

Multivariate Real Analysis

Muchang Bahng

Spring 2025

Contents

1	Banach Spaces	4
1.1	Coordinate Systems	4
2	Differentiation	5
2.1	Rules of Differentiation	9
2.2	Continuously Differentiable Functions	10
2.3	Extrema and Concavity	11
2.4	Optimization with Lagrange Multipliers	13
2.5	Inverse and Implicit Function Theorems	15
3	Higher Order Derivatives	19
3.1	k-Times Continuously Differentiable Functions	20
3.2	Taylor Series	21
3.3	Matrix Calculus	23
4	Vector Fields	25
4.1	Gradients	25
4.2	Divergence	26
4.3	Curl	27
4.4	Conservative, Solenoidal Vector Fields	28
5	Riemann and Darboux Integration	30
5.1	Conditions for Integrability	31
5.2	Iterated Integrals and Fubini's Theorem	32
5.3	Integration over Regions between Curves	33
5.4	Change of Basis	34
5.5	Improper Integrals	37
6	Surfaces	40
7	Integration of Forms	43
7.1	Line Integrals	43
7.2	Conservative Vector Fields	45
7.2.1	Curvature	48
7.3	Surface Integrals	49
7.3.1	2-Dimensional Paramaterizations of Surfaces	49
7.3.2	Orientation of Surfaces	51
7.3.3	Scalar, Vector Surface Integrals	52
7.3.4	Surface Integrals over Graphs	54
7.4	Integral Theorems	55
7.4.1	Green's Theorem	55

7.4.2	Stokes' Theorem	56
7.4.3	Gauss' Theorem	57
8	Sequences of Functions	58

In here, we extend the results of univariate real analysis to multivariate and/or vector-valued functions. In practice, multivariate calculus is used, and there are many new results that arise in the multivariate case. The case for continuity and convergence of multivariate functions is very straightforward, since these are topological properties. However, the definition of the derivative and the integral will need to be generalized.

In continuity, to prove that a limit is something, we just use ϵ - δ . However, to actually *compute* what the limit is, we have multiple ways to do this in practice.

1. Just compute the function assuming it is continuous.
2. Take some sort of path p and take the univariate limit.

We can also show that the multivariate limit doesn't exist by taking two sequences or two path functions where the limits do not equal each other.

1 Banach Spaces

Note that both the domain and codomain of a function $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ are vector spaces. However, Euclidean spaces have a lot of structure on them, and it is nice to identify the essential properties we need from these sets. It is essential that we work in vector spaces because when we define a derivative, we usually see some form that looks like

$$\frac{f(x+h) - f(x)}{h} \quad (1)$$

Note the operations used here. First, we want a notion of addition in the domain $(x+h)$ and the codomain $f(x+h) - f(x)$, along with some scalar multiplication when we multiply by $1/h$. A vector space precisely supports these operations and therefore is a natural choice. It is immediate that to define convergence, we definitely need a topology. We will see later that we want to define multivariate derivatives by adding a norm to this term, requiring the use of a normed vector space. Completion is clearly essential as we have seen in single-variable analysis.

Definition 1.1 (Banach Space)
<p>A Banach space is a normed completed vector space.</p>

Note that by extending the dimension, we have essentially lost the ordering \leq on these spaces, along with the field properties. Therefore, we will need to adapt our definitions accordingly.

1.1 Coordinate Systems

Frenet frame?

2 Differentiation

Definition 2.1 (Frechet Derivative)

A function $f : D \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$ is **differentiable** at $x \in D$ if there exists a unique linear map $Df_x : \mathbb{R}^n \rightarrow \mathbb{R}^m$ —called the **total derivative** or **Frechet derivative**—such that

$$\lim_{h \rightarrow 0} \frac{\|f(x+h) - f(x) - Df_x h\|}{\|h\|} = 0 \quad (2)$$

If such a map Df_x exists, then it is unique.

Proof. The only claim to prove is uniqueness if defined.

The reason we want D to be open is that we require that $x+h$ to also be in D for sufficiently small h , and we can guarantee this since an ϵ -ball around x is guaranteed to be in D . Just as with the univariate case, the fundamental increment lemma also holds.

Lemma 2.1 (Fundamental Increment Lemma)

Suppose the derivative of $f : [a, b] \subset \mathbb{R}^n \rightarrow \mathbb{R}$ at x exists. Then, there exists a function $\Phi : \mathbb{R}^n \rightarrow \mathbb{R}$ such that

$$f(x+h) = f(x) + Df_x h + \Phi(h)\|h\| \quad (3)$$

for sufficiently small nonzero h , and

$$\lim_{h \rightarrow 0} \Phi(h) = 0 \quad (4)$$

Note that when $m = 1$, then Df_x is a linear functional, i.e. a dual vector. A simple way to extract derivatives is to fix all of the input components except for one, and treat f as a single-variable function. This results in—for now—a completely separate notion of a derivative.

Definition 2.2 (Directional, Partial Derivative)

The **directional derivative** of a multivariate function $f : D \subset \mathbb{R}^n \rightarrow \mathbb{R}$ at a point $\mathbf{a} \in D$ in direction $\mathbf{v} \in \mathbb{R}^n$ is the instantaneous rate of change of f when moving along direction \mathbf{v} at \mathbf{a} . Formally,

$$\nabla_{\mathbf{v}} f(\mathbf{a}) := \lim_{h \rightarrow 0} \frac{f(\mathbf{a} + h\mathbf{v}) - f(\mathbf{a})}{h} \quad (5)$$

When computing directional derivatives, it is convenient to normalize the directional vector \mathbf{v} to be unit length so that it coincides with the partial derivatives. We don't technically need to set $\|\mathbf{v}\| = 1$, but if we have two vectors \mathbf{v} and a scaled $c\mathbf{v}$, then the directional derivatives will also be scaled ($\nabla_{c\mathbf{v}} f(\mathbf{a}) = c\nabla_{\mathbf{v}} f(\mathbf{a})$), so we will only work with unit directional vectors. Some say that this restriction is undesirable, since it loses the linearity of the function $\mathbf{v} \mapsto \nabla_{\mathbf{v}} f(\mathbf{a})$.

If \mathbf{v} is a unit basis vector \mathbf{e}_i , then we define this specific instance to be the **partial derivative** of f with respect to argument \mathbf{x}_i .

$$\partial_{x_i} f(\mathbf{a}) = \left. \frac{\partial}{\partial x_i} \right|_{\mathbf{a}} f := \lim_{h \rightarrow 0} \frac{f(\mathbf{a} + h\mathbf{e}_i) - f(\mathbf{a})}{h} \quad (6)$$

which can be calculated by differentiating the function w.r.t. x_i and fixing all other variables. The partial derivative looks at the function as it is approaching \mathbf{a} along an axis, while a directional derivative looks at the function as it is approaching from any direction in the domain.

Therefore, we have sort of three (or two) separate notions of derivatives. A natural question to ask is whether the existence of one derivative implies the existence of another.

Theorem 2.1 (Existence of Derivatives)

Let us have a function $f : D \subset \mathbb{R}^n \rightarrow \mathbb{R}$ and a point $\mathbf{a} \in D$.

1. If f is differentiable at $\mathbf{a} \in D$, then all of its directional derivatives exist. Furthermore, the total derivative $Df_{\mathbf{a}}$ applied to the directional unit vector \mathbf{v} is equal to the directional derivative at \mathbf{a} in direction \mathbf{v} .

$$Df_{\mathbf{a}}\mathbf{v} = \nabla_{\mathbf{v}}f(\mathbf{a}) \quad (7)$$

2. If all directional derivatives exist, then the partials exist (since we can just set the directional vectors to be the unit vectors).

Therefore, we know that differentiability (i.e. existence of the total derivative) is the strongest. If this is the case, then the partial and directional derivatives exist. It turns out that if we know the partial derivative

Furthermore, since our vector spaces come with a basis, we can realize Df_x in a matrix form. Furthermore, we can compute it quite easily!

Definition 2.3 (Jacobian)

Given $f : E \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$ differentiable at x , the **Jacobian** of f at x is the matrix realization of the total derivative, denoted Jf_x .

Theorem 2.2 (Partial Derivatives as Image of Total Derivative)

The partial derivative equals the image of the basis vector

$$\frac{\partial f_j}{\partial x_i} = (Df_x e_i)_j \quad (8)$$

Corollary 2.1 (Directional Derivatives as Image of Total Derivative)

The directional derivative equals the image of the direction vector.

$$\nabla_v f(x) = (Df_x v)_j \quad (9)$$

Proof. By linearity.

Corollary 2.2 (Entries of Jacobian are Partials)

The entries of the Jacobian matrix are precisely the partial derivatives.

Proof. Immediate result of the previous corollary.

Analogous to the univariate case, a nice way to visualize the derivative is by looking at the tangent *plane*. In general, we have an *affine hyperplane*. Define this here. Unfortunately, the only types of functions that we can meaningfully visualize are those mapping $f : \mathbb{R}^2 \rightarrow \mathbb{R}$.

Figure 1

So, to prove that a function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is differentiable at \mathbf{a} , and if so, what its total derivative is, there are essentially two steps.

1. We find a candidate for M by evaluating the partials.

$$M = (\partial_{x_1} f(\mathbf{a}) \quad \partial_{x_2} f(\mathbf{a}) \quad \dots \quad \partial_{x_n} f(\mathbf{a})) \quad (10)$$

2. We check to see if the limit is true.

$$\lim_{\mathbf{h} \rightarrow \mathbf{0}} \frac{f(\mathbf{a} + \mathbf{h}) - f(\mathbf{a}) - M\mathbf{h}}{\|\mathbf{h}\|} = 0 \quad (11)$$

If it is, then $Df_{\mathbf{a}} = M$, and the "tangent plane" of f at \mathbf{a} is defined by the equation $y = f(\mathbf{a}) + M\mathbf{h}$.

Example 2.1 (Computing Total Derivative)

The function $f(x_1, x_2) = x_1^2 + x_2^2$ is differentiable at $(1, 1)$. We let $M = (\partial_{x_1} f(1, 1), \partial_{x_2} f(1, 1)) = (2, 2)$ and see that

$$\lim_{\mathbf{h} \rightarrow \mathbf{0}} \frac{f(\mathbf{a} + \mathbf{h}) - f(\mathbf{a}) - M\mathbf{h}}{\|\mathbf{h}\|} = \lim_{\mathbf{h} \rightarrow \mathbf{0}} \frac{f(1 + h_1, 1 + h_2) - f(1, 1) - 2h_1 - 2h_2}{\sqrt{h_1^2 + h_2^2}} \quad (12)$$

$$= \lim_{\mathbf{h} \rightarrow \mathbf{0}} \frac{(1 + h_1)^2 + (1 + h_2)^2 - 2 - 2h_1 - 2h_2}{\sqrt{h_1^2 + h_2^2}} \quad (13)$$

$$= \lim_{\mathbf{h} \rightarrow \mathbf{0}} \sqrt{h_1^2 + h_2^2} = 0 \quad (14)$$

So, $Df(1, 1) = (2, 2)$.

Example 2.2 (Computing Tangent Plane)

Let us find the equation of the tangent plane to $f(x, y) = \ln(2x + y)$ at $(-1, 3)$. Our total derivative, if it exists, is the covector of partials

$$Df = \left(\frac{\partial f}{\partial x} \quad \frac{\partial f}{\partial y} \right) = \left(\frac{2}{2x+y} \quad \frac{1}{2x+y} \right) \quad (15)$$

which is indeed continuous at a neighborhood of $(-1, 3)$ (in fact, in every neighborhood not containing $\mathbf{0}$). By continuity of partials, f is differentiable at $(-1, 3)$, with $Df_{(-1,3)} = (2, 1)$. The equation of the plane is then

$$z = f(-1, 3) + Df_{(-1,3)} \begin{pmatrix} x+1 \\ y-3 \end{pmatrix} = 2(x+1) + 1(y-3) \implies z = 2x + y - 1 \quad (16)$$

However, the converse is not true for either statements. You cannot just evaluate all the partial derivatives and assume that the total derivative exists!

Example 2.3 (Existence of Directional Derivatives $\not\Rightarrow$ Differentiability)

The function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$, defined

$$f(x_1, x_2) := \begin{cases} 0 & \text{for } (x_1, x_2) = (0, 0) \\ \frac{x_1^3}{x_1^2 + x_2^2} & \text{for } (x_1, x_2) \neq (0, 0) \end{cases}$$

is not differentiable at $(0, 0)$, but all directional derivatives exist. That is, for any (conventionally) unit $\mathbf{v} = (v_1, v_2)$, its directional derivative is always well defined to be

$$\nabla_{\mathbf{v}} f(0, 0) = \lim_{h \rightarrow 0} \frac{\frac{h^3 v_1^3}{h^2(v_1^2 + v_2^2)}}{h} = \frac{v_1^3}{v_1^2 + v_2^2} = v_1^3$$

Now assuming that there is such a linear M , we can find the partials by setting $\mathbf{v} = (1, 0)$ and $\mathbf{v} = (0, 1)$, giving

$$M = (\partial_{x_1} f(0, 0) \quad \partial_{x_2} f(0, 0)) = (1 \quad 0)$$

But

$$\begin{aligned} \lim_{\mathbf{h} \rightarrow \mathbf{0}} \frac{f(a + \mathbf{h}) - f(a) - M\mathbf{h}}{\|\mathbf{h}\|} &= \lim_{\mathbf{h} \rightarrow \mathbf{0}} \frac{f(h_1, h_2) - f(0, 0) - 1 \cdot h_1 - 0 \cdot h_2}{\sqrt{h_1^2 + h_2^2}} \\ &= \lim_{\mathbf{h} \rightarrow \mathbf{0}} \frac{\frac{h_1^3}{h_1^2 + h_2^2} - h_1}{\sqrt{h_1^2 + h_2^2}} \\ &= \lim_{\mathbf{h} \rightarrow \mathbf{0}} -\frac{h_1 h_2^2}{(h_1^2 + h_2^2)^{3/2}} \end{aligned}$$

and taking along the path $\mathbf{h} = (k, k)$ gives

$$\lim_{(k, k) \rightarrow \mathbf{0}} -\frac{k^3}{(2k^2)^{3/2}} = -\frac{1}{2^{3/2}} \neq 0$$

Example 2.4 (Existence of Partial Derivatives $\not\Rightarrow$ Existence of Directional Derivatives)

Consider the function

$$f(x_1, x_2) = \begin{cases} \frac{x_1 x_2}{x_1^2 + x_2^2} & \text{if } (x_1, x_2) \neq (0, 0) \\ 0 & \text{if } (x_1, x_2) = (0, 0) \end{cases}$$

The partial derivatives exist everywhere. Away from the origin we can simply compute

$$\begin{aligned} \frac{\partial f}{\partial x_1} &= \frac{x_2(x_1^2 + x_2^2) - x_1 x_2 \cdot 2x_1}{(x_1^2 + x_2^2)^2} = \frac{-x_1^2 x_2 + x_2^3}{(x_1^2 + x_2^2)^2} \\ \frac{\partial f}{\partial x_2} &= \frac{x_1(x_1^2 + x_2^2) - x_1 x_2 \cdot 2x_2}{(x_1^2 + x_2^2)^2} = \frac{x_1^3 - x_1 x_2^2}{(x_1^2 + x_2^2)^2} \end{aligned}$$

As for the partials at the origin, we must compute using the limit rule.

$$\begin{aligned} \partial_{x_1} f(\mathbf{0}) &= \lim_{h \rightarrow 0} \frac{f(\mathbf{0} + h\mathbf{e}_1) - f(\mathbf{0})}{h} = \lim_{h \rightarrow 0} \frac{f(h, 0)}{h} = 0 \\ \partial_{x_2} f(\mathbf{0}) &= \lim_{h \rightarrow 0} \frac{f(\mathbf{0} + h\mathbf{e}_2) - f(\mathbf{0})}{h} = \lim_{h \rightarrow 0} \frac{f(0, h)}{h} = 0 \end{aligned}$$

However, the directional derivative taken in direction $\mathbf{v} = (1, 1)$ gives

$$\begin{aligned}\nabla_{(1,1)}f(0,0) &= \lim_{h \rightarrow 0} \frac{f(\mathbf{0} + h(1,1)) - f(\mathbf{0})}{h} \\ &= \lim_{h \rightarrow 0} \frac{f(h,h)}{h} \\ &= \lim_{h \rightarrow 0} \frac{1}{2h}\end{aligned}$$

which does not have a limit as $h \rightarrow 0$. To visualize this, let's look at the values of f along various lines in \mathbb{R}^2 .

1. $f = 0$ at the line $x_1 = 0$ and $x_2 = 0$, which is why the partials are 0.
2. $f = \frac{1}{2}$ at the line where $x_1 = x_2$, except for the point $(0,0)$, where $f = 0$, which is why the limit in the direction doesn't exist.

2.1 Rules of Differentiation

Just like single variable calculus, the total derivative behaves in predictable ways: it is linear, product/quotient rules, and the chain rule.

Theorem 2.3 (Linearity of Total Derivatives)

Let $f, g : D \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$ be differentiable at $\mathbf{a} \in D$. Then, the total derivative at \mathbf{a} is linear w.r.t. the function arguments.

1. $D(f + g)_{\mathbf{a}} = Df_{\mathbf{a}} + Dg_{\mathbf{a}}$
2. $D(cf)_{\mathbf{a}} = cDf_{\mathbf{a}}$

Furthermore, if f and g are differentiable over D , then

1. $D(f + g) = Df + Dg$
2. $D(cf) = cDf$

Note that for the product and quotient rules, our scope is only for scalar valued functions.

Theorem 2.4 (Product Rule)

Let $f, g : D \subset \mathbb{R}^n \rightarrow \mathbb{R}$ be differentiable at \mathbf{a} . Then,

$$D(fg)_{\mathbf{a}} = Df_{\mathbf{a}}g(\mathbf{a}) + f(\mathbf{a})Dg_{\mathbf{a}} \quad (17)$$

If f, g are differentiable over D , then

$$D(fg) = Df \cdot g + f \cdot Dg$$

Theorem 2.5 (Quotient Rule)

Let $f, g : D \subset \mathbb{R}^n \rightarrow \mathbb{R}$ be differentiable at \mathbf{a} with $g(\mathbf{a}) \neq 0$. Then,

$$D\left(\frac{f}{g}\right)_{\mathbf{a}} = \frac{Df_{\mathbf{a}}g(\mathbf{a}) - f(\mathbf{a})Dg_{\mathbf{a}}}{g(\mathbf{a})^2}$$

If f, g are differentiable over D and g never vanishes on D , then

$$D\left(\frac{f}{g}\right) = \frac{Df \cdot g - f \cdot Dg}{g^2}$$

Theorem 2.6 (Chain Rule)

Let $f : D \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$ and $g : E \subset \mathbb{R}^p \rightarrow \mathbb{R}^n$ be two functions such that $f \circ g : E \subset \mathbb{R}^p \rightarrow \mathbb{R}^m$ is defined on E . Suppose g is differentiable at $\mathbf{a} \in E$ and f is differentiable at $g(\mathbf{a}) \in D$. Then, $f \circ g$ is differentiable at \mathbf{a} , and

$$D(f \circ g)_{\mathbf{a}} = Df_{g(\mathbf{a})} \circ Dg_{\mathbf{a}}$$

If g is differentiable over E and f over $g(E) \subset D$, then $f \circ g$ is differentiable over E , and

$$D(f \circ g)(\cdot) = Df_{g(\cdot)} \circ Dg_{(\cdot)}$$

Therefore, given the composition of function $f \circ g$, we have two methods of finding the derivative matrix of $f \circ g$ at point x_0 . First is to explicitly compute $f \circ g$ and find its $m \times p$ derivative matrix $D(f \circ g)$, and plug in \mathbf{a} to get $D(f \circ g)_{\mathbf{a}}$. The second way is to use the chain rule to find the individual total derivatives $Df_{g(\mathbf{a})}$ and $Dg_{\mathbf{a}}$, and multiply them together.

2.2 Continuously Differentiable Functions

An even stronger condition beyond differentiability is continuous partials, and we often prove continuity of partials to prove differentiability.

Theorem 2.7 (Continuous Partial \implies Differentiability)

Given a function $f : D \subset \mathbb{R}^n \rightarrow \mathbb{R}$ and a point $\mathbf{a} \in D$, if all the partials $\partial_{x_i} f$ exist and are continuous at \mathbf{a} , then f is differentiable at \mathbf{a} .

Example 2.5 (Differentiability $\not\implies$ Continuous Partial)

The function

$$g(x) \equiv \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

is differentiable, with derivative at $x = 0$ to be $g'(0) = 0$, since $g(h)$ is bounded by h^2 .

$$\lim_{h \rightarrow 0} \frac{h^2 \sin\left(\frac{1}{h}\right) - 0}{h} \leq \lim_{h \rightarrow 0} \frac{h^2}{h} = 0$$

which makes

$$g'(x) \equiv \begin{cases} -\cos\left(\frac{1}{x}\right) + 2x \sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

But because $\cos\left(\frac{1}{x}\right)$ oscillates at $x \rightarrow 0$, $g'(x)$ is not continuous at $x = 0$. Therefore $g(x)$ is differentiable but not in $C^1(\mathbb{R})$.

Definition 2.4 (C^1 Space)

The vector space of all functions $f : D \subset \mathbb{R}^n \rightarrow \mathbb{R}$ with continuous partials is denoted $C^1(D; \mathbb{R})$ or $C^1(D)$. They are called **continuously differentiable**.

We can also visualize this theorem. Since the partials are continuous, then the tangent subspace, which is determined by the span of the tangent vectors determined by the partials, also changes continuously, and therefore the total derivative within a neighborhood of \mathbf{a} exists. Note that from now, whenever we talk

about differentiating a function f , we will assume that it is C^1 . This is overkill, since the set of all k -times differentiable functions is a subset of C^k , but it is conventional to work with C^k functions.

2.3 Extrema and Concavity

Definition 2.5 (Local Extrema)

Given a function $f : D \subset \mathbb{R}^n \rightarrow \mathbb{R}$, a point $\mathbf{x}_0 \in D$ is a *local minimum* if there exists a neighborhood U of \mathbf{x}_0 such that

$$f(\mathbf{x}) \geq f(\mathbf{x}_0) \text{ for every } \mathbf{x} \in U$$

Similarly, \mathbf{x}_0 is a *local maximum* if there exists a neighborhood U of \mathbf{x}_0 such that

$$f(\mathbf{x}) \leq f(\mathbf{x}_0) \text{ for every } \mathbf{x} \in U$$

Theorem 2.8 (1st Derivative Test)

If \mathbf{x}_0 is a local extremum of a differentiable function f , then $Df_{\mathbf{x}_0} = \mathbf{0}$. That is, \mathbf{x}_0 is a critical point of f , i.e. every directional derivative through \mathbf{x}_0 is 0.

Note that even though the converse of this theorem is not true, we can use the contrapositive to determine that every point that has a nonzero derivative cannot be an extremum. A function may also have an infinite amount of critical points (e.g. if they lie in a circle). In order to determine whether a critical point \mathbf{x}_0 is a relative maximum, minimum, or neither, we use the second derivative test.

Theorem 2.9 (2nd Derivative Test)

Let \mathbf{x}_0 be a critical point of C^2 function $f : \mathbb{R}^n \rightarrow \mathbb{R}$. That is, $Df_{\mathbf{x}_0} = \mathbf{0}$. Then,

1. \mathbf{x}_0 is a local minimum if $Hf_{\mathbf{x}_0}$ is positive definite.
2. \mathbf{x}_0 is a local maximum if $Hf_{\mathbf{x}_0}$ is negative definite.
3. \mathbf{x}_0 is a *saddle point* if $Hf_{\mathbf{x}_0}$ is not positive definite nor negative definite.

Visually, this makes sense since given a critical point \mathbf{x}_0 , the derivative matrix would be 0, meaning that the 2nd degree Taylor expansion of f near \mathbf{x}_0 would be in form

$$f(\mathbf{x}) \approx f(\mathbf{x}_0) + \frac{1}{2}(\mathbf{x} - \mathbf{x}_0)^T Hf_{\mathbf{x}_0}(\mathbf{x} - \mathbf{x}_0)$$

If $Hf_{\mathbf{x}_0}$ is positive definite, then by definition $\frac{1}{2}(\mathbf{x} - \mathbf{x}_0)^T Hf_{\mathbf{x}_0}(\mathbf{x} - \mathbf{x}_0) > 0$ for all \mathbf{x} near \mathbf{x}_0 , and so f would increase in every direction within the neighborhood of \mathbf{x}_0 . The logic follows similarly for negative definite matrix $Hf_{\mathbf{x}_0}$. If $Hf_{\mathbf{x}_0}$ is not positive nor negative definite, then $\frac{1}{2}(\mathbf{x} - \mathbf{x}_0)^T Hf_{\mathbf{x}_0}(\mathbf{x} - \mathbf{x}_0)$ could be positive or negative, depending on which direction vector $\mathbf{h} = \mathbf{x} - \mathbf{x}_0$ we choose for computing the directional derivative. Therefore, f will increase for certain \mathbf{h} and decrease for other \mathbf{h} .

Definition 2.6 (Global Extrema)

Given $f : D \subset \mathbb{R}^n \rightarrow \mathbb{R}$, a point $\mathbf{x}_0 \in A$ is said to be an *absolute, or global, maximum* if

$$f(\mathbf{x}_0) \geq f(x) \text{ for all } \mathbf{x} \in D$$

and a *global minimum* if

$$f(\mathbf{x}_0) \leq f(x) \text{ for all } \mathbf{x} \in D$$

Unfortunately, determining whether a point \mathbf{x}_0 is a local extremum requires us to define an open neighborhood

around \mathbf{x}_0 . This means that we can only determine local extrema within open sets in \mathbb{R}^n . Therefore, we must modify our procedure when looking for extrema on functions defined over closed bounded sets. We now describe a method of computing the global extrema. Let $f : D \subset \mathbb{R}^n \rightarrow \mathbb{R}$ be a multivariable function defined on a closed and bounded set $D \equiv U \cup \partial U$, where U is open and ∂U is the boundary of D . To find the global extrema on D , we find all local extrema of

1. f defined over the interior of U , which is open, by locating all points where $Df = \mathbf{0}$
2. f defined over ∂U , preferably defined as a composition of the path function $p : I \subset \mathbb{R} \rightarrow \partial U$ and $f : \partial U \rightarrow \mathbb{R}$. That is, find the values of t such that $D(f \circ p)(t) = 0$ and identify $p(t)$.

We take all these critical points and choose the largest to be the global maximum and the smallest to be the global minimum.

Definition 2.7 (Convex Set)

A subset $D \subset \mathbb{R}^n$ is a **convex set** if for any two points $\mathbf{x}, \mathbf{y} \in D$, the line segment joining them is also in D . That is,

$$\{\theta \mathbf{x} + (1 - \theta) \mathbf{y} \mid 0 \leq \theta \leq 1\} \subset D$$

Definition 2.8 (Convex Function)

Let D be a convex set. A function $f : D \subset \mathbb{R}^n \rightarrow \mathbb{R}$ is a **convex function** if

$$f(\theta \mathbf{x} + (1 - \theta) \mathbf{y}) \leq \theta f(\mathbf{x}) + (1 - \theta) f(\mathbf{y})$$

We can visualize this as the graph of the function in $D \oplus \mathbb{R}$ always being "under" every line segment connecting $(\mathbf{x}, f(\mathbf{x}))$ and $(\mathbf{y}, f(\mathbf{y}))$.

If we assume that f is C^1 or C^2 , we can use additional tools to prove convexity. The theorem for C^1 functions is quite intuitive, since for a convex function, the tangent plane on its graph must never be "above" the graph. In other words, the first order approximation must be a global underestimate of f .

Theorem 2.10 (Convexity of C^1 Functions)

Let D be a convex set and $f : D \subset \mathbb{R}^n \rightarrow \mathbb{R}$ be C^1 . Then, f is convex over D if and only if

$$f(\mathbf{x}_0) + Df_{\mathbf{x}_0}(\mathbf{x} - \mathbf{x}_0) \leq f(\mathbf{x})$$

for all $\mathbf{x}_0, \mathbf{x} \in D$.

Theorem 2.11 (Convexity of C^2 Functions)

Let D be a convex set and $f : D \subset \mathbb{R}^n \rightarrow \mathbb{R}$ be C^2 . Then, f is convex over D if and only if Hf is positive semidefinite over all interior points of D (i.e. all eigenvalues of $Hf_{\mathbf{a}}$ are nonnegative for all $\mathbf{a} \in D$).

The computation of the Hessian now gives us much more information about the graph of the function of interest.

Theorem 2.12 ()

A function $f : D \subset \mathbb{R}^n \rightarrow \mathbb{R}$ defined on a convex set D is convex if and only if its Hessian matrix Hf is positive semidefinite for all $\mathbf{x} \in D$.

Once we have computed the Hessian, let's take the eigendecomposition of it. Since $Hf_{\mathbf{a}}$ is a real symmetric matrix, by the spectral theorem, it will have n real eigenvalues $\lambda_1, \dots, \lambda_n$ (in descending values) and corresponding orthonormal eigenvectors $\mathbf{v}_1, \dots, \mathbf{v}_n$. Now given the gradient $\nabla f(\mathbf{a})$ at \mathbf{a} , we can approximate $\nabla f(\mathbf{a} + \mathbf{h})$ at $\mathbf{a} + \mathbf{h}$ using its total derivative as

$$\nabla f(\mathbf{a} + \mathbf{h}) \approx \nabla f(\mathbf{a}) + D\nabla f_{\mathbf{a}}\mathbf{h} = \nabla f(\mathbf{a}) + Hf(\mathbf{a})\mathbf{h}$$

The eigenvalues of $Hf(\mathbf{a})$ will tell us how "fast" the gradient changes at \mathbf{a} . That is, given a small displacement vector \mathbf{h} , we can take an orthonormal decomposition of it in the form

$$\mathbf{h} = \sum_i h_i \mathbf{v}_i$$

and now the approximate gradient can be written as

$$\nabla f(\mathbf{a} + \mathbf{h}) = \nabla f(\mathbf{a}) + \sum_i h_i \lambda_i \mathbf{v}_i$$

Therefore, bigger λ_i 's will contribute to a greater change in $f(\mathbf{a})$, and smaller ones will contribute less. We can use this information to speed up convergence by scaling along different axes of \mathbf{h} when sampling.

2.4 Optimization with Lagrange Multipliers

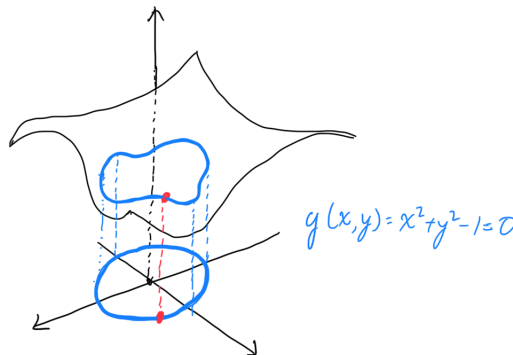
In many cases we are required to find the local extrema of a function $f : D \subset \mathbb{R}^n \rightarrow \mathbb{R}$ subject to a system of equality constraints (i.e. subject to the condition that one or more equations have to be satisfied exactly by the chosen values of the variables) of the form:

$$g_1(\mathbf{x}) = 0, g_2(\mathbf{x}) = 0, \dots, g_c(\mathbf{x}) = 0$$

which can be summarized into the constraint $\mathbf{g} : \mathbb{R}^n \rightarrow \mathbb{R}^c$

$$\mathbf{g}(x) = \begin{pmatrix} g_1(x) \\ \vdots \\ g_c(x) \end{pmatrix} = \mathbf{0}$$

which really just represents a level set of \mathbf{g} at $\mathbf{0}$, i.e. a set described by an implicit representation. In physics, these types of "well-behaved" constraints are known as *holonomic constraints*. Here is an example of a function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ constrained to the unit circle, where $g(x, y) = x^2 + y^2 - 1 = 0$.



To solve this constraint problem, we use the method of Lagrange multipliers. The basic idea is to convert a constrained problem into a form such that the derivative test of an unconstrained problem can still be applied. The relationship between the gradient of the function and gradients of the constraints rather naturally leads to a reformulation of the original problem, known as the *Lagrangian function*. That is, in order to find the maximum/minimum of f subjected to the equality constraint $\mathbf{g}(\mathbf{x}) = \mathbf{0}$, we form the Lagrangian function

$$\mathcal{L}(\mathbf{x}, \lambda) \equiv f(\mathbf{x}) - \lambda^T \mathbf{g}(\mathbf{x})$$

and find the stationary points of \mathcal{L} considered as a function of $\mathbf{x} \in D$ and the Lagrange multiplier $\lambda \in \mathbb{R}$. The main advantage to this method is that it allows the optimization to be solved without explicit parameterization in terms of the constraints.

Theorem 2.13 (Lagrange Multipliers Theorem)

Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a C^1 function and let $\mathbf{g}(\mathbf{x}) = \mathbf{0}$, where $\mathbf{g} : \mathbb{R}^n \rightarrow \mathbb{R}^c$, be a system of C^1 constraint equations: $\mathbf{g} := (g_1, g_2, \dots, g_c)$. Let \mathbf{x}^* be an optimal solution to the optimization problem of maximizing $f(\mathbf{x})$ subject to the constraint $\mathbf{g}(\mathbf{x}) = \mathbf{0}$ such that $\text{rank } D\mathbf{g}_{\mathbf{x}^*} = c < n$. Then, there exists a unique vector λ^* of Lagrange multipliers $\lambda_1^*, \lambda_2^*, \dots, \lambda_c^*$ s.t.

$$Df_{\mathbf{x}^*} = \lambda^{*T} D\mathbf{g}_{\mathbf{x}^*}$$

where $Df_{\mathbf{x}^*}$ can be interpreted as the $1 \times n$ Jacobian matrix of f and $D\mathbf{g}_{\mathbf{x}^*}$ as the $c \times n$ Jacobian of \mathbf{g} . Since both $Df_{\mathbf{x}^*}$ and $\lambda^{*T} D\mathbf{g}_{\mathbf{x}^*}$ are maps from \mathbb{R}^n to \mathbb{R} , we can invoke Riesz representation theorem to turn this into gradients:

$$\nabla f(\mathbf{x}^*) = \nabla \mathbf{g}(\mathbf{x}^*)(\lambda^*)$$

which has a matrix realization of

$$\begin{aligned} \begin{pmatrix} \frac{\partial f}{\partial x_1}(\mathbf{x}^*) \\ \vdots \\ \frac{\partial f}{\partial x_n}(\mathbf{x}^*) \end{pmatrix} &= \begin{pmatrix} \frac{\partial g_1}{\partial x_1}(\mathbf{x}^*) & \dots & \frac{\partial g_c}{\partial x_1}(\mathbf{x}^*) \\ \vdots & \ddots & \vdots \\ \frac{\partial g_1}{\partial x_n}(\mathbf{x}^*) & \dots & \frac{\partial g_c}{\partial x_n}(\mathbf{x}^*) \end{pmatrix} \begin{pmatrix} \lambda_1^* \\ \vdots \\ \lambda_c^* \end{pmatrix} \\ &= \lambda_1^* \begin{pmatrix} \frac{\partial g_1}{\partial x_1}(\mathbf{x}^*) \\ \vdots \\ \frac{\partial g_1}{\partial x_n}(\mathbf{x}^*) \end{pmatrix} + \lambda_2^* \begin{pmatrix} \frac{\partial g_2}{\partial x_1}(\mathbf{x}^*) \\ \vdots \\ \frac{\partial g_2}{\partial x_n}(\mathbf{x}^*) \end{pmatrix} + \dots + \lambda_c^* \begin{pmatrix} \frac{\partial g_c}{\partial x_1}(\mathbf{x}^*) \\ \vdots \\ \frac{\partial g_c}{\partial x_n}(\mathbf{x}^*) \end{pmatrix} \end{aligned}$$

This equation tells us that at any critical points \mathbf{x}^* of f evaluated under the equality constraints, the gradient of f at \mathbf{x}^* can be expressed as a unique linear combination of the gradients of the constraints $\nabla g_i(\mathbf{x}^*)$ (at \mathbf{x}^*), with the Lagrange multipliers acting as coefficients. Therefore, finding the critical points \mathbf{x}^* of f constrained with \mathbf{g} is equivalent to solving the system of $c + n$ equations for the n unknowns in \mathbf{x} and c unknowns in λ :

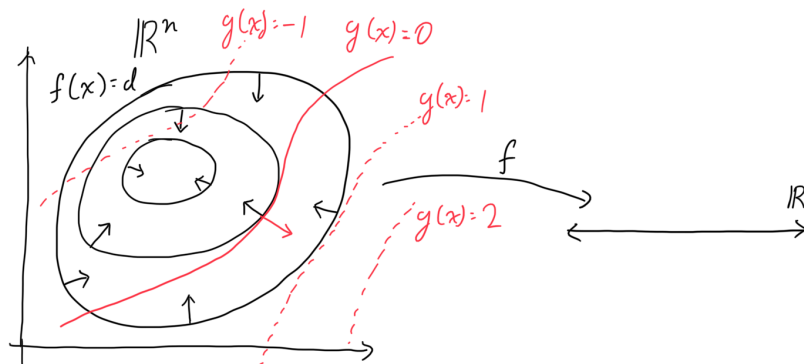
$$\begin{aligned} \mathbf{g}(\mathbf{x}) &= \mathbf{0} \\ \nabla f(\mathbf{x}) &= \nabla \mathbf{g}(\mathbf{x})(\lambda) \end{aligned}$$

which can be rewritten as

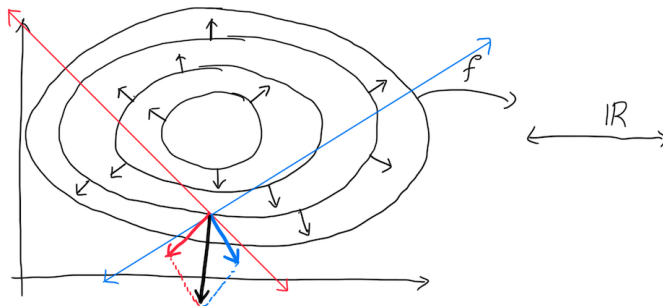
$$\begin{aligned} c \text{ constraint equations} &\begin{cases} g_1(\mathbf{x}) = 0 \\ \dots = 0 \\ g_c(\mathbf{x}) = 0 \end{cases} \\ n \text{ Lagrangian equations} &\begin{cases} \frac{\partial f}{\partial x_1}(\mathbf{x}^*) = \lambda_1^* \frac{\partial g_1}{\partial x_1}(\mathbf{x}^*) + \lambda_2^* \frac{\partial g_2}{\partial x_1}(\mathbf{x}^*) + \dots + \lambda_c^* \frac{\partial g_c}{\partial x_1}(\mathbf{x}^*) \\ \dots = \dots \\ \frac{\partial f}{\partial x_n}(\mathbf{x}^*) = \lambda_1^* \frac{\partial g_1}{\partial x_n}(\mathbf{x}^*) + \lambda_2^* \frac{\partial g_2}{\partial x_n}(\mathbf{x}^*) + \dots + \lambda_c^* \frac{\partial g_c}{\partial x_n}(\mathbf{x}^*) \end{cases} \end{aligned}$$

More abstractly, $Df_{\mathbf{x}^*}$ is the linear functional in $(\mathbb{R}^n)^*$, and $Dg_{\mathbf{x}^*}$, which is a linear map from \mathbb{R}^n to \mathbb{R}^c , can be interpreted as a map from $(\mathbb{R}^c)^*$ to $(\mathbb{R}^n)^*$. Since λ^* "lives" in $(\mathbb{R}^c)^*$, $Dg_{\mathbf{x}^*}(\lambda^*) \in (\mathbb{R}^n)^*$, which is the same space that $f_{\mathbf{x}^*}$ lives in.

Let us introduce a visualization for when there is a single constraint $g : \mathbb{R}^n \rightarrow \mathbb{R}$. From the properties of the gradient, $\nabla f(x_0)$ is orthogonal to the level set of points satisfying $f(x) = f(x_0)$ at point x_0 . Note that the constraint function g also maps $\mathbb{R}^n \rightarrow \mathbb{R}$, and so it has its own level surfaces. We can see that the point where the contour line of $g(x) = 0$ tangentially touches the contours of f is the maximum. Since it intersects it tangentially, the gradient vector at that point $\nabla g(x_0)$ is parallel to $\nabla f(x_0)$.



We can visualize this for multiple constraints as well, where $\nabla f(x_0)$ (the gradient vector of f at x^*) can be expressed as a linear combination of $\nabla g_1(x_0)$ and $\nabla g_2(x_0)$ (gradient vectors of the constraint functions at x^*).



From the properties of the gradient introduced before, $\nabla f(x_0)$ is orthogonal to the level set of points satisfying $f(x) = c$ at the point x_0 . But this level set $f(x) = c$ actually intersects the level set determined by $g(x) = c$ at the point x_0 and is indistinguishable from each other at x_0 . This means that $\nabla g(x_0)$ is normal the level set of $g(x) = c$ at $x_0 \iff$ it is normal to the level set of $f(x) = c$ at x_0 . But $\nabla f(x_0)$ is also normal at that point, so $\nabla f(x_0)$ must be parallel to $\nabla g(x_0)$.

2.5 Inverse and Implicit Function Theorems

Theorem 2.14 (Inverse Function Theorem for Multivariable Functions and its Matrix Realization)

Let $\mathbf{f} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a C^1 function defined on an open neighborhood of \mathbf{x}_0 in the domain. If the total derivative/Jacobian $D\mathbf{f}_{\mathbf{x}_0}$ at \mathbf{x}_0 is invertible, an inverse function of \mathbf{f} is defined on some neighborhood

of $\mathbf{y}_0 = \mathbf{f}(\mathbf{x}_0)$. Given that we are working with a fixed basis, \mathbf{f} can be modeled by the set of n equations

$$\begin{aligned} f_1(x_1, x_2, \dots, x_n) &= y_1 \\ &\dots = \dots \\ f_2(x_1, x_2, \dots, x_n) &= y_2 \end{aligned}$$

This theorem says that this system of n equations has a unique solution for x_1, x_2, \dots, x_n in terms of y_1, \dots, y_n , provided that we restrict \mathbf{x} and \mathbf{y} to small enough neighborhoods of \mathbf{x}_0 and \mathbf{y}_0 . This inverse function $\mathbf{f}^{-1} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is also C^1 , and its total derivative/Jacobian $D\mathbf{f}_{\mathbf{y}_0}^{-1}$ at $\mathbf{y}_0 = \mathbf{f}(\mathbf{x}_0)$ is the inverse linear map of $D\mathbf{f}_{\mathbf{x}_0}$.

$$D\mathbf{f}_{\mathbf{y}_0}^{-1} = (D\mathbf{f}_{\mathbf{x}_0})^{-1}$$

Example 2.6 ()

Consider the vector-valued function $\mathbf{f} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined by

$$\mathbf{f}(x_1, x_2) = \begin{pmatrix} e^{x_1} \cos(x_2) \\ e^{x_1} \sin(x_2) \end{pmatrix}$$

The total derivative/Jacobian matrix is

$$J\mathbf{f}(x_1, x_2) = \begin{pmatrix} e^{x_1} \cos(x_2) & -e^{x_1} \sin(x_2) \\ e^{x_1} \sin(x_2) & e^{x_1} \cos(x_2) \end{pmatrix} \implies \det J\mathbf{f}(x_1, x_2) = e^{2x_1} \cos^2(x_2) + e^{2x_1} \sin^2(x_2) = e^{2x_1}$$

Since the determinant e^{2x_1} is nonzero everywhere, $D\mathbf{f}_{\mathbf{x}}$ is nonsingular. Thus, the theorem guarantees that for every point $\mathbf{x}_0 \in \mathbb{R}^2$, there exists a neighborhood about \mathbf{x}_0 over which \mathbf{f} is invertible. However, this does not mean \mathbf{f} is invertible over its entire domain: in this case \mathbf{f} isn't even injective since it is periodic: e.g. the preimage of $(e, 0)$ contains $(1, 0)$ and $(1, 2\pi)$.

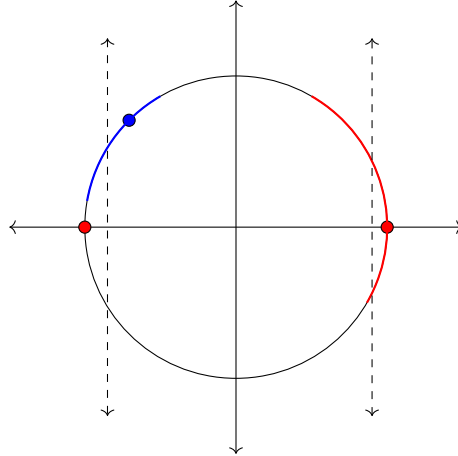
Remember that given an explicit representation of a set $\mathbf{y} = \mathbf{f}(\mathbf{x})$, we can easily find the implicit form as $\mathbf{F}(\mathbf{x}, \mathbf{y}) = \mathbf{y} - \mathbf{f}(\mathbf{x}) = \mathbf{0}$. What about the other way around? That is, given an implicit representation of some surface, what conditions must be met so that it can be represented as the graph of a function? The implicit function theorem is a tool that allows relations between points in \mathbb{R}^n to be converted to functions of several real variables. That is, it states that for sufficiently "nice" points on a n -dimensional surface defined as $\mathbf{F}(\mathbf{x}, \mathbf{y}) = \mathbf{0}$ (where $\mathbf{F} : \mathbb{R}^{n+m} \rightarrow \mathbb{R}^m$), we can locally pretend that this surface is a graph of a function $\mathbf{g} : \mathbb{R}^n \rightarrow \mathbb{R}^m$ whose graph $(\mathbf{x}, \mathbf{g}(\mathbf{x}))$ is precisely the set of all (\mathbf{x}, \mathbf{y}) s.t. $\mathbf{f}(\mathbf{x}, \mathbf{y}) = \mathbf{0}$. When $m = 1$, it basically states that if an implicit surface suffices the vertical line test in a neighborhood, then it can be written as a function.

Example 2.7 (Circle)

Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be defined by $f(x, y) = x^2 + y^2 - 1$. The level set at $z = 0$ would be the set of points satisfying

$$x^2 + y^2 - 1 = 0$$

the unit circle. The derivative of f with respect to y is 0 at the points $(-1, 0)$ and $(1, 0)$, meaning that in any neighborhood of these points, we cannot define a function of y with respect to x . This is true, indeed, since any such function would fail the vertical line test, which can be seen in the red neighborhood around $(1, 0)$. However, the blue neighborhood of the point $(-\sqrt{2}/2, \sqrt{2}/2)$ does indeed define a function of y with respect to x satisfying the vertical line test.


Theorem 2.15 (Special Implicit Function Theorem)

Let $f : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$ be a C^1 function with a point $(\mathbf{a}, b) \in \mathbb{R}^{n+1}$ on the level set $f(\mathbf{x}, y) = 0$. If $\partial_y f(\mathbf{a}, b)$, which can also be thought of as the 1×1 truncated Jacobian matrix $D_y f(\mathbf{a}, b) = \partial_y f(\mathbf{a}, b)$ w.r.t. y , of

$$Df_{(\mathbf{a}, b)} = (D_{\mathbf{x}} f(\mathbf{a}, b) \quad D_y f(\mathbf{a}, b)) = (\partial_{x_1} f(\mathbf{a}, b) \quad \dots \quad \partial_{x_n} f(\mathbf{a}, b) \mid \partial_y f(\mathbf{a}, b))$$

is invertible (in this case nonzero), then there exists an open neighborhood $U_{\mathbf{a}} \subset \mathbb{R}^n$ of \mathbf{a} and a unique C^1 function $y : U \subset \mathbb{R}^n \rightarrow \mathbb{R}$ s.t. $f(\mathbf{x}, y(\mathbf{x})) = 0$ for all $\mathbf{x} \in U$. That is, we can find a $y : U \rightarrow \mathbb{R}$ s.t. the graph $y(\mathbf{x})$ within U coincides with the graph of $f(\mathbf{x}, y) = 0$. Moreover, the total derivative/Jacobian of $y : \mathbb{R}^n \rightarrow \mathbb{R}$ in U is the $1 \times n$ matrix given by the matrix product

$$Dg_{\mathbf{a}} = -(D_y f_{\mathbf{a}})^{-1} D_{\mathbf{x}} f_{\mathbf{a}}$$

Example 2.8 (Circle Example)

Let $n = m = 1$ and $f(x_1, x_2) = x_1^2 + x_2^2 - 1$. We would like to find out at which points \mathbf{a} can this surface be explicitly represented by a function $g : U_{\mathbf{a}} \subset \mathbb{R} \rightarrow \mathbb{R}$ defining x_2 from x_1 . Its Jacobian is

$$Df = (\partial_{x_1} f \quad \partial_{x_2} f) = (2x_1 \quad 2x_2)$$

The truncated Jacobian w.r.t. x_2 is $2x_2$, which is invertible iff $x_2 \neq 0$. By the implicit function theorem, we can locally write the circle in the form $x_2 = g(x_1)$ for all points where $x_2 \neq 0$. This is easy to see. For example, we can choose the point $(0.8, 0.6)$ on the level set, and the appropriate explicit function is

$$x_2 = g(x_1) = \sqrt{1 - x_1^2}$$

within the neighborhood of $x_1 = 0.8$. For $(\pm 1, 0)$, we cannot since every function defined within a neighborhood of $x_1 = \pm 1$ fails the vertical line test. The derivative of g , by the theorem, can be defined implicitly as

$$Dg = -(\partial_{x_2} f)^{-1} D_{x_1} f = -(2x_2)^{-1}(2x_1) = -\frac{x_1}{x_2}$$

which leads to the differential equation

$$g'(x_1) = -\frac{x_1}{g(x_1)} \text{ where we solve for } g$$

If we would have liked to find a function $h : U_{a_{x_2}} \subset \mathbb{R} \rightarrow \mathbb{R}$ defining x_1 from x_2 , then we can redo everything to find that the truncated Jacobian w.r.t. x_1 is $2x_1$, which is invertible iff $x_1 \neq 0$, and the derivative is

$$Dh = -(\partial_{x_1} f)^{-1} D_{x_2} f = -(2x_1)^{-1} (2x_2) = -\frac{x_2}{x_1}$$

which leads to the differential equation

$$h'(x_2) = -\frac{x_2}{h(x_2)} \text{ where we solve for } h$$

Theorem 2.16 (General Implicit Function Theorem)

Let $f : \mathbb{R}^{n+m} \rightarrow \mathbb{R}^m$ be a C^1 function with a point $(\mathbf{a}, \mathbf{b}) \in \mathbb{R}^{n+m}$ on the level set $f(\mathbf{x}, \mathbf{y}) = 0$. If the $m \times m$ truncated Jacobian matrix $D_{\mathbf{y}} f(\mathbf{a}, \mathbf{b})$ w.r.t. \mathbf{y} , of

$$Df_{(\mathbf{a}, \mathbf{b})} = (D_{\mathbf{x}} f_{(\mathbf{a}, \mathbf{b})} \quad D_{\mathbf{y}} f_{(\mathbf{a}, \mathbf{b})})$$

is invertible, then there exists an open neighborhood $U_{\mathbf{a}} \subset \mathbb{R}^n$ of \mathbf{a} and a unique C^1 function $\mathbf{y} : U \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$ s.t. $f(\mathbf{x}, \mathbf{y}(\mathbf{x})) = 0$ for all $\mathbf{x} \in U$. That is, we can find a $\mathbf{y} : U \rightarrow \mathbb{R}^m$ s.t. the graph $\mathbf{y}(\mathbf{x})$ within U coincides with the graph of $f(\mathbf{x}, \mathbf{y}) = 0$. Moreover, the total derivative/Jacobian of \mathbf{y} is the $m \times n$ matrix given by the matrix product

$$Dg_{\mathbf{a}} = -(D_{\mathbf{y}} f_{\mathbf{a}})^{-1} D_{\mathbf{x}} f_{\mathbf{a}}$$

3 Higher Order Derivatives

So far, we have talked about derivatives as linear maps. However, this language is a bit too restrictive because it doesn't allow us to represent the derivatives of higher order using matrices. It turns out that these higher order derivatives are tensors, so let's generalize our definitions. Recall from linear algebra that a linear map $f : V \rightarrow W$ is an element of the tensor product space $V^\dagger \otimes W$, where V^\dagger represents the dual space of V . That is,

$$\text{Hom}(V, W) \simeq V^\dagger \otimes W \quad (18)$$

Definition 3.1 (Frechet Derivative)

A function $f : V \rightarrow W$ is **differentiable** at $x \in V$ if there exists a unique tensor $Df_x \in V^\dagger \otimes W$ —called the **total derivative** or **Frechet derivative**—such that

$$\lim_{h \rightarrow 0} \frac{\|f(x+h) - f(x) - Df_x h\|}{\|h\|} = 0 \quad (19)$$

It's a simple—almost trivial—change, but extremely powerful. Note that if f is differentiable over V , then the transformation $Df : V \rightarrow (V^\dagger \otimes W)$ that maps x_1 to its Frechet derivative Df_{x_1} can also be analyzed. Supposing that it is differentiable, the derivative of it at x is

$$D^2 f_x := D(Df)_x \in V^\dagger \otimes (V^\dagger \otimes W) \simeq (V^\dagger)^{\otimes 2} \otimes W \quad (20)$$

by the definition above. Intuitively this also make sense: if we want to take the second derivative of f , we must first specify the point in the domain of Df (our velocity) plus the point in the domain of f itself (our position) to compute it (the acceleration). Continuing this pattern, we get the following recursive definition.

Definition 3.2 (k th Frechet Derivative)

For $k > 1$, a function $f : V \rightarrow W$ is **k -times differentiable** at $x \in V$ if

1. f is $k-1$ times differentiable at $x \in V$, and
2. there exists a unique rank $(k, 1)$ tensor $D^k f_x \in (V^\dagger)^{\otimes k} \otimes W$ —called the **k th total derivative** or **k th Frechet derivative**—such that

$$\lim_{h \rightarrow 0} \frac{\|D^{k-1} f_x(x+h) - D^{k-1} f_x(x) - D^k f_x h\|}{\|h\|} = 0 \quad (21)$$

where $D^{k-1} f_x$ is the $(k-1)$ th derivative of f at x .

Generally, multivariate calculus courses do not go into tensor algebra, so these higher order derivatives are omitted; you only work with the Hessian for scalar-valued functions. But we won't be scared off and will present this in full generality, and this gives us two major benefits:

1. We can compute higher order derivative as we have seen.
2. We can compute functions that also input or output tensors, such as matrix-valued functions. This knowledge becomes very useful for optimization, e.g. when you optimize a neural network with respect to matrices (in fully-connected layers) and 3-tensors (e.g. convolutions) and is a must-know for building autograd libraries.

Definition 3.3 (k th Partial Derivative)

Directional derivatives lose meaning in these higher order regimes, and we just work with iterated partials.

Definition 3.4 (Hessian)

Given $f : E \subset V \rightarrow \mathbb{R}$ twice-differentiable at x , the **Hessian** of f at x is the matrix realization of the total derivative, denoted Hf_x .

Theorem 3.1 (Entries of Hessian are 2nd Partial)

The entries of the Hessian matrix are precisely the partials.

$$Hf_x := \begin{pmatrix} \partial_{x_1 x_1}(\mathbf{a}) & \cdots & \partial_{x_1 x_n}(\mathbf{a}) \\ \vdots & \ddots & \vdots \\ \partial_{x_n x_1}(\mathbf{a}) & \cdots & \partial_{x_n x_n}(\mathbf{a}) \end{pmatrix} \text{ and } Hf := \begin{pmatrix} \partial_{x_1 x_1} & \cdots & \partial_{x_1 x_n} \\ \vdots & \ddots & \vdots \\ \partial_{x_n x_1} & \cdots & \partial_{x_n x_n} \end{pmatrix} \quad (22)$$

Proof.

3.1 k-Times Continuously Differentiable Functions**Definition 3.5 (C^k Functions)**

A function $f : D \subset \mathbb{R}^n \rightarrow \mathbb{R}$ is said to be a C^k function if all k -times iterated partial derivatives

$$\partial_{x_{i_1} x_{i_2} \dots x_{i_k}} f \quad (23)$$

exist and are continuous. The vector space of all C^k functions is denoted $C^k(D; \mathbb{R})$, or $C^k(D)$.

Whenever we want to get the k th iterated partial derivative of f , we will assume that $f \in C^k$. Again, this is overkill, but it is conventional since we don't really work with the set of functions with existing partial derivatives. Note that mathematicians throw around the word "smooth" a lot. Usually, it means one of three things: it is of class C^1 , C^∞ , or C^k where k is however high it needs to be to satisfy our assumptions. For example, if I say let us differentiate smooth f two times, then I am assuming that $f \in C^2(\mathbb{R}^n)$.

Visualizing C^k -functions is easy for low orders. A C^0 function produces a graph that isn't "ripped" or "punctured," since this is exactly what a discontinuity would look like. A C^1 function requires the surface to be smooth in such a way that there is a well defined affine tangent subspace at every point. This means that there cannot be any sharp "points" or "edges" on the graph since a tangent subspace cannot be well defined.

Theorem 3.2 (Clairut's Theorem)

Given $f \in C^2$ at point \mathbf{a} , its second iterated partials are equal.

$$\partial_{x_i x_j} f(\mathbf{a}) = \partial_{x_j x_i} f(\mathbf{a}) \text{ for } i, j = 1, 2, \dots, n \quad (24)$$

Proof. For clarity, denote x_i, x_j as x, y and ignore the rest of the variables. Then, the partial derivatives $\partial_{xy} f$ and $\partial_{yx} f$ at a point (x_0, y_0) can be expressed as double limits:

$$\partial_{xy} f(x_0, y_0) = \lim_{y \rightarrow y_0} \frac{\partial_x f(x_0, y) - \partial_x f(x_0, y_0)}{y - y_0} \quad (25)$$

where $\partial_x f : D \subset \mathbb{R}^n \rightarrow \mathbb{R}$. We can use the two limit definitions of partial derivatives

$$\partial_x f(x_0, y) = \lim_{x \rightarrow x_0} \frac{f(x, y) - f(x_0, y)}{x - x_0} \text{ and } \partial_x f(x_0, y_0) = \lim_{x \rightarrow x_0} \frac{f(x, y_0) - f(x_0, y_0)}{x - x_0} \quad (26)$$

and substitute them to get the two partials

$$\partial_{xy}f(x_0, y_0) = \lim_{y \rightarrow y_0} \frac{\lim_{x \rightarrow x_0} \frac{f(x, y) - f(x_0, y)}{x - x_0} - \lim_{y \rightarrow y_0} \frac{f(x, y_0) - f(x_0, y_0)}{x - x_0}}{y - y_0} \quad (27)$$

$$= \lim_{y \rightarrow y_0} \lim_{x \rightarrow x_0} \left(\frac{f(x, y) - f(x_0, y) - f(x, y_0) + f(x_0, y_0)}{(x - x_0)(y - y_0)} \right) \quad (28)$$

$$\partial_{yx}f(x_0, y_0) = \lim_{x \rightarrow x_0} \frac{\lim_{y \rightarrow y_0} \frac{f(x, y) - f(x, y_0)}{y - y_0} - \lim_{y \rightarrow y_0} \frac{f(x_0, y) - f(x_0, y_0)}{y - y_0}}{x - x_0} \quad (29)$$

$$= \lim_{x \rightarrow x_0} \lim_{y \rightarrow y_0} \left(\frac{f(x, y) - f(x, y_0) - f(x_0, y) + f(x_0, y_0)}{(y - y_0)(x - x_0)} \right) \quad (30)$$

Now invoking our assumption that f is C^2 , the two limits, which approach (x_0, y_0) along different paths, both exist and are equal to

$$\lim_{(x, y) \rightarrow (x_0, y_0)} \frac{f(x, y) - f(x_0, y) - f(x, y_0) + f(x_0, y_0)}{(x - x_0)(y - y_0)} \quad (31)$$

and therefore $\partial_{xy}f = \partial_{yx}f$.

Corollary 3.1 (Symmetry in k th Iterated Partial)

Given $f \in C^k$, its k th iterated partials are equal. That is, given any permutation σ ,

$$\partial_{x_{i_1} x_{i_2} \dots x_{i_k}} f = \partial_{x_{\sigma(i_1)} x_{\sigma(i_2)} \dots x_{\sigma(i_k)}} f \text{ for } i_1, \dots, i_k = 1, \dots, n \quad (32)$$

Proof. By induction.

Corollary 3.2 ()

The Hessian of a C^2 function is symmetric.

Let's talk more about this. By the spectral theorem, we can eigendecompose it.

3.2 Taylor Series

To talk about convergence, the big-O notation is very useful.

Definition 3.6 (Classes of Infinitesimal Functions)

A function $\alpha : \mathbb{R}^n \rightarrow \mathbb{R}$ is **infinitesimal** if $\alpha \rightarrow 0$ as $\mathbf{x} \rightarrow \mathbf{x}_0$. There are multiple "levels" of infinitesimal functions, i.e. how fast they converge to 0. We can classify them by comparing their limits to polynomials.

1. α is of class $O(1)$ if

$$\lim_{\mathbf{x} \rightarrow \mathbf{x}_0} \frac{\alpha(\mathbf{x})}{1} = 0$$

This means that $\alpha(\mathbf{x})$ tends to 0 infinitely faster than 1 (which just means that it tends to 0).

2. α is of class $O(h)$ if

$$\lim_{\mathbf{x} \rightarrow \mathbf{x}_0} \frac{\alpha(\mathbf{x})}{\|\mathbf{x} - \mathbf{x}_0\|} = 0$$

- This means that $\alpha(\mathbf{x})$ tends to 0 infinitely faster than the linear $\|\mathbf{h}\|$, where $\mathbf{h} = \mathbf{x} - \mathbf{x}_0$.
3. α is of class $O(h^2)$ if

$$\lim_{\mathbf{x} \rightarrow \mathbf{x}_0} \frac{\alpha(\mathbf{x})}{\|\mathbf{x} - \mathbf{x}_0\|^2} = 0$$

- This means that $\alpha(\mathbf{x})$ tends to 0 infinitely faster than the quadratic $\|\mathbf{h}\|^2$, where $\mathbf{h} = \mathbf{x} - \mathbf{x}_0$.
4. α is of class $O(h^k)$ if

$$\lim_{\mathbf{x} \rightarrow \mathbf{x}_0} \frac{\alpha(\mathbf{x})}{\|\mathbf{x} - \mathbf{x}_0\|^k} = 0$$

This means that $\alpha(\mathbf{x})$ tends to 0 infinitely faster than the k th-order $\|\mathbf{h}\|^k$, where $\mathbf{h} = \mathbf{x} - \mathbf{x}_0$. Clearly, $O(h^k) \supset O(h^{k+1})$.

Now given a $f : D \subset \mathbb{R}^n \rightarrow \mathbb{R}$, we present some polynomial approximations of f at $\mathbf{x}_0 \in D$:

1. If $f \in C^0$, the zeroth (constant) approximation is just

$$P_0(\mathbf{x}) = f(\mathbf{x}_0)$$

This is not interesting at all, since it is just constant. Furthermore, the error term $\epsilon_0(\mathbf{x}) = f(\mathbf{x}) - P_0(\mathbf{x})$ is an infinitesimal function as $\mathbf{x} \rightarrow \mathbf{x}_0$ and is of class $O(1)$, since

$$\lim_{\mathbf{x} \rightarrow \mathbf{x}_0} \frac{f(\mathbf{x}) - P_0(\mathbf{x})}{1} = 0$$

2. If $f \in C^1$, the first (linear) approximation requires us to use our total derivative:

$$P_1(\mathbf{x}) = f(\mathbf{x}_0) + Df_{\mathbf{x}_0}(\mathbf{x} - \mathbf{x}_0)$$

and we know that the error $\epsilon_1(\mathbf{x}) = f(\mathbf{x}) - P_1(\mathbf{x})$ is infinitesimal as $\mathbf{x} \rightarrow \mathbf{x}_0$ and is of class $O(h)$, since

$$\lim_{\mathbf{x} \rightarrow \mathbf{x}_0} \frac{f(\mathbf{x}) - P_1(\mathbf{x})}{\|\mathbf{x} - \mathbf{x}_0\|} = 0$$

3. If $f \in C^2$, the second (quadratic) approximation requires us to use a quadratic term (i.e. a bilinear form of $\mathbf{h} = \mathbf{x} - \mathbf{x}_0$) centered at \mathbf{x}_0 . Call it $H_{\mathbf{x}_0} : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$, and our estimation is

$$P_2(\mathbf{x}) = f(\mathbf{x}_0) + Df_{\mathbf{x}_0}(\mathbf{x} - \mathbf{x}_0) + \frac{1}{2}H(\mathbf{x} - \mathbf{x}_0, \mathbf{x} - \mathbf{x}_0)$$

which we would like the error term $\epsilon_2(\mathbf{x}) = f(\mathbf{x}) - P_2(\mathbf{x})$ to be $O(h^2)$, or in limit terms, P_2 must satisfy

$$\lim_{\mathbf{x} \rightarrow \mathbf{x}_0} \frac{f(\mathbf{x}) - P_2(\mathbf{x})}{\|\mathbf{x} - \mathbf{x}_0\|^2} = 0$$

We show that this form H is precisely the Hessian matrix.

Theorem 3.3 (Hessian)

The second order approximation of a C^2 -differentiable function f about a point \mathbf{x}_0 is

$$f(\mathbf{x}) = f(\mathbf{x}_0) + Df_{\mathbf{x}_0}(\mathbf{x} - \mathbf{x}_0) + \frac{1}{2}(\mathbf{x} - \mathbf{x}_0)^T Hf_{\mathbf{x}_0}(\mathbf{x} - \mathbf{x}_0) + O(h^2)$$

where $Df_{\mathbf{x}_0}$ is the total derivative at \mathbf{x}_0 and $Hf_{\mathbf{x}_0}$ is the Hessian matrix at \mathbf{x}_0 .

3.3 Matrix Calculus

Now we will take a look at functions that have either an input or output as matrices. Essentially, matrices are also vectors, so there is nothing new here to learn, but having a concrete set of notation is useful. First, note that when we talk about a total derivative $D\mathbf{f}_a$, we can interpret this as a linear map that takes in some small perturbation \mathbf{h} and gives us the result $D\mathbf{f}_a(\mathbf{h})$. In our column-vector setting, this just corresponded to left matrix multiplication:

$$D\mathbf{f}_a(\mathbf{h}) = D\mathbf{f}_a\mathbf{h} \quad (33)$$

This is not the case in the matrix setting. Let us compare the following:

1. The derivative of $f : \mathbb{R} \rightarrow \mathbb{R}$ at a is a linear function $Df_a : \mathbb{R} \rightarrow \mathbb{R}$ satisfying $f(a+h) \approx f(a) + Df_a(h) + O(h^2)$. But linearity reduces Df_a to simply a scalar, and so our condition reduces to

$$f(a+h) \approx f(a) + Df_a h + O(h^2)$$

2. The derivative of a path function $\mathbf{f} : \mathbb{R} \rightarrow \mathbb{R}^m$ is a linear function $D\mathbf{f}_a : \mathbb{R} \rightarrow \mathbb{R}^m$ satisfying $\mathbf{f}(a+h) = \mathbf{f}(a) + D\mathbf{f}_a(h) + O(h^2)$. Linearity implies that $D\mathbf{f}_a$ is a rank-1 linear map, which reduces to it being a column vector, and our condition reduces to

$$\underbrace{\mathbf{f}(a+h)}_{m \times 1} \approx \underbrace{\mathbf{f}(a)}_{m \times 1} + \underbrace{D\mathbf{f}_a}_{m \times 1} \underbrace{h}_{1 \times 1}$$

3. The derivative of a matrix function $\mathbf{F} : \mathbb{R} \rightarrow \mathbb{R}^{m \times n}$ is a linear map $D\mathbf{F}_a : \mathbb{R} \rightarrow \mathbb{R}^{m \times n}$ satisfying $\mathbf{F}(a+h) \approx \mathbf{F}(a) + D\mathbf{F}_a(h)$. This time, linearity does not reduce it to simple left-hand matrix multiplication. We could just say that this is a left-hand scalar multiplication, but this doesn't generalize well, so we are stuck with just saying that $D\mathbf{F}_a$ is a linear map.

$$\underbrace{\mathbf{F}(a+h)}_{m \times n} \approx \underbrace{\mathbf{F}(a)}_{m \times n} + \underbrace{D\mathbf{F}_a(h)}_{m \times n}$$

Now let us take a look at when we have matrix inputs.

1. The derivative of $f : \mathbb{R}^{m \times n} \rightarrow \mathbb{R}$ is a linear function $Df_{\mathbf{A}} : \mathbb{R}^{m \times n} \rightarrow \mathbb{R}$ satisfying $f(\mathbf{A} + \mathbf{H}) \approx f(\mathbf{A}) + Df_{\mathbf{A}}(\mathbf{H})$. We could let $Df_{\mathbf{A}}$ be some linear map like $\mathbf{M} \mapsto \mathbf{v}^T \mathbf{M} \mathbf{u}$, where \mathbf{v}, \mathbf{u} is fixed. But in generality, we just have the condition

$$\underbrace{f(\mathbf{A} + \mathbf{H})}_{1 \times 1} \approx \underbrace{f(\mathbf{A})}_{1 \times 1} + \underbrace{Df_{\mathbf{A}}(\mathbf{H})}_{1 \times 1}$$

2. The derivative of $\mathbf{f} : \mathbb{R}^{m \times n} \rightarrow \mathbb{R}^d$ is some linear map $D\mathbf{f}_{\mathbf{A}} : \mathbb{R}^{m \times n} \rightarrow \mathbb{R}^d$ satisfying $\mathbf{f}(\mathbf{A} + \mathbf{H}) \approx \mathbf{f}(\mathbf{A}) + D\mathbf{f}_{\mathbf{A}}(\mathbf{H})$. Again, we could construct some form that would give us a linear map in terms of some matrix multiplication, but in generality, we have the condition

$$\underbrace{\mathbf{f}(\mathbf{A} + \mathbf{H})}_{d \times 1} \approx \underbrace{\mathbf{f}(\mathbf{A})}_{d \times 1} + \underbrace{D\mathbf{f}_{\mathbf{A}}(\mathbf{H})}_{d \times 1}$$

Now we present some theorems on basic differentiation. Proving these just requires us to expand the function and compute the derivatives component-wise.

Theorem 3.4 (Derivative of Affine Map)

Given $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ defined $f(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{b}$ (where \mathbf{A}, \mathbf{b} is not dependent on \mathbf{x}), its derivative is

$$Df = \mathbf{A}$$

Theorem 3.5 ()

Given the scalar α defined by

$$\alpha = \mathbf{y}^T \mathbf{A} \mathbf{x}$$

where $\mathbf{y} \in \mathbb{R}^{m \times 1}$, $\mathbf{A} \in \mathbb{R}^{m \times n}$, and $\mathbf{x} \in \mathbb{R}^{n \times 1}$, then

$$\frac{\partial \alpha}{\partial \mathbf{x}} = \mathbf{y}^T \mathbf{A} : \mathbb{R}^n \rightarrow \mathbb{R}$$

and

$$\frac{\partial \alpha}{\partial \mathbf{y}} = \mathbf{x}^T \mathbf{A}^T : \mathbb{R}^m \rightarrow \mathbb{R}$$

Rewritten in the total derivative notation, we can interpret α as a function of both \mathbf{x} and \mathbf{y} and write

$$D\alpha_{(\mathbf{x}, \mathbf{y})} = \begin{pmatrix} \mathbf{y}^T \mathbf{A} \\ \mathbf{x}^T \mathbf{A}^T \end{pmatrix} : \mathbb{R}^{n+m} \rightarrow \mathbb{R}$$

Theorem 3.6 ()

Given the scalar α defined by the quadratic form

$$\alpha = \mathbf{x}^T \mathbf{A} \mathbf{x} \tag{34}$$

where $\mathbf{x} \in \mathbb{R}^{n \times 1}$ and $\mathbf{A} \in \mathbb{R}^{n \times n}$, then

$$\frac{\partial \alpha}{\partial \mathbf{x}} = \mathbf{x}^T (\mathbf{A} + \mathbf{A}^T) \tag{35}$$

or in the total derivative notation,

$$D\alpha_{\mathbf{a}} = \mathbf{a}^T (\mathbf{A} + \mathbf{A}^T) : \mathbb{R}^n \rightarrow \mathbb{R} \tag{36}$$

4 Vector Fields

4.1 Gradients

Definition 4.1 (Gradient)

The gradient of a C^1 scalar-valued function f is the vector field $\nabla f : D \subset \mathbb{R}^n \rightarrow \mathbb{R}^n$ defined

$$\nabla f(\mathbf{a}) := \begin{pmatrix} \partial_{\mathbf{x}_1}(\mathbf{a}) \\ \vdots \\ \partial_{\mathbf{x}_n}(\mathbf{a}) \end{pmatrix}$$

The gradient at a point is a **tangent vector**.

Note that the gradient is a vector field (a bundle of vectors), while the total derivative is a covector field (a bundle of covectors). Since \mathbb{R}^n is an inner product space, we can invoke Riesz Representation theorem and see that they are related in the way that

$$Df_{\mathbf{a}}\mathbf{v} = \nabla f(\mathbf{a}) \cdot \mathbf{v}$$

where \cdot represents the dot product. At this point, it's a bit hard to see the difference between these two, but in more abstract spaces the total derivative generalizes much better than the gradient, which exists for inner product spaces. From this, we can write a coordinate independent definition of the gradient.

Definition 4.2 (Gradient)

The gradient of a scalar valued function $f \in C^1$ is the unique vector field whose dot product with any vector \mathbf{v} at each point is the directional derivative of f along \mathbf{v} . That is,

$$\nabla f_{\mathbf{a}} \cdot \mathbf{v} = Df_{\mathbf{a}}\mathbf{v} \text{ for all } \mathbf{a} \in D$$

Theorem 4.1 (Gradient as Direction of Fastest Increase)

Let f be a real-valued function such that $\nabla f(x) \neq 0$. Then, at the point x , $\nabla f(x)$ points in the direction along which f is increasing the fastest. Equivalently, $-\nabla f(x)$ points in the direction along which f is decreasing the fastest.

Proof. Note that this is a coordinate-independent proof. Given a directional vector \mathbf{v} , we can normalize it since we are only interested in direction. Evaluating it with the total derivative at x gives us $D_{\mathbf{a}}f\mathbf{v}$. But by definition,

$$\nabla f_{\mathbf{a}} \cdot \mathbf{v} = Df_{\mathbf{a}}\mathbf{v}$$

which means that

$$\begin{aligned} \sup_{\|\mathbf{v}\|=1} \{Df_{\mathbf{a}}\mathbf{v}\} &= \sup_{\|\mathbf{v}\|=1} \{\nabla f_{\mathbf{a}} \cdot \mathbf{v}\} \\ &= \sup_{\|\mathbf{v}\|=1} \{\|\nabla f_{\mathbf{a}}\| \|\mathbf{v}\| \cos(\theta)\} \\ &= \sup\{\|\nabla f_{\mathbf{a}}\| \cos(\theta)\} \\ &= \|\nabla f_{\mathbf{a}}\| \text{ when } \theta = 0 \end{aligned}$$

Therefore, \mathbf{v} must point in the direction of $\nabla f_{\mathbf{a}}$.

Therefore, we can interpret the gradient evaluated at a point as the tangent vector that points in the direction

of fastest increase. We can also interpret the gradient ∇f itself as the vector field that determines some sort of "flow" in the domain \mathbb{R}^n . Therefore, if we drop a point in this field, the point will flow through \mathbb{R}^n through a current determined by ∇f and will eventually end up at a local maximum.

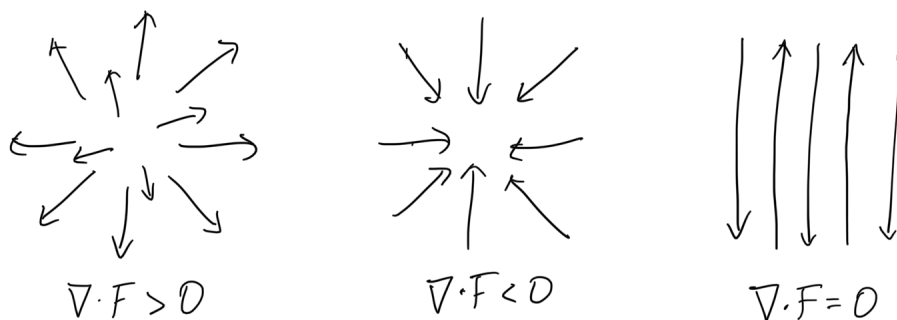
Definition 4.3 (Del Operator)

For convenience, we use the del operator to denote the gradient. The **del operator** $\nabla : f \mapsto \nabla f$ takes in a differentiable function and outputs the gradient of it.

4.2 Divergence

Colloquially, the divergence is an operator div that operates on a vector field and produces a scalar field which provides the quantity of the vector field's source at each point. Technically, the divergence represents the volume density of the outward flux of a vector field from an infinitesimal volume around a given point.

There is a very nice geometric interpretation for divergence. Imagine that the vector field F represents fluid flow in \mathbb{R}^n . Divergence is then the "measure" of the net amount of fluid flowing in and out of an infinitesimally small region, labeled at each point. If the net fluid flow is positive (i.e. more fluid is flowing in than out) at point x_0 , then $\text{div } F(x_0) > 0$. If the net fluid flow is negative (i.e. more fluid is flowing out than in) at point x_0 , then $\text{div } F(x_0) < 0$. This measure assigns a number to every point in the space (creating a scalar field). Therefore, each point either acts as a "source" of fluid emanating from it or as a "sink" that sucks in more fluid than it puts out.



Definition 4.4 (Divergence)

The **divergence** of a vector field $\mathbf{F} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a scalar field defined

$$\text{div } \mathbf{F} := \nabla \cdot \mathbf{F} = \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \vdots \\ \frac{\partial}{\partial x_n} \end{pmatrix} \cdot \begin{pmatrix} F_1 \\ \vdots \\ F_n \end{pmatrix} = \sum_i \frac{\partial F_i}{\partial x_i}$$

When $n = 1$, \mathbf{F} reduces to a regular function and $\text{div } \mathbf{F}$ reduces to the ordinary derivative. Some further properties:

1. By linearity of partials, div is also a linear operator. That is, given two vector fields \mathbf{F}, \mathbf{G} and two scalars α, β ,

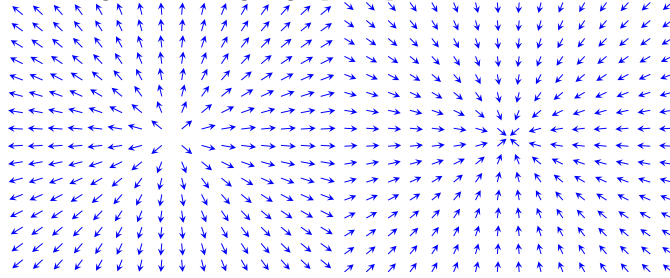
$$\text{div}(\alpha \mathbf{F} + \beta \mathbf{G}) = \alpha \text{div } \mathbf{F} + \beta \text{div } \mathbf{G}$$

2. Divergence satisfies the product rule: Given a vector field $\mathbf{F} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ and a scalar function $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}$.

$$\nabla \cdot (\varphi \mathbf{F}) = \nabla \varphi \cdot \mathbf{F} + \varphi (\nabla \cdot \mathbf{F})$$

Example 4.1 ()

The divergence of the origin in the left graph is clearly negative since the net flow is out of the point, while the divergence of the origin in the right graph is positive since the net fluid flow is in.

**Lemma 4.1 (Divergence in Cylindrical Coordinates)**

For vector field $F : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ expressed in cylindrical coordinates as

$$F = \begin{pmatrix} F_r \\ F_\theta \\ F_z \end{pmatrix}$$

the divergence is

$$\operatorname{div} F = \nabla \cdot F = \frac{1}{r} \frac{\partial}{\partial r} (r F_r) + \frac{1}{r} \frac{\partial F_\theta}{\partial \theta} + \frac{\partial F_z}{\partial z}$$

Note that the condition of locality is important, since in general a global cylindrical coordinate system would be inconsistent.

Lemma 4.2 (Divergence in Spherical Coordinates)

For vector field $F : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ expressed in spherical coordinates (r, θ, ϕ) , the divergence is

$$\operatorname{div} F = \nabla \cdot F = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 F_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta F_\theta) + \frac{1}{r \sin \theta} \frac{\partial F_\phi}{\partial \phi}$$

4.3 Curl

Colloquially, the curl is a vector operator that describes the infinitesimal circulation of a vector field in 3-dimensional Euclidean space, where the curl at each point is represented by a vector whose length and direction denote the magnitude and axis as the maximum circulation. That is, if one drops a twig or a ball with its center of mass at a certain point, the curl measures how much it will spin. In physics, the rotation of a rigid body in 3-dimensions can be described by a vector ω along the axis of rotation. ω is called the *angular velocity vector*, with $\|\omega\|$ denoting the angular speed of the body. The curl of this vector field measured at the center of mass of the body is measured as 2ω . That is, the curl outputs *twice* the angular velocity vector of any rigid body. Note that unlike the gradient and divergence operators, curl does not generalize as simply to other dimensions.

Definition 4.5 (Curl)

The *curl* of a 3-dimensional C^k vector field $F : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is an operator

$$\operatorname{curl} : C^k(\mathbb{R}^3; \mathbb{R}^3) \rightarrow C^{k-1}(\mathbb{R}^3; \mathbb{R}^3)$$

defined

$$\operatorname{curl} F \equiv \nabla \times F \equiv \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \equiv \begin{pmatrix} \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \\ \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} \\ \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \end{pmatrix}$$

Definition 4.6 (Irrotational Vector Fields)

A vector field F is *irrotational* if

$$\operatorname{curl} F = \mathbf{0}$$

Visually, this indicates that there are no "whirlpools" everywhere, meaning that any rigid body placed anywhere, while it may travel along a path, will not rotate around its own axis.

It has been shown that fluid draining from a tub is usually irrotational except for right at the center, which is surprising since the fluid itself is "rotating" around the drain.

Theorem 4.2 ()

For any C^2 vector field F ,

$$\operatorname{div} \operatorname{curl} F = \nabla \cdot (\nabla \times F) = 0$$

That is, the divergence of any curl is 0.

Proof. Proved by equality of mixed partials.

Definition 4.7 ()

The *Laplace operator*, or *Laplacian*, of a function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is the divergence of the gradient.

$$\nabla^2 f \equiv \nabla \cdot (\nabla f) \equiv \sum_{i=1}^n \frac{\partial^2 f}{\partial x_i^2}$$

4.4 Conservative, Solenoidal Vector Fields

Definition 4.8 (Conservative Vector Fields)

A vector field $F : U \subset \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a *conservative vector field* if and only if there exists a scalar field $f : U \subset \mathbb{R}^n \rightarrow \mathbb{R}$ such that

$$F = \nabla f$$

on U .

Conservative vector fields appear naturally in mechanics: they are vector fields representing forces of physical systems in which energy is conserved.

Theorem 4.3 ()

Given a C^2 -function $f : \mathbb{R}^3 \rightarrow \mathbb{R}$,

$$\nabla \times (\nabla f) = \mathbf{0}$$

That is, the curl of any gradient vector field is the zero vector.

Proof. $\nabla \times \nabla f$ can be expanded to

$$\left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}, \frac{\partial^2 f}{\partial z \partial x} - \frac{\partial^2 f}{\partial x \partial z}, \frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x} \right) = (0, 0, 0)$$

by equality of mixed partials.

Definition 4.9 (Solenoidal Vector Fields)

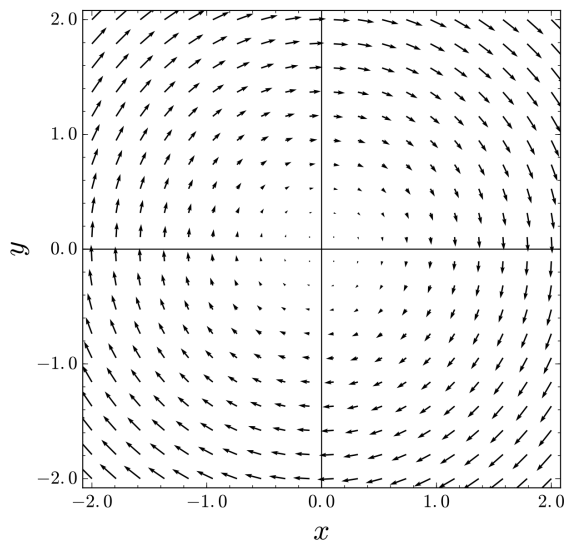
A *solenoidal*, or *incompressible*, vector field is a vector field $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that

$$\operatorname{div} F = \nabla \cdot F = 0$$

at all point in the field. That is, the field has no sources or sinks.

Example 4.2 ()

The vector field $F : (x, y) \mapsto (y, -x)$ is solenoidal.



5 Riemann and Darboux Integration

We have done integration over closed intervals $[a, b]$. The natural extension is to define integration over *boxes* $B = \prod_i [a_i, b_i]$. Essentially the construction is exactly the same.

Definition 5.1 (Partition/Mesh)

Let $B = \prod_i [a_i, b_i] \subset \mathbb{R}^n$ be a box. Then, a **partition**—or **mesh**—of B is a finite set of points $P = \{x_{ij}\}_{1 \leq i \leq n, 0 \leq j \leq m_i}$ ^a s.t.

$$a_i = x_{i,0} \leq x_{i,1} \leq x_{i,2} \leq \dots \leq x_{i,m_i-1} \leq x_{i,m_i} = b_i \quad \forall i = 1, \dots, n \quad (37)$$

We denote each **box** within the partition as

$$\Delta_J = \Delta_{j_1, j_2, \dots, j_n} = \prod_i [x_{i, j_i-1}, x_{i, j_i}] \quad \forall j_i = 1, \dots, m_i \quad (38)$$

of volume $|\Delta_J| := \prod_i (x_{i, j_i} - x_{i, j_i-1})$.

^aTherefore, for each dimension i , we want to take the interval $[a_i, b_i]$ and subdivide it into $m_i + 1$ subintervals.

This nearly¹ partitions B into a grid of $\prod_i m_i$ smaller boxes, and can be seen as a discretization of the integral which we will construct.

Definition 5.2 (Riemann Sums with Respect to Partition)

Let P be a partition of $B \in \mathbb{R}^n$ and let $f : B \subset \mathbb{R}^n \rightarrow \mathbb{R}$ be bounded. Then, the **upper and lower Riemann sums** of f with respect to P are defined

$$U(P, f) := \sum_J \sup_{x \in \Delta_J} f(x) |\Delta_J|, \quad L(P, f) := \sum_J \inf_{x \in \Delta_J} f(x) |\Delta_J| \quad (39)$$

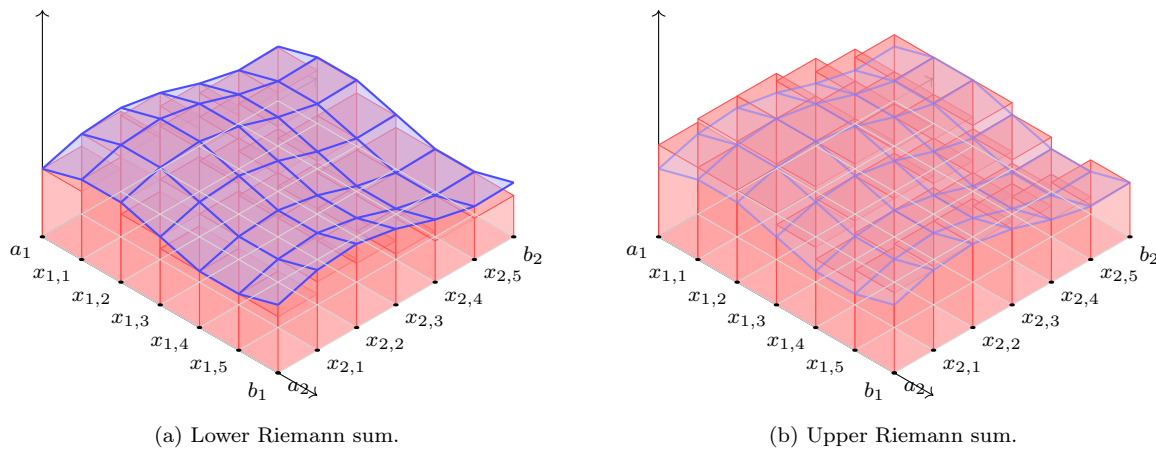


Figure 2

¹since boundaries overlap

Definition 5.3 (Riemann Integral)

Given that $f : B \subset \mathbb{R}^n \rightarrow \mathbb{R}$ is bounded, the **upper and lower Riemann integrals** of f are defined

$$\overline{\int}_B f(x) dx := \inf_P U(P, f), \quad \underline{\int}_B f(x) dx := \sup_P L(P, f) \quad (40)$$

If the upper and lower Riemann integrals are equal, then the **Riemann integral** of f

$$\int_B f(x) dx \quad (41)$$

is defined as such. Furthermore, f is said to be **Riemann integrable** over B .

5.1 Conditions for Integrability**Theorem 5.1 ()**

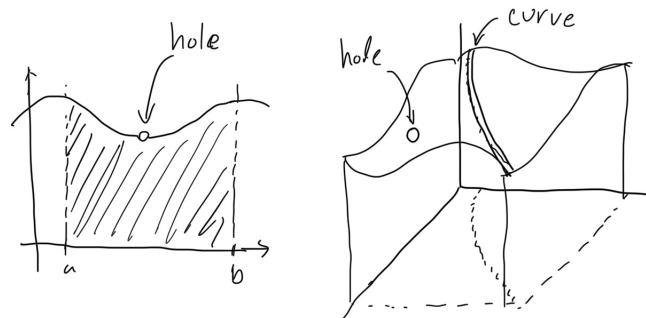
$f : B \subset \mathbb{R}^n \rightarrow \mathbb{R}$ continuous means f is integrable over B .

However, there are some functions with discontinuities that are in fact integrable.

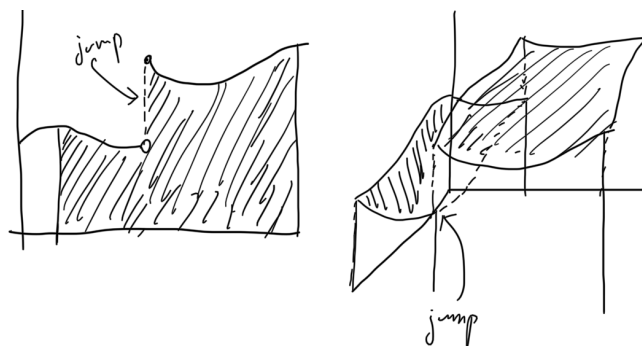
1. Given that there is a subset N in B with volume 0 over which f is not defined, we can integrate over $B \setminus N$. In the one and two dimensional cases,

$$\int_{B \setminus N} f(x) dx \text{ and } \iint_{B \setminus N} f(x) dA$$

are well-defined. Visually,



2. The function is defined for all values in the region, but there is a jump in the value of the function.



Informally, if we can visualize the Riemann sum converging to a well-defined area as the rectangles get thinner and thinner, then a discontinuous function is integrable. Indeed, all continuous functions (over a bounded set) are integrable since their Riemann sums are well defined.

5.2 Iterated Integrals and Fubini's Theorem

The construction of the integral is one step, but we should know how to practically compute such an integral. We can do this by recursively "reducing" the dimension of the region of the integral until we can work with one dimension. This is known as *Cavalieri's principle*: Let S be a bounded n -dimensional solid in \mathbb{R}^n . Define an $n - 1$ subspace P in \mathbb{R}^n and given the quotient space \mathbb{R}^n/P with elements P_x , let

$$S \subset \bigcup_{a \leq x \leq b} P_x \quad (42)$$

That is, S is "in between" affine subspaces P_a and P_b . Given the cross section of S with P_x , defined $P_x \cap S$, denote the integral of this cross section as $A(x)$. Then, colloquially, the volume of S can be represented as the integral $\int_a^b A(x) dx$. This idea basically says that the volume of S is the sum of the areas of its infinitesimal cross sections.

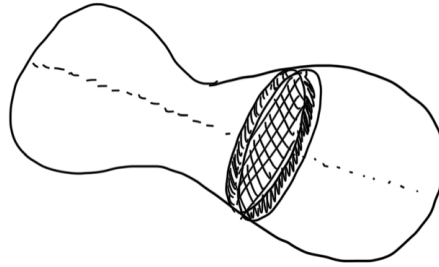


Figure 3: Visual of Cavalieri's principle.

Given a solid $S \subset \mathbb{R}^n$, it is easy to see that no matter what subspace P we choose—that is, no matter what orientation we choose to "cut" the solid—the sum of all of its cross sections should be equal to the true volume of S . In the case when S is a box in \mathbb{R}^n , Fubini's theorem states that whether we cut S up along the x_1 -axis, x_2 -axis, ..., or the x_n -axis, the symmetry in volume is always preserved. This theorem is really just a specific case of this general symmetry in volume.

Theorem 5.2 (Fubini's Theorem)

Given a function $f : \mathbb{R}^n \rightarrow \mathbb{R}$, let

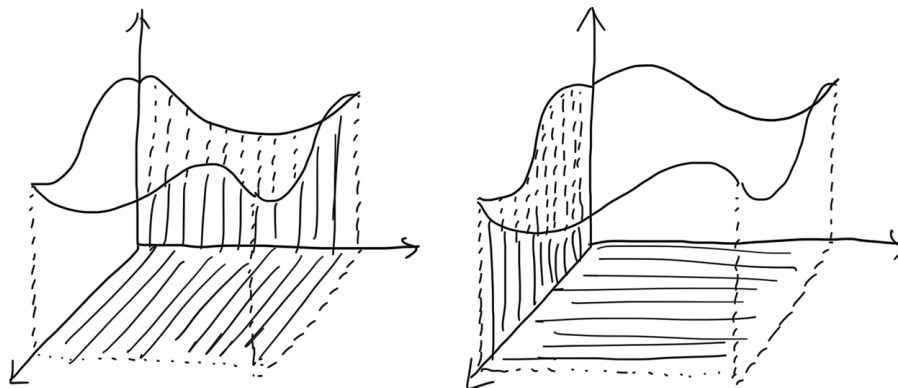
$$B \equiv \prod_{i=1}^n [\alpha_i, \beta_i]$$

and let p be any permutation of the elements $\{x_1, x_2, \dots, x_n\}$. Then

$$\begin{aligned} \int_B f dV &= \int_{\alpha_n}^{\beta_n} \dots \int_{\alpha_1}^{\beta_1} f(x_1, x_2, \dots, x_n) dx_1 \dots dx_n \\ &= \int_{p(\alpha_n)}^{p(\beta_n)} \dots \int_{p(\alpha_1)}^{p(\beta_1)} f(x_1, x_2, \dots, x_n) dp(x_1) \dots dp(x_n) \end{aligned}$$

In the two dimensional case, we have

$$\iint_B f \, dA = \int_c^d \int_a^b f(x, y) \, dx \, dy = \int_a^b \int_c^d f(x, y) \, dy \, dx$$



and in the three dimensional case, we can use any of the 6 permutations. Computation of these integrals is simple. You do the innermost integral first with respect to the corresponding variable, while treating the rest of the variables constant. Evaluating each integral outputs a formula for a higher dimensional cross section of the solid S . It is clear that computing iterated integrals is really just doing Cavalieri's principle repeatedly.

5.3 Integration over Regions between Curves

Definition 5.4 (Simple Regions w.r.t. a Variable)

A bounded region D in \mathbb{R}^n is said to be x_i -simple if it is bounded by the graphs of two continuous functions $u_1, u_2 : \mathbb{R}^{n-1} \rightarrow \mathbb{R}$ of the variables

$$x_1, x_2, \dots, x_{i-1}, x_{i+1}, \dots, x_n$$

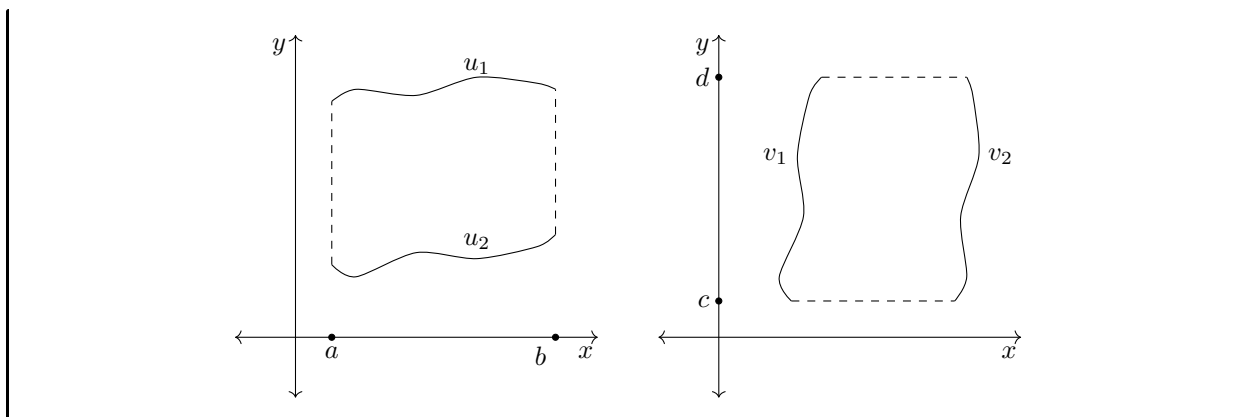
That is, D can be expressed in the form

$$\{x \in \mathbb{R}^n \mid u_1(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n) \leq x_i \leq u_2(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)\}$$

If a region is simple in all of its variables, it is simply called *simple*. Note that n -dimensional boxes are simple regions.

Example 5.1 ()

In \mathbb{R}^2 , the region on the left graph is an y -simple region and the region on the right is a x -simple region.



We now describe the method of calculating double integrals over elementary regions.

Theorem 5.3 ()

The double integral over a y -simple region D bounded by functions u_1 and u_2 in \mathbb{R}^2 and the x -values a and b (as shown in the left graph of example 2.1) is

$$\iint_D f(x, y) = \int_a^b \int_{u_2(x)}^{u_1(x)} f(x, y) dy dx$$

The double integral over an x -simple region D bounded by functions v_1 and v_2 in \mathbb{R}^2 and the y -values c and d (shown in the right of graph of example 2.1) is

$$\iint_D f(x, y) = \int_c^d \int_{v_2(y)}^{v_1(y)} f(x, y) dx dy$$

Example 5.2 ()

Integrating $f(x, y)$ over the unit disk would have the form

$$\int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} f(x, y) dy dx \text{ or } \int_{-1}^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} f(x, y) dx dy$$

Note that the unit disk is both x and y simple.

5.4 Change of Basis

Sometimes, integrating a region over a different basis would make the integral computation much more simpler. In this case, we may be able to transform more complicated regions into elementary regions. We first introduce a change of basis in 2 dimensions and then generalize it into higher dimensions.

Let \mathbb{R}^2 have the standard orthonormal basis e_1, e_2 , commonly known as the x, y basis. Now, let us construct new basis vectors of \mathbb{R}^2 , denoted f_1, f_2 such that f_1, f_2 are functions of e_1, e_2 . Since they are both bases that span \mathbb{R}^2 , we can equally represent e_1, e_2 as functions of f_1, f_2 .

$$\begin{aligned} e_1 &= g(f_1, f_2) \\ e_2 &= h(f_1, f_2) \end{aligned}$$

Note that this change of basis does not necessarily have to be linear, as in the context of passive transfor-

mation in linear algebra. Then, every point (x, y) in the (e_1, e_2) -basis can be rewritten as

$$\begin{aligned}(x, y) &= xe_1 + ye_2 \\ &= xg(f_1, f_2) + yh(f_1, f_2) \\ &= uf_1 + vf_2\end{aligned}$$

Note that it is customary to denote x, y as the coefficients in the e_1, e_2 basis and u, v as the coefficients in the new f_1, f_2 basis. This way, we can not only write e_1 and e_2 as functions of f_1 and f_2 , but we can also write the coefficients x, y as functions of the coefficients u, v ! That is,

$$\begin{aligned}x &= x(u, v) \\ y &= y(u, v)\end{aligned}$$

which is really just a function

$$B : \mathbb{R}^2 \longrightarrow \mathbb{R}^2, \quad B(u, v) = \begin{pmatrix} x(u, v) \\ y(u, v) \end{pmatrix}$$

Notice that B changes the u, v coordinates to the x, y coordinates, and B^{-1} changes the x, y coordinates to the u, v coordinates.

$$B^{-1} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2, \quad B^{-1}(x, y) = \begin{pmatrix} u(x, y) \\ v(x, y) \end{pmatrix}$$

Note that these coefficients actually change *contravariantly*, that is, they change inversely with respect to how the basis vectors are changed. In vector calculus, it is conventional to represent a change of basis with functions that relate the coefficients x, y with u, v , rather than the bases f_1, f_2 with e_1, e_2 .

Theorem 5.4 (Integration over Change of Bases in \mathbb{R}^2)

Let \mathbb{R}^2 have the standard orthonormal basis e_1, e_2 . Now, let us construct new basis vectors of \mathbb{R}^2 , denoted f_1, f_2 such that the coefficients of the vectors in \mathbb{R}^2 are related by the change of basis function

$$B = \begin{pmatrix} x \\ y \end{pmatrix} \implies B(u, v) = \begin{pmatrix} x(u, v) \\ y(u, v) \end{pmatrix}$$

Given region $D \subset \mathbb{R}^2$ and $S = B(D)$ is the region transformed by B , the integral of function $f(x, y)$ over region D can be expressed as

$$\iint_D f(x, y) dA = \iint_S f(x(u, v), y(u, v)) |JB(u, v)| d\bar{A}$$

where $|JB(u, v)|$ is the determinant of the Jacobian matrix of B . Expanding the Jacobian determinant gives

$$|JB(u, v)| = \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u}$$

Theorem 5.5 (Integration over Change of Bases in \mathbb{R}^3)

Given that we have the change of basis function

$$B : \mathbb{R}^3 \longrightarrow \mathbb{R}^3, \quad B(u, v, w) = \begin{pmatrix} x(u, v, w) \\ y(u, v, w) \\ z(u, v, w) \end{pmatrix}$$

a region $D \in \mathbb{R}^3$ and $S = B(D)$, the region transformed by B , the integral of $f(x, y, z)$ over region D

can be expressed as

$$\iiint_D f(x, y, z) dV = \iiint_S f(x(u, v, w), y(u, v, w), z(u, v, w)) |JB(u, v, w)| d\bar{V}$$

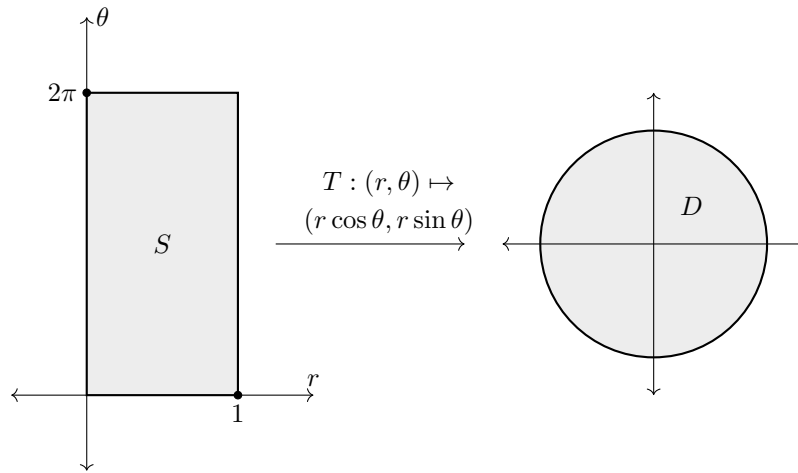
where $|JB(u, v, w)|$ is the Jacobian determinant of B .

Example 5.3 ()

Given a real-valued function f defined over the region $D \subset \mathbb{R}^2$, we can perform a change of basis of the x, y coordinates into polar ones within a new region S . The change of basis

$$x = r \cos \theta$$

$$y = r \sin \theta$$



Theorem 5.6 (Integration over Change of Bases in \mathbb{R}^n)

Let \mathbb{R}^n have the standard orthonormal basis e_1, e_2, \dots, e_n , and let us construct a new basis f_1, f_2, \dots, f_n such that the coefficients of the vectors in \mathbb{R}^n are related with the functions

$$B : \mathbb{R}^n \longrightarrow \mathbb{R}^n, \quad B(u_1, u_2, \dots, u_n) = \begin{pmatrix} x_1(u_1, \dots, u_n) \\ x_2(u_1, \dots, u_n) \\ \vdots \\ x_n(u_1, \dots, u_n) \end{pmatrix}$$

Given that the region $D \subset \mathbb{R}^n$ is transformed into a new region $S = B(D) \subset \mathbb{R}^n$ under this basis transformation, the integral of function $f(x_1, \dots, x_n)$ over region D can be expressed as

$$\int_D f(x) dH = \int_S f(x_1(u), x_2(u), \dots, x_n(u)) |JB(u_1, \dots, u_n)| d\bar{H}$$

where the integral on both the left and right hand side represents integration over an n -dimensional region, x represents the n -tuple (x_1, \dots, x_n) , u represents the n -tuple (u_1, \dots, u_n) , and $|JB(u_1, \dots, u_n)|$ represents the Jacobian determinant of function B .

We now describe some common change of basis formulas for polar, cylindrical, and spherical coordinates.

Theorem 5.7 (Integration in Polar Coordinates)

$$\iint_D f(x, y) dx dy = \iint_S f(r \cos \theta, r \sin \theta) r dr d\theta \quad (43)$$

Definition 5.5 (Cylindrical, Spherical Coordinates)

In \mathbb{R}^3 , *cylindrical coordinates* have the following relation to rectangular coordinates.

$$x = r \cos \theta \quad (44)$$

$$y = r \sin \theta \quad (45)$$

$$z = z \quad (46)$$

In \mathbb{R}^3 , *spherical coordinates* have the following relation to rectangular coordinates.

$$x = \rho \sin \phi \cos \theta$$

$$y = \rho \sin \phi \sin \theta$$

$$z = \rho \cos \phi$$

Corollary 5.1 (Integration in Cylindrical Coordinates)

$$\iiint_D f(x, y, z) dx dy dz = \iiint_S f(r \cos \theta, r \sin \theta, z) r dr d\theta dz$$

Corollary 5.2 (Integration in Spherical Coordinates)

$$\iiint_D f(x, y, z) dx dy dz = \iiint_S f(\rho \sin \phi \cos \theta, \rho \sin \phi \sin \theta, \rho \cos \phi) \rho^2 \sin \theta d\rho d\theta d\phi$$

Example 5.4 (Gaussian Integral)

The following is the (un-normalized) probability distribution function of the Gaussian distribution.

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$$

5.5 Improper Integrals

There are generally two types of improper integrals.

1. The region D integrated over is unbounded.
2. The function f that is integrated is unbounded within the region D .

These types of improper integrals are usually evaluated using a limiting process. When the interval I is unbounded, say $(1, \infty)$, the integral can be evaluated as

$$\int_1^{\infty} \frac{1}{x^2} dx = \lim_{b \rightarrow \infty} \int_1^b \frac{1}{x^2} dx = \lim_{b \rightarrow \infty} \left(1 - \frac{1}{b}\right) = 1$$

In case 2, we can add a limit at the point where the function f diverges as such.

$$\int_0^1 \frac{1}{\sqrt{x}} dx = \lim_{a \rightarrow 0} \int_a^1 \frac{1}{\sqrt{x}} dx = \lim_{a \rightarrow 0} (2 - 2\sqrt{a}) = 2$$

We now describe how to integrate over a certain path p embedded in a higher dimensional space \mathbb{R}^n , possibly with a scalar or vector field f . We must first go over oriented paths.

Extending the previous case, we use a multivariate limiting process in \mathbb{R}^2 . We will first work with case 2, when f is unbounded within the region D . Let us define an elementary region D in \mathbb{R}^2 ; without loss of generality, we will make it y -simple, meaning that D can be expressed as

$$D \equiv \{(x, y) \in \mathbb{R}^2 \mid a \leq x \leq b, \phi_1(x) \leq y \leq \phi_2(x)\}$$

We can actually assume that the region in which f is unbounded lies in the boundary ∂D . This is because if it lied in the interior of D , we could split D into pieces across a path that intersects this region with divergent values, evaluate the integrals over the pieces separately, and then sum the integrals. For example, in the rectangular region below, let the dashed line represent the values where the function f diverges. Then, we can split the region into two rectangular regions shown in the right.



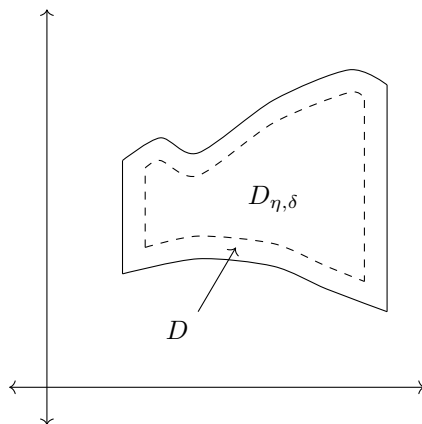
Therefore, assuming that f is unbounded in ∂D , we can construct a new region

$$D_{\eta, \delta} \equiv \{(x, y) \in \mathbb{R}^2 \mid a + \eta \leq x \leq b - \eta, \phi_1(x) + \delta \leq y \leq \phi_2(x) - \delta\}$$

for some arbitrarily small numbers $\eta, \delta > 0$, meaning that the integral (reduced to iterated integrals using Fubini's theorem)

$$F(\eta, \delta) \equiv \iint_{D_{\eta, \delta}} f(x, y) dA = \int_{a+\eta}^{b-\eta} \int_{\phi_1(x)+\delta}^{\phi_2(x)-\delta} f(x, y) dy dx$$

is well defined.



Clearly, the function $F(\eta, \delta)$ is a function of two variables η and δ . So, if the limit

$$\lim_{(\eta, \delta) \rightarrow (0, 0)} F(\eta, \delta)$$

is well defined, then so is the improper integral. For it to exist, the iterated limits must both equal to a well-defined real number \mathcal{L} (and to each other). That is,

$$\lim_{\eta \rightarrow 0} \lim_{\delta \rightarrow 0} F(\eta, \delta) = \lim_{\delta \rightarrow 0} \lim_{\eta \rightarrow 0} F(\eta, \delta) = \mathcal{L} \implies \lim_{(\eta, \delta) \rightarrow (0, 0)} F(\eta, \delta) = \mathcal{L}$$

It is also worthwhile to note that functions unbounded at isolated points can be evaluated using the methods above using a change of basis. Consider the example below.

Example 5.5 ()

In the unit disk $D \subset \mathbb{R}^2$, let the function f be defined as

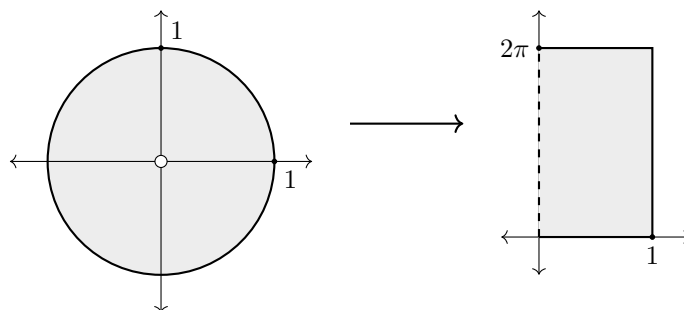
$$f(x, y) \equiv \frac{1}{\sqrt{x^2 + y^2}}$$

Clearly, f is continuous at every point except $0 = (0, 0)$, meaning that

$$\iint_{D \setminus \{0\}} f(x, y) dA$$

is well-defined. In order to solve the integral over the entire disk, we convert to polar coordinates and evaluate the limit

$$\iint_{D \setminus \{0\}} f(x, y) dA = \lim_{\delta \rightarrow 0} \int_{\delta}^1 \int_0^{2\pi} r f(r \cos \theta, r \sin \theta) d\theta dr$$



If we are given an unbounded region $D \subset \mathbb{R}^2$, we can first create a bounded region and expand that region using a limit to cover all of D .

6 Surfaces

We can represent a K -dimensional subset $S \subset \mathbb{R}^N$ in multiple ways, where $K < N$. There are three conventional ways to do this.

1. We can parameterize it with a function $f : D \subset \mathbb{R}^k \rightarrow \mathbb{R}^n$ to create a **parameterized set** defined as the image of an *injective* f under D . Letting $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{u} \in \mathbb{R}^k$, the parameterization is defined

$$\mathbf{u} \mapsto f(\mathbf{u}) = (f_1(\mathbf{u}), f_2(\mathbf{u}), \dots, f_n(\mathbf{u}))$$

2. A function $\mathbf{f} : \mathbb{R}^n \rightarrow \mathbb{R}^m$ of the form $\mathbf{y} = \mathbf{f}(\mathbf{x})$ creates an **explicit representation** by defining all $(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^{n+m}$ satisfying

$$\mathbf{y} = \mathbf{f}(\mathbf{x})$$

3. A **level set** of the form $\mathbf{F}(\mathbf{x}) = \mathbf{0}$ creates an **implicit representation** by defining all $\mathbf{x} \in \mathbb{R}^n$ satisfying

$$\mathbf{F}(\mathbf{x}) = \mathbf{0}$$

Now if \mathbf{F} was scalar valued, then the equation $F(\mathbf{x}) = 0$ defines a hypersurface in \mathbb{R}^n with codimension 1.

1. If \mathbf{F} is a k -vector valued function, then the implicit surface generally has codimension k , since we can interpret $\mathbf{F}(\mathbf{x}) = \mathbf{0}$ as a system of k constraint equations.

Generally, the change of representations is simple only when the explicit representation $\mathbf{y} = \mathbf{f}(\mathbf{x})$ is given. The implicit form is $\mathbf{F}(\mathbf{x}, \mathbf{y}) = \mathbf{y} - \mathbf{f}(\mathbf{x}) = \mathbf{0}$, and the parameterized form is the map $\mathbf{x} \mapsto (\mathbf{x}, \mathbf{f}(\mathbf{x}))$. However, the explicit representation is very limited in usefulness, because it can only describe sets that are graphs of functions that pass the vertical line test. The implicit function theorem, stated later, states conditions under which an equation $\mathbf{F}(\mathbf{x}) = \mathbf{0}$ can be solved explicitly for any of the x_i 's. The other two representations are much more versatile, with the implicit representation being slightly more general, but the parametric form being more useful, since we can directly compute points on the S . Some examples are:

1. a 1-dimensional path/curve in \mathbb{R}^n
2. a 2-dimensional surface in \mathbb{R}^3
3. a k -dimensional set in \mathbb{R}^n

If these surfaces are smooth enough, then there must exist geometric tangent vectors, geometric tangent planes, and geometric orthogonal vectors on them. We say "geometric" to distinguish them from the vectors in the tangent space $T_{\mathbf{x}_0}\mathbb{R}^n$. It is important to know how to derive them.

Theorem 6.1 (Explicit Representation)

Let us have the surface $S \subset \mathbb{R}^{n+1}$ defined by $y = f(\mathbf{x})$ and a point on the surface $(\mathbf{x}_0, f(\mathbf{x}_0))$.

1. To get the equation of the set of affine points forming the geometric tangent plane, we look at all points (\mathbf{x}, y) satisfying

$$y = f(\mathbf{x}_0) + Df_{\mathbf{x}_0}(\mathbf{x} - \mathbf{x}_0)$$

and to get an arbitrary tangent vector protruding from \mathbf{x}_0 , we look at all vectors (\mathbf{v}, w) of form

$$w = Df_{\mathbf{x}_0}\mathbf{v}$$

i.e. all vectors of form $(\mathbf{v}, Df_{\mathbf{x}_0}\mathbf{v})$.

2. To get the equation of the orthogonal vector, convert this to the implicit representation $g(\mathbf{x}, y) = y - f(\mathbf{x}) = 0$, and see that the gradient is orthogonal to the tangent plane. So, the orthogonal vector at \mathbf{x}_0 is

$$\nabla g(\mathbf{x}_0, f(\mathbf{x}_0)) = \begin{pmatrix} -\nabla f(\mathbf{x}_0) \\ 1 \end{pmatrix}$$

Note that indeed, dotting this with an arbitrary tangent vector of the form above gives

$$\begin{pmatrix} -\nabla f(\mathbf{x}_0) \\ 1 \end{pmatrix} \cdot \begin{pmatrix} \mathbf{v} \\ Df_{\mathbf{x}_0}\mathbf{v} \end{pmatrix} = -\nabla f(\mathbf{x}_0) \cdot \mathbf{v} + Df_{\mathbf{x}_0}\mathbf{v} = 0$$

Given a level set $S = \{\mathbf{x} \in \mathbb{R}^n \mid f(\mathbf{x}) = c\}$, a vector \mathbf{v} is a **tangent vector** of S at \mathbf{a} if the directional derivative (if it exists) satisfies

$$\nabla_{\mathbf{v}} f(\mathbf{a}) = 0$$

If f is differentiable at \mathbf{a} , then this condition is equivalent to

$$Df_{\mathbf{a}}\mathbf{v} = 0$$

Intuitively, $Df_{\mathbf{a}}\mathbf{v}$ answers the question: "If I move infinitesimally in the direction \mathbf{v} , what happens to f ?" We would want this direction to preserve the value of $f = c$, and so the derivative should be 0. Therefore, we look for the vectors \mathbf{v} satisfying $Df_{\mathbf{a}}\mathbf{v} = 0$, i.e. the annihilator $(Df_{\mathbf{a}})^0 \subset \mathbb{R}^n$. This result is precisely the well-known theorem that states that "gradients are orthogonal to level sets." It is intuitive to claim that if we have some sort of directional vector \mathbf{v} , then this \mathbf{v} must be "tangent" if the directional derivative towards \mathbf{v} must be 0, essentially staying within the level set of value c .

Theorem 6.2 (Implicit Representation)

Let us have the surface $S \subset \mathbb{R}^n$ defined by $F(\mathbf{x}) = 0$ and a point $\mathbf{a} \in S$.

1. The gradient $\nabla F(\mathbf{x}_0)$ is simply the orthogonal vector at \mathbf{x}_0 .
2. The set of all directional tangent vectors protruding from \mathbf{x}_0 is defined by the set of directional vectors \mathbf{v} satisfying

$$\nabla F(\mathbf{x}_0) \cdot \mathbf{v} = 0$$

and the set of all affine points forming the geometric tangent plane are all $\mathbf{x} \in \mathbb{R}^n$ satisfying

$$\nabla F(\mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0) = 0$$

Proof. This is trivial since we can invoke Reisz representation theorem and see that

$$Df_{\mathbf{a}}\mathbf{v} = 0 \implies \nabla_{\mathbf{a}} f \cdot \mathbf{v} = 0$$

This theorem now simplifies our derivation of tangent planes of a function $f : \mathbb{R}^n \rightarrow \mathbb{R}$. To find the equation of a tangent plane of $y = f(\mathbf{x})$ at $\mathbf{x} = \mathbf{a}$, we can simply write the one-line equation as

$$y = f(\mathbf{a}) + Df_{\mathbf{a}}(\mathbf{x} - \mathbf{x}_0)$$

However, if we had an implicit function of the form $g(\mathbf{x}, y) = c$, then separating this into an explicit function of y is hard. Therefore, we can simply treat g itself as a function of the $n + 1$ variables (\mathbf{x}, y) , and treat $g(\mathbf{x}, y) = c$ as a level set.

Theorem 6.3 (Parametric Representation)

Let us have $f : D \subset \mathbb{R}^k \rightarrow \mathbb{R}^n$, with injective f defining a surface $f(D) \subset \mathbb{R}^n$. Let us have $\mathbf{u}_0 \in D$ with $f(\mathbf{u}_0) = \mathbf{x}_0 \in f(D)$. Our idea is this: we compute k directional derivatives of f in k linearly independent direction vectors at \mathbf{u}_0 , which will give us k (linearly independent, due to injectiveness of f , but not necessarily orthogonal) geometric tangent vectors protruding from \mathbf{x}_0 that span the tangent space $T_{\mathbf{x}_0}$. If $k = n - 1$, then the orthogonal vector is uniquely defined to be the vector spanning $T_{\mathbf{x}_0}^\perp$, and is $k < n - 1$, there is no unique orthogonal vector defined.

1. The set of all directional tangent vectors \mathbf{v} protruding from \mathbf{x}_0 is represented by the set of all

linear combinations of the partials, aka the image of the Jacobian of f

$$\{c_1 \partial_{u_1} f(\mathbf{u}_0) + \dots + c_k \partial_{u_k} f(\mathbf{u}_0) \mid \mathbf{c} \in \mathbb{R}^n\} = \text{Im} \begin{pmatrix} \left| \partial_{u_1} f(\mathbf{u}_0) \right. & \dots & \left| \partial_k f(\mathbf{u}_0) \right. \\ \vdots & & \vdots \end{pmatrix} = \text{Im} Df_{\mathbf{u}_0}$$

The tangent space is the space of all $\mathbf{x} \in \mathbb{R}^n$ of the form

$$f(\mathbf{x}_0) + Df_{\mathbf{x}_0} \mathbf{u} \text{ for all } \mathbf{u} \in \mathbb{R}^k$$

2. If $k = n - 1$, the orthogonal vector is the unique vector that is orthogonal to all $\partial_{u_i} f(\mathbf{u}_0)$, which can be computed using linear algebra techniques (e.g. kernel of $Df_{\mathbf{x}_0}$). If $n = 3, k = 2$, then this can simply be computed using the cross product.

7 Integration of Forms

7.1 Line Integrals

Definition 7.1 (Orientations, Simple Curves, Closed Curves)

A path function $p : [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}^n$ determines a curve in \mathbb{R}^n with endpoints $p(a)$ and $p(b)$. The direction the curve p takes, that is from $p(a)$ to $p(b)$ in \mathbb{R}^n is called the *orientation* of p . A path or a curve with a defined orientation is called an *oriented curve*.

A *simple curve* C to be the image of an injective piecewise C^1 map $c : I \subset \mathbb{R} \rightarrow \mathbb{R}^n$. Since it is injective, it does not intersect itself, and C is piecewise smooth in \mathbb{R}^n . If $I = [a, b]$, then $c(a)$ and $c(b)$ are the endpoints of the curve. A simple curve with an orientation is called an *oriented simple curve*.

A closed curve C is the image of piecewise C^1 map $c : [a, b] \rightarrow \mathbb{R}^n$ such that $c(a) = c(b)$. That is, the endpoints of C are equal. A *simple closed curve* is a closed curve that is injective over the interval $[a, b)$. Note that a closed curve has two possible orientations.

If C is an oriented simple curve or an oriented simple closed curve, then we can unambiguously define line integrals along them.

Definition 7.2 ()

Let h be an injective function that takes $[\alpha, \beta] \subset \mathbb{R}$ to the interval $[a, b] \subset \mathbb{R}$. Given an oriented simple path function $p : [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}^n$, the composition

$$\rho = p \circ h : [\alpha, \beta] \rightarrow \mathbb{R}^n$$

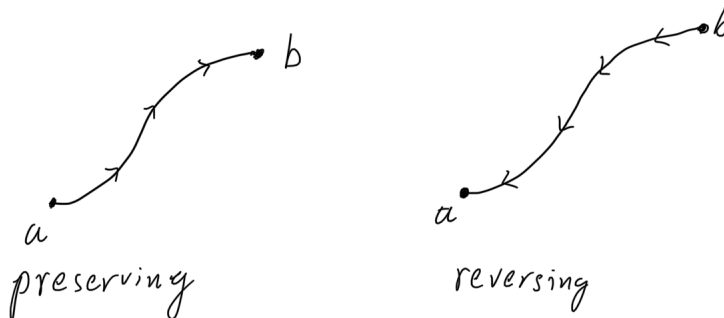
is called a *reparamaterization* of p . Note that since h is injective, it takes endpoints to endpoints. If h preserves the direction in which the path travels, that is, if

$$(p \circ h)(\alpha) = a \text{ and } (p \circ h)(\beta) = b$$

then h is *orientation preserving*. If

$$(p \circ h)(\alpha) = b \text{ and } (p \circ h)(\beta) = a$$

then h is *orientation reversing*. Note that a path c having the same image as p does not imply that c is a reparamaterization of p , since c may not be injective.



Definition 7.3 (Scalar Line Integral)

Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$, which can be interpreted as a scalar field. Now define a C^1 path function

$$c : [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}^n$$

such that the composition of functions

$$f \circ c : [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}$$

is continuous. Then, the *path integral*, or *scalar line integral*, of f along the path c is defined

$$\begin{aligned} \int_c f \, ds &= \int_a^b f(c(t)) \|c'(t)\| \, dt \\ &= \int_a^b f(x_1(t), x_2(t), \dots, x_n(t)) \|c'(t)\| \, dt \end{aligned}$$

If $c(t)$ is only piece-wise C^1 , we can define the path integral by breaking $[a, b]$ into pieces over which $f(c(t)) \|c'(t)\|$ is continuous and then summing the integrals over the pieces. That is,

$$\int_a^b f(c(t)) \|c'(t)\| \, dt = \sum_{i=0}^{n-1} \int_{\alpha_i}^{\alpha_{i+1}} f(c(t)) \|c'(t)\| \, dt$$

Note that since f is a scalar-valued function, we can interpret a path integral as the sum of infinitesimal segments of the path c having a weight determined by f at each section. If f is a constant function outputting 1 at every point, then the path integral just outputs the length of the path c in \mathbb{R}^n .

$$L = \int_a^b f(c(t)) \|c'(t)\| \, dt = \int_a^b \|c'(t)\| \, dt$$

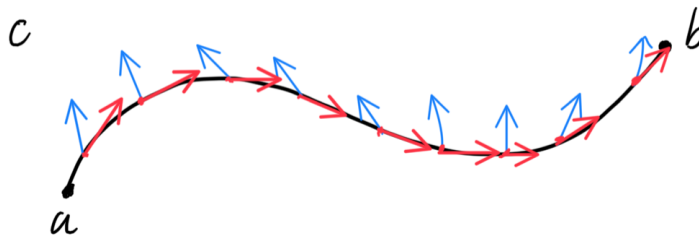
Definition 7.4 (Vector Line Integral)

Let $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a vector field on \mathbb{R}^n that is continuous on the C^1 oriented path $c : [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}^n$. The *line integral* of F along c is defined by the formula

$$\int_c F \cdot ds = \int_a^b F(c(t)) \cdot c'(t) \, dt$$

where \cdot represents the dot product of F with c' over the interval $[a, b]$. It is also commonly written in differential notation,

$$\int_c F \cdot ds = \int_c F \cdot (dx_1, \dots, dx_n) = \int_c F_1 dx_1 + F_2 dx_2 + \dots + F_n dx_n$$



Similarly with path integrals, we can also define line integrals as the sum of integrals over piece-wise continuous sections of c . That is, given an oriented curve C made up of several oriented component curves C_i , $i = 1, 2, \dots, k$, we can parameterize C by parameterizing the pieces C_i 's separately. Thus, we can treat $C = C_1 + \dots + C_k$ and get

$$\int_C F \cdot ds = \sum_{i=1}^k \int_{C_i} F \cdot ds$$

Note that a vector line integral is a generalization of scalar line integrals, so any results holding for vector line integrals also holds for their scalar counterpart.

Example 7.1 (Work)

In mechanics, work W is defined as

$$W = F \cdot d$$

where F is force and d is displacement. With this knowledge, the reader can easily see that the work done by vector field F on a particle traveling along a path c from time a to time b can be calculated by the line integral

$$\begin{aligned} W &= \int_a^b F(c(t)) \cdot c'(t) dt \\ &= \int_c F_1 dx + F_2 dy + F_3 dz \end{aligned}$$

Theorem 7.1 (Invariance of Path Parameterizations on Vector Line Integrals)

Let F be a vector field and f be a scalar field, both continuous on the C^1 path function $p : [a, b] \rightarrow \mathbb{R}^n$ and let $q : [\alpha, \beta] \rightarrow \mathbb{R}^n$ be a reparameterization of p . Then,

$$\begin{aligned} q \text{ is orientation preserving} &\implies \int_p F \cdot ds = \int_q F \cdot ds \\ q \text{ is orientation reversing} &\implies \int_p F \cdot ds = - \int_q F \cdot ds \end{aligned}$$

7.2 Conservative Vector Fields

We now introduce a fundamental theorem about line integrals over gradient fields. Recall the fundamental theorem of calculus and its equivalent form.

Theorem 7.2 (Fundamental Theorem of Single Variable Calculus)

Let function $\nabla g : \mathbb{R} \rightarrow \mathbb{R}$ be the gradient of the single variable C^1 function $g : \mathbb{R} \rightarrow \mathbb{R}$; that is, ∇g is a conservative vector field on \mathbb{R} . Then,

$$\int_a^b \nabla g(x) dx = g(b) - g(a)$$

Note that in the single variable case,

$$\frac{d}{dx} g(x) = \nabla g(x)$$

This means that the value of the integral of ∇g only depends on the value of g at the endpoints of the interval $[a, b]$.

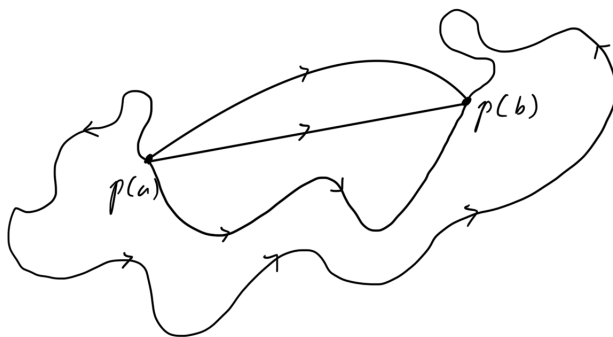
We can extend this to line integrals for functions mapping \mathbb{R}^n to \mathbb{R} .

Theorem 7.3 (Invariance of Line Integrals in Conservative Vector Fields)

Given that $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a C^1 conservative vector field with $\nabla f = F$ for C^2 function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ and path function $p : [a, b] \rightarrow \mathbb{R}^n$ is a piecewise C^1 path, then

$$\int_p F \cdot ds = \int_p \nabla f \cdot ds = f(p(b)) - f(p(a))$$

That is, the line integral of any path in a conservative vector field is dependent on the value of f at the endpoints $p(a)$ and $p(b)$.



In physics, calculating the work done by a force represented by a vector field requires us to know the path that it travels through.

$$W = \int_p F \cdot ds$$

However, in many cases F is assumed to be conservative, so it is only necessary that we find the displacement of the particle from its endpoints, resulting in the simplification of the formula.

$$W = \int_p \nabla f \cdot ds = f(p(b)) - f(p(a))$$

Corollary 7.1 (Equivalent Conditions for Vector Field to be Conservative)

The following conditions are equivalent:

1. $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a conservative vector field.
2. The line integral of $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ in curve C is path independent; that is, if C_1 and C_2 are two parametrizations of C ,

$$\int_{C_1} F \cdot ds = \int_{C_2} F \cdot ds$$

3. Given that C is a closed loop, the line integral of $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ across C is 0.

$$\oint_C F \cdot ds = 0$$

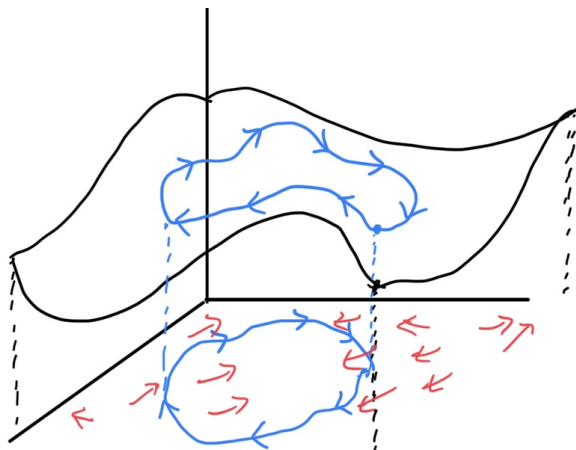
4. The curl of $F : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ vanishes

$$\text{curl } F = \nabla \times F = \begin{pmatrix} \frac{\partial F}{\partial x} \\ \frac{\partial F}{\partial y} \\ \frac{\partial F}{\partial z} \end{pmatrix} \times \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix} = 0$$

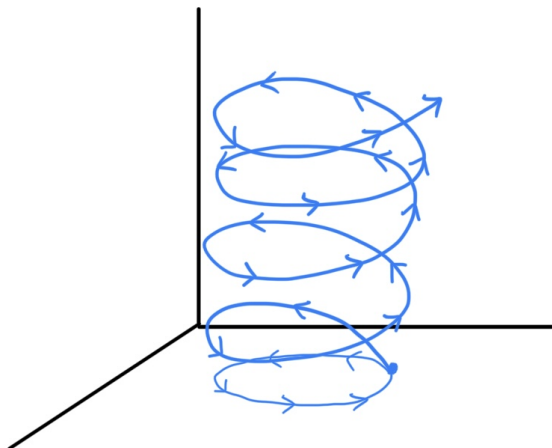
5. The following partial derivatives of $F : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ are equal

$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}$$

We can develop a bit of intuition to determine whether a vector field is conservative or not. If vector field F is conservative, then there exists a smooth scalar field f such that $\nabla f = F$. For each latitude and longitude on a certain map, we can give it an altitude as a function of those coordinates (picture a map with a bunch of hills and valleys). The gradient and thus the vector field is all the vectors that point in the direction of highest ascent. The vector field is all the vectors that point in the direction of highest ascent. Extending the metaphor the path integral is like starting on at a point and climbing the hills and valleys, creating work as you go up a hill (proportional to the steepness and thus the dot product of your motion vector with the gradient vector field in the path integral) and decreasing the work you put in by going down a hill. Since the path is closed, it is like you are going up and down the same amount overall, so the path integral is zero. Following this analogy, the vector field determined by this function (marked as arrows in the x, y plane) is conservative.



If we can construct a closed loop around F where the line integral is nonzero, then it means that we have ended up at a "higher" or "lower" (altitude) at the same point. This means that rather than being a certain landscape, there exist different "levels" of values at one point, like a spiraling staircase. For example, look at the solenoidal vector field below, where we can construct a closed loop (a circle going around the origin counterclockwise). There is no "surface" that can be defined such that it contains the solenoid.



Clearly, as a particle travels through the vector field along the path, it does positive work while it has zero displacement, and clearly, there exists no function that can output both these values as determined by vector field F .

Theorem 7.4 (Helmholtz Decomposition)

Let $F : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be a C^2 vector field. Then, F can be decomposed into a curl-free component and a divergence-free component. That is, there exists vector fields A and Φ

$$F = -\nabla \cdot \Phi + \nabla \times A$$

7.2.1 Curvature

Definition 7.5 (Curvature at a Point)

Let $c : [a, b] \rightarrow C \subset \mathbb{R}^3$ be a unit-speed parameterization of C , meaning that $\|c'(t)\| = 1$ for all $t \in [a, b]$, and let $p = c(t_0)$ be a point in C . The *curvature* $\kappa(p)$ at p is a mapping defined

$$\kappa : C \rightarrow \mathbb{R}, \quad \kappa(p) \equiv \|c''(t_0)\|$$

Notice that since we require a unit speed parameterization of C , we do not need to worry about how a given curve is parameterized.

Since the curvature is defined pointwise for each point in curve C , we can integrate over all the curvatures in C to define the total curvature.

Definition 7.6 (Total Curvature)

The *total curvature* of a curve $c : [a, b] \rightarrow C \subset \mathbb{R}^3$ is the scalar line integral

$$\int_C \kappa ds$$

We now present an important theorem in differential geometry.

Theorem 7.5 (Fary-Milnor Theorem)

Given a unit speed parameterization $c : [a, b] \rightarrow C \subset \mathbb{R}^3$, if C is closed (that is, $c(a) = c(b)$), then

$$\oint_C \kappa \, ds \geq 2\pi$$

and equals 2π only when C is a circle. Furthermore, if C is a closed space curve with

$$\oint_C \kappa \, ds \leq 4\pi$$

then C is "unknotted." That is, C can be continuously deformed without ever intersecting itself into a planar circle. Therefore, for knotted curves C , we have

$$\oint_C \kappa \, ds > 4\pi$$

7.3 Surface Integrals

Surface integrals are the 2-dimensional analogue, or the double integral version, of line integrals. It is the integration of surfaces.

7.3.1 2-Dimensional Parameterizations of Surfaces

Just like how we create path functions using a parameterization function $p : [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}^n$, we can parameterize surfaces by defining a function

$$\varphi : D \subset \mathbb{R}^2 \rightarrow \mathbb{R}^n, \quad \varphi(u, v) \equiv \begin{pmatrix} x_1(u, v) \\ \vdots \\ x_n(u, v) \end{pmatrix}$$

The surface

$$S = \varphi(D)$$

corresponding to the function φ is its image. If φ is differentiable or is of class C^1 , then we call S a *differentiable* or C^1 surface, respectively.

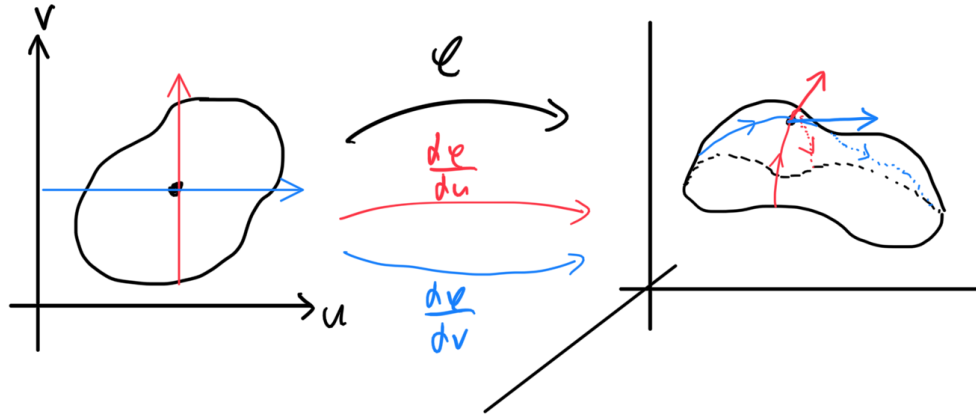
For those that are familiar with differential geometry, this makes every parameterized surface a 2-manifold induced by the single homeomorphism φ . In fact, it is more than just locally homeomorphic; it is *globally* homeomorphic.

Definition 7.7 (Tangent Vectors of Surfaces Embedded in \mathbb{R}^3)

Given surface parameterization

$$\varphi : \mathbb{R}^2 \rightarrow \mathbb{R}^3, \quad \varphi(u, v) \equiv \begin{pmatrix} x(u, v) \\ y(u, v) \\ z(u, v) \end{pmatrix}$$

it is visually clear that there can be up to two linearly independent tangent vectors at a point on the surface S . We can calculate these two vectors by embedding two nonparallel paths in $D \subset \mathbb{R}^2$ and taking the derivative with respect to a point traveling through these paths, which would give us a tangent vector on S . To keep things simple, we take the partial derivatives with respect to u and v .



Clearly, these paths are functions

$$\frac{\partial \varphi}{\partial u} \equiv \begin{pmatrix} \frac{\partial x}{\partial u} \\ \frac{\partial y}{\partial u} \\ \frac{\partial z}{\partial u} \end{pmatrix} : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$$

$$\frac{\partial \varphi}{\partial v} \equiv \begin{pmatrix} \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial v} \end{pmatrix} : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$$

where

$$\frac{\partial \varphi}{\partial u}(u_0, v_0), \frac{\partial \varphi}{\partial v}(u_0, v_0)$$

represent two vectors in \mathbb{R}^3 that are tangent to S at the point $\varphi(u_0, v_0) \in \mathbb{R}^3$.

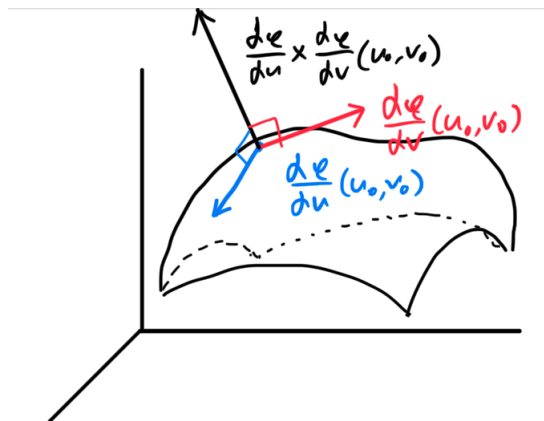
We must make sure that the surface S is smooth in the sense that (informally) there aren't any wrinkles, points, folds, or self-intersections in such a way that the tangent plane to the surface is not well-defined.

Definition 7.8 (Regular Surfaces)

To formalize this concept, we say that S is *regular*, or *smooth*, at point (u_0, v_0) if

$$\frac{\partial \varphi}{\partial u} \times \frac{\partial \varphi}{\partial v} \neq 0$$

where \times is the Euclidean cross product. That is, if the vector that is orthogonal to the two tangent vectors is well defined at a point, the surface is said to be smooth at that point. Note that $\frac{\partial \varphi}{\partial u}$ is parallel to $\frac{\partial \varphi}{\partial v}$ if and only if their cross product is 0.



It is quite clear that $(\frac{\partial \varphi}{\partial u} \times \frac{\partial \varphi}{\partial v})(u_0, v_0) \neq 0 \implies \frac{\partial \varphi}{\partial u}$ and $\frac{\partial \varphi}{\partial v}$ are linearly independent. This means that an entire span of tangent vectors, i.e. a tangent plane, of the surface S at $\varphi(u_0, v_0)$ exists. S is said to be *regular* if it is regular at all points $\varphi(u_0, v_0) \in S$.

In fact, the tangent plane at $\varphi(u_0, v_0)$ is the set of points

$$\{\varphi(u_0, v_0) + \frac{\partial \varphi}{\partial u}(u_0, v_0)c_1 + \frac{\partial \varphi}{\partial v}(u_0, v_0)c_2 \mid c_1, c_2 \in \mathbb{R}\}$$

which is precisely the affine tangent plane spanned by T_u and T_v . Note also that the vector $T_u \times T_v$, if nonzero, is normal to this plane, which leads to this equivalent definition.

Definition 7.9 (Tangent Planes of Surfaces)

Given a parameterized surface $\varphi : D \subset \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ that is regular at $\varphi(u_0, v_0)$, the tangent plane of the surface S at $\varphi(u_0, v_0) = (x_0, y_0, z_0)$ is defined

$$\{(x, y, z) \in \mathbb{R}^3 \mid (x - x_0, y - y_0, z - z_0) \cdot n = 0\}$$

where $n = (\frac{\partial \varphi}{\partial u} \times \frac{\partial \varphi}{\partial v})(u_0, v_0)$.

We finally construct the concept of signed areas before defining surface integration. We have all the tools we need to calculate surface areas, but remember that integration also covers the concept of *signed areas*, which could be negative. In order to define this, we define the concept of orientation on surfaces.

7.3.2 Orientation of Surfaces

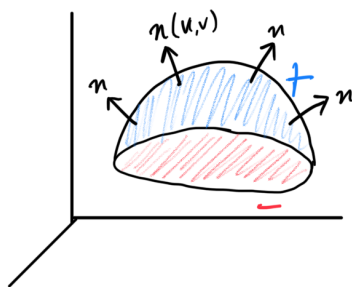
Definition 7.10 (Oriented Surfaces)

An *oriented surface* is a two-sided surface with one side specified as the *outside/positive* side and the other side as the *inside/negative* side. Note that an oriented surface is not guaranteed to have two sides (e.g. a Möbius strip). To ensure that there exist two sides, S must be regular.

Surprisingly, a parameterization does not have an intrinsic orientation. Rather, we determine the orientation ourselves by choosing a unit vector that generally points towards the outside of the surface S . Again, this choice is arbitrary, but it is customary to choose a vector that generally points "out." Either way, the orientation (unit) vector at every point $\varphi(u, v) \in S$, denoted as n , is

$$n(\varphi(u, v)) = \pm \frac{\frac{\partial \varphi}{\partial u} \times \frac{\partial \varphi}{\partial v}}{\left\| \frac{\partial \varphi}{\partial u} \times \frac{\partial \varphi}{\partial v} \right\|}$$

which can be visually calculated using the right hand rule.


Definition 7.11 (Orientation Preserving, Reversing Paramaterizations)

Given an oriented surface S with its positive side determined by the direction of unit vector $n(\varphi(u, v))$, the paramaterization φ is said to be *orientation preserving* if

$$n(\varphi(u, v)) = \frac{\frac{\partial \varphi}{\partial u} \times \frac{\partial \varphi}{\partial v}}{\left\| \frac{\partial \varphi}{\partial u} \times \frac{\partial \varphi}{\partial v} \right\|}$$

and *orientation reversing* if

$$n(\varphi(u, v)) = -\frac{\frac{\partial \varphi}{\partial u} \times \frac{\partial \varphi}{\partial v}}{\left\| \frac{\partial \varphi}{\partial u} \times \frac{\partial \varphi}{\partial v} \right\|}$$

So, to find whether a paramaterization is orientation preserving or reversing, it suffices to find the cross product $T_u \times T_v$ and see if it points in the same direction of the normal vector n (which should have already been determined when deciding the orientation of S).

Given a paramaterization φ and an un-oriented surface S , we can also just construct φ to be orientation-preserving (or reversing) by *defining* the normal vector n to be

$$n(\varphi(u, v)) = \frac{\frac{\partial \varphi}{\partial u} \times \frac{\partial \varphi}{\partial v}}{\left\| \frac{\partial \varphi}{\partial u} \times \frac{\partial \varphi}{\partial v} \right\|} \quad \left(\text{or } n(\varphi(u, v)) = -\frac{\frac{\partial \varphi}{\partial u} \times \frac{\partial \varphi}{\partial v}}{\left\| \frac{\partial \varphi}{\partial u} \times \frac{\partial \varphi}{\partial v} \right\|} \right)$$

So rather than finding out whether a paramaterization φ is orientation preserving or reversing by comparing $T_u \times T_v$ with n , we have defined n in a way such that φ must be orientation preserving (or reversing). We can utilize these tools of paramaterization to now define the surface integral.

7.3.3 Scalar, Vector Surface Integrals

A physical interpretation of a scalar surface integral is the weighted surface area of a certain surface.

Definition 7.12 (Scalar Surface Integrals)

Let $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ be a C^1 scalar field defined on a paramaterized surface $S \subset \mathbb{R}^3$ with paramaterization $\varphi : D \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3$. That is, $\varphi(D) = S$. We define the integral f over S to be

$$\begin{aligned} \iint_S f \, dS &= \iint_S f(x, y, z) \, dS \\ &= \iint_D f(\varphi(u, v)) \left\| \frac{\partial \varphi}{\partial u} \times \frac{\partial \varphi}{\partial v} \right\| \, du \, dv \end{aligned}$$

Note that this will require us to transform f , a function of x, y, z , into the function $f \circ \varphi$ of u, v . Additionally, if the parameterization of the surface S is not defined, then it one must be constructed. It is also clear that if S is a union of surfaces S_i , then its surface integral is the sum of the surface integrals of the S_i 's.

Letting the scalar field f be the constant field equal to 1, the scalar surface integral measures the surface area of S .

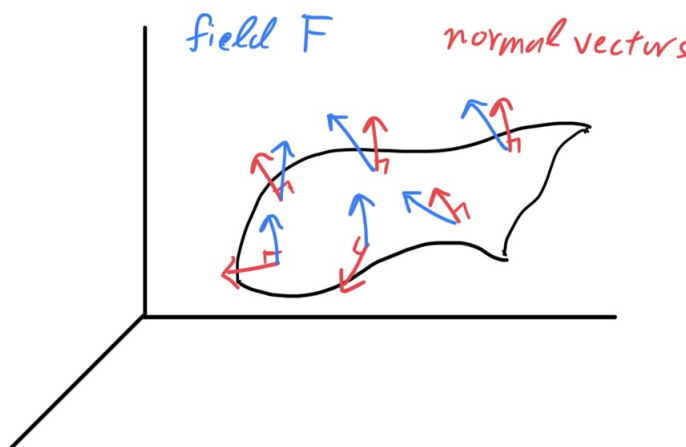
$$A(S) = \iint_S dS = \iint_D \left\| \frac{\partial \varphi}{\partial u} \times \frac{\partial \varphi}{\partial v} \right\| du dv$$

It is easy to see that the orientation of the parameterization φ does not affect scalar surface integrals, since the sign of the orientation gets nullified by the absolute value sign over $\left\| \frac{\partial \varphi}{\partial u} \times \frac{\partial \varphi}{\partial v} \right\|$.

Its physical interpretation is to measure the rate at which a fluid (determined by a vector field F) is crossing a given surface S . It also has many applications in electromagnetism.

Definition 7.13 (Vector Surface Integrals)

Let F be a vector field defined on surface S , the image of a parameterized surface φ . The *surface integral* of F over S is defined below, which is equivalent to summing up the dot product of the vector field and the normal vector to the surface.



It can be calculated with the following formulas by converting it into a scalar surface integral where the scalar field is the value of the dot product of the vector field with the normal vectors of the surface.

$$\begin{aligned} \iint_S F \cdot dS &= \iint_S (F \cdot n) dS \\ &= \iint_D \left(F(\varphi(u, v)) \cdot \frac{\frac{\partial \varphi}{\partial u} \times \frac{\partial \varphi}{\partial v}}{\left\| \frac{\partial \varphi}{\partial u} \times \frac{\partial \varphi}{\partial v} \right\|} \right) \left\| \frac{\partial \varphi}{\partial u} \times \frac{\partial \varphi}{\partial v} \right\| du dv \\ &= \iint_D F(\varphi(u, v)) \cdot \left(\frac{\partial \varphi}{\partial u} \times \frac{\partial \varphi}{\partial v} \right) du dv \end{aligned}$$

Since we are now talking about vector fields, the orientation of the parameterization is now significant. Visually, if the orientation of the surface S generally aligns with the vector field F , then the integral will be positive (since two vectors α, β generally pointing in the same direction implies that $\alpha \cdot \beta > 0$). The orientation of the parameterization, which is dependent on $\frac{\partial \varphi}{\partial u} \times \frac{\partial \varphi}{\partial v}$, determines the direction of the normal vector n (since it is defined to be $(\frac{\partial \varphi}{\partial u} \times \frac{\partial \varphi}{\partial v}) / \left\| \frac{\partial \varphi}{\partial u} \times \frac{\partial \varphi}{\partial v} \right\|$). Therefore, changing the orientation of φ will reverse the direction of n , which will then reverse the sign of the integral since n now points in the opposite direction.

of the vector field F than it previously did (by reversing the sign of the dot products). This is formalized in the theorem below.

Theorem 7.6 (Invariance of Surface Paramaterizations on Vector Surface Integrals)

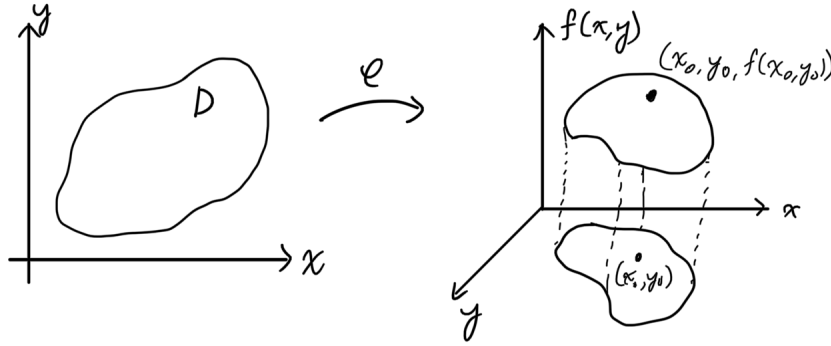
Let S be an oriented surface and let φ_1 and φ_2 be two regular paramaterizations with F a continuous vector field defined on S . Then, assuming φ_1 is orientation preserving,

$$\begin{aligned}\varphi_2 \text{ is orientation preserving} &\implies \iint_{\varphi_1} F \cdot dS = \iint_{\varphi_2} F \cdot dS \\ \varphi_2 \text{ is orientation reversing} &\implies - \iint_{\varphi_1} F \cdot dS = \iint_{\varphi_2} F \cdot dS\end{aligned}$$

7.3.4 Surface Integrals over Graphs

Given that we have the graph of a function $g : \mathbb{R}^2 \rightarrow \mathbb{R}$ rather than a general surface, we can paramaterize it simply as

$$\varphi(u, v) \equiv (u, v, g(u, v))$$



This means that

$$\frac{\partial \varphi}{\partial u} \times \frac{\partial \varphi}{\partial v} = \begin{pmatrix} -\frac{\partial g}{\partial u} \\ -\frac{\partial g}{\partial v} \\ 1 \end{pmatrix} \implies \left\| \frac{\partial \varphi}{\partial u} \times \frac{\partial \varphi}{\partial v} \right\| = \sqrt{1 + \left(\frac{\partial g}{\partial u}\right)^2 + \left(\frac{\partial g}{\partial v}\right)^2}$$

So we can simplify the equation for the surface area S of the graph of g over the region D in the xy -plane, as

$$\begin{aligned}A(S) &= \iint_S dS = \iint_D \left\| \frac{\partial \varphi}{\partial u} \times \frac{\partial \varphi}{\partial v} \right\| dA \\ &= \iint_D \sqrt{1 + \left(\frac{\partial g}{\partial u}\right)^2 + \left(\frac{\partial g}{\partial v}\right)^2} du dv\end{aligned}$$

With the same g , we can find the weighed surface area of S over the scalar function $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ with the formula

$$\iint_S f dS = \iint_D f(u, v, g(u, v)) \sqrt{1 + \left(\frac{\partial g}{\partial u}\right)^2 + \left(\frac{\partial g}{\partial v}\right)^2} du dv$$

Finally, with the same graph g , the surface integral over the vector field F is

$$\begin{aligned}\iint_S F \cdot dS &= \iint_D F(\varphi(u, v)) \cdot \left(\frac{\partial \varphi}{\partial u} \times \frac{\partial \varphi}{\partial v} \right) du dv \\ &= \iint_D \left(F_1(u, v) \left(-\frac{\partial g}{\partial u} \right) + F_2(u, v) \left(-\frac{\partial g}{\partial v} \right) + F_3(u, v) \right) du dv\end{aligned}$$

7.4 Integral Theorems

Recall the differential notation for writing line integrals. For 2 and 3 dimensions, it is written as

$$\begin{aligned}\int_C F \cdot ds &= \int_C F \cdot (dx, dy) = \int_C F_1 dx + F_2 dy \\ \int_C F \cdot ds &= \int_C F \cdot (dx, dy, dz) = \int_C F_1 dx + F_2 dy + F_3 dz\end{aligned}$$

7.4.1 Green's Theorem

Green's Theorem gives the relationship between a line integral around a simple closed curve C and a double integral over the plane region D bounded by C .

Theorem 7.7 (Green's Theorem in \mathbb{R}^2)

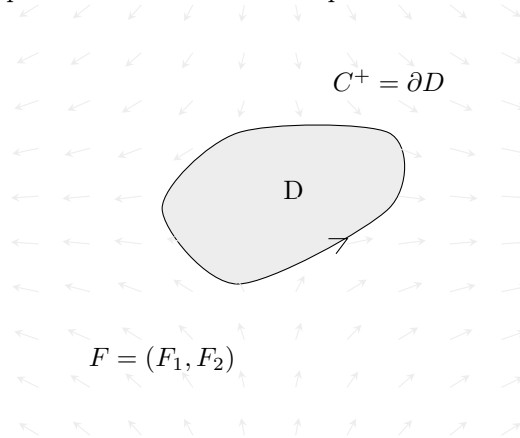
Let there be a 2-dimensional C^1 vector field F on \mathbb{R}^2 defined on a simple oriented closed piecewise-smooth curve C and its bounded region $D \subset \mathbb{R}^2$ (that is, $C = \partial D$). Let the orientation of the path of C be such that it is traveling *counterclockwise*, i.e. a point traveling through C would see the region D to its *left*, denoted as C^+ and the clockwise orientation as C^- . Then,

$$\oint_{C^+} F_1 dx + F_2 dy = \iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dx dy$$

By reversing the orientation, it is clear that we have

$$\oint_{C^-} F_1 dx + F_2 dy = - \iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dx dy$$

Note that this theorem is expressed in terms of the components of the vector field F .



Green's theorem has many applications in physics. For example, in order to solve two-dimensional flow integrals measuring the sum of fluid outflowing from a volume, Green's theorem allows us to calculate the total outflow summed about an enclosing area .

Corollary 7.2 ()

Let D be a region for which Green's theorem applies with positively oriented boundary ∂D . Then, the area of D can be computed with the formula

$$A(D) = \frac{1}{2} \oint_{\partial D} x dy - y dx$$

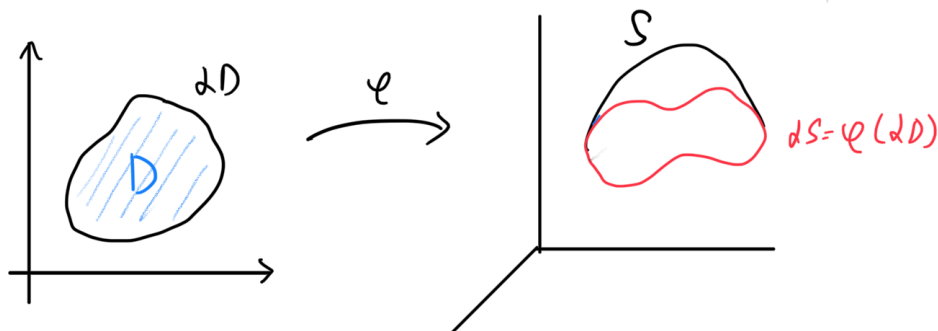
Green's theorem can be used to determine the area and centroid of plane figures solely by integrating over the perimeter.

7.4.2 Stokes' Theorem

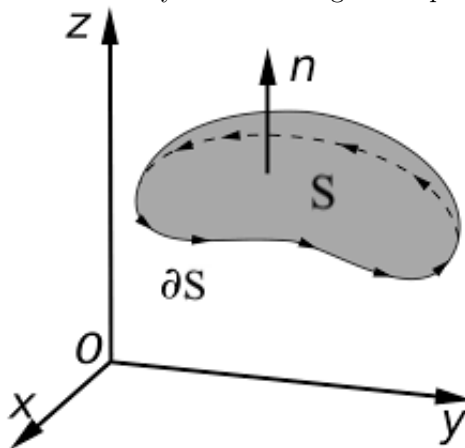
Green's theorem relates line integrals to double integrals. Stokes' theorem generalizes Green's theorem by relating line integrals to surface integrals of 2-dimensional surfaces embedded in \mathbb{R}^3 .

Theorem 7.8 (Stokes' Theorem)

Let S be an oriented regular surface defined by parameterization $\varphi : D \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3$, and let the image of the boundary ∂D under φ be the boundary ∂S of S . We can interpret ∂S as a path mapping from $\mathbb{R} \rightarrow S \subset \mathbb{R}^3$.



The orientation unit vector n of S induces the positive orientation of ∂S , denoted ∂S^+ . Visually, if you are walking along the curve with your head pointing in the same direction as the unit normal vectors while the surface is on the left then you are walking in the positive direction on ∂S .



Given that F is a C^1 vector field defined on S , then

$$\iint_S \text{curl } F \cdot dS = \iint_S (\nabla \times F) \cdot dS = \oint_{\partial S^+} F \cdot ds$$

If S has no boundary, that is, if the image of $p' = \partial S$ is not a simple closed curve, then the integral is 0.

The above theorem implies that the vector surface integral of a surface without a boundary (i.e. a closed graph, such as a sphere) is always 0 along the curl of any C^1 field. Geometrically, this means that given a closed solid S with field $\nabla \times F$, the rate of flow of the vector field into S is equal to the flow out of S .

7.4.3 Gauss' Theorem

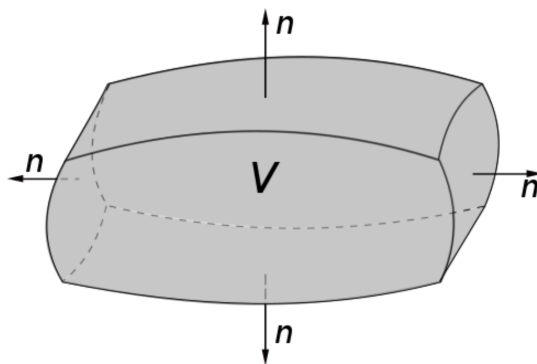
The divergence theorem relates the flux of a vector field through a closed surface to the divergence of the field in the volume enclosed.

Theorem 7.9 (Gauss' Divergence Theorem)

Let V be a subset of \mathbb{R}^3 . Denote by ∂V the oriented closed surface that bounds V (with outward pointing normal orientation vectors), and let F be a C^1 vector field defined on a neighborhood of V . Then,

$$\iiint_V \operatorname{div} F \, dV = \iiint_V (\nabla \cdot F) \, dV = \oint_{\partial V} F \cdot dS = \oint_{\partial V} (F \cdot n) \, dS$$

where the two left-most integrals are volume integrals, and the two right-most integrals are surface integrals. Intuitively, this makes sense; the volume integrals represent the total of the sources in volume V , and the right hand side represents the total flow across the boundary ∂V .



8 Sequences of Functions