Set Theory

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1 Intro

1.1 Natural Numbers and Induction

Definition 1.1 (Inductive Set, Natural Numbers)

A set $X \subset \mathbb{R}$ is inductive if for each number $x \in X$, it also contains $x+1$. The set of *natural numbers*j, denoted $\mathbb{N}$, is the smallest inductive set containing 1.

We can use this inductive property of natural numbers to prove properties of them. Note that this can only be used to prove for finite (yet unbounded) numbers!

Lemma 1.1 (Induction Principle)

Given $P(n)$, a property depending on positive integer n,
1. if $P(n_0)$ is true for some positive integer n_0, and
2. if for every $k \geq n_0$, $P(k)$ true implies $P(k+1)$ true,

then $P(n)$ is true for all $n \geq n_0$.

Lemma 1.2 (Strong Induction Principle)

Given $P(n)$, a property depending on a positive integer n,
1. if $P(n_0), P(n_0+1), \ldots, P(n_0+m)$ are true for some positive integer n_0, and nonnegative integer m, and
2. if for every $k > n_0 + m, P(j)$ is true for all $n_0 \leq j \leq k$ implies $P(k)$ is true,

then $P(n)$ is true for all $n \geq n_0$.

The idea behind the strong induction principle leads to the proof using infinite descent. Infinite descent combines strong induction with the fact that every subset of the positive integers has a smallest element, i.e. there is no strictly decreasing infinite sequence of positive integers.

Lemma 1.3 (Infinite Descent)

Given $P(n)$, a property depending on positive integer, assume that $P(n)$ is false for a set of integers $\mathcal{S}$. Let the smallest element of $\mathcal{S}$ be n_0. If $P(n_0)$ false implies $P(k)$ false, where $k < n_0$, then by contradiction $P(n)$ is true for all n.

1.2 Countable and Uncountable Sets

Definition 1.2 (Equipotence)

Two sets A and B are **equipotent**, written $A \approx B$, if there exists a bijective map $f : A \to B$. This implies that their cardinalities are the same: $|A| = |B|$. It has the following properties:
1. Reflexive: $A \approx A$
2. Symmetric: $A \approx B$ implies $B \approx A$
3. Transitive: $A \approx B$ and $B \approx C$ implies $A \approx C$

Definition 1.3 ()

For any positive integer n, let J_n be the set whose elements are the integers $1, 2, \ldots, n$. For any set A, we define
1. A is **finite** if $A \approx J_n$ for some n. The empty set is also considered to be finite.

2. A is **infinite** if it is not finite.
3. A is countable if $A \approx \mathbb{N}$.
4. A is uncountable if A is neither finite nor countable.
5. A is at most countable if A is finite or countable.

At this point, we may already be familiar with the fact that $\mathbb{Q}$ is countable and $\mathbb{R}$ is uncountable. Let us formalize the statement that a countable infinity is the smallest type of infinity. We can show this by taking a countable set and showing that every infinite subset must be countable. If it was uncountable, then this would mean that a countable set contains an uncountable set.

Theorem 1.1 ()

Every infinite subset of a countable set A is countable.

Theorem 1.2 ()

An at most countable union of countable sets is countable.

Theorem 1.3 ()

A finite Cartesian product of countable sets is countable.

Corollary 1.1 ()

$\mathbb{Q}$ is countable.

Now, how do we prove that a set is uncountable? We can't really use the contrapositive of Theorem 1.2, since to prove that an arbitrary set A is uncountable, then we must find an infinite subset that is not countable. But now we must prove that this subset itself is not countable, too! Therefore, we can use this theorem.

Theorem 1.4 ()

Given an arbitrary set A, if every countable subset B is a proper subset of A, then A is uncountable.

Proof.

Assume that A is countable. Then A itself is a countable subset of A, but by the assumption, A should be a proper subset of A, which is absurd. Therefore, A is uncountable.

Theorem 1.5 ()

Let A be the set of all sequences whose elements are the digits 0 and 1. Then, A is uncountable.