Point Set Topology

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Contents

Given a set, what are some fundamental structures that you can put on a set? We can first talk about relations, such as ordering, or functions, such as a norm or a distance. These are constructed as subsets of a Cartesian product of finite X's. Another structure on set X is to define a set of subsets $\mathscr{T} \subset 2^X$ that allow us to interpret how certain elements of a set are "nearby" each other without the notion of a metric.[1] This set of subsets is called a *topology*, with its elements being *open sets*. So how do you define such a thing? Well intuitively, given two elements $x, y \in X$, if there exists two disjoint open sets U_1, U_2 such that $x \in U_1$ and $y \in U_2$, then we can *distinguish* these points in such a way. If this is true for all points in X, then this gives us a nice *Hausdorff* property to work with. If there exists no open sets that can do this, then x and y, although distinct in X, may be *indistinguishable* in the topological sense.

If this notion of nearness can be rigorously defined, we may be able to characterize the elements and subsets of X. One nice notion is the concept of *limit points* which asks whether x is "infinitesimally close" to a certain set. This allows us to define limits without the notion of a metric, and with this foundation we build the notion of continuity.

A trivial way to construct such a topology is to take the power set 2^X itself. However, this may be "too big" in a sense that no interesting properties can be deduced. But this doesn't mean we can take any subset of 2^X. We compromise by defining topologies to be a subset of 2^X with certain properties, which we will mention in the next section.

The construction of the topology allows us to study properties of these spaces. Moreover, if we have a function that maps from one topological space to another, how do we know what kinds of properties will be preserved and what will be lost? It turns out that these topological properties are invariant under certain mappings called *homeomorphisms*. Therefore, topology can also be seen as a method to study spaces and properties that are preserved under homeomorphisms.

[1] In ZFC set theory, a topology may be more fundamental in the sense that it is a subset of the power set, while the other structures are subsets of a Cartesian product, which itself is a construction from the power set.

1 Topologies

The first thing to define is a topology.

Definition 1.1 (Topology)

Let X be a set and $\mathscr{T}$ be a family of subsets of X. Then $\mathscr{T}$ is a **topology** on X[a] if it satisfies the following properties.

1. *Contains Empty and Whole Set*:
$$\emptyset, X \in \mathscr{T} \tag{1}$$
2. *Closure Under Union.* If $\{U_\alpha\}_{\alpha \in A}$ is a class of sets in $\mathscr{T}$, then
$$\bigcup_{\alpha \in A} U_\alpha \in \mathscr{T} \tag{2}$$
3. *Closure Under Finite Intersection*: If $U_1, \ldots, U_n$ is a finite class[b] of sets in $\mathscr{T}$, then
$$\bigcap_{i=1}^{n} U_i \in \mathscr{T} \tag{3}$$

A **topological space** is denoted $(X, \mathscr{T})$.

[a] I will use script letters to denote topologies and capital letters to denote sets.

[b] Note that we restrict property 3 to be a *finite* intersection because it turns out that the finiteness of intersection allows us to prove many nice properties about topologies, which we will mention later. Another reason is that if we remove this finite restriction, the open ball topology on $\mathbb{R}$ would imply that $\cap_{i=1}^{\infty}(-1/i, +1/i) = 0$ is an open set $\implies$ all points are open sets too, which is generally not what we want in analysis.

For the sake of giving at least one nontrivial example, here is an example of a finite topology.

Example 1.1 (Topologies of a Set of Cardinality 3)

There are a total of 29 topologies that we can construct on $\{1, 2, 3\}$. Two such examples are

1. $\{\emptyset, \{1, 2\}, \{1, 2, 3\}\}$
2. $\{\emptyset, \{3\}, \{2, 3\}, \{1, 2, 3\}\}$

When we define a new topology, we must first prove that they are topologies, and so these definitions are really theorems. However, I will introduce them as definitions and reserve the theorem environment for actual theorems.

Definition 1.2 (Discrete, Indiscrete Topologies)

Given a set X,

1. 2^X is a topology, called the **discrete topology**.
2. $\{\emptyset, X\}$ is a topology, called the **indiscrete topology**.

Proof. Listed.

1. The first property is trivially proven. From the theorems of set theory, $U_\alpha \subset X \implies \cup U_\alpha \subset X \implies \cup U_\alpha \in 2^X$. Finally the same logic holds for intersection as well.
2. The first property is trivially proven. We can check for the 4 combinations of unions and intersections and see that they all result in either $\emptyset$ or X.

Here is our first nontrivial example of a topology. It's nice that it can be directly constructed given any set, with no additional structure.

Definition 1.3 (Cofinite Topology)

Given a set X, the set of all subsets U, satisfying the property that $X \setminus U$ is finite, is a topology, called the **cofinite topology** or the **finite complement topology**.[a]

[a] While this definition may seem a bit arbitrary, this is very similar to the Zariski topology, which is used in algebraic topology.

Proof. Let us denote this set $\mathscr{T}_c$.
1. By definition $\emptyset \in \mathscr{T}_c$. It is clear that $X \setminus X = \emptyset$ has cardinality 0, and therefore is in $\mathscr{T}_c$.
2. Let $\{U_\alpha\}_{\alpha \in I} \in \mathcal{T}_c$ by a collection of open sets of X. Then by deMorgan's laws,
$$X \setminus \bigcup_{\alpha \in I} U_\alpha = \bigcap_{\alpha \in I} (X \setminus U_\alpha) \tag{4}$$
$X \setminus U_\alpha$ is countable for all $\alpha \in I$, so let us fix some α'. Then
$$\bigcap_{\alpha \in I} (X \setminus U_\alpha) \subset U_{\alpha'} \implies \left| \bigcap_{\alpha \in I} (X \setminus U_\alpha) \right| \leq |U_{\alpha'}| \tag{5}$$
and so the intersection is also countable.
3. Let $\{U_i\}_{i=1}^n$ by a finite collection of open sets of X. Then by deMorgan's laws,
$$X \setminus \bigcap_{i=1}^n U_i = \bigcup_{i=1}^n (X \setminus U_i) \tag{6}$$
Since U_i are open, $X \setminus U_i$ are countable, and since the finite union of countable sets are countable, the RHS is countable, which implies the LHS is countable and so $\cap_{i=1}^n U_i$ is open as well.

Slightly modifying the definition does not result in a topology.

Example 1.2 (Countable Complement is Not A Topology)

Given a set X, consider the collection
$$\mathscr{T}_\infty := \{U \subset X \mid X \setminus U \text{ is infinite or empty or all of } X\} \tag{7}$$
This is not a topology. Let us take $X = \mathbb{R}$, and look at the sets $\mathbb{Z}_{\geq 0}, \mathbb{Z}_{\leq 0}$ consisting of all the non-negative and non-positive integers. They are both infinite, and so $\mathbb{R} \setminus \mathbb{Z}_{\geq 0}$ and $\mathbb{R} \setminus \mathbb{Z}_{\leq 0}$ are in $\mathcal{T}_\infty$. Consider their union.
$$(\mathbb{R} \setminus \mathbb{Z}_{\geq 0}) \cup (\mathbb{R} \setminus \mathbb{Z}_{\leq 0}) = \mathbb{R} \setminus (\mathbb{Z}_{\geq 0} \cap \mathbb{Z}_{\leq 0}) = \mathbb{R} \setminus \{0\} \tag{8}$$
But $\mathbb{R} \setminus (\mathbb{R} \setminus \{0\}) = \{0\}$, and so $\mathbb{R} \setminus \{0\}$ is not open. Therefore $\mathcal{T}_c$ doesn't satisfy the definition of a topology.

Definition 1.4 (Finer, Coarser Topologies)

Suppose that $\mathscr{T}$ and $\mathscr{T}'$ are two topologies on a given set X. If $\mathscr{T} \subset \mathscr{T}'$, we say that $\mathscr{T}'$ is **finer** than $\mathscr{T}$, or equivalently, we say that $\mathscr{T}$ is **coarser** than $\mathscr{T}'$.

We can think of the topology of a set X as a truck full of gravel as the open sets. If the gravel is smashed into smaller, finer pieces, then the amount of stuff that we can make from the finer gravel increases, which corresponds to a bigger topology. Clearly, the indiscrete topology is the coarsest topology and the discrete topology is the finest.

Theorem 1.1 (Intersection of Topologies)

Given a family of topologies $\{\mathscr{T}\}_{\alpha \in A}$, the set

$$\mathscr{T} = \bigcap_{\alpha \in A} \mathscr{T}_\alpha \tag{9}$$

is a topology.

Corollary 1.2 (Unique Coarsest and Finest Topology)

Given a family of topologies $\{\mathscr{T}\}_{\alpha \in A}$, there exists
1. a unique smallest topology on X containing all the collections $\mathscr{T}_\alpha$.
2. a unique largest topology on X contained in each $\mathscr{T}_\alpha$.

Example 1.3

Let $X = \{a, b, c\}$, and let

$$\mathscr{T}_1 = \{\emptyset, X, \{a\}, \{a, b\}\} \tag{10}$$
$$\mathscr{T}_2 = \{\emptyset, X, \{a\}, \{b, c\}\} \tag{11}$$

We claim that the
1. smallest topology containing $\mathscr{T}_1, \mathscr{T}_2$ is

   $$\mathscr{T}_{1\cup 2} = \{\emptyset, X\{a\}, \{b\}, \{a, b\}, \{b, c\}\} \tag{12}$$

   Note that this is not simply the union of topologies. The union wouldn't have $\{b\}$, making it not a topology.
2. largest topology contained in $\mathscr{T}_1, \mathscr{T}_2$ is

   $$\mathscr{T}_{1\cap 2} = \{\emptyset, X, \{a\}\} \tag{13}$$

   Note that this is simply the intersection of the two topologies.

1.1 Open Sets

This leads to the most general definition of an open set. Note that an open set doesn't really mean anything without talking about with respect to its topology.

Definition 1.5 (Open Set)

The elements of $\mathscr{T}$ are called **open sets** in X.[a]
1. An open set U which contains a point x is called an **open neighborhood** of x, denoted U_x.
2. Given an open neighborhood U_x of x, the set $U_x \setminus \{x\}$ is called the **punctured open neighborhood** of x.

[a] As implied from the definition of a topology, the arbitrary union and finite intersection of any number of open sets is an open set.

Lemma 1.3 (A Set Full of Open Sets is Open)

Given X with a topology $\mathscr{T}$, let $S \subset X$. Then S is open if for every $x \in S$, there exists an open neighborhood U_x satisfying $x \in U_x \subset S$.

Proof. Since we can set

$$S = \bigcup_{x \in S} U_x \tag{14}$$

it is an arbitrary union of open sets and therefore must be open.

1.2 Limit Points and Closed Sets

First, we need to learn what it generally means for a point to be infinitesimally close to a set. If we take a point and draw smaller and smaller circles around it, the circle itself should still overlap with S, no matter how small it gets.

Definition 1.6 (Limit Point)

Given a topological space $(X, \mathscr{T})$, let $x \in X$ be a point and $S \subset X$ a subset. x is a **limit point of** S if every punctured neighborhood of x intersects S. The set of all limit points of a set S is denoted S'.

Note that limit point are generally used to talk about points that are infinitesimally close to a set S. A limit point may not necessarily be in S, and a point of S may not necessarily be a limit point. This is why we use a *punctured neighborhood*, rather than an open neighborhood. For continuity as we will see later, we just talk about neighborhoods since we also claim that the limit exists and the function value is the limit.

Example 1.4 (Examples of Limit Points)

What about the limit points that are not in S? Generally, there are two instances.

1. Let S represent the gray area. B is in the "interior" of S and therefore is a limit point. A and C are on the "boundary" of S yet not in S, and we can show that they are limit points as well.

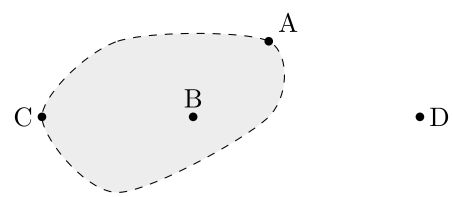


Figure 1: Points A, B, C are limit points of the open set.

2. A point can be at the "convergence point" of a sequence.

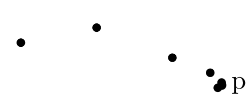


Figure 2: Note that if S is a sequence of points in $\mathbb{R}^2$ that converges to p without ever hitting it, we can say that $p \notin S$ is a limit point of S.

Example 1.5 (Examples of Non-Limit Points)

There are generally two instances of non-limit points. Let $X = \mathbb{R}$ and $S = (0, 1) \cup \{2\}$.
1. 5 is clearly not a limit point.
2. 2, although in S, is not a limit point since we are talking about the punctured neighborhood. A point in S that is not a limit point is called an **isolated point**.

Example 1.6 (Counterintuitive Limit Points in Lower Limit Topology)

Note that given an interval $(a, b) \subset \mathbb{R}$ in the lower limit topology, a is a limit point but b is *not* a limit point!

Theorem 1.4 (Union of Limit Points is Limit Points of Union)

Let $A_1, \ldots, A_n$ be a finite collection of sets. Then

$$\bigcup_{i=1}^{n} A_i' = \bigg(\bigcup_{i=1}^{n} A_i\bigg)' \tag{15}$$

Proof. Let the LHS be W and the RHS be V. If $x \in W$, $x \in A_i'$ for some i, and so for all $\epsilon > 0$, there exists a $B_\epsilon^\circ(x)$ s.t.

$$B_\epsilon^\circ(x) \cap A_i \neq \emptyset \implies B_\epsilon^\circ(x) \cap \bigg(\bigcup_{i=1}^{n} A_i\bigg) \neq \emptyset \tag{16}$$

which means that $x \in V$. Now assume that $x \in V$. Then for all $\epsilon > 0$, there exists a $B_\epsilon^\circ(x)$ s.t.

$$B_\epsilon^\circ(x) \cap \bigg(\bigcup_{i=1}^{n} A_i\bigg) \neq \emptyset \tag{17}$$

which implies that $B_\epsilon^\circ(x) \cap A_i \neq \emptyset$ for some i, which means that $x \in A_i' \subset W$.

Note that this is clearly not true for infinite unions. Look at the countable set $\mathbb{Q} \subset \mathbb{R}$. Each $\{q\}' = \emptyset$, but $\mathbb{Q}' = \mathbb{R}$. Look at the uncountable set $\mathbb{R}$. Each $\{x \in \mathbb{R}\}' = \emptyset$, but $\mathbb{R}' = \mathbb{R}$.

Definition 1.7 (Closed Set)

A set $S \subset X$ is **closed**[a] if it satisfies either of the equivalent properties.
1. Its complement $X \setminus S$ is open in $\mathscr{T}$.
2. It contains all of its limit points.

[a]Note that open and closed sets are not mutually exclusive. A set might be open, closed, both, or neither. A set that is both open and closed is called **clopen**.

Proof. We prove bidirectionally.
1. ($\rightarrow$) Given that S contains all its limit points, then let $x \in S^c$. x is not a limit point of S since if it were, then it would be in S, and so there exists a punctured open neighborhood $B_\epsilon^\circ(x)$ of x s.t. $S \cap B_\epsilon^\circ(x) = \emptyset$. Since $x \notin S$, we also have $S \cap B_\epsilon(x) = \emptyset$, which implies that $B_\epsilon(x) \subset S^c$. Since for every $x \in S^c$, there exists a $B_\epsilon(x) \subset S^c$, S^c is open.
2. ($\leftarrow$) For simplicity, it suffices to prove if S open, then S^c is closed. Given that S is open, we have for every $x \in S$, there exists $B_\epsilon(x) \subset S$, which implies that $B_\epsilon(x) \cap S^c = \emptyset$. Since there exists an

$B_\epsilon(x)$ that does not contain points in S^c, x cannot be a limit point of S^c, and so there exists no limit points of S^c in S. Therefore, all limit points of S^c are in S^c, proving that S^c is closed.

Theorem 1.5 (Topological Space wrt Closed Sets)

Let X be a topological space. Then, the following conditions hold

1. $\emptyset$ and X are clopen.
2. Arbitrary intersections of closed sets are closed.
3. Finite unions of closed sets are closed.

Proof. Listed.

1. Let x be a limit point of $\cap_\alpha F_\alpha$, and we want to show that $x \in \cap_\alpha F_\alpha$. By definition of limit points, for every $\epsilon > 0$, we have
$$B_\epsilon(x) \cap \left(\bigcap_\alpha F_\alpha \right) \tag{18}$$
which means that $B_\epsilon(x) \cap F_\alpha \neq \emptyset$ for all α. This means that x is a limit point for every F_α, and since they are all closed, $x \in F_\alpha$ for all α, which implies that $x \in \cap_\alpha F_\alpha$.

1.3 Dense Subsets

Definition 1.8 (Dense Subsets)

Let $S \subset (X, \mathscr{T})$. S is **dense** in X if every point $p \in X$ is a limit point of S. In other words, for any point $p \in X$ and any open neighborhood U_p of p, $U_p \cap S$ is nontrivial. Otherwise, p is a point of S.

The following example is a crucial fact for proving further properties of topological spaces.

Example 1.7

$\mathbb{Q}^n$ is a dense set of $\mathbb{R}^n$ with the open ball topology. If we have the discrete topology of $\mathbb{R}^2$, an open neighborhood of a point is the point itself, so no limit points would exist beyond the points in S itself. So $\mathbb{Q}^n$ is not dense in $\mathbb{R}^n$ with this topology.

1.4 Interiors and Closures

Now that we've determined limit points, we would like to extend sets into their limit points. The process of doing this is called the *closure* of a set.

Definition 1.9 (Closure)

The **closure** of set S is $\overline{S}$ is defined in the following equivalent ways.

1. $\overline{S} = S \cup S'$, i.e. the union of itself and its limit points.
2. $\overline{S}$ is the intersection of all closed sets C containing S.

Proof.

Theorem 1.6

If X is a metric space and $E \subset X$, then
1. $\overline{E}$ is closed.
2. $E = \overline{E}$ if and only if E is closed.
3. $\overline{E} \subset F$ for every closed set $F \subset X$ such that $E \subset F$. That is, if $E \subset F$ closed, then "increasing" the size of E to its closure will not make it greater than F.

Proof. Listed.
1. Let x be a limit point of $\overline{E}$. Then, for every $\epsilon > 0$, we have $B_\epsilon(x) \cap \overline{E} \neq \emptyset$, which means that either $B_\epsilon(x) \cap E \neq \emptyset$ (in which case $x \in E' \implies x \in \overline{E}$ and we are done) or $B_\epsilon(x) \cap E' \neq \emptyset$. We wish to prove that in the latter case, x being a limit point of E' still implies that x is a limit point of E. Since $B_\epsilon(x) \cap E' \neq \emptyset$, there must exist a $y \in B_\epsilon(x) \cap E'$. Since $y \in E'$, we can construct an open ball $B_\delta(y)$ containing elements of E, and since $B_\epsilon(x)$ is open, we can contain $B_\delta(y)$ entirely within $B_\epsilon(x)$. Therefore,
$$B_\delta(y) \cap E \neq \emptyset \implies B_\epsilon(x) \cap E \neq \emptyset$$
therefore, $x \in E' \implies x \in \overline{E}$.
2. If E is closed, then $E' \subset E \implies \overline{E} = E \cup E' = E$. If $E = \overline{E} = E \cup E'$, then $E' \subset E \implies E$ is closed.
3. Since $E \subset F$, it suffices to prove that $E' \subset F$. Consider a limit point x of E. Then every punctured open neighborhood of x satisfies $B^\circ_\epsilon(x) \cap E \neq \emptyset$. But since $E \subset F$, we have
$$B^\circ_\epsilon(x) \cap F \neq \emptyset$$
and so x is also a limit point of F. But since F is closed, $x \in F$. Therefore, $\overline{E} = E \cup E' \subset F$.

The first two statements (1) and (2) imply the following.

Corollary 1.7

The closure of the closure of E is equal to the closure of E.

Proof. We know that $\overline{\overline{E}} \supset \overline{E}$, so we must prove that $\overline{\overline{E}} \subset \overline{E}$, which is equivalent to proving that $\overline{E}' \subset \overline{E}$. Let $x \in \overline{E}'$, i.e. is a limit point of $\overline{E}$. Then, for every $\epsilon > 0$, we have $B_\epsilon(x) \cap \overline{E} \neq \emptyset$. Pick a point y from this intersection, and since $B_\epsilon(x)$ is open, we can construct an open ball $B_\delta(y)$ fully contained in $B_\epsilon(x)$. Since $y \in \overline{E}$, y is a limit point of E, which implies
$$B_\delta(y) \cap E \neq \emptyset \implies B_\epsilon(x) \cap E \neq \emptyset \tag{19}$$
and therefore x is a limit point of E, $x \in \overline{E}$.

Example 1.8

If S is an open ball, $\bar{S}$ is the closed ball.

From semantics, it may seem like the interior and exterior (defined later) are related, but from a mathematical point of view, the interior and closure are dual notions.

Definition 1.10 (Interior)

Let $S \subset X$. Then, the following definitions of the **interior** of S, denoted S°, are equivalent.

1. $x \in S^\circ$ if $\exists U_x \ni x$ s.t. $U_x \subset S$.
2. S° is the union of all open sets contained in S.
3. S^o is the complement of the closure of the complement of S.

$$S^o = \left(\overline{S^c}\right)^c \tag{20}$$

Proof.

An interior point means that we can always contain the point in S with some "breathing room." By definition an open set is a set where all of its points are interior points. A set is then said to be open if every point has this breathing room. This can be useful when defining differentiation at a point within an open set, since we can always find a neighborhood to take limits on.

Lemma 1.8 (Open and Closed in Terms of Interiors and Closures)

Let S be a subset of some topological space X.

1. S is open iff $S = S^o$. S^o is always open.
2. S is closed iff $S = \overline{S}$. $\overline{S}$ is always closed.

Theorem 1.9 (Clopen sets in Reals)

There are no proper clopen sets in $\mathbb{R}$.

1.5 Exteriors and Boundaries

Definition 1.11 (Exteriors)

Let $S \subset X$. The **exterior** of S, denoted S^e, is defined in the following equivalent ways.[a]

1. S^e is the complement of the closure of S.
2. S^e is the interior of the complement of S.

[a] We can informally think of the exterior being strictly outside of S and its boundary.

Proof.

Definition 1.12 (Boundary)

Let $S \subset X$. The **boundary** of S, denoted ∂S, is defined in the following equivalent ways.

1. ∂S is the closure minus the interior of S in X.
2. ∂S is the intersection of the closure of S with the closure of its complement, i.e the set of all points x such that every neighborhood U_x intersects both the interior and exterior.
3. ∂S is the set of points that are neither in the exterior nor the interior.
4. $x \in \partial S$ if every neighborhood of x intersects both the interior and exterior of S.

Proof.

From the above, we get the intuitive notion that these three parts divide up the whole space.

Theorem 1.10 (Partitioning of Space)

Given $S \subset X$, X is partitioned into the interior, boundary, and exterior of S.

$$X = S^\circ \sqcup \partial S \sqcup S^e \tag{21}$$

Proof. The fact that

One counterintuitive result is the Lakes of Wada, which are three disjoint connected open sets of the open unit square $(0,1)^2$ with the property that they *all* have the same boundary. In other words, for any point selected on the boundary of one of the lakes, the other two lakes' boundaries also contain that point.

1.6 Basis

We want to continue analyzing the properties of a topology, but sometimes working with the entire topology is a bit thorny. There is a tamer representation of a topology, which can also give us the starting point to *construct* topologies.

Definition 1.13 (Basis)

If X is a set, a **basis** on X is a collection $\mathscr{B}$ of subsets of X (called **basis elements**) such that

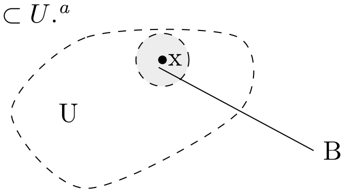
1. For each $x \in X$, there is at least one basis element $B \in \mathscr{B}$ containing x. That is, the elements of $\mathscr{B}$ covers X.
2. If x belongs to the intersection of two basis elements B_1 and B_2, then there is a basis element B_3 containing x such that $B_3 \subset (B_1 \cap B_2)$.

The name gives away the clue that a topology may be created from this basis.

Definition 1.14 (Basis to Topology)

Given a basis $\mathscr{B}$ on a set X, we can define a topology $\mathscr{T}$, called the **topology generated by** $\mathscr{B}$, in the following equivalent ways.

1. $\mathscr{T}$ consists of subsets U of X satisfying the property that for each $x \in U$, there exists a basis element $B \in \mathscr{B}$ such that $x \in B \subset U$.[a]



2. $\mathscr{T}$ consists of all possible unions of elements in $\mathscr{B}$.

$$\mathscr{T} := \left\{ \bigcup_i B_i \mid B_i \in \mathscr{B} \right\} \tag{22}$$

[a]Note that since we can always set $U = \emptyset$, the basis doesn't need to contain $\emptyset$.

Proof. We prove that the 2 methods generate a topology, and then finally prove that it they are the same topology.

1. Clearly, $\emptyset$ and X itself are in $\mathscr{T}$. To prove property 2, given a certain indexed family of subsets $\{U_\alpha\}_{\alpha \in I}$ of $\mathscr{T}$, we must show that

$$U = \bigcup_{\alpha \in I} U_\alpha \in \mathscr{T} \tag{23}$$

Given $x \in U$, there exists at least one index α such that $x \in U_\alpha$. Since $U_\alpha \in \mathscr{T}$ already, there exists a basis element $b \in \mathscr{B}$ such that $x \in b \subset U_\alpha$. But

$$U_\alpha \subseteq U \implies b \subset U \tag{24}$$

So, by definition, any arbitrary union of U of these subsets is also in $\mathscr{T}$.
To prove property 3, we must show that

$$W = \bigcap_{\alpha \in I} U_\alpha \in \mathscr{T} \tag{25}$$

Given $x \in W$, by definition of a basis element, there exists a $b \in \mathscr{B}$ such that

$$x \in b \subset (U_\beta \cap U_\gamma) \forall \beta, \gamma \in I \implies \text{ there exists } \tilde{b} \in \mathscr{B} \text{ s.t. } x \in \tilde{b} \subset \bigcap_{\alpha \in I} U_\alpha \tag{26}$$

By definition, W is also open. Since this arbitrary set of subsets $\mathscr{T}$ suffices the 3 properties, it is a topology of X by definition.

2. ($\rightarrow$) Given a collection of elements in $\mathscr{B}$, they are also elements of $\mathscr{T}$. Since $\mathscr{T}$ is a topology, their union in also in $\mathscr{T}$.
($\leftarrow$) Given an open set $U \in \mathscr{T}$, for every point $x \in U$, by definition we can choose a basis element $b \in \mathscr{B}$ such that $x \in b \subset U$. Then, the union of all these basis elements is by definition b.

We have learned how to go from a basis to a topology. The following lemma tells us how to identify a basis within a topology.

Theorem 1.11 (Topology to Basis)

Let $(X, \mathscr{T})$ be a topological space, and let $\mathscr{B}$ be a collection of open subsets of X such that for every open set U and each $x \in U$, there exists an element $B \in \mathscr{B}$ such that

$$x \in B \subset U \tag{27}$$

Then, $\mathscr{B}$ is a basis for the topology of X.

Proof. Note that there are two claims here: $\mathscr{B}$ is a basis and the topology that $\mathscr{B}$ generates is equal to $\mathscr{T}$.

1. To prove that $\mathscr{B}$ is a basis, note that X is an open set, and by assumption, for every $x \in X$, there exists a $B \in \mathscr{B}$ s.t. $x \in B \subset X$. Therefore $\mathscr{B}$ covers X. Now take two basis elements $B_1, B_2 \in \mathscr{B}$ with $x \in B_1 \cap B_2$. Since we know that B_1, B_2 are open, $B_1 \cap B_2$ is open and so for each $x \in B_1 \cap B_2$, there exists a basis element B_3 s.t. $x \in B_3 \in (B_1 \cap B_2)$. Thus $\mathscr{B}$ is a basis.
2. Let us call $\mathscr{T}'$ the topology generated by $\mathscr{B}$. Then, given $U \in \mathscr{T}$, by assumption for any $x \in U$, there exists a basis element $B \in \mathscr{B}$ s.t. $x \in B \subset U$, so $x \in \mathscr{T}'$. Conversely, if $U \in \mathscr{T}'$, then U is an arbitrary union of elements $B \in \mathscr{B}$ where each B is open in $\mathscr{T}$, so $U \in \mathscr{T}$. So $\mathscr{T} = \mathscr{T}'$.

Characterizing topologies in terms of basis is quite effective since we can work with more manageable sets.

Lemma 1.12 (Fineness w.r.t. Basis)

Given two topologies $\mathscr{T}$ and $\mathscr{T}'$ with their bases $\mathscr{B}$ and $\mathscr{B}'$, respectively, the following are equivalent.

1. $\mathscr{T}'$ is finer than $\mathscr{T}$.
2. For each $x \in X$ and basis element $B \in \mathscr{B}$ containing x, there exists a basis element $B' \in \mathscr{B}'$ such that $x \in B' \subset B$.

So we have seen how we can take a collection of sets satisfying the basis properties and construct a topology as the union of the sets in this collection. What happens if we can relax some of these conditions? Note that the first condition was that the basis elements must cover X. This is non-negotiable. However, if we remove the second requirement that a basis element must be contained in an intersection of basis elements, we can get a *subbasis*.

Definition 1.15 (Subbasis)

A **subbasis** $\mathscr{S}$ for a topology on X is a collection of subsets of X whose union is equal to X.

Theorem 1.13 (Subbasis to Topology)

Given a subbasis $\mathscr{S}$ on a set X, the **topology generated by** $\mathscr{S}$ is defined to be the collection $\mathscr{T}$ of all unions of finite intersections of elements of $\mathscr{S}$.

Proof. It suffices to show that the collection of finite intersections of elements form a basis.

Now we show three extremely common topologies that can be constructed with bases.

1.6.1 Order Topology

Definition 1.16 (Order Topology)

Let X be a set with a simple order relation. Let $\mathscr{B}$ be the collection of all sets of the following types.[a]

1. All open intervals $(a, b) \coloneqq \{x \in X \mid a < x < b\} \subset X$
2. All half-open intervals $[a_0, b)$, where a_0 is the minimum element of X
3. All half-open intervals $(a, b_0]$, where b_0 is the maximum element of X.

This set $\mathscr{B}$ is a basis for the **order topology** of X.

[a] If X has no minimum or maximum, then there are no sets of type of 2 or 3, respectively.

Proof. We prove that this set $\mathscr{B}$ is a basis.

1. It covers X. If $x \in X$ is the maximum or minimum we can cover it with $(a, b_0]$ and $[a_0, b)$, respectively. If not, then x is bounded above and below, and so there exists $a, b \in X$ s.t. $a < x < b \implies x \in (a, b)$.
2. Let $x \in (a_1, b_1)$ and $x \in (a_2, b_2)$. Then,

$$x \in (\max\{a_1, a_2\}, \min\{b_1, b_2\}) \in \mathscr{B} \tag{28}$$

Therefore, the generated collection is indeed a topology.

Example 1.9 (Standard Order Topology on $\mathbb{R}$)

The standard topology on $\mathbb{R}$ is precisely the order topology derived from the usual order on $\mathbb{R}$. Since $\mathbb{R}$ has no minimum or maximum, the basis consists of open intervals $(a, b) \subset \mathbb{R}$ with $a, b \in \mathbb{R}$.

Example 1.10 (Basis of Open Intervals with Rational Endpoints)

We can however get away with smaller basis that generate the same topology on $\mathbb{R}$. If we take the set of all open intervals $(a, b) \subset \mathbb{R}$ with $a, b \in \mathbb{Q}$, this is also a basis for the same standard order topology. Too see why, let us denote this basis as $\mathscr{B}'$ and the basis of all open intervals with real endpoints be

$\mathscr{B}$. Then, clearly $B' \subset \mathscr{B} \implies \mathscr{T}' \subset \mathscr{T}$. As for the other, way, let us take an open interval $(a, b) \in \mathscr{B}$. Then we can see that

$$(a, b) = \bigcup_{\substack{p,q \in \mathbb{Q} \\ a<p,q<b}} (p, q) \tag{29}$$

where equality follows from density of rationals in $\mathbb{R}$.

Example 1.11 ($\mathbb{R}^2$ with Dictionary Order)

Given $\mathbb{R} \times \mathbb{R}$ with the dictionary order, then $\mathbb{R} \times \mathbb{R}$ has neither a largest nor smallest element. Therefore, the order topology on $\mathbb{R} \times \mathbb{R}$ consists of all "intervals" of form

$$\big((a, b), (c, d)\big) \coloneqq \{(x, y) \in \mathbb{R}^2 \mid (a, b) < (x, y) < (c, d)\} \tag{30}$$

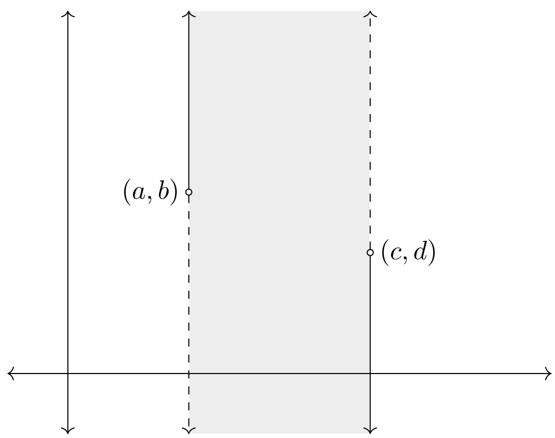


Figure 3: This means that open rays and lines are also a part of the topology of $\mathbb{R} \times \mathbb{R}$.

Example 1.12 (Positive Integers)

The set of positive integers $\mathbb{Z}_+$ form an ordered set with a smallest element. The order topology for $\mathbb{Z}_+$ is precisely the discrete topology since every one-point set is an open set.

$$\{n\} = (n - 1, n + 1) \tag{31}$$

Example 1.13 (Two Copies of Positive Integers)

The dictionary order topology on $\{1, 2\} \times \mathbb{Z}_+$ results in every one point set being open, except for the point $(2, 1)$. Since every neighborhood of $(2, 1)$ must contain some point of form $(1, n)$ for arbitrarily large n, $\{(2, 1)\}$ is not open.

Definition 1.17

If X is an ordered set a $a \in X$, then there are 4 subsets of X called rays determined by a.
1. $(a, +\infty)$
2. $(-\infty, a)$
3. $[a, +\infty)$

4. $(-\infty, a]$

The first two sets are called **open rays**, and the latter two sets are called **closed rays**.

We can extend the basis of open intervals to some other basis based on the order, which generates other topologies.

Example 1.14 (Lower/Upper Limit Topology)

Given a totally ordered set $(X, \leq)$,

1. the **lower limit topology** is the topology generated by the basis of all half-closed half-open intervals of form
$$[a, b) := \{x \in X \mid a \leq x < b\} \tag{32}$$
2. the **upper limit topology** is the topology generated by the basis of all half-open half-closed intervals of form
$$(a, b] := \{x \in X \mid a < x \leq b\} \tag{33}$$

Example 1.15 (Nested Interval Topology)

In the space $X = (0, 1)$, the **nested interval topology** is the topology generated by the basis of nested intervals of the form
$$\mathscr{B}_{ni} := \{(0, 1 - \frac{1}{n}) \mid n \in \mathbb{N}\} \tag{34}$$

A topology generated by closed intervals can also be a topology!

Example 1.16 (Closed Interval Topology)

In the set $X = [-1, 1]$, the following set
$$\mathscr{B}_{ci} := \{[-1, a) \mid a > 0\} \bigcup \{(b, 1] \mid b < 0\} \tag{35}$$
is a basis. The topology it generates is called the **closed interval topology**, denoted $\mathscr{T}_{ci}$.

Finally, we talk about a seemingly arbitrary topology called the K-topology, but it is useful for counterexamples.

Example 1.17 (K-Topology)

In $\mathbb{R}$, let us denote $K = \{1/n\}_{n \in \mathbb{N}}$. Then the **K-topology** on $\mathbb{R}$ is the topology generated by the basis consisting of

1. all open intervals (a, b) with $a, b \in \mathbb{R}$.
2. all sets of the form $(a, b) \setminus K$ with $a, b \in \mathbb{R}$.

Now that we have some collection of topologies, let's try to compare them. We claim the following.

Theorem 1.14 (Comparison of Topologies of the Real Line)

1.6.2 Metric Topology

For common sets like $\mathbb{R}^n$, which has an inner product, or $\mathbb{Q}$, which has an order, it is easy to build these topologies with set-builder notation. Consider the following.

Definition 1.18 (Metric Topology)

Given a metric space (X, d), let us denote the **metric topology**, or **open-ball topology**, as the set of subsets U satisfying the property that for all $x \in U$, there exists a positive $r \in \mathbb{R}$ such that $B(x, r) \subset U$, where $B(x, r) \coloneqq \{y \in X \mid d(x, y) < r\}$ is the open ball of radius r around x. We claim that this is a topology.

Proof. We show that the properties of a topology hold.

1. For the empty set, the inclusion of an open ball for a point in $\emptyset$ is vacuously satisfied. For the whole set, we choose any point x and any r, and the open ball is trivially a subset of X.
2. Let $\{U_\alpha\}_{\alpha \in I}$ be a collection of open subsets of X. Let their union be denoted U. We claim U is open. Pick any point $x \in U$. Then by definition of union, there exists some $\alpha \in I$ s.t. $x \in U_\alpha$. Since U_α is open, there exists a $r > 0$ s.t. $B(x, r) \subset U_\alpha \subset U$. Therefore U is open.
3. Let $U_1, \ldots, U_k$ be open, and let us denote their intersection as U. We claim U is open. Pick a point $x \in U$. Then for each $i = 1, \ldots, k$, $x \in U_i$ and there exists a corresponding $r_i > 0$ such that the open ball $B(x, r_i) \subset U_i$. Take the set $R = \{r_i\}$, which is a finite set living in $\mathbb{R}$. We will take for granted that every finite subset of an ordered set has a minimum.[a] Let us denote $r^* = \min R$, and we claim that r^* gives us a ball that can fit inside U. Assume $y \in B(x, r^*)$. Then

$$\begin{aligned} y \in B(x, r^*) &\implies d(x, y) < r^* && (36) \\ &\implies d(x, y) < \min R && (37) \\ &\implies d(x, y) < r_i \text{ for } i = 1, \ldots, k && (38) \\ &\implies y \in B(x, r_i) \text{ for } i = 1, \ldots, k && (39) \end{aligned}$$

Since $B(x, r_i)$ by construction is contained within U_i, $y \in U_i$ for all i. This means by definition of intersection that $y \in U$, and we have proven that $B(x, r^*)$ completely fits inside U.

[a] If we wish to prove it, we can start with a singleton set, claim that its minimum is the only element. Then we use induction by assuming for a set R of size k that a minimum exists, and by adding 1 more element r we update the minimum to be $\min\{r, \min R\}$ and show that this is indeed the minimum.

Note that while open balls are used to define whether a set is open or not, the definition doesn't state whether open balls themselves are open sets. It turns out that it is easy to prove that they are.

Lemma 1.15 (Open Balls are Open Sets)

The open ball wrt any metric d is an open set wrt the metric topology.

Proof. Let $y \in B(x, r)$. Then $d(x, y) < r \implies 0 < r - d(x, y)$. To show that $B(x, r)$ is open, we would like to show that there exists some $r' > 0$ s.t. $y \in B(y, r') \subset B(x, r)$. Set $r' = r - d(x, y)$. Then

$$\begin{aligned} z \in B(y, r') &\implies d(y, z) < r - d(x, y) && (40) \\ &\implies d(x, y) + d(x, y) < r && (41) \\ &\implies d(x, z) < r && (42) \\ &\implies z \in B(x, r) && (43) \end{aligned}$$

and so $B(y, r') \subset B(x, r)$. We are done.

Example 1.18 (Discrete Metric Induces Discrete Topology)

Given a set X, induce the metric d defined

$$d(x, y) := \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases} \tag{44}$$

This metric induces the discrete topology on X, since the basis elements of the open balls

$$B_r(x) := \{y \in X \mid d(x, y) < r\} \tag{45}$$

consists of two types of open sets. When $r \leq 1$, then $B_r(x) = x$ (since the radius is 0). If $r > 1$, then the open set is the entire space X.

While the behavior for finite sets are predictable under the metric topology, as soon we we get into infinite sets, the properties of the metric topology may differ.

Example 1.19 (Metric Topologies on $\mathbb{Z}$ and $\mathbb{Q}$)

$\mathbb{Z}$ and $\mathbb{Q}$ are countable sets, so there is a bijection between them. If we give each of them the metric topology, $\mathbb{Z}$ ends up having the discrete topology (take the 0.5-ball around each integer), whereas for $\mathbb{Q}$, we will see later that by the density of the rationals there are an infinite number of rationals in $(q - r, q + r)$ for $q \in \mathbb{Q}$. Therefore, the metric topology may or may not induce the discrete topology.

Example 1.20 (Supremum Norm in $\mathbb{R}^3$)

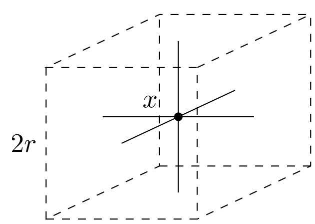


Figure 4: In $\mathbb{R}^3$, each basis element is a cube centered at x with side lengths $2r$.

Theorem 1.16 (Metric Topologies on Finite Sets)

If (X, d) is a finite metric space, then the metric topology on it is the discrete topology.

Proof. Take all pairwise points and compute $\epsilon = \min_{x \neq y}\{d(x, y)\}$. Since X is finite, all pairs are finite and therefore the minimum exists. Now let us take the ϵ-ball around x. Then every $y \neq x$ has distance $d(x, y) \geq \epsilon$, and therefore $y \notin B(x, \epsilon)$. So all single points are open sets, which induces the discrete topology.

A finite set S of points does not have any limit points, since if we draw small enough circles around a $p \in S$, then at some point the circle will not contain any more points (remember that we're talking about deleted neighborhoods). Following this, we can deduce that a limit point must always have an infinite number of points close to it, as in no matter how small the circle gets, there are always an infinite number of points contained within that circle. This also means that if p is a limit point, then we can construct a sequence of points in S that converges to p, since every open ball with smaller and smaller radii will still have points in

S.

Theorem 1.17 (Neighborhood of Limit Point Contains Infinite Points in Metric Space)

Let X be a metric space. If p is a limit point of S, then every neighborhood of p contains infinitely many points of S. The converse is also true trivially.

Proof. Assume p is a limit point and that there exists a finite number of points within a deleted neighborhood $B_r^\circ(p)$. Then, we can enumerate them $p_1, p_2, \ldots, p_n$ by their distances to p, with

$$d(p_1, p) \leq d(p_2, p) \leq \ldots \leq d(p_n, p) \tag{46}$$

Since $p_1 \neq p$, we have $d(p_1, p) > 0$ and so, we can choose an $0 < \epsilon < d(p_1, p)$ s.t. $B_\epsilon^\circ(p)$ does not contain any of the p_i's. This neighborhood does not contain any elements of S and so p is not a limit point.

Corollary 1.18 (Finite Set in Metric Space has No Limit Points)

Let X be a metric space and $S = \{s_i\}_{i=1}^n$ be a finite set. Then, $S' = \emptyset$.

Proof. If S is a finite set, then every neighborhood of every point p in $\mathbb{R}^n$ will have at most finite points, which, by the previous theorem, is not a limit point.

Lemma 1.19 (Fineness of Metric Topologies)

Let d and d' be two metrics on the set X with their respective induced topologies $\mathscr{T}, \mathscr{T}'$. We claim that $\mathscr{T} \subset \mathscr{T}'$ iff there exists a $M > 0$ s.t.

$$d'(x, y) < M \cdot d(x, y) \tag{47}$$

for all $x, y \in X$. That is, we can bound d' with a constant multiple of d.

Proof.

More specifically, the metric topology generated by the L_2-metric on $\mathbb{R}^n$ is called the **Euclidean topology**. Note that the topological property of stability under countable intersection was required to show that the minimum of R existed. This is not true for infinite sets in general. This gives us some motivation as to why we need the *finite* intersection rather than an infinite one.

Lemma 1.20 (Singletons are Not Open in $\mathbb{R}^n$)

A singleton set is not open in $\mathbb{R}^n$ with the Euclidean topology.

Proof. We claim that the singleton set $S = \{0\}$ is not open under the Euclidean metric. We pick a point in S, which can only be 0. Assume that there exists an $r > 0$ s.t. $B(x, r) \subset S$. $\mathbb{R}$ is Archimedean, so there exists a natural number N s.t. $0 < 1/N < r$. We construct the vector $v = (v_1, \ldots, v_n)$ s.t. $v_1 = 1/N$ and $v_i = 0$ everywhere else. The distance between 0 and v is

$$\|v - 0\| = \|v\| = \sqrt{(1/N)^2} = 1/N < r \tag{48}$$

so $v \in B(x, r)$. But $v \neq 0$, and by contradiction such an r cannot exist. In $\mathbb{R}^n$ we consider the countable intersection of open balls (which we have proved in class are open sets) around 0 of radius $1/N$ for

$n \in \mathbb{N}$. We claim that

$$\bigcap_{n\in\mathbb{N}} B(0, 1/n) = \{0\} \tag{49}$$

We see that $1/n$ must always be positive and so $\|0-0\| = 0 < 1/n$. Therefore the LHS $\supset$ RHS. To see that the intersection contains no other element, consider any vector $v \neq 0$. Then by definition of the metric, $d(v, 0) > 0$. By the Archimedean property, there exists a natural $N \in \mathbb{N}$ s.t. $0 < 1/N < d(v, 0)$, which means that $v \notin B(0, 1/N)$, and so v cannot be in the intersection. Therefore, the intersection must be $\{0\}$, and we have shown that B_0 is not open, so we are done.

Theorem 1.21 (Metric Topology is Always Finer than Cofinite)

For a metric space (X, d), the metric topology is finer than the cofinite topology.

Proof. Note that if X is finite, then both are reduced to the discrete topologies.

While it is not surprising that a basis uniquely generates a topology, it is not immediately obvious *what* the generated topology looks like. It turns out that many different bases may generate the same topology, and the concept of fineness allows us to compare these topologies more effectively. For example, if two topologies are both finer than the other, then they must be equal.

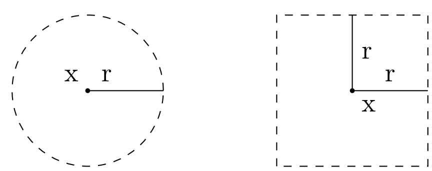
Theorem 1.22 (Euclidean Topology on $\mathbb{R}^n$)

L^p norms all generate the same topology on $\mathbb{R}^n$.

Proof. We can show that

$$n^q d_\infty \leq n^q d_2 \leq n^q d_1 \leq d_p \leq n^{-p} d_\infty \tag{50}$$

where q is the holder conjugate of p. Visually, we can see that every open ball in $(\mathbb{R}^n, d)$ (with the Euclidean metric) is the form to the left, while an open ball in $(\mathbb{R}^n, \rho)$ (with the square metric) is of form on the right.



Clearly, we can form any open set of any "shape" using any arbitrary combination of these "circles" or "squares," indicating that they generate the same topology.

2 Limits and Continuity

2.1 Limits of Sequences

Recall what a sequence is.

Definition 2.1 (Convergence of Sequence in Topological Space)

A sequence (x_n) of points in topological space $(X, \mathscr{T})$ is said to **converge** to the point $x \in X$ if $\forall$ open neighborhoods $U_x \ni x$, $\exists N \in \mathbb{N}$ such that

$$n \geq N \implies x_n \in U \tag{51}$$

If there exists no limit, then (x_n) is said to **diverge**.

Note that x being the limit of a sequence (x_i) is stronger than the claim that x is a limit point of $\{x_i\}$. If we consider the sequence $0, 1, 0, 1, \ldots$, we can see that both 0 and 1 are limit points, but the limit does not exist. We would like to define some notion of limit points in the language of sequences. We can precisely do this by treating a sequence as a set and talking about subsequential limits.

Definition 2.2 (Subsequences)

A **subsequence** of $(x_n)_n$ is a sequence $(x_{n_k})_k$, where $(n_k)_k$ is a strictly increasing infinite subsequence of $1, 2, 3, \ldots$.

Definition 2.3 (Partial Limits)

The **partial limit** of a sequence (x_n) is the limit of any of its subsequence.

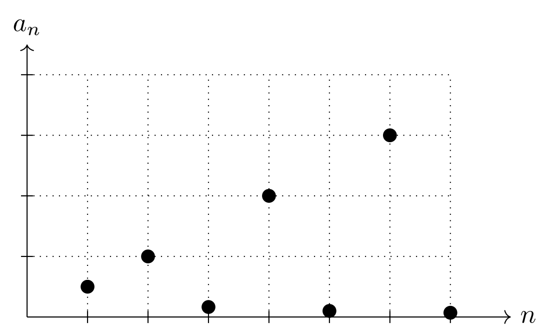


Figure 5: Two partial limits of the sequence $a_n = 1/n$ for n odd and $n/2$ for n even, is $+\infty$ and 0.

Lemma 2.1 (Partial Limits Equivalent to Limit Point)

Given a sequence (x_n), x is a limit point of (x_n) iff there exists a subsequence of (x_n) that converges to x.

Theorem 2.2 (Convergence of Sequence in Metric Space)

Let (X, d) be a metric space. x is the limit of (x_n) if it satisfies one of the two equivalent conditions:

1. if $x_n \to x$ under the metric topology generated by d

2. $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ such that
$$n \geq N \implies d(x_n, x) < \epsilon \tag{52}$$

2.2 Limits of Functions

Now we talk about limits of functions. We will talk about a variable x approaching a particular value $a \in X$, denoted $x \to a$. But this isn't clear. When we talk about the concept of something approaching another thing, we have established two definitions.

1. A *sequence* can approach to its limit, which is a *point*.
2. A *point* can be a limit point of a *set*.

When we write $x \to p$, we are talking about some indeterminate variable x and a point p, it isn't immediately clear what this means. As we will soon define, this will refer to a neighborhood of p or equivalently to *all* sequences converging to p. So informally, we can think of $x \to p$ as notation for all sequences $(x_n) \to p$.

Definition 2.4 (Limit of Function between Topological Spaces)

Let $f : E \subset X \to Y$ be a map between topological spaces and $p \in X$ be a limit point of E. The limit of f at x is any[a] point $q \in Y$ satisfying the following: For all open neighborhoods $V_q \subset Y$ of q, there exists a punctured open neighborhood $\mathring{U}_p \subset X$ of x s.t.
$$f\big(\mathring{U}_p \cap E\big) \subset V_q \tag{53}$$

[a] Note that this limit q may not be unique unless Y is Hausdorff.

Note that while the definition may look technically complicated, it makes sense. First, we want p to be a limit point since a function can "tend toward" some boundary. We also want to take the punctured open neighborhood to ensure that $x \neq p$, since functions can jump at p. Finally, we want to map $\mathring{U}_p \cap E$ since that is what f is defined over.

Theorem 2.3 (Limit of a Function between Metric Spaces)

Let $f : E \subset X \to Y$ be a map between metric spaces and $p \in X$ be a limit point of E. We say $f(x) \to q$ as $x \to p$, i.e.
$$\lim_{x \to p} f(x) = q \tag{54}$$
if it meets the following equivalent conditions.

1. *ϵ-δ Definition.* If $\forall \epsilon > 0$, $\exists \delta > 0$ s.t. $0 < d_X(x, p) < \delta \implies d_Y(f(x), q)) < \epsilon$.

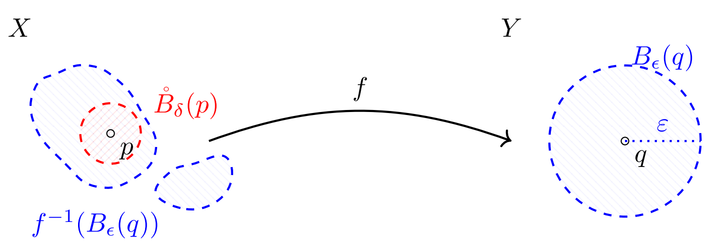


Figure 6: Said in one line, the preimage of any open ball around $y = f(x)$ must contain some open deleted open ball around x.

2. *Sequential Definition.* If for all sequences $(x_n) \to p$, $f(x_n) \to q$.

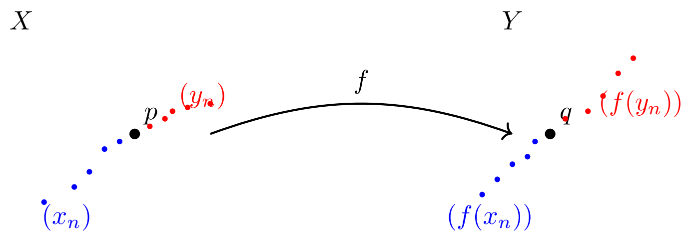


Figure 7: For every sequence that converges to the left, the new sequence mapped through f converges to q. Note that we choose the points x_n to be in the "deleted" neighborhood $E \setminus a$ (neighborhood E with point a removed) to force us to choose a sequence that is not $a, a, \ldots$. That is, it forces us to choose different points for the sequence.

Proof. We prove equivalence.

1. ($\rightarrow$). Assume $\lim_{x \to p} f(x) = q$. Let $(x_n) \in E$ s.t. $x_n \to p$ with $x_n \neq p$. We wish to show that $f(x_n) \to q$. Let $\epsilon > 0$. Then $\exists \delta > 0$ s.t. $0 < d_X(x, p) < \delta \implies d_Y(f(x), q) < \epsilon$. Since $\delta > 0$, by definition $\exists N \in \mathbb{N}$ s.t. if $n \geq N$, $d_X(x_n, p) < \delta \implies d_Y(f(x_n), q)$.

Sometimes, the ϵ-δ definition is good, but a lot of the times the sequential definition is good enough and more insightful. Note also that the topological definition of a limit does not include the sequential definition because it is not true.[2]

Example 2.1 (Counterexample)

2.3 Continuous Functions

Note that from set theory, we can construct functions as a subset of Cartesian product of two spaces X, Y. There is nothing new here.

Definition 2.5 (Continuous Function)

A function f between 2 topological spaces $(X, \mathscr{T}_X)$ and $(Y, \mathscr{T}_Y)$ is **continuous at** $x \in X$ if the preimage of every open neighborhood of $f(x) \in Y$ is an open neighborhood of $x \in X$.

$$U_{f(x)} \in \mathscr{T}_Y \implies x \in f^{-1}(U_{f(x)}) \in \mathscr{T}_X \tag{55}$$

f is said to be **continuous** (at all points) if the preimage of every open set in Y is an open set in X.[a]

[a]Note that continuity of a function f is not only determined by the function itself, but also by the topologies of X and Y.

Note that it is easier for f to be continuous when the $\mathscr{T}_X$ is finer (since there are more open sets in X for the preimage of $V \subset Y$ to map to) or $\mathscr{T}_Y$ is coarser (since there are fewer open sets that we have to check to map to open sets of X).

Theorem 2.4 (Sufficient Properties for Continuity)

Let X, Y, be topological spaces and let $f : X \longrightarrow Y$. Then, the following are equivalent to f being continuous.

[2]https://math.stackexchange.com/a/3151525/616717

1. The preimage of every basis element $B \in \mathscr{T}_Y$ is open in X.
2. For every closed set B in Y, the set $f^{-1}(B)$ is closed in X.
3. For every subset A of X, $f(\bar{A}) \subset f(\bar{A})$.

Proof. Listed.

1. An arbitrary open set V of Y can be written as $V = \cup_{\alpha \in J} b_\alpha$. Then,

$$f^{-1}(V) = f^{-1}\Big(\bigcup_{\alpha \in J} b_\alpha\Big) = \bigcup_{\alpha \in J} f^{-1}(b_\alpha) \tag{56}$$

Great, so we have a few ways in which we can check continuity of a function. There are a few special cases.

Lemma 2.5 (Trivially Continuous Functions)

We have the following for general topological spaces.

1. The identity function $\text{Id} : (X, \mathscr{T}_1) \to (X, \mathscr{T}_2)$ is continuous if $\mathscr{T}_1 \supset \mathscr{T}_2$.
2. A constant function $f : (X, \mathscr{T}) \to (Y, \mathscr{T}_2)$ is always continuous, regardless of the topologies.

Proof. If we take an open set $U \in \mathscr{T}_2$, its preimage is the same set U, which is guaranteed to be in $\mathscr{T}_1$ since $\mathscr{T}_1$ is finer than $\mathscr{T}_2$.

2.4 Construction of Continuous Functions

Theorem 2.6 (Arithmetic on Real Continuous Functions)

If X is a topological space, and if $f, g : X \longrightarrow \mathbb{R}$ are continuous, then $f+g$, $f-g$, and $f \cdot g$ are also continuous. f/g is continuous if $g(x) \neq 0$ for all $x \in X$.

Theorem 2.7 (Analytic Continuity = Topological Continuity)

Given metric spaces with their induced metric topologies $(X, \mathscr{T}_X, d_X)$ and $(Y, \mathscr{T}_Y, d_Y)$. The following are equivalent.

1. $f : X \to Y$ is continuous at x.
2. For every $\delta > 0$, there exists an $\epsilon = \epsilon(\delta) > 0$ such that for all $z \in X$, $d_X(x, z) < \epsilon \implies d_Y(f(x), f(x)) < \delta$.[a]

[a] This is the definition of continuity at a point in analysis.

Proof. ($\rightarrow$) Assume f is continuous according to the $\epsilon - \delta$ definition. Let U be any open set in Y containing the point y, and let x be an element in $f^{-1}(U)$ such that $y = f(x)$. We must prove that $f^{-1}(U)$ is also open. Since open sets contain neighborhoods (e.g. open balls) of all of its points, we can claim that, since U is open by assumption, there exists an open ball B_y around y with radius $\epsilon > 0$. This guarantees the existence of a point $z \in U$ such that $\rho(y, z) < \epsilon$ for any $\epsilon > 0$ that we choose. Since f is continuous, for every $\epsilon > 0$ that we chose previously, there exists a $\delta > 0$ such that $d(x, w) \implies \rho(f(x), f(w)) < \epsilon$. Since $\rho(f(x), f(w)) < \epsilon$, we can conclude that $f(w) \in B_y \subset U$ when $d(x, w) < \delta$. Therefore, $d(x, w) < \delta \implies w \in f^{-1}(U)$. But this is equivalent to saying that if $w \in B_(x, \delta)$, then $w \in f^{-1}(U)$, which means that every single point $x \in f^{-1}(U)$ contains an open ball neighborhood fully contained in $f^{-1}(U)$. So, by definition, $f^{-1}(U)$ is open.
($\leftarrow$) Assume $f^{-1}(U)$ is open when U is an open set in Y, i.e. f is continuous under the topological

definition. Let us define the open ball

$$B(f(x), \epsilon) \coloneqq \{y \in Y \mid \rho(f(x), y) < \epsilon\} \in \mathscr{T}_Y \tag{57}$$

By our assumption, $f^{-1}\big(B(f(x), \epsilon)\big)$ is an open set in $\mathscr{T}_X$, and clearly, $x \in f^{-1}\big(B(f(x), \epsilon)\big)$ since f^{-1} maps the point $f(x) \in B(f(x), \epsilon)$ to $x \in f^{-1}\big(B(f(x), \epsilon)\big)$. But since $f^{-1}\big(B(f(x), \epsilon)\big)$ is open, we can construct an open ball around x with radius δ fully contained within the open set. Moreover, by selecting a point $p \in B(f(x), \delta) \subset f^{-1}\big(B(f(x), \epsilon)\big)$, we can guarantee that $f(p) \in B(f(x), \epsilon)$. This is precisely the $\epsilon - \delta$ definition of continuity. That is, given an $\epsilon > 0$ to be the radius of an open ball $B(f(x), \epsilon)$ in Y, we can always choose a $\delta > 0$ to be the radius of the open ball $B(x, \delta)$ in X that is fully contained within the preimage of $B(f(x), \epsilon)$. In mathematical notation,

$$p \in B(x, \delta) \subset f^{-1}\big(B(f(x), \epsilon)\big) \implies f(p) \in f\big(B(x, \delta)\big) \subset B(f(x), \epsilon) \tag{58}$$

or equivalently in terms of metrics,

$$d(x, p) < \delta \implies \rho(f(x), f(p)) < \epsilon \tag{59}$$

Lemma 2.8 (Composition of Continuous Functions)

If $f : X \to Y$ and $g : Y \to Z$ is continuous, then $g \circ f : X \to Z$ is continuous.

2.5 Open and Closed Maps

Open and closed functions map open/closed sets to open/closed sets, unlike continuous functions which take the preimage. However, they do are not natural and most maps are not open nor closed, so this is a pretty special condition.

Definition 2.6 (Open, Closed Maps)

A map $f : X \to Y$ is said to be
1. **open** if it maps open sets of X to open sets of Y.
2. **closed** if it maps open sets of X to closed sets of Y.

Note that open and closed maps are completely independent. A map may be open, closed, neither, or both.

Example 2.2 (Open but Not Closed)

The projection $\pi_1 : X \times Y \to X$ is an open map but but closed. Consider $\pi_1 : \mathbb{R}^2 \to \mathbb{R}$ with $S = \{(x, y) \in \mathbb{R}^2 \mid xy = 1\}$. Then $\pi_1(S) = \mathbb{R} \setminus \{0\}$, which is not closed.[a]

[a] In open maps, the typical behavior is that points are "copied," i.e. for projections, the preimage of $\pi_1^{-1}(x) = x \times Y$, where all $y \in Y$ are copied.

Example 2.3 (Closed but Not Open)

$f : \mathbb{R} \to \mathbb{R}$ with $f(x) = x^2$ is closed but not open since $f(\mathbb{R}) = [0, +\infty)$ which is not open.

2.6 Homeomorphisms

Definition 2.7 (Homeomorphism)

A bijective, bicontinuous function $f : X \longrightarrow Y$ between two topological spaces is called a **homeomorphism** between X and Y. If there exists at least one homeomorphism between X and Y, then X is said to be **homeomorphic** to Y, denoted $X \cong Y$.

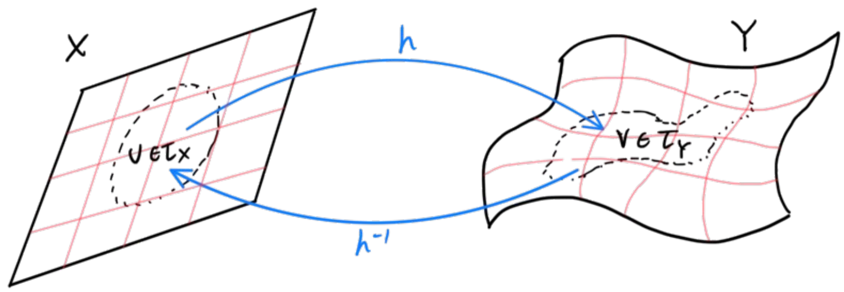


Figure 8: The visual below shows a homeomorphism between the plane X and the surface Y.

Theorem 2.9 (Sufficient Properties of Homeomorphism)

Suppose $f : X \to Y$ is a bijection. TFAE.
1. $U \subset Y$ is open iff $f^{-1}(U)$ is open.
2. $U \subset X$ is open iff $f(U)$ is open.
3. f is a homeomorphism.

Note that we may have functions that are bijective and continuous, but not bicontinuous. In order to construct such examples one of the easiest things we can do is just endow the codomain with the discrete topology, which guarantees continuity.

Example 2.4 (Bijective and Continuous but not Homeomorphism)

$\mathbb{Z}$ and $\mathbb{Q}$ are countable sets, so there is a bijection between them. If we give each of them the metric topology, $\mathbb{Z}$ ends up having the discrete topology (take the 0.5-ball around each integer), whereas for $\mathbb{Q}$, we will see later that by the density of the rationals there are an infinite number of rationals in $(q-r, q+r)$ for $q \in \mathbb{Q}$. Note that this bijection $f : \mathbb{Z} \to \mathbb{Q}$ is continuous (since $\mathbb{Z}$ has discrete topology) but not bicontinuous.

Example 2.5 (Comparability and Homeomorphic Spaces)

Consider the set $X = \{a, b\}$ with the two topologies $\mathscr{T}_3 = \{\emptyset, \{a\}, X\}$ and $\mathscr{T}_4 = \{\emptyset, \{b\}, X\}$. They are not comparable but they seem "similar" in a way in that if we swap all the a's and b's in $\mathscr{T}_3$, then we get $\mathscr{T}_4$. We can make this rigorous by defining $f : (X, \mathscr{T}_3) \to (X, \mathscr{T}_4)$ with $f(a) = b, f(b) = a$, and showing that it is a homeomorphism.

In fact, a homeomorphism f is an equivalence relation between two topological spaces. This partitions the set of all topological spaces into **homeomorphism classes**. Analogous to how isomorphisms preserve algebraic structures, homeomorphisms preserve topological structure between topological spaces.

Example 2.6 (Homeomorphism Classes of 2D Manifolds)

There is an infinite family of 2-dimensional manifolds, call them M and N, and each set in each family is not homeomorphic to another.

1. $M_0 = S^2$ (sphere). $M_1 = T^2$ (torus). M_2 is a donut with two holes. M_3 has three holes, and so on.
2. N_1 is the Mobius strip. N_2 is the Klein bottle.

Additionally, not only does a homeomorphism give a bijective correspondence between points in X and Y, but it also determines a bijection between **the set of all open sets in** X **and** Y (that is, a bijection between their topologies)! This bijection then allows two spaces that are homeormophic to have the same topological properties.

Theorem 2.10 (Preservation of Topological Properties)

A homeomorphism f between two topological spaces $(X, \mathscr{T}_x)$ and $(Y, \mathscr{T}_Y)$ preserves all topological properties (e.g. separability, countability, compactness, (path) connectedness) of X onto Y and Y onto X.

Definition 2.8 (Embedding)

Suppose that $f : X \longrightarrow Y$ an injective continuous map with X, Y topological spaces. Let $Z \coloneqq \operatorname{Im} f$. Then, the function

$$f' : X \longrightarrow Z \subset Y \tag{60}$$

obtained by restricting the codomain of f is bijective. If f' happens to be a homeomorphism of X with Z, then we say that the map

$$f : X \longrightarrow Y \tag{61}$$

is a **topological embedding**, or more simply an **embedding**, of X in Y.

A homeomorphism can be useful, but we can work a lot more flexibly with it by knowing that the restriction of a homeomorphism is a homeomorphism.

Theorem 2.11 (Restriction of Homeomorphism is Homeomorphism)

If $f : X \to Y$ is a homeomorphism, then for any $x \in X$, the restriction

$$f|_{X \setminus \{x\}} : X \setminus \{x\} \to Y \setminus \{f(x)\} \tag{62}$$

is also a homeomorphism.

2.7 Local Homeomorphisms

3 Induced Topologies

3.1 Initial and Final Topologies

We have seen some examples of how to create topologies. They can be created without any assumptions on the set, such as the discrete, indiscrete, and the cofinite topologies. More often, we want to consider how a certain structure like the order or a metric induces a topology. Now, we will consider how *functions* can induce a topology. The uniqueness of such induced topologies is called the *universal property*.

Definition 3.1 (Initial Topology)

Given a space X and a family of topological spaces $\{Y_\alpha\}_{\alpha \in A}$

$$f_i : X \to (Y_\alpha, \mathscr{T}_\alpha) \tag{63}$$

the **initial topology** on X is the coarsest topology $\mathscr{T}$ on X s.t. that each

$$f_i(X, \mathscr{T}) \to (Y_\alpha, \mathscr{T}_\alpha) \tag{64}$$

is continuous.

Definition 3.2 (Final Topology)

Given a space Y and a family of topological spaces $\{X_\alpha\}_{\alpha \in A}$

$$f : (X, \mathscr{T}_\alpha) \to Y \tag{65}$$

the **final topology** on Y is the finest topology $\mathscr{T}$ on Y s.t. each

$$f : (X, \mathscr{T}_\alpha) \to (Y, \mathscr{T}) \tag{66}$$

is continuous.

Note that it makes sense to talk about the coarsest topology on the domain and the finest topology on the codomain. If it were the other way around, i.e. the finest topology on the domain, then the initial topology on X would be the discrete topology, making every function defined on X continuous. In the same logic, the coarsest topology on Y would trivially be the trivial topology, making all Y-valued functions continuous. With these current definitions, if $\mathscr{T}_Y$ is too fine (e.g. if $\mathscr{T}_Y = 2^Y$), then the open sets of $\mathscr{T}_Y$ would be too fine and therefore would have a preimage that may not be open in X.

3.2 Subspace Topology

The reason we want to do this is because we want to think of Y as its own entity, independent of X.

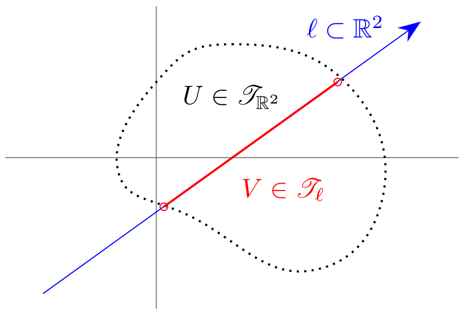
Definition 3.3 (Subspace Topology)

Given topological space X and subspace $Y \subset X$, the **subspace topology** on Y is defined in the equivalent ways.

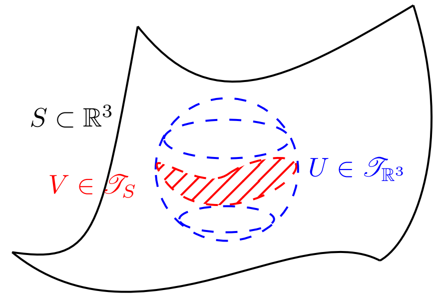
1. It is the initial topology on the subspace Y with respect to the inclusion map $\iota : Y \to X$.
2. It is the topology consisting of X-open sets intersection Y.

$$\mathscr{T}_Y = \{(U \cap Y) \subset Y \mid U \in \mathscr{T}_X\} \tag{67}$$

We can also say that $A \subset Y$ is closed in Y iff it is the inersection of Y with a closed set of X.



(a) The subspace topology of a line l in $\mathbb{R}^2$.



(b) The subspace topology of a surface $\mathcal{L}$ in $\mathbb{R}^3$.

Figure 9: Visual of subspace topology.

Proof. We prove the properties.

1. *Trivial.* We see that $\emptyset = \emptyset \cap Y$ and $Y = X \cap Y$.
2. *Stability under Union.* Suppose $\{V_\alpha\}_{\alpha \in A}$ are setes that are open in Y. Then for each α there exists an open set $U_\alpha \subset X$ that is open in X. Therefore,

$$\bigcup_{\alpha \in A} V_\alpha = \bigcup_{\alpha \in A} (U_\alpha \cap Y) \tag{68}$$

$$= Y \cap \left(\bigcup_{\alpha \in A} U_\alpha \right) \tag{69}$$

where $\cup_\alpha U_\alpha$ is open in X, and therefore we shown that there exists such an open set.
3. *Stability under Finite Intersection.* Suppose $\{V_i\}_{i=1}^n$ are open in Y. Then we can do the same thing.

Furthermore, we can immediately retrieve the basis of the subspace topology.

Theorem 3.1 (Induced Basis of Subspace Topologies)

If $\mathscr{B}$ is a basis for the topology of X, then

$$\mathscr{B}_Y \coloneqq \{B \cap Y \mid B \in \mathscr{B}\} \tag{70}$$

is a basis for the subspace topology of Y.

Proof.

Now if the subspace Y were to be an open or closed subset of X, the properties of openness and closedness carry over nicely.

Lemma 3.2 (Open/Closed Subspaces)

Let X be a topological space, $Y \subset X$ have the subspace topology, and $S \subset Y$.

1. If Y is open in X, then S open in Y iff S open in X.
2. If Y is closed in X, then S closed in Y iff S closed in X.

Since the subspace is so natural to consider, we will by default imply that if X is a topological space and

$Z \subset X$, Z is endowed the subspace topology.

Lemma 3.3 (Restrictions and Injections are Continuous)

The results immediately follow:

1. Given $f : X \to Y$ and $Z \subset X$, $f|_Z : Z \to Y$ is continuous.
2. Given X and $Z \subset X$, the canonical injection $\iota : Z \to X$ is continuous.

Proof. Listed.

1. Let us take an open set U in Y. Then it is of the form $V \cap Y$ for some V open in X. Therefore taking the preimage gives
$$f|_Z^{-1}(U) = f^{-1}(U) = f^{-1}(V \cap Y) = f^{-1}(V) \cap f^{-1}(Y) = f^{-1}(V) \cap Z \tag{71}$$
where $f^{-1}(V)$ is open by continuity of f, and so the intersection is open.
2. This is true by definition.

Given these results, one may wonder whether—just like how we restricted a continuous function to a smaller continuous function—we can "extend" a function to a larger function. However, this is not always true.

Example 3.1 (Combining Continuous Functions May not be Continuous)

Let us take $\mathbb{R}$ and divide it into $\mathbb{Q}$ and $(\mathbb{R} \setminus \mathbb{Q}) \setminus \{0\}$. Then let us define

$$f : \mathbb{Q} \to \mathbb{R} f(x) = 0 \tag{72}$$
$$g : (\mathbb{R} \setminus \mathbb{Q}) \setminus \{0\} \to \mathbb{R} g(x) = x \tag{73}$$

Then f and g are trivially continuous, but taking the function

$$h(x) \coloneqq \begin{cases} f(x) = 0 & \text{if } x \in \mathbb{Q} \\ g(x) = x & \text{if } x \notin \mathbb{Q} \end{cases} \tag{74}$$

which is not continuous.[a]

[a] Inspired from here.

But not all hope is lost. It does turn out that under certain conditions, we can in fact construct such continuous functions.

Lemma 3.4 (Pasting Lemma, Gluing Lemma)

Let $X = A \cup B$, where A, B are closed in X. Let $f : A \longrightarrow Y$ and $g : B \longrightarrow Y$ be continuous. If

$$f(x) = g(x) \text{ for all } x \in A \cap B \tag{75}$$

Then f and g can be combined to form a continuous function $h : X \longrightarrow Y$, defined

$$h(x) \coloneqq \begin{cases} f(x) & x \in A \setminus B \\ f(x) \text{ or } g(x) & x \in A \cap B \\ g(x) & x \in B \setminus A \end{cases} \tag{76}$$

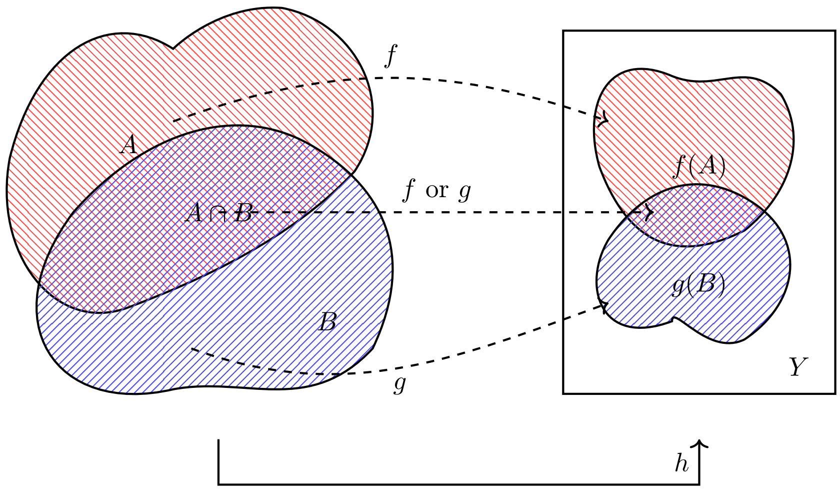


Figure 10: Visual of the pasting lemma.

Consider any set $U \subset Y$. Note that if U is an open set in X that happens to be contained in Y, then we can set $U = U \cap Y$, so U is open in Y. However, we have seen that being open in Y does not necessarily imply that it is open in X.

Example 3.2 (Non-Open Sets may be Open in Subspace)

Let $X = \mathbb{R}$ with the Euclidean topology and let $Y = [0, 1]$.
1. $[0, 1]$ is open in Y but not open in X.
2. Intervals of the form $(a, 1]$ and $[0, b)$ are open in Y but not open (nor closed) in X.

Example 3.3 (Singleton Sets in Subspace Topologies)

Consider $X = \mathbb{R}$ with the lower limit topology with $Y = [0, 1]$. The following
1. $[1/2, 1] = Y \cap [1/2, 2)$, and
2. $\{1\} = Y \cap [1, 2)$

are open in the subspace topology. It turns out that $\{1\}$ is the only singleton set open in Y.

Let's go through a few examples.

Example 3.4 (Closed Unit Interval in $\mathbb{R}$)

The basis for the subspace topology of $[0, 1] \subset \mathbb{R}$ with the Euclidean topology consists of the intervals
1. (a, b) where $0 \leq a < b \leq 1$.
2. $[0, b)$ where $0 < b \leq 1$.
3. $(a, 1]$ where $0 \leq a < 1$.

Example 3.5 (Unit Sphere in $\mathbb{R}^n$)

Let $S^n \subset \mathbb{R}^{n+1}$ be the unit **n-sphere** defined $S^n := \{x \in \mathbb{R}^{n+1} \mid ||x||^2 = 1\}$. When thinking about S^n as a space itself, we use the subspace topology coming from the standard topology of $\mathbb{R}^n$.

Example 3.6 ($S^1 \subset \mathbb{R}^2$)

Let's focus on $n = 1$. For $a < b$, let

$$A_{a,b} = \{(\cos t, \sin t) \mid a < t < b\} \tag{77}$$

Then, we can see that
1. if $b - a > 2\pi$, then $A_{a,b} = S^1$.
2. If $b - a \leq 2\pi$, then $A_{a,b}$ is an "open arc" from $(\cos a, \sin a)$ to $(\cos b, \sin b)$.

Given that we have an equivalence class defined

$$A_{a,b} \sim A_{a+2\pi k, b+2\pi k} \text{ for all } k \in \mathbb{Z} \tag{78}$$

We claim that $\{A_{a,b}\}$ is a basis for the subspace topology of S^1. We can see that the open arc covering the top right quadrant in $\mathbb{R}^2$ is

$$S^1 \cap (0,1)^2 = S^1 \cap B_\infty\big((\frac{1}{2}, \frac{1}{2}), \frac{1}{2}\big) \tag{79}$$

Now let's focus more on metric spaces. Note that if we want to construct topologies of subspaces of metric spaces, there are two ways to do it. It would be quite bad if these resulted in different topologies, but fortunately we have the following theorem.

Theorem 3.5 (Topologies on Subspaces of Metric Spaces Coincide)

Let (X, d_X) be a metric space, with $Y \subset X$. There are 2 ways we can define a topology on Y.
1. Take the metric topology $\mathscr{T}_X$ on X, and then take the subspace topology on Y.
2. Induce a metric $d_Y = d_{X|Y}$ on Y which is a restriction of d_X to Y, and then take the metric topology of it.

We claim that these two constructions give the same topology, as shown in the commutative diagram.

$$\begin{tikzcd} d_X \arrow[r] \arrow[d] & d_Y \arrow[d] \\ \mathscr{T}_X \arrow[r] & \mathscr{T}_Y \end{tikzcd}$$

Figure 11

Proof. The basis for the subspace topology on Y is

$$\mathscr{B}_1 = \{B_{d_X}(x, r) \cap Y \mid x \in X, r > 0\} \tag{80}$$

and the basis for the (induced) metric topology on Y is

$$\mathscr{B}_2 = \{B_{d_Y}(y, r) \cap Y \mid y \in Y, r > 0\} = \{B_{d_X}(y, r) \cap Y \mid y \in Y, r > 0\} \tag{81}$$

It is immediate that $\mathscr{B}_2 \subset \mathscr{B}_1$ since it goes over all $x \in X$ rather than $y \in Y$. To see why $\mathscr{B}_1 \subset \mathscr{B}_2$, TBD.

Theorem 3.6 (Closures in Subspace Topologies)

Let $A \subset Y \subset X$. Let $\bar{A}$ denote the closure of A in X. Then, the closure of A in Y equals $\bar{A} \cap Y$.

3.3 Box Topology

There are multiple ways to define the box and product topologies, but their construction with basis elements is most simple.

Definition 3.4 (Box Topology)

Given a family of topological spaces $\{(X_\alpha, \mathscr{T}_\alpha)\}_{\alpha \in A}$, the **box topology** on the space $\prod_{\alpha \in A} X_\alpha$ is the topology generated by the basis

$$\mathscr{B} = \left\{ \prod_{\alpha \in A} U_\alpha \mid U_\alpha \in \mathscr{T}_\alpha \right\} \tag{82}$$

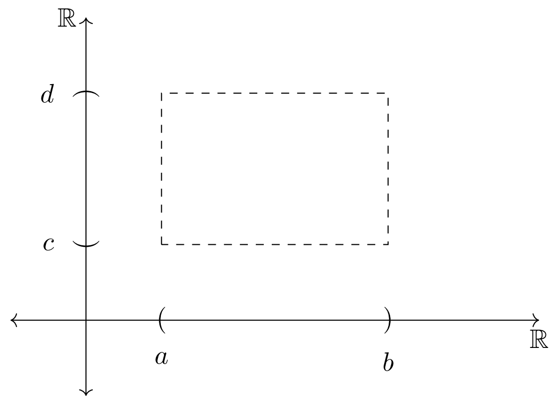


Figure 12: We can visualize the elements of the box topology with the product space $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$, where each $\mathbb{R}$ has an open ball topology. From the visual below, we can see why this is called the "box" topology.

Proof. It is easy to prove that the box topology indeed satisfies the 3 properties of topologies in general.

3.4 Product Topology

While the box topology may seem quite "intuitive" for the first learner, the box topology however, has serious limitations when extending to infinite Cartesian products of spaces. To motivate the product topology, let's try to "reverse engineer" a topology on $X \times Y$ such that the projection mappings $\pi_1 : X \times Y \to X$ is always continuous. We want

1. $U \times Y$ to be open for $U \subset X$ open.
2. $X \times V$ to be open for $V \subset Y$ open.

This implies that $(U \times Y) \cap (X \times V) = U \times V$ should be open. This is how we will define the product topology. The main difference between the construction of open sets in the box topology vs the product topology is that the box topology merely describes open sets as direct products of open sets from each coordinate space while the construction of the product topology is completely dependent on the projection mappings $\pi_\beta : \prod_{\alpha \in I} X_\alpha \to X_\beta$ to be continuous (and nothing more) so that (by definition) the preimages of open sets in X_β under π_β are open sets in $\prod X_\alpha$. Therefore, the construction of the continuous π_β's canonically constructs a basis of open sets in $\prod X_\alpha$.

Definition 3.5 (Product Topology)

Given a family of topological spaces $\{(X_\alpha, \mathscr{T}_\alpha)\}_{\alpha \in A}$, the **product topology** on the space $\prod_{\alpha \in A} X_\alpha$ is defined in the following equivalent ways.

1. It is the initial topology on the product space wrt the family of projections $p_\alpha : \prod_{\alpha \in A} X_\alpha \to X_\alpha$.
2. It is the topology generated by the basis of elements

$$\prod_\alpha U_\alpha \tag{83}$$

where U_α is a proper open subset for at most finitely many α's, and $U_\alpha = X_\alpha$ for all other α.

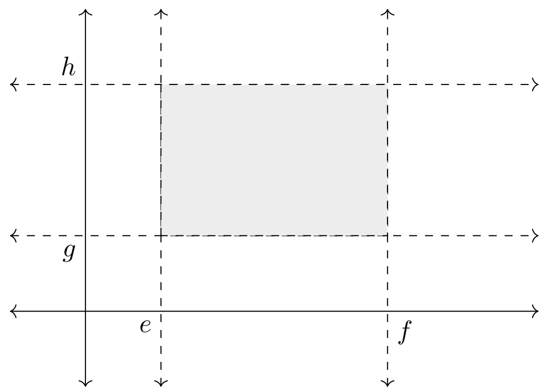


Figure 13: Visually, we can interpret each $\mathscr{S}(U_\beta)$ as a "strip" in the total product space. For example in $\mathbb{R}^2$, there are two "strips" $(e, f) \times \mathbb{R}$ and $\mathbb{R} \times (g, h)$ that intersect. Note that each strip is the preimage of the projection mapping.

We can deduce some conclusions comparing these topologies. First, the product and box topologies are precisely the same if we work in finite Cartesian products of spaces, since any element of the box topology (left hand side) can be expressed as a finite intersection of some open sets (in the right hand side). That is, if $\operatorname{card} I < \infty$, then

$$\prod_{\alpha \in I} U_i = \bigcap_{\alpha \in I} \Big\{ \prod_{\gamma \in I} W_\gamma \mid W_\gamma = U_\gamma \text{ if } \gamma = \alpha,\, W_\gamma = X_\gamma \text{ if } \gamma \neq \alpha \Big\} \tag{84}$$

Secondly, we can see that the box topology is finer than the product topology (strictly finer if working in infinite product spaces).

Example 3.7

The set $(0,1)^{\mathbb{N}} \subset \mathbb{R}^{\mathbb{N}}$ is clearly open in the box topology, but it is considered "too tight" to be in the product topology. However,

$$(0,1) \times \mathbb{R} \times \mathbb{R} \times \dots \tag{85}$$

is open in the product topology since only one (a finite amount) of the factors is not the whole space.

The following theorem reveals why the product topology is superior than the box topology in product spaces.

Theorem 3.7 (Continuity of Functions Mapped to Product Topology)

Given the function

$$f : A \to \prod_{\alpha \in I} X_\alpha,\; f(a) \coloneqq \big(f_\alpha(a)\big)_{\alpha \in I} \tag{86}$$

with its component functions $f_\alpha : A \to X_\alpha$. Let $\prod X_\alpha$ have the product topology. Then the function f is continuous if and only if each function f_α is continuous.

Proof. We prove both directions. Let π_β be the projection of this product onto the βth component space. By construction π_β is continuous $\implies \pi_\beta^{-1}(U_\beta)$ is a basis element of the product topology of $\prod X_\alpha$.

1. ($\rightarrow$) f is continuous, so $f_\beta \coloneqq \pi_\beta \circ f$, as the composition of continuous functions, is also continuous.
2. ($\leftarrow$) Assume that each f_β is continuous. Let there be an open set $U_\beta \subset X_\beta$. Then, the canonical open set π_β^{-1} in the product space $\prod X_\alpha$ is also open. Now, the preimage of $\pi_\beta^{-1}(U_\beta)$ under f is

$$\begin{aligned} f^{-1}\big(\pi_\beta^{-1}(U_\beta)\big) &= (f^{-1} \circ \pi_\beta^{-1})(U_\beta) \\ &= (\pi_\beta \circ f)^{-1}(U_\beta) \\ &= f_\beta^{-1}(U_\beta) \end{aligned}$$

Since f_β is already assumed to be continuous, $f_\beta^{-1}(U_\beta)$ is open in A.

This theorem also works for the box topology only if we are working with finite product spaces. But in general, this theorem fails for the box topology. Consider the following example.

Example 3.8

Let $\mathbb{R}^\omega$ be the countably infinite product of $\mathbb{R}$'s. Let us define the function

$$f : \mathbb{R} \to \mathbb{R}^\omega \tag{87}$$

with coordinate function defined $f_n(t) \coloneqq t$ for all $n \in \mathbb{N}$. Clearly, each f_n is continuous. Given the box topology, we consider one basis element of $\mathbb{R}^\omega$

$$B = \prod_{i=1}^\infty (-\frac{1}{i}, \frac{1}{i}) \tag{88}$$

Assume that f is continuous, that is $f^{-1}(B)$ is open in $\mathbb{R}$. Then, it would contain some finite interval $(-\delta, \delta)$ about 0, meaning that $f\big((-\delta,\delta)\big) \subset B$. This implies that for each $n \in \mathbb{N}$,

$$f_n\big((-\delta,\delta)\big) = (-\delta,\delta) \subset \Big(-\frac{1}{n}, \frac{1}{n}\Big) \tag{89}$$

which contradicts the fact that B is open, since the interval $(-1/n, 1/n)$ converges onto a point 0.

However, there is no useful criterion for the continuity of a mapping $f : X \times Y \longrightarrow A$ even if we have the product topology on $X \times Y$. One might conjecture that this f is continuous if it is continuous in each variable separately, but this is in fact not true.

Theorem 3.8 (Topologies on Products of Metric Spaces Coincide)

Given a metric space

Corollary 3.9

The Euclidean topology on $\mathbb{R}^n$ is equivalent to the product topology of the Euclidean topologies on $\mathbb{R}$.

Theorem 3.10 (Subspace of Products and Products of Subspaces are Equivalent)

If $A \subset X$ and $B \subset Y$, then the following topologies are equivalent.

1. The subspace topology on the product topology of $X \times Y$.
2. The product topology on the subspace topologies of A, B.

Example 3.9 (Sorgenfrey Plane)

The Cartesian product of two real lines with the lower limit topology is called the **Sorgenfrey plane**.

$$\mathbb{R}_\ell \times \mathbb{R}_\ell \tag{90}$$

Lemma 3.11

The addition, subtraction, and multiplication operations are continuous functions from $\mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$, and the quotient operation is a continuous function from $\mathbb{R} \times (\mathbb{R} \setminus \{0\}) \longrightarrow \mathbb{R}$.

Proof. Standard $\epsilon - \delta$ proof.

Now that we have defined what it means for binary operations to be continuous, we can talk about *topological algebra*, which is the study of algebraic structures such that their algebraic operations and inverses are continuous. One important such concept is a *topological group*, which will be mentioned later.

3.5 Quotient Topologies

We have established natural topologies on sets that are constructed from other sets, namely by subsets and Cartesian products. Another way to construct a set is by taking an equivalence relation, which partitions the set into its equivalence classes. The method in which we construct such a topology on this quotient space, called the *quotient topology*, is slightly less straightforward.

3.5.1 Quotient Maps

Definition 3.6 (Quotient Map)

A function $p : X \to Y$ is said to be a **quotient map** if it is surjective and

$$U \text{ is open in } Y \iff p^{-1}(U) \text{ is open in } X \tag{91}$$

Note that we could have also replaced open with closed sets and the definitions are equivalent.

Definition 3.7 (Saturation)

A subset $S \subset X$ is **saturated** with respect to the surjective map $p : X \to Y$ if for every $p^{-1}(A)$ (where $A \subset Y$) that intersects S, $p^{-1}(A)$ is completely contained within S. That is,

$$p^{-1}\big(p(S)\big) = S \tag{92}$$

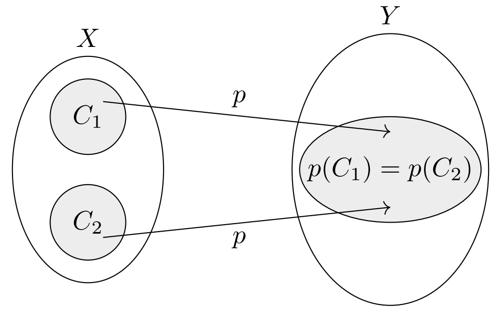


Figure 14: We can see that C_1 and C_2 alone are not saturated, but $C_1 \cup C_2$ is saturated. Visually, for a given set $C \subset X$ to be saturated, there cannot be any points $q \notin C$ such that $q \in p(C)$.

We now introduce an alternative, equivalent definition of quotient maps.

Theorem 3.12 (Quotient Maps w.r.t. Mapping Saturated Sets)

$p : X \to Y$ is a quotient map if and only if p is continuous and p maps saturated open sets of X to open sets of Y (or saturated closed sets of X to closed sets of Y).

The first property is that quotient maps behave nicely under compositions.

Theorem 3.13 (Composition of Quotient Maps)

The composition of two quotient maps is a quotient map.

Proof. We immediately know that the composition of surjective maps are surjective and that of continuous maps are continuous.

However, they do not behave nicely under subspace or products. If $p : X \to Y$ is a quotient map and A is a subspace of X, then the map $p' : A \to p(A)$ obtained by restricting both the domain and codomain of P need not be a quotient map. The product of two quotient maps is not necessarily a quotient map. That is, given $p : A \to B$ and $q : C \to D$ are quotient maps, the map

$$p \times q : A \times C \to B \times D,\ (p \times q)(a \times c) := p(a) \times q(c) \tag{93}$$

is not necessarily a quotient map.

Example 3.10 (Restriction of Quotient Maps are Not Quotient Maps)

Example 3.11 (Products of Quotient Maps are Not Quotient Maps)

Additionally, quotient maps are clearly not homeomorphisms, so topological properties are not preserved.

Example 3.12

However, there is just one extra condition on a quotient map that will make it a homeomorphism.

Lemma 3.14 (Bijective Quotient Maps)

A quotient map that is injective (and hence bijective) is a homeomorphism.

Definition 3.8 (Retraction)

Given topological space X and $A \subset X$, a **retraction** $f : X \to A$ is a continuous map such that $f(a) = a$ for all $a \in A$.

So, the restriction of retractions onto their codomain is the identity map.

Lemma 3.15

Let $p : X \to Y$ be a continuous map. If there is a continuous map $f : Y \to X$ s.t. $p \circ f$ is the identity map, then p is a quotient map.

Proof. Let $U \subset Y$ and let $p^{-1}(U)$ be open. Then,

$$\begin{aligned} p^{-1}(U) \text{ open } &\implies f^{-1}(p^{-1}(U)) \subset Y \text{ open} && (94) \\ &\implies (p \circ f)^{-1}(U) = U \subset Y \text{ open} && (95) \end{aligned}$$

and since p is continuous, p is a quotient map. Since $p \circ f$ is the identity, it must follow that p is surjective.

Theorem 3.16 (Retractions are Quotient Maps)

A retraction r is a quotient map.

Proof. We know that the canonical injection $A \subset X$ is continuous, and $r \circ \iota = \mathrm{id}_A$. Therefore, by the lemma, r must be a quotient map.

3.5.2 Open and Closed Maps

Note that an open map or a closed map (with continuous and surjective) are trivially quotient maps. Since given a $U \subset Y$ with $f^{-1}(U)$ open, then by definition $U = f(f^{-1}(U))$ is open by definition.

Theorem 3.17 (Open/Closed Maps are Stronger than Quotient Maps)

If $p : X \to Y$ is a surjective, continuous map that is either open or closed (that is, maps open sets to open sets or closed sets to closed sets), then p is a quotient map.[a]

[a]Note however, that the converse is not true; there exists quotient maps that are neither open nor closed.

Example 3.13 (Quotient Maps that are Neither Open Nor Closed)

3.5.3 Quotient Topology

Now that we have defined the quotient map, we are ready to define the quotient topology.

Definition 3.9 (Quotient Topology)

Let $p : (X, \mathscr{T}_X) \to Y$ be a surjective map.[a] Then, the **quotient topology** induced by p is defined in the following equivalent ways.

1. It is the final topology on the quotient set $X/\sim$ wrt the projection map p.
2. It is the topology of all subsets U of Y s.t. p^{-1} is open in X.

$$U \text{ open in } X/\sim \iff p^{-1}(U) \text{ saturated and open in } X \tag{96}$$

3. It is the unique topology $\mathscr{T}_Y$ relative to which p is a quotient map.[b]

The quotient set $X/\sim$ with its quotient topology is called the **quotient space**.

[a] A natural surjective map that we can construct is by taking an equivalence relation $\sim$ on X, setting $Y = X/\sim$, and taking $p : x \mapsto [x]$. Every surjective map can be thought of as a map induced by an equivalence relation, since we can set $x \sim x'$ iff $f(x) = f(x')$, so these are equivalent.

[b] We claim that this topology exists and is unique.

Proof. The topology $\mathscr{T}_Y$ on Y is defined by letting it consist of all subsets U of Y such that $p^{-1}(U)$ is open in X. This is indeed a topology since

1. $p^{-1}(\emptyset) = \emptyset$ and $p^{-1}(Y) = X$
2. $p^{-1}\Big(\bigcup_{\alpha \in J} U_\alpha\Big) = \bigcup_{\alpha \in J} p^{-1}(U_\alpha)$
3. $p^{-1}\Big(\bigcap_{i=1}^n U_i\Big) = \bigcap_{i=1}^n p^{-1}(U_i)$

The intuition is the following. The topology on X is fixed, and we must somehow find some topology on Y that makes p a quotient map. If we make $\mathscr{T}_Y$ too coarse, satisfiying continuity of p is easy but it may not necessarily mean that $p^{-1}(U)$ open in X implies U open in Y. However, if we make $\mathscr{T}_Y$ too fine, then continuity may not be satisifed. The theorem states that there is a middle point—in fact exactly one topology—in which cases both directions are satisfied.

Example 3.14

Let $p : (\mathbb{R}, \mathscr{T}_\mathbb{R}) \to \mathbb{R}/2\pi\mathbb{R}$. Then, the final topology of $\mathbb{R}/2\pi\mathbb{R}$ would be simply defined

$$\mathscr{T}_{\mathbb{R}/2\pi\mathbb{R}} \coloneqq \{U \subset \mathbb{R}/2\pi\mathbb{R} \mid U = p(O), O \in \mathscr{T}_\mathbb{R}\} \tag{97}$$

That is, the quotient topology is merely the set of all images of open sets in $\mathbb{R}$ under f. However, if $\mathbb{R}/2\pi\mathbb{R}$ has the discrete topology 2^X, then a single equivalence class, say $[0]$, will get mapped to the collection of points $\{2\pi k \mid k \in \mathbb{Z}\}$, which is clearly not open in $\mathbb{R}$. Note that the final topology (or the quotient topology) is endowed onto the codomain in order to make f continuous (or a quotient mapping).

Example 3.15

Let $X \coloneqq [0, 1] \cap [2, 3] \subset \mathbb{R}$ and $Yy \coloneqq [0, 2] \subset \mathbb{R}$. Then, we define $p : X \to Y$ as

$$p(x) \coloneqq \begin{cases} x & x \in [0,1] \\ x-1 & x \in [2,3] \end{cases} \tag{98}$$

p is continuous (under subspace topology of $X \subset \mathbb{R}$), surjective, and closed, meaning that it is a quotient map. However, it is not open, since the image of the open set $[0, 1]$ of X is $[0, 1]$, which is not open in Y.

Example 3.16 (Finite Sets)

Let $p : \mathbb{R} \to \{a, b, c\}$ be defined as

$$p(x) \coloneqq \begin{cases} a & x > 0 \\ b & x < 0 \\ c & x = 0 \end{cases} \tag{99}$$

Then, the quotient topology of $\{a, b, c\}$ consists of

$$\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\} \tag{100}$$

Okay, so we've learned yet another way to construct topologies. However, things become interesting when we start to compare quotient spaces to other topological spaces that we already know of. The following series of theorems will help in our analysis.

Theorem 3.18 (Induced Maps from Quotient Space)

Let $p : X \to Y$ be a quotient map (e.g. $Y = X/\sim$ for some ER $\sim$). Let $f : X \to Z$ be a function such that if $p(x) = p(x')$, then $f(x) = f(x')$, i.e. $x \sim x' \iff f(x) = f(x')$. Then,

1. f induces the map $\bar{f}$ satisfying $f = \bar{f} \circ p$.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "p"] & Z \\ X/\sim \arrow[ur, "\bar{f}"] & \end{tikzcd}$$

Figure 15: The theorem states that the diagram commutes.

2. f continuous iff $\bar{f}$ continuous.
3. f quotient map iff $\bar{f}$ quotient map.

Proof. Listed.

1.
2. Suppose f is continuous. Let $U \subset Z$ be open. Then we need to show that $\bar{f}^{-1}(U)$ is open. But we can see that $p^{-1}(\bar{f}^{-1}(U)) = f^{-1}(U)$ is open since f is continuious. Therefore $\bar{f}^{-1}(U)$ is open since p is a QM.If $\bar{f}$ is continuous, then $f = \bar{f} \circ p$ is continuous as the composition of continuous maps.
3. Suppose f is a quotient map with $U \subset Z$ s.t. $\bar{f}^{-1}(U)$ is open. We need to show that U is open. Then $p^{-1}(\bar{f}^{-1}(U))$ is open since p is continuous $\implies f^{-1}(U)$ is open. But f is a quotient map, so U is open.

Corollary 3.19

If f is a quotient map, then $\bar{f}$ is a homeomorphism.

Proof. Show that $\bar{f}$ is injective, and so its bijective. Since it's a quotient map, it's a homeomorphism.

Therefore we can just make up any equivalence relation (surjective map) on X which gives us a quotient space $X/\sim$. To figure out whether this quotient space is homeomorphic to a topological space that we already know, we first (cleverly) choose a candidate space Z and try to write a quotient map $f : X \to Z$ that "agrees" with the equivalence relation, i.e. $x \sim x' \iff f(x) = f(x')$. We don't need to worry too much about surjectivity, since if we have continuity and the "reverse continuity" conditions satisfied then we can just restrict Z to the image of f to make it surjective anyways. Once we have found such a quotient map f, using the theorem above we can conclude that $X/\sim$ is homeomorphic to Z, and we are done! We show various examples in the next section.

We end this section with a warning. It was the case that a lot of the properties get passed down to subspaces, but this is not the case for quotient maps.

Example 3.17 (Quotient Topology Reduced to Trivial Topology)

Let $X = \mathbb{R}$ with $x \sim y \iff x - y \in \mathbb{Q}$. We claim that $X/\sim$ is uncountable. If we wish to find open sets in $X/\sim$, we can do this by finding saturated open sets in X. Let $U \subset \mathbb{R}$ be open and saturated. Since it is open, by the density of $\mathbb{Q}$ in $\mathbb{R}$, U must contain a rational number $\implies \mathbb{Q} \subset U$, and so $U = \mathbb{R}$. Therefore the only saturated open sets are $\emptyset, \mathbb{R}$, meaning that $X/\sim$ has the trivial topology.

This has a lot of consequences, and even very mild topological properties can be broken in quotient spaces.

Example 3.18 (Quotient of Hausdorff Space Need not be Hausdorff)

Given $X = \mathbb{R}^2 \setminus \{0\}$ with the ER defined $(x, y) \sim (x', y')$ iff $x = x'$, and if $x = x' = 0$, then $\text{sign}(y) = \text{sign}(y')$. This is the line with 2 origins with the quotient map

$$f(x,y) = \begin{cases} x \text{ if } x \neq 0 \\ a \text{ if } x = 0, y > 0 \\ b \text{ if } x = 0, y < 0 \end{cases} \tag{101}$$

3.5.4 Quotient Spaces

Example 3.19 (1-Sphere)

Let $X = [0, 1]$ with $\sim$ defined only with $0 \sim 1$. Then our intuition may tell us that by "sticking" the endpoints together, we can get a unit circle S^1.

(a) Unit interval $[0, 1]$

(b) Unit circle S^1 in $\mathbb{R}^2$

Figure 16: Visual of the homeomorphism between $[0, 1]$ and S^1.

So can I come up with a function $f : [0, 1] \to S^1$ s.t. $f(0) = f(1)$? Yes, we can define

$$\bar{f}(x) = (\cos 2\pi x, \sin 2\pi x) \tag{102}$$

which indeed satisfies $f(0) = f(1) \iff 0 \sim 1$.[a] Therefore, by the theorem above, $X/\sim \cong S^1$, defined by the homeomorphism $\bar{f}(x) = (\cos 2\pi x, \sin 2\pi x)$.

[a]Note that we could just chosen $\mathbb{R}^2$ and restricted the image to S^1 at the end as well.

Example 3.20 (Alternative Construction of 1-Sphere)

Let $X = \mathbb{R}$ with $\sim$ defined $x \sim y$ iff $x - y \in \mathbb{Z}$. We call this $\mathbb{R}/\mathbb{Z}$. Our intution tells us that by "folding" the real number line into overlapping unit intervals, we can get the unit circle. We will show that

$$\frac{\mathbb{R}}{\mathbb{Z}} \cong S^1 \tag{103}$$

Let us construct the set $(\mathbb{R}, \mathscr{T}_{\mathbb{R}})$ with paramater t. We define maps

$$p : \mathbb{R} \to \mathbb{R}/\mathbb{Z}, \;\; p(t) \coloneqq t \text{ (mod 1)}$$
$$q : [R] \to S^1 \subset \mathbb{C}, \;\; g(t) \coloneqq e^{2\pi i t}$$

We claim that p and q are both quotient mappings. Clearly, p is a quotient mapping. As for q, it it easy to see that it is surjective (but not injective) and continuous ($\mathscr{T}_{S^1}$ has the basis of open intervals on S^1). It is also easy to notice that given an open interval $U \subset S^1$, $q^{-1}(U)$ will be the union of open intervals equally spaced in $\mathbb{R}$. Additionally, given any open interval in $\mathbb{R}$, it maps to an open interval in S^1 (note that S^1 itself is also open). These three conditions imply that q is a quotient map. We now define maps

$$q \circ p^{-1} : \mathbb{R}/\mathbb{Z} \to S^1 \tag{104}$$
$$p \circ q^{-1} : S^1 \to \mathbb{R}/\mathbb{Z} \tag{105}$$

and claim that these maps are homeomorphisms. We can clearly see that the mapping from an open set in $\mathbb{R}/\mathbb{Z}$ to the union of spaced open intervals in $\mathbb{R}$ is an injection, and the mapping from this union of open intervals to the union of open intervals in S^1 is a surjection. The composition of these two mappings clearly defines a bijection. Therefore, $q \circ p^{-1}$ is proven to be a bicontinuous bijective mapping between open sets $U \subset \mathbb{R}/\mathbb{Z}$ and $V \subset S^1 \implies q \circ p^{-1}$ is a homeomorphism.
This result clearly makes sense since

$$\frac{\mathbb{R}}{\mathbb{Z}} \cong \frac{[0,1]}{\sim} \tag{106}$$

where the relation $\sim$ maps every point $x \in (0,1)$ to its own equivalence class and the points $0, 1$ to one equivalence class $\{0\}$. Therefore, it is informally said that the quotient space of the real line is a circle. One may attempt to construct a simpler set by replacing S^1 with the half-open interval $[0,1)$. However, while $[0,1)$ is bijective to $\mathbb{R}/\mathbb{Z}$,

$$\frac{\mathbb{R}}{\mathbb{Z}} \ncong [0,1) \tag{107}$$

That is, the two sets are not homeomorphic because the topologies of $[0,1)$ and $\mathbb{R}/\mathbb{Z}$ are not compatible. For instance, when we attempt to map the open set

$$\left\{ [x] \in \mathbb{R}/\mathbb{Z} \mid 0 \leq x \leq \frac{1}{4} \vee x > \frac{1}{2} \right\} \in \mathscr{T}_{\mathbb{R}/\mathbb{Z}} \tag{108}$$

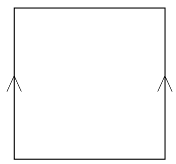
to $\mathscr{T}_{[0,1)}$, it does not return an open set.
Furthermore, this means that

$$S^1 \times S^1 \cong \frac{[0,1]^2}{\sim'} \cong \left(\frac{\mathbb{R}}{\mathbb{Z}}\right)^2 \tag{109}$$

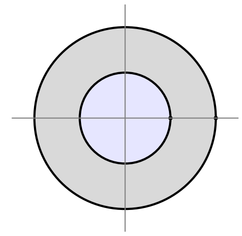
where $\sim'$ is the quotient mapping defined in the previous construction of the torus.

Example 3.21 (Annulus)

Let $X = [0,1]^2$ with $\sim$ defined $(0,y) \sim (1,y)$ for all $y \in [0,1]$. Then our intuition may tell us that by sticking the top and bottom together, we can get an annulus, defined as the closed region $A \subset \mathbb{R}^2$ between the disk of radius 2 and disk of radius 1.



(a) Unit square $[0,1] \times [0,1]$



(b) Annular region between circles of radius 1 and 2

Figure 17: Unit square and annular region

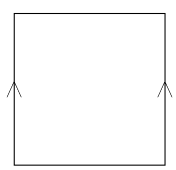
We can indeed come up with the function $f : [0,1]^2 \to A$ defined

$$(x,y) \mapsto \big((1+y)\cos 2\pi x, (1+y)\sin 2\pi x\big) \in \mathbb{R}^2 \tag{110}$$

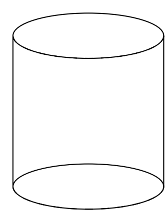
which satisfies $(0,y) \sim (1,y) \iff f(0,y) = f(1,y)$. Therefore by the theorem above $X/\sim \,\cong A$.

Example 3.22 (Cylinder)

Let $X = [0,1]^2$ with $\sim$ defined $(0,y) \sim (1,y)$ for $y \in [0,1]$. Then our intuition may tell us that by sticking the top and bottom together, we can get the cylinder, defined as $C \coloneqq \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2 = 1, z \in [0,1]\}$.



(a) Cylinder as an equivalence class on $[0,1]^2$.



(b) Cylinder as an embedding in $\mathbb{R}^3$.

Figure 18: Two geometric figures: a square with labeled sides and a cylinder

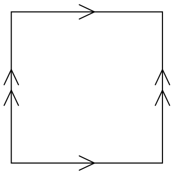
We can indeed come up with the function $f : [0,1]^2 \to C$, defined

$$(x,y) \mapsto \big(\cos 2\pi x, \sin 2\pi x, y\big) \tag{111}$$

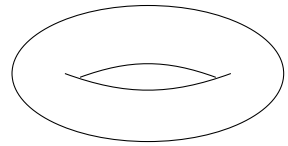
which satisfies $(0,y) \sim (1,y) \iff f(0,y) = f(1,y)$. Therefore by the theorem above $X/\sim \,\cong C$.

Example 3.23 (Torus)

Let $X = [0,1]^2$ with $\sim$ defined $(0,y) \sim (1,y)$ for $y \in [0,1]$ and $(x,0) \sim (x,1)$ for $x \in [0,1]$. Then our intuition may tell us that by first sticking the sides together, we get a cylinder, and by sticking the top with the bottom, we get a torus, denoted T^2.



(a) Torus as an equivalence class on $[0,1]^2$.



(b) Torus as an embedding in $\mathbb{R}^3$.

Figure 19: Representations of a torus.

We can indeed come up with a function $f : [0,1]^2 \to T^2 \subset \mathbb{R}^3$.

$$(x, y) \mapsto \big((2 + \cos 2\pi x) \cos 2\pi y, (2 + \cos 2\pi x) \sin 2\pi y, \sin 2\pi x\big) \tag{112}$$

which is consistent with the relation.

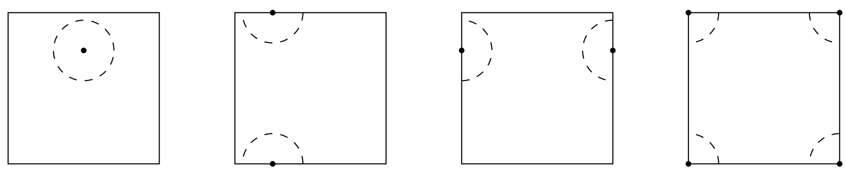
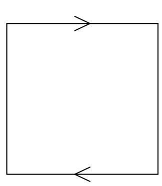


Figure 20: The quotient topology of this quotient space consists of open sets of form.

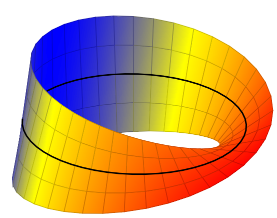
We can check that this mapping is indeed a quotient map. First, it is clearly surjective. By realizing that individual points on the edge of $[0,1]^2$ are open sets themselves (by the subspace topology), we can prove that this map is indeed open and continuous. Therefore, we can see that the induced map $\hat{f} : [0,1]^2/{\sim} \to T^2$ is a homeomorphism.

Example 3.24 (Mobius Strip)

Let $X = [0,1]^2$ with $\sim$ defined $(x, 0) \sim (1 - x, 1)$ for $x \in [0,1]$. Then our intuition may tell us that by flipping over the square and sticking the sides together, we get a weird strip, called the **Mobius strip** and denoted M_1.



(a) Cylinder as an equivalence class on $[0,1]^2$.



(b) Mobius strip has 1 side and is a non-orientable surface.

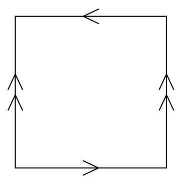
Figure 21: Two geometric figures: a square with labeled sides and a cylinder

We can indeed come up with a function $f : [0,1]^2 \to M_1 \subset \mathbb{R}^3$

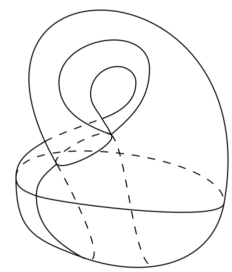
$$(x, y) = \left(\cos y\left[1 + \frac{x}{2}\cos\frac{y}{2}\right], \sin y\left[1 + \frac{x}{2}\cos\frac{y}{2}\right], \frac{x}{2}\sin\frac{y}{2}\right) \tag{113}$$

Example 3.25 (Klein Bottle)

Let $X = [0,1]^2$ with $\sim$ defined $(0, y) \sim (1, y)$ for $y \in [0,1]$ and $(x, 0) \sim (1-x, 1)$ for $x \in [0,1]$. We cannot actually visualize this in $\mathbb{R}^3$, but we may try to construct the cylinder by sticking the left/right sides together, and then trying to glue the top and bottom in opposite direction, which gives us a **Klein Bottle**, denoted K^2.



(a) Klein bottle as an equivalence class on $[0,1]^2$.



(b) Klein bottle cannot be visualized in $\mathbb{R}^3$. Credits here.

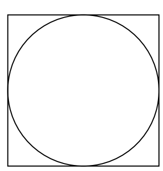
Figure 22: Representations of a torus. Note that we can write a continuous function from K^2 to $\mathbb{R}^3$, but it is not injective and therefore not a homeomorphism, which is why there is a self-intersection.

We just claim that there exists a function $f : [0,1]^2 \to K^2 \subset \mathbb{R}^4$.[a]

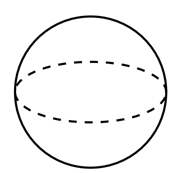
[a] However, an embedding in $\mathbb{R}^3$ is not possible. We will need algebraic topology to prove this.

Example 3.26 (2-Sphere)

Let X be the closed unit disk $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\} \subset \mathbb{R}^2$ with $\sim$ defined as $x \sim y \iff x, y \in S^1$. Then our intuition may tell us that by squishing the edges of the disk into a point, we can construct a **2-sphere**, denoted $S^2 := \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$.



(a) 2-sphere as an equivalence class on the unit disk which is homeomorphic to the unit square.



(b) 2-sphere as an embedding in $\mathbb{R}^3$.

Figure 23: Representation of a sphere.

We can indeed come up with a function $f : [0,1]^2 \to S^2$, defined

$$(x, y) \mapsto \big(\sin x \cos y, \sin x \sin y, \cos x\big) \tag{114}$$

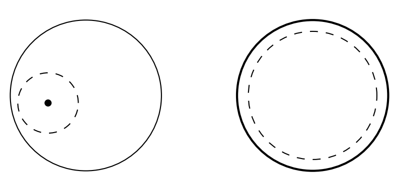


Figure 24: The saturated open sets of X consist of open sets of one of the two forms.

Example 3.27 (Weird Quotient Space)

Given $(\mathbb{R}, \mathscr{T}_{\mathbb{R}})$, let us define the relation $\sim$ determined by the quotient mapping

$$p(x) := \begin{cases} \{x\} & x \notin \mathbb{Z} \\ \mathbb{Z} & x \in \mathbb{Z} \end{cases} \tag{115}$$

In words, this quotient map maps every integer to the equivalence class $[0]$ and maps every other point to its own class.

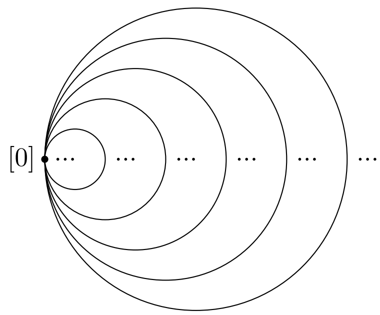


Figure 25: It turns out that every interval $[j, j+1] \subset \mathbb{R},\ j \in \mathbb{Z}$ will get mapped as a closed loop in $\mathbb{R}/\sim$ beginning and ending with $[0]$, since $j, j+1 \mapsto [0]$. So geometrically, $\mathbb{R}/\sim$ consists of an infinite number of nonintersecting closed loops starting and ending with $[0]$.

This wacky mapping is an example of a quotient mapping that does not preserve topological structure. While it will not be proven here, it is known that $(\mathbb{R}, \mathscr{T}_{\mathbb{R}})$ is 1st and 2nd countable, but $\mathbb{R}/\sim$ under this relation is not even 1st countable.

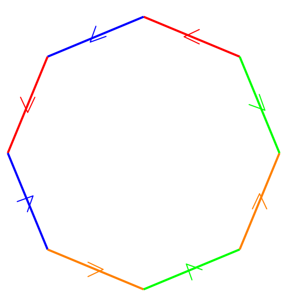
Great, so we've went over the construction of quotient topologies and have identified them with some familiar spaces. It turns out that many of these are examples of *topological manifolds*.

Definition 3.10 (Polygonal Surface)

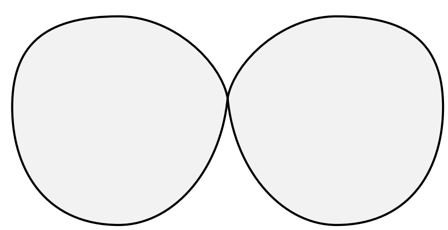
Take a closed polygon in $\mathbb{R}^2$ with an even number of sides. Then we can pair up the edges.

Theorem 3.20

The quotient spaces defined form a polygonal surface is always a topological manifold.



(a)



(b)

Figure 26: Note that in this case (and for some equivalence classes), all vertices of a polygon are equivalent by transitivity.

4 Connectedness

Now that we have seen many examples of topologies, and how one can construct them (e.g. with a metric, subspace, product, quotient), we can begin to talk about some properties of these spaces. The first one is *connectedness*, which is analogous to a topological space being irreducible (e.g. not able to be separated into two smaller topological spaces). This is what we should keep in mind. Let's first go and concretely define what such a separation means.

Definition 4.1 (Separation)

Let X be a topological space. A **separation** of X is a pair U, V of disjoint nonempty open subsets of X whose union is X.

Definition 4.2 (Connected Space)

The space X is said to be **connected** if it satisfies the equivalent definitions

1. there does not exist a separation of X.
2. the only subsets of X that are clopen in X are the empty set and X itself.

and **disconnected** otherwise.

Figure 27: Two examples of spaces $X = A \sqcup B$ that are not connected. In the right, A and B overlap in their boundary but are not connected since they are open.

Proof. If there exists a nontrivial clopen set $U \subsetneq X$, then $U \sqcup U^c$ is a separation of X. If $U \sqcup V$ is a separation, then $V = U^c$, and so U is clopen.

Example 4.1 (Discrete and Indiscrete Topology)

Any set $|S| > 1$ with the discrete topology is disconnected. All subsets in the indiscrete topology is connected.

Example 4.2 (Disconnected Sets can Share Limit Points)

Let Y denote the subspace $[-1, 0) \cup (0, 1]$ of $\mathbb{R}$. Each of the sets $[-1, 0)$ and $(0, 1]$ is nonempty and open in Y (but not in $\mathbb{R}$), so they form a separation of Y. Also, note that neither of these sets contains a limit point of the other (even though they have a common limit point 0).
On the same note, the space $Y = (0, 1) \times (0, 1) \cup (1, 2) \times (0, 1) \subset \mathbb{R}^2$ has the clear separation

$$(0, 1) \times (0, 1) \text{ and } (1, 2) \times (0, 1) \tag{116}$$

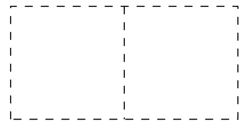


Figure 28: We can visualize the separation of Y as such.

Note that the dashed line is not in Y. Even though the dashed line contains limit points of both the left and right subset of Y, this does not matter.

Note that connectedness is a property of a topological space. Often we will talk about connected subsets $S \subset X$, but this should be taken to mean that S is connected with respect to the subspace topology endowed from X.

Now so far, we have treated connectedness as a property of a space, but we may extend this to talk about whether *points* are connected. Essentially, we want to describe connectedness as an equivalence relation between points. The most nontrivial property is transitivity, which will be established in the following theorem.

Theorem 4.1 (Points are Connected in a Connected Space)

Let X be a topological space. Then X is connected iff $\forall x, y \in X$, $\exists$ a connected subspace A s.t. $x, y \in A$.

Proof. We prove bidirectionally.

1. $(\rightarrow)$. Take $A = X$.
2. $(\leftarrow)$. Let $X = U \cup V$ with $U \cap V = \emptyset$, both nonempty and open. Take $x \in U, y \in V$. So $\exists$ connected $A \subset X$ s.t. $x, y \in A$. But then either $A \subset U$ or $A \subset V$ from the lemma above. Therefore, X is connected by contradiction.

Now, we can define connected components as an equivalence class.

Corollary 4.2 (Connectedness is an Equivalence Relation)

Let $x \sim_c y$ iff $\exists$ a connected subspace $A \subset X$ s.t. $x, y \in A$.

Proof. We prove the 3 properties.

1. *Reflexive.* $x \sim_c x$ clearly since $A = \{x\}$, which is connected.
2. *Symmetric.* Let $x \sim_c y$. Then there exists connected subspace A containing both x, y, i.e. containing y, x, and so $y \sim_c y$.
3. *Transitive.* Let $x \sim_c y, y \sim_c z$. Then there exists connected subspaces $A \ni x, y$ and $B \ni y, z$. Since $y \in A \cap B$, $A \cup B$ is also a connected subspace containing x, z, and so $x \sim_c z$.

Definition 4.3 (Connected Components)

Given a topological space X, the equivalence classes under $\sim_c$ are called **connected components** of X.

With the equivalence class interpretation, it will be much easier to prove many properties.

Theorem 4.3 (Continuous Images of Connected Spaces are Connected)

If X is aconnected and $f : X \to Y$ is continuous, then $f(X)$ is a connected subspace of Y.

Proof. Let $f : X \longrightarrow Y$ be a continuous map, and let X be connected. We wish to prove that the image set $Z = f(X)$ is also connected. Let us denote the restriction of f to Z as

$$\tilde{f} : X \longrightarrow Z \tag{117}$$

which is continuous and surjective. We prove by contradiction. Assume that $Z = A \cup B$ is a separation of Z into 2 disjoint nonempty open sets. Then, $\tilde{f}^{-1}(A)$ and $\tilde{f}^{-1}(B)$ are disjoint open sets whose union is $X \implies \tilde{f}^{-1}(A) \cup \tilde{f}^{-1}(B)$ form a separation of X. This contradicts the hypothesis that X is connected $\implies Z$ is connected.

This establishes the fact that homeomorphisms preserve connectedness, and so connectedness is a topological property.

Corollary 4.4 (Connectedness is a Topological Property)

If $X \cong Y$, then X connected iff Y connected.

Therefore, connectedness is a good way to prove that two spaces are not homeomorphic, whether it is by assuming a homeomorphism itself or taking the restriction of a homeomorphism with one or more points taken off the domain.

Example 4.3 (Intervals of Endpoints 0 and 1)

We claim that $(0, 1)$, $[0, 1)$, and $[0, 1]$ are all pairwise not homeomorphic.

Corollary 4.5 (Quotient Spaces)

If X is connected, then any quotient space of X is connected.

Example 4.4

For $n \geq 1$, S^n and $\mathbb{RP}^n$ is connected.

We have proved some pretty basic results, and just like how we have talked about building new topologies from old ones, we can talk about how to build new connected spaces from old connected spaces. The three that we will mention are the unions, limit point extensions, and products.

4.1 Connectedness

Great, so to prove whether subspaces are connected, we can just think of the subspace topology. But warning: note that since a separation of $Y \subset X$ is a pair of nonempty open sets $A, B \subset Y$ s.t. $A \sqcup B = Y$, since by openness $A = U \cap Y, B = V \cap Y$ for U, V open in X, it may be the case that the larger open sets U, V actually have an intersection.

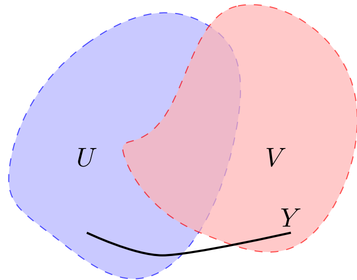


Figure 29: We can see that there is a separation of $A = U \cap Y, B = V \cap Y$ of Y, but U and V do intersect.

Therefore, we cannot rely on the topology on X to deduce connectivity on Y. However, there is another simpler lemma that checks for connectivity of subspaces.

Lemma 4.6 (Closures of One Separated Component is Disjoint from Other)

If A, B is a separation of subspace $Y \subset X$, then $A' \cap B = \emptyset$ and $A \cap B' = \emptyset$. It immediately follows that

$$\overline{A} \cap B = A \cap \overline{B} = \emptyset \tag{118}$$

Proof. If A, B is a separation, then $A \subset U, B \subset V$ for U, V open in X, and $A \cap V = \emptyset, B \cap U = \emptyset$. So points on $B \subset V$ are not limit points of A, and same with points on A.

Lemma 4.7 (Connected Subsets Must be Completely Contained in a Separated Component)

If the sets C and D form a separation of X, and if Y is a connected subset of X, then Y lies entirely within either C or D.

Proof. We can see that $A = C \cap Y$ and $B = D \cap Y$are open in the subspace topology of Y. So $A \cap B = \emptyset$ and $A \cap B = Y$, so either $Y = A$ or $Y = B$.

4.2 Preservation of Connectedness

Now we talk about constructing connected sets as the union of connected sets. It is pretty clear that the union and intersection of connected sets are not connected.

Theorem 4.8 (Conditions for Union of Connected Spaces to be Connected)

Suppose $A_\alpha \subset X$ are connected subspaces of X s.t. $\forall \alpha, \beta$, $A_\alpha \cap A_\beta = \emptyset$. Then $\cup_\alpha A_\alpha$ is connected.

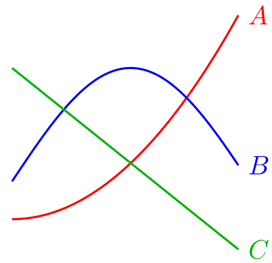


Figure 30: We can see that the connected subspaces $A, B, C \subset \mathbb{R}^2$ intersect pairwise at a point. Therefore their unions is connected.

Proof. Let $A = \cup_\alpha A_\alpha$. Suppose $A = C \sqcup D$ with C, D both open in subspace topology, so $C = A \cap U, D = A \cap V$ for X-open sets U, V. Each A_α is either in C or D, since otherwise I can create a separation. But if $A_\alpha \subset C$ or $A_\alpha \subset D$, then this is impossible since they intersect at one point at least. So either all $A_\alpha \subset C$ or all $A_\alpha \subset D$.

Theorem 4.9 (Connected Sets plus Some/All Limit Points are Connected)

Let A be a connected subset of X. If $A \subset B \subset \bar{A}$, then B is also connected.

Proof. Assume $B = C \cup D$ is a separation of $B \implies A$ must lie entirely within C or D. Without loss of generality, suppose $A \subset C$, which implies that $\bar{A} \subset \bar{C}$. Since $\bar{C}$ and D are disjoint, B cannot intersect $D \implies D = \emptyset$, a contradiction. Therefore, there exists no separation of B.

Theorem 4.10 (Products of Connected Spaces are Connected)

Given connected topological spaces X_α with $\alpha \in J$, the Cartesian product

$$\prod_{\alpha \in J} X_\alpha \tag{119}$$

under the product topology is connected.[a]

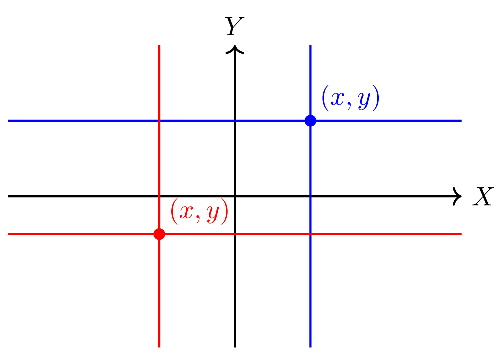


Figure 31: You can see that any two $T_{x \times y}$ have a nontrivial intersection by construction. This is why we need the + shape.

[a]But for infinite products, it is not necessarily connected under the box topology

Proof. We will prove for the finite product case. Given $(x, y) \in X \times Y$, let us define the space

$$T_{x \times y} \coloneqq (\{x\} \times Y) \cup (X \times \{y\}) \tag{120}$$

where both of the components are connected (since they are homeomorphic to Y and X, respectively). We know that $T_{x \times y}$ is connected since it's a union of connected space with nontrivial intersection (x, y), and using the same lemma, the arbitrary union over all points in $X \times Y$ is connected.

$$\bigcup_{(x,y) \in X \times Y} T_{x \times y} = X \times Y \tag{121}$$

is connected.

4.3 Path Connectedness

Now we will be talking about a stronger form called path connectedness. Unlike connectedness—where we began with the topological definition and then claimed that the connected components form an equivalence class—we will introduce path connectedness as an equivalence class from the start. Note that connectedness is about open sets rather than paths.

Definition 4.4 (Path)

A **path** from $x \in X$ to $y \in X$ is a continuous function $f : [a, b] \subset \mathbb{R} \to X$[a] s.t. $f(a) = x, f(b) = y$, and $[a, b]$ is endowed with the Euclidean topology.[b] Two points x, y are said to be **path connected**, denoted $x \sim_p y$, if there exists a path from x to y, i.e. $f(0) = x, f(1) = y$.

[a] Note that we can just reparameterize the path to any other starting and end points with the homeomorphism $[a, b] \cong [c, d$.

[b] So far, we've been pretty agnostic of topologies in definitions, but here we mention a very specific topology.

Lemma 4.11 (Path Components are Equivalence Classes)

$\sim_p$ is an equivalence relation, and the equivalence classes formed by $\sim_p$ on topological space X are called **path components**.

Proof. We prove the 3 properties.

1. *Reflexive.* $x \sim_p x$ since we can choose the constant function $f : t \mapsto x$.
2. *Symmetric.* Let $x \sim_p y$. Then there exists a continuous $f : [0, 1] \to X$ s.t. $f(0) = x, f(1) = y$. We can choose the continuous function $g = f \circ h$, where $h(x) = 1 - x$ is continous connecting y to x.
3. *Transitive.* Let $x \sim_p y$ and $y \sim_p z$. then by the pasting lemma, the overlapping set $\{y\}$ is closed and so we can define the continuous function

$$(f * g)(t) \coloneqq \begin{cases} f(2t) & t \in [0, 1/2] \\ g(2t - 1) & t \in [1/2, 1] \end{cases} \tag{122}$$

which is a path from x to z.

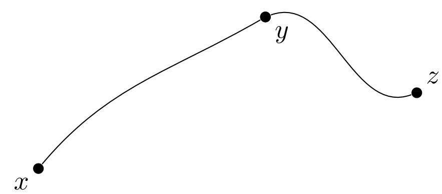


Figure 32: W:

Definition 4.5 (Path Connected Space)

A topological space X is said to be **path connected** for every pair of points $x, y \in X$, $x \sim_p y$.

It seems that path connectedness is conceptually easier to deal with, and we might ask if one implies the other.

Theorem 4.12 (Path Connectedness implies Connectedness)

X is path connected $\implies$ X is connected. That is, each path component is contained in a connected component.

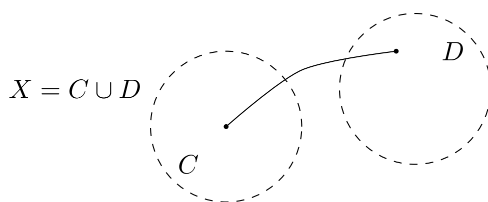


Figure 33: Visually if two spaces are not connected, it doesn't seem like it's path connected.

Proof. We can prove this in two ways.

1. *Directly.* if $x \sim_p y$, then $\exists$ a path $f : [0,1] \to X$ s.t. $f(a) = x, f(b) = y$, and $\Im(f)$ is a connected subspace containing x, y.
2. *Contrapositive.* X not connected implies that there exists disjoint open subsets C, D such that $C \cup D = X$. Assume that X is path connected, i.e. there exists a continuous function $g : [0,1] \longrightarrow X$. Then the preimage of C and D in X must be open sets $g^{-1}(C), g^{-1}(D) \subset [0,1]$ such that $g^{-1}(C) \cup g^{-1}(D) = [0,1]$. But this isn't possible since $[0,1]$ is connected, so by contradiction, X is not path connected. The contrapositive of this statement results in the theorem.

But is the converse true? Intuitively, it seems like it, but a pretty nasty construction of a counterexample is needed. However, note that X connected $\centernot\implies$ X path connected. Note the following example.

Example 4.5 (Topologist's Sine Curve)

Let

$$C = \{(x, y) \in \mathbb{R}^2 \mid x > 0, y = \sin(1/x)\} \tag{123}$$

This is the image of a continuous function. It is both connected and path connected since $C \cong (0, +\infty)$. This is *not* the topologist's sine curve. C is not closed since it doesn't contain the limit points $\{(0, y) \mid 0 \leq y \leq 1\}$ (in red), but if we do cake the closure $\overline{C} = C \cup (\{0\} \times [-1, 1])$, then *this* is the TSC.

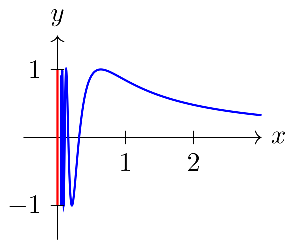


Figure 34: Topologist's sine curve is the union of the image of the oscillating sine curve (blue) with its limit points (red).

$\overline{C}$ is connected since the closure is the union of connected C and some/all limit points of C. Now we claim that $\overline{C}$ is not path connected. Intuitively, given a point on C and point on C', it must zig-zag infinitely many times, and thus cannot get to the red portion in time. Rigorously, suppose $f : [a, b] \to \overline{C}$ were a path from $(0, 0)$ to some (x, y) with $x > 0$. $f^{-1}(\{x\} \times [-1, 1])$ is a closed subset of $[a, b]$ and thus has a max value; call it c. Now we take the restriction on $f : [c, b] \to \overline{C}$. Then

$$f(c) \in C' = \{0\} \times [-1, 1], \qquad f(t) \in C \;\forall t > c \tag{124}$$

Now reparaterize so that $c = 0, b = 1$. Then

$$f(t) = \big(x(t), y(t)\big) = \big(x(t), \frac{1}{\sin x(t)}\big) \tag{125}$$

So we can find some sequence of numbers $t_n \to 0$ s.t. $y(t_n) = (-1)^n \implies \lim_{n\to+\infty} t_n = 0$ but $\lim_{n\to+\infty} y(t_n)$ does not exist, contradicting the fact that f is continuous.

In general, mathematicians like path connectedness better since it makes our lives easier. We can more easily prove path connectedness by just building a path and it also implies connectedness.

Example 4.6 (Euclidean Space)

Let us take $\mathbb{R}^n$ and we claim the following.

1. $\mathbb{R}^n$ is connected since given x, y, we can just draw $f(t) = (1 - t)x + ty$ for $t \in [0, 1]$, i.e. a straight line between them.
2. $\mathbb{R}^n \setminus \{0\}$ is path connected for $n > 1$ since we can draw a line, and if that line passes through the origin, i.e. $y = \lambda x$ for $lambda < 0$, then we choose $z \notin \text{span}(x)$ and choose the path $x \to z \to y$.
3. $\mathbb{R}^n \setminus S$ is path connected for any finite set S.

Theorem 4.13 (Continuous Images of Path Connected Spaces as Path Connected)

If X is path connected and $f : X \to Y$ is continuous, then $f(X) \subset Y$ is path connected.

Proof. Given $y_1, y_2 \in f(X) \subset Y$, we choose x_1, x_2 s.t. $f(x_1) = y_1$ and $f(x_2) = y_2$. Then we choose a path g from x_1 to x_2 since X is path connected, and $f \circ g$ is a path from y_1 to y_2.

Corollary 4.14 (Path Connectedness is a Topological Property)

If $X \cong Y$, then X path connected iff Y path connected.

Corollary 4.15 (Quotients Path Connected Spaces are Path Connected)

Any quotient of a path connected space is path connected.

Theorem 4.16 (Products of Path Connected Spaces)

Any product of path connected spaces is path connected under the product topology.[a]

[a]Any only for finite products in the box topology.

Proof. Given $\prod_\alpha X_\alpha$ of path connected components X_α, let $x = (x_\alpha), y = (y_\alpha)$ be its elements. Then for each α, choose path $f_\alpha : [0,1] \to X_\alpha$ with $f_\alpha(0) = x_\alpha$ and $f_\alpha(1) = y_\alpha$. Then by definition of product topology, the function

$$f = \prod_\alpha f_\alpha : [0,1] \to \prod_\alpha X_\alpha \tag{126}$$

is continuous.

Example 4.7

$\mathbb{R}^\omega$ is not even connected in the box topology. In fact it is horrifically disconnected. Let

$$B := \{x \in \mathbb{R}^\omega \mid |x_i| < R\} \tag{127}$$

be the set of bounded sequences for some $R > 0$. We claim that B is clopen in the box topology. Given $x \in \mathbb{R}^\omega$, let us define the open set

$$U_x = \prod (x_i - 1, x_i + 1) \tag{128}$$

If x is bounded, then $x \in B$ and every element in U_x is bounded $\implies U_x \subset B$. So B is open. If x is unbounded, then $x \in \mathbb{R}^\omega \setminus B$ and every element of U_x is also unbounded $\implies U_x \subset \mathbb{R}^\omega \setminus B$. So $\mathbb{R}^\omega \setminus B$ is open and B is closed.

4.4 Preservation of Path Connectedness

4.5 Local Connectedness and Path Connectedness

The property of local connectedness is also important for a space to possess. Roughly speaking, local connectivity means that each point has "arbitrarily small" neighborhoods that are connected. It is a property on small scales, i.e. for a property on open sets.

Definition 4.6 (Locally (Path) Connected at a Point)

A space X is said to be **locally (path) connected at** x if for every neighborhood U of x, there is a (path) connected open neighborhood V of x contained in U. If X is locally (path) connected at all of its points, then X is simply said to be **locally (path) connected**.

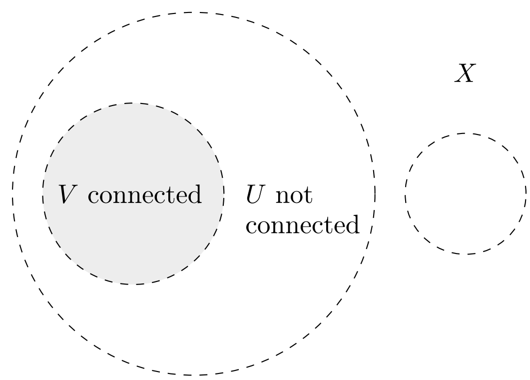


Figure 35: Visually, in the space X, let U be the union of the two open balls shown below. U is clearly open, but not necessarily connected. However, we can form a neighborhood V of x contained in U such that V is connected.

Example 4.8 (Euclidean Space)

$\mathbb{R}^n$ plus any open sets in $\mathbb{R}^n$ is locally connected and locally path connected since open balls of any radius are path connected.

Example 4.9 (Topologist's Sine Curve)

The TSC is not locally connected.

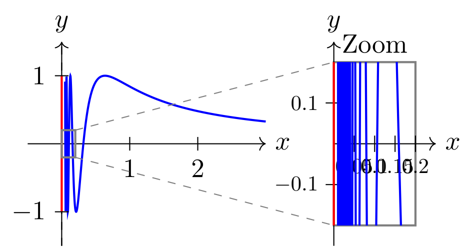


Figure 36: If we take a neighborhood around $(0, 0)$, we can see that the intersection of the image of a function with the open ball around the origin will consists of many almost-vertical lines that are not connected.

Even though local properties alone does not in general allow us to conclude about a global property, there are times when it does.

Theorem 4.17 (LPC + C $\implies$ PC)

If X is locally path connected, connectedness $\iff$ path connectedness, i.e. the connected components and path connected components are the same. In other words,

$$\text{Local Path Connectedness} + \text{Connectedness} \implies \text{Path Connectedness} \tag{129}$$

Equivalently, X is locally connected if there exists a basis for X consisting of connected sets. Local connectedness and connectedness of a space are independent of each other.

Theorem 4.18 (Open Components and Local (Path) Connectedness)

Given space X,

1. X is locally connected iff its connected components are open.
2. X is locally path connected if its path connected components are open.

Proof. For the first claim, we prove bidirectionally.

1. $(\rightarrow)$ Suppose that X is locally connected. Let U be an open set of X and let C be a component of U. If x is any point in C, by definition of local connectedness, there exists a connected neighborhood V of x fully contained in U. Since V is connected, it must additionally lie completely within $C \implies C$ is open in X.
2. $(\leftarrow)$ Suppose that the components of open sets in X are open. Given a point $x \in X$ and neighborhood U of x, let C be the component of U containing x, which means that C is connected. By hypothesis, the components of open sets are alsvo open, so C is also open. Since an open, connected set C exists for all $x \in X$, X is locally connected.

For the second claim, we also prove bidirectionally.

1. TBD.
2. TBD.

5 Separability

Separability comes in different levels.[3] We briefly define some weaker forms of separability.

Definition 5.1 (t_0-Separability)

A topological space X is said to be t_0**-separable** if for each pair of distinct points $x, y \in X$, there exists a neighborhood U that contains x but not y, or a U that contains y but not x.

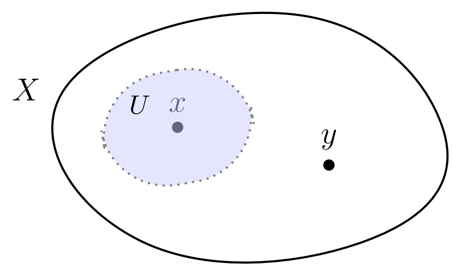


Figure 37: t_0-separability.

Example 5.1 (Nested Interval Topology is Not t_0)

$(0, 1)$ with the nested interval topology is not t_0-separable, since we can't distinguish $\frac{1}{4}$ and $\frac{1}{3}$.

5.1 T1 Separability

Definition 5.2 (t_1-Separability)

A topological space X is said to be t_1**-separable** if for each pair of distinct points $x, y \in X$, we can find two neighborhoods U_x, U_y where $y \notin U_x$ and $x \notin U_y$.

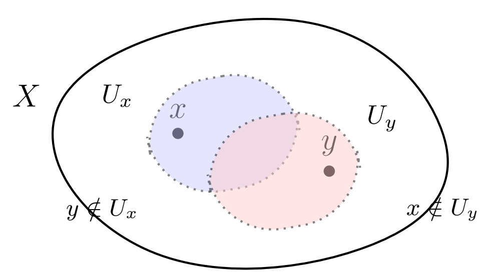


Figure 38: t_1-separability.

Definition 5.3

One point sets are closed is the T_1 axiom. Equivalently, for any pair $x_1 \neq x_2$, there is an open set U s.t. $x_1 \in U$ and $x_2 \notin U$.

[3] Note that this is not to be confused with the separation of a space, which is a completely different topological property.

Example 5.2 (Cofinite is t_1)

$(0,1)$ with the cofinite topology is t_0-separable, since given distinct $x_1, x_2 \in (0,1)$, we can see that $x_1 \in X \setminus x_2$ and $x_2 \in X \setminus x_1$, which are both elements of the cofinite topology. By existence of these elements, $(0,1)$ is t_1-separable.

5.2 Preservation of T1 Separability

5.3 T2 Hausdorff Spaces

Generally, mathematicians consider the Hausdorff condition as a mild extra conditions on topological spaces that make it much easier to deal with. We will assume that most of the topological spaces we work with are Hausdorff.

Definition 5.4 (Hausdorff Space)

A topological space X is called a **Hausdorff space**, or t_2**-separable**, if for each pair of distinct points $x, y \in X$, there exists neighborhoods U_x, U_y that are disjoint.

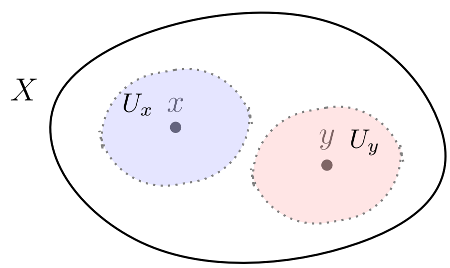


Figure 39: Every pair of distinct points must satisfy this separability condition in a Hausdorff space.

Theorem 5.1 (Limit Points in Hausdorff Spaces)

Given Hausdorff space X and subset $A \subset X$ a point x is a limit point of A if and only if every neighborhood of x contains infinitely many point of A. It immediately follows that every finite point set in a Hausdorff space X is closed.

Proof. We prove both directions

1. ($\rightarrow$) Assume that x is a limit point of A with some neighborhood U_x intersecting A in finitely many points. Then, let the points of intersections be

$$\{x_1, ..., x_n\} = A \cap \{U_x \setminus \{x\}\} \tag{130}$$

   But $U_x \setminus \{x\}$ is open $\implies H \coloneqq \{U_x \setminus (\{x\} \cup \{x_1, ..., x_n\})\}$ is open. But $H \cap A = \emptyset$, contradicting the assumption that x is a limit point.
2. ($\leftarrow$) Simple.

It suffices to show that every one point set $\{x_0\}$ is closed. If x and x_0 are distinct points, then by definition of Hausdorff spaces they have disjoint neighborhoods U_x and $U_{x_0} \implies x \notin \overline{\{x_0\}} \implies \{x_0\} = \overline{\{x_0\}}$, so $\{x_0\}$ is closed.

Lemma 5.2 (Product of Hausdorff Spaces)

Arbitrary Cartesian products of Hausdorff spaces is Hausdorff.[a]

[a]Since this is in the product topology, it immediately follows that the product is also Hausdorff in the finer box topology.

Lemma 5.3 (Subspaces of Hausdorff Spaces)

A subspace of a Hausdorff space is Hausdorff.

Theorem 5.4 (Unique Point of Convergence)

If a sequence converges in a Hausdorff space X, it converges to one point.

Proof. For if (x_α) converges to x and if $y \neq x$, then we need only choose disjoint neighborhoods of y and x to prove that (x_α), by definition, is not convergent to y.

Example 5.3

The space $(0, 1)$ with the nested interval topology is not Hausdorff. In fact, it is impossible to distinguish 2 points x, y if $x, y \in (0, \frac{1}{2})$, meaning that the sequence

$$\frac{1}{10}, \frac{2}{10}, \frac{1}{10}, \ldots \tag{131}$$

converges to both $\frac{1}{10}$ and $\frac{2}{10}$.

Theorem 5.5

Every metric topology satisfies the Hausdorff Axiom.

Proof. If x and y are distinct points of (X, d), then letting

$$\varepsilon = \frac{1}{2} d(x, y) \tag{132}$$

the triangle inequality implies that $B_\varepsilon(x)$ and $B_\varepsilon(y)$ are disjoint.

Lemma 5.6

X Hausdorff implies any finite subset is closed.

Example 5.4

X is an infinite set, then finite complement topology is not Hausdorff.

Example 5.5 (Line with Two Origins)

$X = \mathbb{R} \setminus \{0\} \cup \{0_1, 0_2\}$ with basis given by intervals (a, b) for $b < 0$ or $a > 0$, and $(a, 0) \cup \{0_i\} \cup (0, b)$ for $a < 0 < b$.

5.4 Preservation of Hausdorff

5.5 Regular Spaces

Definition 5.5 (Regular Spaces)

Suppose that one-point sets are closed in X. Then, X is said to be **regular**, or t_3**-separable**, if for each pair consisting of a point x and a closed set C disjoint from x, there exist disjoint open sets containing x and C, respectively.

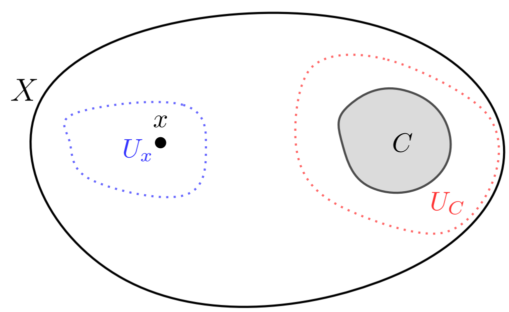


Figure 40: Regular space.

Lemma 5.7 (Product of Regular Spaces)

Arbitrary Cartesian products of regular spaces is regular.

Lemma 5.8 (Subspaces of Regular Spaces)

A subspace of a regular space is regular.

5.6 Preservation of Regularity

5.7 Normal Spaces

Definition 5.6 (Normal Spaces)

Suppose that one-point sets are closed in X. Then, X is said to be **normal**, or t_4**-separable**, if for each pair C, D of disjoint closed sets of X, there exist disjoint open sets containing C and D, respectively.

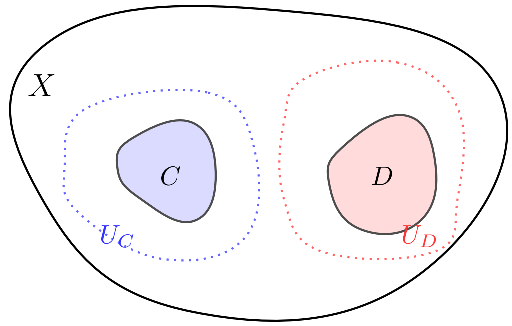


Figure 41: Normal space.

Theorem 5.9

Every regular space with a countable basis is normal.

Theorem 5.10

Every well-ordered set X is normal in the order topology.

Theorem 5.11 (Urysohn Lemma)

Let X be a normal space, and let A, B be disjoint closed subsets of X. Let $[a, b]$ be a closed interval in the real line. Then there exists a continuous map

$$f : X \longrightarrow [a, b] \tag{133}$$

such that $f(x) = a$ for every $x \in A$ and $f(x) = b$ for every $x \in B$.

Definition 5.7 (Separation by Continuous Function)

If A and B are two subsets of the topological space X, and if there is a continuous function $f : X \longrightarrow [0, 1]$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$, it is said that A **and** B **can be separated by a continuous function**.

More colloquially, the lemma states that if every pair of disjoint closed sets in X can be separated by disjoint open sets, then each such pair can be separated by a continuous function.

Theorem 5.12 (Tietze Extension Theorem)

Let X be a normal space and let A be a closed subset of X.

1. Any continuous map of A into the closed interval $[a, b] \subset \mathbb{R}$ may be extended to a continuous map of all X into $[a, b]$.
2. Any continuous map A into the reals $\mathbb{R}$ may be extended to a continuous map of all of X into $\mathbb{R}$.

5.8 Preservation of Normality

However, neither products nor subspaces of normal spaces are necessarily normal.

5.9 Completely Regular Spaces

Definition 5.8 (Completely Regular Spaces)

A space X is **completely regular** if one-point sets are closed in X and if for each point x_0 and each closed set A not containing x_0, there is a continuous function $f : X \longrightarrow [0, 1]$ such that $f(x_0) = 1$ and $f(A) = \{0\}$.

Theorem 5.13

A subspace of a completely regular space is completely regular. A product of completely regular spaces is completely regular.

Theorem 5.14

If X is completely regular, then X can be imbedded in $[0, 1]^J$ for some J.

Corollary 5.15

Let X be a space. The following are equivalent:

1. X is completely regular.
2. X is homeomorphic to a subspace of a compact Hausdorff space.
3. X is homeomorphic to a subspace of a normal space.

6 Compactness

Now we move onto another pillar of topology: compactness. I personally think compactness is the hardest conceptually, so I will provide a lot of explanation on it here. The thing to keep in mind when talking about compactness is that it describes a space being finite or not being able to escape to infinity.

Definition 6.1 (Covers)

A collection $\mathscr{C}$ of subsets of a space X is said to **cover** X, or to be a **covering** of X, if the union of the elements of $\mathscr{C}$ is equal to X. It is called an **open covering** of X if its elements are open subsets of X.

6.1 Open Cover Compactness

Definition 6.2 (Compact Space)

A topological space X is said to be **compact** if every open covering of X contains a finite subcovering (i.e. a finite collection of subcovers) of X. If $K \subset X$ is a compact topological space in the subspace topology, then K is said to be a **compact subspace** of X.

Note that compactness is a property of a topological space in its entirety. This is opposed to considering open and closed sets, which exist as a subset of a topological space. While openness behaves differently depending on its embedding space, compactness stays constant. Therefore, we don't have to worry about talking about which space a compact set is embedded in. However, we will provide a definition for which it makes sense to talk about compact *subsets* of a space.

Definition 6.3 (Compact Subspaces)

Given topological space X and $A \subset Y$, we call A a **compact subspace** of X if either of the two equivalent conditions are met.

1. A is a compact space with respect to the subspace topology endowed from X.
2. If every X-open cover of A has a finite subcover of A.

Proof. We can prove bidirectionally.

1. Suppose that K is compact in X. Then given any open cover $\{U_\alpha\}_\alpha$ of K, there exists a finite subcover $\{U_i\}_i$. Now let there exist an open cover $\{V_\alpha\}$ in Y, but every $V_\alpha = U_\alpha \cap Y$ for some U_α open in X. Therefore, we can take the finite subcover $\{V_i = V_i \cap Y\}_i$.
2. Suppose that K is compact in Y. Then given any open cover $\{V_\alpha\}$ of K, there exists a finite subcover $\{V_i\}_i$. Now let there exist an open cover $\{U_\alpha\}$ in X. Then we set $\{V_\alpha = U_\alpha \cap Y\}_\alpha$, which has a finite subcover $\{V_i = U_i \cap Y\}$, and therefore we can take $\{U_i\}$ as our finite subcover in X.

The concept of compactness does not seem intuitive at first glance. The reason why compactness is such an important property for a space to have is because X being compact tells us that we can *always* analyze the entire X using a finite union of open sets, which can simplify the space greatly. That is, it a measure of finiteness of a space. You should realize that it is much easier to prove that a space is *not* compact, since all we have to do is find a *single* covering that doesn't contain a finite subcovering.

Example 6.1 (Non-Compact Spaces)

We show some examples of non-compact spaces.

1. $\mathbb{R}$ is not compact since given the open covering $\mathcal{F} = \{(x, x+1) \subset \mathbb{R} \mid x \in \mathbb{R}\}$, then if we assume that $\mathcal{F}$ has a finite subcovering, it is the case that $\max\{x_1, x_2, \ldots, x_n\} + 1 \not\in \mathcal{F}$, so this is not a covering.
2. For $\mathbb{R}$, you can also get a nested covering $\{(-r, +r) \mid r > 0\}$, which also has no finite subcovering.
3. Take $X = (1, 1)$ and take $\mathcal{F} = \{(\frac{1}{n}, 1] \mid n \in \mathbb{N}\}$, which is a cover but has no finite subcover.

Example 6.2 (Open Square is Not Compact)

The subset $Y \coloneqq (0,1) \times (0,1) \subset \mathbb{R}^2$ is not compact. That is, we can choose to cover the subspace by the finite union of open sets.

$$[0,1]^2 \subset \bigcup_{k=0}^{\infty} \Big(\frac{2^k - 1}{2^k}, \frac{2^{k+1} - 1}{2^{k+1}} \Big) \times (0,1) \tag{134}$$

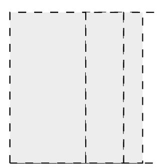


Figure 42: We show the first three elements of the infinite union that covers the open square.

Example 6.3 (Finite Sets are Always Compact)

A finite set S is compact with respect to any topology, because there are only a finite number of open sets to begin with, so every open cover is a finite cover.

According to Terry Tao, a compact set is "small," in the sense that it is easy to deal with. While this may sound counterintuitive at first, since $[0,1]$ is considered compact while $(0,1)$, a subset of $[0,1]$, is considered noncompact. More generally, a set that is compact may be large in area and complicated, but the fact that it is compact means we can interact with it in a finite way using open sets, the building blocks of topology. That finite collection of open sets makes it possible to account for all the points in a set in a finite way. This is easily noticed, since functions defined over compact sets have more controlled behavior than those defined over noncompact sets. Similarly, classifying noncompact spaces are more difficult and less satisfying.

In general, it's pretty easy to prove that a set is not compact. We just need to find one example of an open cover that does not have a finite subcover. To prove that set *is* compact, we must show that for *every* open cover, we can get a finite subcover.

Theorem 6.1 (Closed Subsets of Compact Sets are Compact)

Every closed subset of a compact space is compact.

Proof. This proof is quite trivial. Let Y be a closed subset of compact space X. Given a covering $\mathcal{C}$ of Y by sets open in X, let us form an open covering $\mathscr{B}$ of X by adjoining to $\mathcal{C}$ the single open set $X \setminus Y$. Then, we an see that both $\mathscr{B}$ and $\mathcal{C} \cup (X \setminus Y)$ covers X.

$$\mathscr{B} = \mathcal{C} \cup (X \setminus Y) \tag{135}$$

Since $\mathscr{B}$ is finite, the right hand side must also be expressible as a finite union. Looking through $\mathscr{B}$, we can throw away all the open sets that are entirely in $X \setminus Y$. What remains is a finite covering of Y.

The converse is not necessarily true, unless we have Hausdorff spaces.

Theorem 6.2 (Compact Subsets of Hausdorff Spaces are Closed)

If X is Hausdorff and $A \subset X$ is compact. Then A is closed in X.

Proof. We wish to show that $X \setminus A$ is open. Let $x \in X \setminus A$. Then, for each point $a \in A$, we can choose disjoint neighborhoods $U_a \ni x$ and $V_a \ni a$ (using the Hausdorff condition). The collection

$$\{V_a \mid a \in A\} \tag{136}$$

is an open covering of A. Since A is compact, there must exist a finite subcover $V_1, V_2, \ldots, V_n$. Therefore, $\cup_{i=1}^n V_i$ contains Y and is disjoint from the intersection of open neighborhoods of x

$$U := \bigcap_{i=1}^n U_i \tag{137}$$

Therefore, U is an open neighborhood of x_0, disjoint from $A \implies X \setminus A$ is open $\implies A$ is closed.

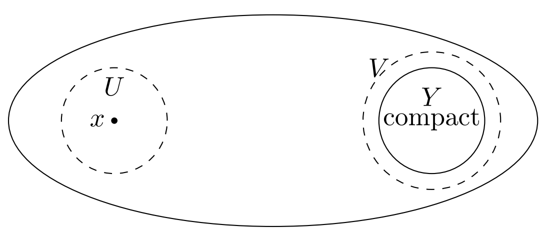
But is it minimal? It turns out that if we don't have Hausdorff, then a compact subset is not necessarily closed.

Example 6.4 (Line with 2 Origins)

Consider the line with 2 origins, $L = (-\infty, 0) \cup (0, +\infty) \cup \{r, s\}$. Then $A = \{r\} \cup (0, 1] = [0, 1]$ is not compact, but it is not closed since r is also a limit point.

Lemma 6.3 (Compact Subsets and Points Can Be Separated in Hausdorff Space)

If Y is a compact subset of a Hausdorff space X and x is not in Y, then there exist disjoint open sets U and V of X containing x and Y, respectively.



Theorem 6.4 (Continuous Mappings from Compact to Hausdorff Spaces)

Let X be compact, Y Hausdorff, and $f : X \to Y$ be a continuous map. Then, f is a closed map, and moreover,

1. f injective $\implies$ f is an embedding.
2. f surjective $\implies$ f is a quotient map.
3. f bijective $\implies$ f is a homeomorphism.

Proof. We prove the four statements.

1. To see that f is a closed map, let $C \subset X$ be closed. Then C is compact, and so $f(C)$ is also compact. But since Y is Hausdorff, $f(C)$ is closed.
2.

The previous theorem is nice in that we can prove a lot of the nice properties of topological spaces without doing much work. Note that compactness and Hausdorff goes very well together in that you can get a lot more out of a space when they are together.

Definition 6.4 (Isolated Point)

A point x is called an **isolated point** if $\{x\}$ is an open set.

Theorem 6.5 (Uncountability of Compact Hausdorff Spaces)

If X is a nonempty, compact, Hausdorff space with no isolated points, then X is uncountable.

Proof.

Thus, any closed interval in $\mathbb{R}$ is uncountable. We didn't need to use decimal expansions to prove it.

Corollary 6.6

If F is closed and K is compact, then $F \cap K$ is compact.

Now lets prove something more set-theoretic about compact sets, namely its cardinality.

Theorem 6.7 (Nested Sequence Theorem)

If X is compact, then for any sequence of closed, nonempty sets $C_1 \supset C_2 \supset \ldots$, we must have

$$\bigcup_{n=0}^{\infty} C_n \neq \emptyset \tag{138}$$

Now we present another theorem in analysis that is actually a topological property.

Theorem 6.8 (Extreme Value Theorem)

Let X be compact and $f : X \to \mathbb{R}$ be continuous. Then f attains both its maximum and minimum in X.

Proof. f continuous must imply that $f(X)$ is compact in $\mathbb{R}$, which means that it is closed and bounded. Since it is bounded, it must have a least upper bound. Since it's closed, it must contain its least upper bound, and so attains its maximum. Similarly for minimum.

The theorem says that $f : [a, b] \to \mathbb{R}$ is bounded, and that bound is *realized* by something. Note that for function on noncompact $(-1, 1)$—say $f(x) = x$—it is bounded but the extrema are not achieved. It doesn't even need to be bounded, e.g. $f(x) = \frac{x}{1-x^2}$.

Theorem 6.9

Every compact Hausdorff space is normal.

6.2 Preservation of Open Cover Compactness

The first property we will state—similar to that for connected spaces—is that continuous images of compact spaces is compact, which establishes compactness as a topological property.

Theorem 6.10 (Continuous Images of Compact Sets are Compact)

The image of a compact space under a continuous map is compact.

Proof. Let $f : X \to Y$ be continuous, and let X be compact. Let $\mathcal{C}$ be a covering of the set $f(X)$ by sets open in Y. Then, the preimage of these sets is the collection

$$\{f^{-1}(\mathcal{A}) \mid \mathcal{A} \in \mathcal{C}\} \tag{139}$$

which clearly covers X. But since X is compact, a finite number of them, say

$$f^{-1}(\mathcal{A}_1), f^{-1}(\mathcal{A}_2), \ldots, f^{-1}(\mathcal{A}_n) \tag{140}$$

covers $X \implies \mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n$ covers $f(X)$.

This establishes that compactness is a topological property. That is, if $X \cong Y$, then X compact $\iff$ Y compact.

Lemma 6.11

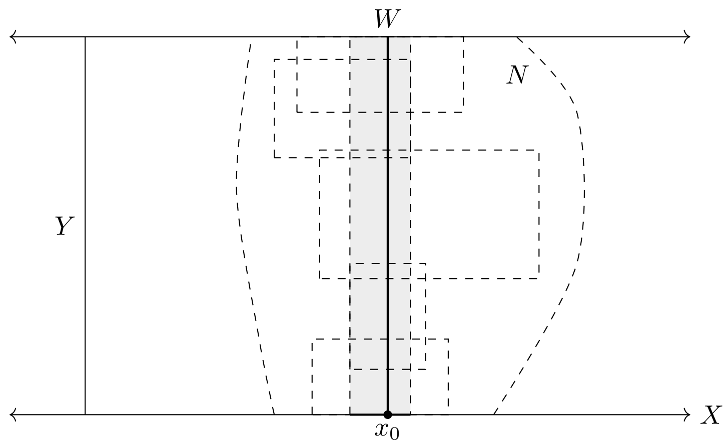
A finite union of compact sets is compact.

Proof. It suffices to prove for two sets A, B by induction. Take an arbitrary cover $\mathscr{L}$ of $A \cup B$. Then $\mathscr{L}$ is a cover of A, so it has a finite subcover $\mathscr{F} \subset \mathscr{L}$. It is also a cover of B, so it has a finite subcover $\mathscr{G} \subset \mathscr{L}$. Therefore, $\mathscr{F} \cup \mathscr{G} \subset \mathscr{L}$ is a cover of $A \cup B$, and since it is the union of finite covers, it is finite.

We now introduce a useful lemma that will come around in many future cases.

Lemma 6.12 (Tube Lemma)

Consider the product space $X \times Y$, where Y is compact. If N is an open set $X \times Y$ containing the slice $x_0 \times Y$ of $X \times Y$, then N contains some tube $W \times Y$ about $x_0 \times Y$, where W is a neighborhood of x_0 in X.



Proof. Let us cover $x_0 \times Y$ by basis elements $U \times V$ (for the topology of $X \times Y$) lying in N. The space x_0 is compact since it is homeomorphic to $Y \implies$ we can cover $x_0 \times Y$ by finitely such basis elements

$$U_1 \times V_1, U_2 \times V_2, \ldots, U_n \times V_n \tag{141}$$

Without loss of generality, we can assume that each $U_i \times V_i$ has a nontrivial intersection with $x_0 \times Y$, since otherwise, it would be superfluous. Now, we define the intersection of all the open neighborhoods of x_0 in X of the basis elements $U_i \times V_i$. That is, let

$$W := \bigcup_{i=1}^{n} U_i \tag{142}$$

As an intersection of open sets, W is also open containing x_0. With this well-defined tube $W \times Y$, we claim that it is entirely contained within N. That is, given a point $x \times y \in W \times Y$, consider the corresponding point $x_0 \times y$ that is the image of the projection of $x \times y$ onto $x_0 \times Y$. Clearly, $x_0 \times y$ belongs to some $U_k \times V_k$ (for some k) $\implies$ $y \in V_k$. Since $x \in W$, x is clearly in U_k, meaning that $x \times y \in U_k \times V_k \subset N$, as desired.

Theorem 6.13 (Finite Products)

The product of finitely many compact spaces is compact.

Proof. Using induction, it suffices to prove that the product of 2 compact spaces is compact. Let X and Y be compact spaces. By the tube lemma, for each $x \in X$, there exists a neighborhood W_x of x such that the tube $W_x \times Y$ can be covered with finitely (by compactness of Y) many open sets in $X \times Y$. The collection of all neighborhoods W_x is an open covering of X. By compactness of X, there exists a finite subcollection

$$W_1, W_2, \ldots, W_k \tag{143}$$

covering X. The finite union of the tubes

$$\bigcup_{i=1}^{k} W_i \times Y \tag{144}$$

clearly covers $X \times Y$, meaning that $X \times Y$ is compact.

Theorem 6.14 (Tychonoff Theorem)

An arbitrary product of compact spaces is compact under the product topology.

Proof.

Definition 6.5 (Finite Intersection Condition)

A collection $\mathcal{C}$ of subsets of X is said to satisfy the **finite intersection condition** if for every finite subcollection

$$\{\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n\} \tag{145}$$

of $\mathcal{C}$, the intersection

$$\bigcap_{i=1}^{n} \mathcal{C}_i \tag{146}$$

is nonempty.

Clearly, the empty sets cannot below to any collection with the finite intersection property. Additionally, the condition is trivially satisfied if the intersection over the entire collection is non-empty or if the collection is nested. However, here is one example that does satisfy the finite intersection condition.

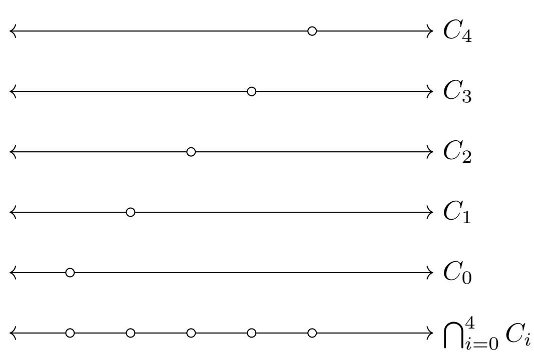
Example 6.5

Let $X = (0, 1)$ and for each positive integer i, X_i is the set of elements of X having a decimal expansion with digit 0 in the ith decimal place. Then, any finite intersection of X_i's is nonempty, but the intersection of all X_i for $i \in \mathbb{N}$ is empty, since no element of $(0, 1)$ has all zero digits.

Here is an analogous example to the previous one.

Example 6.6

In the space $\mathbb{R}$, let us define $C_i \coloneqq \mathbb{R} \setminus \{i\}$. That is, C_i is $\mathbb{R}$ missing a point at i. Then, the collection of all C_i's does satisfy the finite intersection condition. We show below the finite intersection of the five subsets C_0, C_1, C_2, C_3, C_4.



Theorem 6.15

Let X be a topological space. Then x is compact if and only if for any collection $\mathcal{C}$ of closed sets in X satisfying the finite intersection condition, the intersection

$$\bigcap_{C \in \mathcal{C}} C \tag{147}$$

of all the elements of $\mathcal{C}$ is nonempty.

Proof. Given a collection S fo subets of X, let

$$\mathcal{C} := \{X \setminus A \mid A \in S\} \tag{148}$$

be the collection of their complements. Then, the following statements hold

1. S is a collection of open sets if and only if $\mathcal{C}$ is a collection of closed sets.
2. The collection S covers X if and only if the intersection

$$\bigcap_{C \in \mathcal{C}} C \tag{149}$$

   of all the elements of $\mathcal{C}$ is empty.
3. The finite subcollection $\{A_1, A_2, \ldots, A_n\}$ of S covers X if and only if the intersection of the corresponding elements $C_i := X \setminus A$ of $\mathcal{C}$ is empty.

Clearly, (1) is trivial, and (2) and (3) follows from DeMorgan's Law.

$$X \setminus \bigcup_{\alpha \in J} A_\alpha = \bigcap_{\alpha \in J} (X \setminus A_\alpha) \tag{150}$$

Using statement 3, the existence of a finite collection of closed sets C in X satisfying the finite intersection condition is equivalent to its complements (which are open sets) covering X, which is precisely the definition of compactness.

Clearly, the previous example in the real line $\mathbb{R}$ shows that $\mathbb{R}$ is indeed not compact.

Corollary 6.16

The space X is compact if and only if every collection $\mathscr{C}$ of subsets of X satisfying the finite intersection condition, the intersection

$$\bigcap_{A \in \mathscr{C}} \bar{A} \tag{151}$$

of their closures is nonempty.

6.3 Limit Point Compactness

We now state different, weaker types of compactness.

Definition 6.6 (Limit Point Compactness)

A space X is said to be **limit point compact** if every infinite subset of X has a limit point.

Theorem 6.17

Compactness $\implies$ limit point compactness.

Lemma 6.18 (Lebesgue Number Lemma)

Let $\mathscr{C}$ be an open covering of the metric space (X, d). If X is compact, then there is a $\delta > 0$ such that for each subset of X having diameter than δ, there exists an element of $\mathscr{C}$ containing it. This number δ is called a **Lebesgue number** for the covering $\mathscr{C}$.

6.4 Preservation of Limit Point Compactness

6.5 Sequential Compactness

Definition 6.7 (Sequentially Compact)

A space X is said to be **sequentially compact** if every sequence of points in X has a subsequence that converges to a point $x \in X$.

Theorem 6.19 (Equivalence of Compactness in Metrizable Spaces)

Let $(X, \mathscr{T})$ be a metrizable space. Then the following are equivalent:

1. X is compact.
2. X is limit point compact.
3. X is sequentially compact.
4. X is countably compact.

6.6 Preservation of Sequential Compactness

6.7 Local Compactness

Definition 6.8 (Locally Compact)

A space X is said to be **locally compact** at x if there is some compact subset C of X that contains a neighborhood of x. If X is locally compact at each of its points, X is simply to be **locally compact**.

Example 6.7

The real line $\mathbb{R}$ is locally compact since any point $x \in \mathbb{R}$ lies within a certain closed interval $[a, b]$, which is compact. The subspace $\mathbb{Q}$ is not locally compact.

Two of the most well-behaved classes of spcaes to deal with are metrizable spaces and compact Hausdorff spaces. If a given space is not one of these types, the next best thing one can hope for is that it is a subspace of one of these spaces. Clearly, a subspace of a metrizable space is itself metrizable, so one does not get any new spaces this way. However, a subspace of a compact Hausdorff space need not be compact. This leads to the question: Under what conditions is a space homeomorphic to a subspace of a compact Hausdorff space?

Definition 6.9 (Compactification)

Let X be a locally compact Hausdorff space. Take some object outside X, denoted by the symbol ∞, and adjoin it to X, forming the set

$$Y = X \cup \{\infty\} \tag{152}$$

Topologize Y by defining the collection of open sets in Y to be the sets of the following types:
1. U, where U is an open subset of X.
2. $Y \setminus C$, where C is a compact subset of X.

Then, this space Y is called the **one-point compactification of** X. This is in some sense the minimal compactification of X.

We briefly show that this set of open sets on Y is indeed a topology. First, $\emptyset$ is of type 1 and Y itself is of type 2. Given U_i of type 1 and $Y \setminus C_i$ of type 2, we have the intersections of two sets

$$\begin{aligned} & U_1 \cap U_2 && \text{is type 1} \\ & (Y \setminus C_1) \cap (Y \setminus C_2) = Y \setminus (C_1 \cup C_2) && \text{is type 2} \\ & U_1 \cap (Y \setminus C_1) = U_1 \cap (X \setminus C_1) && \text{is type 1} \end{aligned}$$

along with the arbitrary union of sets

$$\begin{aligned} & \bigcup U_\alpha = U && \text{is type 1} \\ & \bigcup (Y \setminus C_\beta) = Y \setminus (\bigcap C_\beta) = Y \setminus C && \text{is type 2} \\ & (\bigcup U_\alpha) \cup (\bigcup (Y \setminus C_\beta)) = U \cup (Y \setminus C) = Y \setminus (C \setminus U) && \text{is type 2} \end{aligned}$$

We now present some properties of one-point compactifications.

Theorem 6.20

Let X be a locally compact Hausdorff space which is not compact, and let Y be a one-point compactification of X. Then Y is a compact Hausdorff space. Additionally, since $X \subset Y$ with $Y \setminus X$ consisting of a single point, $\bar{X} = Y$.

Example 6.8 (Extended Real Number Line)

The one-point compactification of the real line $\mathbb{R}$ is homeomorphic to the circle S^1. That is,

$$\mathbb{R} \cup \{\infty\} \cong S^1 \tag{153}$$

$\mathbb{R} \cup \{\infty\}$ is called the **extended real number line**.

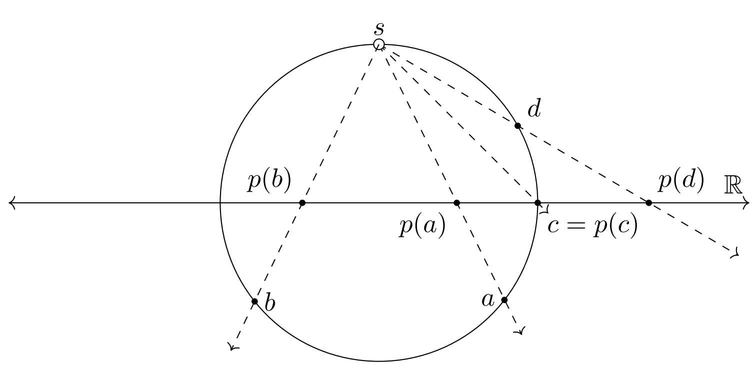


Figure 43: We can visualize this homeomorphism by visualizing the stereographic projection $p : S^1 \setminus \{s\} \to \mathbb{R}$.

Example 6.9 (2-Sphere)

The one point-compactification of the real plane $\mathbb{R}^2$ is homeomorphic to the 2-sphere S^2. That is,

$$\mathbb{R}^2 \cup \{\infty\} \cong S^2 \tag{154}$$

Lemma 6.21

Let X be a Hausdorff space. Then X is locally compact at x if and only if for every neighborhood U of x, there is a neighborhood V of x such that $\bar{V}$ is compact and $\bar{V} \subset U$.

Corollary 6.22

Let X be a locally compact Hausdorff space with Y a subspace of X. If Y is closed in X or open in X, then Y is locally compact.

Corollary 6.23

A space X is homeomorphic to an open subset of a compact Hausdorff space if and only if X is locally compact and Hausdorff.

6.8 Countable Compactness

Definition 6.10 (Countably Compact)

A space X is said to be **countably compact** if every countably open cover has a finite subcover.

6.9 Lindelof Spaces

Definition 6.11 (Lindelof Space)

A space for which every open covering contains a countable subcovering is called a **Lindelof space**.

6.10 Compactifications

Definition 6.12

A **compactification** of a space X is a compact Hausdorff space Y containing X such that X is dense in Y (that is $\bar{X} = Y$). Two compactifications Y_1 and Y_2 of X are said to be **equivalent** if there is a homeomorphism $h : Y_1 \longrightarrow Y_2$ such that $h(x) = x$ for every $x \in X$.

Theorem 6.24

Let X be completely regular, and let $\beta(X)$ be its Stone-Cech compatification. Then every bounded continuous real-valued function on X can be uniquely extended to a continuous real-valued function on $\beta(X)$.

Lemma 6.25

Let $A \subset X$, and let $f : A \longrightarrow Z$ be a continuous map of A into the Hausdorff space Z. There is at most one extension of f to a continuous function $g : \bar{A} \longrightarrow Z$.

Theorem 6.26

Let X be completely regular. Let Y_1, Y_2 be two compactifications of X having the extension property. Then there is a homeomorphism ϕ of Y_1 onto Y_2 such that $\phi(x) = x$ for each $x \in X$.

7 Countability

7.1 1st Countability

Definition 7.1 (1st-Countability)

A space X is said to have a countable basis at x if there exists a sequence $N_1, N_2, ...$ of open neighborhoods of x such that for any neighborhood N of x, there exists an integer i such that $N_i \in N$. That is, the countable basis of neighborhoods get arbitrarily small around x. A space X satisfying this axiom at every point $x \in X$ is said to be a **first-countable space**.

In particular, every metric space is first-countable, since we can construct the sequence of open balls $B(x, \frac{1}{n})$ for each $n \in \mathbb{N}$ which forms a countable basis at x. We now generalize some previous statements about metric spaces to statements about first-countable spaces.

Theorem 7.1

Let X be a space satisfying the first countability axiom, and let $A \subset X$.
1. $x \in \bar{A}$ if and only if there exists a sequence of points in A converging to x.
2. The function $f : X \longrightarrow Y$ is continuous if and only if for every convergent sequence $(x_n) \to x$ in X, the sequence $\big(f(x_n)\big) \to f(x)$ in Y.

7.2 Preservation of 1st Countability

Theorem 7.2 (Subspace)

A subspace of a 1st countable space is 1st countable.

Theorem 7.3 (Finite Products)

Theorem 7.4 (Countable Products)

A countable product of 1st countable spaces is 1st countable.

7.3 2nd Countability

Definition 7.2 (2nd-Countability)

A topological space X is said to satisfy the **second countability axiom** if X has a countable basis for its topology.

Theorem 7.5

Second countability implies first countability.

Proof. If $\mathscr{B}$ is a countable basis for the topology of X, then the subset of $\mathscr{B}$ consisting of elements containing the point x is a countable basis at x.

Example 7.1

The real line $\mathbb{R}$ is second countable. We can contrust a countable basis as the set of all open intervals (a, b) with rational end points. Likewise, $\mathbb{R}^n$ has a countable basis, which is the collection of all products of intervals having rational end points. Additionally, $\mathbb{R}^\omega$ has a countable basis. It is the collection of all products

$$\prod_{n \in \mathbb{N}} U_n \tag{155}$$

where U_n is an open interval with rational endpoints for finitely many values of n and $U_n = \mathbb{R}$ for all other values of n.

Example 7.2

In the uniform topology, $\mathbb{R}^\omega$ satisfies the first countability axiom (since it is metrizable).

7.4 Preservation of 2nd Countability

Theorem 7.6 (Subspace)

A subspace of a 2nd countable space is 2nd countable.

Theorem 7.7 (Finite Products)

Theorem 7.8 (Countable Products)

A countable product of 2nd countable spaces is 2nd countable.

Theorem 7.9

Suppose that X has a countable basis. Then,
1. Every open cover of X has a countable subcover.
2. There exists a countable subset of X which is dense in X.

Proof. Listed.
1. Let $\mathscr{B} = \{B_n\}_{n \in \mathbb{N}}$ be a countable basis for X, and let $\mathscr{A}$ be an open covering of X. For each integer $n \in \mathbb{N}$, chose an element $A_n \in \mathscr{A}$ containing the basis element B_n. The newly formed collection $\mathscr{A}'$ of all the A_n's is countable since it is indexed according to a subset of $\mathbb{N}$. Furthermore, since $B_n \subset A_n$ for every B_n in the basis, the A_n clearly covers X.
2. From each nonempty basis element B_n, we choose a point x_n. The set

$$D \coloneqq \{x_n \mid n \in \mathbb{N}\} \tag{156}$$

is dense in X, since given any $x \in X$, every open basis element B_x about x intersects D. That is,

$$B_x \cap D \neq \emptyset \tag{157}$$

meaning that the set of points x_n get arbitrarily close to x.

8 Metric Topologies

In $\mathbb{R}$, note that every open ball is really just an interval. In fact, every open ball $(x - r, x + r)$ can be expressed with just two elements $a, b \in \mathbb{R}$, as (a, b). Notice that this method of expressing an open set does not even require any metric! Extending this to $\mathbb{R}^n$ would indicate that the topologies of $\mathbb{R}^n$ defined by the endpoint of the open intervals would not necessarily induce any metric either. Notice that these induced topologies is **not** the open ball topology, which must have an associated metric to it. Rather, this induced, non-metric topology is the box topology! While the box topology and the open ball topology are really the same topology, they are generated by inherently different bases.

8.1 Convergence

Theorem 8.1 (Sufficient Conditions for Convergence in Metric Space)

Let (x_n) be a sequence in a metric space X.

1. (x_n) converges to $x \in X$ if and only if every neighborhood of x contains x_n for all but finitely many n.
2. If (x_n) converges to x, then x is unique.
3. If (x_n) converges, then (x_n) is bounded.
4. If $E \subset X$ and x is a limit point of E, then there exists a sequence (x_n) in E that converges to x.

Proof. Listed.

1. ($\implies$) Let $x_n \to x \in X$. Then, for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ s.t. $d(x, x_n) < \epsilon$ for all $n > N$. Given neighborhood $B_\epsilon(x), x_n \in B_\epsilon(x)$ for all $n > N \implies$ at most N elements are not in $B_\epsilon(x)$. ($\impliedby$) Now for any $\epsilon > 0$, let every $B_\epsilon(x)$ contain all but finitely many x_n. Enumerate them $\{x_{n_k}\}_{k=1}^K$, and let
$$\alpha = \max\{n_k\} \tag{158}$$
This means that there exists an $\alpha \in \mathbb{N}$ s.t. $x_n \in B_\epsilon(x)$ for all $n > \alpha$, which implies that $x_n \to x$.
2. Assume $\{x_n\}$ converges to $x, x' \in X$, with $x \neq x'$. Then for all $\epsilon > 0$ there exists $N_1, N_2 \in \mathbb{N}$ s.t. $d(x, x_n) < \epsilon$ for all $n > N_1$ and $d(x', x_n) < \epsilon$ for all $n > N_2$. This means that there exists a $N = \max\{N_1, N_2\}$ satisfying the above. Since $x \neq x'$, set $\epsilon = d(x, x')/2$. Then,
$$d(x, x_n) < \frac{d(x, x')}{2} \text{ and } d(x', x_n) < \frac{d(x, x')}{2} \tag{159}$$
which implies that by adding both sides and invoking triangle inequality, we have
$$d(x, x') \leq d(x, x_n) + d(x', x_n) < d(x, x') \tag{160}$$
which is absurd.
3. Choose any $\epsilon > 0$. Then from (a), $B_\epsilon(x)$ contains all but finitely many $V = \{x_{n_k}\}_{k=1}^K$. Take
$$M = \max\{\epsilon, d(x_{n_1}, x), \ldots, d(x_{n_K}, x)\} \tag{161}$$
and so $d(x, x_n) < M$ for all $n \in \mathbb{N}$.[a]
4. We can explicitly construct one. Let $x \in E'$. Then choose $\epsilon = 1, \frac{1}{2}, \frac{1}{3}, \ldots$ and for every $\epsilon > 0$, $B_\epsilon(x) \cap E \neq \emptyset$. Choose a x_n within this intersection for every $\epsilon = \frac{1}{n}$. Then, we have $\{x_n\}$ contained in E. We want to show that this converges to x. Take any $\epsilon > 0$, then there exists $N \in \mathbb{N}$ s.t. $0 < \frac{1}{N} < \epsilon$, and for every $n > N$, $\frac{1}{n} < \frac{1}{N} < \epsilon$.
$$n > N \implies x_n \in B_{1/n}(x) \subset B_\epsilon(x) \tag{162}$$
which means that $x_n \in B_\epsilon(x)$ for all $n > N$, implying that $\lim_{n\to\infty} x_n = x$.

[a] This is also a direct result of every metric topology being Hausdorff.

Lemma 8.2 (Sequence Converges iff Every Subsequence Converges)

(x_n) converges to x if and only if every subsequence of (x_n) converges to x.

Proof. Let $x_n \to x$. Then, take any subsequence (x_{n_k}) of (x_n). For any $\epsilon > 0$, there exists a $N \in \mathbb{N}$ s.t. $d(x, x_n) < \epsilon$ for all $n > N$. Since N is finite and the n_k's are unbounded, there must exist a $K \in \mathbb{N}$ s.t. $n_k > N$ if $k > K$. Therefore, given any $\epsilon > 0$, we have proved the existence of a $K \in \mathbb{N}$ s.t. $k > K \implies n_k > N$, which implies by convergence of x_n, that

$$d(x_{n_k}, x) < \epsilon \tag{163}$$

which by definition means that x_{n_k} converges to x. Now, for the other direction, given (x_n) with every subsequence converging to x, we can take the subsequence (x_n) itself ($n_k = k$), which converges to x.

Theorem 8.3 (Nested Compact Sets)

Listed.

1. If $\overline{E}$ is the closure of a set E in a metric space X, then
$$\operatorname{diam} \overline{E} = \operatorname{diam} E \tag{164}$$
2. If K_n is a sequence of compact sets in X s.t. $K_n \supset K_{n+1}$ for $n \in \mathbb{N}$ and if
$$\lim_{n \to \infty} \operatorname{diam} K_n = 0 \tag{165}$$
then $\cap_{n=1}^\infty K_n$ consists of exactly one point.

Proof.

Theorem 8.4 (Subsequential Limits form Closed Subset)

The subsequential limits of a sequence (x_n) in a metric space form a closed subset of X.

Proof. Let E be the set of subsequential limits and $y \in E'$ be a limit point of E. We must show that $y \in E$.[a] We will construct a subsequence (x_{n_k}) that converges to y. Given any $\epsilon > 0$, we can see that the $B_{\epsilon/2}(y) \cap E \neq \emptyset$, so choose an element z_ϵ. Furthermore, z_ϵ means that it is a limit point of (x_n), and so $B_{\epsilon/2}(z) \cap \{x_n\} \neq \emptyset$, call this $x(\epsilon)$. Therefore, by the triangle inequality,

$$|y - x(\epsilon)| \leq |y - z| + |z - x(\epsilon)| = \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \tag{166}$$

and so we can take a point from the sequence (x_n) for every ϵ. Now we do this for $\epsilon = 1/n$, and choose n_k that is greater than its previous by restricting the sequence to that past n_{k-1}. Doing this gives a subsequence which converges to y. Therefore $y \in E$.

[a] Intuitively, we can see that y is infinitesimally close to E, which consists of points infinitesimally close to (x_n), and so y should be infinitesimally close to (x_n).

8.2 Compactness

As we will see in the following theorems, compact sets behave well with closed sets. In fact, compactness is in a form a stronger notion than closedness.

Theorem 8.5 (Compact Subsets of Metric Spaces are Closed)

Compact subsets of metric spaces are closed.

Proof. Metric spaces are Hausdorff, and compact subsets of Hausdorff spaces are closed. Note that this is what we essentially do in the more elementary 2nd proof below.

Proof. We would like to show that if A is compact in X, then A^c is open. What we would like to do is if we have some $x \in A^c$, then we must prove that there exists some open set $B_\epsilon(x)$ that is disjoint with A. For every point $a \in A$, we can construct an open balls $V_a = B_{d(x,a)/2}(a)$ and $U_a = B_{d(x,a)/2}(x)$. We know that if $y \in B_{d(x,a)/2}(a)$, then assuming $y \in B_{d(x,a)/2}(x)$ will give

$$d(x,a) \leq d(x,y) + d(y,a) < \frac{d(x,a)}{2} + \frac{d(x,a)}{2} = d(x,a) \tag{167}$$

which is absurd. Since $\{V_a\}_{a \in A}$ forms an open covering of A, then by compactness we can take a finite subcover $V_{a_1}, \ldots, V_{a_n}$, along with the respective neighborhoods of x $U_{a_1}, \ldots, U_{a_n}$. Since we have established

$$V_{a_i} \cap U_{a_i} = \emptyset \implies \bigcap_{i=1}^n V_{a_i} \cap \left(\bigcup_{i=1}^n U_{a_i} \right) = \emptyset \tag{168}$$

and since $\cap_{i=1}^n V_{a_i}$ is open (as it is the intersection of open sets) and disjoint from an open cover of A and hence from A, we have proved that A^c is open, and so A is closed.

In fact, compactness actually implies completeness.

Theorem 8.6

Compact metric spaces are complete.

In order to construct new compact spaces from old ones, we must prove compactness for a number of fundamental spaces. The real number line is a good starting point, but before we can fully appreciate the Heine-Borel theorem, we focus on the properties of closedness and boundedness.

Theorem 8.7 (Compact Subsets of Metric Spaces are Bounded)

Let (X, d) be a metric space and $K \subset X$ be compact. Then, K must be bounded.

Proof. Given nonempty K, choose a point $x \in K$. Then, construct the open covering.

$$\mathcal{F} = \{B_n(x) \mid n \in \mathbb{N}\} \tag{169}$$

which is indeed a covering since the naturals are unbounded in $\mathbb{R}$. Then by compactness there is a finite subcovering, and so there must be some maximum n.

Lemma 8.8 (Closed Intervals are Compact)

Any closed interval $[a, b] \subset \mathbb{R}$ is compact.

Proof. Let $\mathcal{F}$ be a covering, and let

$$S = \{x \in [a, b] \mid [a, x] \text{ is contained in finite subcollection of } \mathcal{F}\} \tag{170}$$

We wish to show that $b \in S$. Since S is bounded by b, let $s = \sup\{s\}$. We claim that $s \in S$.

1. We know that $a \in S$ since a is covered by at least one given set in S, so if $s = a$ we are done. Else if $s > a$, it must also be the case that $s \leq b$, so $s \in (a, b)$. We choose some open $U \in \mathcal{F}$ containing S, i.e. there exists $\epsilon > 0$ s.t. $(s - \epsilon, s] \subset U \implies s - \epsilon$ is not an upper bound for S. So $\exists x$ with $s - \epsilon < x \leq s$ with $s \in S$. So $[a, s]$ is contained in finitely many sets of $\mathcal{F}$, so $s \in S$.
2. Now we will claim $s = b$. Suppose otherwise, i.e. $s < b$. Then
$$[a, s] \subset U_1 \cup \ldots \cup U_n \text{ for } U_i \in \mathcal{F} \tag{171}$$
But $s \in U_i$ for some i, and for some ϵ, $[s, s + \epsilon) \subset U$, and so $[a, s + \frac{\epsilon}{2}] \subset \cup_{i=1}^n U_i$, which implies $s + \frac{\epsilon}{2} \in S$, which contradicts that $s \in S$. So $s = b$.

Now we can extend this to $\mathbb{R}^n$.

Theorem 8.9 (Heine-Borel Theorem)

A subset A of $\mathbb{R}^n$ is compact if and only if it is closed and bounded in the Euclidean metric d or the square metric ρ.

Proof. The forwards direction is actually very easy. Since A is compact in Hausdorff $\mathbb{R}^n$, it must be closed, and since $\mathbb{R}^n$ is a metric space, it must be bounded. So we are done. For the other way, we assume that a subset A is closed and bounded. If A is bounded, then $\exists R > 0$ s.t. $d(0, x) < R$ for all $x \in A$. So $A \subset [-R, R]^n = B$, where B is compact and A is a closed subset. Hence A is compact as well.
Note that bounded w.r.t. metric d is equivalent to being bounded w.r.t metric ρ.

The Heine-Borel theorem is like a reverse implication of the fact that compact sets must be closed and—if in a metric space—bounded. However, the double implication is not achieved when working in general metric spaces!

Example 8.1 (Closed and Bounded in Metric Space But Not Compact)

Consider the space $X = \mathbb{Q}$ and $A = [1, \sqrt{2}] \subset X$. A is closed and bounded, but not compact.

Example 8.2

The unit sphere S^{n-1} and the closed ball B^n in $\mathbb{R}^n$ are compact since they are closed and bounded. The set
$$A \coloneqq \{(x, \frac{1}{x}) \mid 0 < x \leq 1\} \tag{172}$$
is closed in $\mathbb{R}^2$, but is not compact since it is not bounded. The set
$$S \coloneqq \{(x, \sin\frac{1}{x}) \mid 0 < x \leq 1\} \tag{173}$$
is bounded in $\mathbb{R}^2$, but it is not compact since it is not closed.

Theorem 8.10 (Uniform Continuity Theorem)

Let $f : X \to Y$ be a continuous map of the compact metric space (X, d_X) to the metric space $(Y, d_Y$. Then, f is uniformly continuous. That is, given $\epsilon > 0$, there exists a $\delta > 0$ such that for any two points $x_1, x_2 \in X$,

$$d_X(x_1, x_2) < \delta \implies d_Y\big(f(x_1), f(x_2)\big) < \epsilon \tag{174}$$

Theorem 8.11 (Compact Metric Spaces are Complete)

A metric space is compact iff it is complete and totally bounded.

Proof.

8.3 Connectedness

Now let's go to our standard space and see which sets are connected. For this, we will need to visit a bit of real analysis and recall the least upper bound property of the reals.

Theorem 8.12 (Convex Subsets of Reals are Connected)

Y is a convex[a] subset of $\mathbb{R}$ iff Y is connected.

[a] We can define it with the order by saying that an ordered set Y is convex is given $a, b \in Y$ and a c s.t. $a < c < b$, then $c \in Y$.

Proof. We prove by contradiction.[a] Suppose $Y = A \cup B$ disjoint, nonempty, and open in the subspace topology. Choose $a \in A, b \in B$ and WLOG let $a < b$. Now let

$$A_0 = A \cap [a, b], \qquad B_0 = B \cap [a, b] \tag{175}$$

Then $A_0 \sqcup A_0 = [a, b]$ with A_0, B_0 open in the subspace topology in $[a, b]$. Furthermore, A_0 is bounded above so it has a least upper bound. Let $c = \sup A_0$.

1. If $c \in A_0$, by A_0 open there exists a right interval $[c, c + \epsilon) \subset A_0 \implies c + \frac{\epsilon}{2} \in A_0$, which means c is not an upper bound.
2. If $c \in B_0$, by B_0 open there exists a left interval $(c - \epsilon, c] \subset B_0 \implies c - \frac{\epsilon}{2} \in B_0$, which means c is not least.

[a] Note that this is sort of similar to proving how if vectors $v_1, \ldots, v_n$ are linearly independent, then $a_1 v_1 + \ldots + a_n v_n = 0 \iff a_1 = a_2 = \ldots = a_n = 0$. This is generally, the method to prove whether a space is connected.

Corollary 8.13

$\mathbb{R}$ with the standard topology is connected. Furthermore all intervals and rays are connected subsets of $\mathbb{R}$.

Example 8.3 (More Topologies on Real Lines)

Note that

1. $\mathbb{R}$ with the lower limit topology is disconnected, since $(-\infty, a) \sqcup [a, +\infty)$ is a separation.
2. $\mathbb{R}$ with the finite complement topology is connected. In this case, no two open sets are even disjoint, and so this space is very very connected.

Example 8.4 (Dense Disconnected Sets)

The rationals $\mathbb{Q} \subset \mathbb{R}$ are not connected since given any irrational number r, we can write Y as the union of sets

$$Y \cap (-\infty, r),\ Y \cap (r, +\infty) \tag{176}$$

which are open in the subspace topology.

Example 8.5 (Connected Components in $\mathbb{R}^n$)

$\mathbb{R}^n$ is connected, since it's a finite product of $\mathbb{R}$, which we proved was connected. Similarly, all n-cells of form $\prod_{i=1}^n (a_i, b_i)$ are also connected as a product of convex sets.

We will expand on this to prove some already intuitive results.

Theorem 8.14

If $n > 1$, then for any $a \in \mathbb{R}^n$, $\mathbb{R}^n \setminus \{a\}$ is connected.

Proof. Let $U = \{x \in \mathbb{R}^n \mid x_n > a_n\}$ and $V = \{x \in \mathbb{R}^n \mid x_n < a_n\}$. These are connected since they are of form $\mathbb{R}^{n-1} \times (a, b)$. Now let

$$U' = U \cup \{x \in \mathbb{R}^n \mid x_n = a_n\} \setminus \{a\} \tag{177}$$

$$V' = V \cup \{x \in \mathbb{R}^n \mid x_n = a_n\} \setminus \{a\} \tag{178}$$

Note that U', V' are the U, V plus some of its limit points, and so they are connected as well. So $U' \cup V' = \mathbb{R}^n \setminus \{a\}$ connected since they have a nontrivial intersection.

Corollary 8.15

$\mathbb{R} \not\cong \mathbb{R}^n$ for $n > 1$.

Finally, we conclude with a theorem often seen in calculus, but is really a theorem in topology, since it only relies on continuity rather than derivatives, like the MVT.

Theorem 8.16 (Intermediate Value Theorem)

Let $f : X \longrightarrow Y$ be a continuous map of the connected space X into the ordered set Y, with the order topology. Given $a, b \in X$ and $r \in Y$ such that $f(a) < r < f(b)$, then there exists a point $c \in X$ such that $f(c) = r$.

Proof. Assuming the hypothesis, the sets

$$A \coloneqq f(X) \cap (-\infty, r), \qquad B \coloneqq f(X) \cap (r, +\infty) \tag{179}$$

are disjoint. They are also nonempty since

$$f(a) \in A,\ f(b) \in B \tag{180}$$

A and B are open since they are the intersection of open sets. Now, assume that there exists no point $c \in X$ such that $f(c) = r$. Then,

$$f(X) = A \cup B \tag{181}$$

would define a separation of X, contradicting the fact that the image of a connected space under a continuous map must be connected. Therefore, c exists.

8.4 Separability

Theorem 8.17

Every metrizable space is normal.

8.5 Bounded Metric

Definition 8.1 (Bounded Set)

Let (X, d) be a metric space with subset A. A is **bounded** if there exists some number M such that

$$d(x, y) \leq M \text{ for all } x, y \in A \tag{182}$$

If A is bounded, the **diameter** of A is defined to be the number

$$\operatorname{diam} A \coloneqq \sup \{d(x, y) \mid x, y \in A\} \tag{183}$$

Note that boundedness on a set is not a topological property since it depends on the particular metric d that is used for X. For example, we can construct the following metric that makes every subset in X bounded.

Definition 8.2 (Standard Bounded Metric)

Let (X, d) be a metric space. We define a second metric $\tilde{d}$ on X such that

$$\tilde{d}(x, y) \coloneqq \min \{d(x, y), 1\} \tag{184}$$

$\tilde{d}$ is called the **standard bounded metric corresponding to** d.

If we construct open balls with this metric, it is easy to see that they consist of all open balls with radius less than or equal to 1. That is, the topology $\mathscr{T}$ consists of all open balls

$$\mathscr{T} \coloneqq \{B_r(x) \mid x \in X, r \leq 1\} \tag{185}$$

It is also clear that the topology induced by $\tilde{d}$ is the same as the topology induced by d! The significance of this construction of the standard bounded metric is that we can now work with a basis consisting of bounded elements, which is much nicer than a basis of open balls that can have arbitrarily large radii.

We now introduce a metrization theorem on $\mathbb{R}^n$.

Theorem 8.18

The topologies on $\mathbb{R}^n$ induced by the Euclidean metric d and the square metric ρ are the same as the product topology on $\mathbb{R}^n$.

Proof. Given $x, y \in \mathbb{R}^n$, simple algebra shows that

$$\begin{aligned} &\rho(x, y) \leq d(x, y) \leq \sqrt{n}\rho(x, y) \\ &\quad \implies \forall x, \epsilon,\ B_d(x, \epsilon) \subset B_\rho(x, \epsilon) \text{ and } B_\rho(x, \frac{\epsilon}{\sqrt{n}}) \subset B_d(x, \epsilon) \end{aligned}$$

But

$$\{B_\rho(x,\epsilon) \mid x \in \mathbb{R}^n, \epsilon \in \mathbb{R}\} = B_\rho(x, \frac{\epsilon}{\sqrt{n}}) \mid x \in \mathbb{R}^n, \epsilon \in \mathbb{R}\} \tag{186}$$

which means that the metric topology induced by d is the same as the metric topology induced by $\rho \implies$ the two topologies are the same. We know that the topology induced by ρ is the same as the product topology since

$$\prod_{i=1}^n (x_i - r, x_i + r) = \bigcup_{k=1}^n \mathbb{R}^{k-1} \times (x_k - r, x_k + r) \times \mathbb{R}^{n-k} \tag{187}$$

With this theorem, we have proved that given a topological space $\mathbb{R}^n$ with the product topology, there exists a metric (the Euclidean and square metric) that induces this product topology. We can attempt to extrapolate these formulas to $\mathbb{R}^\omega$ by defining

$$d(x,y) := \bigg(\sum_{i=1}^\infty (x_i - y_i)^2 \bigg)^{\frac{1}{2}}$$
$$\rho(x,y) := \sup\{|x_i - y_i|\}$$

However, the metrics do not in general map to elements of $\mathbb{R}$, since the sequence $(x_i - y_i)_{i \in \mathbb{N}}$ could diverge. Therefore, we can redefine the metric ρ to the following bounded one.

$$\tilde{\rho}(x,y) := \sup\{\tilde{d}(x_i, y_i)\} \tag{188}$$

where $\tilde{d}$ is the standard bounded metric on $\mathbb{R}$. Clearly,

$$0 \leq \tilde{\rho}(x,y) \leq 1 \tag{189}$$

$\tilde{\rho}$ is indeed a metric on $\mathbb{R}^\omega$, but unfortunately, it does not induce the product topology. We extend this definition to arbitrary $\mathbb{R}^J$.

Definition 8.3 (Uniform Metric)

Given an indexed set J with points $x, y \in \mathbb{R}^J$, we define

$$\tilde{\rho} := \sup\{\tilde{d}(x_\alpha, y_\alpha) \mid \alpha \in J\} \tag{190}$$

with $\tilde{d}$ the standard bounded metric on $\mathbb{R}$. $\tilde{\rho}$ is called the **uniform metric on** $\mathbb{R}^J$, which induces the **uniform topology**.

The uniform topology on $\mathbb{R}^J$ is finer than the product topology, and they are different if J is infinite. Clearly, $0 \leq \tilde{p}(x,y) \leq 1$, meaning that given the open ball

$$B_r(x) := \{y \in \mathbb{R}^J \mid \tilde{p}(y,x) < r\} \tag{191}$$

if $r \geq 1$, then $B_r(x) = \mathbb{R}^J$ and if $r < 1$, then $B_r(x)$ consists of the n-dimensional box with "radius" r, where $n = \dim \mathbb{R}^J$.

The next theorem gives us a metric that induces the product topology on infinite dimensional $\mathbb{R}^\omega$ by slightly modifying the uniform metric on $\mathbb{R}$. However, with the box topology $\mathbb{R}^\omega$ is not metrizable.

Theorem 8.19

Let $\tilde{d}(a,b) \coloneqq \min\{|a-b|, 1\}$ be the standard bounded metric on $\mathbb{R}$. If $x, y \in \mathbb{R}^\omega$, we define

$$D(x,y) \coloneqq \sup\left\{\frac{\tilde{d}(x_i, y_i)}{i}\right\} \tag{192}$$

Then, D is a metric that induces the product topology on $\mathbb{R}^\omega$.

It is easy to see that $0 \leq D(x,y) \leq 1$. So, given the open ball

$$B_r(x) \coloneqq \{y \in \mathbb{R}^\omega \mid D(x,y) < r\} \tag{193}$$

$B_r(x) = \mathbb{R}^\omega$ if $r > 1$. When $r \leq 1$,

$$B_r(x) \coloneqq (y-r, y+r) \times (y-2r, y+2r) \times \ldots = \prod_{k=1}^{\infty} (y-kr, y+kr) \tag{194}$$

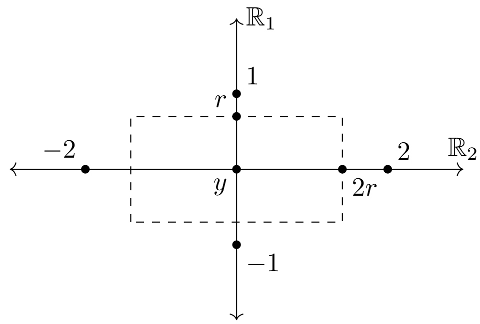


Figure 44: Visually, we take a cross section of this box and look at the slice within $\mathbb{R}_1 \times \mathbb{R}_2$, where the subscripts represent the first and second terms of x.

We can extend the applications of the Bolzano Weierstrass Lemma from analysis to metric spaces in general with the following lemma.

Lemma 8.20 (Sequence Lemma)

If X be a topological space with $A \subset X$. If there exists a sequence of points of A that converges to x, then $x \in \bar{A}$. The converse is true if X is metrizable.

Proof. ($\rightarrow$) Our hypothesis says that x is a limit point of A, which by definition means that $x \in \bar{A}$. ($\leftarrow$) Assuming X is metrizable and $x \in \bar{A}$, let d be a metric for the topology of X. Then, for every $n \in \mathbb{N}$, let us define a sequence of open neighborhoods of x to be

$$\big(B_{\frac{1}{n}}(x)\big) \tag{195}$$

Since $x \in \bar{A}$, there exists a point

$$x_n \in A \cap B_{\frac{1}{n}}(x) \text{ for all } n \in \mathbb{N} \tag{196}$$

This sequence (x_n) that we have proved must exist converges to x.

Theorem 8.21

Let $f : X \longrightarrow Y$ and let X be metrizable. f is continuous if and only if for every convergent sequence $(x_n) \to x$ of X, the following sequence of Y converges to $f(x)$. That is,

$$\big(f(x_n)\big) \longrightarrow f(x) \tag{197}$$

We introduce additional methods of constructing continuous functions.

Definition 8.4 (Uniform Convergence)

Let $f_n : X \longrightarrow Y$ be a sequence of functions from the set X to the metric space (Y, d). The sequence (f_n) is said to **converge uniformly** to the function $f : X \longrightarrow Y$ if, given $\epsilon > 0$, there exists a $N \in \mathbb{N}$ such that

$$d\big(f_n(x), f(x)\big) < \epsilon \tag{198}$$

for all $n \geq N$ and for all $x \in X$.

Theorem 8.22 (Uniform Limit Theorem)

Let $f_n : X \longrightarrow Y$ be a sequence of continuous functions from topological space X to a metric space Y. If f_n converges uniformly to f, then f is continuous.

Proof. $(\rightarrow)$ Trivial.
$(\leftarrow)$ Let V be open in Y, and let x_0 be a point in $f^{-1}(V)$. It suffices to prove that for every $x_0 \in f^{-1}(V)$, there exists a neighborhood U of x_0 such that $U \subset F^{-1}(V)$ or equivalently, $F(U) \subset V$.
Let $y_0 = f(x_0)$. Since Y is a metric space with metric d, we know that there exists an ϵ-ball $B_\epsilon(y_0)$ such that

$$B_\epsilon(y_0) \subset V \tag{199}$$

Then, using uniform convergence, we can choose $N \in \mathbb{N}$ such that for all $n \geq N$ and all $x \in X$,

$$d\big(f_n(x), f(x)\big) < \frac{\epsilon}{4} \tag{200}$$

which also applies at the point $x = x_0$.

$$d\big(f_n(x_0), f(x_0)\big) < \frac{\epsilon}{4} \tag{201}$$

Using continuity of f_n, choose a neighborhood U of x_0 such that f_n carries U into the open $\epsilon/2$-ball centered at $f_n(x_0)$ (note that $f_n(x_0) \neq y_0$), meaning that if $x \in U$

$$d\big(f_n(x), f_n(x_0)\big) < \frac{\epsilon}{2} \tag{202}$$

Adding the three inequalities and using the triangle inequality, we get the fact that if $x \in U$, then

$$d\big(f(x), f(x_0)\big) < \epsilon \tag{203}$$

meaning that the $f(U) \subset B_\epsilon(x_0) \subset V$.

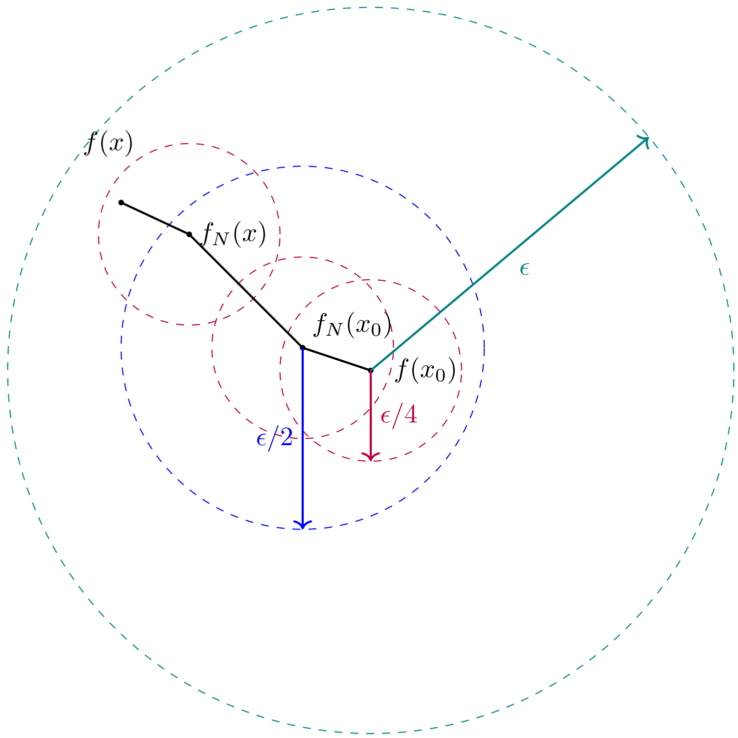


Figure 45: Visually, the three inequalities represent the following open balls in $V \subset Y$.

Theorem 8.23

In a metric space (X, d), a set is **closed** if the limit of every convergent subsequence in X lies in X. That is, X contains all of its limit points.

8.6 Metrization

It is easy to go from a metric to a topology, but a natural question is that given a topology, does there exist a metric that induces this topology? This is precisely the notion of *metrizability*, which is a highly desirable attribute for spaces, and there are many existence theorems that proves metrizability given certain conditions.

Definition 8.5 (Metrizable Space)

If $(X, \mathscr{T})$ is a topological space, $(X, \mathscr{T})$ is said to be **metrizable** if there exists a metric d on X that induces the topology $\mathscr{T}$ of X.

Example 8.6 (Non-Metrizable Finite Spaces)

Let $X = \{a, bc\}$. Then the topology

$$\mathscr{T} = \{\emptyset, \{b\}, \{a, b\}, \{b, c\}, X\} \tag{204}$$

is not metrizable from the theorem above since the only metrizable topologies are discrete.

Theorem 8.24 (Urysohn Metrization Theorem)

Every regular space X with a countable basis is metrizable.

Theorem 8.25 (Imbedding Theorem)

Let X be Hausdorff. Suppose that

$$\{f_\alpha\}_{\alpha \in J},\ f_\alpha : X \longrightarrow \mathbb{R} \tag{205}$$

is a collection of continuous functions satisfying the requirement that for each point $x_0 \in X$ and each neighborhood U of x_0, there is an index α such that f_α is positive at x_0 and vanishes outside U. Then, the function

$$F : X \longrightarrow \mathbb{R}^J, \quad F(x) \coloneqq \big(f_\alpha(x)\big)_{\alpha \in J} \tag{206}$$

is an **imbedding** of X in $\mathbb{R}^J$.

9 Homotopies

9.1 Homotopy

Intuitively, a homotopy means that we can interpolate between f_0 and f_1 with a family of continuous maps f_t varying continuously as a function of t. Sometimes, we slightly modify the definition a homotopy by restricting the family of continuous functions to agree on a certain subset A. We present both definitions below.

Definition 9.1 (Homotopy)

Let X and Y be topological spaces, with $f_0, f_1 : X \to Y$ continuous. Then, a **homotopy** between f_0 and f_1 is a continuous map

$$F : X \times [0,1] \to Y \tag{207}$$

such that $F(x,0) = f_0(x)$ and $f(x,1) = f_1(x)$ for all $x \in X$. In this case, we say f_0 and f_1 are **homotopic**, written $f_0 \simeq f_1$.

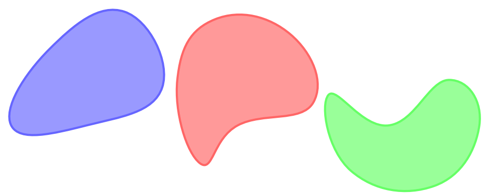


Figure 46: Think of the blobs as images of a certain function residing in Y. They are changing continuously.

Given spaces X, Y with $A \subset X, Y$ and $f_0, f_1 : X \to Y$ continuous maps such that $f_0|_A = f_1|_A$. A **homotopy relative to** A between f_0 and f_1 is a homtopy F with the additional property that

$$F(a,t) = f_0(a,t) = f_1(a,t) \tag{208}$$

for all $a \in A, t \in I$. This is an equivalence relation.

Proof. We prove that this is an equivalence relation.

1. *Reflexive.* Take $F(x,t) = f(x)$.
2. *Symmetric.* If $f_0 \simeq f_1$ via homotopy F, then $G(x,t) = F(x, 1-t)$ is a homotopy from f_1 to f_0.
3. *Transitive.* If $f_0 \simeq f_1$ via F and $f_1 \simeq f_2$ via G, then we define a new function

$$H(x,t) \coloneqq \begin{cases} F(x,2t) & \text{if } 0 \leq t \leq \frac{1}{2} \\ G(x, 2t-1) & \text{if } \frac{1}{2} \leq t \leq 1 \end{cases} \tag{209}$$

which is continuous by the pasting lemma[a] and so is a homotopy.

[a] The maps $t \mapsto F(x,2t)$ and $t \mapsto G(x, 2t-1)$ are continuous on the closed domains $X \times [0, \frac{1}{2}]$ and $X \times [\frac{1}{2}, 1]$ that coincide on $X \times \{\frac{1}{2}\}$.

Example 9.1 (Functions Mapping to Convex Spaces are Null-Homotopic)

If $Y = \mathbb{R}^n$, then for any space X, any two continuous maps $f_0, f_1 : X \to \mathbb{R}^n$ are homotopic by defining

$$F(x,t) = (1-t) f_0(x) + t f_1(x) \tag{210}$$

which is well-defined as Y is convex. Furthermore, it is path connected, and so the value of the constant function does not matter. Any two constant functions are homotopic.

Example 9.2 (Two Non-Homotopic Functions)

Consider $X = S^1, Y = \mathbb{R}^2 \setminus \{0\}$, with $f : S^1 \to Y$ the canonical inclusion, and g any constant map. Then the straight line homotopy no longer works, since the circle gets "stuck" on the hole. That gives some sort of rigorous definition of what a hole can be.

However, note that if two functions are null-homotopic, it *does not* mean that they are homotopic to each other.

Example 9.3 (Null-Homotopic Paths are Not Homotopic)

Given $X = \mathbb{R}, Y = (0, 1) \cup (2, 3)$, say that we are given the functions $f, g : X \to Y$ defined $f(x) = 0.5, f(y) = 2.5$. Then by definition they are null-homotopic, but they are not homotopic to each other.

This gives us a clue that if a space is path connected, then null-homotopic functions are homotopic to each other. We will elaborate on this later. The most fundamental way to construct new homotopies from old ones is by composing them with continuous functions. We can do this either before or after the homotopy.

Lemma 9.1 (Continuous Compositions of Homotopic Maps are Homotopic)

Let $f_0, f_1 : X \to Y$ be continuous maps with $f_0 \simeq_A f_1$.
1. If $g : Y \to Z$ is continuous, then $g \circ f_0 \simeq_A g \circ f_1$.
2. If $h : W \to X$ is continuous, then $f_0 \circ h \simeq_{h^{-1}(A)} f_1 \circ h$.

Proof. $f_0 \simeq f_1$ implies that there is a homotopy

$$F : X \times [0, 1] \to Y \tag{211}$$

s.t. $F(x, 0) = f_0(x), F(x, 1) = f_1(x)$.
1. Now we construct the map
$$G = g \circ F : X \times [0, 1] \to Z \tag{212}$$
Then, g continuous implies G is continuous, and we have
$$G(x, 0) = (g \circ F)(x, 0) = g(f_0(x)) = (g \circ f_0)(x) \tag{213}$$
$$G(x, 1) = (g \circ F)(x, 1) = g(f_1(x)) = (g \circ f_1)(x) \tag{214}$$
which is by definition a homotopy.
2. Given $h : W \to X$ continuous, then $h \times \mathrm{id} : W \times [0, 1] \to X \times [0, 1]$ is continuous under the product topology, and so we define the continuous map
$$H = F \circ (h \times \mathrm{id}) : W \times [0, 1] \to YH \tag{215}$$
which satisfies
$$H(x, 0) = (F \circ (h \times \mathrm{id}))(x, 0) = F(h(x), 0) = f_0(h(x)) = (f_0 \circ h)(x) \tag{216}$$
$$H(x, 1) = (F \circ (h \times \mathrm{id}))(x, 1) = F(h(x), 1) = f_1(h(x)) = (f_1 \circ h)(x) \tag{217}$$

Now we have seen that homotopies form an equivalence class. Now it turns out that under function compositions, homotopies are preserved, making homotopies a congruence class.

Theorem 9.2 (Homotopies is a Congruence Relation Under Composition)

Given $f_0, f_1 : X \to Y$ and $g_0, g_1 : Y \to Z$,

$$f_0 \simeq f_1, g_0 \simeq g_1 \implies g_0 \circ f_0 \simeq g_1 \circ f_1 \tag{218}$$

Proof. We have $F : [0,1] \times X \to Y$ and $G : [0,1] \times Y \to Z$. Then, we can simply define the component-wise composition of maps

$$G \circ (F \times \mathrm{id}) : [0,1] \times X \to Z, \qquad (G \circ (F \times \mathrm{id}))(t, x) \coloneqq G\big(t, F(t,x)\big) \tag{219}$$

which is continuous. Also, it is the case that

$$(G \circ F)(0, x) = G\big(0, F(0,x)\big) = G(0, f_0(x)) = g_0(f_0(x)) = (g_0 \circ f_0)(x) \tag{220}$$
$$(G \circ F)(1, x) = G\big(1, F(1,x)\big) = G(1, f_1(x)) = g_1(f_1(x)) = (g_1 \circ f_1)(x) \tag{221}$$

So this is indeed a homotopy between $g_0 \circ f_0 \simeq g_1 \circ f_1$.

9.2 Path Homotopies and the Fundamental Group

Now we limit our scope to closed paths, which will unlock a nice algebraic structure that we can analyze. This is the beginning of *algebraic topology*. Remember that a continuous map $f : [0,1] \to X$ with $f(0) = x_0, f(1) = x_1$ is called a *path*. A path from x to itself is called a *loop* at x.

Definition 9.2 (Path Homotopy)

Given two paths $f_0, f_1 : [0,1] \to X$, a **path homotopy** is a homotopy relative to $\{0,1\}$, i.e. a continuous map $F : [0,1] \times [0,1] \to X$ s.t.

$$f(s,0) = f_0(s) \qquad\qquad f(s,1) = f_1(s) \tag{222}$$
$$f(0,t) = x \qquad\qquad f(1,t) = x \tag{223}$$

for all s, t.[a]

[a] Note that the first line is the regular homotopy conditions, while the second line is the "relative to".

A corollary of the theorem on continuous compositions is that any path is path homotopic to any reparameterization of the same path.

Corollary 9.3 (Reparameterizations are Still Path Homotopic)

Suppose that $g : [0,1] \to [0,1]$ is a continuous map with $g(0) = 0, g(1) = 1$. Let $f : [0,1] \to X$ be a path, and $f' = f \circ g$. Then $f \simeq_p f'$.

Proof. We can take the homotopy

$$G : [0,1] \times [0,1] \to [0,1], \qquad G(s,t) \coloneqq (1-t)g(s) + ts \tag{224}$$

This is a homotopy between g and the identity map $[0,1]$ relative the endpoints since $G(s,0) = g(s)$ and $G(s,1) = s$. Now we compose this with f to get the new continuous map

$$F = f \circ G : [0,1] \times [0,1] \to X, \tag{225}$$

where

$$F(s,0) = f(G(s,0)) = f(g(s)) = (f \circ g)(s) \tag{226}$$
$$F(s,1) = f(G(s,1)) = f(s) = f(s) \tag{227}$$

and so $f \circ g \simeq_p f$.

The goal here is to try and develop some sort of algebraic structure on these paths. Therefore, let us define the following binary operation.

Definition 9.3 (Concatenation of Paths)

Given two paths f from x_0 to x_1, and g from x_1 to x_2, we can obtain a path from x_0 to x_2 by defining the **concatenation** of the paths as

$$(f * g)(s) \coloneqq \begin{cases} f(2s) & \text{if } 0 \leq s \leq \frac{1}{2} \\ g(2s-1) & \text{if } \frac{1}{2} \leq s \leq 1 \end{cases} \tag{228}$$

This is indeed a path since it is continuous by the pasting lemma.

It turns out that homotopies—which was originally an equivalence relation—is actually a *congruence relation* with respect to concatenation.

Lemma 9.4 (Path Homotopies is a Congruence Relation Under Concatenation)

if f_0, f_1 are paths from x_0 to x_1, and g_0, g_1 are paths from x_1 to x_2, then

$$f_0 \simeq_p f_1, g_0 \simeq_p g_1 \implies f_0 * g_0 \simeq_p f_1 * g_1 \tag{229}$$

Proof. Suppose F and G are path homotopies from f_0 to f_1 and g_0 to g_1. Then, we can define the "concatenation" of homotopies as the map

$$H(s,t) \coloneqq \begin{cases} F(2s,t) & \text{if } 0 \leq s \leq \frac{1}{2} \\ G(2s-1,t) & \text{if } \frac{1}{2} \leq s \leq 1 \end{cases} \tag{230}$$

Since they overlap on a closed set, then by the pasting lemma H is continuous. Furthermore, we can see that

$$H(s,0) = F(0,t) = f_0(s) = (f_0 * g_0)(s) \tag{231}$$
$$H(s,1) = G(1,t) = g_1(s) = (f_1 * g_1)(s) \tag{232}$$

Definition 9.4 (Fundamental Group)

We denote by $\pi_1(X, x_0)$ as the set of path homotopy classes of loops $[f]$ based at x_0. This, along with the binary operation.

$$[f] * [g] \coloneqq [f * g] \tag{233}$$

and inverses is simply $[f]^{-1} \coloneqq [f^{-1}]$ where $f^{-1}(s) = f(1-s)$.

Proof. We prove the group axioms.

1. *Closure.* This is proved since path homotopies is a congruence relation under concatenation.
2. *Associativity.* We prove something a bit more general. Let f, g, h be paths from x_0 to x_1 to x_2 to x_3, respectively. It is not the vase that $(f * g) * h = f * (g * h)$ since $\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{2}\right) \neq \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right)$. But they are reparamaterizations of each other, where φ is some monotonically increasing continuous function from $[0,1]$ to itself. Reparameterizations are path homotopic, and therefore
$$([f] * [g]) * [h] = [h] * ([g] * [h]) \tag{234}$$
3. *Identity.* We claim that the constant path at x_0 (or more formally, the set of all null-homotopic paths at x_0) is the identity element. Since $e_{x_0} : [0,1] \to X$ is defined $e_{x_0}(x) = x_0$, we have established that
$$e_{x_0} * f \simeq_p f, \qquad f * e_{x_0} \simeq_p f \tag{235}$$
4. *Inverses.* If f is a path from x_0 to x_1, then define $\bar{f} \coloneqq f(1-x)$, which is a path from x_1 to x_0. Note that $f * \bar{f}$ is not the constant path e_{x_0}, but it is homotopic since
$$(f * \bar{f})(s) \coloneqq \begin{cases} f(2s) = f(2s) & \text{if } 0 \leq s \leq \frac{1}{2} \\ \bar{f}(2s-1) = f(2-2s) & \text{if } \frac{1}{2} \leq s \leq 1 \end{cases} \tag{236}$$
But we use a sneaky trick and note that $p : [0,1] \to [0,1]$ defined as
$$p(s) \coloneqq \begin{cases} 2s & \text{if } 0 \leq s \leq \frac{1}{2} \\ 2-2s & \text{if } \frac{1}{2} \leq s \leq 1 \end{cases} \tag{237}$$
satisfies $p(0) = p(1) = 0$, which implies that p is homotopic to a constant path at 0, which implies that $f * p$ is also null-homotopic.

Note that the identity element of $\pi_1(X, x_0)$ is the *constant* map $x \mapsto x_0$, not the identity map! This is different from a group of transformations.

9.3 Homomorphisms of Fundamental Groups

Now that we have constructed the fundamental group at a point x_0, the next things to consider are their behavior under group homomorphisms, and how different fundamental groups at different points (of the same topological space) look like.

Definition 9.5 (Induced Group Homomorphism)

Suppose $k : X \to Y$ is a continuous map with $x_0 \in X, y_0 = f(x_0) \in Y$. Define the map, called the **induced group homomorphism** of k as
$$k_* : \pi_1(X, x_0) \to \pi_1(Y, y_0), \qquad k_*([f]) \coloneqq [k \circ f] \tag{238}$$
which is well defined since $f \simeq_p f'$ in $X \implies k \circ f \simeq_p k \circ f'$ in Y as a composition with continuous functions. This is indeed a group homomorphism.

Proof. Let's take a look at $f, g : [0,1] \to X$. Then
$$f * g : [0,1] \to X, \qquad (f * g)(t) \coloneqq \begin{cases} f(2t) & \text{if } 0 \leq t \leq \frac{1}{2} \\ g(2t-1) & \text{if } \frac{1}{2} \leq t \leq 1 \end{cases} \tag{239}$$

Therefore, mapping this through k_* gives us

$$k \circ (f * g) : [0,1] \to Y, \qquad (k \circ (f * g))(t) := \begin{cases} (k \circ f)(2t) & \text{if } 0 \leq t \leq \frac{1}{2} \\ (k \circ g)(2t-1) & \text{if } \frac{1}{2} \leq t \leq 1 \end{cases} \tag{240}$$

Now take $k \circ f, k \circ g : [0,1] \to Y$, and we can construct

$$(k \circ f) * (k \circ g) : [0,1] \to Y, \qquad \big((k \circ f) * (k \circ g)\big) = \begin{cases} (k \circ f)(2t) & \text{if } 0 \leq t \leq \frac{1}{2} \\ (k \circ g)(2t-1) & \text{if } \frac{1}{2} \leq t \leq 1 \end{cases} \tag{241}$$

which matches. Therefore we have $k \circ (f * g) = (k \circ f) * (k \circ g)$, and so under the congruence class

$$k_*([f] * [g]) = k_*([f * g]) = [k \circ f] * [k \circ g] = k_*([f]) * k_*([g]) \tag{242}$$

Lemma 9.5 (Identity Map Induces Identity Group Homomorphism)

If the continuous map between topological spaces is $k = \mathrm{id}_X$, then the induced homomorphism is simply the identity isomorphism.

$$(\mathrm{id}_X)_* = (\mathrm{id}_{\pi_1(X,x_0)}) \tag{243}$$

Proof. Just substituting with the formula above gives

$$k_*([f] * [g]) = [f] * [g] \tag{244}$$

and so k_* must be the identity.

Theorem 9.6 (Composition of Induced Group Homomorphisms)

If $k : X \to Y$ and $l : Y \to Z$ are continuous, then

$$(l \circ k)_* = l_* \circ k_* \tag{245}$$

Proof. We have

$$\begin{aligned} (l \circ k)_*([f]) &= [(l \circ k) \circ f] && (246) \\ &= [l \circ (k \circ f)] && (247) \\ &= l_*([k \circ f]) && (248) \\ &= l_*(k_*([f])) && (249) \\ &= (l_* \circ k_*)([f]) && (250) \end{aligned}$$

Corollary 9.7 (Homeomorphisms Induce Fundamental Group Isomorphisms)

If $X \cong Y$ by a homeomorphism taking x_0 to y_0, then $\pi_1(X, x_0) \cong \pi_1(Y, y_0)$.

Proof. If $k : X \to Y$ is a homeomorphism, then $k : Y \to X$ is also a homeomorphism, both continuous. Then this induces the group homomorphisms $k_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ and $k_*^{-1} : \pi_1(Y, y_0) \to \pi_1(X, x_0)$, and so from the lemma,

$$k^{-1} \circ k = \mathrm{id}_X \implies (k^{-1} \circ k)_* = \mathrm{id}_{\pi_1(X,x_0)} \tag{251}$$

and so $(k^{-1})_* \circ k_* = \mathrm{id}_{\pi_1(X,x_0)}$ and $k_* \circ (k^{-1})_* = \mathrm{Id}_{\pi_1(Y,y_0)}$.

Finally, let's talk about constructing new groups from old ones.

Theorem 9.8 (Direct Products of Fundamental Groups)

For any spaces X, Y, we have

$$\pi_1(X \times Y, x_0 \times y_0) \simeq \pi_1(X, x_0) \times \pi_1(Y, y_0) \tag{252}$$

Moreover, the inclusion and projection maps on the topological spaces induce inclusion and projection maps on the groups.

Proof. A loop $f : I \to X \times Y$ with $f(0) = f(1) = (x_0, y_0)$ is the same as a pair of maps $f_X : I \to X$ and $f_Y : I \to Y$. A homotopy between loops is a continuous map $F : I \times I \to X \times Y$, which is the sames as a pair of homotopies F_X and F_Y.

A fancy way to say this is that at every point, there is a group, and π_1 is a functor from the category of pointed topological spaces to the category of groups.

9.4 Simply Connected Spaces

Now the final thing to mention for now is that there is a group at every single point in a topological space. They may be completely different groups, but the following theorem nicely categorizes them in terms of path components.

Theorem 9.9 (Fundamental Groups of Points in a Path Component are Isomorphic)

If x_0 and x_1 are on the same path component of X, then $\pi_1(X, x_0) \simeq \pi_1(X, x_1)$. The actual isomorphism depends on a choice of a path from x_0 to x_1.

Proof. Let h be a path from x_0 to x_1, and let f be a loop around x_0. We are given a loop in x_0 and want to essentially find a loop around x_1. We can define

$$A_h : \pi_1(X, x_0) \to \pi_1(X_1, x_1), \qquad A_h([f]) \coloneqq [\bar{h} * f * h] \tag{253}$$

Essentially, we want to start at x_1, go to x_0, loop around, and then go back to x_1, which is in total a loop around x_1. We claim that A_h is an isomorphism by showing that it is a group homomorphism and that it has a well-defined inverse.

$$\begin{aligned} A_h([f] * [g]) &= [\bar{h} * f * g * h] && (254) \\ &= [\bar{h} * f] * [g * h] && (255) \\ &= [\bar{h} * f] * [h * \bar{h}] * [g * h] && (256) \\ &= [\bar{h} * f * h] * [\bar{h} * g * h] && (257) \\ &= A_h([f]) * A_h([g]) && (258) \end{aligned}$$

As for inverses, TBD.

Of course, when x_0 and x_1 are in different path components, they may or may not have isomorphic fundamental groups. Therefore, when we know a space X is path connected, we can just drop the point x_0 and just refer to the fundamental group as $\pi_1(X)$.

Therefore, if the space is path connected, the fundamental group at all points are isomorphic. But even if all the groups are isomorphic, they may be nontrivial. This motivates the following definition.

Definition 9.6 (Simply Connected Set)

X is called **simply connected** if it is path connected and $\pi_1(X, x_0) = \{e\}$.

Note the following.

$$\text{Simply Connected} \implies \text{Path Connected} \implies \text{Connected} \tag{259}$$

Example 9.4 (Simply Connected Spaces)

The following are simply connected.

1. $\mathbb{R}^n$
2. Any convex subset of $\mathbb{R}^n$.
3. Any star-shaped subset of $\mathbb{R}^n$, where a star-shaped set is defined as a set where there exists some point $x_0 \in A$ s.t. for all $y \in A$ the line segment from x_0 to y is in A.

Lemma 9.10

If X is simply connected, then any two paths f, g connecting x to y are path-homotopic.

Proof. Given two paths f, g, then $f * \bar{g}$ is defined and is a loop of X based at x. Since X is simply connected, this is homotopic to the constant loop based at x. Then,

$$[f] = [f * \bar{g}] * [g] = [e_x] * [g] = [g] \tag{260}$$

Theorem 9.11 (Union of Simply Connected Spaces)

Suppose $X = U \cup V$, where U, V are open and simply connected and $U \cap V$ is nonempty and path connected. Then X is simply connected.

Proof. Let $x_0 \in U \cap V$. Given a loop f, we can factor f as a composition $f = f_1 * \cdots * f_n$, where each f_i is a path contained either in U or in V, from some $x_{i-1} \in U \cap V$ to $x_i \in U \cap V$, and $x_n = x_0$. For $i = 1, \ldots, n-1$, choose a path g_i from x_0 to x_i in $U \cap V$.

$$f \simeq_p f_1 \overline{g}_1 g_1 f_2 \overline{g}_2 g_2 f_3 \ldots f_{n-1} \overline{g}_{n-1} g_{n-1} f_n \tag{261}$$
$$\simeq_p (f_1 \overline{g}_1)(g_1 f_2 \overline{g}_2)(g_2 f_3 \ldots f_{n-1} \overline{g}_{n-1})(g_{n-1} f_n) \tag{262}$$

Each of the factors is a loop in U or V based at x_0, so it can be contracted to e_{x_0}. Thus, $f \simeq e_{x_0}$, as required.

9.5 Covering Maps

Now it turns out that a lot of sets are simply connected. So we will use this definition to categorize the fundamental groups of familiar spaces. However, the proofs of these requires the knowledge of covering spaces and lifts.

Definition 9.7 (Covering Map)

A function $p : Y \to X$ is a **covering map** if p is
1. surjective,
2. continuous,
3. and for every $x \in X$ there exists an open $U \ni x$ s.t. $p^{-1}(U) = \cup_\alpha V_\alpha$, where V_α are open and disjoint, and $p|_{V_\alpha} : V_\alpha \to U$ is a homeomorphism.

The sets $\{V_\alpha\}$ is a partition into **slices**, and Y is said to be a **covering space** of X.

You can imagine a covering map as like a stack of pancakes V_α that get all squashed down to U. Now let's introduce a few properties of covering maps.

Lemma 9.12 (Preimage of Singletons Have Discrete Topology)

If $p : Y \to X$ is a covering map, then for each $x \in X$, the subspace $p^{-1}(x) \subset X$ has the discrete topology.

Proof. Take some $x \in X$ and look at its preimage $p^{-1}(x)$. Then, we can imagine it as some disjoint union of discrete points, which has the discrete topology. More formally, take some open $U \ni x$ and then its preimage will be the disjoint union of open sets $\sqcup_\alpha V_\alpha$. But each point $y \in p^{-1}(x)$ will be contained in exactly one of the U_α's, and so $y = y \cap U_\alpha$ is open in itself, meaning that all the $\{y_\alpha\}$ will have the discrete topology.

Theorem 9.13 (Covering Maps are Open Maps)

If $p : Y \to X$ is a covering map, then it is open, i.e. maps open sets to open sets.

Proof. Suppose $A \subset Y$ is open. We wish to show that $p(A)$ is open. Choose an arbitrary $x \in p(A)$. By definition of covering maps, there must exist an open $U \ni x$ (but U not necessarily in $p(A)$) s.t. $p^{-1}(U) = \sqcup_\alpha V_\alpha$ for V_α open. There is a point $y \in U$ s.t. $p(y) = x$. Let U_β be the open slice containing y. Then, since A is open by assumption, the set $V_\beta \cap A$ is open in Y and hence open in V_β. Since p maps V_β homeomorphically onto U, it maps open sets to open sets and so $p(V_\beta \cap A)$ is open in U and hence open in X. Furthermore, it is a neighborhood of x contained in $p(A)$. Since for every $x \in p(A)$ has an open neighborhood contained in $p(A)$, $p(A)$ is thus open.

Corollary 9.14 (Covering Maps are Quotient Maps)

A covering map $p : Y \to X$ is a quotient map.

Proof. This is immediate because open maps are quotient maps.[a]

[a]However, open doesn't mean closed!

Example 9.5 (Reals to the Circle)

The following is a covering map.

$$p : \mathbb{R} \to S^1, \qquad p(t) = (\sin 2\pi t, \cos 2\pi t) \tag{263}$$

It is clearly surjective and continuous. Now given any $x \in S^1$, let's take the open neighborhood $U \ni x$. Then, its preimage is the union of disjoint intervals V_n for $n \in \mathbb{N}$, and if we restrict it to each V_n, then it is a homeomorphism.

Example 9.6 (Positive Reals to the Circle)

The map

$$p : \mathbb{R}^+ \to S^1, \qquad p(x) \coloneqq (\cos 2\pi x, \sin 2\pi x) \tag{264}$$

is not a covering map. It is surjective and continuous but the point $x_0 = (1, 0) \in S^1$ has no neighborhood U that is evenly covered by p. We can see that the inverse image of $(0, \epsilon)$ is not homeomorphic to an open U.

These examples might lead you to think that $\mathbb{R}$ is the only connected covering space of S^1. It turns out that a space can be a covering space of itself.

Example 9.7 (Covering Map of S^1 to Itself)

We can consider $S^1 \subset \mathbb{C}$ and claim that the following is a covering map.

$$p : S^1 \to S^1, \qquad p(z) = z^2 \tag{265}$$

Theorem 9.15 (Restriction of Covering Maps)

Let $p : Y \to X$ be a covering map. If $Y_0 \subset Y$ and $X_0 = p^{-1}(Y_0)$, then the restriction $p_0 : Y_0 \to X_0$ is a covering map.

Proof. Given $x_0 \in X_0$, let $U \ni x_0$ be an open set. Then let $\{V_\alpha\}$ be a partition of $p^{-1}(U)$ into slices. We know that $U \cap X_0$ is an open neighborhood of x_0 in X_0, and taking the preimage we have

$$p^{-1}(U \cap X_0) = \sqcup_\alpha V_\alpha \cap p^{-1}(X_0) = \sqcup_\alpha V_\alpha \cap Y_0 \tag{266}$$

The $V_\alpha \cap Y_0$ are disjoint, and V_α open in Y implies that $V_\alpha \cap Y_0$ is open in Y_0. Finally, we can see that p maps V_α homeomorphically onto U, and so p maps $V_\alpha \cap Y_0$ homeomorphically onto $p(V_\alpha) \cap p(Y_0) = U \cap X_0$.

Theorem 9.16 (Product of Covering Maps)

If $p : Y \to X$ and $p' : Y' \to X'$ are covering maps, then

$$p \times p' : X \times X' \to Y \times Y' \tag{267}$$

is a covering map.

Proof. Given $x \in X, x' \in X'$, let $U \ni x, U \ni x'$ be open neighborhoods of x and x' with $\{V_\alpha\}, \{V'_\alpha\}$ the even slices of $p^{-1}, (p')^{-1}$. Then the preimage

$$(p \times p')^{-1}(U \times U') = \bigcup_{\alpha,\beta} V_\alpha \times V'_\beta \tag{268}$$

These are disjoint open sets of $X \times X'$, each of which is mapped homeomorphically onto $U \times U'$ by $p \times p'$.

Theorem 9.17 (Composition of Covering Maps)

Let $q : X \to Y$ and $r : Y \to Z$ be covering maps. If $r^{-1}(z)$ is finite for all $z \in Z$, then $p = r \circ q : X \to Z$ is a covering map.

Proof.

What we should keep in mind is that locally, covering maps look like homeomorphisms.

Example 9.8 (Projection of Products with Discrete Topology)

Let Y have the discrete topology, and $p : X \times Y \to X$ is a projection on the first coordinate. Then we claim that p is a covering map. Note that the open sets are of the form

$$X \times \{y\} \text{ for } y \in Y \tag{269}$$

Example 9.9

Say $k \in \mathbb{Z} \setminus \{0\}$ and $f_k : S^1 \to S^1$ defined

$$f_k(\cos\theta, \sin\theta) := (\cos k\theta, \sin k\theta) \tag{270}$$

This circularly stretches out the circle by a factor of k. We claim that this is a covering map.

Theorem 9.18 (Covering Maps as Homeomorphisms)

Let $p : Y \to X$ be a covering map with X simply connected and Y path connected. Then p is a homeomorphism.

9.6 Lifts

There is a nice relationship between the fundamental group and the covering space.

Definition 9.8 (Lift)

If $p : Y \to X$ is a covering map and $f : Z \to X$ is continuous, then a **lift** of f is a continuous map $\tilde{f} : Z \to Y$ satisfying $p \circ \tilde{f} = f$.[a]

$$\begin{tikzcd} & Y \arrow[d, "p"] \\ Z \arrow[ur, "\tilde{f}"] \arrow[r, "f"] & X \end{tikzcd}$$

Figure 47: Commutative diagram of a lift.

[a] It is called a lift since it "lifts" the function with codomain Z to the larger covering space Y.

Note that not every f has a lift. However, it turns out that the existence of liftings when p is a covering map is an important tool for studying covering spaces and the fundamental group. The path lifting property guarantees that paths can be lifted into a covering space.

Lemma 9.19 (Path Lifting Property)

Suppose $p : Y \to X$ is a covering map with $p(y_0) = x_0$. Given any path $f : [0,1] \to X$ with $f(0) = x_0$, there exists a unique lift $\tilde{f} : [0,1] \to Y$ with $\tilde{f}(0) = y_0$.

Therefore, this essentially means that by taking the covering map p that sends y_0 to x_0 and a path f starting at x_0, we can construct a corresponding path $\tilde{f}$ starting at y_0 such that $f = p \circ \tilde{f}$. Let's do an example.

Example 9.10 (Lift of Real Interval to S^1)

Consider the covering map and path defined as below.

$$p : \mathbb{R} \to S^1, \qquad p(t) = (\cos 2\pi t, \sin 2\pi t) \tag{271}$$

We can see that $p(y_0) = p(0) = (1,0) = x_0$. Now let's consider a few paths $f : [0,1] \to S^1$. We would like to "lift" this path $I \to S^1$ to a path $I \to \mathbb{R}$. The lemma states that there is indeed a unique path.

1. When $f(s) = (\cos \pi s, \sin \pi s)$ which satisfies $f(0) = x_0 = (1,0)$, the lift is $\tilde{f}(t) = t/2$ which satisfies $\tilde{f}(0) = y_0 = 0$.
$$t \xrightarrow{\tilde{f}} \frac{t}{2} \xrightarrow{p} \big(\cos 2\pi \frac{t}{2}, \sin 2\pi \frac{t}{2}\big) = (\cos \pi t, \sin \pi t) = f(t) \tag{272}$$
2. When $f(s) = (\cos \pi s, -\sin \pi s)$ which satisfies $f(0) = x_0 = (1,0)$, the lift is $\tilde{f}(t) = -t/2$ which satisfies $\tilde{f}(0) = y_0 = 0$.
$$t \xrightarrow{\tilde{f}} -\frac{t}{2} \xrightarrow{p} \big(\cos -2\pi \frac{t}{2}, \sin -2\pi \frac{t}{2}\big) = (\cos \pi t, \sin -\pi t) = f(t) \tag{273}$$
since $\cos(t) = \cos(-t)$.

$$\begin{tikzcd} & \mathbb{R} \arrow[d, "p"] \\ {[a,b]} \arrow[ur, "\tilde{f}"] \arrow[r, "f"] & S^1 \end{tikzcd}$$

Figure 48

Then, we show that path homotopies can be lifted as well.

Lemma 9.20 (Homotopy Lifting Property)

Suppose $p : Y \to X$ is a covering map. If $f, g : I \to X$ are paths with $f(0) = g(0), f(1) = g(1)$, and $f \simeq_p g$ (path homotopic), and $\tilde{f}, \tilde{g} : I \to Y$ are lifts starting at the same y_0, then $\tilde{f} \simeq_p \tilde{g}$.

Great, so we have defined covering maps and have shown that paths admit lifts that are consistent with the path homotopy relation. Now, to connect this back to fundamental groups, we introduce the following theorem which allows us to determine the cardinality of fundamental groups through covering maps over simply connected spaces. This will be very useful to us later on.

Theorem 9.21 (Bijection from Fundamental Group to Preimage of Covering Map on Simply Connected Space)

If $p : Y \to X$ is a covering map and Y is simply connected, then $\pi_1(X, x_0)$ is in bijection with $p^{-1}(x_0)$.

Proof. Given a loop $f : [0,1] \to X$ with $f(0) = f(1) = x_0$, fix $y_0 \in p^{-1}(x_0)$. By the path lifting property, we can take the lifted path $\tilde{f} : [0,1] \to Y$ with $\tilde{f}(0) = y_0$.

$$p(\tilde{f}(1)) = f(1) = y_0, \qquad \tilde{f}(1) \in p^{-1}(x_0) \tag{274}$$

By the homotopy lifting property, this only depends on $[f] \in \pi_1(X, x_0)$.

1. To prove surjectivity, since Y is simply connected, this implies Y is path connected. Choose a path g from y_0 to y. Let $f = p \circ g$. Then $L([f]) = g(1) = y$.
2. To prove injectivity, if $L([f]) = L([g])$, then $\tilde{f}$ and $\tilde{g}$ are both paths in Y from y_0 to y_1. Because Y is simply connected, $\tilde{f} \simeq_p \tilde{g} \implies f \simeq_p g \implies [f] = [g]$.

Example 9.11

Take $p : \mathbb{R}^2 \to S^1 \times S^1$ defined

$$(s,t) \mapsto \big((\cos 2\pi s, \sin 2\pi s), (\cos 2\pi t, \sin 2\pi t)\big) \tag{275}$$

Think back to the quotient space representation.

$$T^2 = [0,1]^2/\sim, \qquad (x,0) \sim (x,1) \forall x, (0,y) \sim (1,y) \forall y \tag{276}$$

Then we can think of $p(x,y) = (x \pmod 1, y \pmod 1)$. We claim that this is a covering map.

9.7 Fundamental Groups of Euclidean Space and the Sphere

Now that we have established that $\mathbb{R}^n$ is simply connected, along with some tools on sufficient conditions for simply connectedness and lifts, we can start to categorize fundamental groups of $\mathbb{R}^n$ and S^n, with the corresponding sets minus a point.

Theorem 9.22 (Fundamental Groups of S^n)

We have

1. For $n = 1$, S^1 is path connected but not simply connected.

$$\pi_1(S^1) \simeq \mathbb{Z} \tag{277}$$

2. For $n \geq 2$, S^n is simply connected, and so

$$\pi_1(S^n) \simeq \{e\} \tag{278}$$

Proof. Consider the map $p : \mathbb{R} \to S^1$ given by $t \mapsto (\cos 2\pi t, \sin 2\pi t)$.[a] For each $x \in S^1$, there is an open set U such that $p^{-1}(U)$ is a disjoint union of open sets V_i, where $p|_{V_i}$ is a homeomorphism. Let $x_0 = (1,0)$. Then $p^{-1}(x_0) = \mathbb{Z}$. Then here are the key technical steps.

1. For any continuous path $f : [0,1] \to S^1$ with $f(0) = x_0$, and any choice of $\tilde{x_0} \in \mathbb{R}$ with $p(\tilde{x_0}) = x_0$ (i.e. some integer), there is a unique continuous function $\tilde{f} : [0,1] \to \mathbb{R}$ with $f(0) = \tilde{x_0}$.
2. $\tilde{f}(1) - \tilde{f}(0)$ measures the "net winding" of f. $1/2\pi$ times the net increase in the angle coordinate. In particular, if f is a loop, then $\tilde{f}(1) - \tilde{f}(0)$ is an integer, which we'll call the *winding number* of f, denoted $\omega(f)$. This doesn't depend on which choice of $\tilde{f}$ we made.
3. Given two loops f, g based at x_0, let $\tilde{f}, \tilde{g}$ by their lifts, with say $\tilde{f}(0) = \tilde{g}(0) = 0$. Then $f \simeq_p g$ iff $\tilde{f} \simeq p\tilde{g}$. One direction is easy. If F is a path homotopy between $\tilde{f}$ and $\tilde{g}$, then $p \circ F$ is a path homotopy between f and g. The other direction requires a somewhat elaborate argument using compactness.

4. As a consequence, we see that if $f \simeq_p g$, then $\tilde{f}(1) = \tilde{g}(1)$, and hence $\omega(f) = \omega(g)$.
5. Conversely, if $\omega(f) = \omega(g) = n$, then $\tilde{f}$ and $\tilde{g}$ are both paths in $\mathbb{R}$ from 0 to n. There is a straight line homotopy between them. $F(s,t) = (1-t)\tilde{f}(s) + t\tilde{g}(s)$. So $f \simeq_p g$.
6. We then have $\omega(f * g) = \omega(f) + \omega(g)$.
7. Thus, the winding number gives us an isomorphism from $\pi_1(S^1) \to \mathbb{Z}$.

By using the function $p : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}^2 \setminus \{0\}$, given by $p(r,t) = (r\cos 2\pi t, r \sin 2\pi t)$, we can use the same logic. Another way to see it is that $\mathbb{R}^2 \setminus \{0\} \cong S^1 \times (0, \infty) \cong S^1 \times \mathbb{R}$.

[a] This is a classic example of what's called a covering map, which we will define later.

Now we explore the fundamental groups of S^n minus a point. This becomes very simple once we know the stereographic projection.

Theorem 9.23 (Stereographic Projection)

We claim

$$S^n \setminus \{n\} \cong \mathbb{R}^n \tag{279}$$

Proof. We can construct a formula, but visually, we can take any $x \in S^n \setminus \{n\}$, and there is a unique line L through x and n. Now let $f(x) \coloneqq L_x \cap (\mathbb{R}^{n-1} \times \{0\})$, which is bijective. It also turns out to be a homeomorphism.

Corollary 9.24 (Fundamental Groups of S^n Minus a Point)

We claim that $S^n \setminus \{p\}$ is simply connected, for all $n \in \mathbb{N}$.

$$\pi_1(S^n) \simeq \{e\} \tag{280}$$

So informally, S^n minus a point gives us $\mathbb{R}^n$, so what if we remove a point from $\mathbb{R}^n$? It turns out that by taking n points away from S^n, it is equal to taking $n-1$ points away from $\mathbb{R}^n$.

Theorem 9.25 (Fundamental Groups of $\mathbb{R}^n$ Minus a Point)

We claim that

1. For $n = 1$, $\mathbb{R} \setminus \{0\}$ is not even path connected. For any x_0, we have
$$\pi_1(\mathbb{R} \setminus \{0\}) \simeq \{e\} \tag{281}$$
2. For $n = 2$, $\mathbb{R}^2 \setminus \{0\}$ is path connected, but it is not simply connected.[a]
$$\pi_1(\mathbb{R}^2 \setminus \{x\}) \simeq \mathbb{Z} \tag{282}$$
3. For $n \geq 3$, $\mathbb{R}^n \setminus \{x\}$ is simply connected.[b]

[a] We can think of this as a hole in the middle of the plane that paths cannot unwind through.
[b] We can interpret as if the extra dimensions being added gives more space for a path to unwind.

Proof. For $n = 1$, we can see that $(-\infty, 0)$ and $(0, +\infty)$ are convex subsets, so they are simply connected. For $n = 2$, we can see in polar coordinates,

$$\mathbb{R}^2 \setminus \{0\} \simeq S^1 \times (0, +\infty) \implies \pi_1(\mathbb{R}^2 \setminus \{0\}) \simeq \pi_1(S^1) \times \pi_1((0, +\infty)) \simeq \mathbb{Z} \times \{e\} \simeq \mathbb{Z} \tag{283}$$

For $n = 3$, we can construct it from smaller simply connected spaces. Take

$$U = \mathbb{R}^n \setminus \{(0, \ldots, 0, z) \mid z \geq 0\} \tag{284}$$
$$V = \mathbb{R}^n \setminus \{(0, \ldots, 0, z) \mid z \leq 0\} \tag{285}$$

These are each open (since the axes that we minus out is closed) and simply connected (since they are star convex). Also, $U \cap V = (\mathbb{R}^{n-1} \setminus \{0\}) \times \mathbb{R}$, which is path connected since we assumed that $n \geq 3$. Therefore, by the theorem above, $\mathbb{R}^n \setminus \{0\}$ is simply connected.

Theorem 9.26 (Fundamental Groups of $\mathbb{RP}^n$)

We claim that.

1. For $n = 1$, $\mathbb{RP}^1 \cong S^1$ and so $\pi_1(\mathbb{RP}) \simeq \mathbb{Z}$.
2. For $n \geq 2$, $\pi_1(\mathbb{RP}^n) \simeq \mathbb{Z}_2$

Proof. For $n = 1$, recall that S^1 is homeomorphic to $S^1/\sim$ where $x \sim -x$ through the homeomorhism $z \mapsto z^2$.
For $n \geq 2$, we claim that the projection map $p : S^n \to \mathbb{RP}^n$ defined $-\pm x \to x$ is a covering map. Furthermore, we know that S^n is simply connected (since it's homeomorphic to $\mathbb{R}^n \setminus \{0\}$), and therefore there exists a bijection from $\pi(\mathbb{RP}^n, x_0)$ with $p^{-1}(x_0) = \{\pm x_0\}$. Therefore the order of the fundamental group must be 2, and it can only be $\mathbb{Z}_2$ for all x_0.

At this point it's pretty easy to show the fundamental groups of familiar quotient spaces.

Example 9.12 (Torus)

The torus $T \cong S^1 \times S^1$ is path connected has the fundamental group

$$\pi_1(T) \simeq \pi_1(S^1) \times \pi_1(S^1) \simeq \mathbb{Z} \times \mathbb{Z} \tag{286}$$

So it is not simply connected.

Example 9.13 (Cylinder)

The cylinder is path connected and the fundamental group is

$$\pi_1(S^1 \times [0, 1]) \simeq \pi_1(S^1) \times \pi_1([0, 1]) \simeq \mathbb{Z} \times \{e\} \simeq \mathbb{Z} \tag{287}$$

Example 9.14 (Mobius Strip)

The Mobius strip is not simply connected.

Example 9.15 (Klein Bottle)

The Klein bottle is not simply connected and it is not abelian.

9.8 Retractions

Definition 9.9 (Retraction)

If $A \subset X$, a retraction $r : X \to A$ is a continuous function such that $r(a) = a$ for all $a \in A$.

Example 9.16

The following are retractions.

$$r : \mathbb{R} \to \{x\} \subset \mathbb{R}, \qquad r(x) = x \tag{288}$$

This is also a retraction.

$$r : \mathbb{R} \setminus \{0\} \to S^{n-1}, \qquad x \mapsto \frac{x}{\|x\|} \tag{289}$$

Lemma 9.27 (Group Homomorphism Induced By Retraction)

Given $A \subset X$, a retraction $r : X \to A$, and the canonical injection $\iota : A \to X$,

1. The induced homomorphism $r_* : \pi_1(X, a_0) \to \pi_1(A, a_0)$ is surjective.
2. The induced homomorphism $\iota_* : \pi_1(A, a_0) \to \pi_1(X, a_0)$ is injective.

Proof.

It's pretty easy to show whether a particular example is a retraction or not. However, we will mainly be interested in the existence of retractions.

Theorem 9.28 (Unit Interval to Endpoints)

There is no retraction from $D^1 = [-1, 1]$ to $S^0 = \{-1, 1\}$.

Proof. Let $X = [a, b]$. There is no retraction $f : X \to \{a, b\}$ since $\mathrm{Im}(f)$ must be connected. But $f(a) = a, f(b) = b$. Alternatively, we can use IVT.

Theorem 9.29 (Unit Disk to Circular Boundary)

There is no retraction from $D^2 \to S^1$. Intuitively, we can see that the mapping has to "tear" the disk somewhere!

Proof.

Theorem 9.30

Let $A \subset X$, and $x_0 \in A$, and $r : X \to A$ be a retraction. Then the induced map $r_* : \pi_1(X, x_0) \to \pi_1(A, x_0)$ is surjective.

Proof. Given retraction $r : X \to A$ and the canonical injection $\iota : A \to X$, we have $r \circ \iota = \mathrm{id}_A$ by definition. Now see how it affects the fundamental spaces.

$$\pi_1(A, x_0) \xrightarrow{\iota_*} \pi_1(X, x_0) \xrightarrow{r_*} \pi_1(A, x_0) \tag{290}$$

Therefore $r_* \circ \iota_* = \mathrm{id}_A$. So for any loop $[f] \in \pi_1(A)$, $[f] = r_*(i_*([f]))$, so $[f] \in \mathrm{Im}(r_*)$. Also ι_* is injective.

Corollary 9.31

If X is simply connected and A is not, then there is no retraction.

Extending this a bit, remember that any continuous function $f : [a, b] \to [a, b]$ must have a fixed point by IVT. Thinking $D^1 = [-1, 1]$, we claim the following. Note that a rotation of the disk D^2 has a fixed point in the center.

Theorem 9.32 (Brower's Fixed Point Theorem)

Any continuous function $f : D^2 \to D^2$ has a fixed point, i.e. there exists $x \in D^2$ s.t. $f(x) = x$.[a]

[a] This was first proven in the 1930s based on a non-intersection theorem . But it is non-constructive, i.e. doesn't show you how to find the point.

Proof. Suppose $f : D^2 \to D^2$ has no fixed point. Define a function $g : D^2 \to D^2$ as follows. For x, take $f(x) \neq x$ and draw a ray from $f(x)$ to x (directions matter!). Where it hits the boundary is the output $g(x)$. It is much more tedious to write out in formulas, but if you do, you can see that it's just the composition of continuous functions and is so continuous. If $x \in S^1$, then $g(x) = x$, so I've just constructed a retraction from D^2 to S^1, which is a contradiction!

So what about higher dimensions? If we believe that there is no retraction $D^n \to S^{n-1}$, by the same method we can prove for $D^n \to D^n$. The problem is that for $n \geq 2$ S^{n-1} is simply connected and so $\pi_1(S^{n-1}) = \{e\}$, and same holds for D^n. So we can't prove the non-retraction using fundamental groups π_1. This is where we need higher order homotopy groups.

10 Exercises

10.1 Open, Closed Sets

Exercise 10.1 (Munkres 13.1)

Let X be a topological space; let A be a subset of X. Suppose that for each $x \in A$ there is an open set U containing x such that $U \subset A$. Show that A is open in X.

Solution. Given a $x \in A$, let us label its open neighborhood as $U_x \subset A$. We claim that

$$A = \bigcup_{x \in A} U_x \tag{291}$$

We prove bidirectionally.

1. $A \subset \cup_{x \in A} U_x$. Let $y \in A$. Then there exists an open U_y containing y. Since U_y is in the union by construction.
$$y \in U_y \subset \bigcup_{x \in A} U_x \tag{292}$$
2. $\cup_{x \in A} U_x \subset A$. Let $y \in \cup_{x \in A} U_x$. Then there must be some U_y in this union s.t. $y \in U_y$. But by construction $U_y \subset A$, so $y \in A$.

We are done.

Exercise 10.2 (Munkres 13.2)

Consider the nine topologies on the set $X = \{a, b, c\}$ indicated in Example 1 of §12. Compare them; that is, for each pair of topologies, determine whether they are comparable, and if so, which is the finer.

Solution. Given the figure, we denote $\mathscr{T}_{i,j}$ as the topology in the ith row (from top) and jth column (from left) in the figure below. When I say for all i, j, I mean for all $i, j \in \{1, 2, 3\}$.

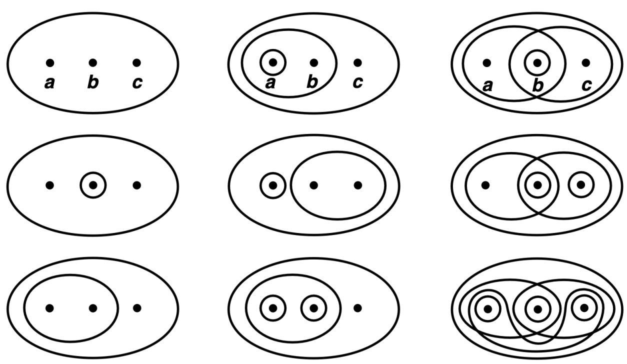


Figure 49

$\mathscr{T}_{11}$ is the indiscrete topology so $\mathscr{T}_{11} \subset \mathscr{T}_{i,j}$ for all i, j, i.e. it is coarsest. $\mathscr{T}_{33}$ is the discrete topology so $\mathscr{T}_{i,j} \subset \mathscr{T}_{3,3}$ for all i, j, i.e. it is the finest. We list all other comparable topologies below.

1. $\mathscr{T}_{1,2} \subset \mathscr{T}_{3,2}, \mathscr{T}_{1,3}, \mathscr{T}_{2,3}$.
2. $\mathscr{T}_{3,1} \subset \mathscr{T}_{1,2}, \mathscr{T}_{3,2}, \mathscr{T}_{1,3}, \mathscr{T}_{2,3}$.
3. $\mathscr{T}_{1,2} \subset \mathscr{T}_{3,2}$.
4. $\mathscr{T}_{1,3} \subset \mathscr{T}_{2,3}$.

Exercise 10.3 (Munkres 13.3)

Show that the collection $\mathcal{T}_c$ given in Example 4 of §12 is a topology on the set X. Is the collection

$$\mathcal{T}_\infty = \{U \mid X - U \text{ is infinite or empty or all of } X\}$$

a topology on X?

Solution. We denote $\mathcal{T}_c$ as the set of all subsets $U \subset X$ such that $X \setminus U$ is countable is all of X. We show the 4 properties:

1. $U = \emptyset \implies X \setminus U = X$, which is by definition in $\mathcal{T}_c$.
2. $U = X \implies X \setminus U = \emptyset$, which has cardinality 0. Therefore it is countable and is in $\mathcal{T}_c$.
3. Let $\{U_\alpha\}_{\alpha \in I} \in \mathcal{T}_c$ by a collection of open sets of X. Then

$$X \setminus \bigcup_{\alpha \in I} U_\alpha = \bigcap_{\alpha \in I} (X \setminus U_\alpha) \tag{293}$$

   $X \setminus U_\alpha$ is countable for all $\alpha \in I$, so let us fix some α'. Then

$$\bigcap_{\alpha \in I} (X \setminus U_\alpha) \subset U_{\alpha'} \implies \left| \bigcap_{\alpha \in I} (X \setminus U_\alpha) \right| \leq |U_{\alpha'}| \tag{294}$$

   and so the intersection is also countable.
4. Let $\{U_i\}_{i=1}^n$ by a finite collection of open sets of X. Then

$$X \setminus \bigcap_{i=1}^n U_i = \bigcup_{i=1}^n (X \setminus U_i) \tag{295}$$

   Since U_i are open, $X \setminus U_i$ are countable, and since the finite union of countable sets are countable, the RHS is countable, which implies the LHS is countable and so $\cap_{i=1}^n U_i$ is open as well.

As for $\mathcal{T}_\infty$, it is not a topology. Let us take $X = \mathbb{R}$, and look at the sets $\mathbb{Z}_{\geq 0}, \mathbb{Z}_{\leq 0}$ consisting of all the non-negative and non-positive integers. They are both infinite, and so $\mathbb{R} \setminus \mathbb{Z}_{\geq 0}$ and $\mathbb{R} \setminus \mathbb{Z}_{\leq 0}$ are in $\mathcal{T}_\infty$. Consider their union.

$$(\mathbb{R} \setminus \mathbb{Z}_{\geq 0}) \cup (\mathbb{R} \setminus \mathbb{Z}_{\leq 0}) = \mathbb{R} \setminus (\mathbb{Z}_{\geq 0} \cap \mathbb{Z}_{\leq 0}) = \mathbb{R} \setminus \{0\} \tag{296}$$

But $\mathbb{R} \setminus (\mathbb{R} \setminus \{0\}) = \{0\}$, and so $\mathbb{R} \setminus \{0\}$ is not open. Therefore $\mathcal{T}_c$ doesn't satisfy the definition of a topology.

Exercise 10.4 (Munkres 13.4)

(a) If $\{\mathcal{T}_\alpha\}$ is a family of topologies on X, show that $\bigcap \mathcal{T}_\alpha$ is a topology on X. Is $\bigcup \mathcal{T}_\alpha$ a topology on X?
(b) Let $\{\mathcal{T}_\alpha\}$ be a family of topologies on X. Show that there is a unique smallest topology on X containing all the collections $\mathcal{T}_\alpha$, and a unique largest topology contained in all $\mathcal{T}_\alpha$.
(c) If $X = \{a, b, c\}$, let

$$\mathcal{T}_1 = \{\emptyset, X, \{a\}, \{a, b\}\} \quad \text{and} \quad \mathcal{T}_2 = \{\emptyset, X, \{a\}, \{b, c\}\}.$$

Find the smallest topology containing $\mathcal{T}_1$ and $\mathcal{T}_2$, and the largest topology contained in $\mathcal{T}_1$ and $\mathcal{T}_2$.

Solution.

Exercise 10.5 (Munkres 13.5)

Show that if $\mathcal{A}$ is a basis for a topology on X, then the topology generated by $\mathcal{A}$ equals the intersection of all topologies on X that contain $\mathcal{A}$. Prove the same if $\mathcal{A}$ is a subbasis.

Solution.

Exercise 10.6 (Munkres 17.1)

Let $\mathcal{C}$ be a collection of subsets of the set X. Suppose that $\emptyset$ and X are in $\mathcal{C}$, and that finite unions and arbitrary intersections of elements of $\mathcal{C}$ are in $\mathcal{C}$. Show that the collection

$$\mathcal{T} = \{X - C \mid C \in \mathcal{C}\}$$

is a topology on X.

Solution.

Exercise 10.7 (Munkres 17.2)

Show that if A is closed in Y and Y is closed in X, then A is closed in X.

Solution.

Exercise 10.8 (Munkres 17.3)

Show that if A is closed in X and B is closed in Y, then $A \times B$ is closed in $X \times Y$.

Solution. It suffices to prove that $(X \times Y) \setminus (A \times B)$ is open.

$$\begin{aligned}
(X \times Y) \setminus (A \times B) &\coloneqq \{(x, y) \in X \times Y \mid (x \notin A) \vee (y \notin B)\} && (297)\\
&= \{(x, y) \in X \times Y \mid x \notin A\} \cup \{(x, y) \in X \times Y \mid y \notin B\} && (298)\\
&= [(X \setminus A) \times Y] \cup [X \times (Y \setminus B)] && (299)
\end{aligned}$$

We know that since A, B are closed, $X \setminus A, Y \setminus B$ are open. Therefore each the expressions under definition of the product topology are open and their union must also be open.

Exercise 10.9 (Munkres 17.4)

Show that if U is open in X and A is closed in X, then $U - A$ is open in X, and $A - U$ is closed in X.

Solution. We know that U^c is closed and A^c is open. Since $U \setminus A = U \cap A^c$, it is the finite intersection of two open sets and therefore is open. Since $A \setminus U = A \cap U^c$, is an intersection of two closed sets, it is closed.

Exercise 10.10 (Munkres 17.5)

Let X be an ordered set in the order topology. Show that $\overline{(a,b)} \subset [a,b]$. Under what conditions does equality hold?

Solution.

Exercise 10.11 (Munkres 17.6)

Let A, B, and A_α denote subsets of a space X. Prove the following:

1. If $A \subset B$, then $\bar{A} \subset \bar{B}$.
2. $\overline{A \cup B} = \bar{A} \cup \bar{B}$.
3. $\bigcup \bar{A}_\alpha \supset \overline{\bigcup A_\alpha}$; give an example where equality fails.

Solution. For the first part, let $x \in \overline{A}$. If $x \in A$, then $x \in B \subset \overline{B}$ and we are done. If $x \in A'$, then by definition every punctured neighborhood U_x° has a nonempty intersection with A, i.e. $U_x^\circ \cap A \neq \emptyset$ for any U_x. Choose $y \in U_x^\circ \cap A$. Since $y \in A \subset B$, this means that $y \in U_x^\circ \cap B$, which proves that $A' \subset B'$. For the second part, we show bidirectionally.

1. $\overline{A} \cup \overline{B} \subset \overline{A \cup B}$. WLOG let $x \in \overline{A}$. If $x \in A$, then $x \in (A \cup B) \subset \overline{A \cup B}$. If $x \notin A$, then $x \in A'$. Therefore for every U_x°, $U_x^\circ \cap A \neq \emptyset$. But this means

$$\emptyset \neq (U_x^\circ \cap A) \cup (U_x^\circ \cap B) = U_x^\circ \cap (A \cup B) \implies x \in (A \cup B)' \subset \overline{A \cup B} \tag{300}$$

2. $\overline{A \cup B} \subset \overline{A} \cup \overline{B}$. Let $x \in \overline{A \cup B}$. If $x \in A \cup B$, then it must be the case that either $x \in A \subset (A \cup A') = \overline{A}$ or $x \in B \subset (B \cup B') = \overline{B}$, which means $x \in \overline{A} \cup \overline{B}$. If not, then $x \in (A \cup B)'$, and therefore for all U_x°,

$$U_x^\circ \cap (A \cup B) \neq \emptyset \implies (U_x^\circ \cap A) \cup (U_x^\circ \cap B) \neq \emptyset \tag{301}$$

Now assume x is not a limit point of A and not a limit point of B. Then there exists open neighborhoods U_x^1 and U_x^2 such that $(U_x^1 \setminus \{x\}) \cap A = \emptyset$ and $(U_x^2 \setminus \{x\}) \cap B = \emptyset$. But since $V_x = U_x^1 \cap U_x^2$ is also open, $V_x^\circ \coloneqq V_x \setminus \{x\}$ is also a existing punctured neighborhood that has a trivial intersection with A and that with B.

$$\begin{aligned} V_x^\circ \cap A &= \big([U_x^1 \cap U_x^2] \setminus \{x\}\big) \cap A && (302)\\ &= \big([U_x^1 \setminus \{x\}] \cap [U_x^2 \setminus \{x\}]\big) \cap A && (303)\\ &= ([U_x^1 \setminus \{x\}] \cap A) \cap ([U_x^2 \setminus \{x\}] \cap A) && (304)\\ &= \emptyset \cap ([U_x^2 \setminus \{x\}] \cap A) = \emptyset && (305) \end{aligned}$$

and the analogous argument follows for B. This means that $V_x^\circ \cap (A \cup B) = (V_x^\circ \cap A) \cup (V_x^\circ \cap B) = \emptyset$, contradicting the fact that $x \in (A \cup B)'$. Therefore x must be a limit point of at least one of A or B.

For the third, assume that $x \in \bigcup \overline{A_\alpha}$. Then there exists an α^* s.t. $x \in \overline{A_{\alpha^*}}$. We know from (1) that

$$A_{\alpha^*} \subset \bigcup_\alpha A_\alpha \implies \overline{A_{\alpha^*}} \subset \overline{\bigcup_\alpha A_\alpha} \tag{306}$$

and so $x \in \overline{\bigcup A_\alpha}$. A counterexample follows from the idea that we've depended on V_x being a *finite* intersection open open sets from before. Consider the set of singletons $A_\alpha = \alpha$ for $\alpha \in (0,1)$. Then

$$\overline{\bigcup_{\alpha \in (0,1)} A_\alpha} = \overline{(0,1)} = [0,1] \neq (0,1) = \bigcup_{\alpha \in (0,1)} \{\alpha\} = \bigcup_{\alpha \in (0,1)} \overline{\{\alpha\}} = \bigcup_{\alpha \in (0,1)} \overline{A_\alpha} \tag{307}$$

Exercise 10.12 (Munkres 17.7)

Criticize the following "proof" that $\bigcup \bar{A}_\alpha \subset \overline{\bigcup A_\alpha}$: if $\{A_\alpha\}$ is a collection of sets in X and if $x \in \bigcup \bar{A}_\alpha$, then every neighborhood U of x intersects $\bigcup A_\alpha$. Thus U must intersect some A_α, so that x must belong to the closure of some A_α. Therefore, $x \in \bigcup \bar{A}_\alpha$.

Solution.

Exercise 10.13 (Munkres 17.8)

Let A, B, and A_α denote subsets of a space X. Determine whether the following equations hold; if an equality fails, determine whether one of the inclusions $\supset$ or $\subset$ holds.

1. $\bar{A} \cap \bar{B} = \overline{A \cap B}$.
2. $\bigcap \bar{A}_\alpha = \overline{\bigcap A_\alpha}$.
3. $\bar{A} - \bar{B} = \overline{A - B}$.

Solution.

Exercise 10.14 (Munkres 17.9)

Let $A \subset X$ and $B \subset Y$. Show that in the space $X \times Y$,

$$\overline{A \times B} = \bar{A} \times \bar{B}.$$

Solution.

Exercise 10.15 (Munkres 17.10)

Show that every order topology is Hausdorff.

Solution.

Exercise 10.16 (Munkres 17.11)

Show that the product of two Hausdorff spaces is Hausdorff.

Solution. Given two Hausdorff spaces $(X, \mathscr{T}_X), (Y, \mathscr{T}_Y)$, we will denote their product space as $(X \times Y, \mathscr{T}_{X \times Y})$. Let us have two points $(x_1, y_1), (x_2, y_2) \in X \times Y$. Then $x_1, x_2 \in X$, and since X is Hausdorff there exists $U_1, U_2 \in \mathscr{T}_X$ containing x_1, x_2 respectively such that $U_1 \cap U_2 = \emptyset$. By similar logic we have $V_1, V_2 \in \mathscr{T}_Y$ containing y_1, y_2. Then, $(x_1, y_1) \in U_1 \times V_1$ open and $(x_2, y_2) \in U_2 \times V_2$ open, and we claim that $(U_1 \times V_1) \cap (U_2 \times V_2) = \emptyset$. If not, then there exists a (x', y') contained in both sets, but this implies that $x' \in U_1 \cap U_2$ contradicting the fact that X is Hausdorff. Therefore $(U_1 \times V_1)$ and $(U_2 \times V_2)$ are disjoint and we have shown such a construction.

Exercise 10.17 (Munkres 17.12)

Show that a subspace of a Hausdorff space is Hausdorff.

Solution. Let $(X, \mathscr{T}_X)$ be Hausdorff and $(Y, \mathscr{T}_Y)$ be a subspace of X with the subspace topology of X. Choose two points $y_1, y_2 \in Y$. Then as elements of X there exists disjoint $U_1, U_2 \in \mathscr{T}_X$ containing y_1, y_2 respectively. Therefore, letting $V_1 = U_1 \cap Y$ and $V_2 = U_2 \cap Y$ be open sets in $\mathscr{T}_Y$, we have

$$V_1 \cap V_2 = (U_1 \cap Y) \cap (U_2 \cap Y) = (U_1 \cap U_2) \cap Y = \emptyset \tag{308}$$

Therefore we have constructed two open sets containing y_1, y_2 that are disjoint, and so $(Y, \mathscr{T}_Y)$ is Hausdorff.

Exercise 10.18 (Munkres 17.13)

Show that X is Hausdorff if and only if the diagonal $\Delta = \{x \times x \mid x \in X\}$ is closed in $X \times X$.

Solution. We prove bidirectionally.

1. $(\rightarrow)$. X is Hausdorff implies $X \times X$ is Hausdorff. We wish to show that a $(X \times X) \setminus \Delta$ is open. So pick a point $x \notin \Delta$, which must be of form (x_1, x_2) for $x_1 \neq x_2$. Since X is Hausdorff, there exists disjoint open sets $U_1 \ni x_1, U_2 \ni x_2$. Therefore, consider the open set $U_1 \times U_2$, which must consist of points (z_1, z_2) where $z_1 \in U_1, z_2 \in U_2$. Therefore $z_1 \neq z_2$, and so $(U_1 \times U_2) \cap \Delta = \emptyset$, and so it is contained within $(X \times X) \setminus \Delta$, i.e. is open.
2. $(\leftarrow)$ Assume that Δ is closed in $X \times X$, i.e. $(X \times X) \setminus \Delta$ is open. We look at the point $(x_1, x_2) \in (X \times X) \setminus \Delta$, which implies $x_1 \neq x_2$. By openness of $(X \times X) \setminus \Delta$, there exists an open set $U_{(x_1, x_2)} \ni (x_1, x_2)$ in $(X \times X) \setminus \Delta$, which we can write as a basis element $U_1 \times U_2$ for open $U_1 \ni x_1, U_2 \ni x_2$ in X.[a] Note that since $(U_1 \times U_2) \cap \Delta = \emptyset$, $U_1 \times U_2$ cannot contain a point of the form (x, x), implying that both U_1 and U_2 cannot contain the same element x, which implies that U_1, U_2 are disjoint. Therefore X is Hausdorff.

[a] Since the basis of the product topology is already defined and choosing (x_1, x_2) by definition of a basis such a basis element must exist containing (x_1, x_2).

Exercise 10.19 (Munkres 17.14)

In the finite complement topology on $\mathbb{R}$, to what point or points does the sequence $x_n = 1/n$ converge?

Solution. It converges to every point $x \in \mathbb{R}$. Take any $x \in \mathbb{R}$, then we wish to show that for every open neighborhood U_x there exists an $N \in \mathbb{N}$ s.t. $x_n \in U_x$ for every $n > N$. Every U_x must be all of $\mathbb{R}$ minus a finite set S. The intersection $S \cap \{x_n\}$ must be finite so there is a maximum index N such that $x_N \notin U_x$. Therefore, for all $n > N$, $x_n \notin S \implies x_n \in U_x$.

Exercise 10.20 (Munkres 17.15)

Show the T_1 axiom is equivalent to the condition that for each pair of points of X, each has a neighborhood not containing the other.

Solution.

Exercise 10.21 (Munkres 17.16)

Consider the five topologies on $\mathbb{R}$ given in Exercise 7 of §13.

1. Determine the closure of the set $K = \{1/n \mid n \in \mathbb{Z}_+\}$ under each of these topologies.
2. Which of these topologies satisfy the Hausdorff axiom? the T_1 axiom?

Solution.

Exercise 10.22 (Munkres 17.17)

Consider the lower limit topology on $\mathbb{R}$ and the topology given by the basis $\mathcal{C}$ of Exercise 8 of §13. Determine the closures of the intervals $A = (0, \sqrt{2})$ and $B = (\sqrt{2}, 3)$ in these two topologies.

Solution.

Exercise 10.23 (Munkres 17.18)

Determine the closures of the following subsets of the ordered square:

$$\begin{aligned} A &= \{(1/n) \times 0 \mid n \in \mathbb{Z}_+\}, \\ B &= \{(1 - 1/n) \times \frac{1}{2} \mid n \in \mathbb{Z}_+\}, \\ C &= \{x \times 0 \mid 0 < x < 1\}, \\ D &= \{x \times \frac{1}{2} \mid 0 < x < 1\}, \\ E &= \{\frac{1}{2} \times y \mid 0 < y < 1\}. \end{aligned}$$

Solution.

Exercise 10.24 (Munkres 17.19)

If $A \subset X$, we define the *boundary* of A by the equation

$$\text{Bd } A = \bar{A} \cap \overline{(X - A)}.$$

1. Show that Int A and Bd A are disjoint, and $\bar{A} = \text{Int } A \cup \text{Bd } A$.
2. Show that Bd $A = \emptyset \Leftrightarrow A$ is both open and closed.
3. Show that U is open $\Leftrightarrow$ Bd $U = \bar{U} - U$.
4. If U is open, is it true that $U = \text{Int}(\bar{U})$? Justify your answer.

Solution.

Exercise 10.25 (Munkres 17.20)

Find the boundary and the interior of each of the following subsets of $\mathbb{R}^2$:

1. $A = \{x \times y \mid y = 0\}$
2. $B = \{x \times y \mid x > 0 \text{ and } y \neq 0\}$
3. $C = A \cup B$
4. $D = \{x \times y \mid x \text{ is rational}\}$
5. $E = \{x \times y \mid 0 < x^2 - y^2 \leq 1\}$
6. $F = \{x \times y \mid x \neq 0 \text{ and } y \leq 1/x\}$

Solution.

Exercise 10.26 (Munkres 17.21)

(Kuratowski) Consider the collection of all subsets A of the topological space X. The operations of closure $A \to \bar{A}$ and complementation $A \to X - A$ are functions from this collection to itself.

1. Show that starting with a given set A, one can form no more than 14 distinct sets by applying these two operations successively.
2. Find a subset A of $\mathbb{R}$ (in its usual topology) for which the maximum of 14 is obtained.

Solution.

10.2 Common Topologies

Exercise 10.27 (Munkres 13.6)

Show that the topologies of $\mathbb{R}_\ell$ and $\mathbb{R}_K$ are not comparable.

Solution. It suffices to show 2 things. Note that $K \coloneqq \{1/n\}_{n \in \mathbb{N}}$.

1. *$\mathcal{T}_\ell$ is not finer than $\mathcal{T}_K$.* Let $U = [-1, 1)$. Let $U = [0, 1) \in \mathscr{B}_\ell$ and $0 \in U$. We would like to show that there is no basis element of $\mathscr{B}_k$ that both contains 0 and is contained in U. Assume that there is. It can be of form (a, b) or $(a, b) \setminus K$. Assume the former. Then $0 \in (a, b) \implies a < 0$. Therefore $-a/2 \in (a, b)$, but $-a/2 \notin [0, 1)$, so $(a, b) \not\subset U$. Therefore it must be of form $(a, b) \setminus K$. However, $K \cap \{0\} = \emptyset$, so $0 \in (a, b) \setminus K \implies 0 \in (a, b)$. But by the same logic as the first case, $(a, b) \setminus K \not\subset U$ since it must contain a negative number. Therefore our assumption is false, and $\mathcal{T}_\ell$ is not finer than $\mathcal{T}_K$.
2. *$\mathcal{T}_K$ is not finer than $\mathcal{T}_\ell$.* Let $U = (-1, 1) \setminus K \in \mathscr{B}_K$ and $0 \in U$. Assume that there is a basis element $[a, b) \in \mathscr{B}_\ell$ such that $0 \in [a, b) \subset U$. $0 \in [a, b) \implies 0 < b$, but since $\mathbb{R}$ is Archimedean, there exists a $N \in \mathbb{N}$ s.t. $0 < 1/N < b$, and therefore $1/N \notin U$. This contradicts the fact that $[a, b) \subset U$, and so no such $[a, b)$ exists.

Therefore by Munkres Lemma 13.3, neither is finer than the other, which by definition means they are incomparable.

Exercise 10.28 (Munkres 13.7)

Consider the following topologies on $\mathbb{R}$:

$$\begin{aligned}
\mathcal{T}_1 &= \text{the standard topology,} \\
\mathcal{T}_2 &= \text{the topology of } \mathbb{R}_K, \\
\mathcal{T}_3 &= \text{the finite complement topology,} \\
\mathcal{T}_4 &= \text{the upper limit topology, having all sets } (a, b] \text{ as basis,} \\
\mathcal{T}_5 &= \text{the topology having all sets } (-\infty, a) = \{x \mid x < a\} \text{ as basis.}
\end{aligned}$$

Determine, for each of these topologies, which of the others it contains.

Solution. We claim

$$\mathscr{T}_3, \mathscr{T}_5 \subset \mathscr{T}_1 \subset \mathscr{T}_2, \mathscr{T}_4,\ \mathscr{T}_3 \not\sim \mathscr{T}_5,\ \mathscr{T}_2 \not\sim \mathscr{T}_4 \tag{309}$$

where $\not\sim$ means that they are not comparable, and $\subset$ means proper subset (strictly finer). We show the following.

1. $\mathscr{T}_3 \subset \mathscr{T}_1$. Let $x \in U_3 \in \mathscr{T}_3$. Then $\mathbb{R} \setminus U_3$ is finite. Therefore the following is defined.

$$r = \min_{y \in (\mathbb{R} \setminus U_3)} |x - y| \tag{310}$$

Therefore, construct the open ball $B(x,r) = (x-r, x+r)$. Since $y \in B(x,r) \implies |y-x| < d$, and every point $z \notin U_3$ must satisfy $|z-x| \geq d$, we proved that $B(x,r) \subset U_3$. By definition this means that $U_3 \subset \mathscr{T}_1$. To show strictness, take the open ball $(0,1) \in \mathscr{T}_1$. $\mathbb{R} \setminus (0,1)$ is infinite, so $\mathscr{T}_1 \not\subset \mathscr{T}_3$.

2. $\mathscr{T}_5 \subset \mathscr{T}_1$. Given $x \in \mathbb{R}$, let us choose a $\mathscr{T}_5$-open neighborhood $(-\infty, a)$ containing x. Then, we construct the $\mathscr{T}_1$-open neighborhood of x as $(x-1, a) \subset (-\infty, a)$, and we are done. To show strictness, take the basis element $(0,1) \in \mathscr{B}_1$ and set $x = 0.5$. Then every basis element of $\mathscr{B}_5$ containing x is of form $(-\infty, a), a > 0.5$ and so contains -1. Therefore it cannot be a subset of $(0,1)$ and so the topologies are not equal.
3. $\mathscr{T}_5 \not\sim \mathscr{T}_3$. Consider $U_5 = (-\infty, 0) \in \mathscr{T}_5$. Its complement $[0, -\infty)$ is infinite so $U_5 \notin \mathscr{T}_3$. Consider $U_3 = (-\infty, 0) \cup (0, +\infty) \in \mathscr{T}_3$. If U_3 is open in $\mathscr{T}_5$, then it must be a union of the basis elements of form $(-\infty, a)$. Since $1 \in U_3$, at least one of the basis elements B must have $a > 1$, but this means $0 \in B \implies 0 \in U_3$. This cannot happen and so U_3 is not open in $\mathscr{T}_5$.
4. $\mathscr{T}_1 \subset \mathscr{T}_2$ is proven in Munkres Lemma 13.4.
5. $\mathscr{T}_1 \subset \mathscr{T}_4$. Let $x \in \mathbb{R}$ and choose a basis element $(a,b) \in \mathscr{B}_1$ containing x. Then we choose the basis element $(a,x] \in \mathscr{B}_4$ which also contains x, and $a < x < b \implies (a,x] \subset (a,b)$. We are done. To show strictness, let us choose $x \in \mathbb{R}$ and choose basis element $(a,x]$. Then we claim there is no $\mathscr{T}_1$-open neighborhood of x contained in $(a,x]$. Assume there was, of form (c,d). Then $x < d$, and by density of rationals there exists a q such that $x < q < d$. Then $q \in (c,d)$ but $q \notin (a,x]$, and $(c,d) \not\subset (a,x]$. By contradiction, there exists no subset, and $\mathscr{T}_1 \not\supset \mathscr{T}_4$.
6. $\mathscr{T}_2 \not\sim \mathscr{T}_4$. Consider $U_2 = (0, 1.1) \setminus K \in \mathscr{T}_2$. Note that $\frac{2}{19}, \frac{21}{20} \in U_2$. Now assume that there is some basis element $(a,b]$ of $\mathscr{T}_4$ that is contained in U_2. It must be the case that $a < 2/19$ and $b \geq 21/20$, but this means that $1/2 \in (a,b]$, which is not in U_2. Therefore, $\mathscr{T}_4$ is not finer than $\mathscr{T}_2$.
   For the other direction, consider $x = 2$ and $U_4 = (1,2] \in \mathscr{B}_2$. We claim that there is no basis element of $\mathscr{T}_4$ that contains x and is contained in U_4. From point 5 above, the basis element cannot be of form (a,b). So it must be of form $(a,b) \setminus K$. Assume that there was such a set. Then $(a,b) \setminus K \subset (1,2]$, but this means that $b > 2$. Therefore by the same reasoning above, there must exist some q s.t. $2 < q < b$, and so $q \in (a,b) \setminus K$ but $q \notin (1,2]$. Therefore $\mathscr{T}_2$ is not finer than $\mathscr{T}_4$.

Exercise 10.29 (Munkres 13.8)

(a) Apply Lemma 13.2 to show that the countable collection

$$\mathcal{B} = \{(a,b) \mid a < b, a \text{ and } b \text{ rational}\}$$

is a basis that generates the standard topology on $\mathbb{R}$.
(b) Show that the collection

$$\mathcal{C} = \{[a,b) \mid a < b, a \text{ and } b \text{ rational}\}$$

is a basis that generates a topology different from the lower limit topology on $\mathbb{R}$.

Solution. Listed.

1. For any open set $U \subset \mathbb{R}$ and $x \in U$, by definition there exists a $r > 0$ s.t. $B(x,r) = (x-r, x+r) \subset U$. But by the density of the rationals in $\mathbb{R}$, there exists an $a, b \in \mathbb{Q}$ s.t.
$$x - r < a < x < b < x + r \implies x \in (a,b) \subset (x-r, x+r) \subset U \tag{311}$$
   Since we can always find such an element $(a,b) \in \mathcal{B}$ satisfying $x \in (a,b) \subset U$, $\mathcal{B}$ is a basis.
2. Call the topology generated by $\mathscr{C}$ to be $\mathscr{T}$, and let the lower limit topology be $\mathscr{T}'$ generated by its corresponding basis, denoted $\mathscr{B}$. Assume that $\mathscr{T} = \mathscr{T}'$. Then, $\mathscr{T}' \subset \mathscr{T}$, and by Lemma

13.2, it must be the case that for each $x \in X$ and basis element $B \in \mathscr{B}$, we can find a $C \in \mathscr{C}$ s.t. $x \in C \subset B$. Consider $x = \sqrt{2}$ and $B = [\sqrt{2}, 2)$. We attempt to find an interval $[a, b)$ with $a, b \in \mathbb{Q}$ such that $\sqrt{2} \in [a, b) \subset [\sqrt{2}, 2)$. Clearly $a \neq \sqrt{2}$ since a is rational. If $a > \sqrt{2}$, then $x \not\in [a, b)$. If $a < \sqrt{2}$, then by density of rationals in $\mathbb{R}$, there exists a $q \in \mathbb{Q}$ satisfying $a < q < \sqrt{2}$. Therefore, there exists a $q \in \mathbb{R}$ s.t. $q \in [a, b)$ but $q \not\in [\sqrt{2}, 2)$, implying that $[a, b) \not\subset [\sqrt{2}, 2)$. Therefore such an interval cannot exist $\implies \mathscr{T}' \not\subset \mathscr{T} \implies \mathscr{T}' \neq \mathscr{T}$.

Exercise 10.30 (Munkres 20.1)

(a) In $\mathbb{R}^n$, define

$$d'(\mathbf{x}, \mathbf{y}) = |x_1 - y_1| + \cdots + |x_n - y_n|.$$

Show that d' is a metric that induces the usual topology of $\mathbb{R}^n$. Sketch the basis elements under d' when $n = 2$.
(b) More generally, given $p \geq 1$, define

$$d'(\mathbf{x}, \mathbf{y}) = \left[\sum_{i=1}^n |x_i - y_i|^p \right]^{1/p}$$

for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Assume that d' is a metric. Show that it induces the usual topology on $\mathbb{R}^n$.

Solution. Let $\mathscr{B}_1, \mathscr{B}_2$ be the basis of open balls with respect to the $d' = d_1$ and d_2 metrics on $\mathbb{R}^n$, with their generated topologies being $\mathscr{T}_1, \mathscr{T}_2$. We show that

$$d_2(x, y) \leq d_1(x, y) \leq \sqrt{n} d_2(x, y) \tag{312}$$

Since all expressions are nonnegative, it suffices to show that

$$(d_2(x, y))^2 \leq (d_1(x, y))^2 \leq n(d_2(x, y))^2 \tag{313}$$

1. $d_2(x, y))^2 \leq (d_1(x, y))^2$. We see that by expanding and seeing that the product of absolute values is always nonnegative,

$$\begin{aligned} (d_1(x, y))^2 &= \left(\sum_i |x_i - y_i| \right)^2 && (314) \\ &= \sum_i (x_i - y_i)^2 + \sum_{i \neq j} |x_i - y_i||x_j - y_j| && (315) \\ &\geq \sum_i (x_i - y_i)^2 && (316) \\ &= (d_2(x, y))^2 && (317) \end{aligned}$$

2. For the second part, we use the Schwartz inequality.

$$\begin{aligned} d_1(x, y) &= \sum_i |x_i - y_i| && (318) \\ &= \sum_i |x_i - y_i| \cdot 1 && (319) \\ &\leq \sqrt{\sum_i (x_i - y_i)^2} \cdot \sqrt{\sum_i 1} && (320) \\ &= d_2(x, y) \cdot \sqrt{n} && (321) \end{aligned}$$

Now we show that $\mathscr{T}_1 = \mathscr{T}_2$.

1. $\mathscr{T}_2 \subset \mathscr{T}_1$. Given an $\mathscr{T}_2$-open neighborhood U and $x \in U$, by definition there exists a $r > 0$ s.t. $x \in B_2(x, r) \subset U$. We claim that there exists a $r' > 0$ such that $B_1(x, r') \subset U$, making this also a $\mathscr{T}_1$-open neighborhood. Set $r' = r$. Then
$$\begin{aligned} y \in B_1(x,r) &\implies d_1(x,y) < r && (322)\\ &\implies d_2(x,y) \leq d_1(x,y) < r && (323)\\ &\implies d_2(x,y) < r && (324)\\ &\implies y \in B_2(x,r) && (325)\end{aligned}$$
and so $x \in B_1(x,r) \subset B_2(x,r) \subset U$.
2. $\mathscr{T}_1 \subset \mathscr{T}_2$. Given an $\mathscr{T}_1$-open neighborhood U and $x \in U$, by definition there exists a $r > 0$ s.t. $x \in B_1(x, r) \subset U$. We claim that there exists a $r' > 0$ such that $B_2(x, r') \subset U$, making this also a $\mathscr{T}_2$-open neighborhood. Set $r' = rn^{-1/2}$. Then
$$\begin{aligned} y \in B_2(x, rn^{-1/2}) &\implies d_2(x,y) < rn^{-1/2} && (326)\\ &\implies n^{1/2} d_2(x,y) < r && (327)\\ &\implies d_1(x,y) < r && (328)\\ &\implies y \in B_1(x,r) && (329)\end{aligned}$$
and so $x \in B_2(x,r) \subset B_1(x,r) \subset U$.

Solution. Let the metric be denoted $d_p(x, y)$. Let q be the Holder conjugate of p, i.e. the unique $q \in \mathbb{R}$ s.t. $(1/p) + (1/q) = 1$. If $p = 1$, then we have proved the equivalence in (a). If $p > 1$, then by the ordered field properties of $\mathbb{R}$, $0 < 1/p < 1 \implies 0 < 1/q < 1 \implies q > 1$. Given that we can always define q, we show two things.

1. $d_1(x,y) \leq n^{1/q} d_p(x,y) \iff n^q d_1(x,y) \leq d_p(x,y)$.
$$\begin{aligned} d_1(x,y) &= \sum_i |x_i - y_i| && (330)\\ &= \sum_i |x_i - y_i| \cdot 1 && (331)\\ &\leq \bigg(\sum_i |x_i - y_i|^p \bigg)^{1/p} \cdot \bigg(\sum_i 1^q \bigg)^{1/q} && (332)\\ &= d_p(x,y) \cdot n^{1/q} && (333)\end{aligned}$$
2. $d_p(x,y) \leq n^{1/p} d_\infty(x,y) \iff n^p d_p(x,y) \leq d_\infty(x,y)$. Since both expressions are nonnegative (since we assumed that it's a metric), it suffices to prove that $(d_p(x,y))^p \leq (d_\infty(x,y))^p$.
$$\begin{aligned} (d_p(x,y))^p &= \sum_i |x_i - y_i|^p && (334)\\ &\leq \sum_i \big(\max_i \{|x_i - y_i|\} \big)^p && (335)\\ &= n \cdot \big(\max_i \{|x_i - y_i|\} \big)^p && (336)\\ &= n \cdot (d_\infty(x,y))^p && (337)\\ d_p(x,y) &= n^{1/p} \cdot d_\infty(x,y) && (338)\end{aligned}$$

Now we prove the following. Since $p = 1$ is proved in (a), we assume $p > 1$, and $q > 1$ is always defined. For notation, let $\mathscr{T}_p$ denote the topology generated by the basis of open balls $B_p(x, r)$ with respect to the d_p metric.

1. $\mathscr{T}_1 \subset \mathscr{T}_p$. Let U be open in $\mathscr{T}_1$ and $x \in U$. Then by definition there exists a $r > 0$ s.t. $x \in B_1(x, r) \subset U$. We claim that there exists a r' s.t. $B_p(x, r') \subset U$, making this also a $\mathscr{T}_p$-open neighborhood. Set $r' = rn^q$.

$$\begin{aligned} y \in B_p(x, r') &\implies d_p(x, y) < rn^q && (339)\\ &\implies n^q \cdot d_1(x, y) \leq d_p(x, y) < rn^q && (340)\\ &\implies d_1(x, y) < r && (341)\\ &\implies y \in B_1(x, y) && (342)\end{aligned}$$

Therefore $x \in B_p(x, r') \subset B_1(x, y) \subset U$.

2. $\mathscr{T}_p \subset \mathscr{T}_\infty$. Let U be open in $\mathscr{T}_p$ and $x \in U$. Then by definition there exists a $r > 0$ s.t. $x \in B_p(x, r) \subset U$. We claim that there exists r' s.t. $B_\infty(x, r') \subset U$, making this also a $\mathscr{T}_\infty$-open neighborhood. Set $r' = rn^p$.

$$\begin{aligned} y \in B_\infty(x, r') &\implies d_\infty(x, y) < rn^q && (343)\\ &\implies n^q d_p(x, y) \leq d_\infty(x, y) < rn^q && (344)\\ &\implies d_p(x, y) < r && (345)\\ &\implies y \in B_p(x, r) && (346)\end{aligned}$$

Therefore, $x \in B_\infty(x, rn^q) \subset B_p(x, r) \subset U$.

From (a) and the previous homework, we know that $\mathscr{T}_1 = \mathscr{T}_\infty = \mathscr{T}_2$, denote this $\mathscr{T}$. Therefore, we have proved that $\mathscr{T} \subset \mathscr{T}_p$ and $\mathscr{T}_p \subset \mathscr{T}$, which means $\mathscr{T} = \mathscr{T}_p$.

10.3 Continuity

Exercise 10.31 (Munkres 18.1)

Prove that for functions $f : \mathbb{R} \to \mathbb{R}$, the ϵ-δ definition of continuity implies the open set definition.

Exercise 10.32 (Munkres 18.2)

Suppose that $f : X \to Y$ is continuous. If x is a limit point of the subset A of X, is it necessarily true that $f(x)$ is a limit point of $f(A)$?

Solution. No. Consider $X = Y = \mathbb{R}$ with the Euclidean topology and let $f(x) = 0$. Consider $A = (-1, 1) \implies f(A) = \{0\}$. 1 is a limit point of A but $f(1) = 0$ is not a limit point of $\{0\}$ since it's an isolated point, i.e. for any punctured open neighborhood $U_0^\circ = U_0 \setminus \{0\}$,

$$U_0^\circ \cap f(A) = (U_0 \setminus \{0\}) \cap \{0\} = \emptyset \tag{347}$$

Exercise 10.33 (Munkres 18.3)

Let X and X' denote a single set in the two topologies $\mathcal{T}$ and $\mathcal{T}'$, respectively. Let $i : X' \to X$ be the identity function.

1. Show that i is continuous $\Leftrightarrow$ $\mathcal{T}'$ is finer than $\mathcal{T}$.
2. Show that i is a homeomorphism $\Leftrightarrow$ $\mathcal{T}' = \mathcal{T}$.

Exercise 10.34 (Munkres 18.4)

Given $x_0 \in X$ and $y_0 \in Y$, show that the maps $f : X \to X \times Y$ and $g : Y \to X \times Y$ defined by

$$f(x) = x \times y_0 \quad \text{and} \quad g(y) = x_0 \times y$$

are imbeddings.

Exercise 10.35 (Munkres 18.5)

Show that the subspace (a, b) of $\mathbb{R}$ is homeomorphic with $(0, 1)$ and the subspace $[a, b]$ of $\mathbb{R}$ is homeomorphic with $[0, 1]$.

Exercise 10.36 (Munkres 18.6)

Find a function $f : \mathbb{R} \to \mathbb{R}$ that is continuous at precisely one point.

Exercise 10.37 (Munkres 18.7)

1. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is "continuous from the right," that is,

$$\lim_{x \to a^+} f(x) = f(a),$$

for each $a \in \mathbb{R}$. Show that f is continuous when considered as a function from $\mathbb{R}_\ell$ to $\mathbb{R}$.
2. Can you conjecture what functions $f : \mathbb{R} \to \mathbb{R}$ are continuous when considered as maps from $\mathbb{R}$ to $\mathbb{R}_\ell$? As maps from $\mathbb{R}_\ell$ to $\mathbb{R}_\ell$? We shall return to this question in Chapter 3.

Solution. This definition of continuity at a point in analysis means that for all $\epsilon > 0$ there exists a $\delta > 0$ s.t. $x \in [a, a + \delta) \implies |f(x) - f(a)| < \epsilon$. Assuming this definition holds, let $U_{f(a)} \in \mathscr{T}$ be an open neighborhood of $f(a)$ (w.r.t. Euclidean topology). We wish to show that the preimage is open in $\mathscr{T}_\ell$.
By definition of open sets in $\mathbb{R}$, there exists an open ball $(f(a) - \epsilon, f(a) + \epsilon) \subset U_{f(a)}$, and from our analytical definition of continuity there exists a $\delta > 0$ s.t. $f([a, a+\delta)) \subset (f(a) - \epsilon, f(a) + \epsilon)$. Therefore taking the preimage we have

$$[a, a + \delta) \subset f^{-1}\big((f(a) - \epsilon, f(a) + \epsilon)\big) \subset f^{-1}\big(U_{f(a)}\big) \tag{348}$$

Since we can construct an open ball (in $\mathscr{T}_\ell$) $[a, a + \delta) \subset f^{-1}(U_{f(a)})$, by definition $f^{-1}(U_{f(a)})$ is open in $\mathbb{R}_\ell$, making f continuous at a. Since this holds for all a, f is continuous.[a]

[a]We could have started off with an arbitrary open set U and chosen an arbitrary $a \in f^{-1}(U)$ to get the same result.

Exercise 10.38 (Munkres 18.8)

Let Y be an ordered set in the order topology. Let $f, g : X \to Y$ be continuous.
1. Show that the set $\{x \mid f(x) \leq g(x)\}$ is closed in X.
2. Let $h : X \to Y$ be the function

$$h(x) = \min\{f(x), g(x)\}.$$

Show that h is continuous. [Hint: Use the pasting lemma.]

Exercise 10.39 (Munkres 18.9)

Let $\{A_\alpha\}$ be a collection of subsets of X; let $X = \bigcup_\alpha A_\alpha$. Let $f : X \to Y$; suppose that $f|A_\alpha$ is continuous for each α.

1. Show that if the collection $\{A_\alpha\}$ is finite and each set A_α is closed, then f is continuous.
2. Find an example where the collection $\{A_\alpha\}$ is countable and each A_α is closed, but f is not continuous.
3. An indexed family of sets $\{A_\alpha\}$ is said to be *locally finite* if each point x of X has a neighborhood that intersects A_α for only finitely many values of α. Show that if the family $\{A_\alpha\}$ is locally finite and each A_α is closed, then f is continuous.

Exercise 10.40 (Munkres 18.10)

Let $f : A \to B$ and $g : C \to D$ be continuous functions. Let us define a map $f \times g : A \times C \to B \times D$ by the equation

$$(f \times g)(a \times c) = f(a) \times g(c). \tag{349}$$

Show that $f \times g$ is continuous.

Solution. Let V be an open set in $B \times D$. Then V can be expressed as the union of basis elements in the product topology, with the form

$$V = \bigcup U_B \times U_D \tag{350}$$

where $U_B \in \mathscr{T}_B$ and $U_D \in \mathscr{T}_D$. Now take the preimage.

$$(f \times g)^{-1}(V) = \bigcup (f \times g)^{-1}(U_B \times U_D) = \bigcup f^{-1}(U_B) \times g^{-1}(U_D) \tag{351}$$

Since f, g are continuous, $f^{-1}(U_B) \in \mathscr{T}_A$ and $g^{-1}(U_D) \in \mathscr{T}_C$. Therefore, $f^{-1}(U_B) \times g^{-1}(U_D)$ is a basis element of the product topology $\mathscr{T}_{A\times C}$, and its arbitrary union is indeed an open set in $A \times C$. Therefore $f \times g$ is continuous.

Exercise 10.41 (Munkres 18.11)

Let $F : X \times Y \to Z$. We say that F is *continuous in each variable separately* if for each y_0 in Y, the map $h : X \to Z$ defined by $h(x) = F(x \times y_0)$ is continuous, and for each x_0 in X, the map $k : Y \to Z$ defined by $k(y) = F(x_0 \times y)$ is continuous. Show that if F is continuous, then F is continuous in each variable separately.

Solution. Let us define $\iota_{y_0} : X \to X \times Y$ as the canonical injection $\iota_{y_0}(x) = (x, y_0)$. We first show that this is continuous. First choose an open set $V \in \mathscr{T}_{X\times Y}$, which is of the form

$$V = \bigcup U_X \times U_Y \tag{352}$$

for open sets $U_X \in \mathscr{T}_X, U_Y \in \mathscr{T}_Y$. Taking the preimage

$$\iota_{y_0}^{-1}(V) = \bigcup \iota_{y_0}^{-1}(U_X \times U_Y) \tag{353}$$

For each term in the union, note that if $y_0 \notin U_Y$, then the preimage is $\emptyset$, which is open. If $y_0 \in U_Y$, then the preimage is U_X which is open. Therefore the union of such open sets is open. With this, note that

$$h = F \circ \iota_{y_0} \tag{354}$$

which is a composition of continuous maps and therefore is continuous. The proof for k is identical.

Exercise 10.42 (Munkres 18.12)

Let $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be defined by the equation

$$F(x \times y) = \begin{cases} xy/(x^2 + y^2) & \text{if } x \times y \neq 0 \times 0, \\ 0 & \text{if } x \times y = 0 \times 0. \end{cases} \tag{355}$$

1. Show that F is continuous in each variable separately.
2. Compute the function $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = F(x \times x)$.
3. Show that F is not continuous.

Solution.

1. Fix $y = y_0$. Then if $y_0 = 0$, $h(x) = 0$ which is continuous. If $y_0 \neq 0$, then

$$h(x) = F(x \times y_0) = \frac{y_0 x}{x^2 + y_0^2} \tag{356}$$

   which is the quotient of two polynomials, which are continuous, and the denominator never vanishes since $x^2 + y_0^2 \geq y_0^2 > 0$. Similarly, if we fix $x = x_0$, $k(y) = 0$ if $x_0 = 0$ and

$$k(y) = F(x_0 \times y) = \frac{x_0 y}{y^2 + x_0^2} \tag{357}$$

   which is the quotient of two polynomials where the denominator never vanishes.
2. We have

$$g(x) = \begin{cases} F(0) = 0 & \text{if } x = 0 \\ F(x \times x) = \frac{x^2}{2x^2} = \frac{1}{2} & \text{if } x \neq 0 \end{cases} \tag{358}$$

3. We see that g above is not continuous since the preimage of the open set $(-0.25, 0.25)$ is $\{0\}$ which is not open. We can write $g = F \circ \iota$, where $\iota(x) = (x, x)$. We claim that ι is continuous. Take any basis element $U \times V \in \mathscr{T}_{X \times X}$ where U, V are open in X. The preimage consists of all points x that are both in U and V, i.e. $\iota^{-1}(U \times V) = U \cap V$, which is open. Therefore ι is continuous. If F was continuous, then $F \circ \iota$ would be continuous, but g is not continuous, so F must not be continuous.

Exercise 10.43 (Munkres 18.13)

Let $A \subset X$; let $f : A \to Y$ be continuous; let Y be Hausdorff. Show that if f may be extended to a continuous function $g : \bar{A} \to Y$, then g is uniquely determined by f.

Solution. Let us consider two such extensions g, h. If $A = \overline{A}$, then $g = f = h$ and this is unique. If $A \subsetneq \overline{A}$ then there exists $x \in A' \setminus A$. Since Y is Hausdorff, there exists disjoint open neighborhoods $U \ni g(x), V \ni h(x)$. It is the case that $x \in g^{-1}(U) \cap h^{-1}(V)$ open (since f, h are continuous and so their preimages are open). Since x is a limit point of A, there exists a $y \in g^{-1}(U) \cap h^{-1}(V) \cap A$ not equal to x. Mapping through through h, g again gives $g(y) \in U$, $h(y) \in V$. But since $y \in A$, the two must agree with f, and so $f(y) = g(y) = h(y)$, which contradicts that Y is Hausdorff.

Exercise 10.44 (Math 411 Spring 2025, PS4)

Suppose X and Y are topological spaces, where Y is Hausdorff, and let f and g be continuous functions from X to Y. Prove that the set $S = \{x \in X \mid f(x) = g(x)\}$ is closed.

Solution. We equivalently wish to show that $X \setminus S$ is open. For any $x \in (X \setminus S)$, we have $f(x) \neq g(x) \in Y$. Since Y is Hausdorff, there exists open $U_{f(x)} \ni f(x)$ and open $U_{g(x)} \ni g(x)$ s.t. $U_{f(x)} \cap U_{g(x)} = \emptyset$. Therefore, we can take their preimage $f^{-1}(U_{f(x)}), g^{-1}(U_{g(x)})$ which is open in X by continuity of f, g. Furthermore we can take their intersection to get another open neighborhood of x.

$$V_x = f^{-1}(U_{f(x)}) \cap g^{-1}(U_{g(x)}) \tag{359}$$

We claim that $V_x \cap S = \emptyset$. Assuming not, we have some $s \in V_x \cap S$. Since $s \in V_x$, $f(s) \in U_{f(x)}$ and $g(s) \in U_{g(x)}$, but since $s \in S$, $f(x) = g(x)$ and these map to the same point, contradicting the fact that $U_{f(x)}$ and $U_{g(x)}$ are disjoint. Therefore, our claim holds true, which implies that $V_x \subset X \setminus S$. Therefore, we have proved that all points $x \in (X \setminus S)$ is an interior point, and thus $X \setminus S$ is open.[a]

[a]This can be shown by letting $X \setminus S$ be the union of all V_x for $x \in (X \setminus S)$ which is open.

Exercise 10.45 (Math 411 Spring 2025, PS5)

Let X be a topological space, and let $f, g : X \to \mathbb{R}$ be continuous maps.

1. Show that the set $\{x \in X \mid f(x) \leq g(x)\}$ is closed in X.
2. Show that the function $h : X \to \mathbb{R}$ given by $h(x) = \max\{f(x), g(x)\}$ is continuous.

Note: If you prefer, instead of $\mathbb{R}$ you can use an arbitrary ordered set Y with the order topology in place of Y, as in §18 #8. (We did not cover the order topology in class; see §14 if you're curious.)

Solution. For any $y \in \mathbb{R}$. The sets $\{x \in X \mid f(x) > y\}$ and $\{x \in X \mid y > g(x)\}$ are open since they are the preimages of $(y, +\infty)$ and $(-\infty, y)$ respectively. Therefore their intersection is open, i.e.

$$U_y = \{x \in X \mid f(x) > y > g(x)\} \tag{360}$$

Therefore their union is also open.

$$U = \bigcup_{y \in \mathbb{R}} U_y = \{x \in X \mid f(x) > g(x)\} \tag{361}$$

where we can always identify a $y = (f(x) + g(x))/2$. Therefore the complement where $f(x) \leq g(x)$ is closed.

As for the second part, we will denote $U_f = \{x \in X \mid f(x) \leq g(x)\}$ and $U_g = \{x \in X \mid g(x) \geq f(x)\}$. They are both closed, where U_g can be proved closed by the same symmetric argument. Let us take any closed set $S \subset X$. Since $X = U_f \cup U_g$, we can denote

$$S = S \cap X = S \cap (U_f \cup U_g) = (S \cap U_f) \cup (S \cap U_g) = V_f \cup V_g \tag{362}$$

where V_f, V_g are closed since we proved before that U_f, U_g are closed. The preimage is

$$\begin{aligned} h^{-1}(S) &= h^{-1}(V_f) \cup h^{-1}(V_g) &\quad (363)\\ &= g^{-1}(V_f) \cup f^{-1}(V_g) &\quad (364) \end{aligned}$$

which is the union of open sets and therefore open.

10.4 Induced Topologies

Exercise 10.46 (Munkres 16.1)

Show that if Y is a subspace of X, and A is a subset of Y, then the topology A inherits as a subspace of Y is the same as the topology it inherits as a subspace of X.

Solution. We have $A \subset (Y, \mathscr{T}_Y) \subset (X, \mathscr{T}_X)$, with $\mathscr{T}_Y$ the subspace topology from $\mathscr{T}_X$. Let us denote $\mathscr{T}_{A|Y}$ and $\mathscr{T}_{A|X}$ the subspace topologies on A when considered its superset as Y and X, respectively. We show the following.

1. $\mathscr{T}_{A|Y} \subset \mathscr{T}_{A|X}$. Let $U \in \mathscr{T}_{A|Y}$. Then $U = V \cap A$ for some $V \in \mathscr{T}_Y$, and $V = W \cap Y$ for some $W \in \mathscr{T}_X$. Therefore, $U = (W \cap Y) \cap A = W \cap (Y \cap A) = W \cap A$ for some $W \in \mathscr{T}_X$, which by definition means $U \in \mathscr{T}_{A|X}$.
2. $\mathscr{T}_{A|X} \subset \mathscr{T}_{A|Y}$. Let $U \in \mathscr{T}_{A|X}$. Then $U = V \cap A$ for some $V \in \mathscr{T}_X$. But note that $A = A \cap Y$, and so $U = V \cap (A \cap Y) = (V \cap Y) \cap A$. Denote $W = V \cap Y$. Since V is open in X, W is open in Y, and therefore we have found such a $W \in \mathscr{T}_Y$ where $U = W \cap A$, which by definition means $U \in \mathscr{T}_{A|Y}$.

Exercise 10.47 (Munkres 16.2)

If $\mathcal{T}$ and $\mathcal{T}'$ are topologies on X and $\mathcal{T}'$ is strictly finer than $\mathcal{T}$, what can you say about the corresponding subspace topologies on the subset Y of X?

Exercise 10.48 (Munkres 16.3)

Consider the set Y = [-1, 1] as a subspace of $\mathbb{R}$. Which of the following sets are open in Y? Which are open in $\mathbb{R}$?

$$A = \{x \mid \frac{1}{2} < |x| < 1\} \tag{365}$$

$$B = \{x \mid \frac{1}{2} < |x| \leq 1\} \tag{366}$$

$$C = \{x \mid \frac{1}{2} \leq |x| < 1\} \tag{367}$$

$$D = \{x \mid \frac{1}{2} \leq |x| \leq 1\} \tag{368}$$

$$E = \{x \mid 0 < |x| < 1 \text{ and } \frac{1}{x} \notin \mathbb{Z}_+\} \tag{369}$$

Solution. We list the supersets which each set is open in.

1. A is open in Y and $\mathbb{R}$. $A_1 = (-1, -\frac{1}{2})$ and $A_2 = (\frac{1}{2}, 1)$ are open in $\mathbb{R}$, and they are also open in Y since $A_1 = A_1 \cap Y$ and $A_2 = A_2 \cap Y$. Therefore, $A = A_1 \cup A_2$ is by definition open.
2. B is open in Y. $(-2, -\frac{1}{2})$ and $(\frac{1}{2}, 2)$ are open in $\mathbb{R}$ and so $(-2, -\frac{1}{2}) \cap Y = [-1, -\frac{1}{2})$ and $(\frac{1}{2}, 2) \cap Y = (\frac{1}{2}, 1]$ is open in Y. It is not open in $\mathbb{R}$ because consider the point 1. Assume that there exists an $\epsilon > 0$ s.t. $(1-\epsilon, 1+\epsilon) \subset B$. This means that $1 + \frac{\epsilon}{2} \in (1-\epsilon, 1+\epsilon)$ but since $1 < 1 + \frac{\epsilon}{2}$, $1 + \epsilon \notin B$.
3. C. Neither. Consider the point $x = \frac{1}{2}$ and assume that there exists a $\epsilon > 0$ s.t. $B(x, \epsilon) = (\frac{1}{2} - \epsilon, \frac{1}{2} + \epsilon) \subset C$. If $\epsilon > \frac{1}{2}$, then $0 \in B(x, \epsilon)$ and so $B(x, \epsilon) \not\subset C$. If $0 < \epsilon \leq \frac{1}{2}$, then this means that $\frac{1}{2} - \frac{\epsilon}{2} \in B(x, \epsilon)$. But $0 \leq \frac{1}{2} - \frac{\epsilon}{2} < \frac{1}{2}$, and so $\frac{1}{2} - \frac{\epsilon}{2} \notin C$, which also implies $B(x, \epsilon) \not\subset C$. Therefore there exists no such open neighborhood around $\frac{1}{2}$. This argument applies to both Y and $\mathbb{R}$, and so C is not open in both.
4. D. Neither. We repeat the same argument as that for C and show that there exists no open

neighborhood around 1/2 contained in D.

5. E is open in Y and $\mathbb{R}$. We prove a small fact: for every $x \in \mathbb{R}$, there exists an integer $z \in \mathbb{Z}$ s.t. $z - 1 < x \leq z$. Let's take $x > 0$. The reals is Archimedean and so for any $x \in \mathbb{R}$ there exists a natural $s \in \mathbb{N}$ s.t. $x < n$. Consider the set $N = \{n \in \mathbb{Z} \mid x < n\}$ of all upper bounds of x, which we proved is nonempty. By the well-ordering principle, this set must have a minimum, which we call z. It must be the case by upper bound that $x \leq z$, and $z - 1$ not an upper bound implies $z - 1 < x$. If $x = 0$ this result is trivial and if $x < 0$ we can found the integral bounds $0 \leq z - 1 < -x < z$ and swap the signs to get $-z < x < -z + 1 \leq 0$.
   We claim that
   $$G = \{x \in \mathbb{R} \mid 1/x \notin \mathbb{N}\} = (-\infty, 0) \cup \bigcup_{n \in \mathbb{N}} \left(\frac{1}{n+1}, \frac{1}{n} \right) = H \tag{370}$$
   If $x \in G$, then it can be either positive or negative. If negative, $x \in (-\infty, 0)$. If positive, then $1/x \in \mathbb{R}$ since it's a field. By the proof above there exists a $n \in \mathbb{N}$ s.t. $n < 1/x < n + 1$ (strict inequality since $1/x$ is not natural) and so by ordered field properties $\frac{1}{n} < x < \frac{1}{n+1} \implies z \in \left(\frac{1}{n+1}, \frac{1}{n}\right)$ for some $n \in \mathbb{N}$. Therefore $x \in H$.
   If $x \in H$, then either x is negative or positive. If $x \in (-\infty, 0)$, then $x \in G$ trivially since it cannot be the case that $1/x > 0$. If x is positive then $x \neq 1/n$ for all $n \in \mathbb{N}$, so $x \in G$. Therefore $H = G$. G is an arbitrary union of known open sets in $\mathbb{R}$, and so G is open. This means that $E = ((-1, 0) \cup (0, 1)) \cap H$ is also open by definition, and so E is open in $\mathbb{R}$. For Y, note that
   $$((-1, 0) \cup (0, 1)) \cap Y = (-1, 0) \cup (0, 1) \tag{371}$$
   and so $E = E \cap Y$, which implies that E is open in Y as well.

Exercise 10.49 (Munkres 16.4)

A map f : X → Y is said to be an open map if for every open set U of X, the set f(U) is open in Y. Show that $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ are open maps.

Solution. Let U be open in $X \times Y$. Then U is a union of some basis elements of the product topology on $X \times Y$, which are of the form $U_X \times U_Y$ where U_X, U_Y are open sets in the topologies of X, Y.
$$U = \bigcup_{\alpha \in A} (U_X)_\alpha \times (U_Y)_\alpha \tag{372}$$

1. We see that π_1 maps all $(x, y) \in U_X \times U_Y$ to x, so it acts on U as
   $$\pi_1(U) = \bigcup_{\alpha \in A} (U_X)_\alpha \tag{373}$$
   Since the union of open sets are open, $\pi_1(U)$ is open in X.
2. We see that π_2 maps all $(x, y) \in U_X \times U_Y$ to y, so it acts on U as
   $$\pi_2(U) = \bigcup_{\alpha \in A} (U_Y)_\alpha \tag{374}$$
   Since the union of open sets are open, $\pi_2(U)$ is open in Y.

Exercise 10.50 (Munkres 16.5)

Let X and X' denote a single set in the topologies $\mathcal{T}$ and $\mathcal{T}'$, respectively; let Y and Y' denote a single set in the topologies $\mathcal{U}$ and $\mathcal{U}'$, respectively. Assume these sets are nonempty.

1. Show that if $\mathcal{T}' \supseteq \mathcal{T}$ and $\mathcal{U}' \supseteq \mathcal{U}$, then the product topology on X' $\times$ Y' is finer than the product topology on X $\times$ Y.
2. Does the converse of (a) hold? Justify your answer.

Solution. Let us denote the set as $(X, \mathscr{T}), (X, \mathscr{T}')$ and $(Y, \mathscr{U}), (Y, \mathscr{U}')$ where $\mathscr{T} \subset \mathscr{T}'$ and $\mathscr{U} \subset \mathscr{U}'$. We wish to show that $\mathscr{T}_{X\times Y} \subset \mathscr{T}_{X'\times X'}$. By Munkres Lemma 13.3, this is equivalent to showing that for any $(x, y) \in X \times Y$ and basis element $U_X \times U_Y \in \mathscr{T} \times \mathscr{U}$ containing (x, y), there is a basis element $U_{X'} \times U_{Y'} \in \mathscr{T}' \times \mathscr{U}'$ such that

$$x \in (U_{X'} \times U_{Y'}) \subset (U_X \times U_Y) \tag{375}$$

Say we have $(x, y) \in X \times Y$, and choose such a U_X, U_Y containing x, y respectively.[a] Then $U_X \times U_Y \in \mathscr{T} \times \mathscr{U}$ is a basis element of $\mathscr{T}_{X\times Y}$ by definition. We see that $U_X \in \mathscr{T} \subset \mathscr{T}'$ and $U_Y \in \mathscr{U} \subset \mathscr{U}'$, so $U_X \times U_Y \in \mathscr{T}' \times \mathscr{U}'$, meaning that it is also a basis element of $\mathscr{T}_{X'\times Y'}$. Therefore, we set $U_{X'} = U_X$ and $U_{Y'} = U_Y$, and we are done.

[a] This is always possible since $X \in \mathscr{T}$ and $Y \in \mathscr{U}$.

Solution. Yes, the converse is true. Let us denote the set as $(X, \mathscr{T}), (X, \mathscr{T}')$ and $(Y, \mathscr{U}), (Y, \mathscr{U}')$ where $\mathscr{T}_{X\times Y} \subset \mathscr{T}_{X'\times Y'}$. We wish to show that $\mathscr{T} \subset \mathscr{T}'$ and $\mathscr{U} \subset \mathscr{U}'$. By Munkres Lemma 13.3, it suffices to show that for any $x \in X$ and basis element $B_X \in \mathscr{T}$, there exists a basis element $B_{X'} \in \mathscr{T}'$ such that

$$x \in B_{X'} \subset B_X \tag{376}$$

We construct $B_{X'}$ as such. Given $x \in X$ and basis element $B_X \in \mathscr{T}$, choose any $y \in Y$ to get $(x, y) \in X \times Y$, along with the basis element $B_X \times B_Y \in \mathscr{T}_{X\times Y}$.[a] Since $\mathscr{T}_{X'\times Y'}$ is finer, there exists a basis element $U = U_{X'} \times U_{Y'} \in \mathscr{T}_{X'\times Y'}$, where $U_{X'} \in \mathscr{T}', U_{Y'} \in \mathscr{U}'$, such that

$$(x, y) \in U_{X'} \times U_{Y'} \subset B_X \times B_Y \tag{377}$$

Consider the projection map $\pi_1 : X \times Y \to X$, which we have shown in 16.4 to be open. Therefore, by mapping the three expressions through π_1, we have $x \in U_{X'} \subset B_X$, where $U_{X'} \in \mathscr{T}'$ and $B_X \in \mathscr{T}$. Since open sets are an arbitrary union of basis elements, there exists a basis element $B_{X'} \subset \mathscr{T}'$ satisfying $x \in B_{X'} \subset U_{X'} \subset B_X$, and we are done.

Since we have shown that the projection π_2 is also an open map, we can do the exact same argument by choosing any $x \in X$ and a basis element B_X containing x, giving us $\mathscr{U} \subset \mathscr{U}'$.

[a] Such a basis element B_Y is guaranteed to exist by definition of a basis.

Exercise 10.51 (Munkres 16.6)

Show that the countable collection

$$\{(a, b) \times (c, d) \mid a < b \text{ and } c < d, \text{ and } a, b, c, d \text{ are rational}\}$$

is a basis for $\mathbb{R}^2$.

Solution. Let's denote the collection as $\mathcal{C}$. For every open set U and each $x \in U$, we know that there exists a $r > 0$ such that the L_∞-open ball $B_\infty(x, r) \subset U$, since the set of such open balls forms a basis.

This in $\mathbb{R}^2$ is denoted

$$(x_1 - r, x_1 + r) \times (x_2 - r, x_2 + r) \tag{378}$$

By the denseness of $\mathbb{Q}$ in $\mathbb{R}$, we can choose an $a, b, c, d \in \mathbb{Q}$ such that $x_1 - r < a < 0 < b < x_1 + r$ and $x_2 - r < c < 0 < d < x_2 + r$, which immediately satisfies.

$$x \in (a, b) \times (c, d) \subset (x_1 - r, x_1 + r) \times (x_2 - r, x_2 + r) = B_\infty(x, r) \subset U \tag{379}$$

Therefore, by Munkres Lemma 13.2 $\mathcal{C}$ is a basis for the Euclidean topology.

Exercise 10.52 (Munkres 16.7)

Let X be an ordered set. If Y is a proper subset of X that is convex in X, does it follow that Y is an interval or a ray in X?

Exercise 10.53 (Munkres 16.8)

If L is a straight line in the plane, describe the topology L inherits as a subspace of $\mathbb{R}_\ell \times \mathbb{R}$ and as a subspace of $\mathbb{R}_\ell \times \mathbb{R}_\ell$. In each case it is a familiar topology.

Solution. The basis of the product topology on $\mathbb{R}_\ell \times \mathbb{R}$ are all sets of form $[a, b) \times (c, d)$.

1. If L is vertical, then it is the standard topology.

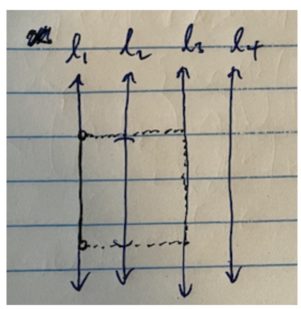


Figure 50: ℓ_1, ℓ_2 intersect the open set at open intervals. ℓ_3, ℓ_4 does not but there are always open sets for which the intersection are open intervals.

2. If L is not vertical, then it is the lower limit topology (or upper limit topology depending on how you parameterize the line).

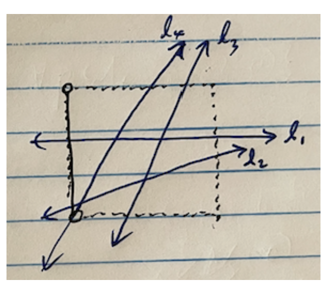


Figure 51: All nonvertical lines will intersect the left "closed" side of some open set and will therefore induce the lower/upper limit topology on the line.

The basis of the product topology on $\mathbb{R}_\ell \times \mathbb{R}_\ell$ are all sets of form $[a, b) \times [c, d)$.

1. if L has a negative slope (as in the graph represented by $y = mx + b$ where $m < 0$), then it is discrete topology since we can imagine the line intersecting the square at the lower-left corner.[a]

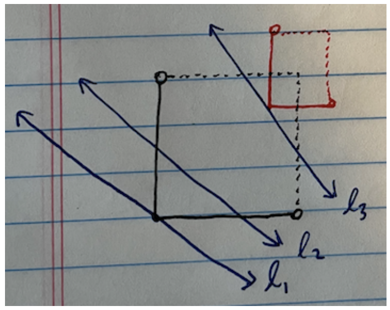


Figure 52: We can construct an open set $[a, b) \times [c, d)$ where the point (a, c) lies on any point on a negatively sloping line. This means that points are open sets, which generates the discrete topology.

2. if L vertical, horizontal, or has a positive slope, then it is the lower limit topology (or upper limit topology depending on how you parameterize the line).

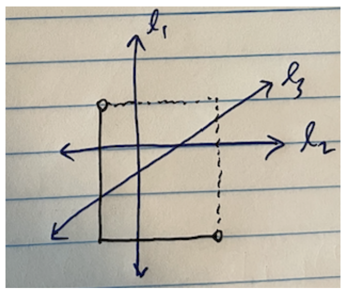


Figure 53: If there is not a negative slope (vertical, horizontal, or positive sloping), then there is no way for a rectangle to intersect the line exactly at the lower-left corner without "going through" the rectangle. Therefore these examples generate half-open half-closed intervals on L.

[a] We can also see it as the topology generated by the basis of closed intervals, but since $[a, b] \cap [b, c] = \{b\}$, this is equivalent to the discrete topology.

Exercise 10.54 (Munkres 16.9)

Show that the dictionary order topology on the set $\mathbb{R} \times \mathbb{R}$ is the same as the product topology $\mathbb{R}_d \times \mathbb{R}$, where $\mathbb{R}_d$ denotes $\mathbb{R}$ in the discrete topology. Compare this topology with the standard topology on $\mathbb{R}^2$.

Exercise 10.55 (Munkres 16.10)

Let $I = [0, 1]$. Compare the product topology on $I \times I$, the dictionary order topology on $I \times I$, and the topology $I \times I$ inherits as a subspace of $\mathbb{R} \times \mathbb{R}$ in the dictionary order topology.

Exercise 10.56 (Math 411 Spring 2025, PS3)

Let P_n denote the set of polynomials in n variables with real coefficients. Any such polynomial defines a function on $\mathbb{R}^n$. If A is any subset of P_n, let $V(A) = \{x \in \mathbb{R}^n \mid p(x) = 0 \text{ for all } p \in A\}$. A subset

$S \subset \mathbb{R}^n$ is called algebraic if it is equal to V(A) for some $A \subset P_n$.

1. Show that $\emptyset$ and $\mathbb{R}^n$ are both algebraic.
2. Show that if A_α are subsets of P_n (indexed by $\alpha \in I$ for some set I), then

$$V\left(\bigcup_{\alpha \in I} A_\alpha\right) = \bigcap_{\alpha \in I} V(A_\alpha).$$

   In other words, any intersection of algebraic sets is algebraic.
3. Suppose $A_1, \ldots, A_k$ are subsets of P_n. Let B be the set of polynomials that can be factored as $f = f_1 \cdots f_k$, where $f_i \in A_i$. Prove that

$$V(B) = V(A_1) \cup \cdots \cup V(A_k).$$

   In other words, any finite union of algebraic sets is algebraic. (Hint: For the inclusion $V(B) \subset V(A_1) \cup \cdots \cup V(A_k)$, it may be easier to show that if $x \notin V(A_1) \cup \cdots \cup V(A_k)$, then $x \notin V(B)$.)
4. Show that $\mathcal{T} = \{U \subset \mathbb{R}^n \mid \mathbb{R}^n - U \text{ is algebraic}\}$ is a topology on $\mathbb{R}^n$. This is known as the Zariski topology, named for the mathematician Oscar Zariski (1899-1986). It is very important in algebraic geometry and related fields.
5. Show that for n = 1, the Zariski topology on $\mathbb{R}^1$ is precisely the finite complement topology.

Note: Instead of doing this with $\mathbb{R}$, you could also do it with $\mathbb{C}$, $\mathbb{Q}$, or any other field.

Solution. Listed.

1. Consider $A = \{f(x) = 1\}$. Then $V(A) = \emptyset$ since f never vanishes.
2. Consider $A = \{f(x) = 0\}$.Then $V(A) = \mathbb{R}^n$ since f always vanishes.

Solution. We see that

$$\begin{aligned}
V\left(\bigcup_{\alpha \in I} A_\alpha\right) &= \big\{x \in \mathbb{R}^n \mid \forall p \in \cup_{\alpha \in I} A_\alpha (p(x) = 0)\big\} && (380)\\
&= \big\{x \in \mathbb{R}^n \mid \forall \alpha \in I \forall p \in A_\alpha (p(x) = 0)\big\} && (381)\\
&= \bigcap_{\alpha \in I} \big\{x \in \mathbb{R}^n \mid \forall p \in A_\alpha (p(x) = 0)\big\} && (382)\\
&= \bigcap_{\alpha \in I} V(A_\alpha) && (383)
\end{aligned}$$

Solution. We prove bidirectionally.

1. $\cup_i V(A_i) \subset V(B)$. Let $x \in \cup_i V(A_i)$. Then $x \in V(A_j)$ for some $1 \leq j \leq k$, which implies that $f_j(x) = 0$ for all $f_j \in A_j$. Therefore by the field properties of $\mathbb{R}$,

$$f(x) = f_1(x) \ldots \underbrace{f_j(x)}_{=0} \ldots f_k(x) = 0 \tag{384}$$

   and therefore $f(x) = 0$ for all $f \in B$, which means that $x \in V(B)$.
2. $\cup_i V(A_i) \supset V(B)$. Assume that $x \notin \cup_i V(A_i)$. Then for all $1 \leq i \leq k$, $x \notin V(A_i)$, which implies that for all i there exists some $f_i^* \in A_i$ s.t. $f_i^*(x) \neq 0$. Now construct the function $f^* = \prod_i f_i^*$, where $f_i^* \in A_i$, $f^* \in B$. But

$$f^*(x) = \prod_{i=1}^k f_i^*(x) \neq 0 \tag{385}$$

   since $f_i^*(x) \neq 0$ for all i, and so we have shown the existence of a function $f^* \in B$ such that $f^*(x) \neq 0$. Therefore $x \notin V(B)$.

Solution. We prove the properties of a topology.

1. From (a), $\emptyset$ is algebraic $\implies \mathbb{R}^n \setminus \emptyset = \mathbb{R}^n$ is in $\mathscr{T}$. Also, $\mathbb{R}^n$ is algebraic $\implies \mathbb{R}^n \setminus \mathbb{R}^n = \emptyset$ is in $\mathscr{T}$.
2. Let $\{U_\alpha\}_{\alpha \in I}$ be a collection of open sets in $\mathscr{T}$. Then by definition $\mathbb{R}^n \setminus U_\alpha$ is algebraic, and
$$\mathbb{R}^n \setminus \left(\bigcup_{\alpha \in I} U_\alpha \right) = \bigcap_{\alpha \in I} (\mathbb{R}^n \setminus U_\alpha) \tag{386}$$
we know from (b) that arbitrary intersections of algebraic sets is algebraic, so the LHS is also algebraic, which by definition means the union is in $\mathscr{T}$.
3. Let $U_1, \ldots, U_k$ be a collection of open sets in $\mathscr{T}$. Then by definition $\mathbb{R}^n \setminus U_i$ is algebraic for $1 \leq i \leq k$, and
$$\mathbb{R}^n \setminus \left(\bigcap_{i=1}^k U_i \right) = \bigcup_{i=1}^k (\mathbb{R}^n \setminus U_i) \tag{387}$$
we know from (c) that finite unions of algebraic sets is algebraic, and so the LHS is also algebraic, which by definition means the finite intersection is in $\mathscr{T}$.

Solution. For an open set U, inclusion in the finite complement topology asserts that $\mathbb{R} \setminus U$ must be finite or $U = \emptyset$, and inclusion in the Zariski topology asserts that $\mathbb{R} \setminus U$ must be algebraic. Therefore it satisfies to show that the complements (within the universe of the power set) are equal, i.e. that the set of all finite subsets of $\mathbb{R}$ plus $\mathbb{R}$ itself and the set of all algebraic subsets of $\mathbb{R}$ is equal. Let us denote the former S and the latter T.

1. $S \subset T$. Let $Y \in S$ (Y is a set). If $Y = \mathbb{R}$, then it is algebraic as shown in (a). Otherwise, it is finite and we can enumerate it as $Y = \{y_1, \ldots, y_n\}$, and define the singleton subset of polynomials
$$A = \left\{ f(x) = \prod_{i=1}^n (x - y_i) \right\} \tag{388}$$
$V(A)$ consists of all reals where $f(x) = 0$, which happens exactly when $x = y_i$ for some i.[a] Therefore, $Y = V(A) \implies Y$ is algebraic $\implies Y \subset T$.
2. $T \subset S$. Let $Y \in T$. Then we know that there exists some subset of polynomials A such that $Y = V(A)$. If A is empty, then $V(A) = \mathbb{R}$ since the predicate in the set-builder notation is vacuously true, and $\mathbb{R}$ is contained in the S. If A is not empty, then there exists some polynomial f from A. By definition it must be the case that for all $y \in Y$, $f(y) = 0$. We consider two cases.
   (a) f is constant. If $f \neq 0$, then $V(A) = \emptyset$, which is in S. If $f = 0$, then $V(A) = \mathbb{R}$, which is in S.
   (b) f has degree $n \geq 1$. Since this is a polynomial ring over a field $\mathbb{F}[x]$, $f(y)$ cannot have more than n real roots.[b] Therefore $|Y| \leq n$ and Y must be finite.

   We have shown that $V(\{f\})$ is finite. Since $V(A)$ consists of y's that hold for all $f \in A$, $V(A) \subset V(\{f\})$.[c] The subset of finite sets is finite, and therefore $Y = V(A) \in S$.

[a] If not, then every $(x - y_i)$ is nonzero, and the product is nonzero.

[b] Using the single factor theorem of commutative rings we can use induction to prove that the set of roots cannot go beyond n. For linear polynomials $f(x) = mx + b$ there is one root $x = -b/m$ which satisfies the base case, and for higher degrees of n we show that if there is a root where $p(a) = 0$, then $p(x) = (x - a)q(x)$ where q is of degree $n - 1$ which cannot have more than $n - 1$ factors.

[c] To see why, look at my solution for (b).

Exercise 10.57 (Munkres 19.4)

Show that $(X_1 \times \cdots \times X_{n-1}) \times X_n$ is homeomorphic with $X_1 \times \cdots \times X_n$.

Exercise 10.58 (Munkres 19.5)

One of the implications stated in Theorem 19.6 holds for the box topology. Which one?

Exercise 10.59 (Munkres 19.6)

Let $\mathbf{x}_1, \mathbf{x}_2, \ldots$ be a sequence of the points of the product space $\prod X_\alpha$. Show that this sequence converges to the point $\mathbf{x}$ if and only if the sequence $\pi_\alpha(\mathbf{x}_1), \pi_\alpha(\mathbf{x}_2), \ldots$ converges to $\pi_\alpha(\mathbf{x})$ for each α. Is this fact true if one uses the box topology instead of the product topology?

Exercise 10.60 (Munkres 19.7)

Let $\mathbb{R}^\infty$ be the subset of $\mathbb{R}^\omega$ consisting of all sequences that are "eventually zero," that is, all sequences $(x_1, x_2, \ldots)$ such that $x_i \neq 0$ for only finitely many values of i. What is the closure of $\mathbb{R}^\infty$ in $\mathbb{R}^\omega$ in the box and product topologies? Justify your answer.

Solution. In the box topology, $\overline{\mathbb{R}^\infty} = \mathbb{R}^\infty$. We show this by taking any sequence not in $\mathbb{R}^\infty$ and showing that there exists an open neighborhood that has a trivial intersection with $\mathbb{R}^\infty$. Consider a non-eventually zero sequence $y \in \mathbb{R}^\omega \setminus \mathbb{R}^\infty$, which must have an infinite number of nonzero terms. y is contained within the open set (in the box topology)

$$U = U_1 \times U_2 \times \ldots, \qquad U_i = \begin{cases} (0, +\infty) & \text{if } y_i > 0 \\ (-1, 1) & \text{if } y_i = 0 \\ (-\infty, 0) & \text{if } y_i < 0 \end{cases} \tag{389}$$

This is an open set that clearly contains y, but it has an empty intersection with $\mathbb{R}^\omega$ since there are an infinite number of nonzero terms y_i and so there are an infinite number of U_i's that do not contain 0. In the product topology, basis elements are all sets of the form $\prod U_i$ where U_i is open in $\mathbb{R}$ and $U_i = \mathbb{R}$ except for finitely many values of i. Therefore, $U_i \neq \mathbb{R}$ for finite values. Now given any sequence $y \in \mathbb{R}^\omega$, we claim that it is a limit point of $\mathbb{R}^\infty$. An open neighborhood of y of form $U_y = \prod U_i$ must have some maximum index N for which $U_i = \mathbb{R}$ when $i > N$, and so the eventually-zero sequence $x = (y_1, y_2, \ldots, y_N, 0, 0, \ldots)$ must be in U, implying that $U_y \cap \mathbb{R}^\infty \neq \emptyset$, and so y is a limit point of U_y. Therefore all sequences are limit points, which means $\overline{\mathbb{R}^\infty} = \mathbb{R}^\omega$.

Exercise 10.61 (Munkres 19.8)

Given sequences $(a_1, a_2, \ldots)$ and $(b_1, b_2, \ldots)$ of real numbers with $a_i > 0$ for all i, define $h : \mathbb{R}^\omega \to \mathbb{R}^\omega$ by the equation

$$h((x_1, x_2, \ldots)) = (a_1 x_1 + b_1, a_2 x_2 + b_2, \ldots). \tag{390}$$

Show that if $\mathbb{R}^\omega$ is given the product topology, h is a homeomorphism of $\mathbb{R}^\omega$ with itself. What happens if $\mathbb{R}^\omega$ is given the box topology?

Solution. We first show h is a bijection. Indeed, the element-wise mappings are bijections and the inverse is

$$h^{-1}((x_1, \ldots,)) = \left(\ldots, \frac{x_i - b_i}{a_i}, \ldots \right) \tag{391}$$

Now we show that it is continuous. Given an open set in $\mathbb{R}^\omega$ in the product topology, it has the form

$$U = (l_1, u_1) \times (l_2, u_2) \times \ldots \times (l_n, u_n) \times \mathbb{R} \times \ldots \tag{392}$$

The preimage under h is

$$h^{-1}(U) = \left(\frac{l_1 - b_1}{a_1}, \frac{u_1 - b_1}{a_1}\right) \times \ldots \times \left(\frac{l_n - b_n}{a_n}, \frac{u_n - b_n}{a_n}\right) \times \mathbb{R} \times \ldots \tag{393}$$

where $l_1 < u_1 \implies (l_1 - b_1)/a_1 < (u_1 - b_1)/a_1$. This is also of the form of open sets in the product topology, and therefore is continuous. Now considering the continuity of the inverse function, we can see that taking the same U as before, we have

$$h(U) = (a_1 l_1 + b_1, a_1 u_1 + b_1) \times \ldots \times (a_n l_n + b_n, a_n u_n + b_n) \times \mathbb{R} \ldots \tag{394}$$

which is also open in the product topology. Therefore h is a homeomorphism under the product topologies.
Now we consider the box topology. Given an box-topology open set of the form

$$U = (l_1, u_1) \times (l_2, u_2) \times \ldots \tag{395}$$

then the preimage and image are

$$h^{-1}(U) = \left(\frac{l_1 - b_1}{a_1}, \frac{u_1 - b_1}{a_1}\right) \times \left(\frac{l_2 - b_2}{a_2}, \frac{u_2 - b_2}{a_2}\right) \times \ldots \tag{396}$$

$$h(U) = h(U) = (a_1 l_1 + b_1, a_1 u_1 + b_1) \times (a_2 l_2 + b_2, a_2 u_2 + b_2) \times \tag{397}$$

which are both open in the box topology and so h and h^{-1} are continuous. Thus h is also a homeomorphism.

Exercise 10.62 (Munkres 19.9)

Show that the choice axiom is equivalent to the statement that for any indexed family $\{A_\alpha\}_{\alpha \in J}$ of nonempty sets, with $J \neq \emptyset$, the cartesian product

$$\prod_{\alpha \in J} A_\alpha$$

is not empty.

Exercise 10.63 (Munkres 19.10)

Let A be a set; let $\{X_\alpha\}_{\alpha \in J}$ be an indexed family of spaces; and let $\{f_\alpha\}_{\alpha \in J}$ be an indexed family of functions $f_\alpha : A \to X_\alpha$.

1. Show there is a unique coarsest topology $\mathcal{T}$ on A relative to which each of the functions f_α is continuous.
2. Let
$$S_\beta = \{f_\beta^{-1}(U_\beta) \mid U_\beta \text{ is open in } X_\beta\},$$
and let $S = \bigcup S_\beta$. Show that S is a subbasis for $\mathcal{T}$.
3. Show that a map $g : Y \to A$ is continuous relative to $\mathcal{T}$ if and only if each map $f_\alpha \circ g$ is continuous.
4. Let $f : A \to \prod X_\alpha$ be defined by the equation
$$f(a) = (f_\alpha(a))_{\alpha \in J};$$
let Z denote the subspace $f(A)$ of the product space $\prod X_\alpha$. Show that the image under f of each element of $\mathcal{T}$ is an open set of Z.

Exercise 10.64 (Munkres 22.2)

(a) Let $p : X \to Y$ be a continuous map. Show that if there is a continuous map $f : Y \to X$ such that $p \circ f$ equals the identity map of Y, then p is a quotient map.
(b) If $A \subset X$, a *retraction* of X onto A is a continuous map $r : X \to A$ such that $r(a) = a$ for each $a \in A$. Show that a retraction is a quotient map.

Solution. Listed.

1. Let $U \subset Y$, and let $p^{-1}(U)$ be open. Then,

$$\begin{aligned} p^{-1}(U) \subset X \text{ open } &\implies f^{-1}(p^{-1}(U)) \subset Y \text{ open} && (398) \\ &\implies (p \circ f)^{-1}(U) = U \subset Y \text{ open} && (399) \end{aligned}$$

and since p is continuous, p is a quotient map. Since $p \circ f$ equals the identity map, it must be the case that p is surjective.

2. We know that the canonical injection $\iota : A \to X$ is continuous, and $r \circ \iota = I$, the identity map of A. Therefore by (a), p is a quotient map.

Exercise 10.65 (Munkres 22.3)

Let $\pi_1 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be projection on the first coordinate. Let A be the subspace of $\mathbb{R} \times \mathbb{R}$ consisting of all points $x \times y$ for which either $x \geq 0$ or $y = 0$ (or both); let $q : A \to \mathbb{R}$ be obtained by restricting π_1. Show that q is a quotient map that is neither open nor closed.

Solution. The restriction of a continuous function is always continuous, so q is continuous. Furthermore q is surjective since given any $x \in \mathbb{R}$, we can always see that $(x, 0) \in q^{-1}(\{x\})$. Finally, let $U \subset \mathbb{R}$ s.t. $q^{-1}(U) \subset A$ is open. Then, for any $x \in U$, we know $(x, 0) \subset q^{-1}(U)$, and so $\exists \epsilon > 0$ s.t. $B((x, 0), \epsilon) \cap A) \subset q^{-1}(U)$. Mapping both sides through q again gives

$$(x - \epsilon, x + \epsilon) = q\big(B((x, 0), \epsilon) \cap A)\big) \subset q(q^{-1}(U)) = U \tag{400}$$

and so U is open. Therefore, q is a quotient map. To see why it is not open, consider the open set of A

$$U = [0, 1) \times (0, 1) = [(-1, 1) \times (0, 1)] \cap A \tag{401}$$

Then $q(U) = [0, 1)$ which is not open in $\mathbb{R}$. To see why not closed, consider the closed set $C = \{(x, y) \in \mathbb{R}^2 \mid xy = 1, x > 0\}$.[a] Then $p(C) = (0, +\infty)$ which is not closed in $\mathbb{R}$.

[a] Also stated to be closed by example 2 of chapter 22.

Exercise 10.66 (Munkres 22.4)

(a) Define an equivalence relation on the plane $X = \mathbb{R}^2$ as follows:
$$x_0 \times y_0 \sim x_1 \times y_1 \quad \text{if } x_0 + y_0^2 = x_1 + y_1^2.$$
Let X^* be the corresponding quotient space. It is homeomorphic to a familiar space; what is it?
(b) Repeat (a) for the equivalence relation
$$x_0 \times y_0 \sim x_1 \times y_1 \quad \text{if } x_0^2 + y_0^2 = x_1^2 + y_1^2.$$

Solution. Listed.

1. Graphing this shows that X^* is the set of all horizontal parabolas of the same scale and opening leftwards with the vertex on the x-axis, i.e. the elements are just left-right shifts of one another.

We claim that it is homeomorphic to $\mathbb{R}$. Let $p : \mathbb{R}^2 \to X^*$ be the quotient map and $f : \mathbb{R}^2 \to \mathbb{R}$ be defined $f(x, y) = x + y^2$, which satisfies $(x, y) \sim (x', y') \iff f(x, y) = f(x', y')$. Therefore, this induces a function $\bar{f} : X^* \to \mathbb{R}$ s.t. $\bar{f} \circ p = f$. Since f is continuous, $\bar{f}$ is continuous. Also, we can see that $\bar{f}$ is a bijection since we can map every element in the equivalence class of $[(x, y)]$ s.t. $x + y^2 = c$ to $c \in \mathbb{R}$. Finally, we show that $\bar{f}^{-1}$ is continuous. We can think of $\bar{f}^{-1}$ as the composition of maps $c \mapsto (c, 0) \mapsto [(c, 0)]$, where the first map is trivially continuous and the second map is p, which is also continuous. Therefore $\bar{f}^{-1}$ is continuous, and it is a homeomorphism.

2. We can see that X^* is the set of all circles of nonnegative radius (the origin is a circle of radius 0), and we claim that it is homeomorphic to $[0, +\infty)$. Let $p : \mathbb{R}^2 \to X^*$ be the quotient map and $f : \mathbb{R}^2 \to [0, +\infty)$ be defined $f(x, y) = x^2 + y^2$, which satisfies $(x, y) \sim (x', y') \iff f(x, y) = f(x', y')$. Therefore, this induces a function $\bar{f} : X^* \to [0, +\infty)$ s.t. $\bar{f} \circ p = f$. Since f is continuous, $\bar{f}$ is continuous. Also, we can see that $\bar{f}$ is a bijection since we can map every element in the equivalence class of $[(x, y)]$ s.t. $x^2 + y^2 = c$ to $c \in [0, +\infty)$. Finally, we show that $\bar{f}^{-1}$ is continuous. We can think of $\bar{f}^{-1}$ as the composition of maps $c \mapsto (\sqrt{c}, 0) \mapsto [(\sqrt{c}, 0)]$, where the first map is continuous[a] and the second map is p, which is also continuous. Therefore $\bar{f}^{-1}$ is continuous, and it is a homeomorphism.

[a] The square root function from $\mathbb{R}_0^+$ to itself is continuous since the preimage of an open interval (a, b) is $(\sqrt{a}, \sqrt{b})$ which is also open in $\mathbb{R}_0^+$, and for $[0, b)$ the preimage is $[0, \sqrt{b})$.

Exercise 10.67 (Munkres 22.5)

Let $p : X \to Y$ be an open map. Show that if A is open in X, then the map $q : A \to p(A)$ obtained by restricting p is an open map.

Exercise 10.68 (Munkres 22.6)

Recall that $\mathbb{R}_K$ denotes the real line in the K-topology. (See §13.) Let Y be the quotient space obtained from $\mathbb{R}_K$ by collapsing the set K to a point; let $p : \mathbb{R}_K \to Y$ be the quotient map.

(a) Show that Y satisfies the T_1 axiom, but is not Hausdorff.

(b) Show that $p \times p : \mathbb{R}_K \times \mathbb{R}_K \to Y \times Y$ is not a quotient map. [Hint: The diagonal is not closed in $Y \times Y$, but its inverse image is closed in $\mathbb{R}_K \times \mathbb{R}_K$.]

Exercise 10.69 (Math 411 Spring 2025, PS6)

For each integer $n \geq 1$, let us consider the following spaces:

$$S^n = \{\mathbf{x} \in \mathbb{R}^{n+1} \mid \|\mathbf{x}\| = 1\}$$
$$X^{n+1} = \mathbb{R}^{n+1} - \{\mathbf{0}\}$$

(Note: the superscript is just a decoration that indicates the "dimensionality" of the space; it does *not* indicate raising something to a power.)

(1) Show that the function $r : X^{n+1} \to S^n$ defined by $r(\mathbf{x}) = \mathbf{x}/\|\mathbf{x}\|$ is a quotient map, and describe the corresponding equivalence relation. (Hint: Use #2 from §22.)

(2) We now define a space called *real projective n-space*, or $\mathbb{RP}^n$. We can define it in either of two ways:

  (a) $X^{n+1}/\sim$, where $\mathbf{x} \sim \mathbf{y}$ iff $\mathbf{x} = \lambda \mathbf{y}$ for some $\lambda \in \mathbb{R} - \{0\}$.

  (b) $S^n/\sim$, where $\mathbf{x} \sim \mathbf{y}$ iff $\mathbf{x} = \pm\mathbf{y}$.

Let $p : X^{n+1} \to \mathbb{RP}^n$ be the quotient map. Prove that $p|_{S^n} : S^n \to \mathbb{RP}^n$ is also a quotient map, and hence that the two descriptions above produce homeomorphic spaces. (Hint: It may help to observe that $p|_{S^n} \circ r = p$.)

(3) Using the description of $\mathbb{RP}^n$ as a quotient of S^n, prove that $\mathbb{RP}^n$ is a Hausdorff space. (Hint: note that for $\mathbf{x} \in S^n$ and $\epsilon > 0$, the set $(B(\mathbf{x}, \epsilon) \cup B(-\mathbf{x}, \epsilon)) \cap S^n$ is open and saturated.)

(4) Let D^n_+ denote the "upper hemisphere" in S^n: $D^n_+ = \{\mathbf{x} \in S^n \mid x_{n+1} \geq 0\}$. Show that D^n_+ is homeomorphic to $D^n = \{\mathbf{x} \in \mathbb{R}^n \mid \|\mathbf{x}\| \leq 1\}$, the unit disk in $\mathbb{R}^n$.

(5) The restriction of p to D^n, viewed as a map $D^n \to \mathbb{RP}^n$, can be shown to be a quotient map. Assuming this, describe an equivalence relation on D^n whose quotient space is homeomorphic to $\mathbb{RP}^n$. In the case where $n = 2$, describe how we may concretely describe $\mathbb{RP}^2$ as a "video game."

Solution. Listed.

1. We claim r is a retraction, since $r(x) = x$ for all $x \in S^n \subset \mathbb{R}^{n+1}$. r is also continuous since a basis element of the topology of S^n can be written $B = \big\{(a_1, b_1) \times \ldots \times (a_{n+1}, b_{n+1})\big\} \cap S^n$. Therefore the preimage of such a set is

$$r^{-1}(B) = \bigcup_{c>0} \big\{(ca_1, cb_1) \times \ldots \times (ca_{n+1}, cb_{n+1})\big\} \cap S^n \tag{402}$$

which is the union of open sets in S^n and therefore is open in S^n. From the first exercise, r is a quotient map. It consists of each ray starting from the origin, and maps each direction to the unit vector, creating a sphere.

2. $p|_{S^n} \circ r = p$. Since p is a quotient map, it is continuous and its restriction $p|_{S^n}$ is also continuous. Since p is a quotient map, it is surjective and therefore $p|_{S^n}$ must be surjective.[a] Now let $U \subset \mathbb{RP}^n$ s.t. $p|_{S^n}^{-1}(U)$ is open in S^n. Then,

$$r^{-1}\big(p|_{S^n}^{-1}(U)\big) = (p|_{S^n} \circ r)^{-1}(U) = p^{-1}(U) \tag{403}$$

is open since r is continuous. But since p is a quotient map, U is open, implying that $p|_{S^n}$ is a quotient map. Therefore, we have shown that

$$X^{n+1}/\sim \cong \mathbb{RP}^n \text{ and } S^n/\sim \cong \mathbb{RP}^n \implies X^{n+1}/\sim \cong S^n/\sim \tag{404}$$

3. Let us have $x, y \in \mathbb{RP}^n$ with $x \neq y$. Then there exists $u, v \in S^n$ s.t. $p|_{S^n}^{-1}(x) = \{u, -u\}$ and $p|_{S^n}^{-1}(y) = \{v, -v\}$. We can let $\epsilon = \frac{1}{2}\min\{\|u - v\|, \|u + v\|\}$, which means that

$$B(\pm u, \epsilon) \cap B(\pm v, \epsilon) = \emptyset \tag{405}$$

in $\mathbb{R}^n$, and so the open neighborhoods around $\pm u, \pm v$, denoted $U_\pm, V_\pm$ are all pairwise disjoint in S^n. Note that $U_+ \cup U_-$ and $V_+ \cup V_-$ are open, saturated, and disjoint, and so $p|_{S^n}$, as a quotient map, maps both of these sets to disjoint open neighborhoods of x, y, proving that $\mathbb{RP}^n$ is Hausdorff.

4. We can visualize this as if we were just "pushing up" the disk onto the hemisphere. More formally, let us define $f : D^n \to D^n_+$ as

$$f(x) = f(x_1, \ldots, x_n) = (x_1, \ldots, x_n, \sqrt{1 - \|x\|}) \tag{406}$$

This is well-defined since we assume that $\|x\| \leq 1$. This is trivially injective since changing any of the x_i's will result in a different values in the first n terms of the output. It is surjective since given any element $y = (y_1, \ldots, y_{n+1}) \in D_+$, we know that $\|y\| = 1$ and so we can find the preimage to be $x = (y_1, \ldots, y_n)$ which will satisfy $\|x\| \leq 1$. Therefore, f is bijective, with the inverse function

$$f^{-1}(y) = f^{-1}(y_1, \ldots, y_{n+1}) = (y_1, \ldots, y_n) \tag{407}$$

To prove continuity of f, consider the basis element of the form

$$(a_1, b_1) \times \ldots \times (a_{n+1}, b_{n+1}) \cap D^n_+ \tag{408}$$

the preimage is

$$f^{-1}\big((a_1, b_1) \times \ldots \times (a_{n+1}, b_{n+1})\big) \cap f^{-1}(D^n_+) = (a_1, b_1) \times (a_n, b_n) \cap D^n \tag{409}$$

which is a basis element of D^n. To prove continuity of f^{-1}, we consider its extension $f^{-1} : \mathbb{R}^{n+1} \to \mathbb{R}^n$, which is simply a projection of the first n elements and is therefore continuous. The restriction of this function $f^{-1} : D^n_+ \to \mathbb{R}^n$ is therefore continuous, and since the image of f^{-1} only hits D^n, every open set in D^n is of the form $U = V \cap D^n$ for V open in $\mathbb{R}^n$, where the preimage will simply be $f(U) = f(V) \cap f(D^n)$. We know $f(V)$ is open[b] and $f(D^n) = D^n_+$, $f(U)$ is open in D^n_+.

5. In D^n, we define the equivalence relation $\sim$ where $x \sim -x$ if $\|x\| = 1$, and every other point (in the interior of the disk) is equivalent to itself only. When $n = 2$, we can imagine a player walking in a circular disk in $\mathbb{R}^2$, and when it crosses the boundary it will end up at the antipodal point.

[a] If $p|_{S^n}$ wasn't surjective, then $p|_{S^n} \circ r$ is not surjective and p is not surjective, a contradiction.
[b] by continuity of f^{-1} with codomain $\mathbb{R}^n$

10.5 Metric Topologies

Exercise 10.70 (Munkres 20.1)

(a) In $\mathbb{R}^n$, define

$$d'(\mathbf{x}, \mathbf{y}) = |x_1 - y_1| + \cdots + |x_n - y_n|.$$

Show that d' is a metric that induces the usual topology of $\mathbb{R}^n$. Sketch the basis elements under d' when $n = 2$.

(b) More generally, given $p \geq 1$, define

$$d'(\mathbf{x}, \mathbf{y}) = \left[\sum_{i=1}^n |x_i - y_i|^p\right]^{1/p}$$

for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Assume that d' is a metric. Show that it induces the usual topology on $\mathbb{R}^n$.

Exercise 10.71 (Munkres 20.2)

Show that $\mathbb{R} \times \mathbb{R}$ in the dictionary order topology is metrizable.

Exercise 10.72 (Munkres 20.3)

(a) Let X be a metric space with metric d. Show that $d : X \times X \to \mathbb{R}$ is continuous.

(b) Let X' denote a space having the same underlying set as X. Show that if $d : X' \times X' \to \mathbb{R}$ is continuous, then the topology of X' is finer than the topology of X.

One can summarize the result of this exercise as follows: If X has a metric d, then the topology induced by d is the coarsest topology relative to which the function d is continuous.

Exercise 10.73 (Munkres 20.4)

Consider the product, uniform, and box topologies on $\mathbb{R}^\omega$.

(a) In which topologies are the following functions from $\mathbb{R}$ to $\mathbb{R}^\omega$ continuous?

$$\begin{aligned} f(t) &= (t, 2t, 3t, \ldots), \\ g(t) &= (t, t, t, \ldots), \\ h(t) &= (t, \frac{1}{2}t, \frac{1}{3}t, \ldots). \end{aligned}$$

(b) In which topologies do the following sequences converge?

$$\begin{aligned} &\mathbf{w}_1 = (1, 1, 1, 1, \ldots), && \mathbf{x}_1 = (1, 1, 1, 1, \ldots), \\ &\mathbf{w}_2 = (0, 2, 2, 2, \ldots), && \mathbf{x}_2 = (0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \ldots), \\ &\mathbf{w}_3 = (0, 0, 3, 3, \ldots), && \mathbf{x}_3 = (0, 0, \frac{1}{3}, \frac{1}{3} \ldots), \\ &\ldots && \ldots \\ &\mathbf{y}_1 = (1, 0, 0, 0, \ldots), && \mathbf{z}_1 = (1, 1, 0, 0, \ldots), \\ &\mathbf{y}_2 = (\frac{1}{2}, \frac{1}{2}, 0, 0, \ldots), && \mathbf{z}_2 = (\frac{1}{2}, \frac{1}{2}, 0, 0, \ldots), \\ &\mathbf{y}_3 = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0, \ldots), && \mathbf{z}_3 = (\frac{1}{3}, \frac{1}{3}, 0, 0, \ldots), \\ &\ldots && \ldots \end{aligned}$$

Exercise 10.74 (Munkres 20.5)

Let $\mathbb{R}^\infty$ be the subset of $\mathbb{R}^\omega$ consisting of all sequences that are eventually zero. What is the closure of $\mathbb{R}^\infty$ in $\mathbb{R}^\omega$ in the uniform topology? Justify your answer.

Exercise 10.75 (Munkres 20.6)

Let $\bar{\rho}$ be the uniform metric on $\mathbb{R}^\omega$. Given $\mathbf{x} = (x_1, x_2, \ldots) \in \mathbb{R}^\omega$ and given $0 < \epsilon < 1$, let

$$U(\mathbf{x}, \epsilon) = (x_1 - \epsilon, x_1 + \epsilon) \times \cdots \times (x_n - \epsilon, x_n + \epsilon) \times \cdots.$$

(a) Show that $U(\mathbf{x}, \epsilon)$ is not equal to the ϵ-ball $B_{\bar{\rho}}(\mathbf{x}, \epsilon)$.
(b) Show that $U(\mathbf{x}, \epsilon)$ is not even open in the uniform topology.
(c) Show that

$$B_{\bar{\rho}}(\mathbf{x}, \epsilon) = \bigcup_{\delta < \epsilon} U(\mathbf{x}, \delta).$$

Exercise 10.76 (Munkres 20.7)

Consider the map $h : \mathbb{R}^\omega \to \mathbb{R}^\omega$ defined in Exercise 8 of §19; give $\mathbb{R}^\omega$ the uniform topology. Under what conditions on the numbers a_i and b_i is h continuous? a homeomorphism?

Exercise 10.77 (Munkres 20.8)

Let X be the subset of $\mathbb{R}^\omega$ consisting of all sequences $\mathbf{x}$ such that $\sum x_i^2$ converges. Then the formula

$$d(\mathbf{x}, \mathbf{y}) = \left[\sum_{i=1}^{\infty} (x_i - y_i)^2\right]^{1/2}$$

defines a metric on X. (See Exercise 10.) On X we have the three topologies it inherits from the box, uniform, and product topologies on $\mathbb{R}^\omega$. We have also the topology given by the metric d, which we call the ℓ^2-topology. (Read "little ell two.")

(a) Show that on X, we have the inclusions

$$\text{box topology} \supset \ell^2\text{-topology} \supset \text{uniform topology}.$$

(b) The set $\mathbb{R}^\infty$ of all sequences that are eventually zero is contained in X. Show that the four topologies that $\mathbb{R}^\infty$ inherits as a subspace of X are all distinct.

(c) The set

$$H = \prod_{n \in \mathbb{Z}_+} [0, 1/n]$$

is contained in X; it is called the Hilbert cube. Compare the four topologies that H inherits as a subspace of X.

Exercise 10.78 (Munkres 20.9)

Show that the euclidean metric d on $\mathbb{R}^n$ is a metric, as follows: If $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and $c \in \mathbb{R}$, define

$$\begin{aligned} \mathbf{x} + \mathbf{y} &= (x_1 + y_1, \ldots, x_n + y_n), \\ c\mathbf{x} &= (cx_1, \ldots, cx_n), \\ \mathbf{x} \cdot \mathbf{y} &= x_1 y_1 + \cdots + x_n y_n. \end{aligned}$$

(a) Show that $\mathbf{x} \cdot (\mathbf{y} + \mathbf{z}) = (\mathbf{x} \cdot \mathbf{y}) + (\mathbf{x} \cdot \mathbf{z})$.

(b) Show that $|\mathbf{x} \cdot \mathbf{y}| \leq \|\mathbf{x}\| \|\mathbf{y}\|$. [Hint: If $\mathbf{x}, \mathbf{y} \neq 0$, let $a = 1/\|\mathbf{x}\|$ and $b = 1/\|\mathbf{y}\|$, and use the fact that $\|a\mathbf{x} \pm b\mathbf{y}\| \geq 0$.]

(c) Show that $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$. [Hint: Compute $(\mathbf{x} + \mathbf{y}) \cdot (\mathbf{x} + \mathbf{y})$ and apply (b).]

(d) Verify that d is a metric.

Exercise 10.79 (Munkres 20.10)

Let X denote the subset of $\mathbb{R}^\omega$ consisting of all sequences $(x_1, x_2, \ldots)$ such that $\sum x_i^2$ converges. (You may assume the standard facts about infinite series. In case they are not familiar to you, we shall give them in Exercise 11 of the next section.)

(a) Show that if $\mathbf{x}, \mathbf{y} \in X$, then $\sum |x_i y_i|$ converges. [Hint: Use (b) of Exercise 9 to show that the partial sums are bounded.]

(b) Let $c \in \mathbb{R}$. Show that if $\mathbf{x}, \mathbf{y} \in X$, then so are $\mathbf{x} + \mathbf{y}$ and $c\mathbf{x}$.

(c) Show that

$$d(\mathbf{x}, \mathbf{y}) = \left[\sum_{i=1}^{\infty} (x_i - y_i)^2\right]^{1/2}$$

is a well-defined metric on X.

Exercise 10.80 (Munkres 20.11)

Show that if d is a metric for X, then

$$d'(x, y) = d(x, y)/(1 + d(x, y))$$

is a bounded metric that gives the topology of X. [Hint: If $f(x) = x/(1 + x)$ for $x > 0$, use the mean-value theorem to show that $f(a + b) - f(b) \leq f(a)$.]

Exercise 10.81 (Munkres 21.1)

Let $A \subset X$. If d is a metric for the topology of X, show that $d|A \times A$ is a metric for the subspace topology on A.

Exercise 10.82 (Munkres 21.2)

Let X and Y be metric spaces with metrics d_X and d_Y, respectively. Let $f : X \to Y$ have the property that for every pair of points x_1, x_2 of X,

$$d_Y(f(x_1), f(x_2)) = d_X(x_1, x_2).$$

Show that f is an imbedding. It is called an *isometric imbedding* of X in Y.

Exercise 10.83 (Munkres 21.3)

Let X_n be a metric space with metric d_n, for $n \in \mathbb{Z}_+$.

1. Show that
$$\rho(x, y) = \max\{d_1(x_1, y_1), \ldots, d_n(x_n, y_n)\} \tag{410}$$
is a metric for the product space $X_1 \times \cdots \times X_n$.
2. Let $\tilde{d}_i = \min\{d_i, 1\}$. Show that
$$D(x, y) = \sup\{\tilde{d}_i(x_i, y_i)/i\} \tag{411}$$
is a metric for the product space $\prod X_i$.

Solution. For the first part, we prove the properties of the metric.

1. Nonnegativity. Note that ρ is the maximum of a finite set of metrics, which must be nonnegative, and so $\rho(x, y) \geq 0$. Second,
$$\begin{aligned} \rho(x, y) = 0 &\iff \max\{d_i(x_i, y_i)\} = 0 && (412)\\ &\iff d_i(x_i, y_i) = 0 \text{ for all } i = 1, \ldots, n && (413)\\ &\iff x_i = y_i \text{ for all } i && (414)\\ &\iff x = y && (415) \end{aligned}$$
2. Symmetricity.
$$\rho(x, y) = \max\{d_i(x_i, y_i)\} = \max\{d_i(y_i, x_i)\} = \rho(y, x) \tag{416}$$

3. Triangle inequality.

$$\begin{aligned}
\rho(x, y) + \rho(y, z) &= \max_i \{d_i(x_i, y_i)\} + \max_j \{d_j(y_j, z_j)\} && (417)\\
&\geq \max_i \{d_i(x_i, y_i) + d_i(y_i, z_i)\} && (418)\\
&\geq \max_i \{d_i(x_i, z_i)\} && (419)\\
&= \rho(x, z) && (420)
\end{aligned}$$

For the second part, we do the same.

1. Nonnegativity. Since $\tilde{d}_i \geq 0$, $\tilde{d}_i/i \geq 0/i = 0$ and so the supremum must be at least 0 (if it's negative then it will not bound d_1.) Second,

$$\begin{aligned}
D(x, y) = 0 &\iff \sup\{\tilde{d}_i(x_i, y_i)/i\} = 0 && (421)\\
&\iff \tilde{d}_i(x_i, y_i)/i = 0 \text{ for all } i && (422)\\
&\iff \tilde{d}_i(x_i, y_i) = 0 \text{ for all } i && (423)\\
&\iff \min\{d_i(x_i, y_i), 1\} = 0 \text{ for all } i && (424)\\
&\iff d_i(x_i, y_i) = 0 \text{ for all } i && (425)\\
&\iff x_i = y_i \text{ for all } i && (426)\\
&\iff x = y && (427)
\end{aligned}$$

2. Symmetricity.

$$D(x, y) = \sup\{\tilde{d}_i(x_i, y_i)/i\} = \sup\{\tilde{d}_i(y_i, x_i)/i\} = D(y, x) \tag{428}$$

3. Triangle inequality.

$$\begin{aligned}
D(x, y) + D(y, z) &= \sup_i \{\tilde{d}_i(x_i, y_i)/i\} + \sup_j \{\tilde{d}_j(y_j, z_j)/j\} && (429)\\
&\geq \sup_i \left\{\frac{\tilde{d}_i(x_i, y_i) + \tilde{d}_i(y_i, z_i)}{i}\right\} && (430)\\
&= \sup_i \left\{\frac{\min\{d_i(x_i, y_i), 1\} + \min\{d_i(y_i, z_i), 1\}}{i}\right\} && (431)\\
&\geq \sup_i \left\{\frac{\min\{d_i(x_i, y_i) + d_i(y_i, z_i), 1\}}{i}\right\} && (432)\\
&\geq \sup_i \left\{\frac{\min\{d_i(x_i, z_i), 1\}}{i}\right\} && (433)\\
&= D(x, z) && (434)
\end{aligned}$$

Exercise 10.84 (Munkres 21.4)

Show that $\mathbb{R}_\ell$ and the ordered square satisfy the first countability axiom. (This result does not, of course, imply that they are metrizable.)

Exercise 10.85 (Munkres 21.5)

Theorem. Let $x_n \to x$ and $y_n \to y$ in the space $\mathbb{R}$. Then

$$\begin{aligned} x_n + y_n &\to x + y, \\ x_n - y_n &\to x - y, \\ x_n y_n &\to xy, \end{aligned}$$

and provided that each $y_n \neq 0$ and $y \neq 0$,

$$x_n / y_n \to x/y.$$

[Hint: Apply Lemma 21.4; recall from the exercises of §19 that if $x_n \to x$ and $y_n \to y$, then $x_n \times y_n \to x \times y$.]

Exercise 10.86 (Munkres 21.6)

Define $f_n : [0, 1] \to \mathbb{R}$ by the equation $f_n(x) = x^n$. Show that the sequence $(f_n(x))$ converges for each $x \in [0, 1]$, but that the sequence (f_n) does not converge uniformly.

Exercise 10.87 (Munkres 21.7)

Let X be a set, and let $f_n : X \to \mathbb{R}$ be a sequence of functions. Let $\bar{\rho}$ be the uniform metric on the space $\mathbb{R}^X$. Show that the sequence (f_n) converges uniformly to the function $f : X \to \mathbb{R}$ if and only if the sequence (f_n) converges to f as elements of the metric space $(\mathbb{R}^X, \bar{\rho})$.

Exercise 10.88 (Munkres 21.8)

Let X be a topological space and let Y be a metric space. Let $f_n : X \to Y$ be a sequence of continuous functions. Let x_n be a sequence of points of X converging to x. Show that if the sequence (f_n) converges uniformly to f, then $(f_n(x_n))$ converges to $f(x)$.

Exercise 10.89 (Munkres 21.9)

Let $f_n : \mathbb{R} \to \mathbb{R}$ be the function

$$f_n(x) = \frac{1}{n^3[x - (1/n)]^2 + 1}.$$

See Figure 21.1. Let $f : \mathbb{R} \to \mathbb{R}$ be the zero function.

(a) Show that $f_n(x) \to f(x)$ for each $x \in \mathbb{R}$.

(b) Show that f_n does not converge uniformly to f. (This shows that the converse of Theorem 21.6 does not hold; the limit function f may be continuous even though the convergence is not uniform.)

Exercise 10.90 (Munkres 21.10)

Using the closed set formulation of continuity (Theorem 18.1), show that the following are closed subsets of $\mathbb{R}^2$:

$$A = \{x \times y \mid xy = 1\},$$
$$S^1 = \{x \times y \mid x^2 + y^2 = 1\},$$
$$B^2 = \{x \times y \mid x^2 + y^2 \leq 1\}.$$

Exercise 10.91 (Munkres 21.11)

Prove the following standard facts about infinite series:

(a) Show that if (s_n) is a bounded sequence of real numbers and $s_n \leq s_{n+1}$ for each n, then (s_n) converges.

(b) Let (a_n) be a sequence of real numbers; define

$$s_n = \sum_{i=1}^{n} a_i.$$

If $s_n \to s$, we say that the *infinite series*

$$\sum_{i=1}^{\infty} a_i$$

converges to s also. Show that if $\sum a_i$ converges to s and $\sum b_i$ converges to t, then $\sum (ca_i + b_i)$ converges to $cs + t$.

(c) Prove the *comparison test* for infinite series: If $|a_i| \leq b_i$ for each i, and if the series $\sum b_i$ converges, then the series $\sum a_i$ converges. [Hint: Show that the series $\sum |a_i|$ and $\sum c_i$ converge, where $c_i = |a_i| + a_i$.]

(d) Given a sequence of functions $f_n : X \to \mathbb{R}$, let

$$s_n(x) = \sum_{i=1}^{n} f_i(x).$$

Prove the *Weierstrass M-test* for uniform convergence: If $|f_i(x)| \leq M_i$ for all $x \in X$ and all i, and if the series $\sum M_i$ converges, then the sequence (s_n) converges uniformly to a function s. [Hint: Let $r_n = \sum_{i=n+1}^{\infty} M_i$. Show that if $k > n$, then $|s_k(x) - s_n(x)| \leq r_n$; conclude that $|s(x) - s_n(x)| \leq r_n$.]

Exercise 10.92 (Munkres 21.12)

Prove continuity of the algebraic operations on $\mathbb{R}$, as follows: Use the metric $d(a, b) = |a - b|$ on $\mathbb{R}$ and the metric on $\mathbb{R}^2$ given by the equation

$$\rho((x, y), (x_0, y_0)) = \max\{|x - x_0|, |y - y_0|\}.$$

(a) Show that addition is continuous. [Hint: Given ϵ, let $\delta = \epsilon/2$ and note that

$$d(x + y, x_0 + y_0) \leq |x - x_0| + |y - y_0|.$$

]

(b) Show that multiplication is continuous. [Hint: Given (x_0, y_0) and $0 < \epsilon < 1$, let

$$3\delta = \epsilon/(|x_0| + |y_0| + 1)$$

and note that

$$d(xy, x_0y_0) \leq |x_0||y - y_0| + |y_0||x - x_0| + |x - x_0||y - y_0|.$$

]

(c) Show that the operation of taking reciprocals is a continuous map from $\mathbb{R} - \{0\}$ to $\mathbb{R}$. [Hint: Show the inverse image of the interval (a, b) is open. Consider five cases, according as a and b are positive, negative, or zero.]

(d) Show that the subtraction and quotient operations are continuous.

10.6 Connectedness

10.7 Compactness

Exercise 10.93 (Munkres 27.2)

Let X be a metric space with metric d; let $A \subset X$ be nonempty.

(a) Show that $d(x, A) = 0$ if and only if $x \in \overline{A}$.

(b) Show that if A is compact, $d(x, A) = d(x, a)$ for some $a \in A$.

(c) Define the ϵ-neighborhood of A in X to be the set

$$U(A, \epsilon) = \{x \mid d(x, A) < \epsilon\}.$$

Show that $U(A, \epsilon)$ equals the union of the open balls $B_d(a, \epsilon)$ for $a \in A$.

(d) Assume that A is compact; let U be an open set containing A. Show that some ϵ-neighborhood of A is contained in U.

(e) Show the result in (d) need not hold if A is closed but not compact.

Solution. For (a), we prove bidirectionally.

1. $(\rightarrow)$. We prove the contrapositive. Let $x \notin \overline{A} \implies \exists\, B(x, \epsilon)$ satisfying $B(x, \epsilon) \cap A \neq \emptyset$. This means that $d(x, A) \geq \epsilon > 0$. Note that $x \notin A$ means that the actual point x can't intersect A, while $x \notin A'$ means that the punctured ball can't intersect A. If $x \notin A'$, then $d(x, A) = 0$ if x is an isolated point.
2. $(\leftarrow)$. If $x \in \overline{A}$, then for every $\epsilon > 0$, $B(x, \epsilon) \cap A \neq 0$. Therefore, $0 \leq d(x, A) < \epsilon$ for all positive ϵ, which implies that $d(x, A) = 0$.

For (b). Since the metric d is continuous, the function $f(y) = d(x, y)$ is continuous over compact A, and by EVT, f achieves its minimum in A. Call this point $a \in A$, and so we have $d(x, a) \leq d(x, A)$. However, $d(x, A) \leq d(x, a)$ for all $a \in A$ by definition, so $d(x, a) = d(x, A)$.

For (c), we prove bidirectionally.

1. $U(A, \epsilon) \subset \cup_{a \in A} B(a, \epsilon)$. Let $x \in U(A, \epsilon)$, and so we have $d(x, A) < \epsilon$. We see that $d(x, A) = \inf\{d(x, a) \mid a \in A\}$, so $\epsilon > d(x, A)$ cannot be a lower bound. Therefore by definition there exists some $a^* \in A$ satisfying $d(x, a^*) < \epsilon \implies x \in B(a^*, \epsilon)$.
2. $\cup_{a \in A} B(a, \epsilon) \subset U(A, \epsilon)$. Let $x \in \cup_{a \in A} B(a, \epsilon)$. Then we pick a particular a^* such that $x \in B(a^*, \epsilon)$, which satisfies $d(x, a^*) < \epsilon$. Since $a^* \in A$, $d(x, A) \leq d(x, a^*) < \epsilon \implies x \in U(A, \epsilon)$

For (d), we first construct the open cover $\mathscr{B} = \{B(a, \epsilon(a))\}_{a \in A}$ consisting of all neighborhoods of a with some $\epsilon = \epsilon(a)$ (that may be dependent on a) such that $B(a, 2\epsilon(a)) \subset U$. This is possible since U is open. $\mathscr{B}$ is an open cover of compact A, so take a finite subcover $\mathscr{F} = \{B(a_i, \epsilon(a_i))\}_{i=1}^n$ with the minimum $\epsilon = \min_i\{\epsilon_i\}$. We claim that

$$A \subset \bigcup_i B(a_i, \epsilon) \subset U(A, \epsilon) \subset \bigcup_i B(a_i, 2\epsilon) \subset U \tag{435}$$

The first inclusion is trivial since $\mathscr{F}$ is an open cover. The second follows directly from (c) since $U(A, \epsilon)$ contains the union of all open balls of the form $B(a, \epsilon)$, which contains those with center a_i. The fourth subset is by construction. As for the third subset, we claim that if $x \in U(A, \epsilon)$, then by (b) there exists some $y \in A$ s.t. $d(x, y) < \epsilon$. Also, since $y \in A \subset \cup_i B(a_i, \epsilon)$, we have $d(y, a_i) < \epsilon$ for some a_i. Therefore, by the triangle inequality,

$$d(x, a_i) < d(x, y) + d(y, a_i) = \epsilon + \epsilon = 2\epsilon \tag{436}$$

and so $x \in B(a_i, 2\epsilon)$ for some a_i, and so $x \in \cup_i B(a_i, 2\epsilon) \subset U$.
For (e), consider the y-axis $\{0\} \times \mathbb{R}$ (a product of closed sets) and the open set from Munkres's Tube Lemma example

$$U = \{(x, y) \in \mathbb{R}^2 \mid |x| < 1/(y^2 + 1)\} \tag{437}$$

U is open, but given any $\epsilon > 0$, $U(A, \epsilon) = (-\epsilon, +\epsilon) \times \mathbb{R}$ is not contained in U since by the Archimidean property, we can set y arbitrarily large so that

$$\frac{1}{y^2 + 1} < \epsilon \tag{438}$$

and letting δ be a number between them, we have $(\delta, y) \in U(A, \epsilon)$ but not in U since $1/(y^2 + 1) < \delta$.

Exercise 10.94 (Munkres 27.6)

Let A_0 be the closed interval $[0, 1]$ in $\mathbb{R}$. Let A_1 be the set obtained from A_0 by deleting its "middle third" $(\frac{1}{3}, \frac{2}{3})$. Let A_2 be the set obtained from A_1 by deleting its "middle thirds" $(\frac{1}{9}, \frac{2}{9})$ and $(\frac{7}{9}, \frac{8}{9})$. In general, define A_n by the equation

$$A_n = A_{n-1} - \bigcup_{k=0}^{\infty} \left(\frac{1 + 3k}{3^n}, \frac{2 + 3k}{3^n} \right).$$

The intersection

$$C = \bigcap_{n \in \mathbb{Z}_+} A_n$$

is called the Cantor set; it is a subspace of $[0, 1]$.

(a) Show that C is totally disconnected.
(b) Show that C is compact.
(c) Show that each set A_n is a union of finitely many disjoint closed intervals of length $1/3^n$; and show that the end points of these intervals lie in C.
(d) Show that C has no isolated points.
(e) Conclude that C is uncountable.

Solution.

1. For (a), pick distinct $x, y \in C$ and WLOG let $x < y$. Then $y - x > 0$, and so by the Archimidean property we can find some n s.t.

$$y - x > \frac{1}{3^{n-2}} \tag{439}$$

Furthermore, we can multiply both sides of the left equation by 3^n to get

$$3^n y - 3^n x > 9 \tag{440}$$

which implies that there exists a natural $a \in \mathbb{N}$ s.t.

$$3^n x < a < a + 1 < \ldots < a + 8 < 3^n y \tag{441}$$

There must be a divisor amongst the $a, a + 1, a + 2$, so call it $a' = 3k$. Then we must have

$$3^n x < 3k < 3k + 1 < 3k + 2 < 3^n y \implies x < \frac{3k}{3^n} < \frac{1 + 3k}{3^n} < \frac{2 + 3k}{3^n} < y \tag{442}$$

and so we know that the middle interval $(\frac{1+3k}{3^n}, \frac{2+3k}{3^n})$ is not included in C. Pick any point $y \in (\frac{1+3k}{3^n}, \frac{2+3k}{3^n})$ and so $[0, y) \cap C$ and $(y, 0] \cap C$ is a separation. Since x, y were arbitrary, C is totally disconnected.

2. For (b), C is bounded, so it suffices to show that it is closed. We claim that A_n is closed for all $n \in \mathbb{N}$. This is because A_0 is closed. Now given that A_{n-1} is closed, the intervals that are taken out are each open since they are open intervals, and their arbitrary union is also open. Therefore, let's call this U_{n-1}, and so

$$A_{n-1} \setminus \bigcup_{k=0}^{\infty} \left(\frac{1+3k}{3^n}, \frac{2+3k}{3^n} \right) = A_{n-1} \setminus U_{n-1} = A_{n-1} \cap (U_{n-1})^c \tag{443}$$

where the complement (in $[0, 1]$) is closed and so the intersection of closed sets is closed. Finally, C which is the intersection of closed sets A_n is closed.

3. We prove using induction. Clearly $A^0 = [0, 1]$ has a length of $1/3^0 = 1$ with endpoints $0, 1$, and $A_1 = [0, 1/3] \cup [2/3, 1]$ consists of two intervals of length $1/3^1$ with endpoints $\{0, 1/3, 2/3, 1\}$ which contains the endpoints of A_0 and cover all multiples of $1/3^1$. Now given that A_n is a finite disjoint union of intervals of length $1/3^n$ of the form

$$I = \left[\frac{a}{3^n}, \frac{a+1}{3^n} \right] = \left[\frac{3a}{3^{n+1}}, \frac{3a+3}{3^{n+1}} \right] \tag{444}$$

with endpoints that have a 1-to-1 correspondence with all multiples of $1/3^n$, we label the left and right endpoints as $l = l_I, u = u_I$ for convenience. Now for A_{n+1} we claim that for each I,

$$I \cap \left(\bigcup_{k=0}^{\infty} \left(\frac{1+3k}{3^{n+1}}, \frac{2+3k}{3^{n+1}} \right) \right) = \left(\frac{1+3a}{3^{n+1}}, \frac{2+3a}{3^{n+1}} \right) \tag{445}$$

because

$$\ldots < 3a-1 = 2+3(a-1) < 3a < 1+3a < 2+3a < 3a+3 < 1+3(a+1) = 3a+4 < \ldots \tag{446}$$

and so every interval I would be divided into two disjoint intervals

$$I = \left[\frac{3a}{3^{n+1}}, \frac{1+3a}{3^{n+1}} \right] \sqcup \left[\frac{2+3a}{3^{n+1}}, \frac{3+3a}{3^{n+1}} \right] = I_1 \sqcup I_2 \tag{447}$$

each of length $1/3^{n+1}$ and endpoints changed from

$$\left\{ \frac{3a}{3^{n+1}}, \frac{3+3a}{3^{n+1}} \right\} \mapsto \left\{ \frac{3a}{3^{n+1}}, \frac{1+3a}{3^{n+1}}, \frac{2+3a}{3^{n+1}}, \frac{3+3a}{3^{n+1}} \right\} \tag{448}$$

which are also consecutive multiples of $1/3^{n+1}$ that will cover all multiples of $1/3^{n+1}$ when taking all intervals I. Therefore, the number of intervals will double, which is still finite. The two subintervals also still disjoint from each other since $1 + 3a < 2 + 3a$, and they are disjoint from all other intervals since their extensions (before the middle interval was taken out) was disjoint. Finally, l is the left endpoint of I_1 and u is the right endpoint of I_2, and so an endpoint of any A_n is an endpoint of A_m for $m \geq n$, which implies that every endpoint lies in C.

4. Given $x \in C$, let us take the ϵ-neighborhood $(x - \epsilon, x + \epsilon)$. Then by Archimidean property, we can choose a large $n \in \mathbb{N}$ s.t. $\frac{1}{3^{n-2}} < \epsilon$, and so

$$9 < 3^n \epsilon \implies 9 < 3^n (x + \epsilon) - 3^n x \tag{449}$$

and so there are naturals $a, a + 1, \ldots, a + 8$ in between $3^n x$ and $3^n (x + \epsilon)$. One of $a, a + 1, a + 2$ must be a multiple of three, and call it $a^* = 3k$. Then

$$3^n x < 1 + 3k < 2 + 3k < 3^n (x + \epsilon) \implies x < \frac{1+3k}{3^n} < \frac{2+3k}{3^n} < 3^n (x + \epsilon) \tag{450}$$

But $(1+3k)/3^n$ is an endpoint of an interval in A_n, particularly $\left[\frac{1+3k}{3^n}, \frac{2+3k}{3^n}\right]$, which is guaranteed to exist since we've proved that the endpoints of the disjoint intervals cover all multiples of $1/3^n$ and also are in C.

5. We have proved that C is compact and has no isolated points. C as a subset of Hausdorff $[0,1]$ is also Hausdorff. So as a compact Hausdorff space with no isolated points it is uncountable.

Exercise 10.95 (Munkres 28.2)

Show that $[0,1]$ is not limit point compact as a subspace of $\mathbb{R}_\ell$.

Solution. Consider the set $A = \{a_n\} = \{1 - \frac{1}{n}\}_{n\in\mathbb{N}}$. $a \in A$ is not a limit point since we can see that if $a = 1 - \frac{1}{k}$, we can see that setting $\epsilon = \frac{1}{2k(k+1)}$ gives

$$a + \epsilon = 1 - \frac{2(k+1)}{2k(k+1)} + \frac{1}{2k(k+1)} = 1 - \frac{2k+1}{2k(k+1)} < 1 - \frac{2k}{2k(k+1)} < 1 - \frac{1}{k+1} \tag{451}$$

and the opposite side with $a - \epsilon$ also holds since $d(a_{k-1}, a_k) > d(a_k, a_{k+1})$.

$$\left(1 - \frac{1}{k-1}\right) - \left(1 - \frac{1}{k}\right) = \frac{1}{k(k-1)} > \frac{1}{k(k+1)} = \left(1 - \frac{1}{k}\right) - \left(1 - \frac{1}{k+1}\right) \tag{452}$$

and so $[a-\epsilon, a+\epsilon)$ does not intersect A. Now pick $x \in (0,1) \setminus A$. Then there exists unique natural $n \in \mathbb{N}$ such that

$$n < \frac{1}{x} < n+1 \implies \frac{1}{n+1} < x < \frac{1}{n} \tag{453}$$

and so we can set $\epsilon = \min\{\frac{1}{n}, \frac{1}{n+1}\}$ to create an open ball around x not intersecting A. Now 0 is not a limit point since we can choose $\epsilon = 1/3$, and so $B(0, 1/3) \cap [0,1] = [0, 1/3)$ does not intersect A which has all elements at least $1/2$. Finally 1 is not a limit point since we can choose the singleton set $\{1\} = [1,2) \cap [0,1]$ that does not intersect A. Therefore $[0,1]$ has an infinite subset that does not have a limit point, so it is not limit point compact.

10.8 Countability

10.9 Separation

10.10 Homotopies

Exercise 10.96 (Munkres 51.1)

Show that if $h, h' : X \to Y$ are homotopic and $k, k' : Y \to Z$ are homotopic, then $k \circ h$ and $k' \circ h'$ are homotopic.

Solution. From the assumptions there exists homotopies $H : X \times I \to Y$ and $K : Y \times I \to Z$. We define the composition homotopy as

$$G : X \times I \to Z, \qquad G(x,t) := K(H(x,t), t) \tag{454}$$

Note that since $(x,t) \mapsto H(x,t)$ is continuous, by the product topology the map $(x,t) \mapsto (H(x,t), t)$ is continuous, and it composed with continuous K makes it also continuous. Finally, we have

$$G(x,0) = K(H(x,0),0) = K(h(x),0) = k(h(x)) = (k \circ h)(x) \tag{455}$$
$$G(x,1) = K(H(x,1),1) = K(h'(x),1) = k'(h'(x),1) = (k' \circ h')(x) \tag{456}$$

Exercise 10.97 (Munkres 51.2)

Given spaces X and Y, let $[X, Y]$ denote the set of homotopy classes of maps of X into Y.
(a) Let $I = [0, 1]$. Show that for any X, the set $[X, I]$ has a single element.
(b) Show that if Y is path connected, the set $[I, Y]$ has a single element.

Solution. Listed.

1. Take continuous $f, g : X \to I$ and define the homotopy $F : X \times I \to I$ between them as the convex combination
$$F(x, t) = (1 - t)f(x) + tg(x) \tag{457}$$
We have $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$, and the products/sums of continuous maps are continuous. Finally, we can see that $F(x, t) \in I$ because given any x,
$$f(x) \leq g(x) \implies f(x) \leq F(x, t) = (1 - t)f(x) + tg(x) \leq g(x) \tag{458}$$
$$f(x) \geq g(x) \implies g(x) \leq F(x, t) = (1 - t)f(x) + tg(x) \leq f(x) \tag{459}$$
for all $t \in [0, 1]$, and since $[0, 1] \subset \mathbb{R}$ is convex, $F(x, t) \in [0, 1]$ for all t and x.
2. Given any continuous function $f : I \to Y$, this is by definition a path. We wish to show that this is homotopic to a constant map, say the value at $f(0) \in Y$. We construct the homotopy
$$F(x, t) : I \times I \to Y, \qquad F(x, t) = f((1 - t)x) \tag{460}$$
This satisfies $F(x, 0) = f(x)$ and $F(x, 1) = f(0)$. This is also continuous since it is the composition of continuous $(x, t) \mapsto (1 - t)x$ and continuous f. Therefore, given two functions $f, g : I \to Y$, we know that $f \simeq f(0)$ and $g \simeq g(0)$. Now since homotopy is an equivalence relation, it suffices to show that $f(0) \simeq g(0)$, i.e. any two constant functions are homotopic. Since Y is path connected, there exists a path $h : I \to Y$ such that $h(0) = f(0), h(1) = g(0)$. Now we define the homotopy
$$H : I \times I \to Y, \qquad H(x, t) = h(t) \tag{461}$$
which is certainly continuous since path h is continuous by definition, and $h(0) = f(0), h(1) = g(0)$.

Exercise 10.98 (Munkres 51.3)

A space X is said to be *contractible* if the identity map $i_X : X \to X$ is nullhomotopic.
(a) Show that I and $\mathbb{R}$ are contractible.
(b) Show that a contractible space is path connected.
(c) Show that if Y is contractible, then for any X, the set $[X, Y]$ has a single element.
(d) Show that if X is contractible and Y is path connected, then $[X, Y]$ has a single element.

Solution. Listed.

1. For I, we define the homotopy $F(x, t) = (1 - t)x$, which is continuous (since it is a scalar multiple of the continuous identity map) and satisfies $F(x, 0) = x = i_X(x)$, and $F(x, 1) = 0$ which is a constant map. For $\mathbb{R}$ we can use the same homotopy and the same logic follows.
2. Given that X is nullhomotopic let $z \in X$ be the value of the constant map that is homotopic to the identity $i : X \to X$, with homotopy $F : X \times I \to X$ where $F(x, 0) = x$ and $F(x, 1) = z$. Then we claim that for any $x \in X$, x is path connected to z since by fixing x, the function $\gamma : I \to X$ defined $\gamma(t) = F(x, t)$ is a path: it is continuous, $\gamma(0) = F(x, 0) = x$, and $\gamma(1) = F(x, 1) = z$. Therefore we have found such a path, and since path connectedness is an equivalence relation, for any $x, y \in X$, $x \sim z \sim y$, making X path connected.
3. Since Y is contractible, let us construct the homotopy of the constant map to the identity map

$F : Y \times I \to Y$ satisfying $F(y, 0) = z$ for some $z \in Y$ and $F(y, 1) = y$. Now given continuous $g : X \to Y$, we wish to show that it is homotopic to the constant map with value z. Consider the homotopy $G : X \times I \to Y$ defined

$$G(x, t) = F(g(x), t) \tag{462}$$

This is clearly continuous as the composition of continuous maps. Furthermore, $G(x, 0) = F(g(x), 0) = z$ and $G(x, 1) = F(g(x), 1) = g(x)$, making this indeed a homotopy. Since homotopy is an equivalence relation for any continuous $f, g : X \to Y$, $f \simeq z \simeq g$, and so there is only one equivalence class.

4. We wish to show that given any two functions $f, g : X \to Y$, $f \simeq g$. Since X is contractible, there exists a homotopy $F : X \times I \to X$ where $F(x, 0) = z$ for some $z \in X$ and $F(x, 1) = x$. We can take f, g and see that $(f \circ F), (g \circ F) : X \times I \to Y$ is a homotopy, as the composition of continuous maps satisfying

$$(f \circ F)(x, 0) = f(F(x, 0)) = f(z) \tag{463}$$
$$(f \circ F)(x, 1) = f(F(x, 1)) = f(x) \tag{464}$$
$$(g \circ F)(x, 0) = g(F(x, 0)) = g(z) \tag{465}$$
$$(g \circ F)(x, 1) = g(F(x, 1)) = g(x) \tag{466}$$

Therefore, $f(z) \sim f$ and $g(z) \sim g$. We know that since Y is path connected, $f(z) \sim g(z)$ as given in my argument in Munkres 51.2.b, and so $f \sim f(z) \sim g(z) \sim g$, and every continuous function is a part of the same equivalence class.