Logic and Set Theory

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1 Propositional Logic

Philosophers still debate about what a proposition really means. As a complete beginner, I mention some interpretations of it, but I by no means claim that this is the definitive definition.

Definition 1.1 (Possible World)

A **possible world** is a complete and consistent way the world is or could have been.

The **language** of propositional logic consists of just two things: propositions and connectives.

Definition 1.2 (Proposition)

A **proposition** does not have a formal definition, but we can describe it in the following ways.

1. They can be understood as an indicator function $f : W \to \{T, F\}$[a] that takes in a possible world and returns a truth value. We can also model it with the preimage of f under T, i.e. the characteristic set of f.
2. They deal with **statements**, which are defined as declarative sentences having a truth value.

Propositions are either true or false.

Example 1.1 ()

The proposition that *the sky is blue* is represented as the function that returns T for every possible world where the sky is blue.

These declarative sentences are contrasted with questions, such as *how are you doing?* and imperative statements such as *please run my models.* Such non-declarative sentences have no truth value.

A statement can contain one or more other statements as parts. For example, compound sentences form simpler sentences.

Definition 1.3 (Connectives)

Statements are combined with **logical connectives**.

Connective	**Symbols**
AND	$A \wedge B$, $A \cdot B$, AB, $A\&B$, $A\&\&B$
OR	$A \vee B$, $A + B$, $A \mid B$, $A \parallel B$
NOT	$\neg A$, $-A$, $\overline{A}$, $\sim A$
NAND	$\overline{A \wedge B}$, $A \mid B$, $\overline{A \cdot B}$
NOR	$\overline{A \vee B}$, $A \downarrow B$, $\overline{A + B}$
XOR	$A \veebar B$, $A \oplus B$
XNOR	$A \odot B$
IMPLIES	$A \Rightarrow B$, $A \supset B$, $A \to B$
EQUIVALENT	$A \equiv B$, $A \Leftrightarrow B$, $A \leftrightarrow B$
NONEQUIVALENT	$A \not\equiv B$, $A \not\Leftrightarrow B$, $A \not\leftrightarrow B$

Table 1: Logical Connectives and Their Symbols

[a] T, F stands for True, False.

Definition 1.4 (Propositional Formula)

Propositions, represented by letters and denoted **propositional variables**, along with these symbols for connectives, combine to make a **propositional formula**.

Propositional logic is not concerned with the structures of propositions beyond the point where they cannot be decomposed any more by logical connectives.

1.1 Arguments

At this point we may look at a set of propositions $P_1, \ldots, P_n$ and try come to a logical conclusion Q. This is called an argument.

Definition 1.5 (Argument)

Let P be a set of propositions, called the **premises**. Let Q be a proposition, called the **conclusion**. Then an **argument** is an attempt to deduce Q from P. It is written in the forms

1. If P, then Q.
2. $P \implies Q$

An argument is **valid** if and only if

1. It is necessary that if P is true, Q is true.
2. It is impossible for P to be true, while Q is false.

Example 1.2 ()

The following is an argument.

If *it is raining*, then *it is cloudy*.

Logic in general aims to specify valid arguments. This is done by defining a valid argument as one in which its conclusion is a logical consequence of its premises. Determining whether a proposition is a a logical consequence of another proposition is the process of **deductive argument**, which has rules. These rules, called **rules of inference**, determines the "legal moves" from one or more premises to the conclusion. We give 2 familiar ones.

Definition 1.6 (Modus Ponens)

Modus ponens is a deductive argument form and rule of inference.[a] The argument states that given the premises

1. $P \implies Q$
2. P

Then our conclusion is Q.

The next one is the familiar statement that a statement is equivalent to its contrapositive.

Definition 1.7 (Modus Tollens)

Modus tollens is a deductive argument form and a rule of inference. The argument states that given the premises

1. $P \implies Q$
2. Q

Then our conclusion is not P.

[a] In some literature it is treated as an axiom, though most people think of it as a rule.

2 First-Order Logic

In propositional logic, we deal with simple declarative propositions. **First-order logic** extends this by covering predicates and quantification. Let's motivate them.

We can think of predicates as properties. If we say *Socrates is a philosopher* and *Plato is a philosopher*, in propositional logic both these statements, represented as P and Q, as utterances that are either true or false, and they are completely independent from one another. However, we may want to view them as an application of a predicate $*$ *is a philosopher* on the entities *Socrates* and *Plato*. This motives the formalism of the domain of discourse and the predicate.

Definition 2.1 (Domain of Discourse)

Given an individual x, its **domain of discourse** is the set over which certain variables of interest in some formal treatment may range.

Definition 2.2 (Predicate)

A **predicate** P is a symbol that represents a property or a relation of a certain individual x in a domain of discourse. Using predicates, $P(x)$ can be viewed as a proposition about the individual x.

Note that a predicate itself is not a proposition, since saying $*$ *is a philosopher* doesn't have any truth or false meaning to it, akin to a sentence fragment. But it is a placeholder $P(\cdot)$ upon which if an individual x is put in, it makes sense to ask whether $P(x)$ is true.

Definition 2.3 (Formula)

A **formula** is a string of propositions, connectives, predicates, and variables ϕ that turns into a proposition once all free variables have been instantiated.

With predicates alone, all we have really done is add notational convenience. However, if we want to state a proposition not just about x, but its domain of discourse, then we can use quantifiers.

Definition 2.4 (Quantifier)

A **quantifier** is an operator that specifies how many individuals in the domain of source satisfy a proposition. The two most used quantifiers are

1. *Universal Quantification.* $\forall$, which means *for every.*
2. *Existential Quantification.* $\exists$, which means *there exists.*

3 Second-Order Logic

First order logic can quantify over individuals, but not over properties. That is, while we can state something like

There exists x such that x is a cube.

we cannot quantify over a predicate. That is, the statement

There exists a property P such that a cube satisfies P.

This statement does not make sense in first-order logic, but makes sense in second-order logic.

4 Naive Set Theory

Unlike axiomatic set theories, which are defined using formal logic, naive set theory was defined informally at the end of the 19th century by Cantor, in natural language (like English). It describes the aspects of mathematical sets using words (e.g. *satisfying, such as, ...*) and suffices for the everyday use of set theory in modern mathematics. However, as we will see, this leads to paradoxes.

Definition 4.1 (Set)

A **set** is a well-defined collection of distinct objects, called **elements**.

This definition tells us *what* a set is, but does not define *how* sets can be formed, and what operations on sets will again produce a set. The term *well-defined* cannot by itself guarantee the consistency and unambiguity of what exactly constitutes and what does not constitute a set, and therefore this is not a formal definition. Attempting to achieve this will be done in axiomatic set theory, like ZFC.

Definition 4.2 (Membership)

If x is a member of A, we write $x \in A$. For any x, it must be the case that either $x \in A$ (exclusive or) $x \notin A$.

Definition 4.3 (Equality)

Two sets A and B are defined to be equal, denoted as $A = B$, when they have precisely the same elements. That is, if $x \in A \iff x \in B$. This means that a set is completely determined by its elements, and the description is immaterial.

Definition 4.4 (Empty Set)

There exists an empty set, denoted $\emptyset$ or $\{\}$, which is a set with no members at all. Because a set is described by its elements, there can only be one empty set.

Now we show how to construct sets.

Definition 4.5 (Set-Builder Notation)

We can construct a set in two ways.

1. We list its elements between curly braces.
   (a) The set $\{1, 2\}$ denotes the set containing 1 and 2. By equality $\{1, 2\} = \{2, 1\}$.
   (b) Repetition/multiplicity is irrelevant, and so $\{1, 2, 2\} = \{1, 1, 1, 2\} = \{1, 2\}$
2. We denote
$$S = \{x | P(x)\} \tag{1}$$
   where P is a property. If x satisfies this property, then $x \in S$.

Naive set theory claims that this construction *always* produces a set. Therefore, a well-defined property is enough to always produce a set of elements satisfying P.

Example 4.1 (Empty Set)

Let $S = \{x \mid x \neq x\}$. For any x, $P(x)$ is false and so S contains no elements. Therefore $S = \emptyset$.

Example 4.2 (Singleton Set)

The set $\{x \mid x = a\} = \{a\}$.

Example 4.3 (Russell Set)

Let $R = \{x \mid x \notin x\}$, i.e. the set of all sets that do not contain themselves as elements.

Theorem 4.1 (Russell's Paradox)

The Russell set exists and does not exist.

Proof.

We will determine if R is an element of itself.
1. If $R \in R$, then by it does contain itself, so it does not satisfy the property and $R \notin R$.
2. If $R \notin R$, then it satisfies the property, so $R \in R$.

Therefore, it is both the case that $x \in R$ and $x \notin R$, which contradicts the membership definition. Therefore, R is both a set from set-builder construction and not a set due to the membership definition.

Theorem 4.2 (Existence of Universe)

Let U be the set of everything, known as the **universal set**. The universal set does exist and does not exist.

Proof.

We can define $U' = \{x \mid \{\} = \{\}\}$, which defines a set. Then the property P that $\{\} = \{\}$ is always true, and U' would contain everything, and by the definition of equality $U = U'$. Now since the Russell set R is both a set and not a set from Russell's paradox, we have $R \in U$ and $R \notin U$, which means that U cannot exist. Therefore U does not exist.

So the sufficiency a well-defined property to be able to construct a set is *too powerful* in that we can construct *any* set we want. This leads us to construct the Russell set, which opens up a lot of paradoxes. Therefore, we would like to restrict the notion of well-defined in a way, which leads to axiomatic set theories.

Definition 4.6 (Subsets)

Given two sets A and B, A is a **subset** of B if every element of A is also an element of B. A subset of B that is not equal to B is called a **proper subset**.

Theorem 4.3 (Equality)

It follows from the definition of equality that

$$A \subset B \text{ and } B \subset A \iff A = B \tag{2}$$

Definition 4.7 (Power Set)

The set of all subsets of a set A is called the **power set** of A, denoted by 2^A.

We could define other things like the union, etc., but I won't bother with it when I will define them for ZFC later.

5 Zermelo-Fraenkel-Choice (ZFC) Set Theory

So with these paradoxes in mind, we would like to construct an axiomatic formulation of set theory. Let's start with the definition, which is similar.

Definition 5.1 (Set)

A **set** S is a collection of **elements** x.

5.1 Axioms

Now we state the 8 axioms, which is the foundation of ZF set theory.

Axiom 5.1 (Axiom of Extensionality)

Two sets are equal (are the same set) if they have the same elements.

Axiom 5.2 (Axiom of Regularity)

Every non-empty set x contains a member y such that x and y are disjoint sets.

Axiom 5.3 (Axiom Schema of Specification/Separation/Restricted Comprehension)

Axiom 5.4 (Axiom of Pairing)

Axiom 5.5 (Axiom of Union)

Axiom 5.6 (Axiom Schema of Replacement)

Axiom 5.7 (Axiom of Infinity)

Axiom 5.8 (Axiom of Power Set)

The axiom of choice is still disputed, but with this axiom we have ZFC set theory.

Axiom 5.9 (Axiom of Choice)

How do we even know that these axioms aren't contradictory? We don't, and that is why we take them as axioms rather than provable theorems. Fortunately, from the formulation in the early 20th century up until now, no contradictions have been found.

6 Intro

6.1 Natural Numbers and Induction

Definition 6.1 (Inductive Set, Natural Numbers)

A set $X \subset \mathbb{R}$ is inductive if for each number $x \in X$, it also contains $x+1$. The set of *natural numbers*j, denoted $\mathbb{N}$, is the smallest inductive set containing 1.

We can use this inductive property of natural numbers to prove properties of them. Note that this can only be used to prove for finite (yet unbounded) numbers!

Lemma 6.1 (Induction Principle)

Given $P(n)$, a property depending on positive integer n,

1. if $P(n_0)$ is true for some positive integer n_0, and
2. if for every $k \geq n_0$, $P(k)$ true implies $P(k+1)$ true,

then $P(n)$ is true for all $n \geq n_0$.

Lemma 6.2 (Strong Induction Principle)

Given $P(n)$, a property depending on a positive integer n,

1. if $P(n_0), P(n_0+1), \ldots, P(n_0+m)$ are true for some positive integer n_0, and nonnegative integer m, and
2. if for every $k > n_0 + m, P(j)$ is true for all $n_0 \leq j \leq k$ implies $P(k)$ is true,

then $P(n)$ is true for all $n \geq n_0$.

The idea behind the strong induction principle leads to the proof using infinite descent. Infinite descent combines strong induction with the fact that every subset of the positive integers has a smallest element, i.e. there is no strictly decreasing infinite sequence of positive integers.

Lemma 6.3 (Infinite Descent)

Given $P(n)$, a property depending on positive integer, assume that $P(n)$ is false for a set of integers $\mathcal{S}$. Let the smallest element of $\mathcal{S}$ be n_0. If $P(n_0)$ false implies $P(k)$ false, where $k < n_0$, then by contradiction $P(n)$ is true for all n.

6.2 Countable and Uncountable Sets

Definition 6.2 (Equipotence)

Two sets A and B are **equipotent**, written $A \approx B$, if there exists a bijective map $f : A \to B$. This implies that their cardinalities are the same: $|A| = |B|$. It has the following properties:

1. Reflexive: $A \approx A$
2. Symmetric: $A \approx B$ implies $B \approx A$
3. Transitive: $A \approx B$ and $B \approx C$ implies $A \approx C$

Definition 6.3 ()

For any positive integer n, let J_n be the set whose elements are the integers $1, 2, \ldots, n$. For any set A, we define

1. A is **finite** if $A \approx J_n$ for some n. The empty set is also considered to be finite.

2. A is **infinite** if it is not finite.
3. A is countable if $A \approx \mathbb{N}$.
4. A is uncountable if A is neither finite nor countable.
5. A is at most countable if A is finite or countable.

At this point, we may already be familiar with the fact that $\mathbb{Q}$ is countable and $\mathbb{R}$ is uncountable. Let us formalize the statement that a countable infinity is the smallest type of infinity. We can show this by taking a countable set and showing that every infinite subset must be countable. If it was uncountable, then this would mean that a countable set contains an uncountable set.

Theorem 6.1 ()

Every infinite subset of a countable set A is countable.

Theorem 6.2 ()

An at most countable union of countable sets is countable.

Theorem 6.3 ()

A finite Cartesian product of countable sets is countable.

Corollary 6.1 ()

$\mathbb{Q}$ is countable.

Now, how do we prove that a set is uncountable? We can't really use the contrapositive of Theorem 6.2, since to prove that an arbitrary set A is uncountable, then we must find an infinite subset that is not countable. But now we must prove that this subset itself is not countable, too! Therefore, we can use this theorem.

Theorem 6.4 ()

Given an arbitrary set A, if every countable subset B is a proper subset of A, then A is uncountable.

Proof.

Assume that A is countable. Then A itself is a countable subset of A, but by the assumption, A should be a proper subset of A, which is absurd. Therefore, A is uncountable.

Theorem 6.5 ()

Let A be the set of all sequences whose elements are the digits 0 and 1. Then, A is uncountable.