Point Set Topology

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Contents

Given a set, what are some fundamental structures that you can put on a set? We can first talk about relations, such as ordering, or functions, such as a norm or a distance. These are constructed as subsets of a Cartesian product of finite X's. Another structure on set X is to define a set of subsets $\mathscr{T} \subset 2^X$ that allow us to interpret how certain elements of a set are "nearby" each other without the notion of a metric.[1] This set of subsets is called a *topology*, with its elements being *open sets*. So how do you define such a thing? Well intuitively, given two elements $x, y \in X$, if there exists two disjoint open sets U_1, U_2 such that $x \in U_1$ and $y \in U_2$, then we can *distinguish* these points in such a way. If this is true for all points in X, then this gives us a nice *Hausdorff* property to work with. If there exists no open sets that can do this, then x and y, although distinct in X, may be *indistinguishable* in the topological sense.

If this notion of nearness can be rigorously defined, we may be able to characterize the elements and subsets of X. One nice notion is the concept of *limit points* which asks whether x is "infinitesimally close" to a certain set. This allows us to define limits without the notion of a metric, and with this foundation we build the notion of continuity.

A trivial way to construct such a topology is to take the power set 2^X itself. However, this may be "too big" in a sense that no interesting properties can be deduced. But this doesn't mean we can take any subset of 2^X. We compromise by defining topologies to be a subset of 2^X with certain properties, which we will mention in the next section.

The construction of the topology allows us to study properties of these spaces. Moreover, if we have a function that maps from one topological space to another, how do we know what kinds of properties will be preserved and what will be lost? It turns out that these topological properties are invariant under certain mappings called *homeomorphisms*. Therefore, topology can also be seen as a method to study spaces and properties that are preserved under homeomorphisms.

[1] In ZFC set theory, a topology may be more fundamental in the sense that it is a subset of the power set, while the other structures are subsets of a Cartesian product, which itself is a construction from the power set.

1 Open and Closed Sets

The first thing to define is a topology.

Definition 1.1 (Topology)

Let X be a set and $\mathscr{T}$ be a family of subsets of X. Then $\mathscr{T}$ is a **topology** on X[a] if it satisfies the following properties.

1. *Contains Empty and Whole Set*:
$$\emptyset, X \in \mathscr{T} \tag{1}$$
2. *Closure Under Union.* If $\{U_\alpha\}_{\alpha \in A}$ is a class of sets in $\mathscr{T}$, then
$$\bigcup_{\alpha \in A} U_\alpha \in \mathscr{T} \tag{2}$$
3. *Closure Under Finite Intersection*: If $U_1, \ldots, U_n$ is a finite class[b] of sets in $\mathscr{T}$, then
$$\bigcap_{i=1}^{n} U_i \in \mathscr{T} \tag{3}$$

A **topological space** is denoted $(X, \mathscr{T})$.

[a] I will use script letters to denote topologies and capital letters to denote sets.

[b] Note that we restrict property 3 to be a *finite* intersection because it turns out that the finiteness of intersection allows us to prove many nice properties about topologies, which we will mention later. Another reason is that if we remove this finite restriction, the open ball topology on $\mathbb{R}$ would imply that $\cap_{i=1}^{\infty}(-1/i, +1/i) = 0$ is an open set $\implies$ all points are open sets too, which is generally not what we want in analysis.

This leads to the most general definition of an open set. Note that an open set doesn't really mean anything without talking about with respect to its topology.

Definition 1.2 (Open Set)

The elements of $\mathscr{T}$ are called **open sets** in X.[a]

1. An open set U which contains a point x is called an **open neighborhood** of x, denoted U_x.
2. Given an open neighborhood U_x of x, the set $U_x \setminus \{x\}$ is called the **punctured open neighborhood** of x.

[a] As implied from the definition of a topology, the arbitrary union and finite intersection of any number of open sets is an open set.

For the sake of giving at least one nontrivial example, here is an example of a finite topology.

Example 1.1 (Topologies of a Set of Cardinality 3)

There are a total of 29 topologies that we can construct on $\{1, 2, 3\}$. Two such examples are

1. $\{\emptyset, \{1, 2\}, \{1, 2, 3\}\}$
2. $\{\emptyset, \{3\}, \{2, 3\}, \{1, 2, 3\}\}$

When we define a new topology, we must first prove that they are topologies, and so these definitions are really theorems. However, I will introduce them as definitions and reserve the theorem environment for actual theorems.

Definition 1.3 (Discrete, Indiscrete Topologies)

Given a set X,
1. 2^X is a topology, called the **discrete topology**.
2. $\{\emptyset, X\}$ is a topology, called the **indiscrete topology**.

Proof.

Listed.
1. The first property is trivially proven. From the theorems of set theory, $U_\alpha \subset X \implies \cup U_\alpha \subset X \implies \cup U_\alpha \in 2^X$. Finally the same logic holds for intersection as well.
2. The first property is trivially proven. We can check for the 4 combinations of unions and intersections and see that they all result in either $\emptyset$ or X.

Definition 1.4 (Finer, Coarser Topologies)

Suppose that $\mathscr{T}$ and $\mathscr{T}'$ are two topologies on a given set X. If $\mathscr{T} \subset \mathscr{T}'$, we say that $\mathscr{T}'$ is **finer** than $\mathscr{T}$, or equivalently, we say that $\mathscr{T}$ is **coarser** than $\mathscr{T}'$.

We can think of the topology of a set X as a truck full of gravel as the open sets. If the gravel is smashed into smaller, finer pieces, then the amount of stuff that we can make from the finer gravel increases, which corresponds to a bigger topology. Clearly, the indiscrete topology is the coarsest topology and the discrete topology is the finest.

Theorem 1.1 (Intersection of Topologies)

Given a family of topologies $\{\mathscr{T}\}_{\alpha \in A}$, the set

$$\mathscr{T} = \bigcap_{\alpha \in A} \mathscr{T}_\alpha \tag{4}$$

is a topology.

Corollary 1.1 (Unique Coarsest and Finest Topology)

Given a family of topologies $\{\mathscr{T}\}_{\alpha \in A}$, there exists
1. a unique smallest topology on X containing all the collections $\mathscr{T}_\alpha$.
2. a unique largest topology on X contained in each $\mathscr{T}_\alpha$.

Example 1.2 ()

Let $X = \{a, b, c\}$, and let

$$\mathscr{T}_1 = \{\emptyset, X, \{a\}, \{a, b\}\} \tag{5}$$

$$\mathscr{T}_2 = \{\emptyset, X, \{a\}, \{b, c\}\} \tag{6}$$

We claim that the
1. smallest topology containing $\mathscr{T}_1, \mathscr{T}_2$ is

$$\mathscr{T}_{1\cup 2} = \{\emptyset, X\{a\}, \{b\}, \{a, b\}, \{b, c\}\} \tag{7}$$

   Note that this is not simply the union of topologies. The union wouldn't have $\{b\}$, making it not a topology.

2. largest topology contained in $\mathscr{T}_1, \mathscr{T}_2$ is

$$\mathscr{T}_{1\cap 2} = \{\emptyset, X, \{a\}\} \tag{8}$$

Note that this is simply the intersection of the two topologies.

1.1 Basis and Fineness

So far so good. We want to continue analyzing the properties of a topology, but sometimes working with the entire topology is a bit thorny. There is a tamer representation of a topology, which can also give us the starting point to *construct* topologies.

Definition 1.5 (Basis)

If X is a set, a **basis** on X is a collection $\mathscr{B}$ of subsets of X (called **basis elements**) such that

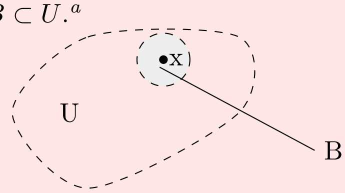
1. For each $x \in X$, there is at least one basis element $B \in \mathscr{B}$ containing x. That is, the elements of $\mathscr{B}$ covers X.
2. If x belongs to the intersection of two basis elements B_1 and B_2, then there is a basis element B_3 containing x such that $B_3 \subset (B_1 \cap B_2)$.

The name gives away the clue that a topology may be created from this basis.

Theorem 1.2 (Basis to Topology)

Given a basis $\mathscr{B}$ on a set X, we can define a topology $\mathscr{T}$, called the **topology generated by** $\mathscr{B}$, in the following equivalent ways.

1. $\mathscr{T}$ consists of subsets U of X satisfying the property that for each $x \in U$, there exists a basis element $B \in \mathscr{B}$ such that $x \in B \subset U$.[a]



2. $\mathscr{T}$ consists of all possible unions of elements in $\mathscr{B}$.

$$\mathscr{T} \equiv \left\{ \bigcup_i b_i \;\middle|\; b_i \in \mathscr{B} \right\} \tag{9}$$

[a]Note that since we can always set $U = \emptyset$, the basis doesn't need to contain $\emptyset$.

Proof.

We prove that the 2 methods generate a topology, and then finally prove that it they are the same topology.

1. Clearly, $\emptyset$ and X itself are in $\mathscr{T}$. To prove property 2, given a certain indexed family of subsets $\{U_\alpha\}_{\alpha \in I}$ of $\mathscr{T}$, we must show that

$$U = \bigcup_{\alpha \in I} U_\alpha \in \mathscr{T} \tag{10}$$

Given $x \in U$, there exists at least one index α such that $x \in U_\alpha$. Since $U_\alpha \in \mathscr{T}$ already, there exists a basis element $b \in \mathscr{B}$ such that $x \in b \subset U_\alpha$. But

$$U_\alpha \subseteq U \implies b \subset U \tag{11}$$

So, by definition, any arbitrary union of U of these subsets is also in $\mathscr{T}$.
To prove property 3, we must show that

$$W = \bigcap_{\alpha \in I} U_\alpha \in \mathscr{T} \tag{12}$$

Given $x \in W$, by definition of a basis element, there exists a $b \in \mathscr{B}$ such that

$$x \in b \subset (U_\beta \cap U_\gamma) \forall \beta, \gamma \in I \implies \text{ there exists } \tilde{b} \in \mathscr{B} \text{ s.t. } x \in \tilde{b} \subset \bigcap_{\alpha \in I} U_\alpha \tag{13}$$

By definition, W is also open. Since this arbitrary set of subsets $\mathscr{T}$ suffices the 3 properties, it is a topology of X by definition.

2. $(\rightarrow)$ Given a collection of elements in $\mathscr{B}$, they are also elements of $\mathscr{T}$. Since $\mathscr{T}$ is a topology, their union in also in $\mathscr{T}$.
   $(\leftarrow)$ Given an open set $U \in \mathscr{T}$, for every point $x \in U$, by definition we can choose a basis element $b \in \mathscr{B}$ such that $x \in b \subset U$. Then, the union of all these basis elements is by definition b.

We have learned how to go from a basis to a topology. The following lemma tells us how to identify a basis within a topology.

Theorem 1.3 (Topology to Basis)

Let X be a topological space, and let $\mathscr{C}$ be a collection of subsets of X such that for every open set U and each $x \in U$, there exists an element $C \in \mathscr{C}$ such that

$$x \in C \subset U \tag{14}$$

Then, $\mathscr{C}$ is a basis for the topology of X.

Proof.

Characterizing topologies in terms of basis is quite effective since we can work with more manageable sets.

Lemma 1.1 (Fineness w.r.t. Basis)

Given two topologies $\mathscr{T}$ and $\mathscr{T}'$ with their bases $\mathscr{B}$ and $\mathscr{B}'$, respectively, the following are equivalent.

1. $\mathscr{T}'$ is finer than $\mathscr{T}$.
2. For each $x \in X$ and basis element $B \in \mathscr{B}$ containing x, there exists a basis element $B' \in \mathscr{B}'$ such that $x \in B' \subset B$.

1.2 Limit Points and Closed Sets

First, we need to learn what it generally means for a point to be infinitesimally close to a set.

Definition 1.6 (Limit Point)

Given a topological space $(X, \mathscr{T})$, let $x \in X$ be a point and $S \subset X$ a subset. x is a **limit point of** S if every punctured neighborhood of x intersects S.[a] The set of all limit points of a set S is denoted S'.

[a]Note that limit point are generally used to talk about points that are infinitesimally close to a set S. A limit point may not necessarily be in S, and a point of S may not necessarily be a limit point. This is why we use a punctured

neighborhood, rather than an open neighborhood. For continuity as we will see later, we just talk about neighborhoods since we also claim that the limit exists and the function value is the limit.

Example 1.3 (Examples of Limit Points)

What about the limit points that are not in S? Generally, there are two instances.

1. Let S represent the gray area. B is in the "interior" of S and therefore is a limit point. A and C are on the "boundary" of S yet not in S, and we can show that they are limit points as well.

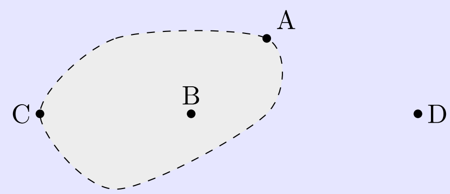


Figure 1: Points A, B, C are limit points of the open set.

2. A point can be at the "convergence point" of a sequence.

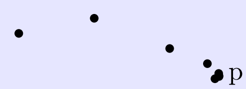


Figure 2: Note that if S is a sequence of points in $\mathbb{R}^2$ that converges to p without ever hitting it, we can say that $p \notin S$ is a limit point of S.

Example 1.4 (Examples of Non-Limit Points)

There are generally two instances of non-limit points. Let $X = \mathbb{R}$ and $S = (0, 1) \cup \{2\}$.

1. 5 is clearly not a limit point.
2. 2, although in S, is not a limit point since we are talking about the punctured neighborhood. A point in S that is not a limit point is called an **isolated point**.

Definition 1.7 (Closed Set)

A set $S \subset X$ is **closed** if its complement $X \setminus S$ is open in $\mathscr{T}$.[a]

[a] Note that open and closed sets are not mutually exclusive. A set might be open, closed, both, or neither. A set that is both open and closed is called **clopen**.

Another property, which is often used as the definition of a closed set, is that it contains all of its limit points.

Lemma 1.2 (Closed Sets Contain Limit Points)

A set $S \subset X$ is closed iff it contains all of its limit points.

Proof.

We prove bidirectionally.

Theorem 1.4 (Topological Space wrt Closed Sets)

Let X be a topological space. Then, the following conditions hold

1. $\emptyset$ and X are clopen.
2. Arbitrary intersections of closed sets are closed.
3. Finite unions of closed sets are closed.

Definition 1.8 (Dense Subsets)

Let $S \subset (X, \tau_X)$. S is **dense** in X if every point $p \in X$ is a limit point of S. In other words, for any point $p \in X$ and any open neighborhood U_p of p, $U_p \cap S$ is nontrivial. Otherwise, p is a point of S.

The following example is a crucial fact for proving further properties of topological spaces.

Example 1.5 ()

$\mathbb{Q}^n$ is a dense set of $\mathbb{R}^n$ with the open ball topology. If we have the discrete topology of $\mathbb{R}^2$, an open neighborhood of a point is the point itself, so no limit points would exist beyond the points in S itself. So $\mathbb{Q}^n$ is not dense in $\mathbb{R}^n$ with this topology.

1.3 Interiors and Closures

Now that we've determined limit points, we would like to extend sets into their limit points. The process of doing this is called the *closure* of a set.

Definition 1.9 (Closure)

The **closure** of set S is $\overline{S} = S \cup S'$, i.e. the union of itself and its limit points.

Example 1.6 ()

If S is an open ball, $\bar{S}$ is the closed ball.

From semantics, it may seem like the interior and exterior (defined later) are related, but from a mathematical point of view, the interior and closure are dual notions.

Definition 1.10 (Interior)

Let $S \subset X$. Then, the following definitions of the **interior** of S, denoted S°, are equivalent.

1. $x \in S^\circ$ if $\exists U_x \ni x$ s.t. $U_x \subset S$.
2. S° is the union of all open sets contained in S.
3. S^o is the complement of the closure of the complement of S.

$$S^o = \left(\overline{S^c}\right)^c \tag{15}$$

Proof.

Lemma 1.3 (Open and Closed in Terms of Interiors and Closures)

Let S be a subset of some topological space X.

1. S is open iff $S = S^o$. S^o is always open.
2. S is closed iff $S = \overline{S}$. $\overline{S}$ is always closed.

1.4 Exteriors and Boundaries

Definition 1.11 (Exteriors)

Let $S \subset X$. The **exterior** of S, denoted S^e, is defined in the following equivalent ways.[a]

1. S^e is the complement of the closure of S.
2. S^e is the interior of the complement of S.

[a]We can informally think of the exterior being strictly outside of S and its boundary.

Proof.

Definition 1.12 (Boundary)

Let $S \subset X$. The **boundary** of S, denoted ∂S, is defined in the following equivalent ways.

1. ∂S is the closure minus the interior of S in X.
2. ∂S is the intersection of the closure of S with the closure of its complement, i.e the set of all points x such that every neighborhood U_x intersects both the interior and exterior.
3. ∂S is the set of points that are neither in the exterior nor the interior.
4. $x \in \partial S$ if every neighborhood of x intersects both the interior and exterior of S.

Proof.

From the above, we get the intuitive notion that these three parts divide up the whole space.

Theorem 1.5 (Partitioning of Space)

Given $S \subset X$, X is partitioned into the interior, boundary, and exterior of S.

$$X = S^\circ \sqcup \partial S \sqcup S^e \tag{16}$$

Proof.

The fact that

One counterintuitive result is the Lakes of Wada, which are three disjoint connected open sets of the open unit square $(0, 1)^2$ with the property that they *all* have the same boundary. In other words, for any point selected on the boundary of one of the lakes, the other two lakes' boundaries also contain that point.

2 Common Topologies

We have given some examples of how we can construct topologies from scratch given an arbitrary set X, possibly with some structure. Now given a collection of 1 or more topological spaces, we will talk about how we can construct new topologies. Note that the topologies introduced in this section don't really require us to talk about functions yet. They can be constructed and completely described in terms of sets.

2.1 Order Topologies

Definition 2.1 (Dictionary Topology)

Let X be a set with a simple order relation. Let $\mathscr{B}$ be the collection of all sets of the following types.

1. All open intervals $(a, b) \subset X$
2. All half-open intervals $[a_0, b)$, where a_0 is the minimum element of X
3. All half-open intervals $(a, b_0]$, where b_0 is the maximum element of X.

This set $\mathscr{B}$ is a basis for the **order topology** of X. If X has no minimum or maximum, then there are no sets of type of 2 or 3, respectively.

Example 2.1 ()

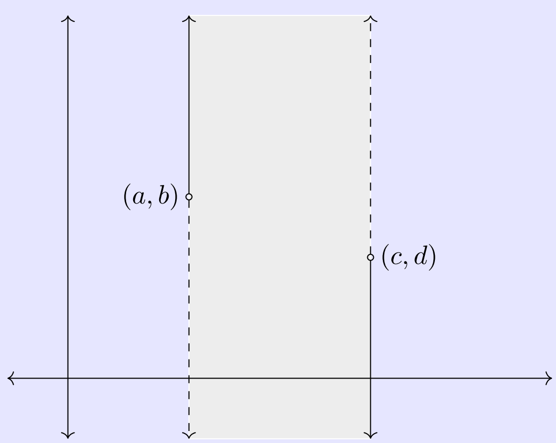
The standard topology on $\mathbb{R}$ is precisely the order topology derived from the usual order on $\mathbb{R}$.

Example 2.2 ()

Given $\mathbb{R} \times \mathbb{R}$ with the dictionary order, then $\mathbb{R} \times \mathbb{R}$ has neither a largest nor smallest element. Therefore, the order topology on $\mathbb{R} \times \mathbb{R}$ consists of all "intervals" of form

$$\big((a, b), (c, d)\big) \equiv \{(x, y) \in \mathbb{R}^2 \mid (a, b) < (x, y) < (c, d)\}$$

A visual diagram is shown below. This means that open rays and lines are also a part of the topology of $\mathbb{R} \times \mathbb{R}$.



Example 2.3 ()

The set of positive integers $\mathbb{Z}_+$ form an ordered set with a smallest element. The order topology for $\mathbb{Z}_+$ is precisely the discrete topology since every one-point set is an open set.

$$\{n\} = (n - 1, n + 1) \tag{17}$$

Example 2.4 ()

The dictionary order topology on $\{1,2\} \times \mathbb{Z}_+$ results in every one point set being open, except for the point $(2,1)$. Since every neighborhood of $(2,1)$ must contain some point of form $(1,n)$ for arbitrarily large n, $\{(2,1)\}$ is not open.

Definition 2.2 ()

If X is an ordered set a $a \in X$, then there are 4 subsets of X called rays determined by a.

1. $(a, +\infty)$
2. $(-\infty, a)$
3. $[a, +\infty)$
4. $(-\infty, a]$

The first two sets are called **open rays**, and the latter two sets are called **closed rays**.

We can extend the basis of open intervals to some other basis, which generates other topologies.

Definition 2.3 (Lower/Upper Limit Topology)

Given a totally ordered set $(X, \leq)$,

1. the **lower limit topology** is the topology generated by the basis of all half-closed half-open intervals of form
$$[a,b) \coloneqq \{x \in X \mid a \leq x < b\} \tag{18}$$
2. the **upper limit topology** is the topology generated by the basis of all half-open half-closed intervals of form
$$(a,b] \coloneqq \{x \in X \mid a < x \leq b\} \tag{19}$$

Theorem 2.1 (Nested Interval Topology)

In the space $X = (0,1)$, the **nested interval topology** is the topology generated by the basis of nested intervals of the form
$$\mathscr{B}_{ni} \coloneqq \{(0, 1 - \frac{1}{n}) \mid n \in \mathbb{N}\} \tag{20}$$

Theorem 2.2 (Closed Interval Topology)

In the set $X = [-1,1]$, the following set
$$\mathscr{B}_{ci} \coloneqq \{[-1,a) \mid a > 0\} \bigcup \{(b,1] \mid b < 0\} \tag{21}$$
is a basis. The topology it generates is called the **closed interval topology**, denoted $\mathscr{T}_{ci}$.

It turns out that since all open balls are in $\tau_{\mathbb{R}^2}$, we can build any shape using the union/intersections of these open balls, such as an open square. Thus all open subsets in $\mathbb{R}^n$ are open sets.

2.2 Metric Topology

For common sets like $\mathbb{R}^n$, which has an inner product, or $\mathbb{Q}$, which has an order, it is easy to build these topologies with set-builder notation. Consider the following.

Definition 2.4 (Metric Topology)

Given a metric space (X, d), let us denote the **metric topology**, or **open-ball topology**, as the set of subsets U satisfying the property that for all $x \in U$, there exists a positive $r \in \mathbb{R}$ such that $B(x, r) \subset U$, where $B(x, r) \coloneqq \{y \in X \mid d(x, y) < r\}$ is the open ball of radius r around x. We claim that this is a topology.

Proof.

We show that the properties of a topology hold.

1. For the empty set, the inclusion of an open ball for a point in $\emptyset$ is vacuously satisfied. For the whole set, we choose any point x and any r, and the open ball is trivially a subset of X.
2. Let $\{U_\alpha\}_{\alpha \in I}$ be a collection of open subsets of X. Let their union be denoted U. We claim U is open. Pick any point $x \in U$. Then by definition of union, there exists some $\alpha \in I$ s.t. $x \in U_\alpha$. Since U_α is open, there exists a $r > 0$ s.t. $B(x, r) \subset U_\alpha \subset U$. Therefore U is open.
3. Let $U_1, \ldots, U_k$ be open, and let us denote their intersection as U. We claim U is open. Pick a point $x \in U$. Then for each $i = 1, \ldots, k$, $x \in U_i$ and there exists a corresponding $r_i > 0$ such that the open ball $B(x, r_i) \subset U_i$. Take the set $R = \{r_i\}$, which is a finite set living in $\mathbb{R}$. We will take for granted that every finite subset of an ordered set has a minimum.[a] Let us denote $r^* = \min R$, and we claim that r^* gives us a ball that can fit inside U. Assume $y \in B(x, r^*)$. Then

$$\begin{aligned} y \in B(x, r^*) &\implies d(x, y) < r^* && (22) \\ &\implies d(x, y) < \min R && (23) \\ &\implies d(x, y) < r_i \text{ for } i = 1, \ldots, k && (24) \\ &\implies y \in B(x, r_i) \text{ for } i = 1, \ldots, k && (25) \end{aligned}$$

Since $B(x, r_i)$ by construction is contained within U_i, $y \in U_i$ for all i. This means by definition of intersection that $y \in U$, and we have proven that $B(x, r^*)$ completely fits inside U.

[a] If we wish to prove it, we can start with a singleton set, claim that its minimum is the only element. Then we use induction by assuming for a set R of size k that a minimum exists, and by adding 1 more element r we update the minimum to be $\min\{r, \min R\}$ and show that this is indeed the minimum.

Note that while open balls are used to define whether a set is open or not, the definition doesn't state whether open balls themselves are open sets. It turns out that it is easy to prove that they are.

Lemma 2.1 (Open Balls are Open Sets)

The open ball wrt any metric d is an open set wrt the metric topology.

Proof.

Let $y \in B(x, r)$. Then $d(x, y) < r \implies 0 < r - d(x, y)$. To show that $B(x, r)$ is open, we would like to show that there exists some $r' > 0$ s.t. $y \in B(y, r') \subset B(x, r)$. Set $r' = r - d(x, y)$. Then

$$\begin{aligned} z \in B(y, r') &\implies d(y, z) < r - d(x, y) && (26) \\ &\implies d(x, y) + d(x, y) < r && (27) \\ &\implies d(x, z) < r && (28) \\ &\implies z \in B(x, r) && (29) \end{aligned}$$

and so $B(y, r') \subset B(x, r)$. We are done.

Example 2.5 (Discrete Metric Induces Discrete Topology)

Given a set X, induce the metric d defined

$$d(x, y) \equiv \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases} \tag{30}$$

This metric induces the discrete topology on X, since the basis elements of the open balls

$$B_r(x) \equiv \{y \in X \mid d(x, y) < r\} \tag{31}$$

consists of two types of open sets. When $r \leq 1$, then $B_r(x) = x$ (since the radius is 0). If $r > 1$, then the open set is the entire space X.

Example 2.6 (Supremum Norm in $\mathbb{R}^3$)

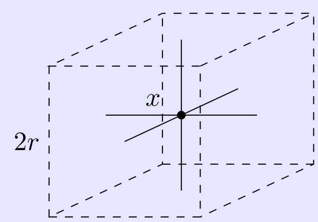


Figure 3: In $\mathbb{R}^3$, each basis element is a cube centered at x with side lengths $2r$.

Theorem 2.3 (Metric Topologies on Finite Sets)

If (X, d) is a finite metric space, then the metric topology on it is the discrete topology.

Proof.

Take all pairwise points.

It is easy to go from a metric to a topology, but a natural question is that given a topology, does there exist a metric that induces this topology? This is precisely the notion of *metrizability*, which is a highly desirable attribute for spaces, and there are many existence theorems that proves metrizability given certain conditions.

Definition 2.5 (Metrizable Space)

If $(X, \mathscr{T})$ is a topological space, $(X, \mathscr{T})$ is said to be **metrizable** if there exists a metric d on X that induces the topology $\mathscr{T}$ of X.

Example 2.7 (Non-Metrizable Finite Spaces)

Let $X = \{a, bc\}$. Then the topology

$$\mathscr{T} = \{\emptyset, \{b\}, \{a, b\}, \{b, c\}, X\} \tag{32}$$

is not metrizable from the theorem above since the only metrizable topologies are discrete.

Lemma 2.2 (Fineness of Metric Topologies)

Let d and d' be two metrics on the set X with their respective induced topologies $\mathscr{T}, \mathscr{T}'$. We claim that $\mathscr{T} \subset \mathscr{T}'$ iff there exists a $M > 0$ s.t.

$$d'(x, y) < M \cdot d(x, y) \tag{33}$$

for all $x, y \in X$. That is, we can bound d' with a constant multiple of d.

Proof.

2.3 Euclidean Topology

More specifically, the metric topology generated by the L_2-metric on $\mathbb{R}^n$ is called the **Euclidean topology**. Note that the topological property of stability under countable intersection was required to show that the minimum of R existed. This is not true for infinite sets in general. This gives us some motivation as to why we need the *finite* intersection rather than an infinite one.

Lemma 2.3 (Singletons are Not Open in $\mathbb{R}^n$)

A singleton set is not open in $\mathbb{R}^n$ with the Euclidean topology.

Proof.

We claim that the singleton set $S = \{0\}$ is not open under the Euclidean metric. We pick a point in S, which can only be 0. Assume that there exists an $r > 0$ s.t. $B(x, r) \subset S$. $\mathbb{R}$ is Archimedean, so there exists a natural number N s.t. $0 < 1/N < r$. We construct the vector $v = (v_1, \ldots, v_n)$ s.t. $v_1 = 1/N$ and $v_i = 0$ everywhere else. The distance between 0 and v is

$$\|v - 0\| = \|v\| = \sqrt{(1/N)^2} = 1/N < r \tag{34}$$

so $v \in B(x, r)$. But $v \neq 0$, and by contradiction such an r cannot exist. In $\mathbb{R}^n$ we consider the countable intersection of open balls (which we have proved in class are open sets) around 0 of radius $1/N$ for $n \in \mathbb{N}$. We claim that

$$\bigcap_{n \in \mathbb{N}} B(0, 1/n) = \{0\} \tag{35}$$

We see that $1/n$ must always be positive and so $\|0 - 0\| = 0 < 1/n$. Therefore the LHS $\supset$ RHS. To see that the intersection contains no other element, consider any vector $v \neq 0$. Then by definition of the metric, $d(v, 0) > 0$. By the Archimedean property, there exists a natural $N \in \mathbb{N}$ s.t. $0 < 1/N < d(v, 0)$, which means that $v \notin B(0, 1/N)$, and so v cannot be in the intersection. Therefore, the intersection must be $\{0\}$, and we have shown that B_0 is not open, so we are done.

Theorem 2.4 ()

For a metric space (X, d), the metric topology is finer than the cofinite topology.

Proof.

Note that if X is finite, then both are reduced to the discrete topologies.

While it is not surprising that a basis uniquely generates a topology, it is not immediately obvious *what* the generated topology looks like. It turns out that many different bases may generate the same topology, and the concept of fineness allows us to compare these topologies more effectively. For example, if two topologies are both finer than the other, then they must be equal.

Theorem 2.5 (Euclidean Topology on $\mathbb{R}^n$)

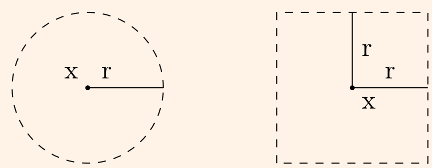
L_p norms all generate the same topology on $\mathbb{R}^n$.

Proof.

We can show that

$$n^q d_\infty \leq n^q d_2 \leq n^q d_1 \leq d_p \leq n^{-p} d_\infty \tag{36}$$

where q is the holder conjugate of p. Visually, we can see that every open ball in $(\mathbb{R}^n, d)$ (with the Euclidean metric) is the form to the left, while an open ball in $(\mathbb{R}^n, \rho)$ (with the square metric) is of form on the right.



Clearly, we can form any open set of any "shape" using any arbitrary combination of these "circles" or "squares," indicating that they generate the same topology.

2.4 Cofinite Topology

Definition 2.6 (Cofinite Topology)

Given a set X, the set of all subsets U, satisfying the property that $X \setminus U$ is finite, is a topology, called the **cofinite topology** or the **finite complement topology**.[a]

[a] While this definition may seem a bit arbitrary, this is very similar to the Zariski topology, which is used in algebraic topology.

Proof.

Let us denote this set $\mathscr{T}_c$.

1. By definition $\emptyset \in \mathscr{T}_c$. It is clear that $X \setminus X = \emptyset$ has cardinality 0, and therefore is in $\mathscr{T}_c$.
2. Let $\{U_\alpha\}_{\alpha \in I} \in \mathcal{T}_c$ by a collection of open sets of X. Then by deMorgan's laws,
$$X \setminus \bigcup_{\alpha \in I} U_\alpha = \bigcap_{\alpha \in I} (X \setminus U_\alpha) \tag{37}$$
$X \setminus U_\alpha$ is countable for all $\alpha \in I$, so let us fix some α'. Then
$$\bigcap_{\alpha \in I} (X \setminus U_\alpha) \subset U_{\alpha'} \implies \bigg| \bigcap_{\alpha \in I} (X \setminus U_\alpha) \bigg| \leq |U_{\alpha'}| \tag{38}$$
and so the intersection is also countable.
3. Let $\{U_i\}_{i=1}^n$ by a finite collection of open sets of X. Then by deMorgan's laws,
$$X \setminus \bigcap_{i=1}^n U_i = \bigcup_{i=1}^n (X \setminus U_i) \tag{39}$$
Since U_i are open, $X \setminus U_i$ are countable, and since the finite union of countable sets are countable, the RHS is countable, which implies the LHS is countable and so $\cap_{i=1}^n U_i$ is open as well.

Slightly modifying the definition does not result in a topology.

Example 2.8 (Countable Complement is Not A Topology)

Given a set X, consider the collection

$$\mathscr{T}_\infty := \{U \subset X \mid X \setminus U \text{ is infinite or empty or all of } X\} \tag{40}$$

This is not a topology. Let us take $X = \mathbb{R}$, and look at the sets $\mathbb{Z}_{\geq 0}, \mathbb{Z}_{\leq 0}$ consisting of all the non-negative and non-positive integers. They are both infinite, and so $\mathbb{R} \setminus \mathbb{Z}_{\geq 0}$ and $\mathbb{R} \setminus \mathbb{Z}_{\leq 0}$ are in $\mathcal{T}_\infty$. Consider their union.

$$(\mathbb{R} \setminus \mathbb{Z}_{\geq 0}) \cup (\mathbb{R} \setminus \mathbb{Z}_{\leq 0}) = \mathbb{R} \setminus (\mathbb{Z}_{\geq 0} \cap \mathbb{Z}_{\leq 0}) = \mathbb{R} \setminus \{0\} \tag{41}$$

But $\mathbb{R} \setminus (\mathbb{R} \setminus \{0\}) = \{0\}$, and so $\mathbb{R} \setminus \{0\}$ is not open. Therefore $\mathcal{T}_c$ doesn't satisfy the definition of a topology.

3 Functions and Continuity

Note that from set theory, we can construct functions as a subset of Cartesian product of two spaces X, Y. There is nothing new here.

Definition 3.1 (Continuous Function)

A function f between 2 topological spaces $(X, \mathscr{T}_X)$ and $(Y, \mathscr{T}_Y)$ is **continuous at** $x \in X$ if the preimage of every open neighborhood of $f(x) \in Y$ is an open neighborhood of $x \in X$.

$$U_{f(x)} \in \mathscr{T}_Y \implies x \in f^{-1}(U_{f(x)}) \in \mathscr{T}_X \tag{42}$$

f is said to be **continuous** (at all points) if the preimage of every open set in Y is an open set in X.[a]

[a] Note that continuity of a function f is not only determined by the function itself, but also by the topologies of X and Y.

Note that it is easier for f to be continuous when the $\mathscr{T}_X$ is finer (since there are more open sets in X for the preimage of $V \subset Y$ to map to) or $\mathscr{T}_Y$ is coarser (since there are fewer open sets that we have to check to map to open sets of X).

Theorem 3.1 (Sufficient Properties for Continuity)

Let X, Y, be topological spaces and let $f : X \longrightarrow Y$. Then, the following are equivalent to f being continuous.

1. The preimage of every basis element $B \in \mathscr{T}_Y$ is open in X.
2. For every closed set B in Y, the set $f^{-1}(B)$ is closed in X.
3. For every subset A of X, $f(\bar{A}) \subset \overline{f(A)}$.

Proof.

Listed.

1. An arbitrary open set V of Y can be written as $V = \cup_{\alpha \in J} b_\alpha$. Then,

$$f^{-1}(V) = f^{-1}\Big(\bigcup_{\alpha \in J} b_\alpha\Big) = \bigcup_{\alpha \in J} f^{-1}(b_\alpha) \tag{43}$$

Great, so we have a few ways in which we can check continuity of a function. There are a few special cases.

Lemma 3.1 (Trivially Continuous Functions)

We have the following for general topological spaces.

1. The identity function $\mathrm{Id} : (X, \mathscr{T}_1) \to (X, \mathscr{T}_2)$ is continuous if $\mathscr{T}_1 \supset \mathscr{T}_2$.
2. A constant function $f : (X, \mathscr{T}) \to (Y, \mathscr{T}_2)$ is always continuous, regardless of the topologies.

Proof.

If we take an open set $U \in \mathscr{T}_2$, its preimage is the same set U, which is guaranteed to be in $\mathscr{T}_1$ since $\mathscr{T}_1$ is finer than $\mathscr{T}_2$.

3.1 Construction of Continuous Functions

Theorem 3.2 (Arithmetic on Real Continuous Functions)

If X is a topological space, and if $f, g : X \longrightarrow \mathbb{R}$ are continuous, then $f + g$, $f - g$, and $f \cdot g$ are also continuous. f/g is continuous if $g(x) \neq 0$ for all $x \in X$.

Theorem 3.3 (Analytic Continuity = Topological Continuity)

Given metric spaces with their induced metric topologies $(X, \mathscr{T}_X, d_X)$ and $(Y, \mathscr{T}_Y, d_Y)$. The following are equivalent.

1. $f : X \to Y$ is continuous at x.
2. For every $\delta > 0$, there exists an $\epsilon = \epsilon(\delta) > 0$ such that for all $z \in X$, $d_X(x, z) < \epsilon \implies d_Y(f(x), f(x)) < \delta$.[a]

[a]This is the definition of continuity at a point in analysis.

Proof.

($\rightarrow$) Assume f is continuous according to the $\epsilon - \delta$ definition. Let U be any open set in Y containing the point y, and let x be an element in $f^{-1}(U)$ such that $y = f(x)$. We must prove that $f^{-1}(U)$ is also open. Since open sets contain neighborhoods (e.g. open balls) of all of its points, we can claim that, since U is open by assumption, there exists an open ball B_y around y with radius $\epsilon > 0$. This guarantees the existence of a point $z \in U$ such that $\rho(y, z) < \epsilon$ for any $\epsilon > 0$ that we choose. Since f is continuous, for every $\epsilon > 0$ that we chose previously, there exists a $\delta > 0$ such that $d(x, w) \implies \rho(f(x), f(w)) < \epsilon$. Since $\rho(f(x), f(w)) < \epsilon$, we can conclude that $f(w) \in B_y \subset U$ when $d(x, w) < \delta$. Therefore, $d(x, w) < \delta \implies w \in f^{-1}(U)$. But this is equivalent to saying that if $w \in B_(x, \delta)$, then $w \in f^{-1}(U)$, which means that every single point $x \in f^{-1}(U)$ contains an open ball neighborhood fully contained in $f^{-1}(U)$. So, by definition, $f^{-1}(U)$ is open.
($\leftarrow$) Assume $f^{-1}(U)$ is open when U is an open set in Y, i.e. f is continuous under the topological definition. Let us define the open ball

$$B(f(x), \epsilon) \equiv \{y \in Y \mid \rho(f(x), y) < \epsilon\} \in \tau_Y$$

By our assumption, $f^{-1}\big(B(f(x), \epsilon)\big)$ is an open set in τ_X, and clearly, $x \in f^{-1}\big(B(f(x), \epsilon)\big)$ since f^{-1} maps the point $f(x) \in B(f(x), \epsilon)$ to $x \in f^{-1}\big(B(f(x), \epsilon)\big)$. But since $f^{-1}\big(B(f(x), \epsilon)\big)$ is open, we can construct an open ball around x with radius δ fully contained within the open set. Moreover, by selecting a point $p \in B(f(x), \delta) \subset f^{-1}\big(B(f(x), \epsilon)\big)$, we can guarantee that $f(p) \in B(f(x), \epsilon)$. This is precisely the $\epsilon - \delta$ definition of continuity. That is, given an $\epsilon > 0$ to be the radius of an open ball $B(f(x), \epsilon)$ in Y, we can always choose a $\delta > 0$ to be the radius of the open ball $B(x, \delta)$ in X that is fully contained within the preimage of $B(f(x), \epsilon)$. In mathematical notation,

$$p \in B(x, \delta) \subset f^{-1}\big(B(f(x), \epsilon)\big) \implies f(p) \in f\big(B(x, \delta)\big) \subset B(f(x), \epsilon)$$

or equivalently in terms of metrics,

$$d(x, p) < \delta \implies \rho(f(x), f(p)) < \epsilon$$

3.2 Sequences

Definition 3.2 (Sequence)

A sequence (x_α) of points in topological space $(X, \mathscr{T})$ is said to **converge** to the point $x \in X$ if for every neighborhood U of x there exists a $N \in \mathbb{N}$ such that

$$x_n \in U \text{ for all } n \geq N \tag{44}$$

3.3 Homeomorphisms

Definition 3.3 (Homeomorphism)

A bijective, bicontinuous function $f : X \longrightarrow Y$ between two topological spaces is called a **homeomorphism** between X and Y. If there exists at least one homeomorphism between X and Y, then X is said to be **homeomorphic** to Y, denoted $X \cong Y$.

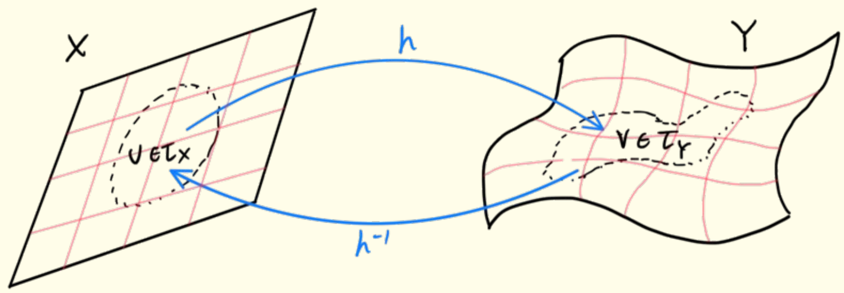


Figure 4: The visual below shows a homeomorphism between the plane X and the surface Y.

Theorem 3.4 (Sufficient Properties of Homeomorphism)

Suppose $f : X \to Y$ is a bijection. TFAE.

1. $U \subset Y$ is open iff $f^{-1}(U)$ is open.
2. $U \subset X$ is open iff $f(U)$ is open.
3. f is a homeomorphism.

Example 3.1 (Comparability and Homeomorphic Spaces)

Consider the set $X = \{a, b\}$ with the two topologies $\mathscr{T}_3 = \{\emptyset, \{a\}, X\}$ and $\mathscr{T}_4 = \{\emptyset, \{b\}, X\}$. They are not comparable but they seem "similar" in a way in that if we swap all the a's and b's in $\mathscr{T}_3$, then we get $\mathscr{T}_4$. We can make this rigorous by defining $f : (X, \mathscr{T}_3) \to (X, \mathscr{T}_4)$ with $f(a) = b, f(b) = a$, and showing that it is a homeomorphism.

In fact, a homeomorphism f is an equivalence relation between two topological spaces. This partitions the set of all topological spaces into **homeomorphism classes**. Analogous to how isomorphisms preserve algebraic structures, homeomorphisms preserve topological structure between topological spaces.

Example 3.2 (Homeomorphism Classes of 2D Manifolds)

There is an infinite family of 2-dimensional manifolds, call them M and N, and each set in each family is not homeomorphic to another.

1. $M_0 = S^2$ (sphere). $M_1 = T^2$ (torus). M_2 is a donut with two holes. M_3 has three holes, and so

on.

2. N_1 is the Mobius strip. N_2 is the Klein bottle.

Additionally, not only does a homeomorphism give a bijective correspondence between points in X and Y, but it also determines a bijection between **the set of all open sets in X and Y** (that is, a bijection between their topologies)! This bijection then allows two spaces that are homeormophic to have the same topological properties.

Theorem 3.5 (Preservation of Topological Properties)

A homeomorphism f between two topological spaces (X, τ_x) and (Y, τ_Y) preserves all topological properties (e.g. separability, countability, compactness, (path) connectedness) of X onto Y and Y onto X.

Definition 3.4 (Embedding)

Suppose that $f : X \longrightarrow Y$ an injective continuous map with X, Y topological spaces. Let $Z \equiv \operatorname{Im} f$. Then, the function

$$f' : X \longrightarrow Z \subset Y \tag{45}$$

obtained by restricting the codomain of f is bijective. If f' happens to be a homeomorphism of X with Z, then we say that the map

$$f : X \longrightarrow Y \tag{46}$$

is a **topological embedding**, or more simply an **embedding**, of X in Y.

4 Induced Topologies

4.1 Initial and Final Topologies

We have seen some examples of how to create topologies. They can be created without any assumptions on the set, such as the discrete, indiscrete, and the cofinite topologies. More often, we want to consider how a certain structure like the order or a metric induces a topology. Now, we will consider how *functions* can induce a topology. The uniqueness of such induced topologies is called the *universal property*.

Definition 4.1 (Initial Topology)

Given a space X and a family of topological spaces $\{Y_\alpha\}_{\alpha \in A}$

$$f_i : X \to (Y_\alpha, \mathscr{T}_\alpha) \tag{47}$$

the **initial topology** on X is the coarsest topology $\mathscr{T}$ on X s.t. that each

$$f_i(X, \mathscr{T}) \to (Y_\alpha, \mathscr{T}_\alpha) \tag{48}$$

is continuous.

Definition 4.2 (Final Topology)

Given a space Y and a family of topological spaces $\{X_\alpha\}_{\alpha \in A}$

$$f : (X, \mathscr{T}_\alpha) \to Y \tag{49}$$

the **final topology** on Y is the finest topology $\mathscr{T}$ on Y s.t. each

$$f : (X, \mathscr{T}_\alpha) \to (Y, \mathscr{T}) \tag{50}$$

is continuous.

Note that it makes sense to talk about the coarsest topology on the domain and the finest topology on the codomain. If it were the other way around, i.e. the finest topology on the domain, then the initial topology on X would be the discrete topology, making every function defined on X continuous. In the same logic, the coarsest topology on Y would trivially be the trivial topology, making all Y-valued functions continuous. With these current definitions, if $\mathscr{T}_Y$ is too fine (e.g. if $\mathscr{T}_Y = 2^Y$), then the open sets of $\mathscr{T}_Y$ would be too fine and therefore would have a preimage that may not be open in X.

4.2 Subspace Topology

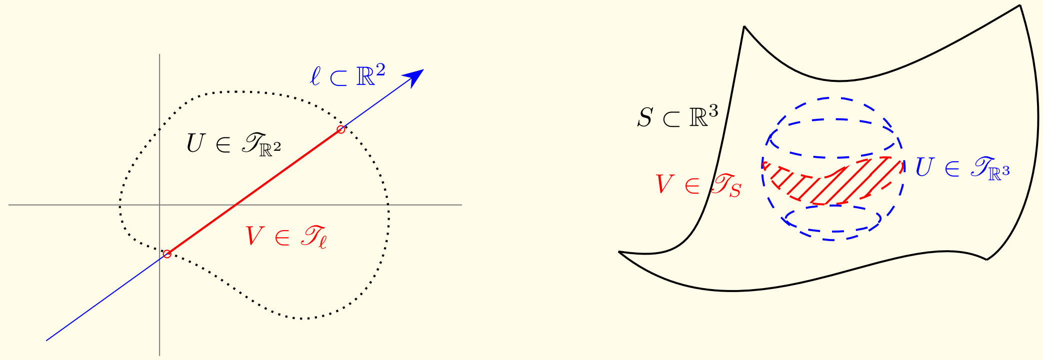
The reason we want to do this is because we want to think of Y as its own entity, independent of X.

Definition 4.3 (Subspace Topology)

Given topological space X and subspace $Y \subset X$, the **subspace topology** on Y is defined in the equivalent ways.

1. It is the initial topology on the subspace Y with respect to the inclusion map $\iota : Y \to X$.
2. It is the topology consisting of X-open sets intersection Y.

$$\mathscr{T}_Y = \{(U \cap Y) \subset Y \mid U \in \mathscr{T}_X\} \tag{51}$$



(a) The subspace topology of a line l in $\mathbb{R}^2$.

(b) The subspace topology of a surface $\mathcal{L}$ in $\mathbb{R}^3$.

Figure 5: Visual of subspace topology.

Proof.

We prove the properties.

1. *Trivial.* We see that $\emptyset = \emptyset \cap Y$ and $Y = X \cap Y$.
2. *Stability under Union.* Suppose $\{V_\alpha\}_{\alpha \in A}$ are setes that are open in Y. Then for each α there exists an open set $U_\alpha \subset X$ that is open in X. Therefore,
$$\bigcup_{\alpha \in A} V_\alpha = \bigcup_{\alpha \in A} (U_\alpha \cap Y) \tag{52}$$
$$= Y \cap \left(\bigcup_{\alpha \in A} U_\alpha \right) \tag{53}$$
where $\cup_\alpha U_\alpha$ is open in X, and therefore we shown that there exists such an open set.
3. *Stability under Finite Intersection.* Suppose $\{V_i\}_{i=1}^n$ are open in Y. Then we can do the same thing.

Furthermore, we can immediately retrieve the basis of the subspace topology.

Theorem 4.1 (Induced Basis of Subspace Topologies)

If $\mathscr{B}$ is a basis for the topology of X, then
$$\mathscr{B}_Y \coloneqq \{B \cap Y \mid B \in \mathscr{B}\} \tag{54}$$
is a basis for the subspace topology of Y.

Proof.

Since the subspace is so natural to consider, we will by default imply that if X is a topological space and $Z \subset X$, Z is endowed the subspace topology.

Lemma 4.1 (Restrictions and Injections are Continuous)

The results immediately follow:

1. Given $f : X \to Y$ and $Z \subset X$, $f|_Z : Z \to Y$ is continuous.
2. Given X and $Z \subset X$, the canonical injection $\iota : Z \to X$ is continuous.

Proof.

Listed.

1. Let us take an open set U in Y. Then it is of the form $V \cap Y$ for some V open in X. Therefore taking the preimage gives
$$f|_Z^{-1}(U) = f^{-1}(U) = f^{-1}(V \cap Y) = f^{-1}(V) \cap f^{-1}(Y) = f^{-1}(V) \cap Z \tag{55}$$
where $f^{-1}(V)$ is open by continuity of f, and so the intersection is open.
2. This is true by definition.

Given these results, one may wonder whether—just like how we restricted a continuous function to a smaller continuous function—we can "extend" a function to a larger function. However, this is not always true.

Example 4.1 (Combining Continuous Functions May not be Continuous)

Let us take $\mathbb{R}$ and divide it into $\mathbb{Q}$ and $(\mathbb{R} \setminus \mathbb{Q}) \setminus \{0\}$. Then let us define
$$f : \mathbb{Q} \to \mathbb{R} f(x) = 0 \tag{56}$$
$$g : (\mathbb{R} \setminus \mathbb{Q}) \setminus \{0\} \to \mathbb{R} g(x) = x \tag{57}$$
Then f and g are trivially continuous, but taking the function
$$h(x) \coloneqq \begin{cases} f(x) = 0 & \text{if } x \in \mathbb{Q} \\ g(x) = x & \text{if } x \notin \mathbb{Q} \end{cases} \tag{58}$$
which is not continuous.[a]

[a] Inspired from here.

But not all hope is lost. It does turn out that under certain conditions, we can in fact construct such continuous functions.

Lemma 4.2 (Pasting Lemma, Gluing Lemma)

Let $X = A \cup B$, where A, B are closed in X. Let $f : A \longrightarrow Y$ and $g : B \longrightarrow Y$ be continuous. If
$$f(x) = g(x) \text{ for all } x \in A \cap B \tag{59}$$
Then f and g can be combined to form a continuous function $h : X \longrightarrow Y$, defined
$$h(x) \equiv \begin{cases} f(x) & x \in A \setminus B \\ f(x) \text{ or } g(x) & x \in A \cap B \\ g(x) & x \in B \setminus A \end{cases} \tag{60}$$

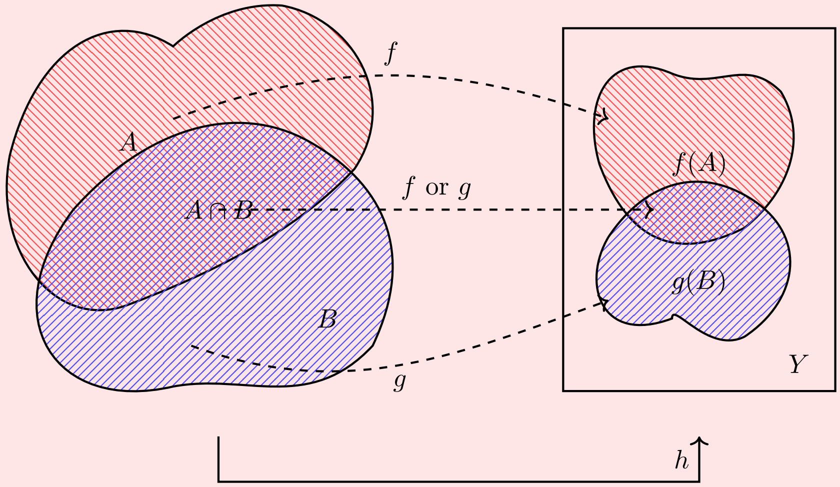


Figure 6: Visual of the pasting lemma.

Consider any set $U \subset Y$. Note that if U is an open set in X that happens to be contained in Y, then we can set $U = U \cap Y$, so U is open in Y. However, we have seen that being open in Y does not necessarily imply that it is open in X.

Example 4.2 (Non-Open Sets may be Open in Subspace)

Let $X = \mathbb{R}$ with the Euclidean topology and let $Y = [0, 1]$.
1. $[0, 1]$ is open in Y but not open in X.
2. Intervals of the form $(a, 1]$ and $[0, b)$ are open in Y but not open (nor closed) in X.

Example 4.3 (Singleton Sets in Subspace Topologies)

Consider $X = \mathbb{R}$ with the lower limit topology with $Y = [0, 1]$. The following
1. $[1/2, 1] = Y \cap [1/2, 2)$, and
2. $\{1\} = Y \cap [1, 2)$

are open in the subspace topology. It turns out that $\{1\}$ is the only singleton set open in Y.

Let's go through a few examples.

Example 4.4 (Closed Unit Interval in $\mathbb{R}$)

The basis for the subspace topology of $[0, 1] \subset \mathbb{R}$ with the Euclidean topology consists of the intervals
1. (a, b) where $0 \leq a < b \leq 1$.
2. $[0, b)$ where $0 < b \leq 1$.
3. $(a, 1]$ where $0 \leq a < 1$.

Example 4.5 (Unit Sphere in $\mathbb{R}^n$)

Let $S^n \subset \mathbb{R}^{n+1}$ be the unit **n-sphere** defined $S^n := \{x \in \mathbb{R}^{n+1} \mid ||x||^2 = 1\}$. When thinking about S^n as a space itself, we use the subspace topology coming from the standard topology of $\mathbb{R}^n$.

Example 4.6 ($S^1 \subset \mathbb{R}^2$)

Let's focus on $n = 1$. For $a < b$, let

$$A_{a,b} = \{(\cos t, \sin t) \mid a < t < b\} \tag{61}$$

Then, we can see that
1. if $b - a > 2\pi$, then $A_{a,b} = S^1$.
2. If $b - a \leq 2\pi$, then $A_{a,b}$ is an "open arc" from $(\cos a, \sin a)$ to $(\cos b, \sin b)$.

Given that we have an equivalence class defined

$$A_{a,b} \sim A_{a+2\pi k, b+2\pi k} \text{ for all } k \in \mathbb{Z} \tag{62}$$

We claim that $\{A_{a,b}\}$ is a basis for the subspace topology of S^1. We can see that the open arc covering the top right quadrant in $\mathbb{R}^2$ is

$$S^1 \cap (0,1)^2 = S^1 \cap B_\infty\big((\frac{1}{2}, \frac{1}{2}), \frac{1}{2}\big) \tag{63}$$

Now let's focus more on metric spaces. Note that if we want to construct topologies of subspaces of metric spaces, there are two ways to do it. It would be quite bad if these resulted in different topologies, but fortunately we have the following theorem.

Theorem 4.2 (Topologies on Subspaces of Metric Spaces Coincide)

Let (X, d_X) be a metric space, with $Y \subset X$. There are 2 ways we can define a topology on Y.
1. Take the metric topology $\mathscr{T}_X$ on X, and then take the subspace topology on Y.
2. Induce a metric $d_Y = d_{X|Y}$ on Y which is a restriction of d_X to Y, and then take the metric topology of it.

We claim that these two constructions give the same topology, as shown in the commutative diagram.

$$\begin{tikzcd} d_X \arrow[r] \arrow[d] & d_Y \arrow[d] \\ \mathscr{T}_X \arrow[r] & \mathscr{T}_Y \end{tikzcd}$$

Figure 7

Proof.

The basis for the subspace topology on Y is

$$\mathscr{B}_1 = \{B_{d_X}(x, r) \cap Y \mid x \in X, r > 0\} \tag{64}$$

and the basis for the (induced) metric topology on Y is

$$\mathscr{B}_2 = \{B_{d_Y}(y, r) \cap Y \mid y \in Y, r > 0\} = \{B_{d_X}(y, r) \cap Y \mid y \in Y, r > 0\} \tag{65}$$

It is immediate that $\mathscr{B}_2 \subset \mathscr{B}_1$ since it goes over all $x \in X$ rather than $y \in Y$. To see why $\mathscr{B}_1 \subset \mathscr{B}_2$, TBD.

Theorem 4.3 (Closures in Subspace Topologies)

Let $A \subset Y \subset X$. Let $\bar{A}$ denote the closure of A in X. Then, the closure of A in Y equals $\bar{A} \cap Y$.

4.3 Box Topology

There are multiple ways to define the box and product topologies, but their construction with basis elements is most simple.

Definition 4.4 (Box Topology)

Given a family of topological spaces $\{(X_\alpha, \mathscr{T}_\alpha)\}_{\alpha \in A}$, the **box topology** on the space $\prod_{\alpha \in A} X_\alpha$ is the topology generated by the basis

$$\mathscr{B} = \left\{ \prod_{\alpha \in A} U_\alpha \mid U_\alpha \in \mathscr{T}_\alpha \right\} \tag{66}$$

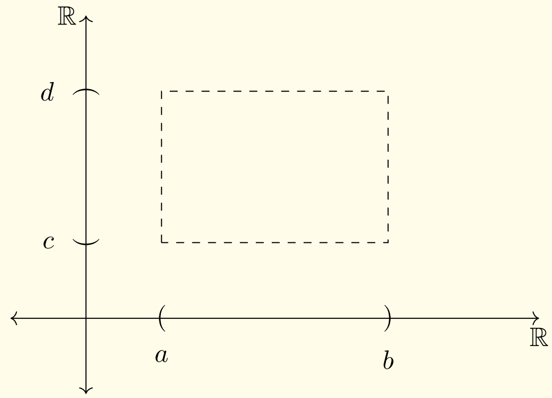


Figure 8: We can visualize the elements of the box topology with the product space $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$, where each $\mathbb{R}$ has an open ball topology. From the visual below, we can see why this is called the "box" topology.

Proof.

It is easy to prove that the box topology indeed satisfies the 3 properties of topologies in general.

4.4 Product Topology

While the box topology may seem quite "intuitive" for the first learner, the box topology however, has serious limitations when extending to infinite Cartesian products of spaces. The main difference between the construction of open sets in the box topology vs the product topology is that the box topology merely describes open sets as direct products of open sets from each coordinate space while the construction of the product topology is completely dependent on the construction of the projection mappings

$$\pi_\beta : \prod_{\alpha \in I} X_\alpha \to X_\beta \tag{67}$$

to be continuous (and nothing more) so that (by definition) the preimages of open sets in X_β under π_β are open sets in $\prod X_\alpha$. Therefore, the construction of the continuous π_β's canonically constructs a basis of open sets in $\prod X_\alpha$.

Definition 4.5 (Product Topology)

Given a family of topological spaces $\{(X_\alpha, \mathscr{T}_\alpha)\}_{\alpha \in A}$, the **product topology** on the space $\prod_{\alpha \in A} X_\alpha$ is defined in the following equivalent ways.

1. It is the initial topology on the product space wrt the family of projections $p_\alpha : \prod_{\alpha \in A} X_\alpha \to X_\alpha$.

2. It is the topology generated by the basis of elements

$$\prod_{\alpha} U_\alpha \tag{68}$$

where U_α is a proper open subset for at most finitely many α's, and $U_\alpha = X_\alpha$ for all other α.

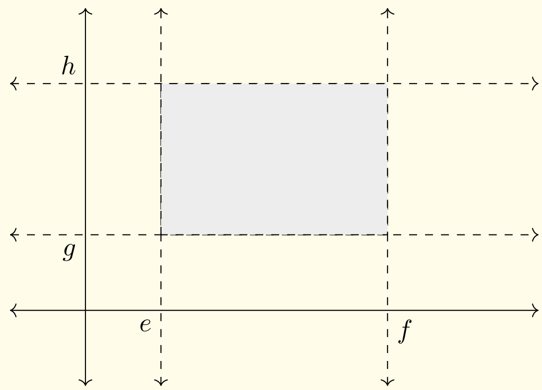


Figure 9: Visually, we can interpret each $\mathscr{S}(U_\beta)$ as a "strip" in the total product space. For example in $\mathbb{R}^2$, there are two "strips" $(e, f) \times \mathbb{R}$ and $\mathbb{R} \times (g, h)$ that intersect. Note that each strip is the preimage of the projection mapping.

We can deduce some conclusions comparing these topologies. First, the product and box topologies are precisely the same if we work in finite Cartesian products of spaces, since any element of the box topology (left hand side) can be expressed as a finite intersection of some open sets (in the right hand side). That is, if $\text{card}\, I < \infty$, then

$$\prod_{\alpha \in I} U_i = \bigcap_{\alpha \in I} \big\{ \prod_{\gamma \in I} W_\gamma \mid W_\gamma = U_\gamma \text{ if } \gamma = \alpha,\, W_\gamma = X_\gamma \text{ if } \gamma \neq \alpha \big\} \tag{69}$$

Secondly, we can see that the box topology is finer than the product topology (strictly finer if working in infinite product spaces).

Example 4.7 ()

The set $(0,1)^{\mathbb{N}} \subset \mathbb{R}^{\mathbb{N}}$ is clearly open in the box topology, but it is considered "too tight" to be in the product topology. However,

$$(0, 1) \times \mathbb{R} \times \mathbb{R} \times \dots \tag{70}$$

is open in the product topology since only one (a finite amount) of the factors is not the whole space.

The following theorem reveals why the product topology is superior than the box topology in product spaces.

Theorem 4.4 (Continuity of Functions Mapped to Product Topology)

Given the function

$$f : A \to \prod_{\alpha \in I} X_\alpha,\ f(a) \equiv \big(f_\alpha(a)\big)_{\alpha \in I} \tag{71}$$

with its component functions $f_\alpha : A \to X_\alpha$. Let $\prod X_\alpha$ have the product topology. Then the function f is continuous if and only if each function f_α is continuous.

Proof.

We prove both directions. Let π_β be the projection of this product onto the βth component space. By construction π_β is continuous $\implies \pi_\beta^{-1}(U_\beta)$ is a basis element of the product topology of $\prod X_\alpha$.

1. $(\rightarrow)$ f is continuous, so $f_\beta \equiv \pi_\beta \circ f$, as the composition of continuous functions, is also continuous.
2. $(\leftarrow)$ Assume that each f_β is continuous. Let there be an open set $U_\beta \subset X_\beta$. Then, the canonical open set π_β^{-1} in the product space $\prod X_\alpha$ is also open. Now, the preimage of $\pi_\beta^{-1}(U_\beta)$ under f is

$$\begin{aligned} f^{-1}\big(\pi_\beta^{-1}(U_\beta)\big) &= (f^{-1} \circ \pi_\beta^{-1})(U_\beta) \\ &= (\pi_\beta \circ f)^{-1}(U_\beta) \\ &= f_\beta^{-1}(U_\beta) \end{aligned}$$

Since f_β is already assumed to be continuous, $f_\beta^{-1}(U_\beta)$ is open in A.

This theorem also works for the box topology only if we are working with finite product spaces. But in general, this theorem fails for the box topology. Consider the following example.

Example 4.8 ()

Let $\mathbb{R}^\omega$ be the countably infinite product of $\mathbb{R}$'s. Let us define the function

$$f : \mathbb{R} \to \mathbb{R}^\omega \tag{72}$$

with coordinate function defined $f_n(t) \equiv t$ for all $n \in \mathbb{N}$. Clearly, each f_n is continuous. Given the box topology, we consider one basis element of $\mathbb{R}^\omega$

$$B = \prod_{i=1}^{\infty} (-\frac{1}{i}, \frac{1}{i}) \tag{73}$$

Assume that f is continuous, that is $f^{-1}(B)$ is open in $\mathbb{R}$. Then, it would contain some finite interval $(-\delta, \delta)$ about 0, meaning that $f\big((-\delta,\delta)\big) \subset B$. This implies that for each $n \in \mathbb{N}$,

$$f_n\big((-\delta,\delta)\big) = (-\delta,\delta) \subset \Big(-\frac{1}{n}, \frac{1}{n}\Big) \tag{74}$$

which contradicts the fact that B is open, since the interval $(-1/n, 1/n)$ converges onto a point 0.

However, there is no useful criterion for the continuity of a mapping $f : X \times Y \longrightarrow A$ even if we have the product topology on $X \times Y$. One might conjecture that this f is continuous if it is continuous in each variable separately, but this is in fact not true.

Theorem 4.5 (Topologies on Products of Metric Spaces Coincide)

Given a metric space

Corollary 4.1 ()

The Euclidean topology on $\mathbb{R}^n$ is equivalent to the product topology of the Euclidean topologies on $\mathbb{R}$.

Lemma 4.3 ()

The addition, subtraction, and multiplication operations are continuous functions from $\mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$, and the quotient operation is a continuous function from $\mathbb{R} \times (\mathbb{R} \setminus \{0\}) \longrightarrow \mathbb{R}$.

Proof.

Standard $\epsilon - \delta$ proof.

Now that we have defined what it means for binary operations to be continuous, we can talk about *topological algebra*, which is the study of algebraic structures such that their algebraic operations and inverses are continuous. One important such concept is a *topological group*, which will be mentioned later.

4.5 Quotient Topologies

Definition 4.6 (Quotient Map)

Let X and Y be topological spaces, and let $p : X \to Y$ be a surjective map. The map p is said to be a **quotient map** if

$$U \text{ is open in } Y \iff p^{-1}(U) \text{ is open in } X \tag{75}$$

Note that the conditions for being a quotient map is stronger then regular continuity. It is sometimes called **strong continuity**. An equivalent condition for p to be a quotient map is to require that given $A \subset Y$,

$$A \text{ closed in } Y \iff p^{-1}(A) \text{ closed in } X \tag{76}$$

This equivalence follows from the fact that

$$f^{-1}(Y \setminus B) = X \setminus f^{-1}(B) \tag{77}$$

Definition 4.7 (Saturation)

A subset $C \subset X$ is **saturated** with respect to the surjective map $p : X \to Y$ if for every $p^{-1}(A)$ (where $A \subset Y$) that intersects C, $p^{-1}(A)$ is completely contained within C. That is,

$$p^{-1}\big(p(C)\big) = C \tag{78}$$

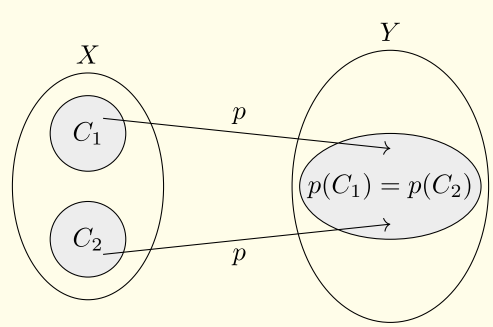


Figure 10: We can see that C_1 and C_2 alone are not saturated, but $C_1 \cup C_2$ is saturated. Visually, for a given set $C \subset X$ to be saturated, there cannot be any points $q \notin C$ such that $q \in p(C)$.

We now introduce an alternative, equivalent definition of quotient maps.

Theorem 4.6 ()

$p : X \to Y$ is a quotient map if and only if p is continuous and p maps saturated open sets of X to open sets of Y (or saturated closed sets of X to closed sets of Y).

Proposition 4.1 ()

If $p : X \to Y$ is a surjective, continuous map that is either open or closed (that is, maps open sets to open sets or closed sets to closed sets), then p is a quotient map.[a]

[a]Note however, that the converse is not true; there exists quotient maps that are neither open nor closed.

Definition 4.8 ()

If X is a space and A is a set and if $p : X \to A$ is a surjective map, then there exists exactly one topology $\mathscr{T}$ on A relative to which p is a quotient map. $\mathscr{T}$ is called the **quotient topology** induced by p.

To construct the quotient topology for the surjective map p, we must make p continuous. Therfore, the topology $\mathscr{T}$ on A is defined by letting it consist of all subsets U of A such that $p^{-1}(U)$ is open in X. This is indeed a topology since

1. $p^{-1}(\emptyset) = \emptyset$ and $p^{-1}(A) = X$
2. $p^{-1}\Big(\bigcup_{\alpha \in J} U_\alpha\Big) = \bigcup_{\alpha \in J} p^{-1}(U_\alpha)$
3. $p^{-1}\Big(\bigcap_{i=1}^n U_i\Big) = \bigcap_{i=1}^n p^{-1}(U_i)$

Example 4.9 ()

Let $p : (\mathbb{R}, \mathscr{T}_\mathbb{R}) \to \mathbb{R}/2\pi\mathbb{R}$. Then, the final topology of $\mathbb{R}/2\pi\mathbb{R}$ would be simply defined

$$\mathscr{T}_{\mathbb{R}/2\pi\mathbb{R}} \equiv \{U \subset \mathbb{R}/2\pi\mathbb{R} \mid U = p(O), O \in \mathscr{T}_\mathbb{R}\} \tag{79}$$

That is, the quotient topology is merely the set of all images of open sets in $\mathbb{R}$ under f. However, if $\mathbb{R}/2\pi\mathbb{R}$ has the discrete topology 2^X, then a single equivalence class, say $[0]$, will get mapped to the collection of points $\{2\pi k \mid k \in \mathbb{Z}\}$, which is clearly not open in $\mathbb{R}$. Note that the final topology (or the quotient topology) is endowed onto the codomain in order to make f continuous (or a quotient mapping).

Proposition 4.2 ()

Given a relation $\sim$ on a topological space $(X, \mathscr{T}_X)$, the quotient topology of the quotient space $X/\sim$, is precisely the final topology on the quotient set with respect to the quotient map $p : X \to X/\sim$. That is,

$$\mathscr{T}_{X/\sim} \equiv \big\{U \subseteq X/\sim \ \mid \{x \in X \mid p(x) \in U\} \in \mathscr{T}_X \big\} \tag{80}$$

which is the topology whose open sets are the subsets of $X/\sim$ that have an open preimage under the surjective map $p : x \mapsto [x]$.

Example 4.10 ()

Let $X \equiv [0,1] \cap [2,3] \subset \mathbb{R}$ and $Yy \equiv [0,2] \subset \mathbb{R}$. Then, we define $p : X \to Y$ as

$$p(x) \equiv \begin{cases} x & x \in [0,1] \\ x-1 & x \in [2,3] \end{cases} \tag{81}$$

p is continuous (under subspace topology of $X \subset \mathbb{R}$), surjective, and closed, meaning that it is a quotient map. However, it is not open, since the image of the open set $[0,1]$ of X is $[0,1]$, which is not open in Y.

Example 4.11 ()

Let $p : \mathbb{R} \to \{a,b,c\}$ be defined as

$$p(x) \equiv \begin{cases} a & x > 0 \\ b & x < 0 \\ c & x = 0 \end{cases} \tag{82}$$

Then, the quotient topology of $\{a,b,c\}$ consists of

$$\emptyset, \{a\}, \{b\}, \{a,b\}, \{a,b,c\} \tag{83}$$

Definition 4.9 (Quotient Space)

Let X be a topological space, and let $\tilde{X}$ be a partition of X into disjoint subsets whose union is X. Let $p : X \to \tilde{X}$ be the surjective map mapping every point $x \in X$ to the subset that it is in. In the quotient topology induced by p, $\tilde{X}$ is called the **quotient space** of X.

To construct a quotient space, we can equivalently define a relation on X. That is, a subset U of $\tilde{X}$ is a collection of equivalence classes, and the set $p^{-1}(U)$ is the union of equivalence classes belonging to U. Therefore, the typical open set of $\tilde{X}$ is a collection of equivalence classes whose union is an open set in X.

Example 4.12 (Construction of a Torus)

Let $X \equiv [0,1] \times [0,1] \subset \mathbb{R}^2$. We define an equivalence relation Y consisting of the equivalence classes

$$\begin{gathered} \big\{\{(x,y)\} \mid 0 < x,y < 1\big\} \cup \big\{\{(x,0),(x,1)\} \mid 0 < x < 1\big\} \cup \\ \big\{\{(0,y),(1,y)\} \mid 0 < y < 1\big\} \cup \big\{\{(0,0),(0,1),(1,0),(1,1)\}\big\} \end{gathered}$$

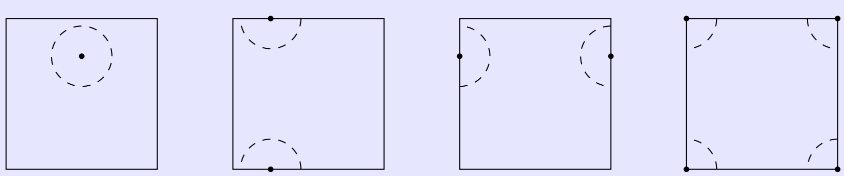


Figure 11: The quotient topology of this quotient space consists of open sets of form.

This quotient space X/Y is homeomorphic to the torus $S^1 \times S^1$, denoted

$$\frac{X}{Y} \cong S^1 \times S^1 \tag{84}$$

We can visualize the construction of the equivalence relation Y as a "gluing" of the rectangle X by its edges and corners.
We can check that this mapping is indeed a quotient map. First, it is clearly surjective. By realizing that individual points on the edge of $[0,1]^2$ are open sets themselves (by the subspace topology), we can prove that this map is indeed open and continuous.

Example 4.13 (Construction of the 2-Sphere)

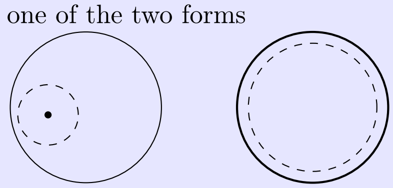
Let X be the closed unit ball

$$X \equiv \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\} \tag{85}$$

and define the equialence classes R as

$$R \equiv \Big\{\{(x,y)\} \mid x^2 + y^2 < 1\Big\} \cup \{S^1\} \tag{86}$$

which will consist of open sets of one of the two forms



Then, this quotient space X/R is homeomorphic to the 2-sphere

$$S^2 \equiv \{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\} \tag{87}$$

Visually, we can imagine the disk being glued together by its sides to continuously form the 2-sphere.

Example 4.14 (Construction of the 1-Sphere)

We will show that

$$\frac{\mathbb{R}}{\mathbb{Z}} \cong S^1 \tag{88}$$

Let us construct the set $(\mathbb{R}, \mathscr{T}_\mathbb{R})$ with paramater t. We define maps

$$p : \mathbb{R} \to \mathbb{R}/\mathbb{Z}, \;\; p(t) \equiv t \text{ (mod 1)}$$
$$q : [R] \to S^1 \subset \mathbb{C}, \;\; g(t) \equiv e^{2\pi i t}$$

We claim that p and q are both quotient mappings. Clearly, p is a quotient mapping. As for q, it it easy to see that it is surjective (but not injective) and continuous ($\mathscr{T}_{S^1}$ has the basis of open intervals on S^1). It is also easy to notice that given an open interval $U \subset S^1$, $q^{-1}(U)$ will be the union of open intervals equally spaced in $\mathbb{R}$. Additionally, given any open interval in $\mathbb{R}$, it maps to an open interval in S^1 (note that S^1 itself is also open). These three conditions imply that q is a quotient map. We now define maps

$$q \circ p^{-1} : \mathbb{R}/\mathbb{Z} \to S^1 \tag{89}$$
$$p \circ q^{-1} : S^1 \to \mathbb{R}/\mathbb{Z} \tag{90}$$

and claim that these maps are homeomorphisms. We can clearly see that the mapping from an open set in $\mathbb{R}/\mathbb{Z}$ to the union of spaced open intervals in $\mathbb{R}$ is an injection, and the mapping from this union of open intervals to the union of open intervals in S^1 is a surjection. The composition of these two mappings clearly defines a bijection. Therefore, $q \circ p^{-1}$ is proven to be a bicontinuous bijective mapping between open sets $U \subset \mathbb{R}/\mathbb{Z}$ and $V \subset S^1 \implies q \circ p^{-1}$ is a homeomorphism.
This result clearly makes sense since

$$\frac{\mathbb{R}}{\mathbb{Z}} \cong \frac{[0,1]}{\sim} \tag{91}$$

where the relation $\sim$ maps every point $x \in (0,1)$ to its own equivalence class and the points $0,1$ to one equivalence class $\{0\}$. Therefore, it is informally said that the quotient space of the real line is a circle.
One may attempt to construct a simpler set by replacing S^1 with the half-open interval $[0,1)$. However, while $[0,1)$ is bijective to $\mathbb{R}/\mathbb{Z}$,

$$\frac{\mathbb{R}}{\mathbb{Z}} \ncong [0,1) \tag{92}$$

That is, the two sets are not homeomorphic because the topologies of $[0,1)$ and $\mathbb{R}/\mathbb{Z}$ are not compatible. For instance, when we attempt to map the open set

$$\left\{[x] \in \mathbb{R}/\mathbb{Z} \mid 0 \leq x \leq \frac{1}{4} \vee x > \frac{1}{2}\right\} \in \mathscr{T}_{\mathbb{R}/\mathbb{Z}} \tag{93}$$

to $\mathscr{T}_{[0,1)}$, it does not return an open set.
Furthermore, this means that

$$S^1 \times S^1 \cong \frac{[0,1]^2}{\sim'} \cong \left(\frac{\mathbb{R}}{\mathbb{Z}}\right)^2 \tag{94}$$

where $\sim'$ is the quotient mapping defined in the previous construction of the torus.

Example 4.15 (Construction of the Cylinder)

Let us define the cylinder as

$$C \equiv \{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1, z \in [0,1]\} \tag{95}$$

Then, we can see that

$$C \cong \frac{[0,1]^2}{\sim} \tag{96}$$

where $\sim$ is the equivalence relation defined by the quotient mapping

$$p\big((x,y)\big) \equiv \begin{cases} \{(x,y)\} & x \neq 0, x \neq 1 \\ \{(0,y),(1,y)\} & x = 0 \text{ or } x = 1 \end{cases} \tag{97}$$

Subspaces do not behave well under quotient maps. That is, if $p : X \to Y$ is a quotient map and A is a subspace of X, then the map $p' : A \to p(A)$ obtained by restricting both the domain and codomain of P need not be a quotient map. Additionally, quotient maps are clearly not homeomorphisms, so topological properties are not preserved.

However, composites of maps do behave nicely.

Proposition 4.3 ()

The composition of two quotient maps is a quotient map.

Proof.

Indeed, the composition of surjective, continuous, and open maps is surjective, continuous, and open.

However, the product of two quotient maps is not necessarily a quotient map. That is, given $p : A \to B$ and $q : C \to D$ are quotient maps, the map

$$p \times q : A \times C \to B \times D, \ (p \times q)(a \times c) \equiv p(a) \times q(c) \tag{98}$$

is not necessarily a quotient map.

Example 4.16 ()

Given $(\mathbb{R}, \mathscr{T}_\mathbb{R})$, let us define the relation $\sim$ determined by the quotient mapping

$$p(x) \equiv \begin{cases} \{x\} & x \not\in \mathbb{Z} \\ \mathbb{Z} & x \in \mathbb{Z} \end{cases} \tag{99}$$

In words, this quotient map maps every integer to the equivalence class $[0]$ and maps every other point to its own class.

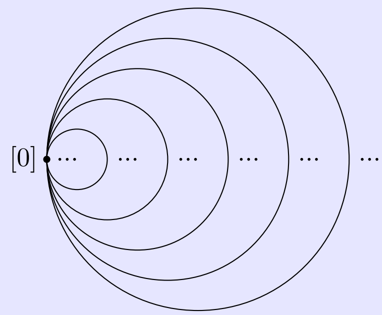


Figure 12: It turns out that every interval $[j, j+1] \subset \mathbb{R}$, $j \in \mathbb{Z}$ will get mapped as a closed loop in $\mathbb{R}/\sim$ beginning and ending with $[0]$, since $j, j+1 \mapsto [0]$. So geometrically, $\mathbb{R}/\sim$ consists of an infinite number of nonintersecting closed loops starting and ending with $[0]$.

This wacky mapping is an example of a quotient mapping that does not preserve topological structure. While it will not be proven here, it is known that $(\mathbb{R}, \mathscr{T}_\mathbb{R})$ is 1st and 2nd countable, but $\mathbb{R}/\sim$ under this relation is not even 1st countable.

We now introduce theorems of continuous maps from quotient spaces inducing other continuous maps.

Theorem 4.7 ()

Let $p : (X, \mathscr{T}_X) \rightarrow (Y, \mathscr{T}_Y)$ be a quotient map. Let $(Z, \mathscr{T}_Z)$ be a topological space and let there exist a function $f : Y \rightarrow Z$. f is continuous if and only if $f \circ p$ is continuous.

$$\begin{tikzcd} X \arrow[d, "p"] \arrow[dr, "f \circ p"] & \\ Y \arrow[r, "f"] & Z \end{tikzcd}$$

Proof.

We prove bidirectionally.

1. $(\rightarrow)$ Assume f is continuous. By definition of the quotient topology, p is continuous $\implies f \circ p$ is continuous.
2. $(\leftarrow)$ Assume $f \circ p$ is continuous $\iff (f \circ p)^{-1}(\omega) \in \mathscr{T}_X$ for every $\omega \in \mathscr{T}_Z \iff p^{-1}\big(f^{-1}(\omega)\big) \in \mathscr{T}_X$, but p is continuous, so $f^{-1}(\omega)$ is open in Y. Therefore, given $\omega \in \mathscr{T}_Z$, $f^{-1}(\omega) \in \mathscr{T}_Y \implies f$ is continuous.

The previous theorem determines continuity of f and $f \circ p$ given a function mapping $Y \rightarrow Z$. The following analogous theorem determines continuity of an induced map f given a function mapping $X \rightarrow Z$.

Theorem 4.8 ()

Let $p : (X, \mathscr{T}_X) \to (Y, \mathscr{T}_Y)$ be a quotient map. Let Z be a space and let $g : X \to Z$ be a map such that g is constant on the elements x of each equivalence class induced by p. That is, if x_1 and x_2 are in the same equivalence class induced by p, i.e.

$$p(x_1) = p(x_2) \tag{100}$$

then $g(x_1) = g(x_2)$. If g is continuous, then g induces a continuous map $f : X \to Z$ such that $g = f \circ p$.

$$\begin{tikzcd} X \arrow[d, "p"] \arrow[dr, "g=f\circ p"] & \\ Y \arrow[r, "f"] & Z \end{tikzcd}$$

Figure 13: The theorem states that the diagram commutes.

5 Connectedness

5.1 Connected Spaces

Definition 5.1 (Separation)

Let X be a topological space. A **separation** of X is a pair U, V of disjoint nonempty open subsets of X whose union is X. The space X is said to be **connected** if it satisfies the equivalent definitions

1. there does not exist a separation of X.
2. the only subsets of X that are clopen in X are the empty set and X itself.

Figure 14: Two examples of spaces $X = A \sqcup B$ that are not connected. In the right, A and B overlap in their boundary but are not connected since they are open.

Connectedness is clearly a topological property since it is completely defined in terms of the collection of open sets in X. Since homeomorphisms preserves topological properties, we can say that if X is connected, every space homeomorphic to X is also connected.

Example 5.1 (Separation of a Rectangle)

The space $Y = (0,1) \times (0,1) \cup (1,2) \times (0,1) \subset \mathbb{R}^2$ has the clear separation

$$(0,1) \times (0,1) \text{ and } (1,2) \times (0,1) \tag{101}$$

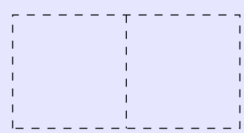


Figure 15: We can visualize the separation of Y as such.

Note that the dashed line is not in Y. Even though the dashed line contains limit points of both the left and right subset of Y, this does not matter.

Example 5.2 ()

Let X denote a two-point space in the indiscrete topology. Clearly, there is no separation of X, so X is connected.

Example 5.3 ()

Let Y denote the subspace $[-1,0) \cup (0,1]$ of $\mathbb{R}$. Each of the sets $[-1,0)$ and $(0,1]$ is nonempty and open in Y (but not in $\mathbb{R}$), so they form a separation of Y. Also, note that neither of these sets contains a limit point of the other (even though they have a common limit point 0).

Example 5.4 ()

$[-1, 1]$, the subspace of $\mathbb{R}$, has no separation, so it is connected.

Example 5.5 ()

The rationals $\mathbb{Q} \subset \mathbb{R}$ are not connected since given any irrational number a, we can write Y as the union of sets

$$Y \cap (-\infty, a),\ Y \cap (a, +\infty) \tag{102}$$

which are open in the subspace topology.

Lemma 5.1 ()

If the sets C and D form a separation of X, and if Y is a connected subset of X, then Y lies entirely within either C or D.

Proof.

Trivial. Easy to visualize.

Theorem 5.1 ()

The union of a collection of connected sets that have a point in common is connected.

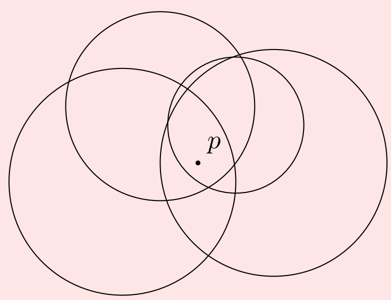


Figure 16: We can visualize this by thinking of overlapping balls.

Proof.

Let $\{A_\alpha\}$ be a collection of connected subsets of a space X, and let

$$p \in \bigcap A_\alpha \tag{103}$$

Then, we claim that

$$Y \equiv \bigcup A_\alpha \tag{104}$$

is connected. Assume Y is not connected, that is, there exists $Y = C \cup D$ as a separation of Y. Then, $p \in C$ or $p \in D$. Without loss of generality, suppose $p \in C$. Since each A_α is connected, it must lie entirely within C (by the previous lemma, since it contains the point $p \in C$) $\implies D = \emptyset$, a contradiction that D must be nonempty.

Theorem 5.2 ()

Let A be a connected subset of X. If $A \subset B \subset \bar{A}$, then B is also connected.

Proof.

Assume $B = C \cup D$ is a separation of $B \implies A$ must lie entirely within C or D. Without loss of generality, suppose $A \subset C$, which implies that $\bar{A} \subset \bar{C}$. Since $\bar{C}$ and D are disjoint, B cannot intersect $D \implies D = \emptyset$, a contradiction. Therefore, there exists no separation of B.

Theorem 5.3 ()

The image of a connected space under a continuous map is connected.

Proof.

Let $f : X \longrightarrow Y$ be a continuous map, and let X be connected. We wish to prove that the image set $Z = f(X)$ is also connected. Let us denote the restriction of f to Z as

$$\tilde{f} : X \longrightarrow Z \tag{105}$$

which is continuous and surjective. We prove by contradiction. Assume that $Z = A \cup B$ is a separation of Z into 2 disjoint nonempty open sets. Then, $\tilde{f}^{-1}(A)$ and $\tilde{f}^{-1}(B)$ are disjoint open sets whose union is $X \implies \tilde{f}^{-1}(A) \cup \tilde{f}^{-1}(B)$ form a separation of X. This contradicts the hypothesis that X is connected $\implies Z$ is connected.

Theorem 5.4 ()

Given connected topological spaces X_α with $\alpha \in J$, the Cartesian products of them is connected. That is,

$$\prod_{\alpha \in J} X_\alpha \tag{106}$$

with the product topology is connected. If J is infinite, then the product space is not necessarily connected under the box topology.

Definition 5.2 (Linear Continuum)

A simply ordered set L having more than one element is called a **linear continuum** if the following hold.

1. L has the least upper bound property.
2. If $x < y$, then there exists z such that $x < z < y$

A classic example of the linear continuum is the real number line and every set homeomorphic to it.

Theorem 5.5 ()

If L is a linear continuum in the order topology, then L is connected and so it every interval and ray in L.

Corollary 5.1 ()

$\mathbb{R}$ is connected, along with every interval and ray in $\mathbb{R}$.

Theorem 5.6 (Intermediate Value Theorem)

Let $f : X \longrightarrow Y$ be a continuous map of the connected space X into the ordered set Y, with the order topology. Given $a, b \in X$ and $r \in Y$ such that $f(a) < r < f(b)$, then there exists a point $c \in X$ such that $f(c) = r$.

Proof.

Assuming the hypothesis, the sets

$$A \equiv f(X) \cap (-\infty, r),\ B \equiv f(X) \cap (r, +\infty) \tag{107}$$

are disjoint. They are also nonempty since

$$f(a) \in A,\ f(b) \in B \tag{108}$$

A and B are open since they are the intersection of open sets. Now, assume that there exists no point $c \in X$ such that $f(c) = r$. Then,

$$f(X) = A \cup B \tag{109}$$

would define a separation of X, contradicting the fact that the image of a connected space under a continuous map must be connected. Therefore, c exists.

5.2 Path Connectedness

Definition 5.3 (Path Connectedness)

Given points x and y of the space x, a **path** in X from x to y is a continuous map $f : [a, b] \longrightarrow X$ of some closed interval in $\mathbb{R}$ to X such that $f(a) = x$ and $f(b) = y$. A space X is said to be **path connected** if every pair of points of X can be joined by a path in X.

Proposition 5.1 (Path Connectedness implies Connectedness)

X is path connected $\implies$ X is connected.

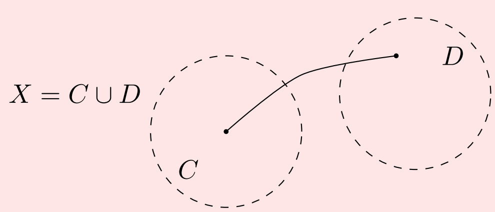


Figure 17

Proof.

X not connected implies that there exists disjoint open subsets C, D such that $C \cup D = X$. Assume that X is path connected, i.e. there exists a continuous function $g : [0, 1] \longrightarrow X$. Then the preimage of C and D in X must be open sets $g^{-1}(C), g^{-1}(D) \subset [0, 1]$ such that $g^{-1}(C) \cup g^{-1}(D) = [0, 1]$. But this isn't possible since $[0, 1]$ is connected, so by contradiction, X is not path connected. The contrapositive of this statement results in the proposition.

However, note that X connected $\not\Longrightarrow$ X path connected. Note the following example.

Example 5.6 (Connected but Not Path Connected)

Given the following function

$$f : [0,1] \longrightarrow [-1,1],\ f(x) = \sin \frac{1}{x} \tag{110}$$

$[-1, 1]$ is connected, but not path connected since the path oscillates infinitely many times as it approaches 0 from both -1 and 1.

The concept of homotopies is dealt with in algebraic topology, but it is worthwhile to mention it now.

Definition 5.4 (Homotopy)

Two continuous paths from x to y in topological space X is **homotopic** if one can be continuously "deformed" into the other, such a deformation being the **homotopy** between two functions. The set of linearly homotopic paths form a relation, and thus **homotopy classes** can be further defined.

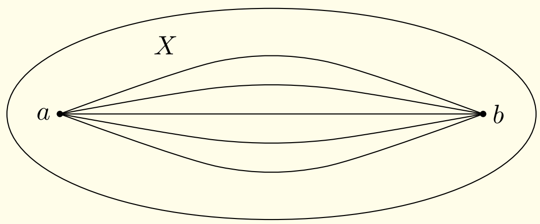


Figure 18: Visually, the set of all the curves in the space X as shown are in a single homotopy class.

It is clear that the space X consists of a single homotopy class of curves from a to b. However, a space may have an infinite number of such classes.

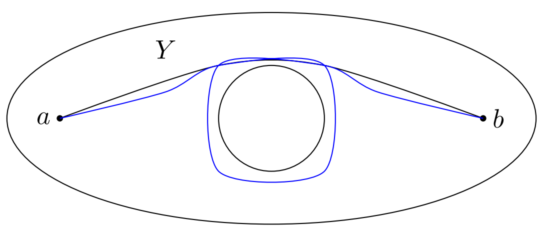


Figure 19: Let us define the space $Y \equiv X \setminus C$ where C is a circular region in X. Then, Y has an infinite number of homotopy classes. We show two curves, that are in two different homotopy classes.

Definition 5.5 (Simply Connected Set)

A **simply connected set** is a set such that all paths between any two given points are homotopic. That is, a simply connected set has one homotopy class.

5.3 Components and Path Components

Definition 5.6 (Connected Components)

Given X, define an equivalance relation on X by setting $x \sim y$ if there is a connected subset of X containing both x and y. The equivalence classes are called the **components**, or **connected components**, of X.

Theorem 5.7 ()

The components of X are connected disjoint subsets of X whose union is X, such that each connected subset of X intersects only one of them.

Proof.

Trivial.

Definition 5.7 (Path Components)

We can define another equivalence relation on the space X by defining $x \sim y$ if there is a path in X from x to y. The equivalence classes are called the **path components** of X. It can be easily shown that this is an equivalence relation.

Theorem 5.8 ()

The path components of X are path connected disjoint subsets of X whose union is X, such that each path connected subset of X intersects only one of them.

Proof.

Trivial.

The property of local connectedness is also important for a space to possess. Roughly speaking, local connectivity means that each point has "arbitrarily small" neighborhoods that are connected.

Definition 5.8 (Locally Connected at a Point)

A space X is said to be **locally connected at** x if for every neighborhood U of x, there is a connected open neighborhood V of x contained in U. If X is locally connected at all of its points, then X is simply said to be **locally connected**.

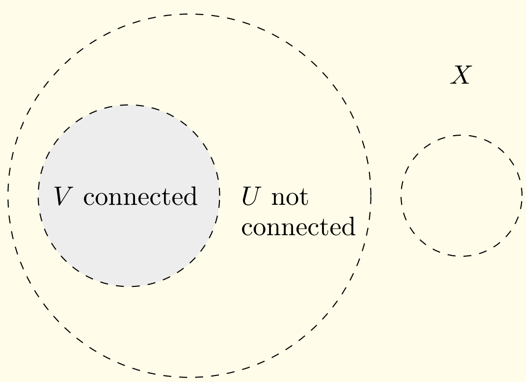


Figure 20: Visually, in the space X, let U be the union of the two open balls shown below. U is clearly open, but not necessarily connected. However, we can form a neighborhood V of x contained in U such that V is connected.

Equivalently, X is locally connected if there exists a basis for X consisting of connected sets. Local connectedness and connectedness of a space are independent of each other.

Definition 5.9 (Locally Path Connected at a Point)

A space X is **locally path connected at** x if for every neighborhood U of x, there is a path connected neighborhood V of x completely contained in U. If X is locally path connected at each of its points, then it is simply said to be **locally path connected**. We can visualize this condition similarly as that of local connectedness.

Theorem 5.9 ()

A space X is locally connected if and only if for every open set U of X, each component of U is open in X.

Proof.

We prove bidirectionally.

1. $(\rightarrow)$ Suppose that X is locally connected. Let U be an open set of X and let C be a component of U. If x is any point in C, by definition of local connectedness, there exists a connected neighborhood V of x fully contained in U. Since V is connected, it must additionally lie completely within $C \implies C$ is open in X.
2. $(\leftarrow)$ Suppose that the components of open sets in X are open. Given a point $x \in X$ and neighborhood U of x, let C be the component of U containing x, which means that C is connected. By hypothesis, the components of open sets are alsvo open, so C is also open. Since an open, connected set C exists for all $x \in X$, X is locally connected.

Theorem 5.10 ()

A space X is locally path connected if and only if for every open set U of X, each path component of U is open in X.

Theorem 5.11 ()

If X is a topological space, each path component of X lies in a component of X. If X is locally path connected, then the components and the path components of X are the same.

6 Compactness

Definition 6.1 (Covers)

A collection $\mathscr{C}$ of subsets of a space X is said to **cover** X, or to be a **covering** of X, if the union of the elements of $\mathscr{C}$ is equal to X. It is called an **open covering** of X if its elements are open subsets of X.

Definition 6.2 (Compactness)

A space X is said to be **compact** if every open covering of X contains a finite subcovering (i.e. a finite collection of subcovers) of X. It may be better to think of compactness as such: If you can find any infinite open covering of the space, then it is not compact.

Lemma 6.1 ()

Let Y be a subspace of X. Then Y is compact if and only if every covering of Y by sets open in X contains a finite subcollection covering Y.

Example 6.1 (Open Square is Not Compact)

The subset $Y \equiv (0,1) \times (0,1) \subset \mathbb{R}^2$ is not compact. That is, we can choose to cover the subspace by the finite union of open sets.

$$[0,1]^2 \subset \bigcup_{k=0}^{\infty} \Big(\frac{2^k - 1}{2^k}, \frac{2^{k+1}-1}{2^{k+1}} \Big) \times (0,1) \tag{111}$$

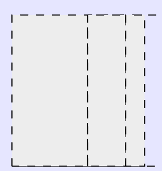


Figure 21: We show the first three elements of the infinite union that covers the open square.

Theorem 6.1 ()

Every closed subset of a compact space is compact.

Proof.

This proof is quite trivial. Let Y be a closed subset of compact space X. Given a covering $\mathcal{C}$ of Y by sets open in X, let us form an open covering $\mathscr{B}$ of X by adjoining to $\mathcal{C}$ the single open set $X \setminus Y$. Then, we an see that both $\mathscr{B}$ and $\mathcal{C} \cup (X \setminus Y)$ covers X.

$$\mathscr{B} = \mathcal{C} \cup (X \setminus Y) \tag{112}$$

Since $\mathscr{B}$ is finite, the right hand side must also be expressible as a finite union. Looking through $\mathscr{B}$, we can throw away all the open sets that are entirely in $X \setminus Y$. What remains is a finite covering of Y.

Theorem 6.2 ()

Every compact subset of a Hausdorff space is closed.

Proof.

Let Y be a compact subset of the Hausdorff Space X. We claim that $X \setminus Y$ is open. Let $x \in X \setminus Y$. Then, for each point $y_i \in Y$, we can choose disjoint neighborhoods U_i of x and V_i of y_i (using the Hausdorff condition). The collection

$$\{V_i \mid y_i \in Y\} \tag{113}$$

is an open covering Y. Since Y is compact, there must exist a finite number of open sets $V_1, V_2, \ldots, V_n$ covering Y. Therefore,

$$\bigcup_{i=1}^{n} V_i \tag{114}$$

contains Y and is disjoint from the intersection of open neighborhoods of x

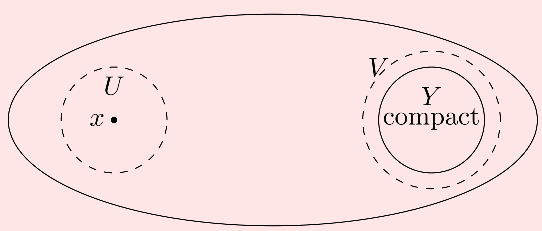
$$U \equiv \bigcap_{i=1}^{n} U_i \tag{115}$$

Therfore, U is an open neighborhood of x_0, disjoint from $Y \implies X \setminus Y$ is open $\implies Y$ is closed.

This results gives the following lemma.

Lemma 6.2 ()

If Y is a compact subset of a Hausdorff space X and x is not in Y, then there exist disjoint open sets U and V of X containing x and Y, respectively.



Theorem 6.3 ()

The image of a compact space under a continuous map is compact.

Proof.

Let $f : X \to Y$ be continuous, and let X be compact. Let $\mathcal{C}$ be a covering of the set $f(X)$ by sets open in Y. Then, the preimage of these sets is the collection

$$\{f^{-1}(\mathcal{A}) \mid \mathcal{A} \in \mathcal{C}\} \tag{116}$$

which clearly covers X. But since X is compact, a finite number of them, say

$$f^{-1}(\mathcal{A}_1), f^{-1}(\mathcal{A}_2), \ldots, f^{-1}(\mathcal{A}_n) \tag{117}$$

covers $X \implies \mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n$ covers $f(X)$.

Theorem 6.4 ()

Let $f : X \to Y$ be a bijective continuous function. If X is compact and Y is Hausdorff, then f is a homeomorphism.

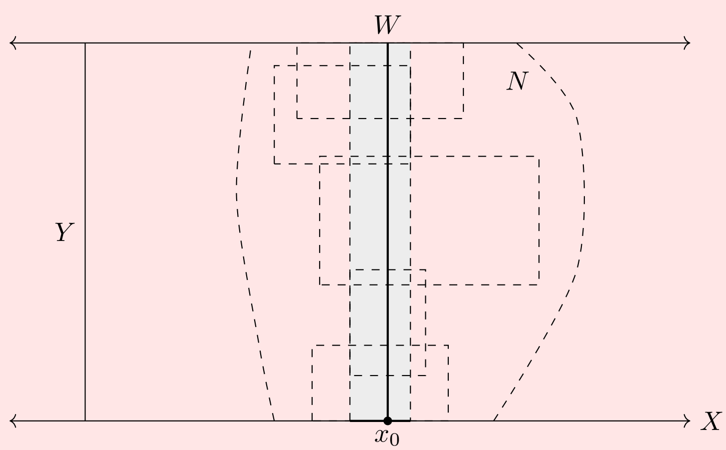
Proof.

It suffices to prove that f is an open or closed mapping. We shall show that f is the latter. Let U be closed in X. By the previous theorems, U is compact $\implies f(U)$ is compact in Hausdorff $Y \implies f(U)$ is closed. Therefore, f is closed.

We now introduce a useful lemma that will come around in many future cases.

Lemma 6.3 (Tube Lemma)

Consider the product space $X \times Y$, where Y is compact. If N is an open set $X \times Y$ containing the slice $x_0 \times Y$ of $X \times Y$, then N contains some tube $W \times Y$ about $x_0 \times Y$, where W is a neighborhood of x_0 in X.



Proof.

Let us cover $x_0 \times Y$ by basis elements $U \times V$ (for the topology of $X \times Y$) lying in N. The space x_0 is compact since it is homeomorphic to $Y \implies$ we can cover $x_0 \times Y$ by finitely such basis elements

$$U_1 \times V_1, U_2 \times V_2, \ldots, U_n \times V_n \tag{118}$$

Without loss of generality, we can assume that each $U_i \times V_i$ has a nontrivial intersection with $x_0 \times Y$, since otherwise, it would be superfluous. Now, we define the intersection of all the open neighborhoods of x_0 in X of the basis elements $U_i \times V_i$. That is, let

$$W \equiv \bigcup_{i=1}^{n} U_i \tag{119}$$

As an intersection of open sets, W is also open containing x_0. With this well-defined tube $W \times Y$, we claim that it is entirely contained within N. That is, given a point $x \times y \in W \times Y$, consider the corresponding point $x_0 \times y$ that is the image of the projection of $x \times y$ onto $x_0 \times Y$. Clearly, $x_0 \times y$ belongs to some $U_k \times V_k$ (for some k) $\implies y \in V_k$. Since $x \in W$, x is clearly in U_k, meaning that $x \times y \in U_k \times V_k \subset N$, as desired.

Theorem 6.5 ()

The product of finitely many compact spaces is compact.

Proof.

Using induction, it suffices to prove that the product of 2 compact spaces is compact. Let X and Y be compact spaces. By the tube lemma, for each $x \in X$, there exists a neighborhood W_x of x such that the tube $W_x \times Y$ can be covered with finitely (by compactness of Y) many open sets in $X \times Y$. The collection of all neighborhoods W_x is an open covering of X. By compactness of X, there exists a finite subcollection

$$W_1, W_2, \ldots, W_k \tag{120}$$

covering X. The finite union of the tubes

$$\bigcup_{i=1}^{k} W_i \times Y \tag{121}$$

clearly covers $X \times Y$, meaning that $X \times Y$ is compact.

Definition 6.3 (Finite Intersection Condition)

A collection $\mathcal{C}$ of subsets of X is said to satisfy the **finite intersection condition** if for every finite subcollection

$$\{\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n\} \tag{122}$$

of $\mathcal{C}$, the intersection

$$\bigcap_{i=1}^{n} \mathcal{C}_i \tag{123}$$

is nonempty.

Clearly, the empty sets cannot below to any collection with the finite intersection property. Additionally, the condition is trivially satisfied if the intersection over the entire collection is non-empty or if the collection is nested. However, here is one example that does satisfy the finite intersection condition.

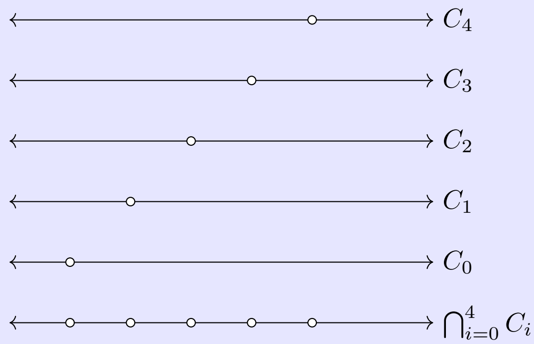
Example 6.2 ()

Let $X = (0, 1)$ and for each positive integer i, X_i is the set of elements of X having a decimal expansion with digit 0 in the ith decimal place. Then, any finite intersection of X_i's is nonempty, but the intersection of all X_i for $i \in \mathbb{N}$ is empty, since no element of $(0, 1)$ has all zero digits.

Here is an analogous example to the previous one.

Example 6.3 ()

In the space $\mathbb{R}$, let us define $C_i \equiv \mathbb{R} \setminus \{i\}$. That is, C_i is $\mathbb{R}$ missing a point at i. Then, the collection of all C_i's does satisfy the finite intersection condition. We show below the finite intersection of the five subsets C_0, C_1, C_2, C_3, C_4.



Theorem 6.6 ()

Let X be a topological space. Then x is compact if and only if for any collection $\mathcal{C}$ of closed sets in X satisfying the finite intersection condition, the intersection

$$\bigcap_{C \in \mathcal{C}} C \tag{124}$$

of all the elements of $\mathcal{C}$ is nonempty.

Proof.

Given a collection S fo subets of X, let

$$\mathcal{C} \equiv \{X \setminus A \mid A \in S\} \tag{125}$$

be the collection of their complements. Then, the following statements hold

1. S is a collection of open sets if and only if $\mathcal{C}$ is a collection of closed sets.
2. The collection S covers X if and only if the intersection

$$\bigcap_{C \in \mathcal{C}} C \tag{126}$$

   of all the elements of $\mathcal{C}$ is empty.
3. The finite subcollection $\{A_1, A_2, \ldots, A_n\}$ of S covers X if and only if the intersection of the corresponding elements $C_i \equiv X \setminus A$ of $\mathcal{C}$ is empty.

Clearly, (1) is trivial, and (2) and (3) follows from DeMorgan's Law.

$$X \setminus \bigcup_{\alpha \in J} A_\alpha = \bigcap_{\alpha \in J} (X \setminus A_\alpha) \tag{127}$$

Using statement 3, the existence of a finite collection of closed sets C in X satisfying the finite intersection condition is equivalent to its complements (which are open sets) covering X, which is precisely the definition of compactness.

Clearly, the previous example in the real line $\mathbb{R}$ shows that $\mathbb{R}$ is indeed not compact.

Corollary 6.1 ()

The space X is compact if and only if every collection $\mathscr{C}$ of subsets of X satisfying the finite intersection condition, the intersection

$$\bigcap_{A \in \mathscr{C}} \bar{A} \tag{128}$$

of their closures is nonempty.

6.1 Compact Sets of the Real Line

In order to construct new compact spaces from old ones, we must prove compactness for a number of fundamental spaces. The real number line is a good starting point, and in order to prove that every closed interval in $\mathbb{R}$ is compact, we only need the following theorem.

Theorem 6.7 ()

Let X be a simply ordered set having the least upper bound property (That is, every nonempty subset of X with an upper bound has a least upper bound). Then, in the order topology, every closed interval in X is compact.

Corollary 6.2 ()

Every closed interval in $\mathbb{R}$ is compact.

Theorem 6.8 (Heine-Borel Theorem)

A subset A of $\mathbb{R}^n$ is compact if and only if it is closed and bounded in the Euclidean metric d or the square metric p.

Example 6.4 ()

The unit sphere S^{n-1} and the closed ball B^n in $\mathbb{R}^n$ are compact since they are closed and bounded. The set

$$A \equiv \{(x, \frac{1}{x}) \mid 0 < x \leq 1\} \tag{129}$$

is closed in $\mathbb{R}^2$, but is not compact since it is not bounded. The set

$$S \equiv \{(x, \sin\frac{1}{x}) \mid 0 < x \leq 1\} \tag{130}$$

is bounded in $\mathbb{R}^2$, but it is not compact since it is not closed.

Theorem 6.9 (Maximum, Minimum Value Theorem)

Let $f : X \to Y$ be continuous, where Y is an ordered set in the order topology. If X is compact, then there exists points c and d in X such that $f(c) \leq f(x) \leq f(d)$ for every $x \in X$. That is, f has a maximum and a minimum at the values d and c, respectively.

6.2 Limit Point Compactness

We now state different, weaker types of compactness.

Definition 6.4 (Sequentially Compact)

A space X is said to be **sequentially compact** if every sequence of points in X has a subsequence that converges to a point $x \in X$.

Definition 6.5 (Countably Compact)

A space X is said to be **countably compact** if every countably open cover has a finite subcover.

Definition 6.6 (Limit Point Compactness)

A space X is said to be **limit point compact** if every infinite subset of X has a limit point.

Theorem 6.10 ()

Compactness $\implies$ limit point compactness.

Lemma 6.4 (Lebesgue Number Lemma)

Let $\mathscr{C}$ be an open covering of the metric space (X, d). If X is compact, then there is a $\delta > 0$ such that for each subset of X having diameter than δ, there exists an element of $\mathscr{C}$ containing it. This number δ is called a **Lebesgue number** for the covering $\mathscr{C}$.

Another theorem of calculus, suitably generalized to topological spaces, is stated.

Theorem 6.11 (Uniform Continuity Theorem)

Let $f : X \to Y$ be a continuous map of the compact metric space (X, d_X) to the metric space $(Y, d_Y$. Then, f is uniformly continuous. That is, given $\epsilon > 0$, there exists a $\delta > 0$ such that for any two points $x_1, x_2 \in X$,

$$d_X(x_1, x_2) < \delta \implies d_Y\big(f(x_1), f(x_2)\big) < \epsilon \tag{131}$$

Theorem 6.12 (Equivalence of Compactness in Metrizable Spaces)

Let $(X, \mathscr{T})$ be a metrizable space. Then the following are equivalent:

1. X is compact.
2. X is limit point compact.
3. X is sequentially compact.
4. X is countably compact.

6.3 Local Compactness

Definition 6.7 (Locally Compact)

A space X is said to be **locally compact** at x if there is some compact subset C of X that contains a neighborhood of x. If X is locally compact at each of its points, X is simply to be **locally compact**.

Example 6.5 ()

The real line $\mathbb{R}$ is locally compact since any point $x \in \mathbb{R}$ lies within a certain closed interval $[a, b]$, which is compact. The subspace $\mathbb{Q}$ is not locally compact.

Two of the most well-behaved classes of spcaes to deal with are metrizable spaces and compact Hausdorff spaces. If a given space is not one of these types, the next best thing one can hope for is that it is a subspace of one of these spaces. Clearly, a subspace of a metrizable space is itself metrizable, so one does not get any new spaces this way. However, a subspace of a compact Hausdorff space need not be compact. This leads to the question: Under what conditions is a space homeomorphic to a subspace of a compact Hausdorff space?

Definition 6.8 (Compactification)

Let X be a locally compact Hausdorff space. Take some object outside X, denoted by the symbol ∞, and adjoin it to X, forming the set

$$Y = X \cup \{\infty\} \tag{132}$$

Topologize Y by defining the collection of open sets in Y to be the sets of the following types:

1. U, where U is an open subset of X.
2. $Y \setminus C$, where C is a compact subset of X.

Then, this space Y is called the **one-point compactification of** X. This is in some sense the minimal compactification of X.

We briefly show that this set of open sets on Y is indeed a topology. First, $\emptyset$ is of type 1 and Y itself is of type 2. Given U_i of type 1 and $Y \setminus C_i$ of type 2, we have the intersections of two sets

$$\begin{aligned} & U_1 \cap U_2 && \text{is type 1} \\ & (Y \setminus C_1) \cap (Y \setminus C_2) = Y \setminus (C_1 \cup C_2) && \text{is type 2} \\ & U_1 \cap (Y \setminus C_1) = U_1 \cap (X \setminus C_1) && \text{is type 1} \end{aligned}$$

along with the arbitrary union of sets

$$\begin{aligned} & \bigcup U_\alpha = U && \text{is type 1} \\ & \bigcup (Y \setminus C_\beta) = Y \setminus (\bigcap C_\beta) = Y \setminus C && \text{is type 2} \\ & (\bigcup U_\alpha) \cup (\bigcup (Y \setminus C_\beta)) = U \cup (Y \setminus C) = Y \setminus (C \setminus U) && \text{is type 2} \end{aligned}$$

We now present some properties of one-point compactifications.

Theorem 6.13 ()

Let X be a locally compact Hausdorff space which is not compact, and let Y be a one-point compactification of X. Then Y is a compact Hausdorff space. Additionally, since $X \subset Y$ with $Y \setminus X$ consisting of a single point, $\bar{X} = Y$.

Example 6.6 (Extended Real Number Line)

The one-point compactification of the real line $\mathbb{R}$ is homeomorphic to the circle S^1. That is,

$$\mathbb{R} \cup \{\infty\} \cong S^1 \tag{133}$$

$\mathbb{R} \cup \{\infty\}$ is called the **extended real number line**.

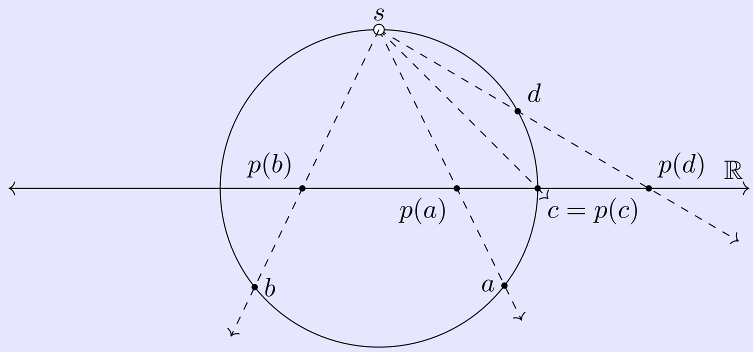


Figure 22: We can visualize this homeomorphism by visualizing the stereographic projection $p : S^1 \setminus \{s\} \to \mathbb{R}$.

Example 6.7 (2-Sphere)

The one point-compactification of the real plane $\mathbb{R}^2$ is homeomorphic to the 2-sphere S^2. That is,

$$\mathbb{R}^2 \cup \{\infty\} \cong S^2 \tag{134}$$

Lemma 6.5 ()

Let X be a Hausdorff space. Then X is locally compact at x if and only if for every neighborhood U of x, there is a neighborhood V of x such that $\bar{V}$ is compact and $\bar{V} \subset U$.

Corollary 6.3 ()

Let X be a locally compact Hausdorff space with Y a subspace of X. If Y is closed in X or open in X, then Y is locally compact.

Corollary 6.4 ()

A space X is homeomorphic to an open subset of a compact Hausdorff space if and only if X is locally compact and Hausdorff.

7 Countability

Definition 7.1 (1st-Countability)

A space X is said to have a countable basis at x if there exists a sequence $N_1, N_2, ...$ of open neighborhoods of x such that for any neighborhood N of x, there exists an integer i such that $N_i \in N$. That is, the countable basis of neighborhoods get arbitrarily small around x. A space X satisfying this axiom at every point $x \in X$ is said to be a **first-countable space**.

In particular, every metric space is first-countable, since we can construct the sequence of open balls $B(x, \frac{1}{n})$ for each $n \in \mathbb{N}$ which forms a countable basis at x. We now generalize some previous statements about metric spaces to statements about first-countable spaces.

Theorem 7.1 ()

Let X be a space satisfying the first countability axiom, and let $A \subset X$.

1. $x \in \bar{A}$ if and only if there exists a sequence of points in A converging to x.
2. The function $f : X \longrightarrow Y$ is continuous if and only if for every convergent sequence $(x_n) \to x$ in X, the sequence $\big(f(x_n)\big) \to f(x)$ in Y.

Definition 7.2 (2nd-Countability)

A topological space X is said to satisfy the **second countability axiom** if X has a countable basis for its topology.

Proposition 7.1 ()

Second countability implies first countability.

Proof.

If $\mathscr{B}$ is a countable basis for the topology of X, then the subset of $\mathscr{B}$ consisting of elements containing the point x is a countable basis at x.

Example 7.1 ()

The real line $\mathbb{R}$ is second countable. We can contrust a countable basis as the set of all open intervals (a, b) with rational end points. Likewise, $\mathbb{R}^n$ has a countable basis, which is the collection of all products of intervals having rational end points. Additionally, $\mathbb{R}^\omega$ has a countable basis. It is the collection of all products

$$\prod_{n \in \mathbb{N}} U_n \tag{135}$$

where U_n is an open interval with rational endpoints for finitely many values of n and $U_n = \mathbb{R}$ for all other values of n.

Example 7.2 ()

In the uniform topology, $\mathbb{R}^\omega$ satisfies the first countability axiom (since it is metrizable).

Theorem 7.2 ()

A subspace of a first and second countable space is first and second countable, respectively. A countable product of first and second countable space is first and second countable, respectively.

Theorem 7.3 ()

A subset A of space X is said to be **dense** in X if $\bar{A} = X$.

Theorem 7.4 ()

Suppose that X has a countable basis. Then,
1. Every open cover of X has a countable subcover.
2. There exists a countable subset of X which is dense in X.

Proof.

Listed.
1. Let $\mathscr{B} = \{B_n\}_{n \in \mathbb{N}}$ be a countable basis for X, and let $\mathscr{A}$ be an open covering of X. For each integer $n \in \mathbb{N}$, chose an element $A_n \in \mathscr{A}$ containing the basis element B_n. The newly formed collection $\mathscr{A}'$ of all the A_n's is countable since it is indexed according to a subset of $\mathbb{N}$. Furthermore, since $B_n \subset A_n$ for every B_n in the basis, the A_n clearly covers X.
2. From each nonempty basis element B_n, we choose a point x_n. The set
$$D \equiv \{x_n \mid n \in \mathbb{N}\} \tag{136}$$
is dense in X, since given any $x \in X$, every open basis element B_x about x intersects D. That is,
$$B_x \cap D \neq \emptyset \tag{137}$$
meaning that the set of points x_n get arbitrarily close to x.

Definition 7.3 (Lindelof Space)

A space for which every open covering contains a countable subcovering is called a **Lindelof space**.

8 Separation

Separability comes in different levels.[2] We briefly define some weaker forms of separability.

Definition 8.1 (t_0-Separability)

A topological space X is said to be t_0**-separable** if for each pair of distinct points $x, y \in X$, there exists a neighborhood U that contains x but not y, or a U that contains y but not x.

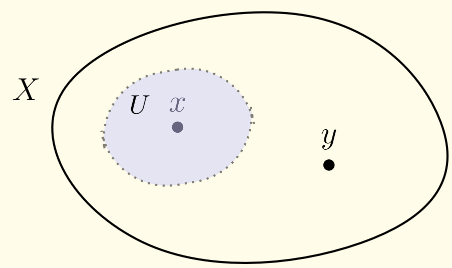


Figure 23: t_0-separability.

Definition 8.2 (t_1-Separability)

A topological space X is said to be t_1**-separable** if for each pair of distinct points $x, y \in X$, we can find two neighborhoods U_x, U_y where $y \notin U_x$ and $x \notin U_y$.

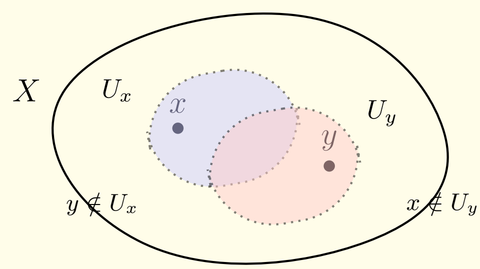


Figure 24: t_1-separability.

Example 8.1 (Nested Interval Topology is Not t_0)

$(0, 1)$ with the nested interval topology is not t_0-separable, since we can't distinguish $\frac{1}{4}$ and $\frac{1}{3}$.

Example 8.2 (Cofinite is t_1)

$(0, 1)$ with the cofinite topology is t_0-separable, since given distinct $x_1, x_2 \in (0, 1)$, we can see that $x_1 \in X \setminus x_2$ and $x_2 \in X \setminus x_1$, which are both elements of the cofinite topology. By existence of these elements, $(0, 1)$ is t_1-separable.

8.1 Hausdorff Spaces

Generally, mathematicians consider the Hausdorff condition as a mild extra conditions on topological spaces that make it much easier to deal with. We will assume that most of the topological spaces we work with are

[2] Note that this is not to be confused with the separation of a space, which is a completely different topological property.

Hausdorff.

Definition 8.3 (Hausdorff Space)

A topological space X is called a **Hausdorff space**, or **t_2-separable**, if for each pair of distinct points $x, y \in X$, there exists neighborhoods U_x, U_y that are disjoint.

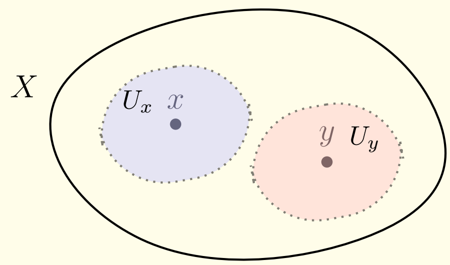


Figure 25: Every pair of distinct points must satisfy this separability condition in a Hausdorff space.

Theorem 8.1 (Limit Points in Hausdorff Spaces)

Given Hausdorff space X and subset $A \subset X$ a point x is a limit point of A if and only if every neighborhood of x contains infinitely many point of A. It immediately follows that every finite point set in a Hausdorff space X is closed.

Proof.

We prove both directions

1. ($\rightarrow$) Assume that x is a limit point of A with some neighborhood U_x intersecting A in finitely many points. Then, let the points of intersections be
$$\{x_1, ..., x_n\} = A \cap \{U_x \setminus \{x\}\} \tag{138}$$
But $U_x \setminus \{x\}$ is open $\implies H \equiv \{U_x \setminus (\{x\} \cup \{x_1, ..., x_n\})\}$ is open. But $H \cap A = \emptyset$, contradicting the assumption that x is a limit point.
2. ($\leftarrow$) Simple.

It suffices to show that every one point set $\{x_0\}$ is closed. If x and x_0 are distinct points, then by definition of Hausdorff spaces they have disjoint neighborhoods U_x and $U_{x_0} \implies x \notin \overline{\{x_0\}} \implies \{x_0\} = \overline{\{x_0\}}$, so $\{x_0\}$ is closed.

Lemma 8.1 (Product of Hausdorff Spaces)

Arbitrary Cartesian products of Hausdorff spaces is Hausdorff.[a]

[a]Since this is in the product topology, it immediately follows that the product is also Hausdorff in the finer box topology.

Lemma 8.2 (Subspaces of Hausdorff Spaces)

A subspace of a Hausdorff space is Hausdorff.

Theorem 8.2 (Unique Point of Convergence)

If a sequence converges in a Hausdorff space X, it converges to one point.

Proof.

For if (x_α) converges to x and if $y \neq x$, then we need only choose disjoint neighborhoods of y and x to prove that (x_α), by definition, is not convergent to y.

Example 8.3 ()

The space $(0, 1)$ with the nested interval topology is not Hausdorff. In fact, it is impossible to distinguish 2 points x, y if $x, y \in (0, \frac{1}{2})$, meaning that the sequence

$$\frac{1}{10}, \frac{2}{10}, \frac{1}{10}, \ldots \tag{139}$$

converges to both $\frac{1}{10}$ and $\frac{2}{10}$.

Theorem 8.3 ()

Every metric topology satisfies the Hausdorff Axiom.

Proof.

If x and y are distinct points of (X, d), then letting

$$\varepsilon = \frac{1}{2} d(x, y) \tag{140}$$

the triangle inequality implies that $B_\varepsilon(x)$ and $B_\varepsilon(y)$ are disjoint.

8.2 Regular Spaces

Definition 8.4 (Regular Spaces)

Suppose that one-point sets are closed in X. Then, X is said to be **regular**, or t_3**-separable**, if for each pair consisting of a point x and a closed set C disjoint from x, there exist disjoint open sets containing x and C, respectively.

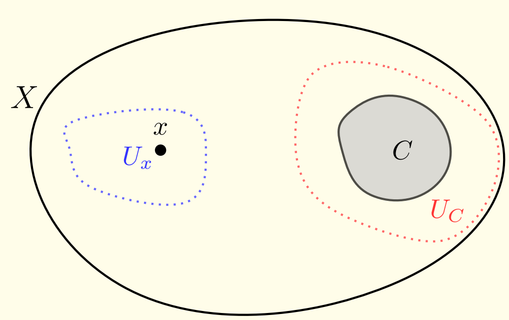


Figure 26: Regular space.

Lemma 8.3 (Product of Regular Spaces)

Arbitrary Cartesian products of regular spaces is regular.

Lemma 8.4 (Subspaces of Regular Spaces)

A subspace of a regular space is regular.

8.3 Normal Spaces

Definition 8.5 (Normal Spaces)

Suppose that one-point sets are closed in X. Then, X is said to be **normal**, or t_4**-separable**, if for each pair C, D of disjoint closed sets of X, there exist disjoint open sets containing C and D, respectively.

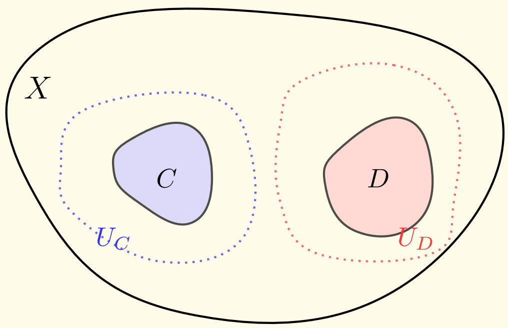


Figure 27: Normal space.

However, neither products nor subspaces of normal spaces are necessarily normal. a subspace of a normal space is not necessarily normal; a product of normal spaces is not necessarily normal.

Theorem 8.4 ()

Every metrizable space is normal.

Theorem 8.5 ()

Every compact Hausdorff space is normal.

Theorem 8.6 ()

Every regular space with a countable basis is normal.

Theorem 8.7 ()

Every well-ordered set X is normal in the order topology.

8.4 The Urysohn Lemma

Theorem 8.8 (Urysohn Lemma)

Let X be a normal space, and let A, B be disjoint closed subsets of X. Let $[a, b]$ be a closed interval in the real line. Then there exists a continuous map

$$f : X \longrightarrow [a, b] \tag{141}$$

such that $f(x) = a$ for every $x \in A$ and $f(x) = b$ for every $x \in B$.

Definition 8.6 (Separation by Continuous Function)

If A and B are two subsets of the topological space X, and if there is a continuous function $f : X \longrightarrow [0, 1]$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$, it is said that A **and** B **can be separated by a continuous function**.

More colloquially, the lemma states that if every pair of disjoint closed sets in X can be separated by disjoint open sets, then each such pair can be separated by a continuous function.

Theorem 8.9 (Tietze Extension Theorem)

Let X be a normal space and let A be a closed subset of X.
1. Any continuous map of A into the closed interval $[a, b] \subset \mathbb{R}$ may be extended to a continuous map of all X into $[a, b]$.
2. Any continuous map A into the reals $\mathbb{R}$ may be extended to a continuous map of all of X into $\mathbb{R}$.

8.5 The Urysohn Metrization Theorem

Theorem 8.10 (Urysohn Metrization Theorem)

Every regular space X with a countable basis is metrizable.

Theorem 8.11 (Imbedding Theorem)

Let X be Hausdorff. Suppose that

$$\{f_\alpha\}_{\alpha \in J},\ f_\alpha : X \longrightarrow \mathbb{R} \tag{142}$$

is a collection of continuous functions satisfying the requirement that for each point $x_0 \in X$ and each neighborhood U of x_0, there is an index α such that f_α is positive at x_0 and vanishes outside U. Then, the function

$$F : X \longrightarrow \mathbb{R}^J,\ F(x) \equiv \big(f_\alpha(x)\big)_{\alpha \in J} \tag{143}$$

is an **imbedding** of X in $\mathbb{R}^J$.

9 Metric Topologies

In $\mathbb{R}$, note that every open ball is really just an interval. In fact, every open ball $(x-r, x+r)$ can be expressed with just two elements $a, b \in \mathbb{R}$, as (a, b). Notice that this method of expressing an open set does not even require any metric! Extending this to $\mathbb{R}^n$ would indicate that the topologies of $\mathbb{R}^n$ defined by the endpoint of the open intervals would not necessarily induce any metric either. Notice that these induced topologies is **not** the open ball topology, which must have an associated metric to it. Rather, this induced, non-metric topology is the box topology! While the box topology and the open ball topology are really the same topology, they are generated by inherently different bases.

Definition 9.1 (Bounded Set)

Let (X, d) be a metric space with subset A. A is **bounded** if there exists some number M such that

$$d(x, y) \leq M \text{ for all } x, y \in A \tag{144}$$

If A is bounded, the **diameter** of A is defined to be the number

$$\operatorname{diam} A \equiv \sup \{d(x, y) \mid x, y \in A\} \tag{145}$$

Note that boundedness on a set is not a topological property since it depends on the particular metric d that is used for X. For example, we can construct the following metric that makes every subset in X bounded.

Definition 9.2 (Standard Bounded Metric)

Let (X, d) be a metric space. We define a second metric $\tilde{d}$ on X such that

$$\tilde{d}(x, y) \equiv \min \{d(x, y), 1\} \tag{146}$$

$\tilde{d}$ is called the **standard bounded metric corresponding to** d.

If we construct open balls with this metric, it is easy to see that they consist of all open balls with radius less than or equal to 1. That is, the topology $\mathscr{T}$ consists of all open balls

$$\mathscr{T} \equiv \{B_r(x) \mid x \in X, r \leq 1\} \tag{147}$$

It is also clear that the topology induced by $\tilde{d}$ is the same as the topology induced by d! The significance of this construction of the standard bounded metric is that we can now work with a basis consisting of bounded elements, which is much nicer than a basis of open balls that can have arbitrarily large radii.

We now introduce a metrization theorem on $\mathbb{R}^n$.

Theorem 9.1 ()

The topologies on $\mathbb{R}^n$ induced by the Euclidean metric d and the square metric ρ are the same as the product topology on $\mathbb{R}^n$.

Proof.

Given $x, y \in \mathbb{R}^n$, simple algebra shows that

$$\begin{aligned} &\rho(x, y) \leq d(x, y) \leq \sqrt{n}\rho(x, y) \\ &\implies \forall x, \epsilon,\ B_d(x, \epsilon) \subset B_\rho(x, \epsilon) \text{ and } B_\rho(x, \frac{\epsilon}{\sqrt{n}}) \subset B_d(x, \epsilon) \end{aligned}$$

But

$$\{B_\rho(x,\epsilon) \mid x \in \mathbb{R}^n, \epsilon \in \mathbb{R}\} = B_\rho(x, \frac{\epsilon}{\sqrt{n}}) \mid x \in \mathbb{R}^n, \epsilon \in \mathbb{R}\} \tag{148}$$

which means that the metric topology induced by d is the same as the metric topology induced by $\rho \implies$ the two topologies are the same. We know that the topology induced by ρ is the same as the product topology since

$$\prod_{i=1}^n (x_i - r, x_i + r) = \bigcup_{k=1}^n \mathbb{R}^{k-1} \times (x_k - r, x_k + r) \times \mathbb{R}^{n-k} \tag{149}$$

With this theorem, we have proved that given a topological space $\mathbb{R}^n$ with the product topology, there exists a metric (the Euclidean and square metric) that induces this product topology. We can attempt to extrapolate these formulas to $\mathbb{R}^\omega$ by defining

$$d(x,y) \equiv \bigg(\sum_{i=1}^\infty (x_i - y_i)^2 \bigg)^{\frac{1}{2}}$$

$$\rho(x,y) \equiv \sup \{|x_i - y_i|\}$$

However, the metrics do not in general map to elements of $\mathbb{R}$, since the sequence $(x_i - y_i)_{i \in \mathbb{N}}$ could diverge. Therefore, we can redefine the metric ρ to the following bounded one.

$$\tilde{\rho}(x,y) \equiv \sup \{\tilde{d}(x_i, y_i)\} \tag{150}$$

where $\tilde{d}$ is the standard bounded metric on $\mathbb{R}$. Clearly,

$$0 \leq \tilde{\rho}(x,y) \leq 1 \tag{151}$$

$\tilde{\rho}$ is indeed a metric on $\mathbb{R}^\omega$, but unfortunately, it does not induce the product topology. We extend this definition to arbitrary $\mathbb{R}^J$.

Definition 9.3 (Uniform Metric)

Given an indexed set J with points $x, y \in \mathbb{R}^J$, we define

$$\tilde{\rho} \equiv \sup \{\tilde{d}(x_\alpha, y_\alpha) \mid \alpha \in J\} \tag{152}$$

with $\tilde{d}$ the standard bounded metric on $\mathbb{R}$. $\tilde{\rho}$ is called the **uniform metric on** $\mathbb{R}^J$, which induces the **uniform topology**.

The uniform topology on $\mathbb{R}^J$ is finer than the product topology, and they are different if J is infinite. Clearly, $0 \leq \tilde{p}(x,y) \leq 1$, meaning that given the open ball

$$B_r(x) \equiv \{y \in \mathbb{R}^J \mid \tilde{p}(y,x) < r\} \tag{153}$$

if $r \geq 1$, then $B_r(x) = \mathbb{R}^J$ and if $r < 1$, then $B_r(x)$ consists of the n-dimensional box with "radius" r, where $n = \dim \mathbb{R}^J$.

The next theorem gives us a metric that induces the product topology on infinite dimensional $\mathbb{R}^\omega$ by slightly modifying the uniform metric on $\mathbb{R}$. However, with the box topology $\mathbb{R}^\omega$ is not metrizable.

Theorem 9.2 ()

Let $\tilde{d}(a,b) \equiv \min\{|a-b|, 1\}$ be the standard bounded metric on $\mathbb{R}$. If $x, y \in \mathbb{R}^\omega$, we define

$$D(x,y) \equiv \sup\left\{\frac{\tilde{d}(x_i, y_i)}{i}\right\} \tag{154}$$

Then, D is a metric that induces the product topology on $\mathbb{R}^\omega$.

It is easy to see that $0 \leq D(x,y) \leq 1$. So, given the open ball

$$B_r(x) \equiv \{y \in \mathbb{R}^\omega \mid D(x,y) < r\} \tag{155}$$

$B_r(x) = \mathbb{R}^\omega$ if $r > 1$. When $r \leq 1$,

$$B_r(x) \equiv (y-r, y+r) \times (y-2r, y+2r) \times \ldots = \prod_{k=1}^{\infty} (y-kr, y+kr) \tag{156}$$

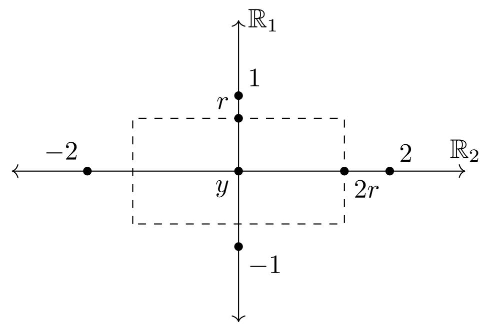


Figure 28: Visually, we take a cross section of this box and look at the slice within $\mathbb{R}_1 \times \mathbb{R}_2$, where the subscripts represent the first and second terms of x.

We can extend the applications of the Bolzano Weierstrass Lemma from analysis to metric spaces in general with the following lemma.

Lemma 9.1 (Sequence Lemma)

If X be a topological space with $A \subset X$. If there exists a sequence of points of A that converges to x, then $x \in \bar{A}$. The converse is true if X is metrizable.

Proof.

($\rightarrow$) Our hypothesis says that x is a limit point of A, which by definition means that $x \in \bar{A}$.
($\leftarrow$) Assuming X is metrizable and $x \in \bar{A}$, let d be a metric for the topology of X. Then, for every $n \in \mathbb{N}$, let us define a sequence of open neighborhoods of x to be

$$\big(B_{\frac{1}{n}}(x)\big) \tag{157}$$

Since $x \in \bar{A}$, there exists a point

$$x_n \in A \cap B_{\frac{1}{n}}(x) \text{ for all } n \in \mathbb{N} \tag{158}$$

This sequence (x_n) that we have proved must exist converges to x.

Theorem 9.3 ()

Let $f : X \longrightarrow Y$ and let X be metrizable. f is continuous if and only if for every convergent sequence $(x_n) \to x$ of X, the following sequence of Y converges to $f(x)$. That is,

$$\big(f(x_n)\big) \longrightarrow f(x) \tag{159}$$

We introduce additional methods of constructing continuous functions.

Definition 9.4 (Uniform Convergence)

Let $f_n : X \longrightarrow Y$ be a sequence of functions from the set X to the metric space (Y, d). The sequence (f_n) is said to **converge uniformly** to the function $f : X \longrightarrow Y$ if, given $\epsilon > 0$, there exists a $N \in \mathbb{N}$ such that

$$d\big(f_n(x), f(x)\big) < \epsilon \tag{160}$$

for all $n \geq N$ and for all $x \in X$.

Theorem 9.4 (Uniform Limit Theorem)

Let $f_n : X \longrightarrow Y$ be a sequence of continuous functions from topological space X to a metric space Y. If f_n converges uniformly to f, then f is continuous.

Proof.

$(\rightarrow)$ Trivial.
$(\leftarrow)$ Let V be open in Y, and let x_0 be a point in $f^{-1}(V)$. It suffices to prove that for every $x_0 \in f^{-1}(V)$, there exists a neighborhood U of x_0 such that $U \subset F^{-1}(V)$ or equivalently, $F(U) \subset V$. Let $y_0 = f(x_0)$. Since Y is a metric space with metric d, we know that there exists an ϵ-ball $B_\epsilon(y_0)$ such that

$$B_\epsilon(y_0) \subset V \tag{161}$$

Then, using uniform convergence, we can choose $N \in \mathbb{N}$ such that for all $n \geq N$ and all $x \in X$,

$$d\big(f_n(x), f(x)\big) < \frac{\epsilon}{4} \tag{162}$$

which also applies at the point $x = x_0$.

$$d\big(f_n(x_0), f(x_0)\big) < \frac{\epsilon}{4} \tag{163}$$

Using continuity of f_n, choose a neighborhood U of x_0 such that f_n carries U into the open $\epsilon/2$-ball centered at $f_n(x_0)$ (note that $f_n(x_0) \neq y_0$), meaning that if $x \in U$

$$d\big(f_n(x), f_n(x_0)\big) < \frac{\epsilon}{2} \tag{164}$$

Adding the three inequalities and using the triangle inequality, we get the fact that if $x \in U$, then

$$d\big(f(x), f(x_0)\big) < \epsilon \tag{165}$$

meaning that the $f(U) \subset B_\epsilon(x_0) \subset V$.

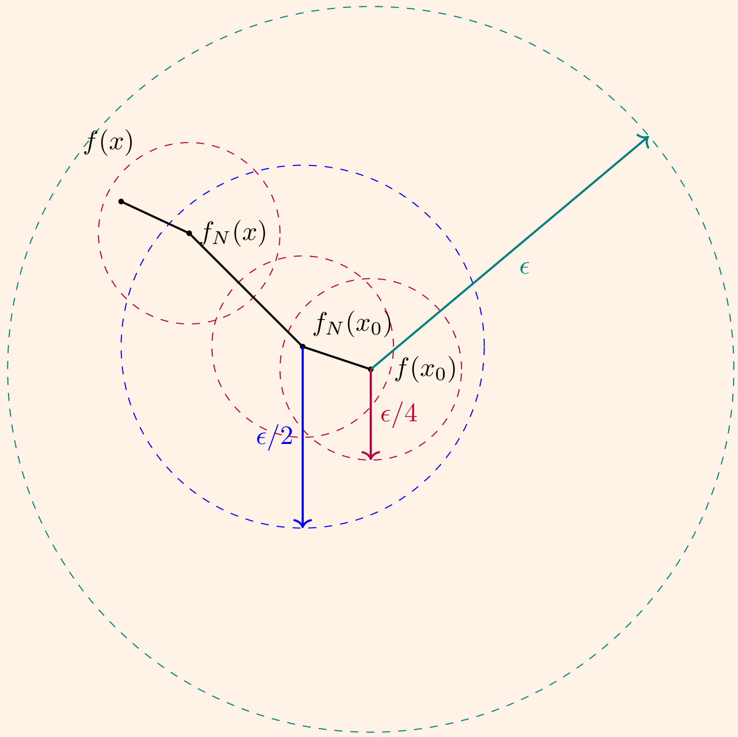


Figure 29: Visually, the three inequalities represent the following open balls in $V \subset Y$.

Theorem 9.5 ()

In a metric space (X, d), a set is **closed** if the limit of every convergent subsequence in X lies in X. That is, X contains all of its limit points.