can be represented with the following commutative diagram, where $I_{\alpha,\beta} = [\alpha, \beta] \subset \mathbb{R}$ and $I_{a,b} = [a,b] \subset \mathbb{R}$.

$$\begin{tikzcd} I_{\alpha,\beta} \arrow[r, "T"] \arrow[dr, "\tilde{\Gamma}"'] & I_{a,b} \arrow[d, "\Gamma"] \\ & \mathbb{R}^3 \end{tikzcd}$$

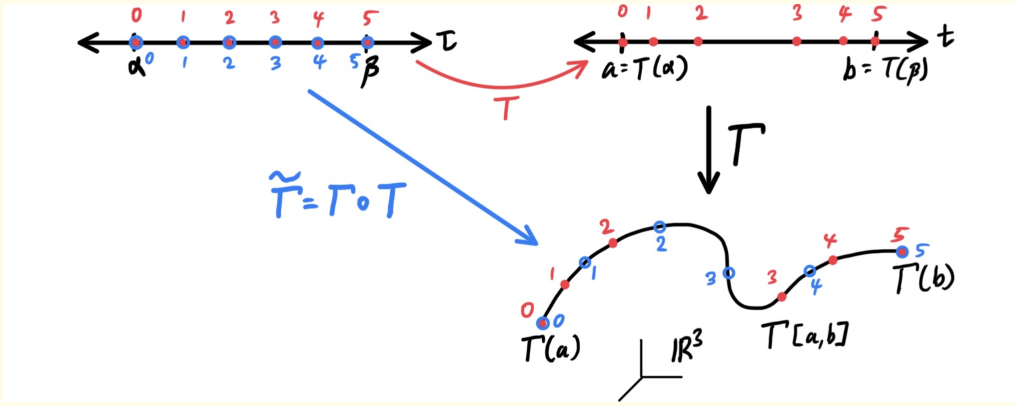


Figure 31: Note that the points are labeled $0, 1, 2, 3, 4, 5$ do not represent numerical values, but rather the order in which the points are paramaterized. We can see from this ordering that T is monotonic.

Theorem 6.20 (Invariance of Arclength Integral under Admissible Change of Parameters)

If a smooth path $\tilde{\Gamma} : [\alpha, \beta] \longrightarrow \mathbb{R}^3$ is obtained from a smooth path $\Gamma : [a, b] \longrightarrow \mathbb{R}^3$ by an admissible change of parameter, then the lengths of the two paths are equal. That is, a

$$\int_a^b |\Gamma'(t)|\, dt = \int_\alpha^\beta |\tilde{\Gamma}'(t)|\, dt \equiv \int_\alpha^\beta |(\Gamma \circ T)'(t)|\, dt \tag{372}$$

6.6 Improper Integrals

Due to some limitations of the Riemann integral, we cannot integrate over "singularities" where either the interval or the function is unbounded. We develop the tools of improper integration to deal with this problem; there are two types of improper integrals.

Definition 6.10 (Improper Integral of Unbounded Interval)

Suppose the function $x \mapsto f(x)$ is defined on the interval $[a, +\infty)$ and is integrable on every closed interval $[a, b]$ contained in that interval. Then, we call the following term

$$\int_a^{+\infty} f(x)\, dx \equiv \lim_{b \to +\infty} \int_a^b f(x)\, dx$$

the **improper Riemann integral of f over the interval** $[a, +\infty)$ and

$$\int_{-\infty}^b f(x)\, dx \equiv \lim_{a \to -\infty} \int_a^b f(x)\, dx$$

the **improper Riemann integral of f over the interval** $(-\infty, b]$.If the limit exists, then we say that the integral **converges** and **diverges** otherwise.

Definition 6.11 (Improper Integral of Unbounded Function)

Suppose the function $x \mapsto f(x)$ is defined on the interval $[a, B)$ and integrable on any closed interval $[a, b] \subset [a, B)$. Then, we call the following term

$$\int_a^B f(x)\,dx \equiv \lim_{b \to B^-} \int_a^b f(x)\,dx$$

the **improper Riemann integral of f over interval** $[a, B)$ and

$$\int_A^b f(x)\,dx \equiv \lim_{a \to A^+} \int_a^b f(x)\,dx$$

the **improper Riemann integral of f over interval** $(A, b]$.

For cohesiveness, we can combine these two definitions of improper integrals into the following one.

Definition 6.12 (Improper Integrals)

Let $[a, \omega)$ be a finite or infinite interval and $x \mapsto f(x)$ a function defined on that interval and integrable over every closed interval $[a, b] \subset [a, \omega)$. Then, by definition

$$\int_a^\omega f(x)\,dx \equiv \lim_{b \to \omega} \int_a^b f(x)\,dx$$

if this limit exists as $b \to \omega, b \in [a, \omega)$. Similarly, given the finite or infinite interval $(\omega, b]$ with f integrable over every closed interval $[a, b] \subset (\omega, b]$, we have

$$\int_\omega^b f(x)\,dx \equiv \lim_{a \to \omega} \int_a^b f(x)\,dx$$

Note that if $\omega \in \mathbb{R}$ and $f \in \mathcal{R}[a, \omega]$, the improper integral is equivalent to the regular Riemann integral.

$$\int_a^\omega f(x) = \lim_{b \to \omega} \int_a^b f(x)\,dx$$

Lemma 6.4 (Properties of the Improper Integral)

Suppose f, g are functions defined on interval $[a, \omega)$ (without loss of generality, we let ω be the upper limit of integration) and integrable on every closed interval $[a, b] \subset [a, \omega)$. Suppose the improper integrals

$$\int_a^\omega f(x)\,dx \text{ and } \int_a^\omega g(x)\,dx$$

are well-defined.

1. For any $\lambda_1, \lambda_2 \in \mathbb{R}$ the function $(\lambda_1 f + \lambda_2 g)(x)$ is integrable in the improper sense on $[a, \omega)$ and

$$\int_a^\omega (\lambda_1 f + \lambda_2 g)(x)\,dx = \lambda_1 \int_a^\omega f(x)\,dx + \lambda_2 \int_a^\omega g(x)\,dx$$

2. For any $c \in [a, \omega)$,

$$\int_a^\omega f(x)\,dx = \int_a^c f(x)\,dx + \int_c^\omega f(x)\,dx$$

3. If $\varphi : [\alpha, \gamma) \longrightarrow [a, \omega)$ is a smooth strictly monotonic mapping with $\varphi(\alpha) = a$ and $\varphi(\beta) \to \omega$ as

$\beta \to \gamma^-$, then the improper integral of the function $t \mapsto (f \circ \varphi)(t)\varphi'(t)$ over $[\alpha, \gamma)$ exists and

$$\int_a^\omega f(x)\,dx = \int_\alpha^\gamma (f \circ \varphi)(t)\varphi'(t)\,dt$$

Convergence of an Improper Integral

Note that by definition, an improper integral

$$\int_a^\omega f(x)\,dx \equiv \lim_{b \to \omega} \int_a^b f(x)\,dx$$

is a limit of the function

$$\mathcal{F}(b) \equiv \int_a^b f(x)\,dx$$

as $b \to \omega$. This means that we can use the Cauchy criterion to determine the convergence of this limit, and hence, existence of this improper integral.

Theorem 6.21 (Cauchy Criterion for Convergence of an Improper Integral)

If the function $x \mapsto f(x)$ is defined on the interval $[a, \omega)$ and integrable on every closed interval $[a, b] \subset [a, \omega)$, then the integral

$$\int_a^\omega f(x)\,dx$$

converges if and only if for every $\epsilon > 0$ there exists $B \in [a, \omega)$ such that the relation

$$\left| \int_{b_1}^{b_2} f(x)\,dx \right| < \epsilon$$

holds for any $b_1, b_2 \in [a, \omega)$ satisfying $B < b_1$ and $B < b_2$.

Proof.

We have

$$\int_{b_1}^{b_2} f(x)\,dx = \int_a^{b_2} f(x)\,dx - \int_a^{b_1} f(x)\,dx = \mathcal{F}(b_2) - \mathcal{F}(b_1)$$

and therefore the condition is simply the Cauchy criterion for the existence of a limit for the function $\mathcal{F}(b)$ as $b \to \omega$.

Definition 6.13 (Absolute Convergence of an Improper Integral)

The improper integral

$$\int_a^\omega f(x)\,dx$$

converges absolutely if the integral

$$\int_a^\omega |f|(x)\,dx$$

converges. Clearly, the inequality

$$\left| \int_{b_1}^{b_2} f(x)\,dx \right| \leq \left| \int_{b_1}^{b_2} |f|(x)\,dx \right|$$

implies that if an improper integral converges absolutely, then it converges.

This study of absolute convergence reduces to the study of convergence of integrals of nonnegative functions. The following lemma is useful in determining convergence of such functions.

Lemma 6.5 ()

Let there be a function f defined on interval $[a, \omega)$ that is also integrable over every closed interval $[a, b] \subset [a, \omega)$. If $f(x) \geq 0$ on $[a, \omega)$, then the improper integral

$$\int_a^\omega f(x)\, dx$$

exists if and only if the function

$$\mathcal{F}(b) \equiv \int_a^b f(x)\, dx$$

is bounded on $[a, \omega)$.

Proof.

It is clear that

$$\int_a^\omega f(x)\, dx = \lim_{b \to \omega} \mathcal{F}(b)$$

If $f(x) \geq 0$, then the function $\mathcal{F}(b)$ is nondecreasing on $[a, \omega)$ and therefore has a limit as $b \to \omega$ only if it is bounded (since every monotonically increasing sequence that is bounded always converges).

This leads to the familiar integral test for convergence of a series.

Theorem 6.22 (Integral Test for Convergence of a Series)

If the function $x \mapsto f(x)$ is defined on the interval $[1, +\infty)$, nonnegative, nonincreasing, and integrable on each closed interval $[1, b] \subset [1, +\infty)$, then the series

$$\sum_{n=1}^\infty f(n) = f(1) + f(2) + \ldots$$

and the integral

$$\int_a^{+\infty} f(x)\, dx$$

either both converge or both diverge.

We can use the comparison test analogue to determine convergence of improper integrals.

Theorem 6.23 (Comparison Test for Convergence of Improper Integrals)

Suppose the functions $f(x), g(x)$ are defined on the interval $[a, \omega)$ and integrable on any closed interval $[a, b] \subset [a, \omega)$. If

$$0 \leq f(x) \leq g(x)$$

on $[a, \omega)$, then

$$\int_a^\omega g(x)\, dx \text{ converges} \implies \int_a^\omega f(x)\, dx \text{ converges}$$

and the inequality

$$\int_a^\omega f(x)\, dx \leq \int_a^\omega g(x)\, dx$$

holds. Also,

$$\int_a^\omega f(x)\,dx \text{ diverges } \implies \int_a^\omega g(x)\,dx \text{ diverges}$$

Improper Integrals with Multiple Singularities

Definition 6.14 (Improper Integral with Both Limits as Singularities)

Given singularities ω_1, ω_2, the improper integral is defined

$$\int_{\omega_1}^{\omega_2} f(x)\,dx \equiv \int_{\omega_1}^{c} f(x)\,dx + \int_{c}^{\omega_2} f(x)\,dx$$

where c is an arbitrary point in (ω_1, ω_2).

Example 6.7 (Gaussian Integral)

The integral

$$\int_{-\infty}^{+\infty} e^{-x^2}\,dx = \sqrt{\pi}$$

7 Sequences of Functions

We now motivate this section by introducing double limits.

Example 7.1 (Double)

Consider the double sequence $\left(\frac{1}{m+n}\right)_{m,n\in\mathbb{N}}$. We can compute the limit as both $n, m \to +\infty$ in many ways.

1. We can first set $m \to +\infty$, then $n \to +\infty$.
2. We can first set $n \to +\infty$, then $m \to +\infty$.
3. We might want to take n twice as slow as m.

All of these converge to the same value of 0, so there is no problem.

In this case, we are considering a double sequence in $\mathbb{R}$. However, if we fix one value, then it becomes a sequence of functions, and we already have established that classes of functions form a vector space. In general, interchanging limits are not allowed.

Example 7.2 ()

Consider the slightly different sequence $\left(\frac{m}{m+n}\right)_{m,n}$.

1. If $m >> n$, i.e. take the sequence of values $(10^k, k)$, then this will approach 1.
2. If $n >> m$, i.e take the sequence of values $(k, 10^k)$, then this will approach 0.
3. In intermediate cases, you can in fact get any number between 0 and 1.

Therefore, in general,

$$\lim_{n\to\infty}\lim_{m\to\infty} a_{mn} \neq \lim_{n\to\infty}\lim_{m\to\infty} a_{nm} \tag{373}$$

To get equality, we need to make a stronger assumption than simple existence of limits.

Definition 7.1 (Pointwise Convergence)

Let $E \subset \mathbb{R}$ and $f, f_n : E \to \mathbb{R}$. We say that $f_n \to f$ **pointwise** if

$$\lim_{n\to\infty} f_n(x) = f(x) \text{ for all } x \in E \tag{374}$$

Example 7.3 ()

Let $f_n(x) = x/n$. Then $f_n \to 0$ pointwise, where 0 is the 0 function. This is true since for every fixed x, we can set n so large that $x/n < \epsilon$ for any ϵ.

Example 7.4 ()

Let $f_n(x) : [0, 1] \to \mathbb{R}$ defined $f_n(x) = x^n$. Then,

$$0 \leq x < 1 \implies x^n \to 0 \text{ as } n \to \infty \tag{375}$$

$$x = 1 \implies x^n \to 1 \text{ as } n \to \infty \tag{376}$$

So,

$$f_n \to f^*(x) := \begin{cases} 0 & \text{if } x < 1 \\ 1 & \text{if } x = 1 \end{cases} \tag{377}$$

Note that all f_n are continuous but f^* is discontinuous since

$$\lim_{x\to 1}\lim_{n\to\infty} f_n(x) = 0 \neq 1 = \lim_{n\to\infty}\lim_{x\to 1} f_n(x) \tag{378}$$

Example 7.5 ()

Consider the function $f_n : [0,1] \to \mathbb{R}$ defined by $f_n(0) = f_n(1/n) = 0$ and $f_n(1/2n) = 2n$, with everything else linearly interpolated. Then $f_n \to 0$ since it is constantly 0 at 0 and for every $x > 0$, there exists $1/N < x$ and so $f_n(x) = 0$ for all $n \geq N$. However, $\int_0^1 f_n(x)\,dx = 1$, so

$$\lim_{n\to\infty} \int_0^1 f_n(x)\,dx \neq \int_0^1 \lim_{n\to\infty} f_n(x)\,dx \tag{379}$$

So integration is not continuous with respect to the topology induced by pointwise convergence.

The problem is not in the construction of the limits or the integral, but with the pointwise convergence. With pointwise convergence,

1. we cannot exchange limits
2. does not preserve continuity
3. cannot exchange integrals
4. cannot exchange sums

Example 7.6 ()

Review.

$$\left(1 - \frac{1}{n}\right)^n \to e^{-1} \tag{380}$$

But with *uniform convergence*, we can do all this.

7.1 Uniform Convergence

Definition 7.2 (Uniform Convergence)

Given $f_n : E \to \mathbb{R}$ of bounded functions, (f_n) is said to **converge uniformly** to a bounded function $f : E \to \mathbb{R}$ if $\forall \epsilon > 0$, there exists $N \in \mathbb{N}$ s.t.

$$n \geq N \implies |f_n(x) - f(x)| < \epsilon \text{ for all } x \in E \tag{381}$$

the "for all $x \in E$" is the uniform part, which is similar to uniform continuity.

Generally, to prove uniform convergence, you will need to find that $|f_n(x) - f(x)|$ is bounded by something that is independent of x, and it goes to 0 as $n \to \infty$.

Example 7.7 ()

We have uniform convergence

$$\frac{\sin(e^n x)}{n} \to 0 \text{ since } \left|\frac{\sin(e^n x)}{n} \leq \frac{1}{n} \to 0 \tag{382}$$

Example 7.8 ()

$f_n(x) = x^n$ does not converge uniformly to *any* function in $[0,1]$. It suffices to find a sequence $(y_n) \subset [0,1]$ s.t. $f_n(y_n) - f(y_n)| \not\to 0$ as $n \to \infty$. Take $y_n = 1 - \frac{1}{n}$. Then $f_n(y_n) \to \frac{1}{e} \neq 0$, but

$f(y_n) \to 1$.

Example 7.9 ()

The triangle functions do not converge uniformly since f_n is unbounded, i.e. $f_n(\frac{1}{2n}) = 2n$, while $f_n(\frac{1}{n}) = 0$. So we can construct two sequences that

$$(1/2n) \to \infty, (1/n) \to 0 \text{ as } n \to \infty \tag{383}$$

Theorem 7.1 (Uniform Convergence Implies Pointwise Convergence)

Uniform convergence implies pointwise convergence.

Theorem 7.2 ()

We claim the following.

$$f_n \to f \text{ uniformly on } E \iff \lim_{n\to\infty} \sup_{x \in E} |f_n(x) - f(x)| = 0 \tag{384}$$

Definition 7.3 (Uniformly Cauchy)

A sequence $f_n : E \to \mathbb{R}$ is called **uniformly Cauchy** if $\forall \epsilon > 0$, there exists $N \in \mathbb{N}$ s.t. $\forall n, m \geq N$,

$$|f_n(x) - f_m(x)| < \epsilon \text{ for all } x \in E \tag{385}$$

Lemma 7.1 ()

(f_n) converges uniformly iff (f_n) is uniformly Cauchy.

Continuity.

Integration.

Differentiability.

Note that uniform convergence may not be met due to some counterexamples. In general, there are 3 ways that uniform convergence can fail to happen.

1. *Concentration.* Note that x^n as $n \to \infty$ almost converges except at one point.
2. *Translation.* Consider $f_n(x) = \sin(x - n)$. Then by increasing n we are shifting it to $+\infty$.
3. *Oscillation.* Consider $f_n(x) = \sin(nx)$. As n increases the function oscillates widely. This is sort of like the worst.[8]

Lemma 7.2 ()

If $f : [0,1] \to \mathbb{R}$ is continuouusly differentiable on $[0,1]$, then

$$\lim_{n\to\infty} \int_0^1 f(x) \sin(nx)\, dx = 0 \tag{386}$$

This means that as $n \to \infty$, the integral becomes small.

[8] It turns out that this is the same as (2) under the Fourier transform.

Proof.

Now we prove a stronger version.

Theorem 7.3 ()

For every $f \in C([0,1])$,

$$\lim_{n \to \infty} \int_0^1 f(x)\, \sin(nx)\, dx = 0 \tag{387}$$

Proof.

Let $\epsilon > 0$. Then by the Weierstrass approximation theorem[a], there exists a polynomial $p : [0,1] \to \mathbb{R}$ for which

$$\sup_{x \in [0,1]} |p(x) - f(x)| < \epsilon \tag{388}$$

Now consider

$$\begin{aligned} \left| \int_0^1 f(x)\, \sin(nx)\, dx \right| &\leq \left| \int_0^1 p(x) \sin(nx)\, dx \right| + \left| \int_0^1 (f(x) - p(x)) \sin(nx)\, dx \right| && (389) \\ &\leq \left| \int_0^1 p(x) \sin(nx)\, dx \right| + \epsilon && (390) \\ &\leq 2\epsilon && (391) \end{aligned}$$

where the first inequality is the triangle inequality, the second is due to the Weierstrass approximation theorem, and the third is due to $p(x)$ being infinitely differentiable, and so by the lemma above it is $\leq \epsilon$.

[a]aka, the set of polynomials is dense in the set of continuous functions with the supremum metric. Remember polynomial interpolation, which is for a finite number of points. This is a little different.

Theorem 7.4 ()

Proof.

Suppose for the sake of contradiction that $\sin(n_k x) \to g(x)$ uniformly for subsequence $(n_k)_k$. Then $g(x)$ must be continuous on $[0,1]$. Then

$$\int_a^b g(x)^2\, dx = \lim_{n \to \infty} \int_0^1 g(x) \sin(n_k x)\, dx = 0 \tag{392}$$

due to the theorem, which implies that $g = 0$. But since $g(nx) = \pm 1$ for some x for all n, we have a contradiction.

We would like uniform convergence, so we want conditions to avoid lack of uniform convergence. Keep in mind to counterexamples. To avoid translation, work with compact space, or if not compact, have the functions decay uniformly. To avoid oscillation, we can bound the derivative, which is a restriction on each function $|f_n'(x)| \leq M$. The Cauchy criterion is too much. To avoid going to infinity, just bound f: $|f_n(x)| \leq M$ for all $n \in \mathbb{N}, x \in X$.

7.2 Equicontinuous Families

The bounding of derivatives can be a bit strong. We aren't always working with differentiable functions, so we introduce a similar concept and introduce the Arzela-Ascolli theorem.

Definition 7.4 (Equicontinuous Family)

A family of functions $\mathcal{F}$ on E is said to be **equicontinuous** if $\forall \epsilon > 0$ there exists a $\delta > 0$ s.t.

$$|x - y| < \delta \implies |f(x) - f(y)| < \epsilon \tag{393}$$

for all $x, y \in E, f \in \mathcal{F}$. So this doesn't even depend on f.

Example 7.10 ()

Fix $M \geq 0$. Then

$$\mathcal{F}_n \coloneqq \{f : [0,1] \to \mathbb{R} \mid |f'(x)| \leq M\} \tag{394}$$

is an equicontinuous family. For any $f \in \mathcal{F}$, the MVT $|f(x) - f(y)| = |f'(c)(x-y)|$ for some $c \in (x, y)$. But since $f'(c)$ is bounded by M, take $\delta = \epsilon/M$.

Example 7.11 ()

$\mathcal{F} = \{\sin(nx)\}_{n\in\mathbb{N}}$ is not equicontinuous on $[0,1]$ since

$$\left| \sin\left(n\frac{\pi}{2n}\right) - \sin\left(n\frac{\pi}{n}\right) \right| = 1 \tag{395}$$

for all n. So setting $x_n = \frac{\pi}{2n}, y_n = \frac{\pi}{n}$, we have $d(x_n, y_n) \to 0$ while $d(f(x_n), f(y_n)) \geq 1$. So this is not equicontinuous.

Now the Ascolli's theorem gives us conditions to get rid of translation, oscillations, and infinity. To prove the second statement, we will need a lemma, so we state it now, along with providing a neat trick for constructing sequences.

Lemma 7.3 ()

Let (f_n) be a sequence of functions on $[0,1]$ that's uniformly bounded. Let $\{q_m\}_{m=1}^\infty$ be a countable set of numbers in $[0,1]$. Then $\exists$ a subsequence (f_{n_k}) for which $f_{n_k}(q_m)$ is convergent for all $m \in \mathbb{N}$.

Proof.

Intuitively, if we find a sequence of functions, we want to look at each point—say 1—and look at $(f_n(1))_n$. $(f_n(1))$ is bounded and so contains a convergent subsequence $(f_{n_k}(1))_k$. Now with this subsequence, we look at $(f_{n_k}(0))_k$ which is bounded and therefore $(f_{n_{k_j}}(0))_j$ converges, and $(f_{n_{k_j}}(1))_j$ must converge as a subsequence of convergent $(f_{n_k}(1))_k$. Now do this for all q's, and we get a single subsequence that converges for all of them.
For ease of notation, let f_{ij} denote the jth term of the ith subsequence. Then there exists $(f_{n,1})_n$ s.t. $(f_{n,1}(q_1))_n$ converges. Take a subsequence $f_{n,2}$ of $f_{n,1}$ s.t. $(f_{n,2}(q_2))_n$ converges. Given $(f_{n,k})_n$, find a subsequence of it, called $(f_{n,k+1})_n$ for which $(f_{n,k+1}(q_{k+1}))_n$ converges. Now $(f_{n,n})_n$ is a subsequence of the original one (nth term of nth subsequence) for which $(f_{q,n})_n$ is eventually a subsequence of $(f_{n,j})_n$ for any fixed j.

As an intuitive example, suppose

1. $(f_{2n}(q_1))$ converges
2. $(f_{3n}(q_2))$ converges
3. $(f_{5n}(q_3))$ converges
4. $(f_{7n}(q_4))$ converges

So combining the first two, we have that $(f_{6n}(q_i))$ converges for $i = 1, 2$. Continuing on, $(f_{30n}(q_i))$ converges for $i = 1, 2, 3$. But you can't do this infinitely. So if you want a single subsequence s.t. all the sequences converges, we can do

$$f_2, f_6, f_{30}, f_{210}, f_{2310}, \ldots \tag{396}$$

Since

1. if you take out f_2, it is a subsequence of (f_{3n}) which converges for q_2, and
2. if you also take out f_6, it is a subsequence of (f_{6n}) which converges for q_1, q_2, and
3. if you also take out f_{30}, it is a subsequence of (f_{30n}) which converges for q_1, q_2, q_3

so $f_{n_k}(q_i)$ converges for all i.

Theorem 7.5 (Arzela-Ascolli's Theorem)

We claim the following.

1. If a sequence of continuous functions $f_n : [0, 1] \to \mathbb{R}$ converges uniformly, then they form an equicontinuous family.
2. If a sequence of functions $f_n : [0, 1] \to \mathbb{R}$ is equicontinuous (and so uniformly bounded), then it has a uniformly convergent subsequence.

Proof.

Assume (f_n) is uniformly convergent. Then it is uniformly Cauchy. To prove equicontinuity, given a $\epsilon > 0$ we need to find a $\delta > 0$ for all the functions. Since f_n is uniformly Cauchy, $\exists N$ s.t. if $n \geq N$, then

$$\sup_{x \in [0,1]} |f_n(x) - f_N(x)| < \epsilon/3 \tag{397}$$

Consider the first N functions $f_1, \ldots, f_N$. They are all continuous on a compact set and so uniformly continuous. So for each f_i, there exists a δ_i s.t. $|x - y| < \delta_i \implies |f_i(x) - f_i(y)| < \epsilon$. So take $\delta = \frac{1}{3} \min_i \delta_i > 0$. So for all $1 \leq i \leq N$,

$$|x - y| < \delta \implies |f_i(x) - f_i(y)| < \epsilon/3 \tag{398}$$

and for $n \geq N$,

$$|f_n(x) - f_n(y)| \leq \underbrace{|f_n(x) - f_N(x)}_{<\epsilon/3} + \underbrace{f_N(x) - f_N(y)}_{<\epsilon/3} + \underbrace{f_N(y) - f_n(y)}_{<\epsilon/3} < \epsilon \tag{399}$$

For the second part, let $E = \mathbb{Q} \cap [0, 1]$. It is a good thing that E is dense in $[0, 1]$. Let (f_n) be an equicontinuous on $[0, 1]$ and uniformly bounded. Due to the lemma, there exists a $(f_{n_k})_k$ so that the f_{n_k} converges pointwise on E (since E is countable). We will not use equicontinuity of $(f_{n_k})_k$ to prove it's uniformly Cauchy on $[0, 1]$, which will imply that it's convergent. To make notation easier we will call $f_{n_k} = g_k$. Let $\epsilon > 0$. Since g_k is equicontinuous, $\exists \delta > 0$ s.t.

$$|x - y| < \delta \implies |g_k(x) - g_k(y)| < \epsilon \tag{400}$$

Since $E = \{q_1, q_2, \ldots\}$ is dense in $[0, 1]$, $\{B_\delta(q_i)\}_{i=1}^\infty$ is an open cover of $[0, 1]$. Since $[0, 1]$ is compact, there exists a finite subcover

$$[0, 1] \subset \bigcup_{j=1}^N B_\delta(q_{i_j}) \tag{401}$$

Since $(g_k(q_{i_j}))$ converges for each $1 \leq j \leq N$, there exists M_j s.t.

$$n, m \geq M_j \implies |g_n(q_{i_j} - g_m(q_{i_j}))| < \epsilon \tag{402}$$

Take $M = \max_j M_j$. Now if $m, n \geq M$, given $x \in [0,1]$ $\exists q_i$ with $1 \leq i \leq N$ so that $x \in B_\delta(q_i)$, and so

$$|g_n(x) - g_m(x)| \leq \underbrace{|g_n(x) - g_n(q_i)|}_{<\epsilon} + \underbrace{|g_n(q_i) - g_m(q_i)|}_{<\epsilon} + \underbrace{|g_m(q_i) - g_m(x)|}_{<\epsilon} < 3\epsilon \tag{403}$$

where the first and third inequalities come from equicontinuity, and the middle come from convergence on E. So by setting $\delta/3$ we are done.

Example 7.12 ()

An important application is in the existence of minimizers/maximizers for optimization problems involving functions. To minimize

$$J(f) = \int_0^1 \sqrt{1 + f'(t)}\, dt \tag{404}$$

is the length of the curve of $f : [0,1] \to \mathbb{R}$. To minimize the length of the curve, we must search over a set of functions. So to use EVT, you must know what the compact subsets of functions.

Ascolli's theorem exactly characterizes these compact subsets. These compact subsets of function spaces is the closure of equicontinuous functions.

Corollary 7.1 ()

A set of functions $K \subset C([0,1])$ is compact iff it is, under the supremum metric $\sup_{x \in [0,1]}$,
1. closed
2. bounded
3. equicontinuous

The first two are needed for finite dimensions. The third condition is for function spaces.

7.3 Approximation of the Identity

Definition 7.5 (Approximation of the Identity)

A family of functions $\{\varphi_\epsilon\}$ parameterized by ϵ is called an **approximation of the identity** if
1. $\int_{-\infty}^{\infty} \varphi_\epsilon(y)\, dy = 1$
2. $\lim_{\epsilon \to 0} \int_{|y|>\delta} \varphi_\epsilon(y)\, dy = 0$ for all $\delta > 0$
3. $\varphi_\epsilon \geq 0$.[a]

[a]This condition is flexible, but it makes things a bit easier for now.

Example 7.13 ()

Consider the functions f_ϵ satisfying $f(-\epsilon) = f(\epsilon) = 0$, $f(0) = 1/\epsilon$, and everything in between linearly interpolated. Then this is an approximation of the identity.

Example 7.14 ()

Take any $\varphi \geq 0$ s.t. $\int_{-\infty}^{\infty} \varphi = 1$ and define

$$\varphi_\epsilon = \frac{1}{\epsilon}\varphi\left(\frac{x}{\epsilon}\right) \tag{405}$$

which we can think of as squeezing the function horizontally to 0 and making the amplitude very large. Then we see that

$$\int_{-\infty}^{\infty} \varphi_\epsilon(y)\,dy = \int_{-\infty}^{\infty} \frac{1}{\epsilon}\varphi\left(\frac{y}{\epsilon}\right) = \int_{-\infty}^{\infty} \varphi(x)\,dx = 1 \tag{406}$$

Fix $\delta > 0$. Then

$$\int_{\delta}^{\infty} \varphi_\epsilon(x)\,dx = \int_{\delta}^{\infty} \frac{1}{\epsilon}\varphi\left(\frac{x}{\epsilon}\right)dx = \int_{\delta/\epsilon}^{\infty} \varphi(y)\,dy \to 0 \text{ as } \epsilon \to 0 \tag{407}$$

Since δ is fixed, we have $\delta/\epsilon \to +\infty$ as $\epsilon \to 0$, and so this integral of the tail above converges to 0.

The approximation of the identity (AoI) has an amazing property.

Theorem 7.6 ()

Let $\{\varphi_\epsilon\}$ be an AoI. Assume $f : \mathbb{R} \to \mathbb{R}$ is bounded and continuous. Then

$$\lim_{\epsilon \to 0} \int_{-\infty}^{\infty} \varphi_\epsilon(y)\,f(y)\,dy = f(0) \tag{408}$$

Figure 32: The triangle has area 1. Now if you integrate the product of $\varphi_\epsilon(y)$ and $f(y)$, it's like taking the product and multiplying by f. But as $\epsilon \to 0$, the triangle's area is 1, and at the end you just multiply by $f(0)$.

Proof.

We have

$$\begin{aligned}
\left|\int_{-\infty}^{\infty} \varphi_\epsilon(y)\,f(y)\,dy - f(0)\right| &= \left|\int_{-\infty}^{\infty} \varphi_\epsilon(y)\,\big(f(y) - f(0)\big)\,dy\right| &(409)\\
&\leq \left|\int_{-\delta}^{\delta} \varphi_\epsilon(y)\big(f(y) - f(0)\big)\,dy\right| + \left|\int_{|y|>\delta} \varphi_\epsilon(y)\big(f(y) - f(0)\big)\,dy\right| &(410)\\
&\leq \sup_{y \in [-\delta, \delta]} |f(y) - f(0)| + 2M \int_{|y|>\delta} \varphi_\epsilon(y)\,dy &(411)
\end{aligned}$$

where the final step follows from the left integral is less than $\sup_{[-\delta,\delta]} |f(y) - f(0)|$, and for the right integral, we have $f(y) - f(0) \leq 2\sup|f| \leq 2M$. Since you don't know the limit exists, you take the limsup,

$$\limsup_{\epsilon \to 0} \left|\int_{-\infty}^{\infty} \varphi_\epsilon(y)\,f(y)\,dy - f(0)\right| \leq \sup_{y \in [-\delta,\delta]} |f(y) - f(0)| \to 0 \text{ as } \delta \to 0 \tag{412}$$

Corollary 7.2 (Convolution)

If $\{f_\epsilon\}$ is an AoI with f bounded and continuous and $x \in \mathbb{R}$, we have

$$\lim_{\epsilon \to 0} \int_{-\infty}^{\infty} \varphi_\epsilon(y)\, f(x-y)\, dy = f(x) \tag{413}$$

This is called the **convolution** of f with φ_ϵ.

Definition 7.6 (Dirac Delta Function)

$\varphi_\epsilon \to \delta_0$ as $\epsilon \to 0$, the Dirac delta function. This is a limit.

Theorem 7.7 ()

Consider the functions

$$\varphi_n(x) = \begin{cases} c_n(1-x^2)^n & \text{if } x \in [-1,1] \\ 0 & \text{else} \end{cases}, \qquad c_n = \bigg(\int_{-1}^{1} (1-x^2)^n\, dx \bigg)^{-1} \tag{414}$$

Then,

$$\int_{-1}^{1} \varphi_n(x) = \int_{-\infty}^{\infty} \varphi_n(x)\, dx = 1 \tag{415}$$

so $\{\varphi_n\}$ is an approximation of the identity.

Proof.

Note that c_n is chosen such that $\int_{-\infty}^{\infty} \varphi_n(x)\, dx = 1$ and $\varphi_n \geq 0$. We want to show

$$\int_{|x|>\delta} \varphi_n(x)\, dx \to 0 \text{ as } n \to \infty \text{ for all } \delta > 0 \tag{416}$$

We claim that $c_n \leq 10\sqrt{n}$. since we wish to upper bound the multiplicative inverse of an integral, it suffices to lower bound the inverse—i.e. the integral itself.

$$\begin{aligned} \int_{-1}^{1} (1-x^2)^n\, dx = 2\int_0^1 (1-x^2)^n\, dx &\geq 2\int_0^{1/\sqrt{n}} (1-x^2)^n\, dx && (417) \\ &\geq 2\int_0^{1/\sqrt{n}} (1-nx^2)\, dx && (418) \end{aligned}$$

where the last inequality follows from the binomial inequality $(1-s)^n \geq 1 - sn$. Therefore, the final integral is now computable, so it equals

$$'' = 2\bigg(x - \frac{nx^3}{3}\bigg)\bigg|_0^{1/\sqrt{n}} = \frac{4}{3}\frac{1}{\sqrt{n}} \implies c_n \leq \bigg(\frac{4}{3}\frac{1}{\sqrt{n}}\big)^{-1} < \sqrt{n} \tag{419}$$

Now we have

$$\int_{|x|>\delta} \varphi_n(x)\,dx = 2\int_{\delta}^{+\infty} \varphi_n(x)\,dx \tag{420}$$
$$= 2\int_{\delta}^{1} c_n(1-x^2)^n\,dx \tag{421}$$
$$\leq 20\sqrt{n}\int_{\delta}^{1} (1-x^2)^n\,dx \tag{422}$$
$$\leq 20\sqrt{n}(1-\delta^2)^n \to 0 \text{ as } n\to\infty \tag{423}$$

Note that we could have claim that the bound be $c_n \leq (10\sqrt{n})^{1000}$ and this would still be true.

Theorem 7.8 (Stone-Weierstrass Theorem)

Let $f \in C([a,b])$. Then $\forall \epsilon > 0$, $\exists$ a polynomial p s.t.

$$d(f,p) \coloneqq \sup_{x\in[a,b]} d\big(f(x), p(x)\big) < \epsilon \tag{424}$$

This is equivalent to saying that $\mathbb{R}[x]$ is dense in $C([a,b])$, or that for any $f \in C([a,b])$, $\exists$ sequence (p_n) of polynomials s.t. $p_n \to p$ uniformly on $[a,b]$.

Proof.

By translation and dilation, it suffices to take $[a,b] = [-1,1]$. This is because translation/dilations are automorphisms (?). It also suffices to consider only f for which $f(-1) > 0$ and $f(1) = k$ for some number k. This is because we can always replace f with $\tilde{f}$ defined by

$$\tilde{f}(x) \coloneqq f(x) - \left(\frac{(1+x)f(1) + (1-x)f(-1)}{2}\right) \tag{425}$$

Let's extend f by 0 outside of $[-1,1]$. f is now a bounded continuous function $\mathbb{R}$ implying that f is uniformly continuous. Then we can take the integral.

$$\varphi_n(x) = \begin{cases} \frac{(1-x^2)^n}{\int (1-x^2)^n\,dx} & \text{if } x \in [-1,1] \\ 0 & \text{else} \end{cases} \tag{426}$$

Since f is bounded, uniformly continuous on $\mathbb{R}$, and since $\{\varphi\}$ is an approximation of the identity,

$$\int_{-\infty}^{\infty} \varphi_n(t) f(x-t)\,dt \to f(x) \tag{427}$$

and so p_n is defined on $[-\frac{1}{2}, \frac{1}{2}]$ with $p_n \to f$ uniformly.

Now we can use the exact same strategy to prove convergence of Fourier series.

Definition 7.7 (L^2 Inner Product)

The L^2 inner product is defined on $C([a,b])$ as

$$\langle f, g\rangle \coloneqq \frac{1}{b-a}\int_a^b f(x)\overline{g(x)}\,dx \tag{428}$$

where f, g are complex valued.

We know that orthonormal bases behave nicely. We present one particularly important one.

Lemma 7.4 ()

The functions $\{e^{inx}\}_{n\in\mathbb{Z}}$ are orthonormal in $C([0, 2\pi])$.

Proof.

We have for $n, m \in \mathbb{Z}$

$$\langle e^{inx}, e^{imx}\rangle = \frac{1}{2\pi}\int_0^{2\pi} e^{inx}\overline{e^{imx}}\,dx = \frac{1}{2\pi}\int_0^{2\pi} e^{inx}e^{-imx}\,dx = \frac{1}{2\pi}\int_0^{2\pi} e^{i(n-m)x}\,dx \tag{429}$$

So if $n = m$, then $\langle f, g\rangle = 1$. If not, then

$$\langle f, g\rangle = \frac{1}{2\pi i(n-m)} e^{i(n-m)x}\bigg|_0^{2\pi} = 0 \tag{430}$$

and we are done.

Lemma 7.5 ()

Define

$$\varphi_N(x) = \sum_{k=-N}^{N} e^{ikx} \tag{431}$$

Then, the family $\{\varphi_N\}_{N=0}^{\infty}$ forms an AoI. This is called a *generalized AoI*.

With this, we can prove that any sufficiently smooth (i.e. C^1) functions can be approximated with Fourier series.

8 Multivarite Functions

8.1 Continuity

8.2 Frechet Derivative

9 Integration of Differential Forms

10 Exercises

10.1 Number Systems

Exercise 10.1 (Math 531 Spring 2025, PS2.1)

Prove that the set of all matrices of the form:

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix}, \tag{432}$$

with $a, b \in \mathbb{R}$ forms a field with the usual sum and product operations of matrices. What does this field resemble? Give extensions to 3×3 and 4×4 matrices.

Solution 10.1

Exercise 10.2 (Math 531 Spring 2025, PS2.2)

Why can't the field of complex numbers (with its usual operations) be made into an ordered field?

Solution 10.2

Solution is shown as theorem.

Exercise 10.3 (Math 531 Spring 2025, PS2.3)

Prove there are no finite ordered fields.

Solution 10.3

Solution is shown as theorem.

Exercise 10.4 (Math 531 Spring 2025, PS2.4)

Prove that if x and n are natural numbers, then

$$x^n - 1 = (x-1)(1 + x + x^2 + \ldots + x^{n-1}). \tag{433}$$

Solution 10.4

We use the commutative addition and multiplication, plus distributive property in $\mathbb{Z}$.

$$\begin{aligned}
(x-1)\left(\sum_{i=0}^{n-1} x^i\right) &= x\sum_{i=0}^{n-1} x^i - \sum_{i=0}^{n-1} x_i && (434)\\
&= \sum_{i=1}^{n} x^i - \sum_{i=0}^{n-1} x^i && (435)\\
&= x^n + \sum_{i=1}^{n-1} x^i - \sum_{i=1}^{n-1} x^i - 1 && (436)\\
&= x^n - 1 && (437)
\end{aligned}$$

Exercise 10.5 (Math 531 Spring 2025, PS2.7)

Prove that there is no $q \in \mathbb{Q}$ for which
$$q^2 + q = 4. \tag{438}$$

Solution 10.5

Assume that such a $q \in \mathbb{Q}$ in canonical form exists. Then by the field properties, since $\frac{1}{4} \in \mathbb{Q}$,
$$q^2 + q + \frac{1}{4} = 4 + \frac{1}{4} = \frac{17}{4} \in \mathbb{Q} \tag{439}$$

But by distributive properties, $(q + \frac{1}{2})^2 \in \mathbb{Q}$. We claim that there exists no rational $x = a/b$ (a, b coprime) s.t. $x^2 = 17/4$. If there were, then clearly $a \neq 0$ and
$$\begin{aligned} \frac{a^2}{b^2} = \frac{17}{4} &\implies 4a^2 = 17b^2 && (440)\\ &\implies 2|b, \text{ and so } b = 2b' \text{ for some } b' \in \mathbb{N} && (441)\\ &\implies a^2 = 17(b')^2 && (442)\\ &\implies 17|a \text{ and so } a = 17a' \text{ for some } a' \in \mathbb{Z} && (443)\\ &\implies 17(a')^2 = (b')^2 && (444) \end{aligned}$$

which implies that $17|b'$, but this contradicts the assumption that a, b are coprime. Therefore $q + \frac{1}{2} \notin \mathbb{Q} \implies q \notin \mathbb{Q}$.

Exercise 10.6 (Math 531 Spring 2025, PS2.8)

Let X be an ordered set with the least upper bound property. Prove that X has the greatest lower bound property.

Solution 10.6

Shown in theorem above.

Exercise 10.7 (Math 531 Spring 2025, PS2.9)

Prove that if $x, y \in \mathbb{Q}$ we have that
$$||x| - |y|| \leq |x - y|. \tag{445}$$

Solution 10.7

By subadditivity of the norm we have
$$\begin{aligned} |x| \leq |x-y| + |y| &\implies |x| - |y| \leq |x-y| && (446)\\ |y| \leq |y-x| + |x| &\implies |y| - |x| \leq |y-x| && (447) \end{aligned}$$

But $|y - x| = |-1(x-y)| = |-1| \cdot |x-y| = |x-y|$, and so
$$\max\{|x| - |y|, |y| - |x|\} \leq |x-y| \tag{448}$$

and the LHS is the definition of the norm $||x| - |y||$ in $\mathbb{Q}$.

Exercise 10.8 (Rudin 1.1)

If r is rational ($r \neq 0$) and x is irrational, prove that $r + x$ and rx are irrational.

Solution 10.8

If we assume that $rx = t$ and $r + x = s$ are rational, then this violates the field axioms of $\mathbb{Q}$ since then $x = tr^{-1}$ and $x = s + (-r)$ are rational.

Exercise 10.9 (Rudin 1.2)

Prove that there is no rational number whose square is 12.

Solution 10.9

Assume that there exists a number p/q such that p and q are both not even. Then,

$$\left(\frac{p}{q}\right)^2 = 12 \implies p^2 = 12q^2 = 3(2q)^2 \tag{449}$$

So p much be even $p = 2p'$. Therefore, $p'^2 = 3q^2$, and q must be odd. This means that p' must be odd. We can rewrite the equation

$$p'^2 - q^2 = 2q^2 \implies (p' + q)(p' - q) = 2q^2 \tag{450}$$

where the left hand side is divisible by 4 but the right hand side is divisible by at most 2, leading to a contradiction.

Exercise 10.10 (Rudin 1.3)

Prove that the axioms of multiplication imply the following.

1. If $x \neq 0$ and $xy = xz$, then $y = z$.
2. If $x \neq 0$ and $xy = x$, then $y = 1$.
3. If $x \neq 0$ and $xy = 1$, then $y = x^{-1}$.
4. If $x \neq 0$, then $(x^{-1})^{-1} = x$.

Solution 10.10

Listed.

1. $xy = xz \implies \frac{1}{x} \cdot xy = \frac{1}{x} xz \implies y = z$
2. $xy = x \implies \frac{1}{x} xy = \frac{1}{x} x \implies y = 1$
3. $xy = 1 \implies \frac{1}{x} xy = \frac{1}{x} 1 \implies y = \frac{1}{x}$
4. $(x^{-1})^{-1} \cdot x^{-1} = 1 \implies (x^{-1})^{-1} \cdot x^{-1} \cdot x = 1 \cdot x \implies (x^{-1})^{-1} = x$

Exercise 10.11 (Rudin 1.4)

Let E be a nonempty subset of an ordered set; suppose α is a lower bound of E and β is an upper bound of E. Prove that $\alpha \leq \beta$.

Solution 10.11

Since E is nonempty, we choose any $x \in E$. By definition, $\alpha \leq x$ and $x \leq \beta$, and by transitive property of orderings, we have $\alpha \leq \beta$.

Exercise 10.12 (Rudin 1.5)

Let A be a nonempty set of real numbers which is bounded below. Let $-A$ be the set of all numbers $-x$, where $x \in A$. Prove that

$$\inf A = -\sup(-A) \tag{451}$$

Solution 10.12

We would like to prove that $\inf A \leq -\sup(-A)$ and $\inf A \geq -\sup(-A)$. For the first part, we start off with the definition of the infimum.

$$\begin{aligned}
\inf A \leq x \;\forall x \in A &\implies -\inf A \geq -x \;\forall x \in A \\
&\implies -\inf A \geq x \forall x \in -A \\
&\implies -\inf A \geq \sup(-A) \\
&\implies \inf A \leq -\sup(-A)
\end{aligned}$$

For the second part, we start with the definition of the supremum.

$$\begin{aligned}
\sup(-A) \geq x \forall x \in -A &\implies \sup(-A) \geq -x \forall x \in A \\
&\implies -\sup(-A) \geq x \forall x \in A \\
&\implies -\sup(-A) \leq \inf A
\end{aligned}$$

Exercise 10.13 (Rudin 1.6)

Fix $b > 1$.

1. If m, n, p, q are integers, $n > 0$, $q > 0$, and $r = m/n = p/q$, prove that
$$(b^m)^{1/n} = (b^p)^{1/q}. \tag{452}$$
Hence it makes sense to define $b^r = (b^m)^{1/n}$.
2. Prove that $b^{r+s} = b^r b^s$ if r and s are rational.
3. If x is real, define $B(x)$ to be the set of all numbers b^t, where t is rational and $t \leq x$. Prove that
$$b^r = \sup B(r) \tag{453}$$
when r is rational. Hence it makes sense to define
$$b^x = \sup B(x) \tag{454}$$
for every real x.
4. Prove that $b^{x+y} = b^x b^y$ for all real x and y.

Solution 10.13

Proved in theorem above.

Exercise 10.14 (Rudin 1.7)

Fix $b > 1$, $y > 0$, and prove that there is a unique real x such that $b^x = y$, by completing the following outline. (This x is called the logarithm of y to the base b.)

1. For any positive integer n, $b^n - 1 \geq n(b-1)$.
2. Hence $b - 1 \geq n(b^{1/n} - 1)$.
3. If $t > 1$ and $n > (b-1)/(t-1)$, then $b^{1/n} < t$.
4. If w is such that $b^w > y$, then $b^{-(1/n)} < y$ for sufficiently large n; to see this, apply part (c) with $t = y \cdot b^{-w}$.
5. If $b^w > y$, then $b^{w-(1/n)} > y$ for sufficiently large n.
6. Let A be the set of all w such that $b^w < y$, and show that $x = \sup A$ satisfies $b^x = y$.
7. Prove that this x is unique.

Solution 10.14

Proved in theorem above.

Exercise 10.15 (Rudin 1.8)

Prove that no order can be defined in the complex field that turns it into an ordered field.

Solution 10.15

Note that if $x \geq 0$, then $-x \leq 0$ for all x of any ordered field. Since if $x \geq 0$ and $-x > 0$, then $x - x > 0$, which is absurd. Therefore, one of either i or $-i$ should be greater than 0. But $i^2 = (-i)^2 = -1$, so this means that $-1 > 0$, which implies that $0 < 1$. But either 1 or -1 must ≥ 0.

Exercise 10.16 (Rudin 1.9)

Equip $\mathbb{C}$ with the dictionary order. That is, given $z = a + bi$ and $w = c + di$, $z < w$ if $a < c$, or if $a = c$ and $b < d$. Does this ordered set have a least upper bound property?

Solution 10.16

No it does not. Consider the set $S = \{a + bi \in \mathbb{C} \mid a \leq 3\}$. S is bounded by 4, but it doesn't have a least upper bound. Given any $3+bi$, this is not an upper bound since we can construct $3+(b+\epsilon)i \in S$. Given any $a+bi$ where $a > 3$, we can always find a lower bound of form $a + (b-\epsilon)i$ that also bounds S.

Exercise 10.17 (Rudin 1.10)

Suppose $z = a + bi$, $w = u + iv$, and

$$a = \left(\frac{|w| + u}{2}\right)^{1/2} \text{ and } b = \left(\frac{|w| - u}{2}\right)^{1/2} \tag{455}$$

Prove that $z^2 = w$ if $v \geq 0$ and that $(\bar{z})^2 = w$ if $v \leq 0$. Conclude that every complex number (with one exception!) has two complex square roots.

Solution 10.17

We can calculate

$$z^2 = (a^2 - b^2) + 2abi = u + \sqrt{v^2} i = \begin{cases} u + vi & \text{if } v \geq 0 \\ u - vi & \text{if } v \leq 0 \end{cases} \tag{456}$$

Since if we assume $v \geq 0$, then we have $z^2 = w$. We also get

$$\bar{z}^2 = (a^2 - b^2) - 2abi = u - \sqrt{v^2} i = \begin{cases} u - vi & \text{if } v \geq 0 \\ u + vi & \text{if } v \leq 0 \end{cases} \tag{457}$$

and assuming $v \leq 0$, we have $\bar{z}^2 = w$. Therefore, every complex number w has both $\pm z$ as its square root if $v \geq 0$, $\pm \bar{z}$ if $v \leq 0$, and just one root if $z = 0$.

Exercise 10.18 (Rudin 1.11)

If z is a complex number, prove that there exists an $r \geq 0$ and a complex number w with $|w| = 1$ s.t. $z = rw$. Are w and r always uniquely determined by z?

Solution 10.18

If $z = 0$, then $r = 0$ and there is no unique w. If $z = a + bi \neq 0$, then define

$$r = |z| = (a^2 + b^2)^{1/2}, \;\; w = \frac{1}{r} z \tag{458}$$

which proves existence. As for uniqueness, assume that there are two forms

$$z = rw = r'w' \tag{459}$$

Then, $w = \frac{r'}{r} w' \implies |w| = \big|\frac{r'}{r}\big| |w'| = 1$, which implies that $r'/r = 1$ and so $r = r'$. This means that $w = w'$.

Exercise 10.19 (Rudin 1.12)

If $z_1, \ldots, z_n$ are complex, prove that

$$|z_1 + z_2 + \ldots + z_n| \leq |z_1| + \ldots + |z_n| \tag{460}$$

Solution 10.19

By induction, it suffices to prove $|z_1 + z_2| \leq |z_1| + |z_2|$. We have

$$\begin{aligned} |z_1 + z_2|^2 &= (z_1 + z_2)(\overline{z_1 + z_2}) \\ &= (z_1 + z_2)(\bar{z_1} + \bar{z_2}) \\ &= z_1 \bar{z_1} + z_1 \bar{z_2} + z_2 \bar{z_1} + z_2 \bar{z_2} \\ &= |z_1|^2 + |z_2|^2 + z_1 \bar{z_2} + z_2 \bar{z_1} \\ &= |z_1|^2 + |z_2|^2 + 2(ac + bd) \\ &\leq |z_1|^2 + |z_2|^2 + 2\sqrt{a^2 + b^2}\sqrt{c^2 + d^2} \;\; \text{(Schwartz)} \\ &= |z_1|^2 + |z_2|^2 + 2|z_1||z_2| \\ &= (|z_1| + |z_2|)^2 \end{aligned}$$

since both sides are positive, we can take their square root to get the desired result.

Exercise 10.20 (Rudin 1.13)

If x, y are complex, prove that

$$\big||x| - |y|\big| \leq |x - y| \tag{461}$$

Solution 10.20

Since both sides are nonnegative, we can square both sides. Note that due to Cauchy Schwartz inequality, $2|x||y| \geq x\bar{y} + y\bar{x}$ since expanding them gives

$$2\sqrt{a^2 + b^2}\sqrt{c^2 + d^2} \geq 2(ac + bd) \tag{462}$$

Therefore, the following inequality is true:

$$|x|^2 + |y|^2 - 2|x||y| \leq x\bar{x} + y\bar{y} - x\bar{y} - y\bar{x} \tag{463}$$

which reduces to form $(|x| - |y|)^2 \leq |x - y|^2$.

Exercise 10.21 (Rudin 1.14)

If z is a complex number s.t. $|z| = 1$, that is such that $z\bar{z} = 1$, compute

$$|1 + z|^2 + |1 - z|^2 \tag{464}$$

Solution 10.21

Compute.

$$(1 + z)(1 + \bar{z}) + (1 - z)(1 - \bar{z}) = 1 + z + \bar{z} + z\bar{z} + 1 - z - \bar{z} + z\bar{z} = 4 \tag{465}$$

Exercise 10.22 (Rudin 1.15)

Under what conditions does equality hold in the Schwarz inequality?

Solution 10.22

If they are antiparallel, since

$$\langle x, y \rangle = ||x||||y|| \cos\theta \tag{466}$$

Exercise 10.23 (Rudin 1.16)

Suppose $k \geq 3$, $\mathbf{x}, \mathbf{y} \in \mathbb{R}^k$, $|\mathbf{x} - \mathbf{y}| = d > 0$, and $r > 0$. Prove:

a) If $2r > d$, there are infinitely many $\mathbf{z} \in \mathbb{R}^k$ s.t.

$$|\mathbf{z} - \mathbf{x}| = |\mathbf{z} - \mathbf{y}| = r$$

b) If $2r = d$, there is exactly one such $\mathbf{z}$.

c) If $2r < d$, there is no such $\mathbf{z}$.

Exercise 10.24 (Rudin 1.17)

Prove that

$$|\mathbf{x}+\mathbf{y}|^2 + |\mathbf{x}-\mathbf{y}|^2 = 2|\mathbf{x}|^2 + 2|\mathbf{y}|^2 \tag{467}$$

Solution 10.23

This is trivial if we simply expand

$$\begin{aligned}
|\mathbf{x}+\mathbf{y}|^2 + |\mathbf{x}-\mathbf{y}|^2 &= \langle \mathbf{x}+\mathbf{y}, \mathbf{x}+\mathbf{y}\rangle + \langle \mathbf{x}-\mathbf{y}, \mathbf{x}-\mathbf{y}\rangle & (468)\\
&= \langle \mathbf{x}, \mathbf{x}\rangle + 2\langle \mathbf{x}, \mathbf{y}\rangle + \langle \mathbf{y}, \mathbf{y}\rangle + \langle \mathbf{x}, \mathbf{x}\rangle - 2\langle \mathbf{x}, \mathbf{y}\rangle + \langle \mathbf{y}, \mathbf{y}\rangle & (469)\\
&= 2\langle \mathbf{x}, \mathbf{x}\rangle + 2\langle \mathbf{y}, \mathbf{y}\rangle & (470)\\
&= 2|\mathbf{x}|^2 + 2|\mathbf{y}|^2 & (471)
\end{aligned}$$

Exercise 10.25 (Rudin 1.18)

If $k \geq 2$ and $\mathbf{x} \in \mathbb{R}^k$, prove that there exists $\mathbf{y} \in \mathbb{R}^k$ s.t. $\mathbf{y} \neq \mathbf{0}$, but $\mathbf{x} \cdot \mathbf{y} = 0$. Is this also true if $k = 1$?

Solution 10.24

Let $x \in \mathbb{R}^k$ and $\ell \in \mathbb{R}^{k*}$, the dual space. By Riesz representation theorem, we can define the canonical isomorphism $\ell \mapsto y$ between these two spaces as

$$\ell(x) = (x, y) \tag{472}$$

Since $y \neq 0$ by assumption, $\ell \neq 0$, and so its rank is at least 1. Since ℓ maps to $\mathbb{R}$, the rank has to be 1. By rank nullity theorem, we have

$$\dim N(\ell) = k - \operatorname{rank}(\ell) = k - 1 \tag{473}$$

and so there exists nontrivial annihilators ℓ of x, which can be mapped to a nontrivial $y \in \mathbb{R}^k$.

Exercise 10.26 (Rudin 1.19)

Suppose $\mathbf{a}, \mathbf{b} \in \mathbb{R}^k$. Find $\mathbf{c} \in \mathbb{R}^k$ and $r > 0$ s.t.

$$|\mathbf{x}-\mathbf{a}| = 2|\mathbf{x}-\mathbf{b}| \tag{474}$$

if and only if $|\mathbf{x}-\mathbf{c}| = r$.

Solution 10.25

If we draw out the circle, it must contain two points on the line drawn by connecting A and B. Since it must be symmetric, its center and radius can then be easily calculated to be

$$r = \frac{2}{3}|b-a|, \quad c = \frac{1}{3}(4b-a) \tag{475}$$

Exercise 10.27 (Zorich 2.2.1)

Using the principle of induction, show that
1. the sum $x_1 + \ldots + x_n$ of real numbers is defined independently of the insertion of parentheses to specify the order of addition.
2. the same is true of the product $x_1 \ldots x_n$
3. $|x_1 + \ldots + x_n| \leq |x_1| + \ldots + |x_n|$
4. $|x_1 \ldots x_n| \leq |x_1| \ldots |x_n|$
5. For any $n, m \in \mathbb{N}$ such that $m < n$, $(n - m) \in \mathbb{N}$.
6. $(1 + x)^n \geq 1 + nx$ for $x > -1$ and $n \in \mathbb{N}$, equality holding for when $n = 1$ or $x = 0$.
7. $(a + b)^n = a^n +_n C_1 a^{n-1} b^1 + \ldots + b^n$ (aka binomial theorem).

Solution 10.26

Listed.
1. Let n denote the number of elements in the sum. We prove by strong law of induction. The base case for when $n = 1, 2, 3$ is trivially true.

$$\begin{aligned} x_1 &= x_1 && \text{(identity)} \\ x_1 + x_2 &= x_1 + x_2 && \text{(identity)} \\ (x_1 + x_2) + x_3 &= x_1 + (x_2 + x_3) && \text{(associativity)} \end{aligned}$$

Then, the sum of $n = k$ parameters is defined by $k - 2$ pairs of parentheses defining the order of the sum. These parentheses define a sequence of $k - 1$ 2-fold additions. Now, assume that the claim is true for

$$S_n \equiv x_1 + \ldots x_n \text{ for } n = 1, 2, \ldots, k \tag{476}$$

Then, for a specific sum S_{k+1} of $k + 1$ elements with $k - 1$ parentheses, we can reduce the sum to its final 2-fold addition

$$S_{k+1} \equiv \underbrace{(x_1 + \ldots + x_i)}_{\varphi_1} + \underbrace{(x_{i+1} + \ldots + x_{k+1})}_{\varphi_2} \tag{477}$$

Since $i, k - i + 1 < k$, by the strong law φ_1 and φ_2 are independent of the order of sum.
2. Exactly identical to (a).
3. By the triangle inequality $|x_1 + x + 2| \leq |x_1| + |x_2|$. Now, assume for $n = k$ is true. Then, let $S_k = x_1 + \ldots + x_k$, so

$$|x_1 + \ldots + x_k + x_{k+1}| = |S_k + x_{k+1}| \leq |S_k| + |x_{k+1}| \leq \sum_{i=1}^{k+1} |x_i| \tag{478}$$

4. Same as (c).
5. Let us fix m to be any element of $\mathbb{N}$. Then, the base case is for $n = m + 1$ (which is in $\mathbb{N}$ since it is inductive), so

$$n - m = (m + 1) - m = 1 \in \mathbb{N} \tag{479}$$

Now, given that for some integer $n \geq m + 1$, $n - m \in \mathbb{N}$ is true, we have

$$\begin{aligned} (n + 1) - m &= n + (1 - m) && \text{(associativity)} \\ &= n + (-m + 1) && \text{(commutativity)} \\ &= (n - m) + 1 && \text{(associativity)} \end{aligned}$$

where $(n - m) + 1 \in \mathbb{N}$ by inductive property of $\mathbb{N}$.

6. We prove by induction. For $n = 1$, it is trivial that $(1+x)^1 \geq 1 + 1 \cdot x$. Now assume that the claim is true for some $k \in \mathbb{N}$. Then,

$$\begin{aligned} (1+x)^{k+1} &= (1+x)^k(1+x) \geq (1+kx)(1+x) \\ &= 1 + (k+1)x + kx^2 \\ &\geq 1 + (k+1)x \end{aligned}$$

where equality holds if $x = 0 \implies 1^{k+1} = 1^k \cdot 1 = 1$ or $n = 1 \implies$ trivial case.

7. The base case for $n = 1$ is trivial since $(a+b)^1 = \binom{1}{0}a + \binom{1}{1}b$. We introduce Newton's identity.

$$\begin{aligned} \binom{k}{j-1} + \binom{k}{j} &= \frac{k!}{(j-1)!(k-j+1)!} + \frac{k!}{j!(k-j)!} \\ &= k!\bigg(\frac{j}{j!(k-j+1)!} + \frac{k-j+1}{j!(k-j+1)!}\bigg) \\ &= k! \cdot \frac{k+1}{j!(k-j+1)!} \\ &= \frac{(k+1)!}{j!(k-j+1)!} = \binom{k+1}{j} \end{aligned}$$

Now assuming that the binomial formula holds for some $n = k$, we have

$$\begin{aligned} (a+b)^{k+1} &= (a+b)^k(a+b) && (480) \\ &= \bigg(\sum_{j=0}^k \binom{k}{j} a^j b^{k-j}\bigg)(a+b) && (481) \\ &= \sum_{j=0}^k \binom{k}{j} a^{j+1} b^{k-j} + \sum_{j=0}^k \binom{k}{j} a^j b^{k-j+1} && (482) \\ &= \binom{k}{0} a^0 b^{k+1} + \binom{k}{k} a^{j+1} b^0 + \sum_{j=0}^{k-1} \binom{k}{j} a^{j+1} b^{k-j} + \sum_{j=1}^k \binom{k}{j} a^j b^{k-j+1} && (483) \\ &= \binom{k+1}{0} a^0 b^{k+1} + \binom{k+1}{k+1} a^{j+1} b^0 + \sum_{j=1}^k \bigg[\binom{k}{j-1} + \binom{k}{j}\bigg] a^j b^{k-j+1} && (484) \\ &= \sum_{j=0}^{k+1} \binom{k+1}{j} a^j b^{k-j+1} && (485) \end{aligned}$$

Exercise 10.28 (Zorich 2.2.3)

Show that an inductive set is not bounded above.

Solution 10.27

Assume that a X is a nonempty inductive set that is bounded above. By definition, there exists a number $B \in \mathbb{R}$ such that $\max X < B$. Then, this means that there exists no numbers in $[B, B+1)$. Since X is inductive, this means that there cannot exist any elements of X in the interval $[B-1, B)$, and similarly for the interval $[B-2, B)$, and so on, meaning that if $x \in X$, then $x \notin [B-k, B-k+1)$ for all $k \in \mathbb{Z}$. By the Archimidean principle, this implies that $X = \emptyset$, contradicting our assumption.

Exercise 10.29 (Zorich 2.2.4)

Prove the following.

1. An inductive set is infinite (that is, equipollent with one of its subsets different from itself).
2. The set $E_n = \{x \in \mathbb{N} \mid x \leq n\}$ is finite.

Solution 10.28

Listed.

1. Assume that an inductive set X is finite $\implies$ X is bounded above (we can choose upper bound $B = \max X + 1$). But from 2.2.3, an inductive set cannot be bounded above, contradicting our assumption.
2. It is trivial that $E_1 = \{1\}$ is finite since $\text{card}E_1 = 1$. Now, if for some k, E_k is finite with cardinality e_k, then $\text{card}E_{k+1} = e_k + 1$, which implies finiteness.

Exercise 10.30 (Zorich 2.2.5)

Listed.

1. Let $m, n \in \mathbb{N}$ and $m > n$. Their greatest common divisor $\gcd(m, n) = d \in \mathbb{N}$ can be found in a finite number of steps using the following algorithm of Euclid involving successive divisions with remainder.

$$\begin{aligned} m &= q_1 n + r_1 \\ n &= q_2 r_1 + r_2 \\ r_1 &= q_3 r_2 + r_3 \\ \ldots &= \ldots \\ r_{k-2} &= q_k r_{k-1} + r_k \\ r_{k-1} &= q_{k+1} r_k + 0 \end{aligned}$$

   Then $d = r_k$
2. If $d = \gcd(m, n)$, one can choose numbers $p, q \in \mathbb{Z}$ such that $pm + qn = d$.

Solution 10.29

Listed.

1.
2. Letting $n = r_0$, notice that the equations above satisfy for $i = 0, 1, \ldots$

$$r_i = q_{i+2} r_{i+1} + r_{i+2} \implies r_i - q_{i+2} r_{i+1} = r_{i+2} \tag{1}$$

   Note that the second-to-last equation allows us to write r_k as a linear combination of r_{k-2} and r_{k-1}: $r_k = r_{k-2} - q_k r_{k-1}$. Now by applying (1), we can reduce the above to a linear combination of r_{k-3} and r_{k-2}.

$$\begin{aligned} r_k &= r_{k-2} - q_k r_{k-1} \\ &= r_{k-2} - q_k (r_{k-3} - q_{k-1} r_{k-3}) \\ &= (1 + q_{k-1} q_k) r_{k-2} - q_k r_{k-3} \end{aligned}$$

   and repeatedly doing this allows us to reduce r_k to a linear combination $q_0 r_0 + q_1 r_1$. By the ring properties of $\mathbb{Z}$, the new linear coefficients are also in $\mathbb{Z}$. Reducing one last time using the first equation in the Euclidean algorithm gives

$$\begin{aligned} r_k &= q_0 r_0 + q_1 r_1 \\ &= q_0 n + q_1 (m - q_1 n) \\ &= q_1 m + (q_0 - q_1) n \\ &= pm + qn \end{aligned}$$

Exercise 10.31 (Zorich 2.2.9)

Show that if the natural number n is not of the form k^m, where $k, m \in \mathbb{N}$, then the equation $x^m = n$ has no rational roots.

Solution 10.30

Assume that there is a rational solution $x = p/q$, with $p, q \in \mathbb{N}$ of the equation. Then,

$$\left(\frac{p}{q}\right)^m = \frac{p^m}{q^m} = n \implies p^m = q^m n \tag{486}$$

By the fundamental theorem of arithmetic, the exponents of the prime factors of p^m must all be multiples of m, and so it must be so for the right hand side $\implies$ x must be of form $x = k^m$ for some k. This is a contradiction.

Exercise 10.32 (Zorich 2.2.12)

Knowing that $\frac{m}{n} \equiv m \cdot n^{-1}$ by definition, where $m \in \mathbb{Z}$ and $n \in \mathbb{N}$, derive the "rules" for addition, multiplication, and division of fractions, and also the condition for two fractions to be equal.

Solution 10.31

We can construct a $\mathbb{Q}$ as a quotient space $\mathbb{Z} \times \mathbb{N} / \sim$, where $\sim$ is an equivalent relation where

$$(q_1, p_1) \sim (q_2, p_2) \text{ iff } q_1 p_2 = p_1 q_2 \tag{487}$$

which is the familiar equivalence relation from "simplifying" a fraction. We define addition and multiplication as the following

$$\begin{aligned} (a, b) + (c, d) &= (ad + bc, bd) \\ (a, b) \cdot (c, d) &= (ac, bd) \end{aligned}$$

which turns out to be algebraically closed in $\mathbb{Q}$. The additive identity is the equivalence class $0 = \{(0, c) \mid c \in \mathbb{N}\}$, and the multiplicative identity is the equivalence class $1 = \{(c, c) \mid c \in \mathbb{N}\}$. It is easy to check that $+$ is commutative, the additive inverse is $-(a, b) = (-a, b)$, and the multiplicative inverse is $(a, b)^{-1} = (b, a)$. We can subtract and divide these elements of $\mathbb{Q}$, called "fractions," as such:

$$\begin{aligned} (a, b) - (c, d) &= (a, b) + (-(c, d)) = (a, b) + (-c, d) = (ad - bc, bd) \\ (a, b) \div (c, d) &= (a, b) \cdot (c, d)^{-1} = (a, b) \cdot (d, c) = (ad, bc) \end{aligned}$$

Exercise 10.33 (Zorich 2.2.13)

Verify that the rational numbers $\mathbb{Q}$ satisfy all the axioms for real numbers except for the axiom of completeness.

Solution 10.32

From continuing the steps of 2.2.14, we can prove $\mathbb{Q}$ is an algebraic field (associativity, commutativity of addition and multiplication, along with distributive property).We can actually define the order relation $\leq_{\mathbb{Q}}$ in two ways:

1. $(a, b) \leq (c, d)$ iff $ad \leq_{\mathbb{Z}} bc$, where $\leq_{\mathbb{Z}}$ is the order relation on $\mathbb{Z}$ (which can be defined much more simply).
2. Recognizing that $\mathbb{Q} \subset \mathbb{R}$, we define the canonical injection map $i : \mathbb{Q} \longrightarrow \mathbb{R}$ and by abuse of language, endow the relation $\leq_{\mathbb{Q}}$ as the restriction of $\leq_{\mathbb{R}}$ onto $\mathbb{Q}$. That is, for $(a, b), (c, d) \in \mathbb{Q}$,
$$(a, b) \leq_{\mathbb{Q}} (c, d) \text{ iff } i(a, b) \leq_{\mathbb{R}} i(c, d) \tag{488}$$

The ordering for the 1st step can be checked for consistency.

1. $(a, b) \leq (a, b)$ since $ab \leq ab$ (true in $\mathbb{Z}$)
2. $(a, b) \leq (c, d), (c, d) \leq (a, b)$ means that $ad \leq bc$ and $bc \leq ad \implies ad = bc$ (true in $\mathbb{Z}$)
3. $(a, b) \leq (c, d) \leq (e, f)$ implies $ad \leq bc, cf \leq de$. Multiplying positive (important that $f > 0$!) to the first inequality gives $adf \leq bcf$, and multiplying positive b to the second gives $bcf \leq bde$, and by interpreting $\leq$ as the ordering defined on $\mathbb{Z}$, we use transitive property of $\leq_{\mathbb{Z}}$ to get $adf \leq bde \implies af \leq be \iff (a, b) \leq (e, f)$.
4. For any $(a, b), (c, d) \in \mathbb{Q}$, $(a, b) \leq (c, d)$ or $(a, b) \geq (c, d)$, which is equivalent to $ad \leq bc$ or $ad \geq bc$, which is true in $\mathbb{Z}$.

It is easy to prove $(a, b) \leq (c, d) \implies (a, b) + (p, q) \leq (c, d) + (p, q)$, and $0_{\mathbb{Q}} \leq (a, b), (c, d) \implies 0_{\mathbb{Q}} \leq (a, b) \cdot (c, d)$. However, $\mathbb{Q}$ is **not** complete. We prove this by showing that the subset $X = \{x \in \mathbb{Q} \mid x^2 \leq 2\} \subset \mathbb{Q}$ does not satisfy the least upper bound property. Assume that there is a least upper bound $c \in \mathbb{Q}$. $c \neq \sqrt{2}$ (you should know how to prove irrationality of $\sqrt{2}$!), we have either $c > \sqrt{2}$ or $c < \sqrt{2}$.

1. Let $c < \sqrt{2} \iff c - \sqrt{2} > 0$. By the Archimedean principle, there exists a $k \in \mathbb{N}$ such that $0 < \frac{1}{k} < c - \sqrt{2}$. Then, $\frac{1}{k} \in \mathbb{Q}$ and $\mathbb{Q}$ is a field, so $c - \frac{1}{k} \in \mathbb{Q}$.
$$c - \frac{1}{k} < c - c + \sqrt{2} = \sqrt{2} \tag{489}$$
So c is not least and so it must be the case that $c < \sqrt{2}$.
2. Let $c < \sqrt{2} \iff \sqrt{2} - c > 0$. By the Archimedean principle, there exists a $k \in \mathbb{N}$ such that $0 < \frac{1}{k} < \sqrt{2} - c$. Then, $c + \frac{1}{k} \in \mathbb{Q}$ and
$$c + \frac{1}{k} < c + (\sqrt{2} - c) = c \tag{490}$$
So c is not an upper bound.

Note that given a well-defined $c = \sup X$ and in the case where $c < \sqrt{2}$, we have $2 - c^2 > 0$, so we can choose a well-defined δ satisfying (by Archimedean principle)
$$0 < \delta < \min\left\{1, \frac{2 - c^2}{2c + 1}\right\} \tag{491}$$
which gives us
$$\begin{aligned} (c + \delta)^2 &= c^2 + \delta(2c + \delta) \\ &< c^2 + \delta(2c + 1) && (\delta < 1) \\ &< c^2 + (2 - c^2) = 2 \end{aligned}$$
meaning that c is not an upper bound. Similarly for when $c > \sqrt{2}$.

Exercise 10.34 (Zorich 2.2.15)

Prove the equivalence of these two statements.

1. If X and Y are nonempty sets of $\mathbb{R}$ having the property that $x \leq y$ for every $x \in X, y \in Y$, then there exists $c \in \mathbb{R}$ such that $x \leq c \leq y$ for all $x \in X$ and $y \in Y$.
2. Every set $X \subset \mathbb{R}$ that is bounded above has a least upper bound.

Solution 10.33

Let S_1 be the first statement and S_2 the second.

1. ($S_2 \implies S_1$). Let $X \subset \mathbb{R}$ be a set that is bounded above, and Y is a set such that $x \leq y$ for all $x \in X, y \in Y$. Then, by LUB principle, there exists $c = \sup X \in \mathbb{R}$. Now, we claim that $c \leq y$ for all $y \in Y$. Assume it doesn't: then there exists $y' \in Y$ such that $y' < c$. But since we assumed $x \leq y$ for all $x \in X, y \in Y$, we have $x \leq y'$ for all $x \in X$, which means that y' is an upper bound of X. But $y' < c$, contradicting the given fact that c was the least upper bound.
2. ($S_1 \implies S_2$). Given a nonempty set $X \subset \mathbb{R}$, we wish to show the existence of $\sup X$. We are guaranteed the existence of nonempty set $Y \subset \mathbb{R}$ such that $x \leq y$ for all $x \in X, y \in Y$, which implies that X must be bounded above. Then, by S_1, there must exist a $c \in \mathbb{R}$ such that

$$x \leq c \leq y \text{ for all } x \in X, y \in Y \tag{492}$$

We claim that $c = \sup X$. It is an upper bound of X since $x \leq c$ for all $x \in X$. It is least since the set of all upper bounds of X is Y, and $c \leq y$ for all $y \in Y$.

Exercise 10.35 (Olmsted 1.15)

Prove **Dedekind's Theorem**: Let the real numbers be divided into two nonempty sets A and B such that (i) if $x \in A$ and if $y \in B$, then $x < y$ and (ii) if $x \in \mathbb{R}$ then either $x \in A$ or $x \in B$, then there exists a number c (which may belong to either A or B) such that any number less than c belongs to A and any number greater than c belongs to B.

Solution 10.34

This is really the same statement as Zorich 2.2.15.a, the original statement of completeness, but with the extra condition that the sets $A = X, B = Y$ must be disjoint.

Exercise 10.36 (Olmsted 1.7)

If x is an irrational number, under what conditions on the rational numbers a, b, c, d is $(ax+b)/(cx+d)$ rational?

Solution 10.35

Note that a trivial solution is $a = b = c = d = 1$ which gives 1. Since

$$\frac{ax+b}{cx+d} = \frac{acx + ad - ad + bc}{cx+d} = a + \frac{bc - ad}{cx+d} \tag{493}$$

for the above to be rational it is necessary that $1/(cx+d)$ is rational. But this cannot be the case, which leaves us with the condition that $bc = ad$.

Exercise 10.37 (Olmsted 1.8)

Prove that the system of integers satisfies the axiom of completeness.

Solution 10.36

Let $S \subset \mathbb{Z}$ be bounded from above. It must have a maximum element (justify?), call it c. Then we claim that $c \in \mathbb{Z}$ is the least upper bound. Being the maximum, it is an upper bound, and c is least since the next smallest element is $c - 1$, which is less than $c \in S$, and therefore cannot be an upper bound.

Exercise 10.38 (Zorich 2.2.16/Olmsted 1.16)

Prove the following.

1. If $A \subset B \subset \mathbb{R}$, then $\sup A \leq \sup B$ and $\inf A \geq \inf B$.
2. Let $\mathbb{R} \supset X \neq \emptyset$ and $R \supset Y \neq \emptyset$. If $x \leq y$ for all $x \in X, y \in Y$, then X is bounded above , Y is bounded below, and $\sup X \leq \inf Y$.
3. If the sets X, Y in (b), are such that $X \cup Y = \mathbb{R}$, then $\sup X = \inf Y$.
4. If X and Y are the sets defined in (c), then either X has a maximal element or Y as a minimal element.
5. Show that Dedekind's theorem is equivalent to the axiom of completeness.

Solution 10.37

Listed.

1. Let
$$A' = \{x \in \mathbb{R} \,|\, x \geq a \;\forall a \in A\}$$
$$B' = \{x \in \mathbb{R} \,|\, x \geq b \;\forall b \in B\}$$
where we can easily verify that $B' \subset A'$. By definition, we get $\sup B = \min B'$ and $\sup A = \min A'$. But since $B' \subset A'$, for any $b' \in B'$, there exists an $a' \in A'$ such that $a' \leq b'$, which implies that $\sup B = \min B' \leq \min A' = \sup A$.
2. X is bounded above by any element of Y. Y is bounded below by any element of X. By the completion axiom, there exists a $c \in \mathbb{R}$ such that
$$x \leq c \leq y \text{ for all } x \in X, y \in Y \tag{494}$$
Since c is an upper bound of X, $\sup X \leq c$ by definition, and since c is a lower bound of Y, $\inf Y \geq c$ by definition. Therefore, $\sup X \leq c \leq \inf Y$.
3. From completeness there exists a $c \in \mathbb{R}$ such that $x \leq c \leq y$ for all $x \in X, y \in Y$. Y is, by definition, the set of *all* upper bounds of X (i.e. *every* upper bound of X is in Y, unlike Y defined in 2.2.16.b). Since $c \leq y$ for all $y \in Y$, c is minimal and so $c = \sup X$. X is the set of all lower bounds of Y by definition, so $c \geq x$ for all $x \in X \implies c = \inf Y$. So, $\inf Y = c = \sup X$.
4. We know that there exists $c = \inf Y = \sup X$. Since $X \cup Y = \mathbb{R}$, c must be in at least X or Y. If $c \in X$, then $c = \sup X = \max X$, and if $c \in Y$, then $c = \inf Y = \min Y$.
5. This is the same statement as Zorich 2.2.15.a (an iff equivalence, not just one way implying).

Exercise 10.39 (Olmsted 1.13)

Let S be a nonempty set of numbers bounded above, and let x be the least upper bound of S. Prove that x has the two properties corresponding to an arbitrary positive number ϵ:

1. every element $s \in S$ satisfies the inequality $s < x + \epsilon$

2. at least one element $s \in S$ satisfies the inequality $s > x - \epsilon$

Solution 10.38

Listed.

1. x is an upper bound $\implies s \leq x$ for all $s \in S$, which implies that $s \leq x < x + \epsilon$.
2. By definition, $x - \epsilon$ cannot be an upper bound, so $x - \epsilon \geq s$ for all $s \in S$ is not true. Therefore, there must exist one $s \in S$ such that $s > x - \epsilon$.

Exercise 10.40 (Zorich 2.2.18)

Let $-A$ be the set of numbers of the form $-a$, where $a \in A \subset \mathbb{R}$. Show that $\sup(-A) = -\inf(A)$.

Solution 10.39

If A is unbounded below, then $-\inf A = \infty$ and $-A$ is unbounded above, implying that $\sup A = \infty$. Now assume that A is bounded below, then by completeness, it must have a greatest lower bound. Let us define the set $B = \{b \in \mathbb{R} \,|\, b \leq a \;\forall a \in A\}$. From 2.2.16.b, we have $b \leq \inf A \leq a$ for all $a \in A, b \in B$. Multiplying by -1 gives $-b \geq -\inf A \geq -a$ for all $a \in A, b \in B$, which is equivalent to saying

$$a \leq -\inf A \leq b \text{ for all } a \in -A, b \in -B \tag{495}$$

by definition of $-A, -B$. $-\inf A$ is clearly an upper bound of $-A$, and since

$$\begin{aligned} B &= \{b \in \mathbb{R} \,|\, b \leq a \;\forall a \in A\} \\ &= \{b \in \mathbb{R} \,|\, -b \geq -a \;\forall a \in A\} \\ &= \{b \in \mathbb{R} \,|\, -b \geq a \;\forall a \in -A\} \end{aligned}$$

implies that $-B = \{b \in \mathbb{R} \,|\, b \geq a \;\forall a \in -A\}$ is the set of all upper bounds of A. So, $-\inf A$ is the least upper bound of $-A$, i.e. $-\inf A = \sup(-A)$.

Exercise 10.41 (Zorich 2.2.21)

Show that the set $\mathbb{Q}(\sqrt{n})$ of numbers of the form $a + b\sqrt{n}$ where $a, b \in \mathbb{Q}$, n is a fixed natural number that is not the square of any integer, is an ordered set satisfying the principle of Archimedes but not the axiom of completeness.

Solution 10.40

The order on $\mathbb{Q}(\sqrt{n})$ can be embedded from the ordering on the reals by defining the canonical injection map $i : \mathbb{Q}(\sqrt{n}) \longrightarrow \mathbb{R}$ and defining for any $x, y \in \mathbb{Q}(\sqrt{n})$,

$$x \leq_{\mathbb{Q}(\sqrt{n})} y \iff i(x) \leq_{\mathbb{R}} i(y) \tag{496}$$

Now, let $h > 0$ be any fixed real number, and $x = (a, b) = a + b\sqrt{n}$. By the Archimidean principle, we can find a $k \in \mathbb{Z}$ such that

$$(k-1)h \leq x \leq kh \text{ for some } x \in \mathbb{Q}(\sqrt{n}) \subset \mathbb{R} \tag{497}$$

We now show that $\mathbb{Q}(\sqrt{n})$ is not complete since it doesn't satisfy the LUB property. Since there are infinite prime numbers in $\mathbb{N}$, choose a prime number p that is not a factor of n. Then, we are guaranteed that pn is not a perfect square, and can define the set

$$X = \{x \in \mathbb{Q}(\sqrt{n}) \,|\, x < \sqrt{pn}\} \subset \mathbb{Q}(\sqrt{n}) \tag{498}$$

and assume that $c = c_1 + c_2\sqrt{n} = \sup X$ exists ($c_1, c_2 \in \mathbb{Q}$). Clearly, $c \neq \sqrt{pn} \notin \mathbb{Q}(\sqrt{n})$.

1. Assume $c < \sqrt{pn} \iff 0 < \sqrt{pn} - c \in \mathbb{R}$. By the Archimidean principle, there exists a $k \in \mathbb{N}$ such that $0 < \frac{1}{k} < \sqrt{pn} - c$. Then, we can verify that $c + \frac{1}{k} = (c_1 + \frac{1}{k}) + c_2\sqrt{n} \in \mathbb{Q}(\sqrt{n})$ and
$$c + \frac{1}{k} < c + \sqrt{pn} - c = \sqrt{pn} \implies c + \frac{1}{k} \in X \tag{499}$$
implies that c is not an upper bound. So we must turn to case 2.
2. Assume $c > \sqrt{pn} \iff c - \sqrt{pn} > 0$. By AP, there exists a $k \in \mathbb{N}$ such that $0 < \frac{1}{k} < c - \sqrt{pn}$. Then, we can verify that $c - \frac{1}{k} \in \mathbb{Q}(\sqrt{n})$ and
$$c - \frac{1}{k} > c - c + \sqrt{pn} = \sqrt{pn} \tag{500}$$
implies that $c - \frac{1}{k}$ is an upper bound of X, so c is not least.

Therefore, by contradiction, c does not exist.

Exercise 10.42 (Zorich 2.2.22)

Let $n \in \mathbb{N}$ and $n > 1$. In the set $E_n = \{0, 1, \ldots, n-1\}$, we define the sum and product of two elements as the remainders when the usual sum and product in $\mathbb{R}$ are divided by n. With these operations on it, the set E_n is denoted $\mathbb{Z}_n$.

1. Show that if n is not a prime number, then there are nonzero numbers $m, k \in \mathbb{Z}_n$ such that $m \cdot k = 0$, i.e. there exist nonzero zero divisors.
2. Show that if p is prime, then there are no zero divisors in $\mathbb{Z}_p$ and $\mathbb{Z}_p$ is a field.
3. Show that, no matter what the prime p, $\mathbb{Z}_p$ cannot be ordered in a way consistent with the arithmetic operations on it.

Solution 10.41

Listed.

1. n is composite implies that there exist $1 < m, k < n$ such that $n = mk$. These factors m, k are precisely the zero divisors of $\mathbb{Z}_n$ since $mk = n \equiv 0 \pmod{n}$.
2. With p prime, assume that there are nontrivial zero divisors $1 < m, k < p$ in $\mathbb{Z}_p$. Then, $mk \equiv 0 \pmod{n} \implies mk = lp$ for some $l \in \mathbb{N}$. But this implies that m or k must divide p, which is impossible since $1 < m, k < p$. Then prove field axioms.
3. For any field, we must have $0 \leq 1$, because if not, then
$$0 > 1 \implies 0 < 1^{-1} \cdot 1 = 1^{-1} \implies 0 \cdot 0 < 1^{-1} \cdot 1^{-1} = 1 \tag{501}$$
So, $0 \leq 1$ implies that $0 \leq 1 \leq 2 \leq \ldots \leq p - 1$. But
$$0 + 1 \leq (p - 1) + 1 = 0 \tag{502}$$
is false, so any ordering is impossible.

Exercise 10.43 (Zorich 2.2.23)

Show that if $\mathbb{R}$ and $\mathbb{R}'$ are two models of the set of real numbers and $f : \mathbb{R} \longrightarrow \mathbb{R}'$ (with $f \not\equiv 0'$) is a mapping such that $f(x + y) = f(x) + f(y)$ and $f(x \cdot y) = f(x) \cdot f(y)$ for any $x, y \in \mathbb{R}$. Prove that f is an order-preserving isomorphism.

Solution 10.42

Let $0, 0'$ be the additive identity of $\mathbb{R}, \mathbb{R}'$, respectively, and $1, 1'$ the multiplicative identity. We claim that $f(0) = 0'$ since

$$\begin{aligned} f(0) &= f(0+0) && \text{(definition of additive identity)} \\ &= f(0) + f(0) && \text{(homomorphism over +)} \end{aligned}$$

which implies that $f(0) + f(0) = f(0) = 0' + f(0)$. Since $f(0)$ lives in field $\mathbb{R}'$, its additive identity $-f(0)$ is well defined, and we get $f(0) = f(0) + f(0) + (-f(0)) = 0' + f(0) + (-f(0)) = 0'$. We also claim that $f(1) = 1'$ since

$$\begin{aligned} f(1) &= f(1 \cdot 1) && \text{(definition of multiplicative identity)} \\ &= f(1) \cdot f(1) && \text{(homomorphism over } \cdot \text{)} \end{aligned}$$

which implies that $f(1) \cdot f(1) = 1' \cdot f(1)$. Since $f(1)$ lives in field $\mathbb{R}'$, its multiplicative identity $f(1)^{-1}$ is well defined, and we get $f(1) = f(1) \cdot f(1) \cdot f(1)^{-1} = 1' \cdot f(1) \cdot f(1)^{-1} = 1'$. Now that we have proved mapping of identities, this implies the mapping of inverses.

$$\begin{aligned} 0' = f(0) = f(x - x) = f(x) + f(-x) &\implies f(-x) = -f(x) \\ 1' = f(1) = f(x \cdot x^{-1}) = f(x) \cdot f(x^{-1}) &\implies f(x^{-1}) = f(x)^{-1} \end{aligned}$$

With these conditions, we have proved that f is a homomorphism of fields. Now we prove that f is a bijection, but first, we claim that $f(x) = 0' \implies x = 0$. Assume that there exists a nonzero $x \in \mathbb{R}$ such that $f(x) = 0'$. Then, x^{-1} is well defined, and

$$\begin{aligned} f(x) \cdot f(x^{-1}) &= f(x) \cdot f(x)^{-1} = 0' \\ f(x) \cdot f(x^{-1}) &= f(x \cdot x^{-1}) = f(1) = 1' \end{aligned}$$

which implies that $0' = 1'$. So, $f(1) = 1' = 0'$, and so for all $k \in \mathbb{R}$, $f(k) = f(k \cdot 1) = f(k) \cdot f(1) = f(k) \cdot 0' = 0' \implies f \equiv 0'$, leading to a contradiction of the assumption that $f' \not\equiv 0'$.

1. (f injective). Assume f is not injective, i.e. there exists distinct $x_1, x_2 \in \mathbb{R}$ s.t. $f(x_1) = f(x_2)$. Then, using that fact $f(x) = 0 \implies x = 0$,
$$0 = f(x_1) - f(x_2) = f(x_1 - x_2) \implies x_1 - x_2 = 0 \implies x_1 = x_2 \tag{503}$$
2. (f surjective). Let y be any nonzero element in $\mathbb{R}'$ (clearly if $y = 0'$ then its preimage is 0) and y^{-1} its multiplicative inverse. Assume there exists no $x \in \mathbb{R}$ satisfying $f(x) = y$, meaning that there exist no x satisfying
$$f(x) \cdot = y \cdot y^{-1} = 1' \tag{504}$$
But since f maps inverses to inverses, we can choose $x = (y^{-1})^{-1}$, which leads to
$$f(x) \cdot y^{-1} = (\tag{505}$$

Finally, we prove that f is order preserving. Assume that $x \leq y \iff 0 \leq y - x$, we wish to prove that
$$f(x) \leq f(y) \iff 0 \leq f(y) - f(x) = f(y - x) \tag{506}$$
Therefore, since this preservation of ordering is really the statement $0 \leq y - x \implies 0 \leq f(y - x)$, it suffices to prove that $0 \leq x \implies 0 \leq f(x)$. Now, assume that we have a x such that $f(x) < 0'$. Adding it with the equation $f(1) = 1'$ gives us
$$f(x + 1) < 1' \tag{507}$$
It is easy to prove that $0 \leq x \iff 0 \leq x^{-1}$. Now assume that $0 > f(x)$. **INCOMPLETE**

Exercise 10.44 (Density of Rationals in $\mathbb{R}$)

Prove that for any two distinct $a < b \in \mathbb{R}$, there exists an infinite number of rational numbers between a and b.

Solution 10.43

Since $a < b$, then $b - a > 0$ and by the Archimidean principle, there exists a $k \in \mathbb{N}$ such that

$$0 < \frac{1}{k} < b - a \implies 1 < kb - ka \tag{508}$$

which implies that the length of $[ka, kb)$ greater than 1. By the inductive property of $\mathbb{Z}$, there must be an integer $p \in [ka, kb)$. If there were not, then this would imply that $[ka+1, kb+1)$ and $[ka-1, kb-1)$ had no integers and repeating would mean that there were no integers in $\mathbb{R}$. Therefore,

$$ka \leq p < kb \implies a \leq \frac{p}{k} < b \tag{509}$$

for all $a, b \in \mathbb{R}$, with $p/k \in \mathbb{Q}$. If a is irrational we can replace the $\leq$ to $<$, leaving $a \leq \frac{p}{k} < b$, and if a is rational, we can construct another rational $a + \frac{1}{k} \in (a, b)$.

Exercise 10.45 (Nested Interval Lemma)

With the fact that $\mathbb{R}$ is complete, prove the following.

1. For a sequence of closed nested intervals $I_1 \supset I_2 \supset \ldots$ of $\mathbb{R}$, there exists a point $c \in \mathbb{R}$ belonging to all these intervals.
2. Furthermore, if the hypothesis also satisfies the fact that for any $\epsilon > 0$, there exists a $k \in \mathbb{N}$ such that $|I_k| < \epsilon$ (i.e. the length of the intervals decreases to 0), then the point c common to all sets is unique.

Solution 10.44

Listed.

1. Let $I_n = [a_n, b_n]$, with $a_n < b_n$ finite for all $n \in \mathbb{N}$. For all $n \in \mathbb{N}$, we have $I_n = [a_n, b_n]$ and can take the two subsets $X_n = (-\infty, a_n)$ and $Y_n = (b_n, \infty)$, where $x \leq y$ for every $x \in X_n, y \in Y_n$. We also have the fact that $\mathbb{R} = X_n \cup I_n \cup Y_n$. Since $\mathbb{R}$ is complete, there exists a c such that $x \leq c \leq y$ for all $x \in X, y \in Y$. But $x \leq c \iff c \notin X_n$ and $c \leq y \iff c \notin Y_n$, so for all $n \in \mathbb{N}$, c must be in I_n.
2. Since we have proved (a), it now suffices to prove uniqueness of c. Let there be two distinct points $c_1, c_2 \in \mathbb{R}$ belonging to these intervals. Without loss of generality, assume $c_1 - c_2 > 0$, and choose
$$\epsilon = \frac{c_1 - c_2}{3} \tag{510}$$
Then, there should exist a $k \in \mathbb{N}$ such that $|I_k| < \epsilon$. Since I_k must contain c_1, it must be a subset of $[c_1 - \epsilon, c_1 + \epsilon]$ (should be able to see why) and similarly for c_2.
$$I_k \subset \left[c_1 - \frac{c_1 - c_2}{3}, c_1 + \frac{c_1 - c_2}{3}\right] = \left[\frac{2c_1 + c_2}{3}, \frac{4c_1 - c_2}{3}\right] = L$$
$$I_k \subset \left[c_2 - \frac{c_1 - c_2}{3}, c_2 + \frac{c_1 - c_2}{3}\right] = \left[\frac{-c_1 + 4c_2}{3}, \frac{c_1 + 2c_2}{3}\right] = M$$
But since $c_1 > c_2 \implies \frac{c_1+2c_2}{3} < \frac{2c_1+c_2}{3}$, L and M are disjoint $\implies I_k$, as a subset of both, leaves us with $I_k = \emptyset$, contradicting that it is a closed interval.

Exercise 10.46 ()

Compactness of Closed Interval in $\mathbb{R}$ Prove that any system of open intervals covering (i.e. an open cover of) a closed interval contains a finite subsystem that covers the closed interval. Another way to state this is by saying that every closed interval of $\mathbb{R}$ is compact.

Solution 10.45

A closed interval with a finite open covering is trivially compact since any subcovering is also finite. We only need to deal with when a closed interval $I = [a, b]$ has an infinite open covering $\{U_\alpha\}_{\alpha \in A}$, which means that the set of indices A is infinite. Assume that there exists no finite covering of I. Then, we divide I into two halves

$$I_1 = \left[a, \frac{a+b}{2}\right], \;\; I_2 = \left[\frac{a+b}{2}, b\right] \tag{511}$$

and define a subcovering for each of them. That is, we can define $A_1 \subset A$ and $A_2 \subset A$ such that $\{U_\alpha\}_{\alpha \in A_1} \subset \{U_\alpha\}_{\alpha \in A}$ is a covering of I_1 and $\{U_\alpha\}_{\alpha \in A_2} \subset \{U_\alpha\}_{\alpha \in A}$ is a covering of I_2. At least one of A_1 or A_2 must be infinite, since if they were both finite, then we can define a finite covering $\{U_\alpha\}_{\alpha \in A_1 \cup A_2}$ of I. Choose the interval with the infinite covering and repeat this procedure, which will result in a nested interval that decreases in length by a half.

$$I \supset I_1 \supset I_2 \supset \ldots \tag{512}$$

By the nested interval lemma, there exists a unique point c common to all these intervals. But since $c \in [a, b]$, the open cover $\{U\}$ should contain an open interval $(c - \delta_1, c + \delta_2)$ containing c. We wish to prove that this interval is a superset of some I_k in the sequence above, contradicting the fact that I_k has an infinite cover. Since the length of each I_i decreases arbitrarily (i.e. we can choose any $\epsilon > 0$ and find a I_k with length less than ϵ), we choose $\epsilon = \frac{1}{2} \min\{\delta_1, \delta_2\}$, and we should be able to find some I_k that is a subinterval of $[c - \epsilon, c + \epsilon]$, which itself is a subinterval of $(c - \delta_1, c + \delta_2)$.

$$I_k \subset \left[c - \frac{1}{2}\min\{\delta_1, \delta_2\}, c + \frac{1}{2}\min\{\delta_1, \delta_2\}\right] \subset (c - \delta_1, c + \delta_2) \tag{513}$$

Therefore, $(c - \delta_1, c + \delta_2)$ is a finite cover of I_k, contradicting the fact that all I_k's have infinite covers.

Exercise 10.47 ()

Bolzano-Weierstrass Theorem Prove that every bounded infinite set of real numbers has at least one limit point. (A limit point p of set X is a point such that every open neighborhood of p contains an infinite number of elements of X).

Solution 10.46

Let the set of points be denoted X, and let a be the lower bound and b be the upper bound. Then, $X \subset [a, b] = I$. Now divide $[a, b]$ into halves $[a, \frac{a+b}{2}] \cup [\frac{a+b}{2}, b]$. At least one of the halves must have an infinite number of points; choose the interval with infinite points as I_1 and doing this repeatedly gives the nested sequence

$$I \supset I_1 \supset I_2 \supset \ldots \tag{514}$$

By the nested interval lemma, there exists at least one point $c \in \mathbb{R}$ that is in all these intervals. Furthermore, since $|I_i| = \frac{1}{2^i}(b - a)$ decreases to 0, we can choose a $\epsilon > 0$ and find an interval I_k with $|I_k| < \epsilon$. We claim that c is a limit point of X. Given an ϵ, we wish to prove that there are an infinite number of points within the ϵ-neighborhood $(c - \epsilon, c + \epsilon)$ of c. Since we can find some I_k with $|I_k| < \epsilon$, we can see that

$$I_k \subset (c - \epsilon, c + \epsilon) \tag{515}$$

and therefore the ϵ-neighborhood of c contains I_k, which contains an infinite number of points in X.

Solution 10.47

We can construct another proof that is dependent on the compactness lemma. This construction will be useful for problem 2.3.4. Let X be a given subset of $\mathbb{R}$, and it follows from the definition of boundedness that X is contained in some closed interval $I \subset \mathbb{R}$. We show that at least one point of I is a limit point of X. Assume that it is not. Then each point $x \in I$ would have a neighborhood $U(x)$ containing at most a finite number of points from X. The totality of such neighborhoods $\{U(x)\}$ constructed for the points $x \in I$ forms an open covering of X. Since I is closed, it is compact and therefore we can find a finite subcovering $\{U_i(x)\}_i$ of open intervals that cover I and therefore cover X. This open cover $\{U_i(x)\}_i$ of X is a finite union of sets that each contain at most a finite number of points from X, so the covering of X contains a finite number of points from X, a contradiction that X contains infinite points.

Exercise 10.48 (Zorich 2.3.1)

Show that

1. if I is any system of nested closed intervals, then

$$\sup\{a \in \mathbb{R} \mid [a, b] \in I\} = \alpha \leq \beta = \inf\{b \in \mathbb{R} \mid [a, b] \in I\}$$

   and

$$[\alpha, \beta] = \bigcap_{[a,b] \in I} [a, b]$$

2. if I is a system of nested open intervals (a, b), the intersection

$$\bigcap_{(a,b) \in I} (a, b)$$

   may happen to be empty.

Solution 10.48

Listed.

1. (May be tempted to say that $a_1 \leq a_2 \leq \ldots$, but this assumes that the indexing set I is countable). We claim that for any two intervals $[a_n, b_n]$ and $[a_m, b_m]$ in I,

$$a_n \leq b_m$$

   Assume that $a_n > b_m$. Then $b_n \geq a_n > b_m \geq a_m$ implies that $[a_n, b_n]$ and $[a_m, b_m]$ are disjoint, contradicting the fact that they are nested. Now given that X is the set of a_n's and Y is the set of b_n's, we have $x \leq y$ for all $x \in X, y \in Y$. So by 2.2.16.b, we have $\sup X \geq \inf Y$.
   To prove the second statement, we show that trying to "expand" the interval $[\alpha, \beta]$ will lead to a contradiction. Since α is the LUB, given any $\epsilon > 0$, there exists a $(a_l, b_l) \in X$ such that $\alpha - \epsilon < a_l < \alpha$, which implies that $[\alpha, \beta] \subset [a_l, \beta] \subset [\alpha - \epsilon, \beta]$. Assuming that this extended interval is the intersection, we should be able to choose any point in $[\alpha - \epsilon, \beta]$ and find that it is in every element of I. We choose a point in $[\alpha - \epsilon, a_l)$, which is not in the interval (a_l, b_l). We do the same for $\beta \mapsto \beta + \epsilon$. We also check that "shrinking" the interval $[\alpha, \beta] \mapsto [\alpha + \epsilon, \beta]$ is no good, since we can find an element in $[\alpha, \alpha + \epsilon)$ that is in every interval in I.

2. Take the system of nested open intervals

$$(0,1) \supset (0, \frac{1}{2}) \supset (0, \frac{1}{3}) \ldots (0, \frac{1}{n}) \supset \ldots$$

Take their infinite intersection, denote it S, and assume that some $\epsilon \in (0,1)$ is in S. Since ϵ is a real number, by the Archimedean principle there exists a $k \in \mathbb{N}$ such that $\frac{1}{k} < \epsilon$. Therefore, $\epsilon \notin (0, \frac{1}{k}) \implies \epsilon \notin S$.

Exercise 10.49 (Zorich 2.3.2)

Show that
1. from a system of closed intervals covering a closed interval it is not always possible to choose a finite subsystem covering the interval.
2. from a system of open intervals covering a open interval it is not always possible to choose a finite subsystem covering the interval.
3. from a system of closed intervals covering a open interval it is not always possible to choose a finite subsystem covering the interval.

Solution 10.49

We show with the interval $(0,1)$ or $[0,1]$. Using linear transformations it is easy to generalize this to any other interval (a,b) or $[a,b]$.
1. Consider the infinite covering

$$[0,1] = [0, \frac{1}{2}] \cup [\frac{1}{2}, \frac{3}{4}] \cup [\frac{3}{4}, \frac{7}{8}] \cup \ldots$$

2. Consider the infinite covering

$$(0,1) = (0, \frac{1}{2}) \cup (\frac{1}{2}, \frac{3}{4}) \cup (\frac{3}{4}, \frac{7}{8}) \cup \ldots$$

3. Consider the infinite covering

$$(0,1) = [0, \frac{1}{2}] \cup [\frac{1}{2}, \frac{3}{4}] \cup [\frac{3}{4}, \frac{7}{8}] \cup \ldots$$

Exercise 10.50 (Zorich 2.3.3)

Show that if we only take the set $\mathbb{Q}$ of rational numbers instead of the complete set $\mathbb{R}$ of real numbers, with the definitions of closed, open, and neighborhood of a point $r \in \mathbb{Q}$ to mean respectively the corresponding subsets of $\mathbb{Q}$, then none of the three lemmas is true.

Solution 10.50

We prove only for the nested interval lemma. We choose the series of nested intervals

$$\left(\sqrt{2} - \frac{1}{n}, \sqrt{2} + \frac{1}{n}\right)$$

with $n \in \mathbb{N}$. Assume that there is a $r \in \mathbb{Q}$ such that

$$r \in \left(\sqrt{2} - \frac{1}{n}, \sqrt{2} + \frac{1}{n}\right) \text{ for all } n \in \mathbb{N}$$

which is equivalent to saying that $\left|r-\sqrt{2}\right| < \frac{1}{n}$ for all $n \in \mathbb{N}$. Clearly, $r \neq \sqrt{2}$, and by the Archimidean principle, there exists a $k \in \mathbb{N}$ such that

$$0 < \frac{1}{k} < |r - \sqrt{2}|$$

which contradicts the above.

Exercise 10.51 (Zorich 2.3.4)

Show that the three lemmas above are equivalent to the axiom of completeness.

Solution 10.51

Note that from the proofs, completeness implies nested interval lemma, which implies compactness of closed intervals, which implies the Bolzano-Weierstrass theorem. So, it is sufficient to prove that Bolzano-Weierstrass theorem implies completeness to determine equivalence. There are not a lot of direct proofs, so we prove that Weierstrass implies nested interval, which implies completeness.

1. (Weierstrass $\implies$ Nested) Assume that we have $\mathbb{R}$ with the Bolzano-Weierstrass theorem. Take the series of nested closed intervals

$$I = [a, b] \supset I_1 = [a_1, b_1] \supset I_2 = [a_2, b_2] \supset \ldots$$

   We see that $a \leq a_i \leq b$, so the infinite sequence of monotonically nondecreasing values a_i is bounded. Therefore, it must have a limit point, which we will denote as c. We claim that $a_i \leq c$ for all a_i. Since if it were not, then $c < a_i$ for some i, and choosing $\epsilon = 0.5(a_i - c)$, the ϵ-neighborhood of c will not contain a_j for $j \geq i$ since

$$c < a_i \implies 0.5c < 0.5a_i \implies c + \epsilon = 0.5c + 0.5a_i < a_i < a_{i+1} < \ldots$$

   . With similar reasoning, we can conclude that $b_i \geq c$ for all b_i. This implies that $a_i \leq c \leq b_i$ for all i which is equivalent to saying that $c \in [a_i, b_i] = I_i$ for all $i \in \mathbb{N}$.

2. (Nested $\implies$ LUB Principle) Let $X \subset \mathbb{R}$ be a set that is bounded above, with b_1 any upper bound. Since X is nonempty, there exists $a_1 \in X$ that is not an upper bound (otherwise, X would be a singleton set and it trivially has a least upper bound). Consider the well-defined interval $[a_0, b_0]$. Take the mean $m_0 = 0.5(a_0 + b_0)$, and if m_0 is an upper bound, set it to b_1 (with $a_1 = a_0$) and a_1 if else (with $b_1 = b_0$). Then, we have a sequence of nested intervals

$$[a_0, b_0] \supset [a_1, b_1] \supset [a_2, b_2] \supset \ldots$$

   of decreasing lengths $|I_k| = \frac{1}{2^k}(b - a)$. All of them must contain a unique common point $c \in \mathbb{R}$ by the nested intervals lemma, which implies that

$$a_0 \leq a_1 \leq a_2 \leq \ldots \leq c \leq \ldots \leq b_2 \leq b_1 \leq b_0$$

   I claim two things:

   (a) c is an upper bound for X. Suppose it were not, then there exists some $x \in X$ such that $c < x$, and let the distance between them be $\epsilon = x - c > 0$. By AP, we can choose $k \in \mathbb{N}$ such that $\frac{1}{k} < \epsilon$. All the b_n are upper bounds of X, so we have $x \leq b_n$. Subtracting c on both sides gives

$$0 < x - c = \epsilon \leq b_n - c \leq |I_n| = \frac{1}{2^n}(b_0 - a_0)$$

   where the last inequality follows from $c \in I_n = [a_n, b_n]$, so the maximum distance it can be from the endpoint b_n is $|I_n|$. The inequality above holds for all $n \in \mathbb{N}$, so increasing n

arbitrarily should decrease $\frac{1}{2^n}(b_0 - a_0)$ past ϵ. To formalize this, we use the inequality

$$\frac{1}{2^n} < \frac{1}{n} \text{ for all } n \in \mathbb{N}$$

and so we have

$$\epsilon \leq b_n - c < \frac{1}{n}(b_0 - a_0)$$

We choose the natural number $n = \lceil \frac{2(b_0-a_0)}{\epsilon} \rceil$, which does not satisfy the inequality above since

$$\epsilon < \frac{1}{n}(b_0 - a_0) = \frac{1}{\lceil 2(b_0 - a_0)/\epsilon \rceil}(b_0 - a_0) \leq \frac{\epsilon}{2(b_0 - a_0)}(b_0 - a_0) = \frac{\epsilon}{2}$$

This leads to a contradiction.

(b) We now prove that c is least. Assume that c is not least $\implies$ there exists an upper bound B such that $B < c$ and $x \leq B$ for all $x \in X$. **INCOMPLETE**

Exercise 10.52 (Zorich 2.4.1)

Show that the set of real numbers has the same cardinality as the points of the interval $(-1, 1)$.

Solution 10.52

We define the bijective map $\rho : (-1, 1) \longrightarrow \mathbb{R}$

$$p(x) = \begin{cases} 0 & \text{if } x = 0 \\ \frac{1}{x} & \text{if } x \neq 0 \end{cases}$$

Exercise 10.53 (Zorich 2.4.2)

Give an explicit one-to-one correspondence between

1. the points of two open intervals
2. the points of two closed intervals
3. the point of a closed interval and an open interval
4. the points of the closed interval $[0, 1]$ and $\mathbb{R}$

Solution 10.53

Listed.

1. $\rho : (a, b) \longrightarrow (c, d)$ defined
$$\rho(x) = \frac{d-c}{b-a}(x - a) + c$$
2. the extension of ρ defined on (a) to $[a, b]$
3. From (a) and (b), it suffices to prove a bijection from $(0, 1)$ to $[0, 1]$. We extract a countably infinite sequence from $(0, 1)$, say
$$x_1 = \frac{1}{3},\ x_2 = \frac{1}{4}, \ldots, x_i = \frac{1}{i+2}$$

Then, we define bijection $\rho : (0,1) \longrightarrow [0,1]$ as

$$\rho(x) = \begin{cases} x & \text{if } x \notin \{x_i\} \\ 0 & \text{if } x = x_1 = \frac{1}{2} \\ 1 & \text{if } x = x_2 = \frac{1}{3} \\ x_{i-2} & \text{if } x = x_i \text{ for } i > 2 \end{cases}$$

Colloquially, we extract a copy of $\mathbb{N}$ from $(0,1)$ and use the bijection $\mathbb{N} \simeq \mathbb{N} \cup \{0,-1\}$ to "shift" the terms.

4. We compose the bijections $\rho_1 : [0,1] \longrightarrow (0,1)$ and $\rho_2 : (0,1) \longrightarrow \mathbb{R}$.

Exercise 10.54 (Zorich 2.4.3)

Show that

1. every infinite set contains a countable subset
2. the set of even integers has the same cardinality as the set of all natural numbers
3. the union of an infinite set and an at most countable set has the same cardinality as the original infinite set.
4. the set of irrational numbers has the cardinality of the continuum
5. the set of transcendental numbers has the cardinality of the continuum

Solution 10.54

Listed.

1. Let A be an infinite set. By axiom of choice, choose $a_0 \in A$. Then, $A \setminus \{a_0\} \neq \emptyset$ since A is infinite. By induction, assume you have chosen $a_0, a_1, \ldots, a_k \in A$. Then, since A is infinite, $A \setminus \{a_0, a_1, \ldots, a_k\} \neq \emptyset$, so we can choose $a_{k+1} \in A \setminus \{a_0, \ldots, a_k\}$. Thus, we have constructed a countable subset $\{a_k\}_{k\in\mathbb{N}}$ of A.
2. Given the quotient ring $2\mathbb{Z}$, define the bijection $\rho : 2\mathbb{Z} \longrightarrow \mathbb{N}$ as

$$p(x) = \begin{cases} x+2 & \text{if } x \geq 0 \\ -x-1 & \text{if } x < 0 \end{cases}$$

3. From (a), we can extract a countable set from original set A, call it X. Since the product of countable sets is countable ($\mathbb{N} \cup \mathbb{N}$ is countable), we can define a bijection $\tilde{\rho} : X \longrightarrow X \cup B$. Therefore, we can define a bijection $\rho : A \longrightarrow A \cup B$ as

$$\rho(x) = \begin{cases} x & \text{if } x \in A \setminus X \\ \tilde{\rho}(x) & \text{if } x \in X \end{cases}$$

4. $\mathbb{Q}$ is countable and $\mathbb{R}$ is uncountable. So, $\mathbb{R} \setminus \mathbb{Q}$ must be uncountable since if it were countable, then the union of the rationals and irrationals, which is $\mathbb{R}$, would be countable.
5. It suffices to prove that the set of algebraic numbers (numbers that are possible roots of a polynomial with integer coefficients with leading coefficient nonzero) is countable, since we can apply (d) right after. The set of all kth degree polynomials with integer coefficients is isomorphic to $\mathbb{Z}^k$ through the map

$$a_k x^k + a_{k-1}x^{k-1} + \ldots + a_2 x^2 + a_1 x^1 + a_0 \mapsto (a^{k-1}, a^{k-2}, \ldots, a_1, a_0)$$

and the union of these countable sets (minus the 0 map)

$$P = \bigg(\bigcup_{k=1}^{\infty} \mathbb{Z}^k \bigg) \setminus \{0\} = \big(\mathbb{Z} \setminus \{0\}\big) \cup \mathbb{Z}^2 \cup \ldots$$

is countable. For any element in $\mathbb{Z}^k$, there are at most k real roots, and so we can define the set of roots of an element $z \in \mathbb{Z}^k \subset P$ as a j-tuple of algebraic numbers, which can have at most $j = k$ roots.

$$r(z) = \underbrace{(r_{1z}, r_{2z}, \ldots, r_{jz})}_{j \leq k}$$

Therefore, the union of all these j-tuples for all $z \in P$

$$\bigcup_{z \in P} r(z) = \bigcup_{k=1}^{\infty} \bigcup_{z \in \mathbb{Z}^k} r(z)$$

is a countable union of a countable union of finite sets, making it countable.

Exercise 10.55 (Zorich 2.4.4)

Show that

1. the set of increasing sequences of natural numbers has the same cardinality as the set of fractions of the form $0.\alpha_1\alpha_2\ldots$
2. the set of all subsets of countable set has cardinality of the continuum

Solution 10.55

Listed.

1. Given a sequence of increasing naturals $S = (n_1, n_2, \ldots)$, we can define a binary expansion $0.\alpha_1\alpha_2\ldots$ where $\alpha_i = 1$ if and only if $i \in \mathbb{N}$ is in S and $\alpha_i = 0$ if not. This is clearly a bijection.
2. The set of all segments of increasing natural is equipotent with $2^{\mathbb{N}}$, since the elements of each sequence define a subset of $\mathbb{N}$. Cantor's diagonalization argument proves that the set of infinite binary expansions is uncountable, and by (a), this proves that $2^{\mathbb{N}}$ is uncountable.

This is very interesting since $\mathbb{N} \simeq \mathbb{R}$, but $2^{\mathbb{N}} \simeq \mathbb{R}$, and the set of all infinite q-ary expansions is equipotent to $\mathbb{R}$ too.

Exercise 10.56 (Zorich 2.4.5)

Show that

1. the set $\mathcal{P}(X)$ of subsets of a set X has the same cardinality as the set of all functions $f : X \longrightarrow \{0, 1\}$.
2. for a finite set X of n elements, $\operatorname{card} \mathcal{P}(X) = 2^n$
3. one can write $\operatorname{card} \mathcal{P}(X) = 2^{\operatorname{card} X}$, which implies $\operatorname{card} \mathcal{P}(\mathbb{N}) = 2^{\operatorname{card} \mathbb{N}} = \operatorname{card} \mathbb{R}$
4. for any set X, $\operatorname{card} X < 2^{\operatorname{card} X}$

Solution 10.56

Listed.

1. An element $Y \in \mathcal{P}(X)$ is a subset of X by definition. Letting

$$f_Y(x) = \begin{cases} 0 & \text{if } x \notin Y \\ 1 & \text{if } x \in Y \end{cases}$$

we can construct the bijective map $Y \mapsto f_Y$.

2. We can prove this using the identity (which can be proved using induction)

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

3. Let $F(X; \{0,1\})$ be the set of all binary valued functions from X to $\{0,1\}$. From (a), $\operatorname{card} \mathcal{P}(X) \simeq F(X; \{0,1\})$. Each binary-valued function f is determined by the assignment $f(x)$ for each $x \in X$. Since $f(x)$ has two possible values, the assignment of $f(x)$ for all $x \in X$ has $\{0,1\}^{\operatorname{card} X}$ possible choices. This gives another bijection $F(X; \{0,1\}) \simeq \{0,1\}^{\operatorname{card} X}$, so

$$\mathcal{P}(X) \simeq \{0,1\}^{\operatorname{card} X} \implies \operatorname{card} \mathcal{P}(x) = \operatorname{card}(\{0,1\}^{\operatorname{card} X}) = 2^{\operatorname{card} X}$$

4. If X is finite, then letting $n = \operatorname{card} X$, we can simply prove $n < 2^n$ by induction (which we will not do here). If X is countable, then $\mathcal{P}(X)$ is uncountable (from 2.4.4) and so using (c),

$$\operatorname{card} X = \operatorname{card} \mathbb{N} < \operatorname{card} \mathbb{R} = \operatorname{card} \mathcal{P}(X) = 2^{\operatorname{card} X}$$

For uncountable sets (and for the two cases mentioned above), we can use Cantor's theorem, which states that $\operatorname{card} X < \operatorname{card} \mathcal{P}(X)$, and so using (c), we have $\operatorname{card} X < \operatorname{card} \mathcal{P}(X) = 2^{\operatorname{card} X}$.

Exercise 10.57 (Zorich 2.4.6)

Let $X_1, \ldots, X_m$ be a finite system of finite sets. Show that

$$\begin{aligned}
\operatorname{card}\left(\bigcup_{i=1}^{m} X_i\right) &= \sum_{i_1} \operatorname{card} X_{i_1} - \sum_{i_1 < i_2} \operatorname{card}(X_{i_1} \cap X_{i_2}) + \ldots \\
&\sum_{i_1 < i_2 < i_3} \operatorname{card}(X_{i_1} \cap X_{i_2} \cap X_{i_3}) - \ldots + (-1)^{m-1} \operatorname{card}(X_1 \cap \ldots \cap X_m) \\
&= \sum_{k=1}^{m} \sum_{1 \leq i_1 \ldots i_k \leq m} (-1)^{k-1} \operatorname{card}\left(\bigcap_{j=1}^{k} X_{i_j}\right)
\end{aligned}$$

Solution 10.57

Ignoring Russell's paradox (defining the universe set of all sets), we can use the commutative, associative, and distributive properties of $\cup, \cap$ on the algebra of sets. We prove using induction on m. For $m = 1$, we trivially have $\operatorname{card} X_1 = \operatorname{card} X_1$, and for $m = 2$, we claim

$$\operatorname{card}(X_1 \cup X_2) = \operatorname{card}(X_1) + \operatorname{card}(X_2) - \operatorname{card}(X_1 \cap X_2)$$

X_1 and $X_2 \setminus X_1$ are clearly exclusive sets by definition, with $X_1 \cup X_2 = X_1 \cup (X_2 \setminus X_1)$, so

$$\operatorname{card}(X_1 \cup X_2) = \operatorname{card}\big(X_1 \cup (X_2 \setminus X_1)\big) = \operatorname{card}(X_1) + \operatorname{card}(X_2 \setminus X_1) \tag{2}$$

By definition, the set $X_2 \setminus X_1$ and $X_1 \cap X_2$ are disjoint and satisfies $X_2 = (X_2 \setminus X_1) \cup (X_1 \cap X_2)$ (also by definition), so

$$\operatorname{card}(X_2) = \operatorname{card}(X_2 \setminus X_1) + \operatorname{card}(X_1 \cap X_2) \tag{3}$$

and substituting (3) into (2) gives the claim for $m = 2$. Assuming that the claim is satisfied for some

m, we have

$$\begin{aligned}
\operatorname{card}\left(\bigcup_{i=1}^{m+1} X_i\right) &= \operatorname{card}\left(\left[\bigcup_{i=1}^{m} X_i\right] \cup X_{m+1}\right) \\
&= \operatorname{card}\left(\bigcup_{i=1}^{k} X_i\right) + \operatorname{card}(X_{m+1}) - \operatorname{card}\left(\left[\bigcup_{i=1}^{m} X_i\right] \cap X_{m+1}\right) \qquad \text{(claim for } m = 2) \\
&= \operatorname{card}\left(\bigcup_{i=1}^{k} X_i\right) + \operatorname{card}(X_{m+1}) - \operatorname{card}\left(\bigcup_{i=1}^{m} (X_i \cap X_{m+1})\right) \qquad \text{(distributive prop.)} \\
&= \sum_{k=1}^{m} \sum_{1 \leq i_1 \ldots i_k \leq m} (-1)^{k-1} \operatorname{card}\left(\bigcap_{j=1}^{k} X_{i_j}\right) + \operatorname{card}(X_{m+1}) \\
&\quad - \sum_{k=1}^{m} \sum_{1 \leq i_1 \ldots i_k \leq m} (-1)^{k-1} \operatorname{card}\left(\bigcap_{j=1}^{k} (X_{i_j} \cap X_{m+1})\right)
\end{aligned}$$

With a bit of thought, we can see that the kth term of the second summation contributes to adding another term to the $k+1$th summation term of the first. Therefore, we must try to shift the summation over by 1 index. Let us simplify this by taking the summations and extracting the first and last term, respectively. We have

$$\begin{aligned}
\sum_{k=1}^{m} \sum_{1 \leq i_1 \ldots i_k \leq m} (-1)^{k-1} \operatorname{card}\left(\bigcap_{j=1}^{k} X_{i_j}\right) &= \sum_{1 \leq i_1 \ldots i_k \leq m} \operatorname{card}(X_{i_1}) \\
&\quad + \sum_{k=2}^{m} \sum_{1 \leq i_1 \ldots i_k \leq m} (-1)^{k-1} \operatorname{card}\left(\bigcap_{j=1}^{k} X_{i_j}\right)
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{k=1}^{m} \sum_{1 \leq i_1 \ldots i_k \leq m} (-1)^{k-1} \operatorname{card}\left(\bigcap_{j=1}^{k} (X_{i_j} \cap X_{m+1})\right) \\
&\qquad = \sum_{k=1}^{m} \sum_{1 \leq i_1 \ldots i_k \leq m} (-1)^{k-1} \operatorname{card}\left(\left[\bigcap_{j=1}^{k} X_{i_j}\right] \cap X_{m+1}\right) \\
&\qquad = \sum_{k=1}^{m-1} \sum_{1 \leq i_1 \ldots i_k \leq m} (-1)^{k-1} \operatorname{card}\left(\bigcap_{j=1}^{k} (X_{i_j} \cap X_{m+1})\right) + (-1)^{m-1} \operatorname{card}\left(\bigcap_{j=1}^{m+1} X_j\right) \\
&\qquad = \sum_{k=2}^{m} \sum_{1 \leq i_1 \ldots i_k \leq m} (-1)^{k-2} \operatorname{card}\left(\bigcap_{j=1}^{k-1} (X_{i_j} \cap X_{m+1})\right) + (-1)^{m-1} \operatorname{card}\left(\bigcap_{j=1}^{m+1} X_j\right)
\end{aligned}$$

So subtracting the summations gives

$$\sum_{k=1}^{m} \sum_{1 \leq i_1 \ldots i_k \leq m} (-1)^{k-1} \operatorname{card}\bigg(\bigcap_{j=1}^{k} X_{i_j}\bigg) - \sum_{k=1}^{m} \sum_{1 \leq i_1 \ldots i_k \leq m} (-1)^{k-1} \operatorname{card}\bigg(\bigcap_{j=1}^{k} (X_{i_j} \cap X_{m+1})\bigg) + |X_{m+1}|$$

$$= \sum_{1 \leq i_1 \ldots i_k \leq m} \operatorname{card}(x_i) + \sum_{k=2}^{m} \sum_{1 \leq i_1 \ldots i_k \leq m} (-1)^{k-1} \operatorname{card}\bigg(\bigcap_{j=1}^{k} X_{i_j}\bigg) + \operatorname{card}(X_{m+1})$$

$$+ \sum_{k=2}^{m} \sum_{1 \leq i_1 \ldots i_k \leq m} (-1)^{k-1} \operatorname{card}\bigg(\bigcap_{j=1}^{k-1} (X_{i_j} \cap X_{m+1})\bigg) + (-1)^m \operatorname{card}\bigg(\bigcap_{j=1}^{m+1} X_j\bigg)$$

$$= \sum_{1 \leq i_1 \ldots i_k \leq m+1} \operatorname{card}(X_i) + \sum_{k=2}^{m} \sum_{1 \leq i_1 \ldots i_k \leq m} (-1)^{k-1} \Bigg[\operatorname{card}\bigg(\bigcap_{j=1}^{k} X_{i_j}\bigg)$$

$$+ \operatorname{card}\bigg(\bigg[\bigcap_{j=1}^{k-1} X_{i_j}\bigg] \cap X_{m+1}\bigg)\Bigg] + (-1)^m \operatorname{card}\bigg(\bigcap_{j=1}^{m+1} X_j\bigg)$$

and since the set of sequences of k terms bounded by $m+1$ (of form $1 \leq i_1 \ldots i_k \leq m+1$) is the set of sequences of k terms bounded by m (of form $1 \leq i_1 \ldots i_k \leq m$) unioned with the set of sequences of k terms with max element $m+1$ (of form $1 \leq i_1 \ldots i_k = m+1$), we have

$$\sum_{1 \leq i_1 \ldots i_k \leq m} (-1)^{k-1} \Bigg[\operatorname{card}\bigg(\bigcap_{j=1}^{k} X_{i_j}\bigg) + \operatorname{card}\bigg(\bigg[\bigcap_{j=1}^{k-1} X_{i_j}\bigg] \cap X_{m+1}\bigg)\Bigg] = \sum_{1 \leq i_1 \ldots i_k \leq m+1} \operatorname{card}\bigg(\bigcap_{j=1}^{k} X_{i_j}\bigg)$$

and therefore, substituting the above and observing that the independent terms are the first and last terms of the summation gives

$$\operatorname{card}\bigg(\bigcup_{i=1}^{m+1} X_i\bigg) = \sum_{1 \leq i_1 \ldots i_k \leq m+1} \operatorname{card}(X_i) + \sum_{k=2}^{m} \sum_{1 \leq i_1 \ldots i_k \leq m+1} (-1)^{k-1} \operatorname{card}\bigg(\bigcap_{j=1}^{k} X_{i_j}\bigg)$$

$$+ \ldots + (-1)^m \operatorname{card}\bigg(\bigcap_{j=1}^{m+1} X_j\bigg)$$

$$= \sum_{k=1}^{m+1} \sum_{1 \leq i_1 \ldots i_k \leq m+1} (-1)^{k-1} \operatorname{card}\bigg(\bigcap_{j=1}^{k} X_{i_j}\bigg)$$

Exercise 10.58 (Zorich 2.4.7)

On the closed interval $[0,1] \subset \mathbb{R}$, describe the sets of numbers $x \in [0,1]$ whose ternary representation $x = 0.\alpha_1 \alpha_2 \ldots, \alpha_i \in \{0,1,2\}$ has the property.

1. $\alpha_1 \neq 1$
2. $\alpha_1 \neq 1$ and $\alpha_2 \neq 1$
3. For all $i \in \mathbb{N}$, $\alpha_i \neq 1$ (the Cantor set)

Solution 10.58

Listed.

1. $[0, \frac{1}{3}) \cup [\frac{2}{3}, 1)$
2. $[0, \frac{1}{9}) \cup [\frac{2}{9}, \frac{3}{9}) \cup [\frac{6}{9}, \frac{7}{9}) \cup [\frac{8}{9}, 1)$
3. Made by recursively removing the middle third of every partitioned intervals.

Exercise 10.59 (Zorich 2.4.8)

Show that
1. the set of numbers $x \in [0,1]$ whose ternary representation does not contain 1 has the same cardinality as the set of all numbers whose binary representation has the form $0.\beta_1\beta_2\ldots$
2. the Cantor set has the same cardinality as the closed interval $[0,1]$

Solution 10.59

Listed.
1. We can define a bijection $0.\alpha_1\alpha_2\ldots \mapsto 0.\beta_1\beta_2\ldots$ as $\alpha_i = 0 \iff \beta_i = 0$ and $\alpha_i = 1 \iff \beta_i = 2$.
2. The map above defines a bijection between the Cantor set and the set of all infinite binary expansions in $[0,1]$, which is uncountable by Cantor's diagonalization theorem.

10.2 Euclidean Topology

Exercise 10.60 (Math 531 Spring 2025, PS5.1)

We know what it means for a metric space (X,d) to be compact. We say that it is sequentially compact if every sequence in (X,d) has a convergent subsequence. Prove that a metric space is compact if and only if it is sequentially compact.

Solution 10.60

We prove bidirectionally.
1. $(\rightarrow)$. Assume that X is compact and let (x_n) be a sequence in X. For any $\epsilon > 0$, let

$$\mathscr{C}_\epsilon \coloneqq \{B_\epsilon(x) \mid x \in X\} \tag{516}$$

   be an open cover of open balls. Then there exists a finite subcover $\mathcal{F}_\epsilon \subset \mathscr{C}_\epsilon$. There is a countable sequence (x_n), with each point in at least one open set in $\mathscr{C}$. By the pigeonhole principle, at least one open set must be hit infinitely many times, call this $B(x^*, \epsilon)$. Now consider for $\epsilon = \frac{1}{n}$ for $n \in \mathbb{N}$.
2. $(\leftarrow)$.

Exercise 10.61 (Math 531 Spring 2025, PS5.2)

Give an example of a sequence of real numbers x_n for which

$$|x_{n+1} - x_n| \rightarrow 0 \tag{517}$$

as $n \rightarrow \infty$, but x_n is not convergent.

Solution 10.61

Consider the sequence

$$x_n = \sum_{i=1}^{n} \frac{1}{i} \tag{518}$$

It is the case that $x_{n+1} - x_n = 1/(n+1)$ which tends to 0, but this is a harmonic series which is not convergent.

Exercise 10.62 (Math 531 Spring 2025, PS5.3)

Let $\{x_n\}_{n=0}^{\infty}$ be a sequence of real numbers. Assume that

$$|x_{n+1} - x_n| \leq c|x_n - x_{n-1}| \tag{519}$$

for all $n \geq 1$, for some fixed $c < 1$. Prove that x_n is convergent. Hint: you may want to use the formula that you proved in a previous homework for $1 + c + c^2 + \cdots + c^N$.

Solution 10.62

Exercise 10.63 (Math 531 Spring 2025, PS5.4)

Let x_n be a sequence of rational numbers defined recursively by:

$$x_0 = 1 \tag{520}$$

$$x_{n+1} = \frac{1}{x_n + 2} \tag{521}$$

when $n \geq 0$. The first few terms of this sequence are $1, \frac{1}{3}, \frac{3}{7}, \ldots$ Prove that the sequence is convergent and find its limit.

Solution 10.63

Exercise 10.64 (Math 531 Spring 2025, PS5.5)

As we know, every bounded sequence of real numbers has a convergence subsequence.

1. Let's say we have two sequences a_n and b_n that are bounded. Find a single sequence of indices $\{n_k\}$ so that *both* a_{n_k} and b_{n_k} are convergent. This is called a common convergent subsequence for a_n and b_n.
2. Show that for any finite number of bounded sequences of real numbers, we can find a common convergent subsequence.
3. Now suppose have a sequence of bounded sequences. Find a common convergent subsequence. What does this remind you of?

Solution 10.64

Exercise 10.65 (Math 531 Spring 2025, PS5.6)

Given a sequence of real numbers x_n, we can define the sequence of its means by:

$$x_1, \frac{x_1 + x_2}{2}, \frac{x_1 + x_2 + x_3}{3}, \ldots \tag{522}$$

Call the sequence of means y_n. Prove that if $x_n \to x$, then $y_n \to x$. Discuss the examples $x_n = (-1)^n$, $x_n = n$, $x_n = (-1)^n n$, and $x_n = (-1)^n \sqrt{n}$. The fourth example shows that it is possible to average out chaotic behavior (so long as it isn't focused in one direction). The third example shows that this is impossible if the system becomes too chaotic.

Solution 10.65

Exercise 10.66 (Math 531 Spring 2025, PS4.1)

Let (X, d) be a metric space. Assume that K is compact and F is closed in (X, d). Assume $K \cap F = \emptyset$. Prove that

$$\inf_{x \in F, y \in K} d(x, y) > 0. \tag{523}$$

Show by an example that this number could be zero if K is only assumed to be closed (rather than compact).

Solution 10.66

$K \cap F = \emptyset \implies K \subset F^c$ with F^c open. This means that for every $x \in K \subset F^c$, there exists a $r_x > 0$ s.t. $B(x, r_x) \subset F^c \iff B(x, r_x) \cap F = \emptyset$.
Now we take the covering $\{B(x, \frac{r_x}{2}) \mid x \in K\}$, and since K is compact there must be a finite subcovering, which we denote $C = \{B(x_i, \frac{r_i}{2}) \mid i = 1, \ldots n\}$. Denote $r^* = \min\{r_i\}$, which is positive since we take the minimum of a finite number of positive elements.
Now for any $x \in K$ and $y \in F$, x must be in some $B(x_i, \frac{r_i}{2}) \iff d(x, x_i) < \frac{r_i}{2} \iff -d(x, x_i) > -\frac{r_i}{2}$. With the same i, since $B(x_i, r_i)$ is disjoint from F, we have $d(x_i, y) \geq r_i$. Therefore, by the triangle inequality,

$$d(x, y) \geq d(x_i, y) - d(x_i, x) = r_i - \frac{r_i}{2} = \frac{r_i}{2} \geq \frac{r^*}{2} \tag{524}$$

and thus we have found a nontrivial lower bound.

Exercise 10.67 (Math 531 Spring 2025, PS4.2)

Consider $\mathbb{R}$ with the usual metric. Find an open cover of $\mathbb{Q}$ that does not cover $\mathbb{R}$.

Solution 10.67

$\mathbb{Q} = (-\infty, \sqrt{2}) \cup (\sqrt{2}, +\infty)$.

Exercise 10.68 (Math 531 Spring 2025, PS4.3)

$\mathbb{R}$ is not compact with the usual metric since it is not bounded. Let us, however, define the following metric on $\mathbb{R}$:

$$d_*(x, y) = \frac{|x - y|}{(1 + |x|)(1 + |y|)}. \tag{525}$$

Verify that $(\mathbb{R}, d_*)$ is a metric space. Prove that all subsets of $(\mathbb{R}, d_*)$ are bounded. Show that $\mathbb{R}$ still isn't compact with this metric. What is the problem?

Solution 10.68

We first verify metric.

1. Since $|x - y| \geq 0, 1 + |x| \geq 1, 1 + |y| \geq 1$, $d_*(x, y) \geq 0$. We also see that

$$d_*(x, y) = 0 \iff |x - y| = 0 \iff x = y \tag{526}$$

2. It is symmetric since

$$d_*(x,y) = \frac{|x-y|}{(1+|x|)(1+|y|)} = \frac{|y-x|}{(1+|y|)(1+|x|)} = d_*(y,x) \tag{527}$$

3. It satisfies triangle inequality since

$$\begin{aligned}
d_*(x,y) + d_*(y,z) &= \frac{|x-y|}{(1+|x|)(1+|y|)} + \frac{|y-z|}{(1+|y|)(1+|z|)} &(528)\\
&= \frac{|x-y|(1+|z|) + |y-z|(1+|x|)}{(1+|x|)(1+|y|)(1+|z|)} &(529)\\
&= \frac{|x-y| + |x-y|\cdot|z| + |y-z| + |y-z|\cdot|x|}{(1+|x|)(1+|y|)(1+|z|)} &(530)\\
&\geq \frac{|x-z| + (|x|-|y|)\cdot|z| + (|y|-|z|)\cdot|x|}{(1+|x|)(1+|y|)(1+|z|)} &(531)\\
&= \frac{|x-z| + |x||y| - |y||z|}{(1+|x|)(1+|y|)(1+|z|)} &(532)\\
&= \frac{(1+|y|)|x-z|}{(1+|x|)(1+|y|)(1+|z|)} &(533)\\
&= \frac{|x-z|}{(1+|x|)(1+|z|)} &(534)\\
&= d_*(x,z) &(535)
\end{aligned}$$

where the inequality comes from $|x-y| + |y-z| \geq |x-z|$, $|x-y| \geq ||x|-|y|| \geq |x|-|y|$, and $|y-z| \geq ||y|-|z|| \geq |y|-|z|$.

This is bounded since for any $x, y \in \mathbb{R}$, we can show that the numerator is bounded by the denominator. Since both are positive from (1), it suffices to prove $|x-y|^2 \leq (1+|x|)^2(1+|y|)^2$.

$$\begin{aligned}
(1+|x|)^2(1+|y|)^2 &= (1+2|x|+|x|^2)(1+2|y|+|y|^2) &(536)\\
&= 1 + 2|x| + 2|y| + 4|x||y| + |x|^2 + |y|^2 + 2|x||y|^2 + 2|y||x|^2 + |x|^2|y|^2 &(537)\\
&\geq |x|^2 + |y|^2 + 2|x||y| &(538)\\
&\geq |x|^2 + |y|^2 - 2xy &(539)\\
&= |x-y|^2 &(540)
\end{aligned}$$

where the first inequality holds since all the terms in the expansion are nonnegative and the second holds since $|x||y| \geq xy$. Therefore, $d_*(x,y) \leq 1$. $\mathbb{R}$ is still not compact since we can construct the set of open balls $B_r(0)$ around 0 w.r.t. d_*. Consider the cover

$$\mathscr{C} = \{B_{1-\frac{1}{n}}(0)\}_{n \in \mathbb{N}, n \geq 2} \tag{541}$$

Now assume that there is a finite subcover. Then there must be a maximum index $N \in \mathbb{N}$ in this cover. I claim that this does not cover $\mathbb{R}$. Consider the element $y = N-1 \in \mathbb{R}$. The distance is

$$d_*(x,y) = \frac{|0-(N-1)|}{(1+|0|)(1+|N-1|)} = \frac{N-1}{N} = 1 - \frac{1}{N} \tag{542}$$

and so $y \notin \mathscr{C}$. Hence $\mathscr{C}$ is not a cover of $\mathbb{R}$.

Exercise 10.69 (Math 531 Spring 2025, PS4.4)

Consider the set $X = \mathbb{R} \cup \{Gandalf\}$. Define a metric d_* on X by:

$$d_*(x, y) = \frac{|x - y|}{(1 + |x|)(1 + |y|)} \tag{543}$$

for $x, y \in \mathbb{R}$, while

$$d_*(Gandalf, x) = \frac{1}{1 + |x|}, \tag{544}$$

for all $x \in \mathbb{R}$. Verify that d_* is a metric on X (you don't have to do much for this, since you already did part of it in the previous problem). Prove that (X, d_*) is a compact metric space.

Solution 10.69

Exercise 10.70 (Math 531 Spring 2025, PS4.5)

Let X be any set and endow it with the metric $d(x, y) = 1$ if $x \neq y$ and $d(x, x) = 0$. Check that this is a metric. Find all compact sets in (X, d).

Solution 10.70

It trivially satisfies nonnegativity since it's either 0 or 1, and $d(x, x) = 0$. It is symmetric as well. As for triangle inequality this is trivial. All compact sets are finite sets.

Exercise 10.71 (Math 531 Spring 2025, PS3.1)

Determine for each of the following sets, whether or not it is countable. Justify your answers

1. The set of all functions $f : \{0, 1\} \to \mathbb{N}$.
2. The set B_n of all functions $f : \{1, ..., n\} \to \mathbb{N}$
3. The set $C = \cup_{n \in \mathbb{N}} B_n$
4. The set of all functions $f : \mathbb{N} \to \{0, 1\}$.
5. The set of all functions $f : \mathbb{N} \to \{0, 1\}$ that are "eventually zero" (We say that f is eventually zero if there exists some $N \geq 1$ so that $f(n) = 0$ for all $n \geq N$.)
6. G the set of all functions $f : \mathbb{N} \to \mathbb{N}$ that are eventually constant.

Solution 10.71

Listed.

1. Countable since bijective to $\mathbb{N} \times \mathbb{N}$. We define the bijection as
$$(a_0, a_1) \in \mathbb{N} \times \mathbb{N} \mapsto f(i) = a_i \tag{545}$$
2. Countable since bijective to $\mathbb{N}^n$. We define the bijection as
$$(a_1, \ldots, a_n) \in \mathbb{N}^n \mapsto f(i) = a_i \tag{546}$$
3. Countable since we proved that B_n is countable, and a countable union of countable sets are countable from Rudin Theorem 2.12.
4. Uncountable since we can create a bijection from the set of all sequences (a_i) of 0 or 1, which from Rudin Theorem 2.14 is uncountable.
$$(a_i)_{i \in \mathbb{N}} \mapsto f(i) = a_i \tag{547}$$

5. Countable. Call this set B, and call the set of functions f that have their final 1 at index k to be A_k. Then,
$$B = \cup_{k=1}^{\infty} A_k \tag{548}$$
where $A_0 = A_1 = 1$, and $|A_k| = 2^{k-1}$ for $k \geq 2$. Since B is the countable union of at most countable sets, B must be countable.
6. Countable. Call this set B. Let A_k be the set of functions that are eventually constant to value k. Let A_{ki} be the set of functions that are always k starting from index i (where i is the smallest element). Since everything is determined to be k at i and beyond, A_{ki} can be divided up into the first $i-1$ elements of any natural number, followed by a sequence of k's. Therefore $|A_{ki}| \approx \mathbb{N}^{i-1}$, where $\approx$ means equipotent, and so A_{ki} is countable. Therefore since countable unions of countable sets are countable,
$$A_k = \bigcup_{i=1}^{\infty} A_{ki} \text{ is countable } \implies B = \sum_{k=1}^{\infty} A_k \text{ is countable} \tag{549}$$

Exercise 10.72 (Math 531 Spring 2025, PS3.2)

Tell if the following subsets $A \subset \mathbb{R}$ (with the usual metric $d(x,y) = |x-y|$) are open or closed. Also, find (i) the limit points of A, (ii) the interior of A, (iii) $\bar{A}$.
1. $A = \mathbb{Q}$
2. $A = (0, 1]$
3. $A = \{1, \frac{1}{2}, \frac{1}{4}, ...\}$
4. $A = \{0, 1, \frac{1}{2}, \frac{1}{4}, ...\}$
5. $A = \mathbb{Z}$

Solution 10.72

Listed. We denote A' as the limit points of A and the interior as A^o.
1. Not open nor closed. $A' = \mathbb{R}$, $A^o = \emptyset$. $\bar{A} = \mathbb{R}$.
2. Not open nor closed. $A' = [0, 1]$, $A^o = (0, 1)$. $\bar{A} = [0, 1]$.
3. Not open nor closed. $A' = \{0\}$, $A^o = \emptyset$. $\bar{A} = \{0\} \cup A$.
4. Closed. $A' = \{0\}$, $A^o = \emptyset$. $\bar{A} = A$.
5. Closed. $A' = \emptyset$. $A^o = \emptyset$. $\bar{A} = A$.

Exercise 10.73 (Math 531 Spring 2025, PS3.3)

Prove the following statements subsets A, B of a general metric space (X, d).
- $\overline{A \cup B} = \bar{A} \cup \bar{B}$.
- Show by example that $\overline{A \cap B} \neq \bar{A} \cap \bar{B}$.

Solution 10.73

For the first part, we show bidirectionally.
1. $\overline{A \cup B} \subset \overline{A} \cup \overline{B}$. Let $x \in \overline{A \cup B}$. If $x \in A \cup B$, then it must be the case that either $x \in A \subset (A \cup A') = \overline{A}$ or $x \in B \subset (B \cup B') = \overline{B}$, which means $x \in \overline{A} \cup \overline{B}$. Now assume not. Then $x \in (A \cup B)'$. Therefore, for any $r > 0$, we know that $B(x, r) \cap (A \cup B) \neq \emptyset$. Now let us take a sequence $(r_n = \frac{1}{n})_{n \in \mathbb{N}}$, and for each r_n we have some element $x_n \in (A \cup B)$. Given that we have a countably infinite sequence of x_n, each which may be in A or B, by the pigeonhole principle either A or B must be hit infinitely many times. If $x_n \in A$ infinitely many times, then $x \in \overline{A}$, and analogous for B.
2. $\overline{A} \cup \overline{B} \subset \overline{A \cup B}$. WLOG let $x \in \overline{A}$. If $x \in A$, then $x \in (A \cup B) \subset \overline{A \cup B}$. If $x \notin A$, then $x \in A'$.

Therefore for every $r > 0$, $B(x,r) \cap A \neq \emptyset$. But this means

$$\emptyset \neq (B(x,r) \cap A) \cup (B(x,r) \cap B) = B(x,r) \cap (A \cup B) \implies x \in (A \cup B)' \subset \overline{A \cup B} \tag{550}$$

For a counterexample, consider the sequences

$$A = (x_n) = \frac{1}{n} \qquad B = (y_n) = -\frac{1}{n} \tag{551}$$

for $n \in \mathbb{N}$. $\overline{A} = A \cup \{0\}, \overline{B} = B \cup \{0\}$, and so $\overline{A} \cap \overline{B} = \{0\}$. However, $A \cap B = \emptyset \implies \overline{A \cap B} = \emptyset$.

Exercise 10.74 (Math 531 Spring 2025, PS3.4)

Consider the set of rationals in canonical form (such that numerator and denominator are relatively prime) with potential distance:

$$d_1(\frac{p_1}{q_1}, \frac{p_2}{q_2}) = |q_1 - q_2|. \tag{552}$$

Is this a metric? Prove that the following defines a metric

$$d_2(\frac{p_1}{q_1}, \frac{p_2}{q_2}) = |p_1 - p_2| + |q_1 - q_2|. \tag{553}$$

Solution 10.74

This is not a metric since

$$d_1\left(\frac{2}{1}, \frac{3}{1}\right) = 1 - 1 = 0 \tag{554}$$

when $2/1 \neq 3/1$. For d_2, we show that it satisfies the three properties.

1. *Nonnegativity.* Since it is the sum of 2 absolute values which are norms and therefore nonnegative, it must be nonnegative by ordered field properties. We see that

$$\begin{aligned} \frac{p_1}{q_1} = \frac{p_2}{q_2} &\iff p_1 = p_2 \text{ and } q_1 = q_2 && (555)\\ &\iff |p_1 - p_2| = |q_1 - q_2| = 0 && (556)\\ &\iff |p_1 - p_2| + |q_1 - q_2| = 0 && (557)\end{aligned}$$

2. For symmetricity, note that

$$d_2\left(\frac{p_1}{q_1}, \frac{p_2}{q_2}\right) = |p_1 - p_2| + |q_1 - q_2| = |p_2 - p_1| + |q_2 - q_1| = d_2\left(\frac{p_1}{q_1}, \frac{p_2}{q_2}\right) \tag{558}$$

3. For triangle inequality, we see that for any $p_1/q_1, p_2/q_2, p_3/q_3$,

$$\begin{aligned} d_2\left(\frac{p_1}{q_1}, \frac{p_3}{q_3}\right) &= |p_1 - p_3| + |q_1 - q_3| && (559)\\ &= |(p_1 - p_2) + (p_2 - p_3)| + |(q_1 - q_2) + (q_2 - q_3)| && (560)\\ &\leq |p_1 - p_2| + |p_2 - p_3| + |q_1 - q_2| + |q_2 - q_3| && \text{(subadditivity of norm)}\\ &= d_2\left(\frac{p_1}{q_1}, \frac{p_2}{q_2}\right) + d_2\left(\frac{p_2}{q_2}, \frac{p_3}{q_3}\right) && (561)\end{aligned}$$

Exercise 10.75 (Math 531 Spring 2025, PS3.5)

Let $M = \{x_1, ..., x_3\}$ be a set with three points. Describe the set of all metrics on M. What if M has four points?

Solution 10.75

If M has 3 points call them x_1, x_2, x_3, then the metric is completely defined by the three values

$$d(x_1, x_2) = d(x_2, x_1) \tag{562}$$
$$d(x_2, x_3) = d(x_3, x_2) \tag{563}$$
$$d(x_3, x_1) = d(x_3, x_1) \tag{564}$$

where $d(x, x) = 0$. We must make sure that the triangle inequality satisfies for these 3 numbers. Therefore we can think of this as the set of all triangles in $\mathbb{R}^2$ (that are equivalent under translation and rotation, but not permutation of points).
Similarly for 4 points, we can visualize the metrics as the set of all tetarhedra in $\mathbb{R}^3$ (since each face is a triangle, and therefore for any three points the triangle inequality is guaranteed to be satisfied), equivalent under translation and rotation, but not permutation of the 4 points.

Exercise 10.76 (Math 531 Spring 2025, PS3.6)

Let P be a polynomial of degree $n \geq 1$. Prove that if $P(0) = 0$, then $P(x) = xQ(x)$, for some polynomial Q of degree $n-1$. Deduce that if $P(a) = 0$, then we can write $P(x) = (x-a)Q(x)$ for some Q of degree $n-1$.

Solution 10.76

A nth degree polynomial will have the form

$$p(x) = \sum_{i=0}^{n} c_i x^i \tag{565}$$

Since $p(0) = c_0 = 0 \implies c_0 = 0$. This means that

$$p(x) = \sum_{i=1}^{n} c_i x^i = x \sum_{i=0}^{n-1} c_{i+1} x^i \text{ where } Q(x) = \sum_{i=0}^{n-1} c_{i+1} x^i \tag{566}$$

If $p(a) = 0$, we can construct $f(x) = p(x + a)$, where f is a polynomial since the expansion does not increase its degree. Since $f(x) = p(a) = 0$, by above f can be factorized $f(x) = xg(x)$ for some $(n-1)$th degree polynomial g, and by substitution this means that $p(x) = f(x-a) = (x-a)g(x-a)$.

Exercise 10.77 (Math 531 Spring 2025, PS3.7)

Consider all polynomials $P : \mathbb{R} \to \mathbb{R}$ of degree less than or equal to n. Call this set $\mathcal{P}_n$. Let's define potential distances on $\mathcal{P}_n$.

$$d_1(p, q) = |p(0) - q(0)|. \tag{567}$$

Show this defines a distance on $\mathcal{P}_0$ but not on $\mathcal{P}_n$ for $n \geq 1$. Now consider

$$d_N(p, q) = \sum_{j=0}^{N} |p(j) - q(j)| \tag{568}$$

Show that this defines a distance on $\mathcal{P}_n$, for every $n \leq N$. What does the solution say about polynomials of degree N?

Solution 10.77

If $n = 0$, $\mathcal{P}_n$ is a set of constant functions P, where each constant function P is determined completely by its value at any point, e.g. 0. We check the properties.

1. $d_1(p, q) \geq 0$ since we take the norm at the end. We can see that

$$\begin{aligned} d_1(p,q) = 0 &\iff |p(0) - q(0)| && (569) \\ &\iff p(0) = q(0) && (570) \\ &\iff p = q && (571) \end{aligned}$$

2. It is clearly symmetric.

$$d_1(p,q) = |p(0) - q(0)| = |q(0) - p(0)| = d_1(q,p) \tag{572}$$

3. It satisfies the triangle inequality by subadditivity of the norm.

$$\begin{aligned} d_1(p,r) &= |p(0) - r(0)| && (573) \\ &= |(p(0) - q(0)) + (q(0) - r(0))| && (574) \\ &\leq |p(0) - q(0)| + |q(0) - r(0)| && (575) \\ &= d_1(p,q) + d_1(q,r) && (576) \end{aligned}$$

It doesn't satisfy for P_n because consider $p(x) = x$ and $q(x) = x^2$. They are not the same function but $d_1(p,q) = |p(0) - q(0)| = 0$. For d_N defined on $\mathcal{P}_n$ for $n \leq N$, we verify the properties.

1. This is the sum of norms, so it must be nonnegative. Now we see that if $p = q$, then $p(x) = q(x) \implies |p(x) - q(x)| = 0 \implies d_N(p,q) = 0$. For the other way around, suppose $d_N(p,q) = 0$. Then from problem 3.8, we are solving the linear equation $0 = Vb - Vc$, where b, c are the vectors representing the coefficients of p, q, and V is the Vandermonde matrix with $a_i = i$. By linearity, this is equivalent to solving $0 = V(b - c)$, and since we showed that V is invertible (since a_i's are distinct), V has a trivial kernel and therefore $b - c = 0 \iff b = c \implies p = q$.
2. Symmetricity is trivial.

$$d_N(p,q) = \sum_{j=0}^N |p(j) - q(j)| = \sum_{j=0}^N |q(j) - p(j)| = d_N(q,p) \tag{577}$$

3. For triangle inequality,

$$\begin{aligned} d_N(p,r) &= \sum_{j=0}^N |p(j) - r(j)| && (578) \\ &= \sum_{j=0}^N |(p(j) - q(j)) + (q(j) - r(j))| && (579) \\ &\leq \sum_{j=0}^N |p(j) - q(j)| + |q(j) - r(j)| && (580) \\ &= \sum_{j=0}^N |p(j) - q(j)| + \sum_{j=0}^N |q(j) - r(j)| && (581) \\ &= d_N(p,q) + d_N(q,r) && (582) \end{aligned}$$

This shows that we need to "sample" more points from higher-degree polynomials to get the metric as they are higher-dimensional.

Exercise 10.78 (Math 531 Spring 2025, PS3.8)

Given distinct numbers $a_0, ..., a_N$ and numbers $b_0, ..., b_N$, prove that there exists a polynomial P of degree N with the property that

$$P(a_i) = b_i, \tag{583}$$

for $0 \leq i \leq N$. The most direct way to solve this problem, in my view, is to write the system equations you are trying to solve as a linear system for the coefficients of P. This will give you some matrix M that depends on the numbers $a_0, ..., a_N$. The key is to show that the determinant of this matrix is non-zero. It turns out that the determinant of this matrix is equal to

$$\prod_{0 \leq i < j \leq N} (a_i - a_j), \tag{584}$$

up to a potential $-$ sign depending on how you defined M. Prove this and deduce the result.

Solution 10.78

We can write the system of equations using the Vandermonde matrix $V \in \mathbb{R}^{(N+1)\times(N+1)}$ and c is the vector of coefficients of P.

$$b = Vc \iff \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_N \end{bmatrix} = \begin{bmatrix} 1 & a_0 & a_0^2 & \ldots & a_0^N \\ 1 & a_1 & a_1^2 & \ldots & a_1^N \\ \vdots & \vdots & \ddots & \vdots & \\ 1 & a_N & a_N^2 & \ldots & a_N^N \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_N \end{bmatrix} \tag{585}$$

To calculate the determinant of V, we prove using induction. Clearly for $N = 1$ we have

$$\det \begin{pmatrix} 1 & a_0 \\ 1 & a_1 \end{pmatrix} = a_1 - a_0 \tag{586}$$

Now assume that this formula holds for some $N - 1 \in \mathbb{N}$. Then for N, we can take V and subtract a_0 times the ith column from the $(i+1)$st column. This gives us

$$V = \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 \\ 1 & a_1 - a_0 & a_1^2 - a_0 a_1 & \ldots & a_1^N - a_0 a_1^{N-1} \\ 1 & a_2 - a_0 & a_2^2 - a_0 a_2 & \ldots & a_2^N - a_0 a_2^{N-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & a_N - a_0 & a_N^2 - a_0 a_N & \ldots & a_N^N - a_0 a_N^{N-1} \end{bmatrix} \tag{587}$$

When calculating the determinant, we can perform the cofactor expansion by the first row, and then for each ith row factor out $(a_i - a_0)$ to get

$$\det V = \prod_{j=1}^N (a_j - a_0) \det \begin{bmatrix} 1 & a_1 & \ldots & a_1^{N-1} \\ 1 & a_2 & \ldots & a_2^{N-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & a_N & \ldots & a_N^{N-1} \end{bmatrix} \tag{588}$$

which is the $(N-1)\times(N-1)$ Vandermonde matrix. Therefore, we can apply our inductive hypothesis to get

$$\det V = \prod_{j=1}^N (a_j - a_0) \prod_{1 \leq i < j \leq N} (a_j - a_i) = \prod_{0 \leq i \leq j \leq N} (a_j - a_i) \tag{589}$$

Note that this has a 0 determinant iff $a_i = a_j$ for some $i \neq j$. Therefore sicne a_i's are distinct, it must be nonzero. Therefore, this matrix is nonsingular, i.e. invertible, and we can solve the matrix equation to get

$$c = V^{-1}b \tag{590}$$

which from linear algebra is guaranteed to exist and is unique.

Exercise 10.79 (Rudin 2.1)

Prove that the empty set is a subset of every set.

Solution 10.79

It must suffice that if $x \in \emptyset$, then $x \in A$ for any arbitrary set A. This is vacuously true, since the initial condition is never met.

Exercise 10.80 ()

Show that the empty function $f : \emptyset \to X$, where X is an arbitrary set, is always injective. If $X = \emptyset$, then f is bijective.

Solution 10.80

Given distinct $x, y \in \emptyset$, $f(x) \neq f(y)$ is vacuously true, but if $X \neq \emptyset$, then there exists a $w \in X$ with no preimage. If $X = \emptyset$, then the statement for all $w \in X$, there exists an $x \in \emptyset$ s.t. $f(x) = w$ is vacuously true.

Exercise 10.81 (Rudin 2.2)

A complex number z is said to be algebraic if there are integers $a_0, a_1, \ldots, a_n$, not all zero, such that

$$a_0 z^n + a_1 z^{n-1} + \ldots + a_{n-1} z + a_n = 0. \tag{591}$$

Prove that the set of all algebraic complex numbers is countable. Hint: For every positive integer N there are only finitely many equations with

$$n + |a_0| + |a_1| + \ldots + |a_n| = N \tag{592}$$

Solution 10.81

Consider all polynomials s.t. $n + \sum_{i=0}^{n} |a_i| = N$. There is only a finite number of them, and each polynomial has at most n distinct complex roots. So this set is finite, an unioning over all $N \in \mathbb{N}$ gives an at most countable set of roots.

Exercise 10.82 (Rudin 2.3)

Prove there exists real numbers which are not algebraic.

Solution 10.82

From the previous exercise, if there were no no real numbers which are not algebraic, then every real number is algebraic. This contradicts the fact that the set of all complex numbers is countable.

Exercise 10.83 (Rudin 2.4)

Is the set of all irrational real numbers countable?

Solution 10.83

No. Assume that it is countable. We have $\mathbb{Q}$ countable. Then, by assumption, we must have $\mathbb{R} = \mathbb{Q} \cup \mathbb{Q}^c$ be the union of countable sets, which must be countable, contradicting the fact that it is uncountable.

Exercise 10.84 (Rudin 2.5)

Construct a bounded set of real numbers which exactly 3 limit points.

Solution 10.84

We can construct the union of 3 sequences that converge onto the limit points $0, 1, 2$.

$$\Big\{\frac{1}{n}\Big\}_{n \in \mathbb{N}} \cup \Big\{\frac{1}{n} + 1\Big\}_{n \in \mathbb{N}} \cup \Big\{\frac{1}{n} + 2\Big\}_{n \in \mathbb{N}} \tag{593}$$

Exercise 10.85 ()

Prove that the union of the limit points of sets is equal to the limit points of the union of the sets.

$$\bigcup_{k=1}^{m} A_k' = \left(\bigcup_{k=1}^{m} A_k\right)' \tag{594}$$

Exercise 10.86 (Rudin 2.6)

Let E' be the set of all limit points of a set E. Prove that E' is closed. Prove that E and $\overline{E}$ have the same limit points. (Recall that $\overline{E} = E \cup E'$). Do E and E' always have the same limit points?

Solution 10.85

Listed.

1. Let x be a limit point of E'. Then, for every $\epsilon > 0$, $U = B_\epsilon(x) \cap E' \neq \emptyset$. Take a $y \in U$. Since $y \in B_\epsilon(x)$, which is open, we can construct an open ball $B_\delta(y) \subset B_\epsilon(x)$. Since $y \in E'$, $B_\delta(y)$ must contain elements of E, which means that $B_\epsilon(x)$ must also contain elements of E, and so x is a limit point of $E \implies x \in E'$ and E' is closed.
2. To prove that $E' \subset \overline{E}'$, we know that if $x \in E'$, then for every $\epsilon > 0$, there exists a $B_\epsilon^\circ(x)$ that has a nontrivial intersection with E which means that it has a nontrivial intersection with $E \cup E'$. To prove that $\overline{E}' \subset E'$, we know that if $y \in \overline{E}'$, then for every $\delta > 0$ there exists a $B_\delta(x)$ that has a nontrivial intersection with $\overline{E}$. If $B_\delta(x)$ intersects E then we are done. If $B_\delta(x)$ intersects E', then we can find a $y \in E' \cap B_\delta(x)$. Since $B_\delta(x)$ is open, we can construct $B_\varepsilon(y) \subset B_\delta(x)$ and since $y \in E'$, we know that $B_\varepsilon(y)$ contains an element of E, which means that $B_\delta(x)$ contains an element of E. Therefore, $E' = \overline{E}'$.
3. No. Consider the set $E = \{1/n\}_{n \in \mathbb{N}}$. $E' = \{0\}$, but $E'' = \emptyset$.

Exercise 10.87 (Rudin 2.7)

Let $A_1, A_2, \ldots$ be subsets of a metric space.

1. If $B_n = \cup_{i=1}^n A_i$, prove that $\bar{B}_n = \cup_{i=1}^n \bar{A}_i$ for $n = 1, 2, 3, \ldots$
2. If $B = \cup_{i=1}^\infty A_i$, prove that $\bar{B} \supset \cup_{i=1}^\infty \bar{A}_i$.

Solution 10.86

Listed.

1. We will prove that $\overline{B_n} \subseteq \cup_{i=1}^n \overline{A_i}$ and $\cup_{i=1}^n \overline{A_i} \subseteq \overline{B_n}$. If $x \in B_n$, then $x \in \cup_{i=1}^n A_i$. Therefore, assume that $x \in B_n'$. Then for every $\epsilon > 0$, there exists a $B_\epsilon^\circ(x)$ s.t.
$$B_\epsilon^\circ(x) \cap B_n \neq \emptyset \implies B_\epsilon^\circ(x) \cap \left(\bigcup_{i=1}^n A_n \right) \neq \emptyset$$
This means that there exists some $i = i(\epsilon)$, a function of ϵ, s.t. $B_\epsilon^\circ(x) \cap A_i \neq \emptyset$. However, this i may change if we unfix ϵ. We have so far proved that just for one $\epsilon > 0$ there exists an i. Now if we take a sequence of $\epsilon = 1, \frac{1}{2}, \frac{1}{3}, \ldots$, we have a sequence of $i(\epsilon)$'s living in $\{1, \ldots, n\}$. By the pigeonhole principle, there must be at least one i that is hit infinitely many times, and so we can choose this i, that works for all $\epsilon > 0 \implies x \in A_i' \subseteq \cup_{i=1}^n \overline{A_i}$. If $x \in \cup_{i=1}^n \overline{A_i}$, then there exists an $\overline{A_i}$ s.t. $x \in \overline{A_i}$. If $x \in A_i$, then we are done. If $x \in A_i'$, then for every $\epsilon > 0$, there exists a $B_\epsilon^\circ(x)$ s.t.
$$B_\epsilon^\circ(x) \cap A_i \neq \emptyset \implies B_\epsilon^\circ(x) \cap \left(\bigcup_{i=1}^n A_i \right) \neq \emptyset$$
and so $x \in B_n' \subset \overline{B_n}$.
2. $x \in \cup_{i=1}^\infty \overline{A_i} \implies x \in \overline{A_i}$ for some i. If $x \in A_i$, then $x \in B$ and we are done. If $x \in A_i'$, then for every $\epsilon > 0$ there exists $B_\epsilon(x)$ s.t.
$$B_\epsilon^\circ(x) \cap A_i \neq \emptyset \implies B_\epsilon^\circ(x) \cap \left(\bigcup_{i=1}^\infty A_i \right) \neq \emptyset$$
and so $B_\epsilon^\circ(x) \cap B \neq \emptyset \implies x \in B' \subset \overline{B}$.

Exercise 10.88 (Rudin 2.8)

Is every point of every open set $E \subset \mathbb{R}^2$ a limit point of E? Answer the same question for closed sets in $\mathbb{R}^2$.

Solution 10.87

Yes for open. Given any $x \in U$ open, there always exists an $\epsilon > 0$ s.t.
$$B_\epsilon^\circ(x) \subset B_\epsilon(x) \subset U \tag{595}$$
and so $B_\epsilon^\circ(x)$ has a nontrivial intersection with U. If U is closed, then no. Note that for closed U, we have that every limit point is in U, but not every point in U is a limit point. Consider the isolated point $U = \{x\}$. x is not a limit point of U.

Exercise 10.89 (Rudin 2.9)

Let E° denote the set of all interior points of E in X. Prove the following:
(a) E° is always open.
(b) E is open if and only if $E^\circ = E$.
(c) If $G \subseteq E$ and G is open, then $G \subset E^\circ$.
(d) Prove that the complement of E° is the closure of the complement of E.
(e) Do E and $\bar{E}$ always have the same interiors?
(f) Do E and E° always have the same closures?

Solution 10.88

Listed.

1. We assume that E° is not open (this does not mean that E° is necessarily closed!). That is, there exists an $x \in E^\circ$ s.t. we can't construct an open ball $B_\epsilon(x) \subseteq E^\circ$. Since $x \in E^\circ \subset E$, by definition of an interior point we can construct a $B_\epsilon(x) \subset E$. But from our assumption $B_\epsilon(x) \not\subset E^\circ$. We choose a $y \in B_\epsilon(x) \setminus E^\circ$. Since $B_\epsilon(x)$ is open, there exists a $\delta > 0$ s.t.
$$B_\delta(y) \subset B_\epsilon(x) \subset E$$
But the fact that we can construct an open ball around y means that $y \in E^\circ$, leading to a contradiction.
2. If E is open, then by definition $E \subset E^\circ$. Now $E^\circ \subset E$ holds for all sets since E° must be composed of points from E. If $E = E^\circ$, then for every $x \in E$, $x \in E^\circ$, so by definition there exists an $\epsilon > 0$ s.t. $B_\epsilon(x) \subset E$, which means that E is open.
3. Let $x \in G$ open. Then there exists an $\epsilon > 0$ s.t. $B_\epsilon(x) \subset G$, and so $B_\epsilon(x) \subset E$. Since we can always construct an open ball around x contained within E, $x \in E^\circ$ and $G \subset E^\circ$.
4. ($(E^\circ)^c \subset \overline{E^c}$) If $x \in (E^\circ)^c$, then there exists no $\epsilon > 0$ s.t. $B_\epsilon(x) \subset E$. Then, for any $\epsilon > 0$, $B_\epsilon(x) \not\subset E \implies B_\epsilon(x) \cap E^c \neq \emptyset \implies x \in E^c \subset \overline{E^c}$. ($\overline{E^c} \subset (E^\circ)^c$)) If $x \in \overline{E^c}$, then $x \in E^c$ or $x \in E^{c\prime}$. If $x \in E^c$, note $E^\circ \subset E \implies (E^\circ)^c \supset E^c \implies x \in (E^\circ)^c$. If $x \in E^{c\prime}$, then for all $\epsilon > 0$ $B_\epsilon(x) \cap E^c \neq \emptyset \implies B_\epsilon(x) \not\subset E \implies x \in E^\circ$.
5. No. Consider the rationals $\mathbb{Q} \subset \mathbb{R}$. $\mathbb{Q}^\circ = \emptyset$ but $\overline{\mathbb{Q}}^\circ = \mathbb{R}^\circ = \mathbb{R}$. It is true and straightforward to prove that $E^\circ \subset \overline{E}^\circ$. Let $x \in E^\circ$. Then there exists an $\epsilon > 0$ s.t. $B_\epsilon(x) \subset E \implies B_\epsilon(x) \subset \overline{E} \implies x \in \overline{E}^\circ$.
6. No. Consider $\mathbb{Q} \subset \mathbb{R}$. Then $\overline{\mathbb{Q}} = \mathbb{R}$ and $\overline{\mathbb{Q}^\circ} = \overline{\emptyset} = \emptyset$.

Exercise 10.90 (Rudin 2.10)

Let X be an infinite set. For $p \in X$ and $q \in X$, define
$$d(p,q) = \begin{cases} 1 & \text{if } p \neq q \\ 0 & \text{if } p = q \end{cases} \tag{596}$$
Prove that this is a metric. Which subsets of the resulting metric space are open? Which are closed? Which are compact?

Solution 10.89

This is a metric since clearly it satisfies symmetry and the fact that $d(p,p) = 0$. The triangle inequality
$$d(p,r) \leq d(p,q) + d(q,r) \tag{597}$$
is trivially satisfied if $p = r$, and if $p \neq r$, then either $p \neq q$ or $q \neq r$, and so the RHS ≥ 1. An open ϵ-ball around $x \in X$ is either X, when $\epsilon > 1$, or $\{x\}$ when $\epsilon \leq 1$. Therefore

Exercise 10.91 (Rudin 2.11)

For $x \in \mathbb{R}$ and $y \in \mathbb{R}$, define

$$\begin{aligned} d_1(x,y) &= (x-y)^2 \\ d_2(x,y) &= \sqrt{|x-y|} \\ d_3(x,y) &= |x^2 - y^2| \\ d_4(x,y) &= |x - 2y| \\ d_5(x,y) &= \frac{|x-y|}{1+|x-y|} \end{aligned}$$

Determine, for each of these, whether it is a metric or not.

Solution 10.90

Listed. Positive semidefiniteness and symmetry are easy to check.

1. The triangle inequality gives

$$\begin{aligned} d_1(x,z) \leq d_1(x,y) + d_1(y,z) &\iff (x-z)^2 \leq (x-y)^2 + (y-z)^2 \\ &\iff 0 \leq (x-y)(y-z) \end{aligned}$$

   which is not satisfied if $x < y < z$, so this is not a valid metric.
2. The triangle inequality gives $\sqrt{|x-z|} \leq \sqrt{|x-y|} + \sqrt{|y-z|}$, and since both sides are positive this inequality is equivalent to squaring both sides to get

$$|x-z| \leq |x-y| + |y-z| + 2\sqrt{|x-y||y-z|}$$

   which is true since $|x-z| \leq |x-y| + |y-z|$ of the Euclidean distance satisfies the triangle inequality and $0 \leq \sqrt{|x-y||y-z|}$.
3. This does not satisfy triangle inequality, as taking $0, 1, 2$ gives

$$d_3(0,2) = 4 > 1 + 1 = d_3(0,1) + d_3(1,2)$$

4. This does not satisfy symmetry.
5. For simplicity, let us set $A = |x-y|, B = |y-z|, C = |x-z|$. Then, we get

$$\frac{C}{1+C} \leq \frac{A}{1+A} + \frac{B}{1+B} \iff C \leq A + B + 2AB + ABC$$

   where $C \leq A + B$ is true by triangle inequality of Euclidean distance, $0 \leq AB$, and $0 \leq ABC$.

Intuitively, we want a metric that doesn't "blow up" the distance between x and y. More precisely, we want a valid metric $d(x,y)$ to be $O(|x-y|)$. Having something like a quadratic growth rate $(x-y)^2$ will blow the distance $d(x,z)$ up too much overpowering the individual $d(x,y) + d(y,z)$.

Exercise 10.92 (Rudin 2.12)

Let $K \subset \mathbb{R}$ consist of 0 and the numbers $1/n$ for $n = 1, 2, 3, \ldots$. Prove that K is compact directly from the definition (without using the Heine-Borel theorem).

Solution 10.91

Every open cover of K must have an open set G s.t. $0 \in G$. Since G is open, there exists an open neighborhood $B_\epsilon(0) \subset G$ around 0. By the Archimidean principle, there exists an $N \in \mathbb{N}$ s.t.

$$\epsilon N > 1 \implies \epsilon > \frac{1}{N} \tag{598}$$

and so, $B_\epsilon(0)$ contains all points $\{1/n\}$ for $n > N$. For the rest of the points $1, 1/2, \ldots, 1/N$, we can simply construct a finite cover over each of them, hence getting a finite cover.

Exercise 10.93 (Rudin 2.13)

Construct a compact set of real numbers whose limit points form a countable set.

Solution 10.92

Consider the set

$$E = \left\{ \left(\frac{1}{10}\right)^n + \left(\frac{1}{10}\right)^{n+k} \; : \; n \in \{0\} \cup \mathbb{N}, k \in \mathbb{N} \right\} \cup \{0\} \tag{599}$$

This is clearly bounded by 0 and 1.1. Let us represent the elements of this set by (n, k). We can show that

$$(n_1, k_1) > (n_2, k_2) \tag{600}$$

if $n_1 < n_2$ or $n_1 = n_2$ and $k_1 < k_2$. Therefore, to prove closedness, we must prove that every limit point is a point in E. We can do this by proving that a point not in E cannot be a limit point. Choose any $x \notin E$. Then, due to the ordering, we can see that there exists a (n, k) s.t.

$$A = \left(\frac{1}{10}\right)^n + \left(\frac{1}{10}\right)^{n+k} < k < \left(\frac{1}{10}\right)^n + \left(\frac{1}{10}\right)^{n+k+1} = B \tag{601}$$

and so we can take $\epsilon = \min\{k - A, B - k\}$ and show that $B_\epsilon(x)$ does not contain A nor B, and so has an empty intersection with E. Therefore, it cannot be a limit point of E and is closed. Since E is bounded and closed in $\mathbb{R}$, it is compact. Its limit points contain $1, 0.1, 0.01, \ldots, 0$ (simply by fixing n and letting $k \to \infty$, and so E' is infinite. We have just shown that since E is closed, $E' \subset E$. But E is countable, so E' is countable.

Exercise 10.94 (Rudin 2.14)

Given an example of an open cover of the segment $(0, 1)$ which has no finite subcover.

Solution 10.93

Consider

$$(0, 1/2) \cup \left(\bigcup_{i=1}^\infty \left[1 - \frac{1}{2^i}, 1 - \frac{1}{2^{i+1}}\right) \right) \tag{602}$$

Exercise 10.95 (Rudin 2.15)

Exercise 10.96 (Rudin 2.16)

Regard $\mathbb{Q}$, the set of all rational numbers, as a metric space, with $d(p,q) = |p - q|$. Let E be the set of all $p \in \mathbb{Q}$ s.t. $2 < p^2 < 3$. Show that E is closed and bounded in $\mathbb{Q}$ but is not compact. Is E open in $\mathbb{Q}$?

Solution 10.94

E is clearly bounded by 0 and 2 since $0^2 < 2 < p^2 < 3 < 2^2$. It is closed and we can show this by showing that E^c is open. Let $x \in E^c$. Then, $x^2 < 2$ or $x^2 > 3$.

1. $x^2 < 2 \iff -\sqrt{2} < x < \sqrt{2}$. Now let $\epsilon = \min\{\sqrt{2} - x, x + \sqrt{2}\} > 0$. Then by the Archimidean property there exists a $n \in \mathbb{N}$ s.t. $0 < \frac{1}{n} < \epsilon$. Therefore, the image of $B_{1/n}(x) \subset \mathbb{Q}$ will map onto $(0,2)$.
2. $x^2 > 3 \iff x > \sqrt{3}$ or $x < -\sqrt{3}$. If $x > \sqrt{3}$, then by AP there exists a $n \in \mathbb{N}$ s.t. $x - \frac{1}{n} > \sqrt{3} \implies (x - \frac{1}{n})^2 > 3$. If $x < -\sqrt{3}$, then by AP there exist $n \in \mathbb{N}$ s.t. $x + \frac{1}{n} < -\sqrt{3} \implies (x + \frac{1}{n})^2 > 3$. Either way, the image of $B_{1/n}(x)$ will map within E^c.

It is not compact because E is not closed in $\mathbb{R}$. The limit points of E in $\mathbb{R}$ is $[\sqrt{2}, \sqrt{3}] \cup [-\sqrt{3}, -\sqrt{2}]$, which contains irrationals and is clearly not a subset of E. Since it is not closed in $\mathbb{R}$, it is not compact in $\mathbb{R}$, and it is not compact in $\mathbb{Q} \subset \mathbb{R}$. It is open because

$$E = \big((\sqrt{2}, \sqrt{3}) \cup (-\sqrt{3}, \sqrt{2})\big) \cup \mathbb{Q} \subset \mathbb{R} \tag{603}$$

which is the union of open $\big(\sqrt{2}, \sqrt{3}) \cup (-\sqrt{3}, \sqrt{2})\big)$ and subset $\mathbb{Q} \subset \mathbb{R}$, and so it is open.

Exercise 10.97 (Rudin 2.17)

Let E be the set of all $x \in [0,1]$ whose decimal expansion consists of only the digits 4 and 7. Is E countable? Is E dense in $[0,1]$? Is E compact? Is E perfect?

Exercise 10.98 (Rudin 2.18)

Is there a nonempty perfect set in $\mathbb{R}$ which contains no rational number?

Exercise 10.99 (Rudin 2.19)

Listed.

1. If A and B are disjoint closed sets in some metric space X, prove that they are separated.
2. Prove the same for disjoint open sets.
3. Fix $p \in X$, $\delta > 0$, define A to be the set of all $q \in X$ for which $d(p,q) < \delta$. Define B similarly, with $>$ in place of $<$. Prove that A and B are separated.
4. Prove that every connected metric space with at least two points is uncountable.

Solution 10.95

Listed.

1. This is trivial with the fact that the closure of the closure of A is the closure of A.
2. Let A, B be open. We wish to show that if $x \in A'$, then $x \notin B$. Assume $x \in B$. Then there exists $\epsilon > 0$ s.t. $B_\epsilon(x) \subset B$. But $B \cap A = \emptyset \implies B_\epsilon(x) \cap A = \emptyset$ and so $x \notin A'$, which is a contradiction.
3. Clearly, $A \cap B = \emptyset$. Not let $x \in A \implies$ there exists $\epsilon > 0$ s.t. $B_\epsilon(x) \subset A \implies B_\epsilon(x) \cap B = \emptyset \implies x \in B'$. We can prove similarly to show that $x \in B \implies x \notin A'$.

4. Assume X is countable (solutionis very similar for finite). Then, we can enumerate a $X = \{x_i\}_{i=1}^\infty$. We wish to show that X can be decomposed into the union of an open ball and the interior of its complement as shown in (3). We fix $p \in X$. Then, we take the set $D = \{d(p,x)\}_{x \neq p} \subset \mathbb{R}$. Since D is a countable subset of $\mathbb{R}$, there must exist some $\alpha > 0$ s.t. $\alpha \notin D$. This α partitions the distances into two sets, and we can define

$$X = \{q \in X \mid d(p,q) < \alpha\} \cup \{q \in X \mid d(p,q) > \alpha\}$$

and by (3), these two sets are separated, which means that X is not connected, leading to a contradiction.

Exercise 10.100 (Rudin 2.20)

Are closures and interiors of connected sets always connected? Look at subsets of $\mathbb{R}^2$.

Solution 10.96

The interiors are not always connected. Consider the two closed balls $\overline{B_1((1,0))}$ and $\overline{B_1((-1,0))}$ as subsets of $\mathbb{R}^2$. They are connected but their interiors, which are the two open balls, are not connected. As for closures, they are always connected. Let W be connected. Then for any partition $A \cup B = W$, $\overline{A} \cap B \neq \emptyset$ WLOG. Consider $\overline{W} = W \cup W'$ and take any partition $\overline{W} = C \cup D$. Then, label $A = C \cap W, A^* = C \cap W', B = D \cap W, B^* = D \cap W'$. This implies that $C = A \cup A^*, D = B \cup B^*$, and $A \cup B = W$ (which is connected). Then, we can show that

$$\begin{aligned} \overline{C} \cap D &= (\overline{A \cup A^*} \cap D) = (\overline{A} \cup \overline{A^*}) \cap D = (\overline{A} \cap D) \cup (\overline{A^*} \cap D) \\ &= (\overline{A} \cap B) \cup (\overline{A} \cap B^*) \cup (\overline{A^*} \cap D) \end{aligned}$$

which cannot be empty since by connectedness of W, $\overline{A} \cap B \neq \emptyset$. Therefore, $\overline{W}$ is connected.

Exercise 10.101 (Rudin 2.21)

Let A and B be separated subsets of some $\mathbb{R}^k$. Suppose $a \in A, b \in B$ and define

$$p(t) = (1-t)a + tb \tag{604}$$

for $t \in \mathbb{R}$. Put $A_0 = p^{-1}(A), B_0 = p^{-1}(B)$.
1. Prove that A_0 and B_0 are separated subsets of $\mathbb{R}$.
2. Prove that there exists a $t_0 \in (0,1)$ s.t. $p(t_0) \notin A \cup B$.
3. Prove that every convex subset of $\mathbb{R}^k$ is connected.

Exercise 10.102 (Rudin 2.22)

A metric space is called *separable* if it contains a countable dense subset. Show that $\mathbb{R}^k$ is separable.

Solution 10.97

Consider the set $\mathbb{Q}^k \subset \mathbb{R}^k$. It is a finite Cartesian product (and hence, a countable union) of countable $\mathbb{Q}$, and so it is countable. $\mathbb{Q}^k$ is dense in $\mathbb{R}^k$ since given any $x \in \mathbb{R}^k$, we claim x is a limit point of $\mathbb{Q}^k$. Given any $\epsilon > 0$, we can construct $B_\epsilon^\circ(x)$. For each coordinate x_i, by density of rationals in $\mathbb{R}$ we can choose a $q_i \in \mathbb{Q}$ s.t. $0 < d(x_i, q_i) < \epsilon/k$. Then, using triangle inequality, we can take the distances between each coordinate changed from x_i to q_i. Let q^k be the vector x with the components

$x_1, \ldots, x_k$ changed to $q_1, \ldots, q_k$, respectively.

$$d(x, q) = d(x, q^1) + d(q^1, q^2) + \ldots + d(q^{k-1}, q_k) < \frac{\epsilon}{k} + \ldots + \frac{\epsilon}{k} = \epsilon \tag{605}$$

and so $q \in B_\epsilon^\circ(x)$. Hence the intersection of $\mathbb{Q}^k$ and $B_\epsilon^\circ(x)$ for any $\epsilon > 0$ is nontrivial, so x is a limit point of $\mathbb{Q}^k$.

Exercise 10.103 (Rudin 2.23)

A collection $\{V_\alpha\}$ of open subsets of X is said to be a *base* for X if the following is true: For every $x \in X$ and every open set $G \subset X$ such that $x \in G$, we have $x \in V_\alpha \subset G$ for some α. In other words, every open set in X is the union of a subcollection of $\{V_\alpha\}$. Prove that every separable metric space has a countable base.

Solution 10.98

Since X is separable it contains a countable dense subset, call it S. Then for every $x \in S$, we can look at the set of all open balls with center x and rational radii, call it $\mathcal{B}$. Then $\mathcal{B}$ is countable. Now consider an open set U. By definition, for every $x \in U$, there exists an $\epsilon > 0$ s.t. $B_\epsilon(x) \subset U$. By AP, we can find a $n \in \mathbb{N}$ s.t. $0 < \frac{1}{n} < \epsilon$, and therefore we can find an open ball $B \in \mathcal{B}$ s.t. $B(x) \subset U$. We claim that

$$W := \bigcup_{x \in U} B(x) = U \tag{606}$$

If $x \in U$, then by construction it is contained in $B(x) \subset \cup_{x \in U} B(x)$, and so $U \subset W$. If $x \in W$, then it is in $B(x)$, which is fully contained in U and so $W \subset U$. Therefore every open set can be constructed by a countable union of open balls in countable $\mathcal{B}$.

Exercise 10.104 (Rudin 2.24)

Let X be a metric space in which every infinite subset has a limit point. Prove that X is separable.

Solution 10.99

We fix $\delta > 0$. Choose $x_1 \in X$. Then choose $x_2 \in X$ s.t. $d(x_1, x_2) \geq \delta$, and keep doing this until we choose $x_{j+1} \in X$ s.t. $d(x_{j+1}, x_i) \geq \delta$ for all $i \in 1, \ldots, j$.

1. We claim that this must stop after a finite number of steps. Assume it doesn't. Then by assumption $V = \{x_i\}_{i=1}^\infty$ should have a limit point in X, denote it x. Choose $\frac{\delta}{2} > 0$. Then, $B_{\delta/2}^\circ(x) \cap V \neq \emptyset$. This intersection can only have one point since if it had two x', x'', then since both are in $B_{\delta/2}(x)$, then
$$d(x', x'') \leq d(x', x) + d(x, x'') \leq \frac{\delta}{2} + \frac{\delta}{2} = \delta$$
and since they are both in V, then $d(x', x'') \geq \delta$, which is a contradiction. Since there is a finite number of points in $B_{\delta/2}(x)$ of V, x cannot be a limit point. So this must terminate at some finite $J < \infty$.
2. Denote $W = \{x_i\}_{i=1}^J$. Then, $\mathscr{B}_\delta = \{B_\delta(x) \mid x \in W\}$ must cover X, since if it didn't, there would exist a $y \in X$ s.t. $d(y, x) \geq \delta$ for all $x \in W$, and we can add another element in W.
3. Consider $\delta = 1, 1/2, 1/3, \ldots$ and construct the same cover
$$\mathscr{B}_k = \{B_{1/k}(x_{ki}) \mid i = 1, \ldots, J_k\}$$
which is finite. Therefore, $\mathscr{B} = \cup_{k=1}^\infty \mathscr{B}_k$ must be countable.

4. We claim that countable $\{x_{ki}\}_{k,i}$ is dense. Consider any $x \in X$. For every $\epsilon > 0$, we can find an arbitrarily large $n \in \mathbb{N}$ s.t. $0 < \frac{1}{n} < \epsilon$. Since $\mathscr{B}_n$ is an open cover, there must exist some x_{ni} s.t. $x \in B_{1/n}(x_{ni})$, which by symmetry implies that $x_{ni} \in B_{1/n}(x) \subset B_\epsilon(x)$. Therefore, there always exists an x_{ni} in every $B_\epsilon(x)$, and so $B_\epsilon(x) \cap \{x_{ki}\} \neq \emptyset \implies x$ is a limit point of $\{x_{ki}\}$ and so it is dense.

Exercise 10.105 (Rudin 2.25)

Prove that every compact metric space K has a countable base, and that K is therefore separable.

Solution 10.100

For every $n \in \mathbb{N}$, let us consider an open covering $\mathscr{F}_n := \{B_{1/n}(x_n) \mid x_n \in K\}$. Since K is compact, it has a finite subcovering

$$\mathscr{G}_n := \{B_{1/n}(x_{ni}) \mid i = 1, \ldots, k(n)\} \tag{607}$$

Now consider the union $\mathscr{G} = \cup_{i=1}^n \mathscr{G}_n$, which is countable. We claim that $\mathscr{G}$ is a base. Consider any open set U. Then for every $x \in U$, we want to show that x is contained in a $B_{1/n}(x_{ni}) \subset U$. Since U is open , there exists a $\epsilon > 0$ s.t. $B_\epsilon(x) \subset U$. Now by AP, there exists a $n \in \mathbb{N}$ s.t. $0 < \frac{1}{n} < \frac{\epsilon}{2}$. Therefore $B_{1/n}(x) \subset B_\epsilon(x)$. Since $\mathscr{G}$ is an open covering, there must exist some $B_{1/n}(x_{ni})$ that contains x. Now we wish to show that $B_{1/n}(x_{ni})$ is fully contained in U. Let $y \in B_{1/n}(x_{ni})$. Then, by triangle inequality,

$$d(y, x) = d(y, x_{ni}) + d(x_{ni}, x) < \frac{1}{n} + \frac{1}{n} < \epsilon \tag{608}$$

and therefore $x \in B_{1/n}(x_{ni}) \subset B_\epsilon(x)$. Therefore, for every $x \in U$, we can construct an open ball of $\mathscr{G}$ containing x and contained in U, proving that this is a base.
We claim that the set of all $\mathscr{P} = \{x_{ni}\}_{n,i}$ forms a countable dense subset. This is clearly countable since $\mathscr{G}$ is countable. We must prove that the closure of $\mathscr{P} = K$. Let $x \in K$. Given any $\epsilon > 0$, we wish to show that $B_\epsilon(x) \cap \mathscr{P} \neq \emptyset$. Since $B_\epsilon(x)$ is open, it can be covered by a subcollection of $\mathscr{G}$, and so their centers must be in $B_\epsilon(x)$, proving that $B_\epsilon(x) \cap \mathscr{P} \neq \emptyset$. Therefore, x is a limit point of $\mathscr{P}$.

Exercise 10.106 (Rudin 2.26)

Exercise 10.107 (Rudin 2.27)

Exercise 10.108 (Rudin 2.28)

Exercise 10.109 (Rudin 2.29)

Prove that every open set in $\mathbb{R}$ is the union of an at most countable collection of disjoint segments.

Solution 10.101

Let $U \subset \mathbb{R}$ be open. Then for all $x \in U$ there exists $\epsilon > 0$ s.t. $(x - \epsilon, x + \epsilon) \subset U$. Now since $\mathbb{R}$ is separable (by exercise Rudin 2.22), it has a countable dense subset $\mathbb{Q}$. Consider all segments of rational radius and rational centers

$$\mathscr{B} = \{(q - p, q + p) \subset \mathbb{R} \mid q, p \in \mathbb{Q}\} \tag{609}$$

This is clearly countable. We claim that every open U can be expressed as the union of a subset of $\mathscr{B}$. Now by AP, there exists $n \in \mathbb{N}$ s.t. $0 < \frac{1}{n} < \frac{\epsilon}{2}$, so for all $x \in U$, there exists $n \in \mathbb{N}$ s.t. $(x - \frac{1}{n}, x + \frac{1}{n}) \subset U$. Now since $\mathbb{Q}$ is dense in $\mathbb{R}$, $x \in \mathbb{Q}' \implies (x - \frac{1}{n}, x + \frac{1}{n}) \cap \mathbb{Q} \neq \emptyset$. Say r is in this intersection. Then, by symmetry of metric, $x \in (r - \frac{1}{n}, r + \frac{1}{n})$. Therefore, for all $x \in U$, we have found an open ball in $\mathscr{B}$ that contains x. Now, we must show that this actually is fully contained in U. This is easy, since if $y \in B_{1/n}(r)$, then

$$d(y, x) \leq d(y, r) + d(r, x) \leq \frac{1}{n} + \frac{1}{n} < \epsilon \tag{610}$$

and so $B_{1/n}(r)$ is complete contained in the ϵ-ball around x, which is a subset of U. So for all $x \in U$, we found an open set $U_x \in \mathscr{B}$ covering x and fully contained in U, which means that $\cup_{x \in U} U_x = U$. Now for some intervals $B_1, B_2 \in \mathscr{B}$, if $B_1 \cap B_2 \neq \emptyset$, take their union, which is another segment, and keep doing this until $B_i \cap B_j \neq \emptyset$ for all i, j. The cardinality of this new pruned set will be less than or equal to $\mathscr{B}$, which is countable, and so this must be at most countable.

10.3 Sequences

Exercise 10.110 (Math 531 Spring 2025, PS4.6)

Consider the set of all bounded sequences of real numbers. That is, we consider sequences $\{x_n\}$ for which

$$\sup_{n \in \mathbb{N}} |x_n| \tag{611}$$

exists. For example, the sequence $\{1, 2, 3, \ldots\}$ does not belong to the set, but the sequence $\{1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \ldots\}$ does. Call this set X. Endow it with a metric:

$$d(\{x_n\}, \{y_n\}) = \sup_{n \in \mathbb{N}} |x_n - y_n|. \tag{612}$$

Explain why this is a metric. Make sure to explain why the supremum on the right hand side exists.

Exercise 10.111 (Math 531 Spring 2025, PS4.7)

Consider the metric space (X, d) from the previous problem. Is $\overline{B_1(\{0\})}$ a compact set? Here, $\{0\}$ is just the sequence of zeros: $\{0, 0, 0, 0, \ldots\}$.

Exercise 10.112 (Rudin 3.1)

Prove that convergence of $\{x_n\}$ implies convergence of $\{|x_n|\}$. Is the converse true?

Solution 10.102

If $\{x_n\}$ converges to x, then for all $\epsilon > 0$, there exists a $N \in \mathbb{N}$ s.t. $|x_n - x| < \epsilon$ if $n > N$. We use the inequality $\big||x_n| - |x|\big| \leq |x_n - x|$ to show that then for every $\epsilon > 0$ there exists a $N \in \mathbb{N}$ s.t.

$$\big||x_n| - |x|\big| \leq |x_n - x| \leq \epsilon$$

and so $\{|x_n|\}$ converges to $|x|$.

Exercise 10.113 (Rudin 3.2)

Calculate

$$\lim_{n\to\infty} \sqrt{n^2+n} - n$$

Solution 10.103

We can compute

$$\lim_{n\to\infty} \sqrt{n^2+n} - n = \lim_{n\to\infty} (\sqrt{n^2+n} - n) \cdot \frac{\sqrt{n^2+n}+n}{\sqrt{n^2+n}+n} = \lim_{n\to\infty} \frac{n}{\sqrt{n^2+n}+n}$$

where

$$A_n = \frac{n}{\sqrt{n^2+2n+1}+n} \leq \frac{n}{2n+1} \leq \frac{n}{\sqrt{n^2+n}+n} \leq \frac{n}{\sqrt{n^2}+n} = \frac{n}{2n} = \frac{1}{2} = C_n$$

C_n is ultimately constant. It suffices to prove that A_n limits to $\frac{1}{2}$ by showing that

$$\frac{n}{2n+1} = \frac{n/n}{(2n+1)/n} = \frac{1}{2+\frac{1}{n}}$$

where $\{\frac{1}{n}\}$ is infinitesimal.

Exercise 10.114 (Rudin 3.3)

If $s_1 = \sqrt{2}$ and

$$s_{n+1} = \sqrt{2+\sqrt{s_n}}$$

for $n = 1, 2, \ldots$, prove that $\{s_n\}$ converges and that $s_n < 2$ for $n = 1, 2, \ldots$.

Solution 10.104

We can show that $s_n < 2$ by induction. $s_1 = \sqrt{2} < 2$, so the base case is proved. Now, given that $s_n < 2$, $\sqrt{s_n} < 2 \implies 2 + \sqrt{s_n} < 2 + \sqrt{2} < 4 \implies s_{n+1} = \sqrt{2+\sqrt{s_n}} < 2$ and we are done.

Exercise 10.115 (Rudin 3.4)

Find the upper and lower limits of the sequence $\{s_n\}$ defined by

$$s_1 = 0; \quad s_{2m} = \frac{s_{2m-1}}{2}; \quad s_{2m+1} = \frac{1}{2} + s_{2m}.$$

Solution 10.105

Exercise 10.116 (Rudin 3.5)

For any two real sequences $\{a_n\}$, $\{b_n\}$, prove that

$$\limsup_{n\to\infty}(a_n+b_n) \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n,$$

provided the sum on the right is not of the form $\infty - \infty$.

Solution 10.106

Exercise 10.117 (Rudin 3.6)

Investigate the behavior (convergence or divergence) of Σa_n if
(a) $a_n = \sqrt{n+1} - \sqrt{n}$;
(b) $a_n = (\sqrt{n+1} - \sqrt{n})/n$;
(c) $a_n = (\sqrt[n]{n} - 1)^n$;
(d) $a_n = \frac{1}{1+z^n}$, for complex values of z.

Solution 10.107

Exercise 10.118 (Rudin 3.7)

Prove that the convergence of Σa_n implies the convergence of

$$\sum \frac{\sqrt{a_n}}{n},$$

if $a_n \geq 0$.

Solution 10.108

Exercise 10.119 (Rudin 3.8)

If Σa_n converges, and if $\{b_n\}$ is monotonic and bounded, prove that $\Sigma a_n b_n$ converges.

Solution 10.109

Exercise 10.120 (Rudin 3.9)

Find the radius of convergence of each of the following power series:
(a) $\sum n^3 z^n$,
(b) $\sum \frac{2^n}{n!} z^n$,
(c) $\sum \frac{2^n}{n^2} z^n$,
(d) $\sum \frac{n^3}{3^n} z^n$.

Solution 10.110

Exercise 10.121 (Rudin 3.10)

Suppose that the coefficients of the power series $\sum a_n z^n$ are integers, infinitely many of which are distinct from zero. Prove that the radius of convergence is at most 1.

Solution 10.111

Exercise 10.122 (Rudin 3.11)

Suppose $a_n > 0$, $s_n = a_1 + \cdots + a_n$, and Σa_n diverges.

(a) Prove that $\sum \frac{a_n}{1+a_n}$ diverges.

(b) Prove that

$$\frac{a_{N+1}}{s_{N+1}} + \cdots + \frac{a_{N+k}}{s_{N+k}} \geq 1 - \frac{s_N}{s_{N+k}}$$

and deduce that $\sum \frac{a_n}{s_n}$ diverges.

(c) Prove that

$$\frac{a_n}{s_n^2} \leq \frac{1}{s_{n-1}} - \frac{1}{s_n}$$

and deduce that $\sum \frac{a_n}{s_n^2}$ converges.

(d) What can be said about

$$\sum \frac{a_n}{1 + na_n} \text{ and } \sum \frac{a_n}{1 + n^2 a_n}?$$

Solution 10.112

Exercise 10.123 (Rudin 3.12)

Suppose $a_n > 0$ and Σa_n converges. Put

$$r_n = \sum_{m=n}^{\infty} a_m.$$

(a) Prove that

$$\frac{a_m}{r_m} + \cdots + \frac{a_n}{r_n} > 1 - \frac{r_n}{r_m}$$

if $m < n$, and deduce that $\sum \frac{a_n}{r_n}$ diverges.

(b) Prove that

$$\frac{a_n}{\sqrt{r_n}} < 2(\sqrt{r_n} - \sqrt{r_{n+1}})$$

and deduce that $\sum \frac{a_n}{\sqrt{r_n}}$ converges.

Solution 10.113

Exercise 10.124 (Rudin 3.13)

Prove that the Cauchy product of two absolutely convergent series converges absolutely.

Solution 10.114

Exercise 10.125 (Rudin 3.14)

If $\{s_n\}$ is a complex sequence, define its arithmetic means σ_n by

$$\sigma_n = \frac{s_0 + s_1 + \cdots + s_n}{n+1} \quad (n = 0, 1, 2, \ldots).$$

(a) If $\lim s_n = s$, prove that $\lim \sigma_n = s$.
(b) Construct a sequence $\{s_n\}$ which does not converge, although $\lim \sigma_n = 0$.
(c) Can it happen that $s_n > 0$ for all n and that $\limsup s_n = \infty$, although $\lim \sigma_n = 0$?
(d) Put $a_n = s_n - s_{n-1}$, for $n \geq 1$. Show that

$$s_n - \sigma_n = \frac{1}{n+1} \sum_{k=1}^{n} k a_k.$$

Assume that $\lim(n a_n) = 0$ and that $\{\sigma_n\}$ converges. Prove that $\{s_n\}$ converges. [This gives a converse of (a), but under the additional assumption that $n a_n \to 0$.]
(e) Derive the last conclusion from a weaker hypothesis: Assume $M < \infty$, $|n a_n| \leq M$ for all n, and $\lim \sigma_n = \sigma$. Prove that $\lim s_n = \sigma$, by completing the following outline:
If $m < n$, then

$$s_n - \sigma_n = \frac{m+1}{n-m}(\sigma_n - \sigma_m) + \frac{1}{n-m} \sum_{i=m+1}^{n} (s_n - s_i).$$

For these i,

$$|s_n - s_i| \leq \frac{(n-i)M}{i+1} \leq \frac{(n-m-1)M}{m+2}.$$

Fix $\varepsilon > 0$ and associate with each n the integer m that satisfies

$$m \leq \frac{n-\varepsilon}{1+\varepsilon} < m+1.$$

Then $(m+1)/(n-m) \leq 1/\varepsilon$ and $|s_n - s_i| < M\varepsilon$. Hence

$$\limsup_{n \to \infty} |s_n - \sigma| \leq M\varepsilon.$$

Since ε was arbitrary, $\lim s_n = \sigma$.

Solution 10.115

Exercise 10.126 (Rudin 3.15)

Definition 3.21 can be extended to the case in which the a_n lie in some fixed $\mathbb{R}^k$. Absolute convergence is defined as convergence of $\Sigma|\mathbf{a}_n|$. Show that Theorems 3.22, 3.23, 3.25(a), 3.33, 3.34, 3.42, 3.45, 3.47, and 3.55 are true in this more general setting. (Only slight modifications are required in any of the proofs.)

Solution 10.116

Exercise 10.127 (Rudin 3.16)

Fix a positive number α. Choose $x_1 > \sqrt{\alpha}$, and define $x_2, x_3, x_4, \ldots$, by the recursion formula

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{\alpha}{x_n}\right).$$

(a) Prove that $\{x_n\}$ decreases monotonically and that $\lim x_n = \sqrt{\alpha}$.

(b) Put $\varepsilon_n = x_n - \sqrt{\alpha}$, and show that

$$\varepsilon_{n+1} = \frac{\varepsilon_n^2}{2x_n} < \frac{\varepsilon_n^2}{2\sqrt{\alpha}}$$

so that, setting $\beta = 2\sqrt{\alpha}$,

$$\varepsilon_{n+1} < \beta\left(\frac{\varepsilon_1}{\beta}\right)^{2^n} \qquad (n = 1, 2, 3, \ldots).$$

(c) This is a good algorithm for computing square roots, since the recursion formula is simple and the convergence is extremely rapid. For example, if $\alpha = 3$ and $x_1 = 2$, show that $\varepsilon_1/\beta < \frac{1}{10}$ and that therefore

$$\varepsilon_5 < 4 \cdot 10^{-16}, \quad \varepsilon_6 < 4 \cdot 10^{-32}.$$

Solution 10.117

Exercise 10.128 (Rudin 3.17)

Fix $\alpha > 1$. Take $x_1 > \sqrt{\alpha}$, and define

$$x_{n+1} = \frac{\alpha + x_n}{1 + x_n} = x_n + \frac{\alpha - x_n^2}{1 + x_n}.$$

(a) Prove that $x_1 > x_3 > x_5 > \cdots$.

(b) Prove that $x_2 < x_4 < x_6 < \cdots$.

(c) Prove that $\lim x_n = \sqrt{\alpha}$.

(d) Compare the rapidity of convergence of this process with the one described in Exercise 16.

Solution 10.118

Exercise 10.129 (Rudin 3.18)

Replace the recursion formula of Exercise 16 by

$$x_{n+1} = \frac{p-1}{p}x_n + \frac{\alpha}{p}x_n^{-p+1}$$

where p is a fixed positive integer, and describe the behavior of the resulting sequences $\{x_n\}$.

Solution 10.119

Exercise 10.130 (Rudin 3.19)

Associate to each sequence $a = \{\alpha_n\}$, in which α_n is 0 or 2, the real number

$$x(a) = \sum_{n=1}^{\infty} \frac{\alpha_n}{3^n}.$$

Prove that the set of all $x(a)$ is precisely the Cantor set described in Sec. 2.44.

Solution 10.120

Exercise 10.131 (Rudin 3.20)

Suppose $\{p_n\}$ is a Cauchy sequence in a metric space X, and some subsequence $\{p_{n_i}\}$ converges to a point $p \in X$. Prove that the full sequence $\{p_n\}$ converges to p.

Solution 10.121

Exercise 10.132 (Rudin 3.21)

Prove the following analogue of Theorem 3.10(b): If $\{E_n\}$ is a sequence of closed nonempty and bounded sets in a complete metric space X, if $E_n \supset E_{n+1}$, and if

$$\lim_{n\to\infty} \text{diam } E_n = 0,$$

then $\bigcap_1^\infty E_n$ consists of exactly one point.

Solution 10.122

Exercise 10.133 (Rudin 3.22)

Suppose X is a nonempty complete metric space, and $\{G_n\}$ is a sequence of dense open subsets of X. Prove Baire's theorem, namely, that $\bigcap_1^\infty G_n$ is not empty. (In fact, it is dense in X.) Hint: Find a shrinking sequence of neighborhoods E_n such that $E_n \subset G_n$, and apply Exercise 21.

Solution 10.123

Exercise 10.134 (Rudin 3.23)

Suppose $\{p_n\}$ and $\{q_n\}$ are Cauchy sequences in a metric space X. Show that the sequence $\{d(p_n, q_n)\}$ converges. Hint: For any m, n,

$$d(p_n, q_n) \leq d(p_n, p_m) + d(p_m, q_m) + d(q_m, q_n);$$

it follows that

$$|d(p_n, q_n) - d(p_m, q_m)|$$

is small if m and n are large.

Solution 10.124

Exercise 10.135 (Rudin 3.24)

Let X be a metric space.

(a) Call two Cauchy sequences $\{p_n\}$, $\{q_n\}$ in X equivalent if

$$\lim_{n\to\infty} d(p_n, q_n) = 0.$$

Prove that this is an equivalence relation.

(b) Let X^* be the set of all equivalence classes so obtained. If $P \in X^*, Q \in X^*$, $\{p_n\} \in P, \{q_n\} \in Q$, define

$$\Delta(P, Q) = \lim_{n\to\infty} d(p_n, q_n);$$

by Exercise 23, this limit exists. Show that the number $\Delta(P, Q)$ is unchanged if $\{p_n\}$ and $\{q_n\}$ are replaced by equivalent sequences, and hence that Δ is a distance function in X^*.

(c) Prove that the resulting metric space X^* is complete.

(d) For each $p \in X$, there is a Cauchy sequence all of whose terms are p; let P_p be the element of X^* which contains this sequence. Prove that

$$\Delta(P_p, P_q) = d(p, q)$$

for all $p, q \in X$. In other words, the mapping φ defined by $\varphi(p) = P_p$ is an isometry (i.e., a distance-preserving mapping) of X into X^*.

(e) Prove that $\varphi(X)$ is dense in X^*, and that $\varphi(X) = X^*$ if X is complete. By (d), we may identify X and $\varphi(X)$ and thus regard X as embedded in the complete metric space X^*. We call X^* the completion of X.

Solution 10.125

Exercise 10.136 (Rudin 3.25)

Let X be the metric space whose points are the rational numbers, with the metric $d(x, y) = |x - y|$. What is the completion of this space? (Compare Exercise 24.)

Solution 10.126

10.4 Continuous Functions

10.5 Differentiation

10.6 Integration

10.7 Cauchy Criterion for Integrability

Remember that the Riemann integral, as complicated as the formula is, is still a limit of a function. That means that we can apply the Cauchy criterion to it to determine convergence.

Lemma 10.1 (Cauchy Criterion on Existence of Riemann Integral)

Given a function f, the integral of f over $[a, b]$, defined

$$\int_a^b f(x)\,dx \equiv \lim_{\lambda(P)\to 0} \sum_{i=1}^{n} f(\xi_i)\lambda x_i$$

exists if and only if for every $\epsilon > 0$, there exists a $\delta > 0$ such that

$$\left|\sigma(f; P', \xi') - \sigma(f; P'', \xi''\right| < \epsilon$$

or, what is the same,

$$\left|\sum_{i=1}^{n'} f(\xi_i')\Delta x_i' - \sum_{i=1}^{n''} f^(\xi_i'')\Delta x_i''\right| < \epsilon$$

for any partitions (P', ξ') and (P'', ξ'') with distinguished points on the interval $[a, b]$ with

$$\lambda(P'), \lambda(P'') < \delta$$

In words, this means that for any $\epsilon > 0$ that we choose, there always exists a $\delta > 0$ such that **any** two Riemann sums with mesh size **both** smaller than δ will have an error difference of less than ϵ.

10.8 Lebesgue's Criterion for Riemann Integrability

We give Lebesgue's version of an intrinsic description of a Riemann integrable function.

Definition 10.1 (Measure)

A set $E \subset \mathbb{R}$ has **(Lebesgue) measure zero** if for every number $\epsilon > 0$ there exists a covering of the set E be an at most countable system $\{I_k\}$ of intervals, the sum of whose lengths

$$\sum_{k=1}^{\infty} |I_k| \leq \epsilon$$

This means that the above series summing up the lengths of the intervals is an absolutely convergent series.

Lemma 10.2 ()

We can deduce measures of basic sets.

1. A finite number of points are sets of measure zero.
2. The union of a finite or countable number of sets of measure zero is a set of measure zero.
3. A subset of a set of measure zero is itself a set of measure zero.
4. A closed interval $[a, b]$ with $a < b$ is not a set of measure zero.

Definition 10.2 ()

If a property holds at all points of a set X except possible the points of a set of measure zero, we say that this property holds **almost everywhere on** X or **at almost every point of** X.

Now, we can state Lebesgue's criterion for integrability, which nicely summarizes what we have so far.

Theorem 10.1 (Lebesgue's Criterion for Integrability)

A function defined on a closed interval is Riemann integrable on that interval if and only if it is bounded and continuous at almost every point.

10.9 Important Algebraic Inequalities

We also introduce various inequalities that may be useful for producing future results. The following lemmas can be proved with elementary algebra.

Lemma 10.3 (Young's Inequalities)

If $a > 0$ and $b > 0$, and the numbers p and p are such that $p \neq 0, 1$ and $q \neq 0, 1$, and $\frac{1}{p} + \frac{1}{q} = 1$, then

$$a^{\frac{1}{p}} b^{\frac{1}{q}} \leq \frac{1}{p} a + \frac{1}{q} b \text{ if } p > 1$$
$$a^{\frac{1}{p}} b^{\frac{1}{q}} \geq \frac{1}{p} a + \frac{1}{q} b \text{ if } p < 1$$

and equality holds in both cases if and only if $a = b$.

Lemma 10.4 (Holder's Inequalities)

Let $x_i \geq 0, y_i \geq 0$ for $i = 1, 2, ..., n$, and let $\frac{1}{p} + \frac{1}{q} = 1$. Then,

$$\sum_{i=1}^n x_i y_i \leq \left(\sum_{i=1} x_i^p \right)^{\frac{1}{p}} \left(\sum_{i=1} y_i^q \right)^{\frac{1}{q}} \text{ for } p > 1$$
$$\sum_{i=1}^n x_i y_i \geq \left(\sum_{i=1} x_i^p \right)^{\frac{1}{p}} \left(\sum_{i=1} y_i^q \right)^{\frac{1}{q}} \text{ for } p < 1, p \neq 0$$

Lemma 10.5 (Minkowski's Inequalities)

Let $x_i \geq 0, y_i \geq 0$ for $i = 1, 2, ..., n$. Then,

$$\left(\sum_{i=1}^n (x_i + y_i)^p \right)^{\frac{1}{p}} \leq \left(\sum_{i=1}^n x_i^p \right)^{\frac{1}{p}} + \left(\sum_{i=1}^n y_i^p \right)^{\frac{1}{p}} \text{ when } p > 1$$
$$\left(\sum_{i=1}^n (x_i + y_i)^p \right)^{\frac{1}{p}} \geq \left(\sum_{i=1}^n x_i^p \right)^{\frac{1}{p}} + \left(\sum_{i=1}^n y_i^p \right)^{\frac{1}{p}} \text{ when } p < 1, p \neq 0$$

Theorem 10.2 (Jensen's Inequality)

If $f : (a, b) \longrightarrow \mathbb{R}$ is a convex function, $x_1, ..., x_n$ are points of (a, b), and $\alpha_1, ..., \alpha_n$ are nonnegative numbers such that $\alpha_1 + ... + \alpha_n = 1$, then

$$f(\alpha_1 x_1 + ... + \alpha_n x_n) \leq \alpha_1 f(x_1) + ... + \alpha_n f(x_n)$$