Other Models

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Contents

Just some other models I've learned that don't fit in nicely to any of existing categories yet.

1 Topological Data Analysis

2 Geodesic Regression

In regression, note that we are finding a function $f : \mathcal{X} \to \mathcal{Y}$. In usual linear regression, both $\mathcal{X}, \mathcal{Y}$ are Euclidean space. However, there are cases where it may not be realistic that one or more of them should be modeled as a vector space. Rather, they may be part of a lower-dimensional manifold. For instance, if we want to use linear regression to predict the top k principal components of a dataset, they must be orthogonal, i.e. must be in a *Stiefl manifold*.

There are way to model this. For instance, we could have a projection operator that maps from $\mathbb{R}^m \to \mathcal{Y}$. This has its issues too, for one not being very efficient since perhaps the dimension of $\mathcal{Y}$ may be much less than m. Therefore, it might be better to directly regress onto a manifold. There were many attempts at this, but the first model to generalize OLS to manifolds was created by Fletcher in 2011 [Fle11] and expanded shortly in [TF13].

We start with the case where there is one covariate (i.e. $\mathcal{X} = \mathbb{R}$) and $\mathcal{Y} = (M, d)$ is a smooth Riemannian manifold with a metric. Recall that for a smooth manifold M, for any $p \in M$ and $v \in T_p M$, the tangent space at p, there is a unique geodesic curve $\gamma : [0, 1] \to M$ satisfying $\gamma(0) = p$, $\gamma'(0) = v$. This geodesic is guaranteed to exist locally, and with this,we can define the exponential map at p in the direction of v as

$$\exp_p(v) = \exp(p, v) = \gamma(1) \tag{1}$$

In other words, the exponential map takes a position and velocity as input and returns the point at time 1 along the geodesic with these initial conditions. With this motivation, we use slightly different notation than regular linear regression, referring p as the bias and v as the coefficient.

Definition 2.1 (Geodesic Regression)

The **geodesic regression** model is a probabilistic model that predicts the conditional distribution of $y \in (M, d)$ given $x \in \mathbb{R}$ as

$$y = \exp(\exp(p, vx), \epsilon) \tag{2}$$

where the parameters are $\theta = \{p, v\}$, and ϵ is a random variable defined over the tangent space at $\exp(p, vx)$.

Note that if we set $\mathcal{Y} = \mathbb{R}^m$, then we get the ordinary linear regression model back.

Definition 2.2 (Least Squares Geodesic Regression)

The least squares geodesic regression aims to minimize the MSE loss

$$L(\theta, (x, y)) = L(p, v, x, y) = d(\exp(p, vx), y)^2 \tag{3}$$

Lemma 2.1 (Risk)

The risk is

$$R(f) = \mathbb{E}_{x,y}\left[d(\exp(p, vx), y)^2\right] \tag{4}$$

and the empirical risk for a dataset $\mathcal{D} = \{(x^{(i)}, y^{(i)})\}_{i=1}^n$ is

$$\hat{R}(f) = \frac{1}{n} \sum_{i=1}^n d(\exp(p, vx^{(i)}), y^{(i)})^2 \tag{5}$$

Unfortunately, minimizing this does not yield an analytic solution.

3 Frechet Regression

References

[Fle11] Thomas Fletcher. Geodesic Regression on Riemannian Manifolds. In Pennec, Xavier, Joshi, Sarang, Nielsen, and Mads, editors, *Proceedings of the Third International Workshop on Mathematical Foundations of Computational Anatomy - Geometrical and Statistical Methods for Modelling Biological Shape Variability*, pages 75–86, Toronto, Canada, September 2011.

[TF13] P. Thomas Fletcher. Geodesic regression and the theory of least squares on riemannian manifolds. *Int. J. Comput. Vision*, 105(2):171–185, November 2013.