

Algebraic Topology

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1 Homotopy

Definition 1.1. Let X, Y be topological space and let $F_0, F_1 : X \rightarrow Y$ be continuous maps. A *homotopy* from F_0 to F_1 is a continuous map (with respect to elements $t \in [0, 1]$)

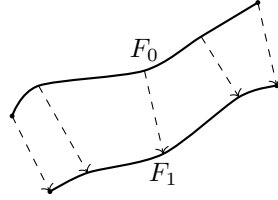
$$H : X \times I \rightarrow Y$$

where $I = [0, 1]$, satisfying

$$H(x, 0) = F_0(x)$$

$$H(x, 1) = F_1(x)$$

for all $x \in X$. We can visualize this homotopy as a continuous deformation of (the images of) F_0 to F_1 . We can also think of the parameter t as a "slider control" that allows us to smoothly transition from F_0 to F_1 as the slider moves from 0 to 1, and vice versa. The figures below represents the homotopies between the one-dimensional curves (left) and 2-dimensional surfaces (right), $\text{Im } F_0$ and $\text{Im } F_1$, with dashed lines.



If there exists a homotopy from F_0 to F_1 , then we say that F_0 and F_1 are *homotopic*, denoted

$$F_0 \simeq F_1$$

Definition 1.2. If the homotopy satisfies

$$H(x, t) = F_0(x) = F_1(x)$$

for all $t \in I$ and $x \in S$, which is a subset of X , then the maps F_0 and F_1 are said to be *homotopic relative to S* .

This is clearly an equivalence relation defined on $C^0(X, Y)$, the set of all continuous functions from X to Y .

1. Identity. Clearly, F is homotopic to itself by setting $H(x, t) \equiv F(x)$ for all $t \in [0, 1]$.
2. Symmetry. Given homotopy $H(x, t)$ from F_0 to F_1 , the homotopy $H^{-1}(x, t) \equiv H(x, 1 - t)$ maps from F_1 to F_0 .
3. Transitivity. Given homotopy H_1 from F_1 to F_2 , and homotopy H_2 from F_2 to F_3 , the homotopy defined

$$H_3(x, t) \equiv \begin{cases} H_1(x, 2t) & 0 \leq t \leq \frac{1}{2} \\ H_2(x, 2t - 1) & \frac{1}{2} \leq t \leq 1 \end{cases}$$

is indeed a homotopy from F_1 to F_3 .

Definition 1.3. The space of homotopy classes from topological space X to Y is denoted

$$[X, Y] \equiv \frac{C^0(X, Y)}{\sim}$$

where \sim is the homotopy relation.

Lemma 1.1. *Homotopy is compatible with function composition in the following sense. If $f_1, g_1 : X \rightarrow Y$ are homotopic, and $f_2, g_2 : Y \rightarrow Z$ are homotopic, then $f_2 \circ f_1$ and $g_2 \circ g_1$ are homotopic. That is, given the two homotopies*

$$\begin{aligned} H_1 : X \times [0, 1] &\rightarrow Y \\ H_2 : Y \times [0, 1] &\rightarrow Z \end{aligned}$$

we can naturally define a third homotopy

$$H_3 : X \times [0, 1] \rightarrow Z, \quad H(x, t) \equiv H_2(x, t) \circ H_1(x, t)$$

which is continuous since compositions of continuous functions are continuous.

Example 1.1. *If $f, g : \mathbb{R} \rightarrow \mathbb{R}^2$ is defined as a*

$$f(x) \equiv (x, x^3), \quad g(x) \equiv (x, e^x)$$

then the map

$$H : \mathbb{R} \times [0, 1] \rightarrow \mathbb{R}^2, \quad H(x, t) \equiv (x, (1-t)x^3 + te^x)$$

is a homotopy between them.

Example 1.2. *Let $id_B : B^n \rightarrow B^n$ be the identity function on the unit n -disk, and let $c_0 : B^n \rightarrow B^n$ be the 0-function sending every vector to 0. Then, id_B and c_0 are homotopic, with homotopy explicitly defined*

$$H : B^n \times [0, 1] \rightarrow B^n, \quad H(x, t) \equiv (1-t)x$$

Example 1.3. *If $C \subseteq \mathbb{R}^n$ is a convex set and $f, g : [0, 1] \rightarrow C$ are paths with the same endpoints, then there exists a linear homotopy given by*

$$H : [0, 1] \times [0, 1] \rightarrow C, \quad (s, t) \mapsto (1-t)f(s) + tg(s)$$

We can extend this example. Let $f, g : \mathbb{R} \rightarrow \mathbb{R}$ be 2 continuous functions. Then $f \simeq g$, since we can construct $F : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ defined

$$F(x, t) \equiv (1-t)f(x) + tg(x)$$

(Note that the set of continuous functions from \mathbb{R} to \mathbb{R} is a convex set.)

This leads to our definition of *path homotopies*, which is just a specific type of homotopy.

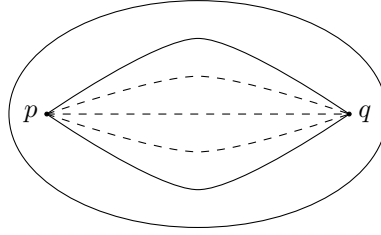
Definition 1.4. Suppose X is a topological space. Two paths $f_0, f_1 : I \rightarrow X$ are said to be *path homotopic*, denoted

$$f_0 \sim f_1$$

if they are homotopic relative to $\{0, 1\}$. This means that there exists a continuous map $H : I \times I \rightarrow X$ satisfying

$$\begin{aligned} H(s, 0) &= f_0(s), \quad s \in I \\ H(s, 1) &= f_1(s), \quad s \in I \\ H(0, t) &= f_0(0) = f_1(0), \quad t \in I \\ H(1, t) &= f_0(1) = f_1(1), \quad t \in I \end{aligned}$$

We can visualize two paths (sharing the same endpoints) being path homotopic if we can "continuously deform" one onto another.



We can notice that for any given points $p, q \in X$, path homotopy is an equivalence class on the set of all paths from p to q .

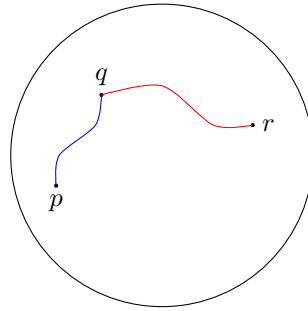
Definition 1.5. The equivalence class of a path f is called a *path class*, denoted $[f]$. Note that in the diagram above, there is only one equivalence class of paths.

We can define a multiplicative structure on paths as such. This is the first step to create a group structure on the set of certain paths.

Definition 1.6. Given two paths f, g such that $f(1) = g(0)$, their product is the path defined

$$(f \cdot g)(s) \equiv \begin{cases} f(2s) & 0 \leq s \leq \frac{1}{2} \\ g(2s - 1) & \frac{1}{2} \leq s \leq 1 \end{cases}$$

It is easy to visualize the product of two paths as the longer path created by "connecting" the two smaller paths.



It is also easy to see that if $f \sim f'$ and $g \sim g'$,

$$f \cdot g \sim f' \cdot g'$$

We can also define the product of these equivalence classes as

$$[f] \cdot [g] \equiv [f \cdot g]$$

Notice that multiplication of paths is not associative in general, but it is associative up to path homotopy. That is,

$$([f] \cdot [g]) \cdot [h] = [f] \cdot ([g] \cdot [h])$$

Definition 1.7. If X is a topological space and $q \in X$, a "loop" in X based at q is a path in X such that

$$f : I \longrightarrow X, f(0) = f(1) = q$$

The set of path classes of loops based at q is denoted

$$\pi_1(X, q)$$

Equipped with the product operation of paths defined before, $(\pi_1(X, q), \cdot)$ is called the *fundamental group of X based at q* . The identity element of this group is the path class of the constant path $c_q(s) \equiv q$, and the inverse of $[f]$ is the path class of

$$f^{-1}(s) \equiv f(1 - s)$$

which is the reverse path of f .

Note that while the fundamental group in general depends on the point q , it turns out that, up to isomorphism, this choice makes no difference as long as the space is path connected.

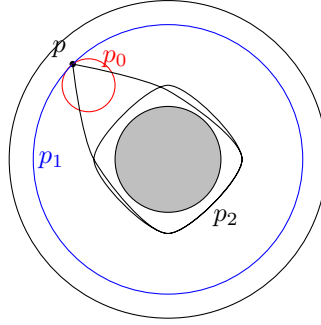
Lemma 1.2. *Let X be a path connected topological space, with $p, q \in X$. Then,*

$$\pi_1(X, p) \simeq \pi_1(X, q)$$

for all p, q .

Therefore, it is conventional to write $\pi_1(X)$ instead of $\pi_1(X, q)$ when X is path connected.

Example 1.4. *Consider the space $X \equiv B_2 \setminus B_1$, which is the 2-disk without the unit disk in \mathbb{R}^2 . Given an arbitrary point $p \in X$, there exists an infinite number of path classes of X at p , denoted $[p_i]$, where i corresponds to how many times the paths loop around the hole. The first three path classes are shown below.*



It is clear that $[p_0]$ is the identity, and the group operation rule is

$$[p_i] \cdot [p_j] = [p_{i+j}]$$

meaning that $\pi_1(X, p)$ is the infinite discrete group generated by $[p_0]$ and $[p_1]$.

Proposition 1.3. *Let \mathcal{A} be a convex subset of \mathbb{R}^n , endowed with the subspace topology, and let X be any topological space. Then, any 2 continuous maps $f, g : X \rightarrow \mathcal{A}$ are homotopic.*

Proof. Since \mathcal{A} is convex, the homotopy defined

$$F(x, t) \equiv (1 - t)f(x) + tg(x)$$

exists. ■

Proposition 1.4. *If X is a path connected space, the fundamental groups based at different points are all isomorphic. That is,*

$$\pi_1(X, p) \simeq \pi_1(X, q)$$

for all $p, q \in X$.

Definition 1.8. If X is path connected and for some $q \in X$, the group $\pi_1(X, q)$ is the trivial group consisting of $[c_q]$ alone, then we say that X is *simply connected*. By definition, this means that every loop is path homotopic to a constant path.

Proposition 1.5. *Let X be a path connected topological space. X is simply connected if and only if any 2 loops based on the same point are path homotopic.*

We can also expect that since homotopy is clearly a topological property, it is preserved under continuous maps. We state this result formally in the following lemma.

Lemma 1.6. *If $F_0, F_1 : X \rightarrow Y$ and $G_0, G_1 : Y \rightarrow Z$ are continuous maps such that $F_0 \simeq F_1$ and $G_0 \simeq G_1$, then*

$$G_0 \circ F_0 \simeq G_1 \circ F_1$$

Similarly, if $f_0, f_1 : I \rightarrow X$ are path homotopic, and $F : X \rightarrow Y$ is a continuous map, then

$$F \circ f_0 \sim F \circ f_1$$

Thus, if $F : X \longrightarrow Y$ is a continuous maps, for each $q \in X$, we can construct a well-defined map

$$F_* : \pi_1(X, q) \longrightarrow \pi_1(Y, F(q))$$

by setting

$$F_*([f]) \equiv [F \circ f]$$

Lemma 1.7. If $F : X \longrightarrow Y$ is a continuous map, then the induced map

$$F_* : \pi_1(X, q) \longrightarrow \pi_1(Y, F(q))$$

is a group homomorphism. ^x That is, F_* preserves multiplicative structure of the loops.

Theorem 1.8 (Properties of the Induced Homomorphism). 1. Let $F : X \longrightarrow Y, G : Y \longrightarrow Z$ be continuous maps. Then for any $q \in X$,

$$(G \circ F)_* = G_* \circ F_* : \pi_1(X, q) \longrightarrow \pi_1(Z, G(F(q)))$$

2. For any space X and any $q \in X$, the homomorphism induced by the identity map $id_X : X \longrightarrow X$ is the identity map

$$id : \pi_1(X, q) \longrightarrow \pi_1(X, q)$$

3. If $F : X \longrightarrow Y$ is a homeomorphism, then

$$F_* : \pi_1(X, q) \longrightarrow \pi_1(Y, F(q))$$

is an isomorphism. That is, homeomorphic spaces have isomorphic fundamental groups.

Example 1.5. The fundamental group of $S^1 \subset \mathbb{C}$ based at 0 is the infinite cyclic group generated by the path class of the loop

$$\alpha : I \longrightarrow S^1, \alpha(s) \equiv e^{2\pi is}$$

Theorem 1.9. If $F : X \longrightarrow Y$ is a homotopy equivalence, then for each $p \in X$,

$$F_* : \pi_1(X, p) \longrightarrow \pi_1(Y, F(p))$$

is an isomorphism.

The following proposition will be revisited when studying manifolds.

Proposition 1.10. The fundamental group of any topological manifold is countable.

1.1 Homotopy Equivalence

Definition 1.9. Given two topological spaces X and Y , a homotopy equivalence between X and Y is a pair of continuous maps $f : X \longrightarrow Y$ and $g : Y \longrightarrow X$ such that

$$g \circ f \simeq id_X \text{ and } f \circ g \simeq id_Y$$

The equivalence classes under \simeq are called *homotopy types*. If such a pair f, g exists, X and Y are said to be *homotopy equivalent*, or of the same homotopy type.

Definition 1.10. Spaces that are homotopy equivalent to a point are called *contractible*. That is, X is contractible if and only if

$$X \simeq \{x_0\}$$

Visually, two spaces are homotopy equivalent if they can be transformed into one another by bending, shrinking, and expanding operations.

Example 1.6. A solid disk is homotopy equivalent to a single point, since one can deform the disk along radial lines to a point.

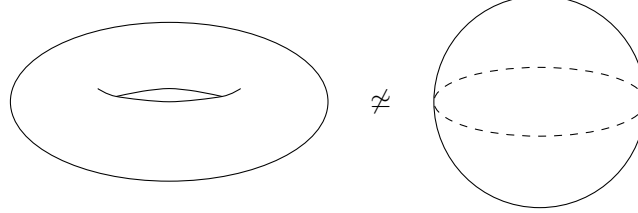
Example 1.7. A mobius strip is homotopy equivalent to a closed (untwisted) strip.

Notice from the visualization of homotopy equivalence the following proposition.

Proposition 1.11. X, Y homeomorphic $\implies X, Y$ homotopy equivalent. However, the converse is not true.

Proof. Just set $f = f$ and $g = f^{-1}$. ■

Example 1.8. A torus is not homotopy equivalent to Y , which also implies that they are not homeomorphic either.



Furthermore, like homeomorphisms, homotopy equivalence is a relation on the set of all topological spaces.

1. Identity. Just set $f, g = id_X$
2. Symmetricity. Given $X \simeq Y$ with $f : X \longrightarrow Y, g : Y \longrightarrow X$, we set $f' \equiv g$ and $g' \equiv f$ and use these functions f', g' to find out that $Y \simeq X$.
3. Transitivity. Let us have $X \simeq Y$ with functions f_1, g_1 and $Y \simeq Z$ with functions f_2, g_2 . Then, we define new functions

$$f_3 \equiv f_2 \circ f_1 : X \longrightarrow Z, g_3 \equiv g_1 \circ g_2 : Z \longrightarrow X$$

which follows to $f_3 \circ g_3 = id_Z$ and $g_3 \circ f_3 = id_X$.

Proposition 1.12. \mathbb{R}^n is homotopically equivalent to a point $\{0\}$.

Proof. We claim that the continuous maps (canonical injection and projection)

$$id_{\mathbb{R}^n} : \{0\} \longrightarrow \mathbb{R}^n, p_0 : \mathbb{R}^n \longrightarrow \{0\}$$

have the property that

$$id_{\mathbb{R}^n} \circ p_0 \simeq id_{\mathbb{R}^n}, p_0 \circ id_{\mathbb{R}^n} \simeq id_{\{0\}}$$

The right-hand homotopy is trivial since $id_{\mathbb{R}^n} \circ p_0 = id_{\mathbb{R}^n}$, and as for the left-hand homotopy, we can explicitly define it as

$$H : [0, 1] \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$$

with

$$H(t, x) \equiv (t)(id_{\mathbb{R}^n} \circ p_0)(x) + (1 - t)id_{\mathbb{R}^n}(x) = (1 - t)id_{\mathbb{R}^n}(x)$$
■

Example 1.9. $S^1 \simeq \mathbb{R}^2 \setminus \{0\}$, and more generally, $S^{n-1} \simeq \mathbb{R}^n \setminus \{0\}$. We can see this with the canonical injection and projections

$$id_{\mathbb{R}^2} : S^1 \longrightarrow \mathbb{R}^2 \setminus \{0\}, \pi_{S^1} : \mathbb{R}^2 \setminus \{0\} \longrightarrow S^1$$

and find that

$$id_{\mathbb{R}^2} \circ \pi_{S^1} \simeq id_{\mathbb{R}^2}, \pi_{S^1} \circ id_{\mathbb{R}^2} \simeq id_{S^1}$$

where the right-hand homotopy is trivial, and the left hand homotopy is defined explicitly as

$$H(x, t) \equiv t(id_{\mathbb{R}^2} \circ \pi_{S^1})(x) + (1 - t)(id_{\mathbb{R}^2})(x)$$

Definition 1.11. A function f is said to be *null homotopic* if it is homotopic to a constant function. This is sometimes called a *null-homotopy*.

Example 1.10. Take a look at a function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$, which represents an arbitrary surface in $\mathbb{R}^2 \oplus \mathbb{R}$. Now, observe the constant function $c(x, y) \equiv c$, which represents a plane parallel to the x, y -plane. Clearly, we can imagine a deformation of the surface of f to the flat surface of c with the homotopy

$$H(x, t) \equiv t f(x) + (1 - t)c(t)$$

which visually represents a linear deformation of c to f . Therefore, f is null-homotopic.

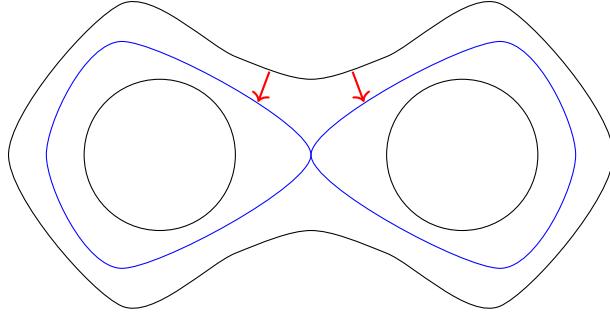
Example 1.11. A map $f : S^1 \rightarrow X$ is null homotopic precisely when it can be continuously extended to a map

$$\tilde{f} : D^2 \rightarrow X$$

that agrees with f on the boundary $\partial D^2 = S^1$. Visually, the existence of \tilde{f} allows us to continuously deform the image of f in $S^1 \oplus X$ to a level curve $f(x) = c$ existing in $S^1 \oplus X$.

Proposition 1.13. A space X is contractible if and only if the identity map from X to itself, which is always a homotopy equivalence, is null homotopic.

Example 1.12. Let Y be the following gray subset of the plane, and let X be the figure-8 shape.



Then $Y \simeq X$, where the corresponding functions are

$F : X \rightarrow Y$, the canonical inclusion

$G : Y \rightarrow X$, the projection onto X

Then, $G \circ F = \text{id}$ and $F \circ G$ is homotopic to the identity, with homotopy defined

$$H(x, t) \equiv t(F \circ G)(x) + (1 - t)(\text{id}_Y)(x)$$

which can be visualized by $H(x, s)$ being the point you get from x by moving a fraction s along the red arrow towards X .

2 Homeomorphism Groups

Definition 2.1. The *homeomorphism group* of a topological space X is the group consisting of all homeomorphisms from X to X , with function composition as the group operation.