AP Calculus AB

Muchang Bahng

Spring 2025

1 Review

So far, we've learned that a **derivative** is the limit defined

$$\frac{df}{dx} = f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \tag{1}$$

An **indefinite integral** is simply the opposite of a derivative in that if the derivative of $f(x)$ is another function $f'(x) = g(x)$, then we can take the indefinite integral on $g(x)$ to get back the original function $f(x)$.[1]

$$\int f'(x)\,dx = \int g(x)\,dx = f(x) + c \tag{2}$$

On the other hand, a **definite integral**, which has two numbers called the **limits of integration** a and b, represent the signed (positive or negative) area under the curve $f(x)$.

$$\int_a^b f(x)\,dx \tag{3}$$

At first glance, it seems like the definite integral and indefinite integral have nothing to do with each other. One is the anti-derivative and the other is used to calculate areas of curvy shapes. But they are connected by the **fundamental theorem of calculus** (which is really two theorems), which I will explain now. First, let's play a game. Say that I give you a function $f(t)$, and I tell you also some starting point $t = a$. I want you to give me a function, let's call it $F(x)$, which computes the area under the curve of $f(t)$ from $t = a$ to $t = x$. That is, it is a function that takes in an input x, and it spits out a number that is equal to the area of $f(t)$ from $t = a$ to $t = b$. We can use definite integrals for this and see that

$$F(x) = \int_a^x f(t)\,dt \tag{4}$$

If you're having a hard time understanding this, look at the figure below. This is the *first fundamental theorem of calculus*. Note that while the lower limit of integration a is a constant, the upper limit is a variable!

[1] Don't forget the $+c$!

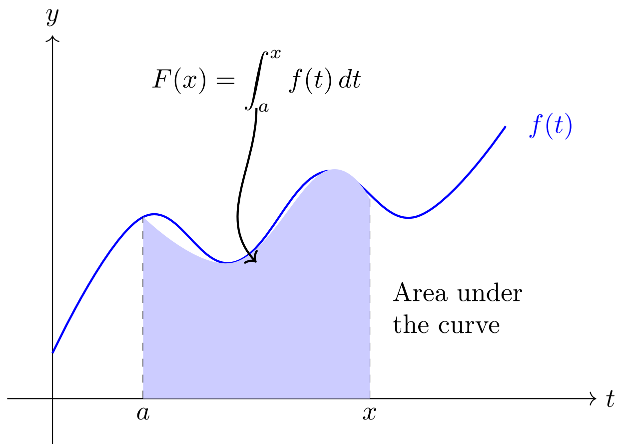

Figure 1: The integral shows the area under the curve. Apologize for the shading of the area not matching exactly as the function.

This is our input to the function $F(x)$, not t! Now let's make the game harder. Say that I give you a function $f(t)$, along with a starting point $t = a$ and an end point $t = b$. I want you to give me a function, let's call it $G(a, b)$, which computes the area under the curve of $f(t)$ from $t = a$ to $t = b$. In other words, it is a function that takes in *two* inputs a and b, and it spits out a number that is equal to the area of $f(t)$ from $t = a$ to $t = b$. This may sound a lot harder than the previous problem, but there is a simple solution! Imagine that there is a number c that is less than a and b, so $c < a < b$. Then, we can use our previous function $F(a)$, which calculates the area of $f(t)$ from $t = c$ to $t = a$, and the same function again $F(b)$, which calculates the area of $f(t)$ from $t = c$ to $t = b$.

$$F(a) = \int_c^a f(t)\,dt \tag{5}$$

$$F(b) = \int_c^b f(t)\,dt \tag{6}$$

Then we can visualize that the area of $f(t)$ from $t = a$ to $t = b$ *must* be $F(b) - F(a)$! Look at the figure below if you are having a hard time visualizing it. Therefore, this gives us our *second fundamental theorem of calculus*. In other words, it states that the definite integral of $f(x)$ can be calculated by taking the indefinite integral of $f(x)$, which we call $F(x)$, and then computing $F(b) - F(a)$.

$$\int_a^b f(x)\,dx = F(x)\big|_a^b = F(b) - F(a) \tag{7}$$

Look at the figure below

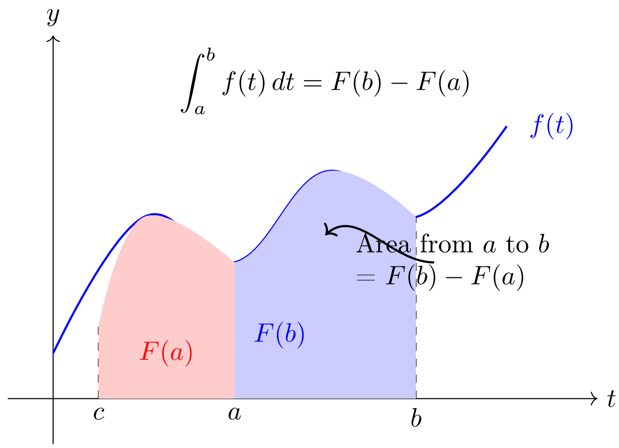

Figure 2

2 Integration By Substitution

Solving areas under curves using definite integrals, believe it or not, is very useful. Therefore, we just want to find a bunch of rules for computing these areas, and **integration by substitution**, also called **u-substitution**, is one such rule. By the fundamental theorem of calculus, we must compute indefinite integrals before computing definite integrals, so let's focus on indefinite integrals.

Remember the chain rule.

$$\frac{d}{dx} f(g(x)) = f'(g(x))\, g'(x) \tag{8}$$

Since the indefinite integral is the antiderivative, if we find a function of form $h(x) = f'(g(x))\, g'(x)$, we can see that

$$\int h(x)\, dx = \int f'(g(x))\, g'(x)\, dx = f(x) + c \tag{9}$$

since we already know the derivative of $f(g(x))$ is $h(x) = f'(g(x))\, g'(x)$.

Definition 2.1 (U-Substitution of Indefinite Integrals)

If we have an integral of the form where $f(x)$ and $u(x)$ are functions.

$$\int h(x)\, dx = \int f(u(x))\, u'(x)\, dx \tag{10}$$

To perform u-substitution, do the following.

1. *Decompose.* The hardest step is *knowing* that $h(x)$ can be written in the form $f(u(x))\, u'(x)$. Therefore, you must identify the functions $f(x)$ and $u(x)$ explicitly. We will show how to do this later.
2. *Differentiate.* Take $u(x)$ and compute the derivative $u' = g'(x)$. We see that

$$\frac{du}{dx} = u' \implies du = u'(x)\, dx \tag{11}$$

   Often, we write $u'(x)$ as just u', so the last line is $du = u'\, dx$.
3. *Substitute.* Substitute the $u(x)$ with the variable u and $u'(x)\, dx$ with du. This gives us

$$\int f(\underbrace{u(x)}_{u})\, \underbrace{u'(x)\, dx}_{du} = \int f(u)\, du \tag{12}$$

4. *Integrate.* Calculate the simplified integral.

$$F(u) = \int f(u)\, du \tag{13}$$

5. *Substitute Back.* Note that $F(u)$ is a function of u, not the original variable x. So we substitute $u(x)$ to get a function of x.

$$F(u(x)) \tag{14}$$

Now let's talk about for definite integrals. It is very similar to that of the indefinite integral the substitute step has one more part.

Definition 2.2 (U-Substitution of Definite Integrals)

If we have an integral of the form where $f(x)$ and $u(x)$ are functions.

$$\int_a^b h(x)\, dx = \int_a^b f(u(x))\, u'(x)\, dx \tag{15}$$

To perform u-substitution, do the following.

1. *Decompose.* The hardest step is *knowing* that $h(x)$ can be written in the form $f(u(x))\, u'(x)$. Therefore, you must identify the functions $f(x)$ and $u(x)$ explicitly. We will show how to do this later.
2. *Differentiate.* Take $u(x)$ and compute the derivative $u' = g'(x)$. We see that

$$\frac{du}{dx} = u' \implies du = u'(x)\, dx \tag{16}$$

Often, we write $u'(x)$ as just u', so the last line is $du = u'\, dx$.

3. *Substitute.* Substitute the $u(x)$ with the variable u and $u'(x)\, dx$ with du. *In addition*, substitute a with $u(a)$ and b with $u(b)$. This gives us

$$\int_a^b f(\underbrace{u(x)}_{u})\, \underbrace{u'(x)\, dx}_{du} = \int_{u(a)}^{u(b)} f(u)\, du \tag{17}$$

Note that the limits of integration have changed!

4. *Integrate.* Calculate the simplified integral. Note that this is just a number, so we don't have to substitute back to the variable x.

$$\int_{u(a)}^{u(b)} f(u)\, du \tag{18}$$

This is it! Finally, note that to compute a definite integral using u-substitution, we can actually do it in 2 ways. Both give you the same answer.

1. Use the u-substitution of definite integrals rule directly. This is the recommended way.
2. Compute the indefinite integral $F(x)$ using the u-substitution of indefinite integrals rule, and then just calculate $F(b) - F(a)$.

Now let's go over some examples.

Exercise 2.1 ()

Calculate
$$\int 5e^{5x}\,dx \tag{19}$$

Solution 2.1

To solve $\int 5e^{5x}\,dx$, we'll use u-substitution. Looking at this integral, we notice that there's both a factor of 5 and e^{5x}, which suggests we should focus on the $5x$ term.

Decompose:

- Our integral $h(x) = 5e^{5x}$ has a form that resembles the chain rule
- We can see $5x$ appears in the exponent, so let's set $u(x) = 5x$
- Then $f(u) = e^u$ will be our outer function
- This decomposition works because $5e^{5x} = e^{5x} \cdot 5 = f(u(x)) \cdot u'(x)$

Differentiate:

$$\begin{aligned} u &= 5x && \text{(our substitution)} \\ \frac{du}{dx} &= 5 && \text{(derivative of } u \text{ with respect to } x\text{)} \\ du &= 5\,dx && \text{(rearranged to solve for } dx\text{)} \end{aligned}$$

Substitute: Now we can rewrite our integral in terms of u:

$$\int 5e^{5x}\,dx = \int e^u\,du \qquad \text{(replaced } 5x \text{ with } u \text{ and } 5\,dx \text{ with } du\text{)}$$

Integrate:

$$\begin{aligned} \int e^u\,du &= e^u + C && \text{(basic integral of } e^u\text{)} \\ &= e^{5x} + C && \text{(substituted back } u = 5x\text{)} \end{aligned}$$

Therefore, $\int 5e^{5x}\,dx = e^{5x} + C$.

Exercise 2.2 ()

Calculate
$$\int (x^2+1)\cdot x\,dx \tag{20}$$

Solution 2.2

To solve $\int (x^2+1)\cdot x\,dx$, we'll use u-substitution. Notice that one factor (x) is the derivative of the other factor (x^2+1) up to a constant, which suggests a substitution.

Decompose:

- Our integral $h(x) = (x^2+1)\cdot x$ has a form where one part appears to be the derivative of another
- Let's set $u(x) = x^2 + 1$ since we see this expression as a factor
- Then $f(u) = u$ will be our function of u
- Notice that the remaining x factor will relate to du (shown in next step)

Differentiate:

$$\begin{aligned}
u &= x^2 + 1 && \text{(our substitution)} \\
\frac{du}{dx} &= 2x && \text{(derivative of } u \text{ with respect to } x\text{)} \\
du &= 2x\,dx && \text{(therefore)} \\
x\,dx &= \frac{1}{2}du && \text{(solved for } x\,dx\text{)}
\end{aligned}$$

Substitute: Now we can rewrite our integral with everything in terms of u:

$$\begin{aligned}
\int (x^2+1)\cdot x\,dx &= \int u \cdot \frac{1}{2}du && \text{(replaced } x^2+1 \text{ with } u \text{ and } x\,dx \text{ with } \tfrac{1}{2}du\text{)} \\
&= \frac{1}{2}\int u\,du && \text{(factored out constant)}
\end{aligned}$$

Integrate:

$$\begin{aligned}
\frac{1}{2}\int u\,du &= \frac{1}{2}\cdot\frac{u^2}{2} + C && \text{(basic power rule)} \\
&= \frac{1}{4}u^2 + C \\
&= \frac{1}{4}(x^2+1)^2 + C && \text{(substituted back } u = x^2+1\text{)}
\end{aligned}$$

Therefore, $\int (x^2+1)\cdot x\,dx = \frac{1}{4}(x^2+1)^2 + C$.

Exercise 2.3 ()

Calculate

$$\int_0^1 (x^2+1)^3 \cdot x\,dx \tag{21}$$

Solution 2.3

To solve $\int_0^1 (x^2+1)^3 \cdot x\,dx$, we'll use u-substitution. This integral is similar to the previous one, but with a cube power and definite bounds.

Decompose:

- Our integral $h(x) = (x^2+1)^3 \cdot x$ has a similar structure to the previous problem
- Let's set $u(x) = x^2 + 1$ since we see this expression being cubed
- Then $f(u) = u^3$ will be our function of u
- Again, the remaining x factor will relate to du

Differentiate:

$$\begin{aligned}
u &= x^2 + 1 && \text{(our substitution)} \\
\frac{du}{dx} &= 2x && \text{(derivative of } u \text{ with respect to } x\text{)} \\
du &= 2x\,dx && \text{(therefore)} \\
x\,dx &= \frac{1}{2}du && \text{(solved for } x\,dx\text{)}
\end{aligned}$$

Substitute: For definite integrals, we also need to change the bounds:

- When $x = 0$: $u = 0^2 + 1 = 1$

- When $x = 1$: $u = 1^2 + 1 = 2$

Now we can rewrite our integral:

$$\begin{aligned} \int_0^1 (x^2+1)^3 \cdot x\,dx &= \int_1^2 u^3 \cdot \frac{1}{2}du && \text{(with new bounds)} \\ &= \frac{1}{2}\int_1^2 u^3\,du && \text{(factored out constant)} \end{aligned}$$

Integrate:

$$\begin{aligned} \frac{1}{2}\int_1^2 u^3\,du &= \frac{1}{2}\left[\frac{u^4}{4}\right]_1^2 && \text{(power rule)} \\ &= \frac{1}{2}\left(\frac{16}{4} - \frac{1}{4}\right) && \text{(evaluated at bounds)} \\ &= \frac{1}{2}\cdot\frac{15}{4} \\ &= \frac{15}{8} \end{aligned}$$

Therefore, $\int_0^1 (x^2+1)^3 \cdot x\,dx = \frac{15}{8}$.

Exercise 2.4 ()

Calculate

$$\int \sqrt{2x-1}\,dx \tag{22}$$

Solution 2.4

To solve $\int \sqrt{2x-1}\,dx$, we'll use u-substitution. The square root suggests making what's inside it our u.

Decompose:

- Our integral $h(x) = \sqrt{2x-1}$ contains a square root
- Let's set $u(x) = 2x - 1$ to simplify what's inside the square root
- Then $f(u) = \sqrt{u} = u^{\frac{1}{2}}$ will be our function of u
- The factor of 2 in $2x - 1$ will relate to du

Differentiate:

$$\begin{aligned} u &= 2x - 1 && \text{(our substitution)} \\ \frac{du}{dx} &= 2 && \text{(derivative of } u \text{ with respect to } x\text{)} \\ du &= 2\,dx && \text{(therefore)} \\ dx &= \frac{1}{2}du && \text{(solved for } dx\text{)} \end{aligned}$$

Substitute:

$$\begin{aligned} \int \sqrt{2x-1}\,dx &= \int \sqrt{u}\cdot\frac{1}{2}du && \text{(replaced } 2x-1 \text{ with } u \text{ and } dx \text{ with } \tfrac{1}{2}du\text{)} \\ &= \frac{1}{2}\int u^{\frac{1}{2}}\,du && \text{(rewrote square root as power)} \end{aligned}$$

Integrate:

$$\begin{aligned}
\frac{1}{2}\int u^{\frac{1}{2}}\,du &= \frac{1}{2}\cdot\frac{2}{3}u^{\frac{3}{2}} + C && \text{(power rule with } \frac{1}{2}+1=\frac{3}{2}\text{)} \\
&= \frac{1}{3}u^{\frac{3}{2}} + C \\
&= \frac{1}{3}(2x-1)^{\frac{3}{2}} + C && \text{(substituted back } u = 2x-1\text{)}
\end{aligned}$$

Therefore, $\int \sqrt{2x-1}\,dx = \frac{1}{3}(2x-1)^{\frac{3}{2}} + C$.

Exercise 2.5 ()

Calculate

$$\int_{\pi/3}^{\pi/2} \sin^2(3x)\,\cos(3x)\,dx \tag{23}$$

Solution 2.5

To solve $\int_{\pi/3}^{\pi/2} \sin^2(3x)\cos(3x)\,dx$, we'll use u-substitution. Notice this looks like a power of sine times its derivative (up to a constant).

Decompose:

- Our integral $h(x) = \sin^2(3x)\cos(3x)$ contains a power of sine times cosine
- Let's set $u(x) = \sin(3x)$ since we see its power
- Then $f(u) = u^2$ will be our function of u
- The $\cos(3x)$ term will relate to du (which makes sense as it's the derivative of sine)

Differentiate:

$$\begin{aligned}
u &= \sin(3x) && \text{(our substitution)} \\
\frac{du}{dx} &= 3\cos(3x) && \text{(derivative using chain rule)} \\
du &= 3\cos(3x)\,dx && \text{(therefore)} \\
\cos(3x)\,dx &= \frac{1}{3}du && \text{(solved for } \cos(3x)\,dx\text{)}
\end{aligned}$$

Substitute: For the definite integral, we need to change the bounds:

- When $x = \frac{\pi}{3}$: $u = \sin(\pi) = 0$
- When $x = \frac{\pi}{2}$: $u = \sin(\frac{3\pi}{2}) = -1$

Now we can rewrite our integral:

$$\begin{aligned}
\int_{\pi/3}^{\pi/2} \sin^2(3x)\cos(3x)\,dx &= \int_0^{-1} u^2 \cdot \frac{1}{3}du && \text{(with new bounds)} \\
&= \frac{1}{3}\int_0^{-1} u^2\,du && \text{(factored out constant)}
\end{aligned}$$

Integrate:

$$\begin{aligned}
\frac{1}{3}\int_0^{-1} u^2\,du &= \frac{1}{3}\left[\frac{u^3}{3}\right]_0^{-1} && \text{(power rule)} \\
&= \frac{1}{3}\left(-\frac{1}{3} - 0\right) && \text{(evaluated at bounds)} \\
&= -\frac{1}{9}
\end{aligned}$$

Therefore, $\int_{\pi/3}^{\pi/2} \sin^2(3x)\cos(3x)\,dx = -\frac{1}{9}$.

Exercise 2.6 ()

Calculate

$$\int_1^5 \frac{x}{\sqrt{2x-1}}\,dx \tag{24}$$

Solution 2.6

To solve $\int_1^5 \frac{x}{\sqrt{2x-1}}\,dx$, we'll use u-substitution. The expression under the square root in the denominator suggests our substitution.

Decompose:

- Our integral $h(x) = \frac{x}{\sqrt{2x-1}}$ has a square root in the denominator
- The expression $2x-1$ appears in the square root, suggesting we let $u(x) = 2x-1$
- We need to express both x and dx in terms of u
- From $u = 2x-1$, we can solve for x: $x = \frac{u+1}{2}$
- Therefore, our integrand will become $\frac{(u+1)/2}{\sqrt{u}}$ after substitution

Differentiate:

$$\begin{aligned}
u &= 2x - 1 && \text{(our substitution)} \\
\frac{du}{dx} &= 2 && \text{(derivative of } u \text{ with respect to } x\text{)} \\
du &= 2\,dx && \text{(therefore)} \\
dx &= \frac{1}{2}du && \text{(solved for } dx\text{)}
\end{aligned}$$

Substitute: For the definite integral, we need to change the bounds:

- When $x = 1$: $u = 2(1) - 1 = 1$
- When $x = 5$: $u = 2(5) - 1 = 9$

Now we can rewrite our integral:

$$\begin{aligned}
\int_1^5 \frac{x}{\sqrt{2x-1}}\,dx &= \int_1^9 \frac{(u+1)/2}{\sqrt{u}} \cdot \frac{1}{2}du && \text{(substituted } x \text{ and } dx\text{)} \\
&= \frac{1}{4}\int_1^9 \frac{u+1}{\sqrt{u}}\,du && \text{(simplified)} \\
&= \frac{1}{4}\int_1^9 (u^{\frac{1}{2}} + u^{-\frac{1}{2}})\,du && \text{(split fraction)}
\end{aligned}$$

Integrate:

$$\begin{aligned}
\frac{1}{4}\int_1^9 (u^{\frac{1}{2}} + u^{-\frac{1}{2}})\,du &= \frac{1}{4}\left[\frac{2}{3}u^{\frac{3}{2}} + 2u^{\frac{1}{2}}\right]_1^9 && \text{(integrated each term)} \\
&= \frac{1}{4}\left(\frac{2}{3}(27) + 2(3) - \frac{2}{3}(1) - 2(1)\right) && \text{(evaluated at bounds)} \\
&= \frac{1}{4}(18 + 6 - \frac{2}{3} - 2) && \text{(simplified)} \\
&= \frac{1}{4}(21\frac{1}{3}) \\
&= \frac{16}{3}
\end{aligned}$$

Therefore, $\int_1^5 \frac{x}{\sqrt{2x-1}}\,dx = \frac{16}{3}$. Note that we were able to integrate this by:

1. Making the substitution $u = 2x - 1$ to simplify the square root
2. Converting x to $\frac{u+1}{2}$ and dx to $\frac{1}{2}du$
3. Breaking up the fraction $\frac{u+1}{\sqrt{u}}$ into $u^{\frac{1}{2}} + u^{-\frac{1}{2}}$
4. Using the power rule on each term