THE STATISTICAL CONCEPTION OF MENTAL FACTORS

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I. The degree of reality of statistical concepts

In a previous paper (1) dealing with the estimation of ability in Spearman's two-factor theory, it was naturally convenient to assume the truth of this theory; with its justification I was not there concerned. But while the amount of experimental justification that is ascribed to it is not a problem with which I am qualified to deal, the actual interpretation of this and other factor theories has statistical aspects on which I should like to be permitted to offer comment.

Whatever degree of objectivity is attributed to the two-factor theory, it is recognized that this theory is not sufficient, group factors being introduced as auxiliaries when this is found necessary. Since a description of tests in terms of a general and specific factors *and* group factors is necessarily always possible, it has been questioned whether we can associate any reality with our estimates of these hypothetical factors, or whether they are merely convenient mathematical descriptions of our observed test scores. Thus Thomson (2) has said: "The factor language is convenient and vivid. But I may, I hope, be allowed to warn against the danger of personifying these factors, which are, I think, only mathematical coefficients." While tending to agree with this warning, I would at the same time point out that there is no *a priori* objection to any coefficient which appeared permanently and satisfactorily measurable in practice being regarded as possessing a certain degree of reality. (The temperature of a body is thought of as something real, but theoretically it appears merely as a mathematical coefficient in an equation of statistical equilibrium.)

There is certainly no reason to believe that general and specific factors are rigid and unanalysable concepts. It was stressed in (1) that general ability must always be a contrast with the statistical average of the group of persons tested; and further, if any statistician adopts the reasonable point of view that when estimating G he must consider specific factors as part of his experimental error, I doubt whether he will regard them as other than syntheses of components which (he hopes) will aggregate into random and normal variates. This statistical interpretation of factors might, however, be a very real one. The very illuminating remarks of Thomson (3) on his sampling theory of abilities seem to me to offer one possible theoretical justification of the factor postulates.

II. Symbolism and estimation of factors

In progressive work aiming at the analysis of test performance and the detection of 'factors', various semi-empirical methods of analysis, such as, for example, Hotelling's analysis of variates into their principal components, have already proved of great value. It is reasonable, however, to regard as one ultimate goal of such work the postulation of a set of factors in terms of which performance can be adequately accounted for, although at the same time factors which have established a sufficiently strong position to be regarded as entities must clearly depend on the results of very extensive experimental research.[1]

[1] In a recent paper Stephenson (6) has attempted to make a distinction between research in type and individualistic psychology, by associating with the latter the usual factor technique and with the former what he has termed "the inverted factor technique". Thus on the postulation of a single general factor, whereas the usual technique enables us to estimate the score for any particular person in an ideal test typical of the tests used, and hence of a larger set of which these tests are a sample (cf. my remarks in § III), the inverted technique would enable us to estimate the score in any particular test for an ideal person typical of the persons tested, and hence of a larger set of which these persons are a sample. It has of course to be shown that such an estimated 'typical person' is a better concept than the 'average person' who scores the average marks in each test (for only a single general factor, I am doubtful whether it would be). Further, to avoid the difficulty of standardizing a set of scores which refer to different tests, Stephenson cited as a *set* of tests the grading of similar articles, such as a series of coloured papers, which grading could almost be regarded as a single test. Ignoring these points, which do not detract from the interest and possible value of the method, I do not think I am sure at present what the distinction drawn by Stephenson is intended to imply.

The chief value of the idea of psychological types must lie, not in an arbitrary classification of men into two or more grades, but in a demonstration that independent classifications of this kind (by, for example, the estimation of factors) are correlated. If from physical tests it was found possible to classify men according to a factor corresponding to Kretschmer's contrast of pyknic and asthenic types (the main general factor emerging from such tests

The symbolism of vector and matrix theory, which has been used extensively by Thomson, is very suited to expositions of factor theories. Thus we write

$$T = MF, \qquad \ldots\ldots(1)$$

where T is a column vector denoting a series of standardized tests, F the postulated column vector of standardized factors, and M the matrix of correlations (supposed known) between tests and factors. Any estimate of F obtained from linear combinations of the tests will be of the form

$$f = AT.$$

From equation (1), and the formula

$$MM' = R,$$

where R is the matrix of test correlations, Thomson (4) has derived the identity

$$M\,(M'R^{-1}T - F) = 0,$$

of which a consistent solution is

$$f = M'R^{-1}T. \qquad \ldots\ldots(2)$$

This he has put forward as an equation to estimate F.

If, however, we adopt the principle used in (1), that specific factors should be introduced only in order to explain discrepancies between observed scores and postulated general or group factors, we write

$$T = M_0 F_0 + M_1 F_1, \qquad \ldots\ldots(3)$$

where F_0 is the set of general and group factors, and F_1 are the specifics. M_1 is thus a diagonal matrix. For two group factors G and H, the ordinary algebraic relation for the pth test is given by

$$T_p = (\rho_{pG} G + \rho_{pH} H) + \kappa_p S_p,$$

where

$$\kappa_p = \sqrt{(1 - \rho^2_{pG} - \rho^2_{pH})}.$$

Minimizing the sum of squares $\Sigma S_p{}^2$, we obtain the equations

$$\Sigma \rho_{pG}\,(T_p - \rho_{pG} g - \rho_{pH} h)/\kappa_p{}^2 = 0,$$
$$\Sigma \rho_{pH}\,(T_p - \rho_{pG} g - \rho_{pH} h)/\kappa_p{}^2 = 0,$$

where g and h are our estimates.

would presumably be one merely of pure size), and it was also found possible to make an independent classification of temperament by means of mental tests, a significant correlation between the physical and mental factors would have provided a statistical justification in ordinary persons of Kretschmer's theory of types. Such a procedure would, it will be noticed, have conformed to orthodox test and factor techniques; it might in fact be regarded as a particular case of the general factor method of showing that two tests, or two series of tests, are not independent, but are better explained in terms of an underlying 'factor', though the case is a very special one because of the natural dichotomy of physical and mental measurements.

In matrix notation for any number of factors,

$$f_0 = (M_0' M_1^{-2} M_0)^{-1} M_0' M_1^{-2} T. \qquad \text{......(4)}$$

This equation represents our estimates for *all* our persons, if more generally we regard T as the matrix set of all the persons' test scores.

It was pointed out in (1) that the principle of estimation adopted there did not completely agree with the solution that has usually been employed, although the difference did not affect the relative weights assigned to the tests in estimating G. When group factors are introduced, however, the discrepancy between equations (2) and (4) is more serious. One point of view appears to have been to consider all the persons with different possible factorial make-ups that would give rise to the observed test scores of a particular person, whereas I have regarded the test scores as a sample of all the possible scores that might have arisen for that person according to the different values of specific factors he may happen to have. If we are postulating a set of factors F_0 and are considering how to estimate this set, given a particular person's scores, this latter point of view seems to me the logical one. We know that the estimate in equation (4) of any group factor has the minimum standard error. If

$$J = M_0' M_1^{-2} M_0,$$

the information (reciprocal of the error variance) on, say, G is given by

$$I_G = |J| / J_{GG},$$

where J_{GG} is the cofactor of the element j_{GG} in $|J|$.

III. The interpretation of factors in the sampling theory of ability

From the idea that the mind is a synthesis of a large number of components of which any test may approximately be regarded as a random sample, Thomson (3) has shown that the tendency to zero tetrad differences that is often observed in practice must necessarily follow theoretically. In a simple form, if for convenience we assign equal weights to all of many statistically independent components, and four tests sample at random fractions p_1, p_2, p_3 and p_4 of these components, then the value of the correlation between tests 1 and 2 will obviously be

$$\rho_{12} = \sqrt{(p_1 p_2)} \qquad \text{......(5)}$$

if the number of components is sufficiently large, and

$$\rho_{12}\rho_{34} = \rho_{13}\rho_{24} = \rho_{14}\rho_{23}.$$

We note further that on the usual Spearman two-factor theory

$$\rho_{1G}=\left(\frac{\rho_{12}\rho_{13}}{\rho_{23}}\right)^{\frac{1}{2}}=\sqrt{p_1}, \qquad \text{......(6)}$$

which is the correlation of test 1 *with an ideal test which samples all the components.*

If, therefore, we are prepared to recognize a statistical interpretation of factors, and in particular of general ability G, and we assume the truth of the sampling theory, the natural *a priori* definition of G on this theory as the average for any person of all his components is the definition to which an application of the two-factor method inevitably leads. Equation (6), for example, indicates that we should proceed to estimate this average from tests by the standard formula.

This point may be stressed by verifying directly how we should attempt to estimate G, which we have defined by

$$G=\lambda\bar{\epsilon}, \qquad \text{......(7)}$$

where $\bar{\epsilon}$ is the mean of all components ϵ, and λ is a standardizing factor. Let us write as our estimate

$$g=a_1T_1+a_2T_2+a_3T_3,$$

where for simplicity we consider three tests, and the test scores T are standardized. Then the average value of $(g-G)^2$ is proportional to

$$\begin{aligned}&(b_1-1)^2 p_1(1-p_2)(1-p_3)+(b_2-1)^2 p_2(1-p_3)(1-p_1)\\&+(b_3-1)^2 p_3(1-p_1)(1-p_2)\\&+(b_1+b_2-1)^2 p_1p_2(1-p_3)+(b_2+b_3-1)^2 p_2p_3(1-p_1)\\&+(b_3+b_1-1)^2 p_3p_1(1-p_2)\\&+(b_1+b_2+b_3-1)^2 p_1p_2p_3+(1-p_1)(1-p_2)(1-p_3),\end{aligned}$$

where $b_1 \propto a_1/\sqrt{p_1}$, etc. The b's are introduced to counteract the effect of standardizing tests comprising different numbers of components. The term $p_1(1-p_2)(1-p_3)$ is the fraction of components in test 1 not entering into tests 2 and 3, and so on (cf. (3)). Minimizing this variance, we obtain the equations

$$\begin{aligned}b_1+b_2p_2+b_3p_3&=1,\\b_1p_1+\ \ b_2+b_3p_3&=1,\\b_1p_1+b_2p_2+\ \ b_3&=1,\end{aligned}$$

of which the solution is $b_1 \propto 1/(1-p_1)$, etc., or

$$a_1 \propto \frac{\sqrt{p_1}}{1-p_1}=\frac{\rho_{1G}}{1-\rho^2_{1G}}.$$

Specific abilities on this theory are simply the differences between the components sampled by particular tests, and the average G of all components. That is, we define for two tests

$$S_1 = \lambda_1 (\bar{\epsilon}_1 - \bar{\eta}_1),$$
$$S_2 = \lambda_2 (\bar{\epsilon}_2 - \bar{\eta}_2), \qquad \ldots\ldots(8)$$

where $\bar{\epsilon}_1$ is the average of components entering into test 1, and $\bar{\eta}_1$ the average of the remaining components. It is obvious by definition that S_1 and S_2 are each statistically independent of G. The correlation between S_1 and S_2 depends on the average value of the product

$$(\bar{\epsilon}_1 - \bar{\eta}_1)(\bar{\epsilon}_2 - \bar{\eta}_2).$$

If the number of components common to tests 1 and 2 is k, this value becomes

$$\frac{k}{p_1 p_2} - \frac{p_1 - k}{(1-p_2)\,p_1} - \frac{p_2 - k}{(1-p_1)\,p_2} + \frac{1 + k - p_1 - p_2}{(1-p_1)(1-p_2)},$$

and $\qquad \propto k - p_1 p_2.$

Thus if the two tests are each random samples from the entire set of components, so that $k = p_1 p_2$, the factors S_1 and S_2 are independent.

The obvious fact that any specific factor must necessarily be a contrast between performance in any particular test and a person's general ability is thus explicitly brought out by this sampling interpretation. It is not fair, for example, to cite the hypothetical case of a person who has specifics above the average in 1000 tests in order to indicate that for such a person our estimate of G would be too high (5), for such a case is a 'statistical impossibility'. The odds against any person being of such a type are $2^{1000}:1$. The common-sense interpretation would be that if a person has high scores in 1000 tests, he has a high general ability rather than high specifics. In practice of course, where few tests are available, we do have to recognize that there will always be some persons for whom our estimates of G will be badly out.

The idea of a specific being merely a contrast may be compared with the point already stressed that a person's general ability is only a statistical contrast with the group of persons with whom he is associated.

It has been assumed above that the tests are samples from all the components, but of course if they were all samples from a particular region only, the tendency for zero tetrad differences would remain, and any postulated G could refer only to the properties of that region. The sampling interpretation has in fact been succinctly expressed by Thomson as follows: "From my point of view these tests are then drawing samples from the same sub-pool of our mind's abilities. If we stray from this pool, and fish in other waters, we shall break the hierarchy; but if we sampled the *whole* pool of a mind, we should again find the tendency to hierarchical order."

If two tests are not completely random, but are sampling from the same fraction p of all the components, they will be correlated to a higher extent than on the two-factor theory, the correlation between them being

$$\rho_{12}=\frac{\sqrt{(p_1 p_2)}}{p}.$$

Suppose five tests have the true correlations

$$R=\begin{pmatrix} 1 & 0{\cdot}80 & 0{\cdot}60 & 0{\cdot}40 & 0{\cdot}30 \\ 0{\cdot}80 & 1 & 0{\cdot}50 & 0{\cdot}33 & 0{\cdot}25 \\ 0{\cdot}60 & 0{\cdot}50 & 1 & 0{\cdot}40 & 0{\cdot}30 \\ 0{\cdot}40 & 0{\cdot}33 & 0{\cdot}40 & 1 & 0{\cdot}20 \\ 0{\cdot}30 & 0{\cdot}25 & 0{\cdot}30 & 0{\cdot}20 & 1 \end{pmatrix}.$$

These give zero tetrads except for ρ_{12} being too high. Since on the random sampling idea

$$\rho_{12}=\sqrt{(p_1 p_2)}=\rho_{1G}\rho_{2G}=0{\cdot}50,$$

we have

$$p=0{\cdot}50/0{\cdot}80=0{\cdot}63.$$

Let us define a group factor

$$H=\mu\,(\bar{\epsilon}_H-\bar{\eta}_H),$$

where $\bar{\epsilon}_H$ is the average of components of the set sampled by tests 1 and 2, and $\bar{\eta}_H$ the average of the remainder. Then by similar arguments as before, H is a factor independent of G; and the modified specific abilities defined as

$$S_1=\mu_1\,(\bar{\epsilon}_1-\bar{\eta}_{1H}),$$
$$S_2=\mu_2\,(\bar{\epsilon}_2-\bar{\eta}_{2H}),$$

where $\bar{\eta}_{1H}$ is the average of components of the set corresponding to H not entering into test 1, etc., are independent of G, H and each other. H thus denotes a contrast between the components sampled by the tests containing this factor, and the rest of the mind, or the part of the mind, sampled by all the tests.

For the set of hypothetical correlations above, since from the definition of H,

$$\rho_{1H}=\sqrt{\left\{\frac{p_1\,(1-p)}{p}\right\}},$$
$$\rho_{2H}=\sqrt{\left\{\frac{p_2\,(1-p)}{p}\right\}},$$

we have

$$\rho_{1H}=0{\cdot}60,\quad \rho_{2H}=0{\cdot}50,$$

from which, as a check, we verify that the actual value of the correlation

$$\rho_{12}=\rho_{1G}\rho_{2G}+\rho_{1H}\rho_{2H}=0{\cdot}80.$$

IV. Summary

A statistical conception of mental factors does not necessarily invalidate the search for factors to which a certain degree of reality may usefully be attributed. A method of estimation is given of a set of postulated factors corresponding to the matrix equation

$$T = MF.$$

A definition and interpretation of general, specific and group factors in terms of the sampling theory of ability is also given.

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