Algebraic Topology

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1 Homotopy

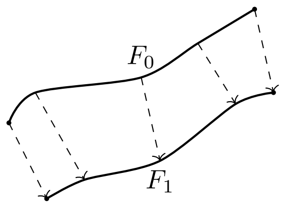
Definition 1.1. Let X, Y be topological space and let $F_0, F_1 : X \longrightarrow Y$ be continuous maps. A *homotopy* from F_0 to F_1 is a continuous map (with respect to elements $t \in [0,1]$)

$$H : X \times I \longrightarrow Y$$

where $I = [0,1]$, satisfying

$$H(x,0) = F_0(x)$$
$$H(x,1) = F_1(x)$$

for all $x \in X$. We can visualize this homotopy as a continuous deformation of (the images of) F_0 to F_1. We can also think of the parameter t as a "slider control" that allows us to smoothly transition from F_0 to F_1 as the slider moves from 0 to 1, and vice versa. The figures below represents the homotopies between the one-dimensional curves (left) and 2-dimensional surfaces (right), $\operatorname{Im} F_0$ and $\operatorname{Im} F_1$, with dashed lines.



If there exists a homotopy from F_0 to F_1, then we say that F_0 and F_1 are *homotopic*, denoted

$$F_0 \simeq F_1$$

Definition 1.2. If the homotopy satisfies

$$H(x,t) = F_0(x) = F_1(x)$$

for all $t \in I$ and $x \in S$, which is a subset of X, then the maps F_0 and F_1 are said to be *homotopic relative to S*.

This is clearly an equivalence relation defined on $C^0(X,Y)$, the set of all continuous functions from X to Y.

1. Identity. Clearly, F is homotopic to itself by setting $H(x,t) \equiv F(x)$ for all $t \in [0,1]$.
2. Symmetry. Given homotopy $H(x,t)$ from F_0 to F_1, the homotopy $H^{-1}(x,t) \equiv H(x, 1-t)$ maps from F_1 to F_0.
3. Transitivity. Given homotopy H_1 from F_1 to F_2, and homotopy H_2 from F_2 to F_3, the homotopy defined
$$H_3(x,t) \equiv \begin{cases} H_1(x,2t) & 0 \leq t \leq \frac{1}{2} \\ H_2(x, 2t-1) & \frac{1}{2} \leq t \leq 1 \end{cases}$$
is indeed a homotopy from F_1 to F_3.

Definition 1.3. The space of homotopy classes from topological space X to Y is denoted

$$[X, Y] \equiv \frac{C^0(X, Y)}{\sim}$$

where $\sim$ is the homotopy relation.

Lemma 1.1. *Homotopy is compatible with function composition in the following sense. If $f_1, g_1 : X \longrightarrow Y$ are homotopic, and $f_2, g_2 : Y \longrightarrow Z$ are homotopic, then $f_2 \circ f_1$ and $g_2 \circ g_1$ are homotopic. That is, given the two homotopies*

$$H_1 : X \times [0,1] \longrightarrow Y$$
$$H_2 : Y \times [0,1] \longrightarrow Z$$

we can naturally define a third homotopy

$$H_3 : X \times [0,1] \longrightarrow Z,\ H(x,t) \equiv H_2(x,t) \circ H_1(x,t)$$

which is continuous since compositions of continuous functions are continuous.

Example 1.1. *If $f, g : \mathbb{R} \longrightarrow \mathbb{R}^2$ is defined as a*

$$f(x) \equiv (x, x^3),\ g(x) \equiv (x, e^x)$$

then the map

$$H : \mathbb{R} \times [0,1] \longrightarrow \mathbb{R}^2,\ H(x,t) \equiv \big(x, (1-t)x^3 + te^x\big)$$

is a homotopy between them.

Example 1.2. *Let $id_B : B^n \longrightarrow B^n$ be the identity function on the unit n-disk, and let $c_0 : B^n \longrightarrow B^n$ be the 0-function sending every vector to 0. Then, id_B and c_0 are homotopic, with homotopy explicitly defined*

$$H : B^n \times [0,1] \longrightarrow B^n,\ H(x,t) \equiv (1-t)x$$

Example 1.3. *If $C \subseteq \mathbb{R}^n$ is a convex set and $f, g : [0,1] \longrightarrow C$ are paths with the same endpoints, then there exists a linear homotopy given by*

$$H : [0,1] \times [0,1] \longrightarrow C,\ (s,t) \mapsto (1-t)f(s) + tg(s)$$

We can extend this example. Let $f, g : \mathbb{R} \longrightarrow \mathbb{R}$ be 2 continuous functions. Then $f \simeq g$, since we can construct $F : [0,1] \times \mathbb{R} \longrightarrow \mathbb{R}$ defined

$$F(x,t) \equiv (1-t)f(x) + tg(x)$$

(Note that the set of continuous functions from $\mathbb{R}$ to $\mathbb{R}$ is a convex set.)

This leads to our definition of *path homotopies*, which is just a specific type of homotopy.

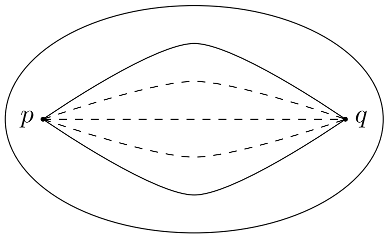
Definition 1.4. Suppose X is a topological space. Two paths $f_0, f_1 : I \longrightarrow X$ are said to be *path homotopic*, denoted

$$f_0 \sim f_1$$

if they are homotopic relative to $\{0, 1\}$. This means that there exists a continuous map $H : I \times I \longrightarrow X$ satisfying

$$H(s,0) = f_0(s),\ s \in I$$
$$H(s,1) = f_1(s),\ s \in I$$
$$H(0,t) = f_0(0) = f_1(0), t \in I$$
$$H(1,t) = f_1(1) = f_0(1), t \in I$$

We can visualize two paths (sharing the same endpoints) being path homotopic if we can "continuously deform" one onto another.



We can notice that for any given points $p, q \in X$, path homotopy is an equivalence class on the set of all paths from p to q.

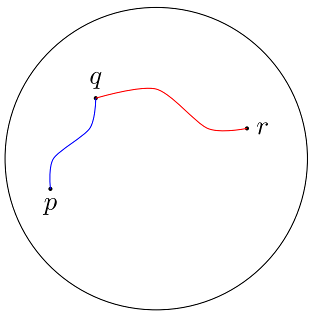
Definition 1.5. The equivalence class of a path f is called a *path class*, denoted $[f]$. Note that in the diagram above, there is only one equivalence class of paths.

We can define a multiplicative structure on paths as such. This is the first step to create a group structure on the set of certain paths.

Definition 1.6. Given two paths f, g such that $f(1) = g(0)$, their product is the path defined

$$(f \cdot g)(s) \equiv \begin{cases} f(2s) & 0 \leq s \leq \frac{1}{2} \\ g(2s-1) & \frac{1}{2} \leq s \leq 1 \end{cases}$$

It is easy to visualize the product of two paths as the longer path created by "connecting" the two smaller paths.



It is also easy to see that if $f \sim f'$ and $g \sim g'$,

$$f \cdot g \sim f' \cdot g'$$

We can also define the product of these equivalence classes as

$$[f] \cdot [g] \equiv [f \cdot g]$$

Notice that multiplication of paths is not associative in general, but it is associative up to path homotopy. That is,

$$([f] \cdot [g]) \cdot [h] = [f] \cdot ([g] \cdot [h])$$

Definition 1.7. If X is a topological space and $q \in X$, a "loop" in X based at q is a path in X such that

$$f : I \longrightarrow X, \ f(0) = f(1) = q$$

The set of path classes of loops based at q is denoted

$$\pi_1(X, q)$$

Equipped with the product operation of paths defined before, $(\pi_1(X, q), \cdot)$ is called the *fundamental group of X based at q*. The identity element of this group is the path class of the constant path $c_q(s) \equiv q$, and the inverse of $[f]$ is the path class of

$$f^{-1}(s) \equiv f(1-s)$$

which is the reverse path of f.

Note that while the fundamental group in general depends on the point q, it turns out that, up to isomorphism, this choice makes no difference as long as the space is path connected.

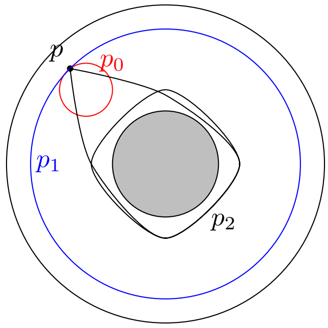
Lemma 1.2. *Let X be a path connected topological space, with $p, q \in X$. Then,*

$$\pi_1(X, p) \simeq \pi_1(X, q)$$

for all p, q.

Therefore, it is conventional to write $\pi_1(X)$ instead of $\pi_1(X, q)$ when X is path connected.

Example 1.4. *Consider the space $X \equiv B_2 \setminus B_1$, which is the 2-disk without the unit disk in $\mathbb{R}^2$. Given an arbitrary point $p \in X$, there exists an infinite number of path classes of X at p, denoted $[p_i]$, where i corresponds to how many times the paths loop around the hole. The first three path classes are shown below.*



It is clear that $[p_0]$ is the identity, and the group operation rule is

$$[p_i] \cdot [p_j] = [p_{i+j}]$$

meaning that $\pi_1(X, p)$ is the infinite discrete group generated by $[p_0]$ and $[p_1]$.

Proposition 1.3. *Let $\mathcal{A}$ be a convex subset of $\mathbb{R}^n$, endowed with the subspace topology, and let X be any topological space. Then, any 2 continuous maps $f, g : X \longrightarrow \mathcal{A}$ are homotopic.*

Proof. Since $\mathcal{A}$ is convex, the homotopy defined

$$F(x, t) \equiv (1 - t)f(x) + tg(x)$$

exists. ■

Proposition 1.4. *If X is a path connected space, the fundamental groups based at different points are all isomorphic. That is,*

$$\pi_1(X, p) \simeq \pi_1(X, q)$$

for all $p, q \in X$.

Definition 1.8. If X is path connected and for some $q \in X$, the group $\pi_1(X, q)$ is the trivial group consisting of $[c_q]$ alone, then we say that X is *simply connected.* By definition, this means that every loop is path homotopic to a constant path.

Proposition 1.5. *Let X be a path connected topological space. X is simply connected if and only if any 2 loops based on the same point are path homotopic.*

We can also expect that since homotopy is clearly a topological property, it is preserved under continuous maps. We state this result formally in the following lemma.

Lemma 1.6. *If $F_0, F_1 : X \longrightarrow Y$ and $G_0, G_1 : Y \longrightarrow Z$ are continuous maps such that $F_0 \simeq F_1$ and $G_0 \simeq G_1$, then*

$$G_0 \circ F_0 \simeq G_1 \circ F_1$$

Similarly, if $f_0, f_1 : I \longrightarrow X$ are path homotopic, and $F : X \longrightarrow Y$ is a continuous map, then

$$F \circ f_0 \sim F \circ f_1$$

Thus, if $F : X \longrightarrow Y$ *is a continuous maps, for each* $q \in X$*, we can construct a well-defined map*

$$F_* : \pi_1(X, q) \longrightarrow \pi_1\big(Y, F(q)\big)$$

by setting

$$F_*([f]) \equiv [F \circ f]$$

Lemma 1.7. *If* $F : X \longrightarrow Y$ *is a contiuous map, then the induced map*

$$F_* : \pi_1(X, q) \longrightarrow \pi_1\big(Y, F(q)\big)$$

is a group homomorphism. ^{x} *That is,* F_* *preserves multiplicative structure of the loops.*

Theorem 1.8 (Properties of the Induced Homomorphism)**.**

1. *Let* $F : X \longrightarrow Y, G : Y \longrightarrow Z$ *be continuous maps. Then for any* $q \in X$*,*

$$(G \circ F)_* = G_* \circ F_* : \pi_1(X, q) \longrightarrow \pi_1\big(Z, G(F(q))\big)$$

2. *For any space* X *and any* $q \in X$*, the homomorphism induced by the identity map* $id_X : X \longrightarrow X$ *is the identity map*

$$id : \pi_1(X, q) \longrightarrow \pi_1(X, q)$$

3. *If* $F : X \longrightarrow Y$ *is a homeomorphism, then*

$$F_* : \pi_1(X, q) \longrightarrow \pi_1\big(Y, F(q)\big)$$

is an isomorphism. That is, homeomorphic spaces have isomorphic fundamental groups.

Example 1.5. *The fundamental group of* $S^1 \subset \mathbb{C}$ *based at* 0 *is the infinite cyclic group generated by the path class of the loop*

$$\alpha : I \longrightarrow S^1, \ \alpha(s) \equiv e^{2\pi i s}$$

Theorem 1.9. *If* $F : X \longrightarrow Y$ *is a homotopy equivalence, then for each* $p \in X$*,*

$$F_* : \pi_1(X, p) \longrightarrow \pi_1\big(Y, F(p)\big)$$

is an isomorphism.

The following proposition will be revisited when studying manifolds.

Proposition 1.10. *The fundamental group of any topological manifold is countable.*

1.1 Homotopy Equivalence

Definition 1.9. Given two topological spaces X and Y, a homotopy equivalence between X and Y is a pair of continuous maps $f : X \longrightarrow Y$ and $g : Y \longrightarrow X$ such that

$$g \circ f \simeq id_X \text{ and } f \circ g \simeq id_Y$$

The equivalence classes under $\simeq$ are called *homotopy types.* If such a pair f, g exists, X and Y are said to be *homotopy equivalent*, or of the same homotopy type.

Definition 1.10. Spaces that are homotopy equivalent to a point are called *contractible.* That is, X is contractible if and only if

$$X \simeq \{x_0\}$$

Visually, two spaces are homotopy equivalent if they can be transformed into one another by bending, shrinking, and expanding operations.

Example 1.6. *A solid disk is homotopy equivalent to a single point, since one can deform the disk along radial lines to a point.*

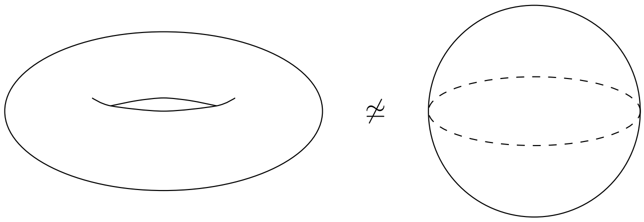
Example 1.7. *A mobius strip is homotopy equivalent to a closed (untwisted) strip.*

Notice from the visualization of homotopy equivalence the following proposition.

Proposition 1.11. *X, Y homeomorphic $\implies$ X, Y homotopy equivalent. However, the converse is not true.*

Proof. Just set $f = f$ and $g = f^{-1}$. ∎

Example 1.8. *A torus is not homotopy equivalent to Y, which also implies that they are not homeomorphic either.*



Furthermore, like homeomorphisms, homotopy equivalence is a relation on the set of all topological spaces.

1. Identity. Just set $f, g = id_X$
2. Symmetricity. Given $X \simeq Y$ with $f : X \longrightarrow Y, g : Y \longrightarrow X$, we set $f' \equiv g$ and $g' \equiv f$ and use these functions f', g' to find out that $Y \simeq X$.
3. Transitivity. Let us have $X \simeq Y$ with functions f_1, g_1 and $Y \simeq Z$ with functions f_2, g_2. Then, we define new functions
$$f_3 \equiv f_2 \circ f_1 : X \longrightarrow Z, \; g_3 \equiv g_1 \circ g_2 : Z \longrightarrow X$$
which follows to $f_3 \circ g_3 = id_Z$ and $g_3 \circ f_3 = id_X$.

Proposition 1.12. *$\mathbb{R}^n$ is homotopically equivalent to a point $\{0\}$.*

Proof. We claim that the continuous maps (canonical injection and projection)
$$id_{\mathbb{R}^n} : \{0\} \longrightarrow \mathbb{R}^n, \; p_0 : \mathbb{R}^n \longrightarrow \{0\}$$
have the property that
$$id_{\mathbb{R}^n} \circ p_0 \simeq id_{\mathbb{R}^n}, \; p_0 \circ id_{\mathbb{R}^n} \simeq id_{\{0\}}$$
The right-hand homotopy is trivial since $id_{\mathbb{R}^n} \circ p_0 = id_{\mathbb{R}^n}$, and as for the left-hand homotopy, we can explicitly define it as
$$H : [0, 1] \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$$
with
$$H(t, x) \equiv (t)(id_{\mathbb{R}^n} \circ p_0)(x) + (1 - t)\, id_{\mathbb{R}^n}(x) = (1 - t)\, id_{\mathbb{R}^n}(x)$$
∎

Example 1.9. $S^1 \simeq \mathbb{R}^2 \setminus \{0\}$*, and more generally,* $S^{n-1} \simeq \mathbb{R}^n \setminus \{0\}$*. We can see this with the canonical injection and projections*
$$id_{\mathbb{R}^2} : S^1 \longrightarrow \mathbb{R}^2 \setminus \{0\}, \; \pi_{S^1} : \mathbb{R}^2 \setminus \{0\} \longrightarrow S^1$$
and find that
$$id_{\mathbb{R}^2} \circ \pi_{S^1} \simeq id_{\mathbb{R}^2}, \; \pi_{S^1} \circ id_{\mathbb{R}^2} \simeq id_{S^1}$$
where the right-hand homotopy is trivial, and the left hand homotopy is defined explicitly as
$$H(x, t) \equiv t(id_{\mathbb{R}^2} \circ \pi_{S^1})(x) + (1 - t)(id_{\mathbb{R}^2})(x)$$

Definition 1.11. A function f is said to be *null homotopic* if it is homotopic to a constant function. This is sometimes called a *null-homotopy.*

Example 1.10. *Take a look at a function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$, which represents an arbitrary surface in $\mathbb{R}^2 \oplus \mathbb{R}$. Now, observe the constant function $c(x, y) \equiv c$, which represents a plane parallel to the x, y-plane. Clearly, we can imagine a deformation of the surface of f to the flat surface of c with the homotopy*
$$H(x, t) \equiv t\, f(x) + (1 - t)c(t)$$
which visually represents a linear deformation of c to f. Therefore, f is null-homotopic.

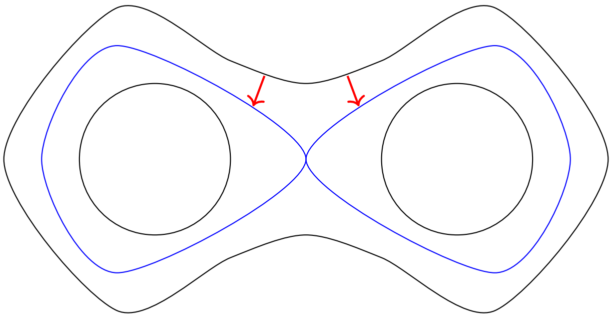
Example 1.11. *A map $f : S^1 \longrightarrow X$ is null homotopic precisely when it can be continuously extended to a map*

$$\tilde{f} : D^2 \longrightarrow X$$

that agrees with f on the boundary $\partial D^2 = S^1$. Visually, the existence of $\tilde{f}$ allows us to continuously deform the image of f in $S^1 \oplus X$ to a level curve $f(x) = c$ existing in $S^1 \oplus X$.

Proposition 1.13. *A space X is contractible if any only if the identity map from X to itself, which is always a homotopy equivalence, is null homotopic.*

Example 1.12. *Let Y be the following gray subset of the plane, and let X be the figure-8 shape.*



Then $Y \simeq X$, where the corresponding functions are

$$\begin{aligned} F : X \longrightarrow Y, &\text{ the canonical inclusion} \\ F : Y \longrightarrow X, &\text{ the projection onto } X \end{aligned}$$

Then, $G \circ F = id$ and $F \circ G$ is homotopic to the identity, with homotopy defined

$$H(x,t) \equiv t(F \circ G)(x) + (1-t)(id_Y)(x)$$

which can be visualized by $H(x,s)$ being the point you get from x by moving a fraction s along the red arrow towards X.

2 Homeomorphism Groups

Definition 2.1. The *homeomorphism group* of a topological space X is the group consisting of all homeomorphisms from X to X, with function composition as the group operation.