Theoria
Combinationis Observationum
Erroribus Minimis Obnoxiae

Theory of the
Combination of Observations
Least Subject to Errors

Theoria Combinationis Observationum Erroribus Minimis Obnoxiae

Pars Prior ◆ Pars Posterior ◆ Supplementum

By Carl Friedrich Gauss

Theory of the Combination of Observations Least Subject to Errors

Part One ◆ Part Two ◆ Supplement

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*The titles of the numbered articles are the translator's and are intended to help orient the reader. They do not appear in the numbered articles.

Translator's Introduction

Although Gauss had discovered the method of least squares during the last decade of the eighteenth century and used it regularly after 1801 in astronomical calculations, it was Legendre who introduced it to the world in an appendix to an astronomical memoir. Legendre stated the principle of least squares for combining observations and derived the normal equations from which least squares estimates may be calculated. However, he provided no justification for the method, other than noting that it prevented extreme errors from prevailing by establishing a sort of equilibrium among all the errors, and he was content to refer the calculator to the methods of the day for solving linear systems.

In 1809, toward the end of his treatise on *The Theory of the Motion of Heavenly Bodies*, Gauss gave a probabilistic justification of the method, in which he essentially showed that if the errors are normal then least squares gives maximum likelihood estimates. However, his reasons for assuming normality were tenuous, and Gauss himself later rejected the approach. In other respects the treatment was more successful. It contains the first mention of Gaussian elimination (worked out in detail in a later publication), which was used to derive expressions for the precision of the estimates. He also described the Gauss–Newton method for solving nonlinear least squares problems and gave a characterization of what we would now call approximations in the ℓ_1 norm.

Shortly thereafter, Laplace turned to the subject and derived the method of least squares from the principle that the best estimate should have the smallest mean error, by which he meant the mean of the absolute value of the error. Since the mean absolute error does not lead directly to the least squares principle, Laplace gave an asymptotic argument based on his central limit theorem.

In the 1820s Gauss returned to least squares in two memoirs, the first in two parts, published by the Royal Society of Gottingen under the common title *Theoria Combinationis Observationum Erroribus Minimis Obnoxiae*. In the *Pars Prior* of the first memoir, Gauss substituted the root mean square error for Laplace's mean absolute error. This enabled him to prove his minimum variance theorem: of all linear combinations of measurements estimating an unknown, the least squares estimate has the greatest precision. The remarkable thing about this theorem is that it does not depend on the distributions of the errors, and, unlike Laplace's result, it is not asymptotic.

The second part of the first memoir is dominated by computational considerations. Among other things Gauss gives several formulas for the residual sum of squares, a technique for adding and deleting an observation from an already solved problem, and new methods for computing variances. The second memoir, called *Supplementum*, is a largely self-contained work devoted to the application of the least squares principle to geodesy. The problem here is to adjust observations so that they satisfy certain constraints, and Gauss shows that the least squares solution is optimal in a very wide sense.

The following work is a translation of the *Theoria Combinationis Observationum* as it appears in Gauss's collected works, as well as the accompanying German notices (*Anzeigen*). The translator of Gauss, or of any author writing in Latin, must make some difficult choices. Historian and classicist Michael Grant quotes Pope's couplet[†]

> O come that easy Ciceronian style,
> So Latin, yet so English all the while.

and goes on to point out that Cicero and English have since diverged. Our language has the resources to render Gauss almost word for word into grammatically correct sentences. But the result is painful to read and does no justice to Gauss's style, which is balanced and lucid, albeit cautious.

In this translation I have aimed for the learned technical prose of our time. The effect is as if an editor had taken a blue pencil to a literal translation of Gauss: sentences and paragraphs have been divided; adverbs and adverbial phrases have been pruned; elaborate turns of phrase have been tightened. But there is a limit to this process, and I have tried never to abandon Gauss's meaning for ease of expression. Moreover, I have retained his original notation, which is not very different from ours and is sometimes revealing of his thought.

Regarding nomenclature, I have avoided technical terms, like "set," that have anachronistic associations. Otherwise I have not hesitated to use the modern term or phrase; e.g., "interval," "absolute value," "if and only if." Borderline cases are continuous for *continuus*, likelihood for *facilitas*, and estimate for *determinatio*. These are treated in footnotes at the appropriate places.[‡]

[†] "Translating Latin prose" in *The Translator's Art*, edited by William Radice and Barbara Williams, Viking Penguin, New York, 1987, p. 83.

[‡] Translator's footnotes are numbered. Gauss's footnotes are indicated by *), as in his collected works.

The cost of all this is a loss of nuance, especially in tone, and historians who need to resolve fine points should consult the original, which accompanies the translation. For the rest, I hope I have produced a free but accurate rendering, which can read with profit by statisticians, numerical analysts, and other scientists who are interested in what Gauss did and how he set about doing it. In an afterword, I have attempted to put Gauss's contributions in historical perspective.

I am indebted to C. A. Truesdell, who made some very useful comments on my first attempt at a translation, and to Josef Stoer, who read the translation of the *Pars Prior*. Urs von Matt read the translation of Gauss's *Anzeigen.* Claudio Beccari kindly furnished his patterns for Latin hyphenation, and Charles Amos provided the systems support to use them. Of course, the responsibility for any errors in conception and execution is entirely mine.

I owe most to Woody Fuller, late professor of Germanic languages at the University of Tennessee and friend to all who had the good fortune to take his classes. He sparked my interest in languages and taught me that science is only half of human learning. This translation is dedicated to his memory.

College Park, Maryland
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