Stock Trading

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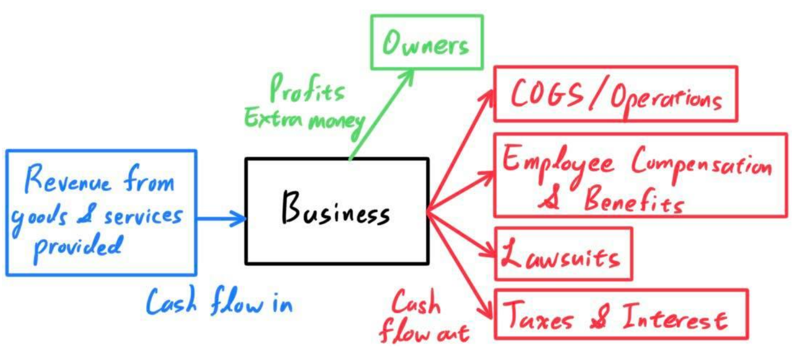
A course on equity and derivative markets and relevant trading strategies. Contains information presented in Duke Math 585: Algorithmic Trading.

1 Equity Markets

Stocks are a type of security that represents a share in the ownership of a corporation. That is, owning a stock is equivalent to owning its relative position in the company and its profits, and possibly voting rights. But what exactly does it mean to "own" a company? To explain this, we can interpret a business to be some sort of entity with input and output cash flows.

1. The majority of the cash flows flowing into the business is the revenue earned from the goods and services that the business provides.
2. The majority of the cash flows flowing out of the business are the costs, such as COGS, employment compensation and benefits, operations, lawsuits, taxes, interest on debts, etc.

Ideally, the company will have a net profit, or a net positive cash flow, which means that the cash flows in is greater than the cash flows out. All this profit, i.e. this extra money, now goes to the owners of this business through **dividends** (or may be reinvested into the company). Note that it is not the case that the profits go to the CEO. The CEO is a glorified version of an employee, but still in the end an employee, with an income just like every other employee, albeit a very high one.



Obviously, the owner has a bigger role to play rather than collect profits. They must also decide on things. But since there are usually many owners, these decisions are made through votes in meetings like the annual shareholder meeting. The ownership of a stock really possible rights to earn dividends and possible rights to vote. There is a much greater variety of stocks, but we will go through common ones.

1. **Common stocks** are the most common type of stock. Common stock shareholders receive dividends in proportion to the amount of profit generated each year, and they have voting rights. They are usually the lowest priority over a company's income, after the creditors and preferred stock shareholders.
2. **Preferred stock** are much less common and liquid. Preferred stock shareholders receive fixed dividends and no voting rights. They receive payments after bondholders but before common shareholders.

Stocks may also have the following characteristics below. Generally, we work with common and preferred stocks, but other ones may be labeled using letters (e.g. Class A, Class B stocks) that have certain qualities mentioned below.

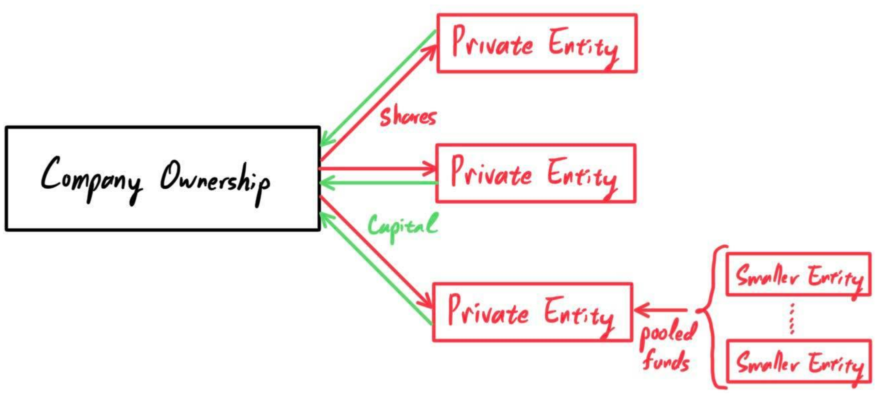
1. **Voting Power**: Nonvoting shares have no voting power while executive shares can be worth 10 votes.
2. **Payment Priority**: Deferred shares are set as a lower priority for dividends and corporate assets.
3. **Cumulative shares** can accumulate dividend payments that have been deferred due to low profits in the past.
4. **Convertible shares** can be converted into different forms of financial assets (e.g. preferred shares may be convertible into common ones, common shares may be convertible into corporate bonds).

Note that if an entity (individual or organization) has the majority ownership or a **controlling interest** (at least 50% in voting rights) of a company, then they have the power to do pretty much whatever they want with the company.

1.1 Primary Market: Private Funding, Public Offerings, Underwriting

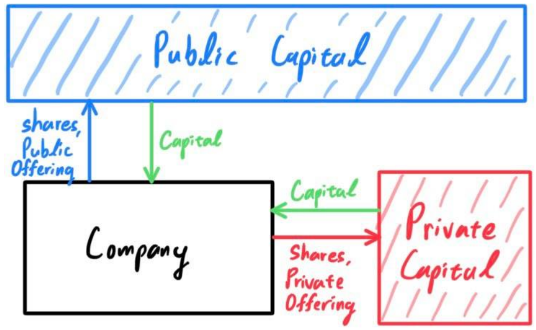
A company gets its funding from debt (e.g. bonds) and equity (stocks). If the prospect of paying interest on debts are not suitable, a company may issue stocks to investors. In other words, the company is selling parts of itself in exchange for capital. The number of shares, type of shares, and the prices for these shares is entirely determined by the seller and buyer. For example, a business may need to raise \$100 million in capital and can choose to issue 10 million shares at \$10 each.

For early-stage businesses, these are mainly funded by **venture capital** firms and **angel investors** (but is not limited to them) through different rounds roughly ordered: seed round, Series A, Series B, Series C, etc. After due diligence, these venture firms, as early investors, may get extra privileges, such as anti-dilution protection, guaranteed board seats, liquidation preferences, priority dividends. Note that upon multiple rounds of issuance, more and more stocks will circulate through more hands, causing **stock dilution**. No investor would want this and therefore stocks must be issued carefully. At this point, the ownership of a company is held within the hands of private individuals and firms.

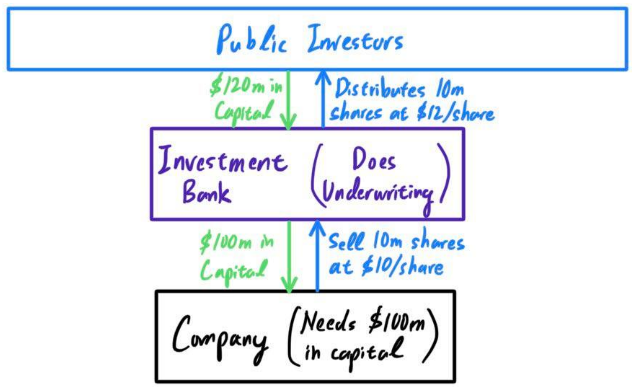


One final characteristic to note is the **nominal value**, or **par value**, of a company's stock. Like the face value of a bond when issued, the nominal value of the stock is its stated value. It is an arbitrary value assigned for balance sheet purposes when the company is issuing share capital, and is typically \$1 or less. It has little to no bearing on the stock's market price, so no need to worry about this number.

Perhaps after a few years, the firm has grown to the point where funding on an even larger scale to support even more expansion is needed. Private sources may be too restrictive or small, and so companies may need to tap into the capital of the general public. Thus, they can do a **public offering**, an issuance of the stock to the general public rather than private entities. This process of an **initial public offering (IPO)** and the company being listed on a stock exchange is what is referred to as a company "going public."



Investment banks, through underwriting services, facilitate public offerings.



1.1.1 Additional Actions on Stocks

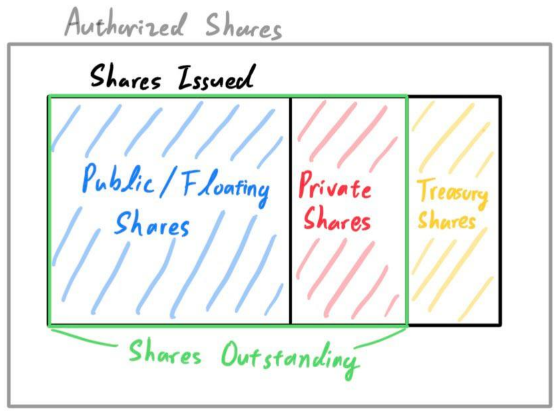
Throughout the years, companies may do the following:

1. Additional **stock issuance** to both public and private investors.
2. **Stock splits** to increase liquidity if the stock price is too high or for other reasons. **Reverse stock splits** may also be done.
3. **Stock buyback** where the company uses its institutional funds to buy back shares from the market to increase its **treasury shares**. This may be used for anti-dilution purposes or to retain voting control. We can think of these as the opposite as a stock issuance, since treasury shares do not have dividend rights nor voting rights.
4. Issue **restricted stock units (RSUs)** to employees. The grant is restricted because it is subject to a vesting schedule, which can be based on length of employment or performance goals.

With these, we can organize the number of stocks circulating in different levels, from biggest to smallest:

1. Authorized shares refers to the maximum number of shares that a corporation is legally permitted to issue. Companies don't usually get close to this number due to market conditions, which will be explained later.
2. Shares issued refers to the total number of shares issued, including all public shares, private shares, and treasury shares.
3. Outstanding shares refer to a company's stock currently held by all shareholders, both public and private shares (including RSUs). It does not include treasury stocks however.
4. Floating shares are the shares considered available for the general public. Moreover, the floating percentage represents the portion of outstanding shares that are floating:

$$\text{Floating Percentage} = \frac{\text{Floating Shares}}{\text{Shares Outstanding}}$$



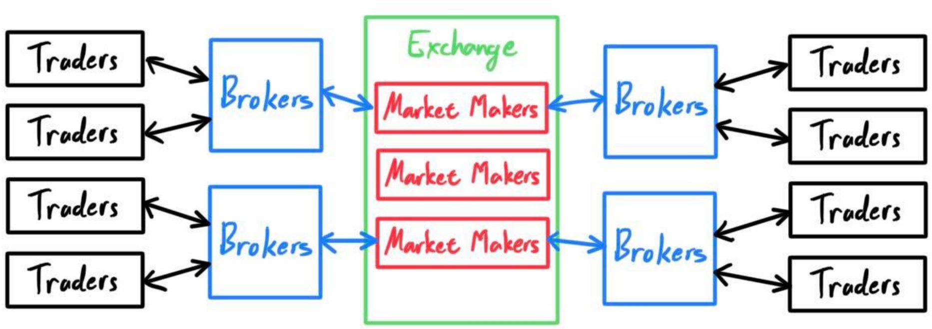
1.2 Secondary Market

1.2.1 Involved Parties and Regulations

Let us go through the basic construction of a market. When people want to buy and sell things, they must go to an **exchange**, which is literally a physical space that is leased to various parties (e.g. large banks, hedge funds) so that they can exchange goods. These exchanges also collect money through fees from companies to be listed on NYSE. There are two big exchanges, the NYSE and the NASDAQ, in the U.S., both located in New York.

Us retail traders are not direct participants of the exchange; the parties that participate there are called the **market makers**, or **liquidity providers**. Market makers, usually large banks or financial institutions (like hedge funds), make sure that there is enough trading volume to ensure liquidity in the market. A buyer and a seller must meet together to complete a deal, and to ensure that this happens smoothly (i.e. provides liquidity), a market maker buys stocks (through the individual's brokerage) and sells them to the corresponding recipient. Essentially, they provide a pool of shares and act as intermediaries between them. They profit from the bid-ask spread, though sudden volatility is always a risk. For example, if a crash happens (i.e. a market maker buys a stock and it tanks before they can sell it), then the market makers get screwed because they are left with an undervalued stock.

The trader cannot directly trade through the market makers. They must contact their **broker**, which is another company that acts as an intermediary. If I want to buy a stock, my buy order gets sent to my broker, which gets sent to the market makers, which pairs me up with a seller through their own broker, and the transaction is completed. These brokers make money through commissions from the traders and from trading in dark pools, which we'll talk about later. They also give access to traders the forecasts of analyst reports for companies and other research.



Clearly, there can be some shady stuff going on here, but luckily, the SEC and the FINRA consistently regulate the markets to ensure fairness for the little players (retail investors). One of the most important regulations is the 2005 Regulation NMS (National Market System), which required exchanges to publish the best bid and offer price for each stock, required them to route orders to the trading venue with the best price, and had set the minimal price quotation increment to \$0.01. This ensured transparency and protection of the investor with the best price execution (but this can be taken advantage of by high frequency traders).

1.2.2 Stock Orders

The two most common orders one can do is the market order and the limit order.

1. A **market order** just buys or sells securities at the market price. This ensures that the transaction will be completed, but the price at which you buy or sell may not be what you want.
2. A **limit order** buys or sells at least at a certain price. It ensures that you get the price you want for a transaction, but it may not be carried out always.
   (a) A buy limit order at \$X tells the broker to buy a stock at \$X or lower.
   (b) A sell limit order at \$x tells the broker to sell a stock at \$ or higher.
3. A **stop loss order** is used to limit an individual's loss or lock in a profit on a stock position.
   (a) If an investor buys a stock at \$100, they can make a stop loss at \$95. This means that when the stock price reaches \$95, then it will automatically place a market order to sell the stock (the price may not be fulfilled at exactly \$95 due to volatility).
   (b) If an investor has a short position, on the stock, then they can put a stop loss order at \$105, essentially telling the broker to buy the stock when the price reaches \$105.

In addition to holding a long position, we can **short sell**, or short, a stock. Given that 100 shares of AAPL are each priced at \$100, we can borrow shares from some investor (probably an institutional investor), sell it on the market for \$100, and hope for the price to drop so we can buy it back. This is pretty symmetric to longing a stock, and many hedge funds use a **long-short equity strategy** involving a combination of longs and shorts, but there are additional risks.

1. You must pay additional interest for borrowing stocks.
2. Your potential losses is not bounded.
3. If you have too much unrealized losses as the stock price goes up, then you may not have enough **margin** (cash) in your account to even buy back to stock. To prevent this, your broker may issue a **margin call**, which forces you to buy the stocks back, locking in your loss, unless you add more capital to your account. This margin call may not happen right at the point when your free cash is not enough to buy back all shorted shares. Rather, your brokerage may run some complicated statistical simulations, calculate some confidence interval, and then decide on a threshold that determines whether you will get a margin call.

The third risk is the deadliest, and at worst, you can get **short-squeezed**. Let's explain how this works. Say you and a bunch of other investors shorted GME. GME prices start going up, and it reaches a point where some investors have to buy back the GME shares due to a margin call. The investors buy back the GME at market price, which adds more orders to the limit order book, causing the price to go even higher. This higher price causes more short-sellers to buy back GME due to additional margin calls, causing GME to go even higher, and so on. This positive feedback loop is extremely deadly, killing off many short sellers.

Sometimes, the ethics of short selling are called into question. People who want to ban shorting state that short selling can drive down the stock price, which can be bad because it doesn't spur the economy and reduces optimism in stock markets. Short selling means that stock is being borrowed and sold on the market, increasing supply, and therefore (all else being equal) decreasing price. However, these short sellers keep the stock price more in line with reality, and bad companies are punished by short sellers.

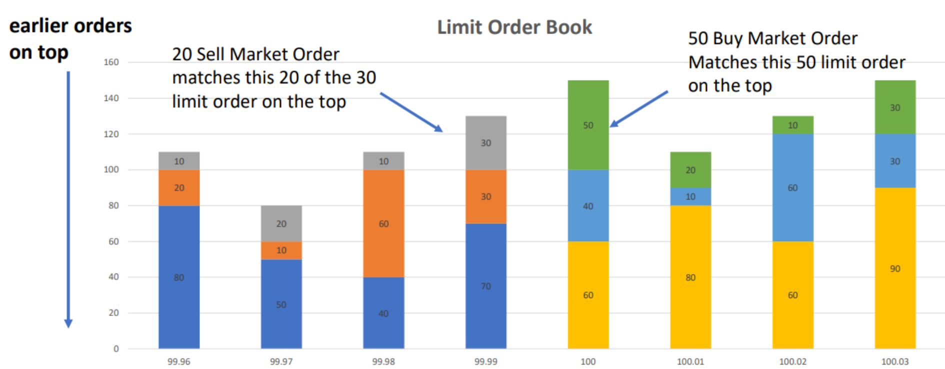
Finally, the entities that drive this short selling are **prime brokers**, which are large financial institutions (usually investment banks) that provide financial management to mainly hedge funds. They act on behalf of the short seller, locating the assets to be sold short for them and providing them with a margin account (which must hold capital to the sum of at least 150% of the value of the initial transaction). They enable hedge funds to borrow large amounts of stocks from institutional investors to short-sell them and allow them to access large amounts of margin from commercial banks. The prime brokerage makes money through commissions. Prime brokers can also loan capital to investors to increase their leverage, and they also provide financial research and analytics to their clients.

1.2.3 Stock Prices and Limit Order Books

Now how is the share price determined? There are two large paradigms for this. The first is the **fundamental approach**, which attempts to calculate the intrinsic value fo the company by calculating its discounted future cash flows.

$$\text{PresentValue} = \sum_{n=1}^{\infty} \frac{\text{Cash Flow}_1}{(1+R)^n}$$

However, we will concern ourselves with the second approach: the **market approach**, which claims that the stock price is determined by supply and demand. This can be seen by looking at the **limit order book** (LoB), which shows all pending limit orders at the current time. This allows us to see where the prices are concentrated at and how strong the demand is versus the supply. In the diagram below, the right side represent the sell limit orders while the left side represents the buy limit orders. If we submit 50 buy market orders, then the best price is executed.



The market makers profit off of the **bid-ask spread**, which is the difference in the buy and sell price. If a seller wants to sell at \$99.99 and a buyer wants to buy at \$100.00, then the market maker can buy the stock from the seller for \$99.99 and sell it to the buyer at \$100.00, making a \$0.01 profit.

1.2.4 High Frequency Trading

A significant mover of markets are **high frequency traders**, or HFTs. HFT is a type of algorithmic trading that are characterized by high speeds and high turnover rates. It is the primary type of algorithmic trading and consists of about 50% of all equity trades in the U.S. There is an average of about 1 trader per 10 milliseconds. Whether HFT is beneficial is very controversial:

1. Some argue that it is good since it increases liquidity and lowers transaction costs for retail investors. Even for potentially risky stocks (e.g. very overvalued), HFTs will always make a market out of them.

2. It can be considered unfair because it gives a huge advantage to HFT firms through front-running and other short-term strategies. This also increases the probability of **flash-crashes** (e.g. 2010 flash crash).

Unsurprisingly, many HFT firms are market makers due to their ability to provide liquidity.

The primary form of HFT is **scalping**, which profits off of small price changes. The multiple benefits and drawbacks are easy to spot:

1. It requires a strict exit strategy since one large loss could eliminate the many small gains the trader worked to obtain.

2. It also requires a much higher ratio of winning trades vs losing ones, while keeping profits roughly equal or slightly bigger than losses.
3. A brief exposure to the market diminishes the probability of running into an adverse event like a crash.
4. Smaller moves are easier to obtain, so there are plenty of opportunities to exploit.

This strategy is extremely sensitive that even the physical location of the firm relative to the exchange matters. Heavy money in infrastructure is invested.

1.2.5 Market Manipulation

Market manipulation is a type of market abuse where there is a deliberate attempt to interfere with the free and fair operation of the market.

1. **Stock Bashing**: Creating fake news about a company to drive prices down and get shares for a cheaper price. The perpetrators sometimes work directly for Investor Relations firms who have convertible notes that convert cor more shares the lower the bid or ask price is. Thus, the bashers can drive the stock price down by convincing shareholders that they have bought a worthless security.
2. **Pump and Dump**: Misleadingly promoting a company to drive share prices up (the "pump", e.g. with bogus emails to investors). When the stock price reaches a target level, the promoter sells their shares (the "dump").
3. **Spoofing / Layering**: When a trader places a bid or offer on a stock with the intent to cancel before execution, these fake order trick other market participants by creating the false impression of heavy buying or selling pressure. Layering is an advanced form of spoofing with multiple orders that are "layered." For example, let's say that I want to buy stock XYZ, which is priced at \$100. If I wanted to use layering, I would put a sell order of 100 stocks at \$101, 100 at \$102, and 100 at \$103. An algorithm or trader might see this and believe that there is selling pressure and jump in front of these orders before they go down. They might sell at the market price of \$100, which may bring the price of XYZ a bit lower, to say \$99.95. I can buy at this price now.
4. **Front Running / Tailgating**: This is the practice of entering into a transaction that takes advantage of nonpublic knowledge of a large "block" pending transaction that will influence the price of an underlying security. Say a broker gets an order from a major client to buy 500,000 shares of XYZ Co. Such a huge purchase is bound to drive up the price of the stock immediately, at least in the short term. The broker sets aside the request for a minute and first buys some XYZ stock for their own personal portfolio or through accounts of relatives. Then the client's order is put through. The broker immediately sells the XYZ shares and pockets a profit. This can be used by HFT firms, since with the knowledge of large block buy orders from institutions, they can "buy up" all the orders in the market and sell it at a slightly higher price to the institutions.
5. **Quote Stuffing**: This is the practice of quickly entering and withdrawing a large number of orders in an attempt to flood the market, used by HFT firms. This can create confusion in the market (by filling up bandwidth and increasing latency of the data feed lines) and trading opportunities for high-speed algorithmic traders.

Another illegal act, which isn't market manipulation, is *insider trading*, which refers to a company insider who trades on advanced knowledge of corporate activities.

1.2.6 Dark Pools

The inefficient aspect about markets that they are sensitive to large **block trades**, which are defined to be a trader involving at least 10,000 shares or at least \$200,000 (though they can get much larger). They are usually made by institutional investors and are often privately negotiated in order to prevent market price changes and fluctuations. When an institutional investor would like to make a sell block trade, they can either

1. sell it on the exchange, where it will cause downwards pressure on the price, causing **slippage** (think of the limit order book: this sell order would clear out all buy orders past the sell price) and perhaps affecting the wider market. Even worse, once an order to sell a huge block has been filled, an investor can submit a buy order at a lower price in hopes that the block sell will hit his lowered price (this is an example of front running).

2. sell it privately off an exchange, but finding other parties to buy such a large amount is difficult.

Either way is quite unfavorable. The market impact of a sale of one million shares in Company XYZ could still be sizable regardless of which option the investor chose since it was not possible to keep the identity or intention of the investor secret in a stock exchange transaction.

These investors can trade in **dark pools**, which are privately organized financial exchanges for trading securities, also an **alternative trading system** (ATS). They were created originally to facilitate block trading by institutional investors who did not wish to impact the markets with their large orders. Most importantly, they keep trades anonymous. They allow traders to make block trades without having to publicize who they are, the buy/sell price, or the number of shares traded.

2 Fundamental Single Stock Analysis

2.1 Earnings Metrics

We will denote company metrics in bold letters and stock metrics in regular font.

Definition 2.1 (P/E Ratio)**.** Given a company XYZ, say that its earnings (profits) are **E** and its market cap **P**. Then, its price to earnings ratio is

$$\frac{\mathbf{P}}{\mathbf{E}}$$

which is a multiple of how much the company is worth given its earnings. Theoretically, it should be much greater than 1, since if it had a PE ratio of 1, then it can earn its entire worth annually, allowing the investor to break even on their investment within a year.

Definition 2.2 (Earnings per Share)**.** Given a company XYZ, say that its earnings are **E** and its number of shares outstanding **S**. Then its earnings per share is

$$\frac{E}{S}$$

which is basically telling us how much a company is earning per share. Note that given the stock price P and the EPS, note that the P/E ratio is simply

$$\frac{\mathbf{P}}{\mathbf{E}} = \frac{\mathbf{P}/\mathbf{S}}{\mathbf{E}/\mathbf{S}} = \frac{P}{EPS}$$

Definition 2.3 (PEG Ratio)**.** The PEG ratio is the P/E ratio, but now it accounts for the growth rate of the company earnings. Say that company XYZ has earnings **E** this year and $\mathbf{E}_0$ last year. Then, its earnings growth can be calculated as $\mathbf{G} = \mathbf{E}/\mathbf{E}_0 - 1$. Then, its PEG ratio is

$$\frac{\mathbf{P}/\mathbf{E}}{\mathbf{G}}$$

Definition 2.4 (P/S Ratio)**.** The P/S ratio is like the P/E ratio, but now we use the company's sales **S**, or revenue. This is particularly useful for when the company does not have net profits every year.

$$\frac{\mathbf{P}}{\mathbf{S}}$$

Definition 2.5 (ROIC)**.**

Definition 2.6 (ROA)**.**

3 Technical Single-Stock Analysis

Let us introduce some valuation metrics that are used for analyzing a portfolio. We will usually work with some sort of time series of the form $\{X_i\}_{i=0}^n$ (i.e. a discrete-time stochastic process), but if we are only worried about the initial and final price X_0 and X_n, then we will just mention them.

Definition 3.1 (Stock Price)**.** Given a stock P, we can construct a discrete-time stochastic process $\{P_i\}$, where P_i are random variables.

Definition 3.2 (Dividend)**.** Given time i, j with $i < j$, the dividend that a stock P pays within time interval $[i, j]$ is the random variable

$$D_{[ij]}$$

Definition 3.3 (Total Return)**.** Given stock price $\{P_i\}$, the amount of money you would have made from time i to J is the random variable

$$P_j - P_i + D_{[i,j]}$$

The **total return** during that period is defined

$$R_{[i,j]} = \frac{P_j - P_i + D}{P_i} = \frac{P_j - P_i}{P_i} + \frac{D_{[i,j]}}{P_i}$$

which is the price return plus the dividend rate. If this is a single-step return, i.e. $j = i + 1$, then we can write this as shorthand:

$$R_i = R_{[i-1,i]} = \frac{P_i - P_{i-1}}{P_{i-1}} \approx \ln\left(\frac{X_i}{X_{i-1}}\right) \text{ for } i = 1, \ldots, n$$

Sometimes, we refer to **return** as the total return without the dividend payment.

Most of the time, we work in **log returns** as an appropriate approximation of the return. This is mathematically justified since if we fix P_i and assume that P_j is close to P_i, we can take the Taylor expansion of $\ln(P_j)$ to be

$$\ln(P_j) \approx \ln(P_i) + \frac{1}{P_i}(P_j - P_i) \implies \frac{P_j - P_i}{P_i} \approx \ln(P_j) - \ln(P_i) = \ln\left(\frac{P_j}{P_i}\right)$$

Almost always, we will work with log returns and $R_{[i,j]}$ will denote the log return from time i to j. It has its advantages:

1. It is time-additive: If we have prices P_i, P_j, P_k for $i < j < k$, we can see that

$$R_{[i,j]} + R_{[j,k]} = \log\left(\frac{P_j}{P_i}\right) + \log\left(\frac{P_k}{P_j}\right) = \log\left(\frac{P_k}{P_i}\right) = R_{[i,k]}$$

2. Symmetricity: It is well known that a stock going down 50% and then going up 50% does not return the stock to its original price, despite it "looking" like it did. This effect is killed when looking at log returns. If $P_j = P_k(1-k)$ for $0 < k < 1$, then we know that it must scale up by a factor of $\frac{1}{1-k} \neq 1 + k$ to get the original price. Indeed, it is clearly the case that

$$\log(1-k) + \log(1+k) = \log(1-k^2) < 0$$

and in fact is always less than 0 (so we always lose money). This is due to the concavity of this function.

3. It focuses on relative change, which is typically invariant on the underlying price of the stock.

These properties allow us to focus on the returns to calculate our metrics.

Definition 3.4 (Expected Return)**.** The **expected return** of a stock from time i to j is simply the expected value

$$\mathbb{E}[R_{[i,j]}] = \mathbb{E}\left[\frac{P_j - P_i}{P_i}\right] \approx \mathbb{E}[\log(P_j) - \log(P_i)]$$

Definition 3.5 (Stock Volatility)**.** Given a stock P, its return $R_{[i,j]}$ is a random variable. Its standard deviation is called the **volatility**.

$$\sigma = \sqrt{\mathrm{Var}(R_{[i,j]})}$$

If we do have data on the stock $\{P_i\}_{i=0}^n$ and its log returns $\{R_i\}_{i=1}^n$, the volatility can be estimated simply by taking the sample standard deviation of this data

$$\sigma = \sigma(\{R_i\}_{i=1}^n) = \sqrt{\frac{1}{n}\sum_{i=1}^n (R_i - \mu)^2} \text{ where } \mu = \frac{1}{n}\sum_{i=1}^n R_i$$

Definition 3.6 (Sharpe Ratio)**.** The **Sharpe ratio** is a measurement of risk-adjusted return. Given that you have some stock price with total return R and volatility σ, the Sharpe ratio is defined as the random variable

$$S = \frac{R - R_f}{\sigma}$$

where R_f is risk-free return (e.g. the return you earn on U.S. Treasury bonds), which can be considered as a constant random variable. Sometimes, R_f may not be constant and may be some time-series data itself.

The Sharpe ratio is one of the most widely used metrics, and it is usually based on an annual horizon. While it is straightforward to compute the annual returns, it is more difficult to compute the annual volatility of a stock because it is sensitive to the time scale (i.e. computing it using daily data will result in much higher volatility than computing in minutely data). In practice, people follow the Square-root-of-Time rule, which takes the daily volatility by a factor of square root of 260, which is approximately the number of trading days in a year. Therefore, the invariant annual Sharpe ratio is

$$S = \frac{R - R_f}{\sigma\sqrt{260}}$$

3.1 Momentum Strategies

Momentum strategies can be divided into momentum trending, which bets that the stock price will continue following the trend, and momentum reversing, which bets that the stock price will revert back to some mean.

3.1.1 Moving Averages

The simple moving average and the exponential moving averages represent where the stock's price is at average.

Definition 3.7 (Simple Moving Average)**.** Let us have some stock price $\{P_i\}_{i=0}^N$ and fix some lookback parameter L. Then, the L-period **simple moving average (SMA)** of the stock is the average of prices of the past L periods, $\{S_i\}_{i=L-1}^N$, where

$$S_i = \frac{1}{L}\sum_{j=i-L+1}^{i} P_j$$

Definition 3.8 (Exponential Moving Average)**.**

We can build momentum strategies by comparing two different MAs of lookback periods $L_1 < L_2$. We can compare the current price to the moving averages, but the current price can be thought of as the 1-period moving average anyways. The L_2-MA is thought of as the long term trend of the stock, while he L_1-MA is the short-term trend. The follow are essentially MA strategies.

1. Momentum Trending:
   (a) If the L_1-MA is above the L_2-MA, then the stock has momentum upwards. $\implies$ Long.
   (b) If L_1-MA is below the L_2-MA, then the stock has momentum downwards. $\implies$ Short.
2. Momentum Reversing:
   (a) If the L_1-MA is above the L_2-MA, then the stock has momentum upwards. $\implies$ Short.
   (b) If L_1-MA is below the L_2-MA, then the stock has momentum downwards. $\implies$ Long.

3.1.2 Bollinger Bands

Definition 3.9 (Bollinger Bands)**.** Let us have some stock price $\{P_i\}_{i=0}^N$ and fix some lookback parameter L, along with some z-score Z. We compute the standard deviation of the prices P_i in the past L periods to get $\{\sigma_i\}_{i=L-1}^N$.

1. The **middle band** is defined to be the L-period SMA, which we will call
$$\{\mathcal{M}_i\}_{i=L-1}^N$$
2. The **upper band** is defined to be the band that is Z standard deviations above the middle band.
$$\{\mathcal{U}_i\}_{i=L-1}^N = \{M_i + Z\sigma_i\}_{i=L-1}^N$$
3. The **upper band** is defined to be the band that is Z standard deviations below the middle band.
$$\{\mathcal{L}_i\}_{i=L-1}^N = \{M_i - Z\sigma_i\}_{i=L-1}^N$$

Now the algorithm for Bollinger bands is very simple.

1. In a momentum following case, if P_i crosses below $\mathcal{L}_i$, we short, and if P_i crosses above $\mathcal{U}_i$, we long.
2. In a momentum reversing case, if P_i crosses below $\mathcal{L}_i$, we long, and if P_i crosses above $\mathcal{U}_i$, we short. The upper band acts as a resistance level, and the lower band acts as a support level.

3.1.3 Relative Strength Index

The relative strength index indicates the momentum or lack of it.

Definition 3.10 (Relative Strength Index)**.** Let us have some stock price $\{P_i\}_{i=0}^N$. Let us define the period changes as $\{D_i\}_{i=1}^N$ with $D_i = P_i - P_{i-1}$. Then, the total gain and total loss can be defined as
$$D_{\text{gain}} = \sum_{D_i > 0} |P_i - P_{i-1}|, \quad D_{\text{loss}} = \sum_{D_i < 0} |P_i - P_{i-1}|$$
Then, the **relative strength index (RSI)** is defined to be
$$RSI = 100 \, \frac{D_{\text{gain}}}{D_{\text{gain}} + D_{\text{loss}}}$$
which ranges in $[0, 100]$, where 0 is extremely bearish and 100 is extremely bullish. Roughly, the RSI being 30 means that for every 1 increase in a period, there are 2 decreases in other periods. The RSI being 70 means that for every 1 decrease, there are 2 increases.

The algorithm is quite simple:

1. In a momentum following case, if the RSI is below 30, we short, and if the RSI is above 70, we long.
2. In a momentum reversing case, if the RSI is below 30, we long, and if the RSI is above 70, we short.

3.1.4 Determining Momentum Trending or Reversing

Now how can we quantitatively decide whether we should use a momentum trending or reverting strategy? We can measure this behavior of a stock by looking at its autocorrelation.

Definition 3.11 (Autocorrelation)**.** Let us have some stock price $\{P_i\}_{i=0}^N$ and fix some lookback parameter L. We construct a lagged version $\{P_i\}_{i=0}^{N-L}$ and compute the correlation of this lagged time series with the original.
$$\text{Corr}(\{P_i\}_{i=0}^{N-L}, \{P_i\}_{i=L}^N)$$
is called the L-period **autocorrelation** of the stock price.

We can determine which strategy to use as follows. Let us assume that L is small.

1. If this autocorrelation is high (near 1), then this indicates that if the stock moves in a certain direction, it is likely to move in that same direction L periods later. So we should use a momentum following strategy.

2. If it is negative (near -1), it indicates that if the stock moves in a certain direction, it is likely to move in the opposite direction L periods later. So we should use a momentum reverting strategy.

Trend following stocks tend to be growth stocks, while trend reversing ones are the traditional conservative companies.

3.2 Pairs Trading

3.2.1 Naive Approach

Intuitively, pairs trading takes two stocks of very similar companies and bets that they will rise and fall together. If one rises and the other falls, then we can long the one that falls and short the one that rises, ultimately betting on the way that they will converge. To do this, take two sets of stock prices

$$\{P_i\}_{i=0}^N \text{ and } \{Q_i\}_{i=0}^N$$

We first talk about the naive approach, which assumes that if P rises by 5%, then Q will also rise by 5%: Beginners just look at their common ratios $\{P_i/Q_i\}_{i=0}^N$ and calculate whether this time series diverge or not. Remember that this is not symmetric, so we must log both of them and look at the log of the quotient, i.e. the difference of the logs.

$$\bigg\{ \log\Big(\frac{P_i}{Q_i}\Big)\bigg\}_{i=0}^N = \{\log(P_i) - \log(Q_i)\}_{i=0}^N$$

If this time series diverges too much from its average, then we can long and short accordingly. This is another way of saying that the time series of returns are

However, this model is too simplistic, since we are limited by the assumption that if P rises by 5%, then Q will also rise by the same 5%. Companies P and Q may use similar supply chains, but perhaps P is more dependent on it than Q. Therefore, if there are supply chain problems, then the effect on P may be say, twice as much as that on Q. So, if P goes down by 5%, then Q may go down by 2.5%.

3.2.2 Sophisticated Pairs Trading

Ultimately, our basis assumption is that the returns are linearly correlated in the following relationship.

$$\frac{\Delta P}{P} = \beta \frac{\Delta Q}{Q} \iff \log(P_j) - \log(P_i) = \beta\big(\log(Q_j) - \log(Q_i)\big)$$

This results in the model

$$\log(P) = \beta \log(Q) + \alpha + \epsilon$$

which can be seen to be equivalent because taking the change over time $[i, j]$ on both sides gives

$$\Delta \log(P) = \beta \Delta \log(Q) + \Delta\epsilon \iff \frac{\Delta P}{P} = \beta \frac{\Delta Q}{Q} + \Delta\epsilon$$

So, if we look at

$$\{\epsilon_i\}_{i=0}^n = \{\log(P_i) - \beta \log(Q_i) - \alpha\}_{i=0}^n$$

we expect this to be a 0-mean time series. Let the standard deviation be $\sigma = \sigma(\{\epsilon_i\}_{i=0}^n)$ and let us fix some Z-score threshhold. Then

1. If $\epsilon_i > Z\sigma$, then short P and long Q.

2. If $\epsilon_i < -Z\sigma$, then long P and short Q.

3.2.3 Long/Short Market Weights

But how much should we long or short? Let's look at a couple scenarios:

1. If $\beta = 1$, and we shorted \$99 of P and long \$1 of Q, then a 5% rise in P would result in 5% rise in Q, but since we shorted much more of P, we would have a net loss. To mitigate this risk, we should have a weight of $P : Q = 1 : 1$.
2. If $\beta = 10$, and we shorted equally \$50 of P and long \$50 of Q, then a 10% rise in P would result in a 1% rise in Q. But since we had equal weights in P and Q, this scenario would cause 10 times more losses in P than gains in Q, resulting in a net loss. To mitigate this risk, we should have a weight $P : Q = 1 : 10$.

So, we need to be careful of setting the ratio of our market weights of the stocks P and Q:

$$\lambda = \frac{\mathrm{MV}_Q}{\mathrm{MV}_P}$$

Intuitively, we can see that our ratio should be $1 : \lambda = 1 : \beta$, i.e. $\lambda = \beta$, but let's formalize this with some mathematical derivation. Our portfolio value is

$$V = \mathrm{MV}_P + \mathrm{MV}_Q$$

where $\mathrm{MV}_P = n_P P$ and $\mathrm{MV}_Q = n_Q Q$, where n_P, n_Q are the number of shares of P, Q. If we are longing P and shorting Q, then $n_P > 0$ and $n_Q < 0$. Then, our change in portfolio value V is

$$\begin{aligned}
\Delta\,\mathrm{V} &= \Delta\,\mathrm{MV}_P + \Delta\,\mathrm{MV}_Q \\
&= \mathrm{MV}_P \left[\frac{\Delta\,\mathrm{MV}_P}{\mathrm{MV}_P} + \frac{\mathrm{MV}_Q}{\mathrm{MV}_P} \frac{\Delta\,\mathrm{MV}_Q}{\mathrm{MV}_Q} \right] \\
&= \mathrm{MV}_P \left[\frac{\Delta P}{P} + \lambda \frac{\Delta Q}{Q} \right] \\
&= \mathrm{MV}_P \left[\beta \frac{\Delta Q}{Q} + \Delta\epsilon + \lambda \frac{\Delta Q}{Q} \right] \\
&= \mathrm{MV}_P \left[(\beta + \lambda) \frac{\Delta Q}{Q} + \Delta\epsilon \right]
\end{aligned}$$

Therefore, our change in portfolio value depends on the terms in the last equation. We don't want the change ΔQ to have any effect on the performance, so we set $\lambda = -\beta$, ultimately resulting in

$$\Delta\,\mathrm{V} = \mathrm{MV}_P\,\Delta\epsilon$$

and now our performance is purely dependent on $\Delta\epsilon$. We would like $\Delta\,\mathrm{V}$ to be positive, so if we are longing P, i.e. $\mathrm{MV}_P > 0$, then we want $\Delta\epsilon$ to also be positive. Likewise, if we are shorting P, then $\mathrm{MV}_P < 0$ and so we want $\Delta\epsilon < 0$.

3.2.4 Choosing Correct Stocks

So, how do we ensure that $\Delta\epsilon$ behaves this way? Remember that $\{\epsilon_i\}$ is 0-mean time series. However, we want to impose the additional condition that it is *mean-reverting* as well. That is, we don't want it to diverge or oscillate too frequently ϵ_i, since if it did then there is the risk of $\{\log(P_i) - \beta \log(Q_i) - \alpha\}_{i=0}^n$ swinging too widely, resulting in losses. In other words, if $\epsilon_i > 0$, then we want $\Delta\epsilon_{i+1} = \epsilon_{i+1} - \epsilon_i < 0$, and if $\epsilon_i < 0$, then $\Delta\epsilon_{i+1} > 0$, so that the ϵ_i's tend to "go back towards 0." More formally, we can plot all ϵ_{i-1}'s with the $\Delta\epsilon_i$'s, and look at a potentially linear relationship

$$\Delta\epsilon_i = \alpha\epsilon_{i-1} + \xi_i$$

To be mean reverting, we want to test that $\alpha < 0$ with adequate statistical significance, i.e. with p-value 5%. By looking at not just the previous ϵ_{i-1} but also the last p ϵ_i's we can develop the general linear relationship

$$\Delta\epsilon_i = \xi_i + \sum_{j=1}^{p} \alpha_j \epsilon_{i-j}$$

This is called the **Augmented Dickey Fuller (ADF)** test, which tells us whether a time series is mean-reverting or not.

3.2.5 Entry and Exit Points

4 Technical Portfolio Analysis

We will define a portfolio as a collection of K stocks with prices labeled as a time series data

$$\{P_{1,i}\}_{i=1}^N, \{P_{2,i}\}_{i=1}^N, \ldots, \{P_{N,i}\}_{i=1}^N$$

with one-step returns $\{R_{k,i}\}_{i=1}^N$ for each $k = 1, \ldots, K$ and total return over the period $[0, N]$ as R^k. For simplicity, let us assume that we buy all stocks at time 0 and hold until time N, and so we can consider a one-period market with $\{P_{k,0}, P_{k,1}\}$ for $k = 1, \ldots, K$ and corresponding returns

$$\{R_k\} = \{\log(P_{k,1}) - \log(P_{k,0})\}$$

all random variables.

Definition 4.1 (Market Value of Portfolio). The **market value of a portfolio** $\{P_1, \ldots, P_K\}$ is the random variable

$$\mathrm{V} = \sum_k \mathrm{MV}_k = \sum_k n_k P_k \text{ which is } \sum_k n_k P_{k,0} \text{ at time } t = 0$$

The market weights for each stock is determined as

$$w_k = \frac{\mathrm{MV}_k}{\mathrm{V}}$$

where $\sum w_k = 1$. We will denote $\mathbf{w} = (w_1, \ldots, w_K)$. If we allow unlimited shorting, then the possible values of of $\mathbf{w}$ are $\{\mathbf{w} \in \mathbb{R}^K \mid \mathbf{w} \cdot \mathbf{1} = 1\}$, but if we only allow all-long positions, then this domain reduces to $\{\mathbf{w} \in \mathbb{R}^K \mid \mathbf{w} \cdot \mathbf{1} = 1,\ 0 \leq w_k \leq 1\}$.

Definition 4.2 (Portfolio Return). The **portfolio return vector** is the random K vector of returns $\mathbf{R} = (R_1, \ldots, R_K)^T$, which may be correlated. We can take its expectation by taking the component expectations over the sample space w, or by integrating the component mappings $e_k : \mathbb{R}^K \to \mathbb{R}$ over the measure λ induced by $\mathbf{R}$

$$\mathbb{E}[\mathbf{R}] \coloneqq \begin{pmatrix} \mathbb{E}[R_1] \\ \vdots \\ \mathbb{E}[R_K] \end{pmatrix} = \begin{pmatrix} \mathbb{E}_\lambda(e_1) \\ \vdots \\ \mathbb{E}_\lambda[e_K] \end{pmatrix}$$

The **portfolio return** is defined as the random variable

$$R = \mathbf{w}^T \mathbf{R} = \sum_{k=1}^K w_k R_k$$

i.e. the weighted sum of the individual total returns of the stocks. The **expected return** of the portfolio is simply

$$\mathbb{E}[R] = \sum_{k=1}^K w_k \mathbb{E}_\lambda[R_k]$$

We can estimate this by sampling the historical returns of the kth stock $R_{k,i}$ of the required length for each stock and then estimating its mean. That is, for each k,

$$R_k \approx \hat{R}_k = \frac{1}{n} \sum_{i=1}^{n} R_{k,i}$$

Definition 4.3 (Portfolio Variance, Volatility)**.** The **covariance matrix** of the portfolio is defined as the covariance matrix of the random vector of returns $\mathbb{R}$.

$$\mathbf{\Sigma} = \mathrm{Cov}(\mathbf{R}) := \begin{pmatrix} \mathrm{Var}(R_1) & \dots & \mathrm{Cov}(R_1, R_K) \\ \vdots & \ddots & \vdots \\ \mathrm{Cov}(R_K, R_1) & \dots & \mathrm{Var}(R_K) \end{pmatrix}$$

We can estimate this by sampling the one-period historical returns R_k and computing the sample variance/covariance

$$\sigma_{k_1 k_2} = \mathrm{Cov}\big(\{R_{k_1,i}\}, \{R_{k_2,i}\}\big)$$

The **portfolio variance** is simply the variance of R.

$$\mathrm{Var}(R) = \mathbf{w}^T \mathbf{\Sigma} \mathbf{w} \approx \sum_{k_1,k_2=1}^{K} w_{k_1} w_{k_2} \sigma_{k_1,k_2}$$

and the **portfolio volatility** is simply the standard deviation of R

$$\sqrt{\mathrm{Var}(R)}$$

Definition 4.4 (Alpha)**.** Alpha is a measure of performance. The **alpha** of a security/portfolio with return R_p compared to some market return R_m is defined as

$$\alpha := \mathbb{E}[R_p - R_m] = \mathbb{E}[R_p] - \mathbb{E}[R_m]$$

If we are given samples of returns $\{R_{p,i}\}$ and $\{R_{m,i}\}$, then we can estimate the α simply as

$$\hat{\alpha} = R_p - R_m = \sum_{k=1}^{j-i} R_{p,i+k} + \sum_{k=1}^{j-i} R_{m,i+k}$$

Definition 4.5 (Beta)**.** Beta is a measure of volatility. The **beta** of a stock compared to some market is defined

$$\beta := \rho_{pm} \frac{\sigma_p}{\sigma_m} = \frac{\mathrm{Cov}(R_p, R_m)}{\mathrm{Var}(R_m)}$$

where ρ_{pm} is the correlation between R_m and R_f and σ represents the volatility of returns. If we are given samples of returns $\{R_{p,i}\}$ and $\{R_{m,i}\}$, then we can estimate the beta using the sample correlation and standard deviation:

$$\hat{\beta} = \hat{\rho}_{pm} \frac{\hat{\sigma}_p}{\hat{\sigma}_m} = \rho(\{R_{p,i}\}, \{R_{m,i}\}) \frac{\sigma(\{R_{p,i}\})}{\sigma(\{R_{m,i}\})}$$

If a stock has beta value of 1, then its price activity is strongly correlated with the market (or the benchmark B_j). A $\beta < 1$ means that the security is less volatile than the market, and $\beta > 1$ means more volatile. A negative β indices that the stock is inversely correlated with the market.

4.1 Markowitz Portfolio Theory - Mean Variance Portfolio

Consider a one-period market with K independent securities which have identical expected returns and variances, i.e. consider $\{P_{k,0}, P_{k,1}\}$ for $k = 1, \dots, K$. Then, the returns

$$R_k = \log(P_{k,1}) - \log(P_{k,0})$$

are random variables such that $\mathbb{E}[R_k] = \mu$ and $\mathrm{Var}(R_k) = \sigma^2$. Let w_k denote the fraction of wealth invested in the kth security. Now consider two portfolios

1. Portfolio A: 100% invested in stock 1 so that $w_1 = 1$ and $w_k = 0$ for $k = 2, \ldots, K$
2. Portfolio B: An equi-weighted portfolio so that $w_k = \frac{1}{K}$ for all k.

Let R_A and R_B denote the portfolio returns of A and B. Then, we have

$$\begin{aligned}
\mathbb{E}[R_A] &= \mathbb{E}[R_B] = \mu \\
\mathrm{Var}(R_A) &= w_1 \mathrm{Var}(R_1) = \sigma^2 \\
\mathrm{Var}(R_B) &= \mathrm{Var}\bigg(\frac{1}{K}\sum_{k=1}^{K} R_k\bigg) = \frac{1}{K^2}\sum_{k=1}^{K}\mathrm{Var}(R_k) = \sigma^2/K
\end{aligned}$$

which means that even though the expected returns of portfolios A and B are the same, the volatility of B is much less than that of A, making it much more advantageous. We can clearly see that the Sharpe ratio of A is μ/σ (assuming the risk-free return is 0), but the Sharpe ratio of B is $\mu\sqrt{K}/\sigma$, which means that as we increase our number of independent stocks K, the ratio goes up by a factor of $\sqrt{K}$.

4.1.1 Efficient Frontier without Risk-Free Asset

Given a portfolio of stocks $\{P_1, \ldots, P_K\}$ with corresponding return vector $\mathbf{R} = (R_1, \ldots, R_K)$, let us take its expected value $\boldsymbol{\mu} = \mathbb{E}[\mathbf{R}]$ and covariance matrix $\boldsymbol{\Sigma} = \mathrm{Cov}(\mathbf{R})$. Its portfolio return is $R = \mathbf{w}^T\mathbf{R}$. We will assume that we must invest all of our cash into something, which will manifest in the constraint equation $\mathbf{w}^T\mathbf{1} = 1$.

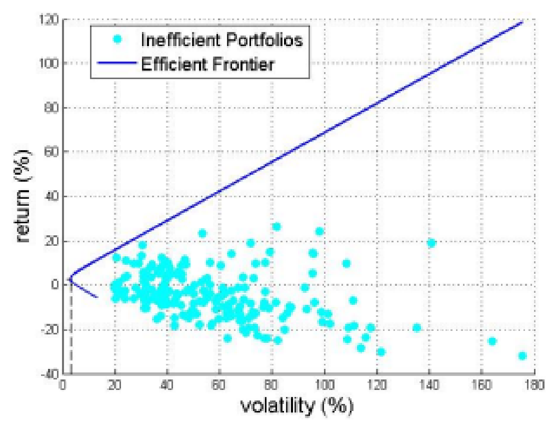
Definition 4.6 (Efficient Frontier without Risk-Free Asset)**.** We would like to construct a risk-return efficient portfolio (by determining the w_k's) so that it has the highest return for the same amount of risk, or the lowest risk for some amount of return. That is, letting $R_{\mathbf{w}}$ be the return of a portfolio with weight $\mathbf{w}$, we would like to find

$$\arg\min_{\mathbf{w}\in\mathbb{R}^K} \mathrm{Var}(R_{\mathbf{w}}) = \arg\min_{\mathbf{w}\in\mathbb{R}^K} \frac{1}{2}\mathbf{w}^T\boldsymbol{\Sigma}\mathbf{w}$$

subject to the constraint equations $\mathbf{w}^T\boldsymbol{\mu} = p$ and $\mathbf{w}^T\mathbf{1} = 1$, where p is our target portfolio returns. The solution to this optimization problem is called the **Markowitz efficient frontier**, which traces out a hyperbola of the form

$$\sigma_R^2 = Ap^2 + Bp + C$$

if we get rid of the $\boldsymbol{\omega}$ parameter. Here we randomly sample a bunch of $\mathbf{w}$'s from $[0,1]^K$ (subject to the constraints, of course), construct the return of the portfolio $R_{\mathbf{w}}$, and then plot the points $(\mathbb{E}[R], \mathrm{Var}(R))$. We can see that none of the points ever cross the frontier.



Note that this efficient portfolio allows us to have unlimited short positions, which may or may not be realistic. If we wanted to work only with all-long portfolios, then we would impose the nonlinear restrictions

$$0 \leq w_k \leq 1 \text{ for } k = 1, \ldots, K$$

which would need to be solved numerically, possibly using nonlinear models.

4.1.2 Efficient Frontier with Risk-Free Asset

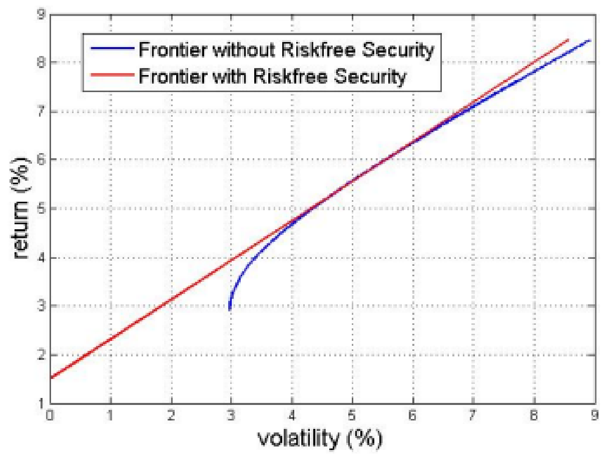
If we include a risk-free asset such as cash C with risk-free return R_f to the portfolio $\{P_1, \ldots, P_K\}$, then we slightly modify our equations. By definition, the risk-free asset has volatility $\sigma_0 = 0$ and its weight must be equal to $w_f = 1 - \sum_{i=1}^K w_k$, so we still want to minimize

$$\sigma^2 = \frac{1}{2}\mathbf{w}^T \mathbf{\Sigma} \mathbf{w}$$

subject to constraint equations

$$R_f \left(1 - \sum_{k=1}^K w_k\right) + \sum_{k=0}^K w_k R_k = R$$

When we allow our portfolio to include the risk-free security, the efficient frontier becomes a straight line that is tangential to the risky efficient frontier and with a y-intercept equal to the risk-free rate.



We can include other linear portfolio constraints, such as no-borrowing, no-short sales, or certain sector constraints. While analytic solutions are generally no longer available, the resulting problems are easy to solve numerically.

4.2 Capital Asset Pricing Model

Let us have some stock P with random variable of return $R_p = R_{P,[i,j]}$ and the market M with random variable of return $R_m = R_{M,[i,j]}$, both within period $[i,j]$. There may be some sort of risk-free return r_f available (e.g. U.S. treasury bonds), so we can observe the returns of these two assets past the risk-free return by considering the joint distribution

$$(R_p - r_f) \times (R_m - r_f)$$

This may or may not be correlated, but the capital asset pricing model shows that there exists a linear relationship between the expected values of these two distributions. The central insight of the CAPM is that in equilibrium the riskiness of an asset is not measured by the standard deviation of its return but by its beta.

Theorem 4.1 (CAPM). *Now let $\overline{R}_m = \mathbb{E}[R_m]$ denote the expected return of the market, and $\overline{R} = \mathbb{E}[R]$ denote the expected return of a security or portfolio. Then, the **capital asset pricing model (CAPM)** asserts that there exists a linear relationship*

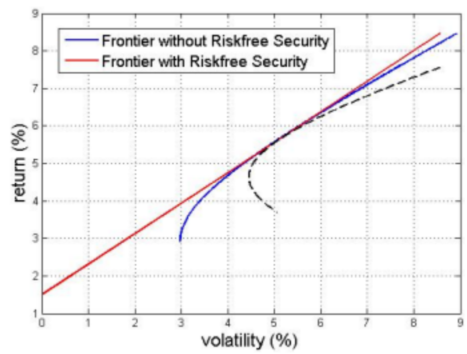
$$\overline{R} = r_f + \beta(\overline{R}_m - r_f)$$

where r_f is the risk-free rate.

Proof. Let us consider a portfolio of weights α and $1-\alpha$ on the risky security and market portfolio, respectively. Let R_α denote the random return of this portfolio as a function of α. We then have

$$\begin{aligned}\mathbb{E}[R_\alpha] &= \alpha\overline{R} + (1-\alpha)\overline{R}_m \\ \mathrm{Var}(R_\alpha) &= \alpha^2\mathrm{Var}(R) + (1-\alpha)^2\mathrm{Var}(R_m) + 2\alpha(1-\alpha)\mathrm{Cov}(R, R_m)\end{aligned}$$

Note that as α varies, the mean and standard deviation $(\mathbb{E}[R_\alpha], \mathrm{Var}(R_\alpha))$ trace out a curve in $\mathbb{R}^2$ that cannot cross the efficient frontier, as shown in the dotted line.



At $\alpha = 0$, the slope of this curve must equal the slope of the capital market line. The slope of the α-curve (where $\sigma(R_\alpha) = \sqrt{\mathrm{Var}(R_\alpha)}$) is

$$\begin{aligned}\frac{d\mathbb{E}[R_\alpha]}{d\sigma(R_\alpha)}\bigg|_{\alpha=0} &= \frac{d\mathbb{E}[R_\alpha]}{d\alpha}\bigg/\frac{d\sigma(R_\alpha)}{d\alpha}\bigg|_{\alpha=0} \\ &= \frac{\sigma(R_\alpha)(\overline{R} - \overline{R}_m)}{\alpha\sigma(R) - (1-\alpha)\,\mathrm{Var}(R_m) + (1-2\alpha)\mathrm{Cov}(R, R_m)}\bigg|_{\alpha=0} \\ &= \frac{\sigma(R_m)(\overline{R} - \overline{R}_m)}{-\mathrm{Var}(R_m) + \mathrm{Cov}(R, R_m)}\end{aligned}$$

The slope of the capital market line is $(\overline{R}_m - r_f)/\sigma(R_m)$, and equating the two

$$\frac{\sigma(R_m)(\overline{R} - \overline{R}_m)}{-\mathrm{Var}(R_m) + \mathrm{Cov}(R, R_m)} = \frac{\overline{R}_m - r_f}{\sigma(R_m)}$$

gives the result. ■