Real Analysis

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Let's first talk about why we need analysis in general in the first place. Algebra allows us to define certain algebraic structures, which are essentially sets with operations. These operations are defined to have a finite number of arguments. For example, let's take a look at the negation $x \mapsto -x$ and the addition $x, y \mapsto x + y$ operations in a group G. We can compose these operations up to any finite length n, removing the parentheses due to associativity, but note that the "sum" below is not a single operation. It is a composition of $n - 1$ operations.

$$x_1 + x_2 + \ldots + x_n \in G \tag{1}$$
$$-(-(\ldots(-x))) \in G \tag{2}$$

This is still well defined due to closure, but what if we wanted to do this an infinite number of times?

$$x_1 + x_2 + \ldots = ? \tag{3}$$
$$\ldots(-(-x)) = ? \tag{4}$$

For someone who has learned about sequences and series in high school, this may not be a big jump in logic, but it is. The objects above are not even well-defined and trying to define them with algebraic tools is equivalent to the famous Zeno's paradox. So we simply need to add more tools in order to define these new mathematical objects, which we call *series*. To define series, we need to first define sequences. Can we do this with algebra? Yes, since we can simply model it as a function.

Definition 0.1 (Sequence)

A sequence is a function $f : \mathbb{N} \to X$. We usually denote a sequence by writing out the first few terms of the sequence, followed by an ellipsis.

$$a_1 = f(1), a_2 = f(2), \ldots \tag{5}$$

or as an indexed set over the naturals $\{a_i\}_{i \in \mathbb{N}}$.

Therefore, we can consider series as a sequence of finite sums, each element which is well-defined.

$$x_1, x_1 + x_2, x_1 + x_2 + x_3, \ldots \tag{6}$$

For any $n \in \mathbb{N}$, we can get the value of $a_n = \sum_{i=1}^{n} a_i$, but can we say something about the limiting behavior of a_n? That is, maybe we can just slap a value x onto this series such that it doesn't "break" any of the rules we have in the finite sense. Unfortunately, it is not possible to define such values for all series, but it is possible for some of them, which we call *convergent series*. To rigorously determine which ones are convergent and which ones are not, we need the tools of topology and analysis. Defining the concept of sequences that model infinitely composed operations is what allows us to define differentiation and integration.

Great, we've motivated the need for analysis, but before jumping straight into real analysis, let's talk about what analysis in general works with. It studies functions of the form $f : X \to Y$, and minimally both X, Y must be *Banach spaces*, i.e. complete normed vector spaces over some field $\mathbb{F}$. Almost all flavors of analysis, including real ($\mathbb{R}$), complex ($\mathbb{C}$), multivariate ($\mathbb{R}^n$), p-adic, and functional (infinite-dimensional Banach spaces) analysis require *at least* a Banach space structure. Why are Banach spaces so great? Well if we were to define convergence in X or Y, then it only makes sense to talk about convergence with respect to a topology. So X, Y must at least be topological spaces. It would also be bad if we were to take a sequence in X and find out that it converges to some element outside of X. Therefore, we want a notion of *completeness* in the sense that all sequences that "get closer," i.e. Cauchy sequences, actually converge in X. Unfortunately, while convergence of sequences is preserved under homeomorphisms (and is thus a topological property), convergence of Cauchy sequences is not.[1] Furthermore, the notion of uniform convergence is a metric space property, not a topological one. Therefore, the concept of distances is crucial to the construction of analysis. As for the norm, I'm still not sure why we need this.[2]

[1] Consider the sequence $a_n = 1/(n + 1)$ in $(0, 1)$ and the map $f(x) = 1/x$ to the set $(1, +\infty)$. a_n is Cauchy but $f(a_n)$ is not.

[2] Aspinwall and Ng told me this, but I'm not sure why. The Frechet derivative seems like it can be purely defined with a metric.

But in college courses such as real and complex analysis, why do we say we work over the *fields* $\mathbb{R}$ and $\mathbb{C}$ rather than the Banach spaces $\mathbb{R}$ and $\mathbb{C}$? This is because of the following theorem.

Theorem 0.1 ()

Every field $\mathbb{F}$ is a 1-dimensional vector space over itself.

Therefore, when we talk about the *field* $\mathbb{R}$, we are really treating it as a vector space $\mathbb{R}$ over the field $\mathbb{R}$.[3] Every other structure beyond this is a "bonus" property that gives us extra tools to prove stronger properties. The most notable is the total ordering on $\mathbb{R}$, which allows us to define upper/lower bounds and other real-analysis specific theorems like the intermediate value theorem or the mean value theorem. Other structures include the inner product or the measure.

Now that we've taken in the big picture, for each type of analysis, we should construct the underlying relevant Banach space. At the very least, we can with the tools of set theory and algebra define the rationals $\mathbb{Q}$ as an ordered field over the quotient space $\mathbb{Z} \times \mathbb{Z}/\sim$. Furthermore, $\mathbb{Q}$ itself is a normed vector space (over $\mathbb{Q}$)[4] and the only thing we need now is completeness.

1. If the norm on $\mathbb{Q}$ is defined as the normal absolute value (Euclidean norm), completing it gives $\mathbb{R}$ as an ordered field which also has a compatible order as that of $\mathbb{Q}$. We study functions mapping to and from $\mathbb{R}$ with *single-variable real analysis*.
2. If we take the *p-adic* norm, then completing it with respect to this gives the *p-adic numbers*, which also forms a field but loses the ordering. We deal with functions over the p-adics with *p-adic analysis*.
3. We can construct $\mathbb{C}$ by taking $\mathbb{R}^2$ and endowing it with a bit more structure. We get *complex analysis*.
4. We can construct $\mathbb{R}^n$ and $\mathbb{C}^n$ by easily defining its vector space structure and then endowing it with a norm, and showing that it is complete with respect to the norm-induced metric. This is known as *multivariate analysis*.
5. With all these defined, we can define Banach function spaces like L^p and perform analysis on operators $f : L^p \to L^q$. This is *functional analysis*.

What we have talked about so far was Cauchy completeness, but there is a different type of completeness called *Dedekind completeness*, also equivalently known as the *least-upper-bound (LUB) property*, defined only on ordered sets (with no other structure). It turns out that in an ordered field, the two forms of completeness are equivalent.[5] Therefore, many real analysis textbooks tend to use Dedekind completeness when constructing the reals, but Cauchy completeness is in a sense more "fundamental." We will go through both independent constructions of $\mathbb{R}$ involving both types of completeness since both are used in future theorems.

1. *Construction from Cauchy Sequences.* We verify that $\mathbb{Q}$ is a field and endow it with the standard Euclidean metric $d(x, y) = |x - y|$. We can then construct a new quotient space S of Cauchy sequences in $\mathbb{Q}$, define all the ordered field operations/relations, and finally show that S satisfies Cauchy completeness. Most would end here and claim that this is $\mathbb{R}$, but we must also prove the Archimedean property with this order. Once done, now we can truly claim $S = \mathbb{R}$.
2. *Construction from Dedekind Cuts.* We verify that $\mathbb{Q}$ is a field, put an order on it, and verify that it is an ordered field. We then construct a new set D of *Dedekind cuts* from $\mathbb{Q}$, define the compatible ordered field operations/relations, and show that this new set D satisfies the least-upper bound property. We claim that $D = \mathbb{R}$.

[3] Thanks to Prof. Lenny Ng for clarifying this.

[4] Note that while we define the norm and metric to usually map to $\mathbb{R}^+$, $\mathbb{R}$ isn't even defined yet and so to avoid circular definitions, we define the norm on the rationals to have codomain $\mathbb{Q}$.

[5] Actually, this is not true. Dedekind completeness is equivalent to Cauchy completeness plus the Archimedean property. An example of a Cauchy-complete non Archimidean field is the field F of rational functions over $\mathbb{R}$, with positive cone consisting of those functions f/g such that the leading coefficients of f, g have the same algebraic sign. The Cauchy completion of this into the equivalence classes of Cauchy sequences in F results in a non-Archimedean field.

1 Number Systems

1.1 The Rationals

1.1.1 Field Properties

Definition 1.1 (Field)

A **field** is an algebraic structure $(\mathbb{F}, +, \cdot)$ where

1. $\mathbb{F}$ is an abelian group under $+$, with 0 being the *additive identity*.
2. $\mathbb{F} \setminus \{0\}$ is an abelian group under $\cdot$, with 1 being the *multiplicative identity*.
3. It connects the two operations through the *distributive property*.

$$x \cdot (y + z) = x \cdot y + x \cdot z \tag{7}$$

Lemma 1.1 (Left = Right Distributivity)

Left and right distributivity are equivalent.

$$x \cdot (y + z) = (y + z) \cdot x \tag{8}$$

Proof. 1.1 ()

$$\begin{aligned} x \cdot (y + z) &= x \cdot y + x \cdot z && \text{(Distributive)} \\ &= y \cdot x + z \cdot x && \text{(Commutative)} \\ &= (y + z) \cdot x && \text{(Distributive)} \end{aligned}$$

Lemma 1.2 (Properties of Addition)

The properties of addition hold in a field.

1. If $x + y = x + z$, then $y = z$.
2. If $x + y = x$, then $y = 0$.
3. If $x + y = 0$, then $y = -x$.
4. $(-(-x)) = x$.

Proof. 1.2 ()

For the first, we have

$$\begin{aligned} x + y = x + z &\implies -x + (x + y) = -x + (x + z) && \text{(addition is a function)} \\ &\implies (-x + x) + y = (-x + x) + z && \text{(+ is associative)} \\ &\implies 0 + y = 0 + z && \text{(definition of additive inverse)} \\ &\implies y = z && \text{(definition of identity)} \end{aligned}$$

For the second, we can set $z = 0$ and apply the first property. For the third, we have

$$\begin{aligned} x + y = 0 &\implies -x + (x + y) = -x + 0 && \text{(addition is a function)} \\ &\implies (-x + x) + y = -x + 0 && \text{(+ is associative)} \\ &\implies 0 + y = -x + 0 && \text{(definition of additive inverse)} \\ &\implies y = -x && \text{(definition of identity)} \end{aligned}$$

For the fourth, we simply follow that if y is an inverse of z, then z is an inverse of y. Therefore, $-x$ being an inverse of x implies that x is an inverse of $-x$. $-(-x)$ must also be an inverse of $-x$. Since inverses are unique[a], $x = -(-x)$.

Lemma 1.3 (Properties of Multiplication)

The properties of multiplication hold in a field.

1. If $x \neq 0$ and $xy = xz$, then $y = z$.
2. If $x \neq 0$ and $xy = x$, then $y = 1$.
3. If $x \neq 0$ and $xy = 1$, then $y = x^{-1}$.
4. If $x \neq 0$, then $(x^{-1})^{-1} = x$.

Proof. 1.3 ()

The proof is almost identical to the first. Since $x \neq 0$, we can always assume that x^{-1} exists. For the first, we have

$$\begin{aligned} xy = xz &\implies x^{-1}(xy) = x^{-1}(xz) && \text{(multiplication is a function)} \\ &\implies (x^{-1}x)y = (x^{-1}x)z && (\times \text{ is associative}) \\ &\implies 1y = 1z && \text{(definition of multiplicative inverse)} \\ &\implies y = z && \text{(definition of identity)} \end{aligned}$$

For the second, we can set $z = 1$ and apply the first property. For the third, we have

$$\begin{aligned} xy = 1 &\implies x^{-1}(xy) = x^{-1}1 && \text{(multiplication is a function)} \\ &\implies (x^{-1}x)y = x^{-1}1 && (\times \text{ is associative}) \\ &\implies 1y = x^{-1}1 && \text{(definition of multiplicative inverse)} \\ &\implies y = x^{-1} && \text{(definition of identity)} \end{aligned}$$

For the fourth, we simply see that x^{-1} is a multiplicative inverse of both x and $(x^{-1})^{-1}$ in the group $(\mathbb{F} \setminus \{0\}, \times)$, and since inverses are unique, they must be equal.

Lemma 1.4 (Properties of Distribution)

For any $x, y, z \in \mathbb{F}$, the field axioms satisfy

1. $0 \cdot x = 0$.
2. If $x \neq 0$ and $y \neq 0$, then $xy \neq 0$.
3. $-1 \cdot x = -x$.
4. $(-x)y = -(xy) = x(-y)$.
5. $(-x)(-y) = xy$.

Proof. 1.4 ()

For the first, note that

$$0x = (0 + 0) \cdot x = 0x + 0x \tag{9}$$

and subtracting $0x$ from both sides gives $0 = 0x$. For the second, we can claim that $xy \neq 0$ equivalently claiming that it will have an identity. Since $x, y \neq 0$, their inverses exists, and we claim

[a] This is proved in algebra.

that $(xy)^{-1} = y^{-1}x^{-1}$ is an inverse. We can see that by associativity,

$$(y^{-1}x^{-1})(xy) = y^{-1}(x^{-1}x)y = y^{-1}y = 1 \tag{10}$$

For the third, we see that

$$0 = 0 \cdot x = (1 + (-1)) \cdot x = 1 \cdot x + (-1) \cdot x = x + (-1) \cdot x \tag{11}$$

which implies that $-1 \cdot x$ is the additive inverse. The fourth follows immediately from the third by the associative property. For the fifth we can see that

$$\begin{aligned}
(-x)(-y) &= (-1)x(-1)y && \text{(property 3)} \\
&= (-1)(-1)xy && \text{($\times$ is commutative)} \\
&= -1 \cdot (-xy) && \text{(property 3)} \\
&= -(-xy) && \text{(property 3)} \\
&= xy && \text{(addition property 4)}
\end{aligned}$$

Now that we've reviewed some fields, let's construct $\mathbb{Q}$ from $\mathbb{Z}$ and verify it's a field.

Definition 1.2 (Rationals)

Given the ordered ring of integers $(\mathbb{Z}, +_\mathbb{Z}, \times_\mathbb{Z}, \leq_\mathbb{Z})$ the **rational numbers** $(\mathbb{Q}, +_\mathbb{Q}, \times_\mathbb{Q})$ are defined as such.

1. $\mathbb{Q}$ is the quotient space on $\mathbb{Z} \times \mathbb{Z} \setminus \{0\}$ with the equivalence relation $\sim$

$$(a, b) \sim (c, d) \iff a \times_\mathbb{Z} d = b \times_\mathbb{Z} c \tag{12}$$

   We denote this class as (a, b), where $b > 0$, since if $b < 0$, we know that $(-a, -b)$ are also in this order.
2. The additive and multiplicative identities are

$$0_\mathbb{Q} := (0_\mathbb{Z}, a), \quad 1_\mathbb{Q} := (a, a) \tag{13}$$

3. Addition on $\mathbb{Q}$ is defined

$$(a, b) +_\mathbb{Q} (c, d) := \big((a \times_\mathbb{Z} d) +_\mathbb{Z} (b \times_\mathbb{Z} c), b \times_\mathbb{Z} d\big) \tag{14}$$

4. The additive inverse is defined

$$-(a, b) := (-a, b) \tag{15}$$

5. Multiplication on $\mathbb{Q}$ is defined

$$(a, b) \times_\mathbb{Q} (c, d) := \big(a \times_\mathbb{Z} c, b \times_\mathbb{Z} d\big) \tag{16}$$

6. The multiplicative inverse is defined

$$(a, b)^{-1} := (b, a) \tag{17}$$

Theorem 1.1 (Rationals are a Field)

$\mathbb{Q}$ is a field.

Proof. 1.5 ()

We do a few things.

1. Verify the additive identity.

$$(a, b) + (0, c) = (ac + 0b, bc) = (ac, bc) \sim (a, b) \tag{18}$$

2. Verify the multiplicative identity.

$$(a, b) \times (c, c) = (ac, bc) \sim (a, b) \tag{19}$$

3. Additive inverse is actually an inverse.

$$(a, b) + (-a, b) = (ab + (-ba), bb) = (0, bb) \sim (0, 1) \tag{20}$$

4. Multiplicative inverse is actually an inverse.

$$(a, b) \times (b, a) = (ab, ba) = (ab, ab) \sim (1, 1) \tag{21}$$

5. Addition is commutative.

$$(a, b) + (c, d) = (ad + bc, bd) = (cb + ad, bd) = (c, d) + (a, b) \tag{22}$$

6. Addition is associative.

$$\begin{aligned}
(a, b) + ((c, d) + (e, f)) &= (a, b) + (cf + de, df) && (23)\\
&= (adf + bcf + bde, bdf) && (24)\\
&= (ad + bc, bd) + (e, f) && (25)\\
&= ((a, b) + (c, d)) + (e, f) && (26)
\end{aligned}$$

7. Multiplication is commutative.

$$(a, b) \times (c, d) = (ac, bd) = (ca, db) = (c, d) \times (a, b) \tag{27}$$

8. Multiplication is associative.

$$\begin{aligned}
(a, b) \times ((c, d) \times (e, f)) &= (a, b) \times (ce, df) && (28)\\
&= (ace, bdf) && (29)\\
&= (ac, bd) \times (e, f) && (30)\\
&= ((a, b) \times (c, d)) \times (e, f) && (31)
\end{aligned}$$

9. Multiplication distributes over addition.

$$\begin{aligned}
(a, b) \times ((c, d) + (e, f)) &= (a, b) \times (c, d) + (a, b) \times (e, f) && (32)\\
&= (ac, bd) + (ae, bf) && (33)\\
&= (abcf + abde, b^2 df) && (34)\\
&= (acf + ade, bdf) \qquad = (a, b) \times (cf + de, df) && (35)
\end{aligned}$$

We have successfully defined the rationals, but now these are almost completely separate elements. We know that all integers are rational numbers, and so to show that the rationals are an extension of $\mathbb{Z}$ we want to identify a *canonical injection* $\iota : \mathbb{Z} \to \mathbb{Q}$. This can't just be any canonical injection; it must preserve the algebraic structure between the two sets and must therefore be a *ring homomorphism*.

Theorem 1.2 (Canonical Injection of $\mathbb{Z}$ to $\mathbb{Q}$ is a Ring Homomorphism)

Let us define the canonical injection $\iota : \mathbb{Z} \to \mathbb{Q}$ to be $\iota(a) = (a, 1)$. This is a ring homomorphism.

Proof. 1.6 ()

We show a few things.

1. Preservation of addition.

$$\begin{aligned} \iota(a) +_{\mathbb{Q}} \iota(b) &= (a, 1) +_{\mathbb{Q}} (b, 1) && (36) \\ &= (1a +_{\mathbb{Z}} 1b, 1^2) && (37) \\ &= (a +_{\mathbb{Z}} b, 1) && (38) \\ &= \iota(a +_{\mathbb{Z}} b) && (39) \end{aligned}$$

2. Preservation of multiplication.

$$\begin{aligned} \iota(a) \times_{\mathbb{Q}} \iota(b) &= (a, 1) \times_{\mathbb{Q}} (b, 1) && (40) \\ &= (a \times_{\mathbb{Z}} b, 1^2) && (41) \\ &= (a \times_{\mathbb{Z}} b, 1) && (42) \\ &= \iota(a \times_{\mathbb{Z}} b, 1) && (43) \end{aligned}$$

3. Preservation of multiplicative identity.

$$\iota(1_{\mathbb{Z}}) = (1, 1) = 1_{\mathbb{Q}} \tag{44}$$

1.1.2 Ordered Field Properties

Great, so we have established that $\mathbb{Q}$ is a field. The next property we want to formalize is order.

Definition 1.3 (Partial, Total/Linear Order)

A **partial order** on a set X is a relation $\leq$ satisfying.

1. Reflexive: $x \leq x$
2. Antisymmetric: $x \leq y, y \leq x \implies x = y$
3. Transitivity: $x \leq y, y \leq z \implies x \leq z$

Note that when we say $x \leq y$, this means "x is related to y" (but does not necessarily mean that y is related to x), or "x is less than or equal to y." A set X with a partial order is called a partially ordered set.

Additionally, given elements x, y of partially order set X, if either $x \leq y$ or $y \leq x$, then x and y are **comparable**. Otherwise, they are **incomparable**. A partial order in which every pair of elements is comparable is called a **total order**, or **linear order**. Note that from this $\leq$ relation, we can similarly define

1. $\leq$: less than or equal to
2. $\geq$: greater than or equal to
3. $<$: strictly less than ($x < y$ iff $x \leq y, x \neq y$)
4. $>$: strictly greater than ($x > y$ iff $x \geq y, x \neq y$)

Example 1.1 (Partially Ordered Sets)

We list some examples of partially ordered sets.

1. The real numbers ordered by the standard "less-than-or-equal" relation $\leq$ (totally ordered set

as well).
2. The set of subsets of a given set X ordered by inclusion. That is, the power set 2^X with the partial order $\subseteq$ is partially ordered.
3. The set of natural numbers equipped with the relation of divisibility.
4. The set of subspaces of a vector space ordered by inclusion.
5. For a partially ordered set P, the sequence space containing all sequences of elements from P, where sequence a precedes sequence b if every item in a precedes the corresponding item in b.

We now want to define the natural ordering of the rationals. There are countless ways to do it, but I just take the difference and claim that it is greater than 0.

Theorem 1.3 (Order on Rationals)

The order $\leq_{\mathbb{Q}}$ defined on the rationals as

$$(a, b) \leq_{\mathbb{Q}} (c, d) \iff ad \leq_{\mathbb{Z}} bc \tag{45}$$

is a total order. Remember that we have defined $b, d > 0$.

Proof. 1.7 ()

We prove the three properties.

1. Reflexive.
$$(a, b) \leq_{\mathbb{Q}} (a, b) \iff ab \leq_{\mathbb{Z}} ab \tag{46}$$
2. Antisymmetric.
$$(a, b) \leq_{\mathbb{Q}} (c, d) \implies ad \leq_{\mathbb{Z}} bc(c, d) \leq_{\mathbb{Q}} (a, b) \qquad \implies bc \leq_{\mathbb{Z}} ad \tag{47}$$
This implies that both $ad = bc$, which by definition means that they are in the same equivalence class.
3. Transitivity. Assume that $(a, b) \leq (c, d)$ and $(c, d) \leq (e, f)$. Then, we notice that $b, d, f > 0$ and therefore by the ordered ring property[a] of $\mathbb{Z}$, we have
$$(a, b) \leq_{\mathbb{Q}} (c, d) \implies ad \leq_{\mathbb{Z}} bc \implies adf \leq_{\mathbb{Z}} bcf \tag{48}$$
$$(c, d) \leq_{\mathbb{Q}} (e, f) \implies cf \leq_{\mathbb{Z}} de \implies bcf \leq_{\mathbb{Z}} bde \tag{49}$$
Therefore from transitivity of the ordering on $\mathbb{Z}$ we have $adf \leq bde$. By the ordered ring property[b] we have $0 \leq bde - adf = d(be - af)$. But notice that $d > 0$ from our definition of rationals, and therefore it must be the case that $0 \leq be - af \implies af \leq_{\mathbb{Z}} be$, which by definition means $(a, b) \leq_{\mathbb{Q}} (e, f)$.

As soon as we define an order the concept of extrema and bounds are well defined. Let's define them too.

Definition 1.4 (Extrema, Bounds)

Given a set X,
1. $x \in X$ is a **maximum** X if $y \leq x$ for all $y \in X$.
2. $x \in X$ is a **minimum** X if $x \leq y$ for all $y \in X$.

Given a totally ordered set X and some subset $S \subset X$.
1. $x \in X$ is an **upper bound** of S if $x \geq y$ for all $y \in S$.

[a] If $a \leq b$ and $0 \leq c$, then $ac \leq bc$.
[b] If $a \leq b$, then $a + c \leq b + c$.

2. $x \in X$ is a **lower bound** of S if $x \leq y$ for all $y \in S$.
3. $x \in X$ is a **supremum**, or **least upper bound**, of S if x is the minimum of the set of all upper bounds of S.
4. $x \in X$ is a **infimum**, or **greatest lower bound**, of S if x is the maximum of the set of all lower bounds of S.

Note that we have defined max/min separately from the concept of bounds. You can define the maximum of a set with just knowing the set, but the bounds require *both* some subset S with respect to an enclosing set X.[a] Intuitively, the main difference between the supremum/infimum and maximum/minimum is that the supremum/infimum accounts for limit points of the subset S.

Note that given a set, we can really put whatever order we want on it. However, consider the field with the following order.

$$\mathbb{F} = \{0, 1\}, \ 0 < 1 \tag{50}$$

This does not behave well with respect to its operations because for example if we have $0 < 1$, then adding the same element to both sides should preserve the ordering. But this is not the case since $0+1 = 1 > 1+1 = 0$. While it may be easy to define an order, we would like it to be an ordered field.

Definition 1.5 (Ordered Field)

An **ordered field** is a field that has an order satisfying

1. $y < z \implies x + y < x + z$ for all $x \in \mathbb{F}$.
2. $x > 0, y > 0 \implies xy > 0$.

Theorem 1.4 (Properties)

In an totally ordered field,

1. $x > 0 \implies -x < 0$.
2. $x \neq 0 \implies x^2 > 0$.
3. If $x > 0$, then $y < z \implies xy < xz$.

Proof. 1.8 ()

The first property is a single-liner

$$0 < x \implies 0 + -x < x + -x \implies -x < 0 \tag{51}$$

For the second property, it must be the case that $x > 0$ or $x < 0$. If $x > 0$, then by definition $x^2 > 0$. If $x < 0$, then

$$x^2 = 1 \cdot x^2 = (-1)^2 \cdot x^2 = (-1 \cdot x)^2 = (-x)^2 \tag{52}$$

and since $-x > 0$ from the first property, we have $x^2 = (-x)^2 > 0$. For the third, we use the distributive property.

$$\begin{aligned} y < z &\implies 0 < z - y && (53)\\ &\implies 0 = x0 < x(z-y) = xz - xy && (54)\\ &\implies xy < xz && (55) \end{aligned}$$

As we have hinted, the rationals is an ordered field.

[a] For example, it makes sense to define the maximum of a set $S = [0, 1]$ by itself, but not an upper bound for it. If $X = \mathbb{Q}$, then the supremum is 1, but if X was the set of all irrationals, then this has no supremum.

Theorem 1.5 (Rationals are an Ordered Field)

$\mathbb{Q}$ is an ordered field.

Proof. 1.9 ()

We show that our defined order satisfies the definition.

1. Assume that $y = (a, b) \leq (c, d) = z$. Let $x = (e, f)$. Then $x + y = (af + be, bf)$, $x + z = (cf + de, df)$. Therefore

$$\begin{aligned} (af + be)df &= adf^2 + bedf && (56) \\ &\leq bcf^2 + bedf && (57) \\ &= (cf + de)bf && (58) \end{aligned}$$

But $(af + be)df = (cf + de)bf$ is equivalent to saying $(af + be, bf) \leq_{\mathbb{Q}} (cf + de, df)$, i.e. $x + y \leq x + z$!

2. Let $x = (a, b), y = (c, d)$. Since $0 < x, 0 < y$, by construction this means that $0 < a, 0 < c$ (since $b, d > 0$ in the canonical rational form). By the ordered ring property of the integers, $0 < ac$. So

$$0 < ac \iff 0 \cdot bd < ac \cdot 1 \iff (0, 1) < (ac, bd) \iff 0_{\mathbb{Q}} < (a, c) \times_{\mathbb{Q}} (b, d) = xy \qquad (59)$$

Not only is it an ordered field, but it also is consistent with the ordering on $\mathbb{Z}$! It's nice how all these properties seem to fit together.

Theorem 1.6 (Preservation of Order)

The canonical injection ι is an *order homomorphism*. That is, for $a, b \in \mathbb{Z}$,

$$a \leq_{\mathbb{Z}} b \iff \iota(a) \leq_{\mathbb{Q}} \iota(b) \qquad (60)$$

Proof. 1.10 ()

$$\begin{aligned} a \leq_{\mathbb{Z}} b &\iff a \cdot 1 \leq_{\mathbb{Z}} b \cdot 1 && (61) \\ &\iff (a, 1) \leq_{\mathbb{Q}} (b, 1) && (62) \\ &\iff \iota(a) \leq_{\mathbb{Q}} \iota(b) && (63) \end{aligned}$$

Note that an order can be used to generate an order topology, which we will define below.

Definition 1.6 (Order Topology on $\mathbb{Q}$)

The order topology on $\mathbb{Q}$ is the topology generated by the set $\mathscr{B}$ of all open intervals

$$(a, b) \coloneqq \{x \in \mathbb{Q} \mid a < x < b\} \qquad (64)$$

Theorem 1.7 (Finite Fields)

There are no finite ordered fields.

1.1.3 Norm

Note that we can also define a norm on the rationals with just the order and algebraic properties.

Theorem 1.8 (Norm on $\mathbb{Q}$)

The following is indeed a norm on $\mathbb{Q}$.

$$|x| := \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases} \tag{65}$$

It is well known that the metric induced by any norm is indeed a metric. Therefore we state the metric as a definition.

Definition 1.7 (Metric on $\mathbb{Q}$)

The Euclidean metric on $\mathbb{Q}$ is defined

$$d(x, y) := |x - y| = \begin{cases} x - y & \text{if } x \geq y \\ y - x & \text{if } x < y \end{cases} \tag{66}$$

Thus we get to what we want: the induced topology of open balls. Again, since we know from point-set topology that metric topologies are indeed topologies, we will state this as a definition rather than a theorem.

Definition 1.8 (Open-Ball Topology on $\mathbb{Q}$)

The Euclidean topology on $\mathbb{Q}$ is the topology generated by the set $\mathscr{B}$ of all open balls

$$B(x, r) := \{y \in \mathbb{Q} \mid |x - y| < r\} \tag{67}$$

Note that this is the same topology as the order topology. This should however be proved.

Theorem 1.9 (Metric and Order Topologies on $\mathbb{Q}$)

The metric and order topologies on $\mathbb{Q}$ are the same topologies.

Proof. 1.11 ()

1.2 The Reals

By constructing the topology of $\mathbb{Q}$ earlier, we can talk about convergence. The first question to ask (if you were the first person inventing the reals) is "how do I know that there exists some other numbers at all?" The first clue is trying to find the side length of a square with area 2. As we see, this number is not rational.

Theorem 1.10 ($\sqrt{2}$ is Not Rational)

There exists no $x \in \mathbb{Q}$ s.t. $x^2 = 2$.

Proof. 1.12 ()

We can “imagine” that a square with area 2 certainly exists, but the fact that its side length is undefined is certainly unsettling. I don’t know about you, but I would try to “invent” $\sqrt{2}$. We can maybe do this in multiple ways.

1. I write out the decimal expansion one by one, which gives our first exposure to sequences.

$$1, 1.4, 1.41, 1.414, \ldots \tag{68}$$

   It is clear that on $\mathbb{Q}$, this sequence does not converge. Our intuition tells that that if the terms get closer and closer to each other, they must be getting closer and closer to *something*, though that something is not in $\mathbb{Q}$. This motivates the definition for *Cauchy completeness*.

2. I would write out maybe some nested intervals so that $\sqrt{2}$ *must* lie within each interval.

$$[1, 2] \supset [1.4, 1.5] \supset [1.41, 1.42] \supset \ldots \tag{69}$$

   This motivates the definition of *nested-interval completeness*.

3. I would define the set of all rationals such that $x^2 < 2$, and try to define $\sqrt{2}$ as the max or supremum of this set. We will quickly find that neither the max nor the supremum exists in $\mathbb{Q}$, and this motivates the definition for *Dedekind completeness*.

All three of these methods points at the same intuition that there should not be any "gaps" or "missing points" in the set that we will construct to be $\mathbb{R}$. This contrasts with the rational numbers, whose corresponding number line has a "gap" at each irrational value.

1.2.1 Dedekind Completeness

Definition 1.9 (Dedekind Cut)

A **Dedekind cut** is a partition of the rationals $\mathbb{Q} = A \sqcup A'$ satisfying the three properties.[a]

1. $A \neq \emptyset$ and $A \neq \mathbb{Q}$.[b]
2. $x < y$ for all $x \in A, y \in A'$.
3. The maximum element of A does not exist in $\mathbb{Q}$.

The minimum of A' may exist in $\mathbb{Q}$, and if it does, the cut is said to be **generated** by $\min A'$.

Note that in $\mathbb{Q}$, there will be two types of cuts:

1. ones that are generated by rational numbers, such as

$$A = \{x \in \mathbb{Q} \mid x < 2/3\}, A' = \{x \in \mathbb{Q} \mid x \geq 2/3\} \tag{70}$$

2. and the ones that are not

$$A = \{x \in \mathbb{Q} \mid x^2 < 2\}, A' = \{x \in \mathbb{Q} \mid x^2 \geq 2\} \tag{71}$$

We can intuitively see that the set of all Dedekind cuts (A, A') will “extend” the rationals into a bigger set. We can then define some operations and an order to construct this into an ordered field, and finally it will have the property that we call “completeness.”

Definition 1.10 (Dedekind Completeness)

A totally ordered algebraic field $\mathbb{F}$ is **complete** if every Dedekind cut of $\mathbb{F}$ is generated by an element of $\mathbb{F}$.

[a] This can really be defined for any totally ordered set.
[b] By relaxing this property, we can actually complete $\mathbb{Q}$ to the extended real number line.

Theorem 1.11 ()

$\mathbb{Q}$ is not Dedekind-complete.

Proof. 1.13 ()

The counter-example is given above for the cut

$$A = \{x \in \mathbb{Q} \mid x^2 < 2\}, A' = \{x \in \mathbb{Q} \mid x^2 \geq 2\} \tag{72}$$

Now we have the tools to define the reals, giving us the beefy theorem.

Theorem 1.12 (Reals as the Dedekind-Completion of Rationals)

Let $\mathbb{R}$ be the set of all Dedekind cuts (A, A') of $\mathbb{Q}$ of $\mathbb{Q}$. For convenience we can uniquely represent (A, A') with just A since $A' = \mathbb{Q} \setminus A$. By doing this we can intuitively think of a real number as being represented by the set of all smaller rational numbers. Let A, B be two Dedekind cuts. Then, we define the following order and operations.

1. *Order.* $A \leq_{\mathbb{R}} B \iff A \subset B$.
2. *Addition.* $A +_{\mathbb{R}} B \coloneqq \{a +_{\mathbb{Q}} b \mid a \in A, b \in B\}$.
3. *Additive Identity.* $0_{\mathbb{R}} \coloneqq \{x \in \mathbb{Q} \mid x < 0\}$.
4. *Additive Inverse.* $-B \coloneqq \{a - b \mid a < 0, b \in (\mathbb{Q} \setminus B)\}$.
5. *Multiplication.* If $A, B \geq 0$, then we define $A \times_{\mathbb{R}} B \coloneqq \{a \times_{\mathbb{Q}} b \mid a \in A, b \in B, a, b \geq 0\} \cup 0_{\mathbb{R}}$. If A or B is negative, then we use the identity $A \times B = -(A \times_{\mathbb{R}} -B) = -(-A \times_{\mathbb{R}} B) = (-A \times_{\mathbb{R}} -B)$ to convert A, B to both positives and apply the previous definition.
6. *Multiplicative Identity.* $1_{\mathbb{R}} = \{x \in \mathbb{Q} \mid x < 1\}$.
7. *Multiplicative Inverse.* If $B > 0$, $B^{-1} \coloneqq \{a \times_{\mathbb{Q}} b^{-1} \mid a \in 1_{\mathbb{R}}, b \in (\mathbb{Q} \setminus B)\}$. If B is negative, then we compute $B^{-1} = -((-B)^{-1})$ by first converting to a positive number and then applying the definition above.

We claim that $(\mathbb{R}, +_{\mathbb{R}}, \times_{\mathbb{R}}, \leq_{\mathbb{R}})$ is a totally ordered field, and the canonical injection $\iota : \mathbb{Q} \to \mathbb{R}$ defined

$$\iota(q) = \{x \in \mathbb{Q} \mid x < q\} \tag{73}$$

is an ordered field isomorphism. Finally, by construction $\mathbb{R}$ is Dedekind-complete.

Definition 1.11 (Least Upper Bound Property)

A totally ordered algebraic field $\mathbb{F}$ (must it be a field?) is complete if every nonempty set of F having an upper bound must have a least upper bound (supremum) in F.

Theorem 1.13 (Dedekind Completeness Equals Least-Upper-Bound Property)

Dedekind completeness is equivalent to the least upper bound property.

Proof. 1.14 ()

Definition 1.12 (Archimidean Principle)

An ordered ring $(X, +, \cdot, \leq)$ that embeds the naturals $\mathbb{N}^a$ is said to obey the **Archimedean principle** if given any $x, y \in X$ s.t. $x, y > 0$, there exists an $n \in \mathbb{N}$ s.t. $\iota(n) \cdot x > y$. Usually, we don't care about the canonical injection and write $nx > y$.

By the canonical injections $\mathbb{N} \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{R}$, we can talk about whether this set has the Archimedean property. In fact Dedekind completeness does imply it.

Theorem 1.14 ()

$\mathbb{R}$ satisfies the Archimedean principle.

Proof. 1.15 ()

Assume that this property doesn't hold. Then for any fixed x, $nx < y$ for all $n \in \mathbb{N}$. Consider the set

$$A = \bigcup_{n \in \mathbb{N}} (-\infty, nx), \quad B = \mathbb{R} \setminus A \tag{74}$$

A by definition is nonempty, and B is nonempty since it contains y. Then, we can show that $a \in A, b \in B \implies a < b$ using proof by contradiction. Assume that there exists $a' \in A, b' \in B$ s.t. $a' > b'$. Since $a' \in A$, there exists a $n' \in \mathbb{N}$ s.t. $a' \in (-\infty, n'x) \iff a' < n'x$. But by transitivity of order, this means $b' < n'x \iff b' \in (-\infty, n'x) \implies b' \in A$.
Going back to the main proof, we see that A is upper bounded by y, and so by the least upper bound property it has a supremum $z = \sup A$.

1. If $z \in A$, then by the induction principle[a] $z + x \in A$, contradicting that z is an upper bound.
2. If $z \notin A$, then by the induction principle[b] $z - x \notin A \implies z - x \in B$. Since every element of B upper bounds A and since $x > 0$, this means that $z - x < z$ is a smaller upper bound of A, contradicting that z is a least upper bound.

Therefore, it must be the case that $nx > y$ for some $n \in \mathbb{N}$.

1.2.2 Cauchy Completeness

Definition 1.13 (Cauchy Sequence)

A sequence a_n in a metric space (X, d) is a **Cauchy sequence** if for every $\epsilon > 0$, there exists an N s.t.

$$d(a_i, a_j) < \epsilon \tag{75}$$

for every $i, j > N$. We call this **Cauchy convergence**.

Note that it is not sufficient to say that a sequence is Cauchy by claiming that each term becomes arbitrarily close to the preceding term. That is,

$$\lim_{n \to \infty} |x_{n+1} - x_n| = 0 \tag{76}$$

For example, look at the sequence

$$a_n = \sqrt{n} \implies a_{n+1} - a_n = \frac{1}{\sqrt{n+1} + \sqrt{n}} < \frac{1}{2\sqrt{n}} \tag{77}$$

However, it is clear that a_n gets arbitrarily large, meaning that a finite interval can contain at most a finite number of terms in $\{a_n\}$.

It is trivial that convergence implies Cauchy convergence, but the other direction is not true. Therefore, we would like to work in a space where these two are equivalent, and this is called completeness.

[a] as in, there exists an ordered ring homomorphism $\iota : \mathbb{N} \to X$
[a] Note that $\mathbb{N}$ is defined recursively as $1 \in \mathbb{N}$ and if $n \in \mathbb{N}$, then $n + 1 \in \mathbb{N}$.
[b] The contrapositive of the recursive definition of $\mathbb{N}$ is: if $n \notin \mathbb{N}$, then $n - 1 \notin \mathbb{N}$.

Definition 1.14 (Cauchy Completeness)

A metric space (X, d) is complete if every Cauchy sequence in that space converges to an element in X.

Theorem 1.15 ()

$\mathbb{Q}$ is not Cauchy-complete.

Proof. 1.16 ()

Let a_n be the largest number x up to the nth decimal expansion such that x^2 does not exceed 2. The first few terms are

$$1.4, 1.41, 1.414, \ldots \tag{78}$$

Therefore, we can construct the reals as equivalence classes over Cauchy sequences. Rather than using the order, we take advantage of the metric.

Theorem 1.16 (Reals as the Cauchy-Completion of the Rationals)

Let $\mathbb{R}$ be the quotient space of all Cauchy sequences (x_n) of rational numbers with the equivalence relation $(x_n) = (y_n)$ iff their difference tends to 0.[a] That is, for every rational $\epsilon > 0$, there exists an integer N s.t. for all naturals $n > N$, $|x_n - y_n| < \epsilon$.

1. *Order.* $(x_n) \leq_{\mathbb{R}} (y_n)$ iff $x = y$ or there exists $N \in \mathbb{N}$ s.t. $x_n \leq_{\mathbb{Q}} y_n$ for all $n > N$.
2. *Addition.* $(x_n) + (y_n) \coloneqq (x_n + y_n)$.
3. *Additive Identity.* $0_{\mathbb{R}} \coloneqq (0_{\mathbb{Q}})$.
4. *Additive Inverse.* $-(x_n) \coloneqq (-x_n)$.
5. *Multiplication.* $(x_n) \times_{\mathbb{R}} (y_n) = (x_n \times_{\mathbb{Q}} y_n)$.
6. *Multiplicative Identity.* $1_{\mathbb{R}} \coloneqq (1)$.
7. *Multiplicative Inverse.* $(x_n)^{-1} \coloneqq (x_n^{-1})$.

We claim that $(\mathbb{R}, +_{\mathbb{R}}, \times_{\mathbb{R}}, \leq_{\mathbb{R}})$ is a totally ordered field, and the canonical injection $\iota : \mathbb{Q} \to \mathbb{R}$ defined

$$\iota(q) = (q) \tag{79}$$

is an ordered field isomorphism. Finally, by construction $\mathbb{R}$ is Cauchy-complete.

1.2.3 Nested Intervals Completeness

The final way we prove is using nested-intervals completeness.

Definition 1.15 (Nested Interval Completeness, Cantor's Intersection Theorem)

Let F be a totally ordered algebraic field. Let $I_n = [a_n, b_n]$ $(a_n < b_n)$ be a sequence of closed intervals, and suppose that these intervals are nested in the sense that

$$I_1 \supset I_2 \supset I_3 \supset \ldots$$

where

$$\lim_{n \to +\infty} b_n - a_n = 0$$

[a] This equivalence class reflects that the same real number can be approximated in many different sequences. In fact, this shows *by definition* that $1, 1, \ldots$ and $0.9, 0.99, 0.999, \ldots$ are the same number!

F is complete if the intersection of all of these intervals I_n contains exactly one point. That is,

$$\bigcup_{n=1}^{\infty} I_n \in F$$

Note that defining nested intervals requires only an ordered field. One may look at this and try to ask if this is a specific instance of the following conjecture: The intersection of a nested sequence of nonempty closed sets in a topological space has exactly 1 point. This claim may not even make sense, actually. If we define nested in terms of proper subsets, then for a finite topological space a sequence cannot exist since we will run out of open sets and so this claim is vacuously true and false. If we allow $S_n = S_{n+1}$ then we can just select $X \supset X \supset \ldots$, which is obviously not true. However, a slightly weaker claim is that every proper nested non-empty closed sets has a non-empty intersection is a consequence of compactness.

Theorem 1.17 ()

$\mathbb{Q}$ is not nested-interval complete.

Proof. 1.17 ()

Consider the intervals $[a_i, b_i]$ where a_i is the largest number x up to the nth decimal expansion such that x^2 does not exceed 2, and b_i is the smallest number x up to the nth decimal expansion such that x^2 is not smaller than 2. The first few terms are

$$[1.4, 1.5], [1.41, 1.42], [1.414, 1.415], \ldots \tag{80}$$

Therefore, we can complete $\mathbb{Q}$. It turns out that this is equivalent to the construction using Dedekind cuts, and by definition this new set is nested interval complete. However, like Cauchy completeness, this actually does not imply the Archimedean property.

1.2.4 Properties of the Real Line

Now that we have completed it, we can define the real numbers.

Definition 1.16 (The Real Numbers)

The **set of real numbers**, denoted $\mathbb{R}$, is a totally ordered complete Archimedean field.

It seems that the real numbers is *any* set that satisfies the definition above. Does this mean that there are multiple real number lines?

Example 1.2 (Multiple Reals?)

For example, let us construct three distinct sets satisfying these axioms:

1. A line $\mathbb{L}$ with $+$ associated with the translation of $\mathbb{L}$ along itself and $\cdot$ associated with the "stretching/compressing" of the line around the additive origin 0.
2. An uncountable list of numbers with possibly infinite decimal points, known as the decimal number system.
$$\ldots, -2.583\ldots, \ldots, 0, \ldots, 1.2343\ldots, \ldots, \sqrt{2}, \ldots \tag{81}$$
3. A circle with a point removed, with addition and multiplication defined similarly as the line.

We will show that there is only one set, up to isomorphism, that satisfies all these properties.

Theorem 1.18 (Uniqueness)

$\mathbb{R}$ is unique up to field isomorphism. That is, if two individuals construct two ordered complete Archimedean fields $\mathbb{R}_A$ and $\mathbb{R}_B$, then

$$\mathbb{R}_A \simeq \mathbb{R}_B \tag{82}$$

Proof. 1.18 ()

The proof is actually much longer than I expected, so I draw a general outline.[a] We want to show how to construct an isomorphism $f : \mathbb{R}_A \to \mathbb{R}_B$.

1. Realize that there are unique embeddings of $\mathbb{N}$ in $\mathbb{R}_A$ and $\mathbb{R}_B$ that preserve the inductive principle, the order, closure of addition, and closure of multiplication, the additive identity, and the multiplicative identity. Call these ordered doubly-monoid (since it's a monoid w.r.t. $+$ and $\times$) homomorphisms ι_A, ι_B.
2. Construct an isomorphism $f_1 : \iota_A(\mathbb{N}) \to \iota_B(\mathbb{N})$ that preserves the inductive principle, order, addition, and multiplication. This is easy to do by just constructing $f_1 = \iota_B \circ \iota_A^{-1}$.
3. Extend f_1 to the ordered ring isomorphism f_2 by explicitly defining what it means to map additive inverses, i.e. negative numbers.
4. Extend f_2 to the ordered field isomorphism f_3 by explicitly defining what it means to map multiplicative inverses, i.e. reciprocals.
5. Extend f_3 to the ordered field isomorphism on the entire domain $\mathbb{R}_A$ and codomain $\mathbb{R}_B$. There is no additional operations that we need to support, but we should explicitly show that this is both injective and surjective, which completes our proof.

Corollary 1.1 ()

Let $\mathbb{R}_D, \mathbb{R}_C$ be the Dedekind and Cauchy completion of $\mathbb{Q}$. Then $\mathbb{R}_D \simeq \mathbb{R}_C$.

The second new property is that the reals are uncountable.

Theorem 1.19 (Cantor's Diagonalization)

The real numbers are uncountable.

Proof. 1.19 ()

We

Provide examples of ordered, Cauchy-complete fields that are not Archimedean.

Theorem 1.20 (Existence of Nth Roots)

Is this true for any complete field?

Theorem 1.21 (Denseness)

1.3 The Extended Reals and Hyperreals

Great! We have officially constructed the reals, and we can finally feel satisfied about defining metrics, norms, and inner products as mappings to the codomain $\mathbb{R}$. Now let's make the concept of infinite numbers

[a] Followed from here.

a bit more rigorous. In short, what we do is just add the numbers $\pm\infty$ to $\mathbb{R}$, which we call the extended reals, and try and extend the properties from $\mathbb{R}$ to the extended reals. We will see that not all properties can be transferred.

Theorem 1.22 (Extended Real Number Line)

The **extended real number line** is the set $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$. We define the following operations.

1. *Order.* $-\infty \leq x$ and $x \leq +\infty$ for all $x \in \overline{\mathbb{R}}$.
2. *Addition.* $+\infty - \infty = 0$. $x + \infty = +\infty$ and $x - \infty = -\infty$ for all $x \in \mathbb{R}$.
3. *Multiplication.*

$$x \times \infty = \begin{cases} +\infty & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -\infty & \text{if } x < 0 \end{cases} \tag{83}$$

It turns out that this is still Dedekind-complete, which is nice. Unfortunately we lose a lot of structure.

1. this is not even a field since the multiplicative inverse of $\pm\infty$ is not defined.
2. the Archimedean principle does not hold
3. we cannot define a metric nor a norm.
4. we can define the order topology, however.

The loss of the field property is quite bad, and we might want to recover this. Therefore, we can add more elements that serve to be the multiplicative inverse of infinity. We call these inverses *infinitesimals* and the new set the *hyperreal numbers*.

Theorem 1.23 (Hyperreals)

The **hyperreals**

In fact, when Newton first invented calculus, the hyperreals were what he worked with, and you can surprisingly build a good chunk of calculus with this. Even though this is a dead topic at this point, a lot of modern notation is based off of this number system, so it's good to see how it works. For example, when we write the integral

$$\int f(x)\, dx \tag{84}$$

we are saying that we are taking the uncountable sum of the terms $f(x)\,dx$, the multiplication of the real number $f(x)$ and the infinitesimal number dx living in the hyperreals. Unfortunately, we cannot fully construct a rigorous theory of calculus with only infinitesimals. However, in practice (especially physics) people tend to manipulate and do algebra with infinitesimals, so having a good foundation on what you can and cannot do with them is practical. While the focus won't be on *smooth infinitesimal analysis (SIA)*, I will include some alternate constructions later purely with infinitesimals.

1.4 Complex Numbers

The next field that will be particularly important is the complex numbers. It is straightforward to construct $\mathbb{C}$, but let's motivate this for a minute.

Example 1.3 (Polynomial Roots)

The roots of the polynomial

$$f(x) = x^2 + 1 \tag{85}$$

does not exist in $\mathbb{R}$.

Therefore, we would like to construct a new space that contains all possible roots for all possible polynomials with real coefficients. We call this $\mathbb{C}$. Clearly, by constructing polynomials of the form $x^2 - r^2$ for some $r \in \mathbb{R}$, we know that $\mathbb{R} \subset \mathbb{C}$. Therefore, we want to create a further extension of $\mathbb{R}$, along with some canonical injection $\iota : \mathbb{R} \to \mathbb{C}$ that is also a field homomorphism. It turns out that once we construct this field, there is no possible way that we can make it an ordered field. However, the norm extends naturally into $\mathbb{C}$ such that ι is isometric. Finally, we can define a new operator called *conjugation* that gives us additional structure.

This is not the only way to construct the complex plane however. Rather than defining all these from scratch, we could just define the addition operations with an isometric vector space isomorphism from $\mathbb{R}^2$ to $\mathbb{C}$ actually, and then define multiplication. Another way is to start again with $\mathbb{Q} \times \mathbb{Q}$, define a norm on it, complete it, and finally define the addition and multiplication operations that satisfy the field property.

Theorem 1.24 (Construction of the Complex Numbers)

Let $\mathbb{C}$ be defined as the space $\mathbb{R} \times \mathbb{R}$ with the following operations.
1. *Addition.* $x = (a, b), y = (c, d) \implies x +_{\mathbb{C}} y = (a + c, b + d)$.
2. *Additive Identity.* $0_{\mathbb{C}} = (0, 0)$.
3. *Additive Inverse.* $x = (a, b) \implies -x = (-a, -b)$.
4. *Multiplication.* $x = (a, b), y = (c, d) \implies x \times_{\mathbb{C}} y = (ac - bd, ad + bc)$.
5. *Multiplicative Identity.* $1_{\mathbb{C}} = (1, 0)$.
6. *Multiplicative Inverse.*

$$x = (a, b) \implies x^{-1} = \left(\frac{a}{a^2 + b^2}, \frac{-b}{a^2 + b^2} \right) \tag{86}$$

Our first claim is that $(\mathbb{C}, +_{\mathbb{C}}, \times_{\mathbb{C}})$ is a field. Furthermore, we define the additional structures
1. *Conjugate.* $x = (a, b) \implies \overline{x} = (a, -b)$.
2. *Norm.* $|x|_{\mathbb{C}} = x \times_{\mathbb{C}} \overline{x} = a^2 + b^2$.
3. *Metric.* This is the norm-induced metric. $d_{\mathbb{C}}(x, y) = |x - y|_{\mathbb{C}}$.
4. *Topology.* This is the metric-induced topology generated by the open balls $B(x, r) := \{y \in \mathbb{C} \mid d(x, y) < r\}$, where $x \in \mathbb{C}, r \in \mathbb{R}$.

Our second claim is that the canonical injection $\iota : \mathbb{R} \to \mathbb{C}$ defined

$$\iota(r) = (r, 0) \tag{87}$$

is an isometric field isomorphism. Our third claim is that $\mathbb{C}$ is Cauchy-complete with respect to this metric.

Note that we do not talk about order $\mathbb{C}$, and so the concepts of Dedekind completeness, least upper bound properties, or Archimedean principle is meaningless in the complex plane.

Definition 1.17 (Imaginary Number)

Let us denote $i = (0, 1)$ which we call the **imaginary number**, which has the property that $i^2 = 1$. With this notation, we can see through abuse of notation that

$$(a, b) = (a, 0) + (0, b) = (a, 0) + (b, 0)(0, 1) = a + bi \tag{88}$$

Therefore, we generally write complex numbers as $z = a + bi$, and we define the real and imaginary components as $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$, respectively.

Theorem 1.25 (Properties of Conjugation)

Conjugation is an isometric field automorphism of $\mathbb{C}$.

Proof. 1.20 ()

1.4.1 Properties of the Complex Numbers

1.5 Dual Numbers

Another similar number system.

1.6 Euclidean Space

Congratulations! We have rigorously constructed both the reals and complex numbers, and this becomes the cornerstone to construct other fundamental sets. Now we consider spaces of the form $\mathbb{R}^n$ or $\mathbb{C}^n$, which we call *Euclidean spaces*, and construct them. This is actually quite easy since we understand linear algebra.

Definition 1.18 (Convex Sets)

A set S is convex if for every point $x, y \in S$, the point

$$z = tx + (1-t)y \in S \tag{89}$$

where $0 \leq t \leq 1$.

1.7 Exercises

Exercise 1.1 (Rudin 1.1)

If r is rational ($r \neq 0$) and x is irrational, prove that $r + x$ and rx are irrational.

Solution 1.1

If we assume that $rx = t$ and $r + x = s$ are rational, then this violates the field axioms of $\mathbb{Q}$ since then $x = tr^{-1}$ and $x = s + (-r)$ are rational.

Exercise 1.2 (Rudin 1.2)

Prove that there is no rational number whose square is 12.

Solution 1.2

Assume that there exists a number p/q such that p and q are both not even. Then,

$$\left(\frac{p}{q}\right)^2 = 12 \implies p^2 = 12q^2 = 3(2q)^2$$

So p much be even $p = 2p'$. Therefore, $p'^2 = 3q^2$, and q must be odd. This means that p' must be odd. We can rewrite the equation

$$p'^2 - q^2 = 2q^2 \implies (p' + q)(p' - q) = 2q^2$$

where the left hand side is divisible by 4 but the right hand side is divisible by at most 2, leading to a contradiction.

Exercise 1.3 (Rudin 1.3)

Prove that the axioms of multiplication imply the following.

1. If $x \neq 0$ and $xy = xz$, then $y = z$.
2. If $x \neq 0$ and $xy = x$, then $y = 1$.
3. If $x \neq 0$ and $xy = 1$, then $y = x^{-1}$.
4. If $x \neq 0$, then $(x^{-1})^{-1} = x$.

Solution 1.3

Listed.

1. $xy = xz \implies \frac{1}{x} \cdot xy = \frac{1}{x} xz \implies y = z$
2. $xy = x \implies \frac{1}{x} xy = \frac{1}{x} x \implies y = 1$
3. $xy = 1 \implies \frac{1}{x} xy = \frac{1}{x} 1 \implies y = \frac{1}{x}$
4. $(x^{-1})^{-1} \cdot x^{-1} = 1 \implies (x^{-1})^{-1} \cdot x^{-1} \cdot x = 1 \cdot x \implies (x^{-1})^{-1} = x$

Exercise 1.4 (Rudin 1.4)

Let E be a nonempty subset of an ordered set; suppose α is a lower bound of E and β is an upper bound of E. Prove that $\alpha \leq \beta$.

Solution 1.4

Since E is nonempty, we choose any $x \in E$. By definition, $\alpha \leq x$ and $x \leq \beta$, and by transitive property of orderings, we have $\alpha \leq \beta$.

Exercise 1.5 (Rudin 1.5)

Let A be a nonempty set of real numbers which is bounded below. Let $-A$ be the set of all numbers $-x$, where $x \in A$. Prove that

$$\inf A = -\sup(-A)$$

Solution 1.5

We would like to prove that $\inf A \leq -\sup(-A)$ and $\inf A \geq -\sup(-A)$. For the first part, we start off with the definition of the infimum.

$$\begin{aligned}
\inf A \leq x \; \forall x \in A &\implies -\inf A \geq -x \; \forall x \in A \\
&\implies -\inf A \geq x \forall x \in -A \\
&\implies -\inf A \geq \sup(-A) \\
&\implies \inf A \leq -\sup(-A)
\end{aligned}$$

For the second part, we start with the definition of the supremum.

$$\begin{aligned}
\sup(-A) \geq x \forall x \in -A &\implies \sup(-A) \geq -x \forall x \in A \\
&\implies -\sup(-A) \geq x \forall x \in A \\
&\implies -\sup(-A) \leq \inf A
\end{aligned}$$

Exercise 1.6 (Rudin 1.8)

Prove that no order can be defined in the complex field that turns it into an ordered field.

Solution 1.6

Note that if $x \geq 0$, then $-x \leq 0$ for all x of any ordered field. Since if $x \geq 0$ and $-x > 0$, then $x - x > 0$, which is absurd. Therefore, one of either i or $-i$ should be greater than 0. But $i^2 = (-i)^2 = -1$, so this means that $-1 > 0$, which implies that $0 < 1$. But either 1 or -1 must ≥ 0.

Exercise 1.7 (Rudin 1.9)

Equip $\mathbb{C}$ with the dictionary order. That is, given $z = a + bi$ and $w = c + di$, $z < w$ if $a < c$, or if $a = c$ and $b < d$. Does this ordered set have a least upper bound property?

Solution 1.7

No it does not. Consider the set $S = \{a + bi \in \mathbb{C} \mid a \leq 3\}$. S is bounded by 4, but it doesn't have a least upper bound. Given any $3+bi$, this is not an upper bound since we can construct $3+(b+\epsilon)i \in S$. Given any $a + bi$ where $a > 3$, we can always find a lower bound of form $a + (b - \epsilon)i$ that also bounds S.

Exercise 1.8 (Rudin 1.10)

Suppose $z = a + bi$, $w = u + iv$, and

$$a = \left(\frac{|w| + u}{2}\right)^{1/2} \text{ and } b = \left(\frac{|w| - u}{2}\right)^{1/2}$$

Prove that $z^2 = w$ if $v \geq 0$ and that $(\bar{z})^2 = w$ if $v \leq 0$. Conclude that every complex number (with one exception!) has two complex square roots.

Solution 1.8

We can calculate

$$z^2 = (a^2 - b^2) + 2abi = u + \sqrt{v^2}i = \begin{cases} u + vi & \text{if } v \geq 0 \\ u - vi & \text{if } v \leq 0 \end{cases}$$

Since if we assume $v \geq 0$, then we have $z^2 = w$. We also get

$$\bar{z}^2 = (a^2 - b^2) - 2abi = u - \sqrt{v^2}i = \begin{cases} u - vi & \text{if } v \geq 0 \\ u + vi & \text{if } v \leq 0 \end{cases}$$

and assuming $v \leq 0$, we have $\bar{z}^2 = w$. Therefore, every complex number w has both $\pm z$ as its square root if $v \geq 0$, $\pm\bar{z}$ if $v \leq 0$, and just one root if $z = 0$.

Exercise 1.9 (Rudin 1.11)

If z is a complex number, prove that there exists an $r \geq 0$ and a complex number w with $|w| = 1$ s.t. $z = rw$. Are w and r always uniquely determined by z?

Solution 1.9

If $z = 0$, then $r = 0$ and there is no unique w. If $z = a + bi \neq 0$, then define

$$r = |z| = (a^2 + b^2)^{1/2}, \quad w = \frac{1}{r}z$$

which proves existence. As for uniqueness, assume that there are two forms

$$z = rw = r'w'$$

Then, $w = \frac{r'}{r} w' \implies |w| = \left|\frac{r'}{r}\right| |w'| = 1$, which implies that $r'/r = 1$ and so $r = r'$. This means that $w = w'$.

Exercise 1.10 (Rudin 1.12)

If $z_1, \ldots, z_n$ are complex, prove that

$$|z_1 + z_2 + \ldots + z_n| \leq |z_1| + \ldots + |z_n|$$

Solution 1.10

By induction, it suffices to prove $|z_1 + z_2| \leq |z_1| + |z_2|$. We have

$$\begin{aligned}
|z_1 + z_2|^2 &= (z_1 + z_2)(\overline{z_1 + z_2}) \\
&= (z_1 + z_2)(\bar{z_1} + \bar{z_2}) \\
&= z_1 \bar{z_1} + z_1 \bar{z_2} + z_2 \bar{z_1} + z_2 \bar{z_2} \\
&= |z_1|^2 + |z_2|^2 + z_1 \bar{z_2} + z_2 \bar{z_1} \\
&= |z_1|^2 + |z_2|^2 + 2(ac + bd) \\
&\leq |z_1|^2 + |z_2|^2 + 2\sqrt{a^2 + b^2}\sqrt{c^2 + d^2} \quad \text{(Schwartz)} \\
&= |z_1|^2 + |z_2|^2 + 2|z_1||z_2| \\
&= (|z_1| + |z_2|)^2
\end{aligned}$$

since both sides are positive, we can take their square root to get the desired result.

Exercise 1.11 (Rudin 1.13)

If x, y are complex, prove that

$$\big||x| - |y|\big| \leq |x - y|$$

Solution 1.11

Since both sides are nonnegative, we can square both sides. Note that due to Cauchy Schwartz inequality, $2|x||y| \geq x\bar{y} + y\bar{x}$ since expanding them gives

$$2\sqrt{a^2 + b^2}\sqrt{c^2 + d^2} \geq 2(ac + bd)$$

Therefore, the following inequality is true:

$$|x|^2 + |y|^2 - 2|x||y| \leq x\bar{x} + y\bar{y} - x\bar{y} - y\bar{x}$$

which reduces to form $(|x| - |y|)^2 \leq |x - y|^2$.

Exercise 1.12 (Rudin 1.14)

If z is a complex number s.t. $|z| = 1$, that is such that $z\bar{z} = 1$, compute

$$|1 + z|^2 + |1 - z|^2$$

Solution 1.12

Compute.

$$(1 + z)(1 + \bar{z}) + (1 - z)(1 - \bar{z}) = 1 + z + \bar{z} + z\bar{z} + 1 - z - \bar{z} + z\bar{z} = 4$$

Exercise 1.13 (Rudin 1.15)

Under what conditions does equality hold in the Schwarz inequality?

Solution 1.13

If they are antiparallel, since

$$\langle x, y \rangle = ||x||||y|| \cos \theta$$

Exercise 1.14 (Rudin 1.16)

Suppose $k \geq 3$, $\mathbf{x}, \mathbf{y} \in \mathbb{R}^k$, $|\mathbf{x} - \mathbf{y}| = d > 0$, and $r > 0$. Prove:

a) If $2r > d$, there are infinitely many $\mathbf{z} \in \mathbb{R}^k$ s.t.

$$|\mathbf{z} - \mathbf{x}| = |\mathbf{z} - \mathbf{y}| = r$$

b) If $2r = d$, there is exactly one such $\mathbf{z}$.

c) If $2r < d$, there is no such $\mathbf{z}$.

Exercise 1.15 (Rudin 1.17)

Prove that

$$|\mathbf{x} + \mathbf{y}|^2 + |\mathbf{x} - \mathbf{y}|^2 = 2|\mathbf{x}|^2 + 2|\mathbf{y}|^2$$

Solution 1.14

This is trivial if we simply expand

$$\begin{aligned}
|\mathbf{x} + \mathbf{y}|^2 + |\mathbf{x} - \mathbf{y}|^2 &= \langle \mathbf{x} + \mathbf{y}, \mathbf{x} + \mathbf{y} \rangle + \langle \mathbf{x} - \mathbf{y}, \mathbf{x} - \mathbf{y} \rangle \\
&= \langle \mathbf{x}, \mathbf{x} \rangle + 2\langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{y}, \mathbf{y} \rangle + \langle \mathbf{x}, \mathbf{x} \rangle - 2\langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{y}, \mathbf{y} \rangle \\
&= 2\langle \mathbf{x}, \mathbf{x} \rangle + 2\langle \mathbf{y}, \mathbf{y} \rangle \\
&= 2|\mathbf{x}|^2 + 2|\mathbf{y}|^2
\end{aligned}$$

Exercise 1.16 (Rudin 1.18)

If $k \geq 2$ and $\mathbf{x} \in \mathbb{R}^k$, prove that there exists $\mathbf{y} \in \mathbb{R}^k$ s.t. $\mathbf{y} \neq \mathbf{0}$, but $\mathbf{x} \cdot \mathbf{y} = 0$. Is this also true if $k = 1$?

Solution 1.15

Let $x \in \mathbb{R}^k$ and $\ell \in \mathbb{R}^{k*}$, the dual space. By Riesz representation theorem, we can define the canonical isomorphism $\ell \mapsto y$ between these two spaces as

$$\ell(x) = (x, y)$$

Since $y \neq 0$ by assumption, $\ell \neq 0$, and so its rank is at least 1. Since ℓ maps to $\mathbb{R}$, the rank has to be 1. By rank nullity theorem, we have

$$\dim N(\ell) = k - \text{rank}(\ell) = k - 1$$

and so there exists nontrivial annihilators ℓ of x, which can be mapped to a nontrivial $y \in \mathbb{R}^k$.

Exercise 1.17 (Rudin 1.19)

Suppose $\mathbf{a}, \mathbf{b} \in \mathbb{R}^k$. Find $\mathbf{c} \in \mathbb{R}^k$ and $r > 0$ s.t.

$$|\mathbf{x} - \mathbf{a}| = 2|\mathbf{x} - \mathbf{b}|$$

if and only if $|\mathbf{x} - \mathbf{c}| = r$.

Solution 1.16

If we draw out the circle, it must contain two points on the line drawn by connecting A and B. Since it must be symmetric, its center and radius can then be easily calculated to be

$$r = \frac{2}{3}|b - a|, \quad c = \frac{1}{3}(4b - a)$$

Exercise 1.18 (Zorich 2.2.1)

Using the principle of induction, show that
1. the sum $x_1 + \ldots + x_n$ of real numbers is defined independently of the insertion of parentheses to specify the order of addition.
2. the same is true of the product $x_1 \ldots x_n$
3. $|x_1 + \ldots + x_n| \leq |x_1| + \ldots + |x_n|$
4. $|x_1 \ldots x_n| \leq |x_1| \ldots |x_n|$
5. For any $n, m \in \mathbb{N}$ such that $m < n$, $(n - m) \in \mathbb{N}$.
6. $(1 + x)^n \geq 1 + nx$ for $x > -1$ and $n \in \mathbb{N}$, equality holding for when $n = 1$ or $x = 0$.
7. $(a + b)^n = a^n +_n C_1 a^{n-1} b^1 + \ldots + b^n$ (aka binomial theorem).

Solution 1.17

Listed.
1. Let n denote the number of elements in the sum. We prove by strong law of induction. The base case for when $n = 1, 2, 3$ is trivially true.

$$\begin{aligned} x_1 &= x_1 && \text{(identity)} \\ x_1 + x_2 &= x_1 + x_2 && \text{(identity)} \\ (x_1 + x_2) + x_3 &= x_1 + (x_2 + x_3) && \text{(associativity)} \end{aligned}$$

   Then, the sum of $n = k$ parameters is defined by $k - 2$ pairs of parentheses defining the order of the sum. These parentheses define a sequence of $k - 1$ 2-fold additions. Now, assume that the claim is true for

$$S_n \equiv x_1 + \ldots x_n \text{ for } n = 1, 2, \ldots, k$$

Then, for a specific sum S_{k+1} of $k+1$ elements with $k-1$ parentheses, we can reduce the sum to its final 2-fold addition

$$S_{k+1} \equiv \underbrace{(x_1 + \ldots + x_i)}_{\varphi_1} + \underbrace{(x_{i+1} + \ldots + x_{k+1})}_{\varphi_2}$$

Since $i, k-i+1 < k$, by the strong law φ_1 and φ_2 are independent of the order of sum.

2. Exactly identical to (a).
3. By the triangle inequality $|x_1 + x + 2| \leq |x_1| + |x_2|$. Now, assume for $n = k$ is true. Then, let $S_k = x_1 + \ldots + x_k$, so

$$|x_1 + \ldots + x_k + x_{k+1}| = |S_k + x_{k+1}| \leq |S_k| + |x_{k+1}| \leq \sum_{i=1}^{k+1} |x_i|$$

4. Same as (c).
5. Let us fix m to be any element of $\mathbb{N}$. Then, the base case is for $n = m+1$ (which is in $\mathbb{N}$ since it is inductive), so

$$n - m = (m+1) - m = 1 \in \mathbb{N}$$

Now, given that for some integer $n \geq m+1$, $n - m \in \mathbb{N}$ is true, we have

$$\begin{aligned}
(n+1) - m &= n + (1-m) && \text{(associativity)} \\
&= n + (-m+1) && \text{(commutativity)} \\
&= (n-m) + 1 && \text{(associativity)}
\end{aligned}$$

where $(n-m) + 1 \in \mathbb{N}$ by inductive property of $\mathbb{N}$.

6. We prove by induction. For $n = 1$, it is trivial that $(1+x)^1 \geq 1 + 1 \cdot x$. Now assume that the claim is true for some $k \in \mathbb{N}$. Then,

$$\begin{aligned}
(1+x)^{k+1} = (1+x)^k (1+x) &\geq (1+kx)(1+x) \\
&= 1 + (k+1)x + kx^2 \\
&\geq 1 + (k+1)x
\end{aligned}$$

where equality holds if $x = 0 \implies 1^{k+1} = 1^k \cdot 1 = 1$ or $n = 1 \implies$ trivial case.

7. The base case for $n = 1$ is trivial since $(a+b)^1 = \binom{1}{0} a + \binom{1}{1} b$. We introduce Newton's identity.

$$\begin{aligned}
\binom{k}{j-1} + \binom{k}{j} &= \frac{k!}{(j-1)!(k-j+1)!} + \frac{k!}{j!(k-j)!} \\
&= k! \bigg(\frac{j}{j!(k-j+1)!} + \frac{k-j+1}{j!(k-j+1)!} \bigg) \\
&= k! \cdot \frac{k+1}{j!(k-j+1)!} \\
&= \frac{(k+1)!}{j!(k-j+1)!} = \binom{k+1}{j}
\end{aligned}$$

Now assuming that the binomial formula holds for some $n = k$, we have

$$\begin{aligned}
(a+b)^{k+1} &= (a+b)^k (a+b) \\
&= \left(\sum_{j=0}^{k} \binom{k}{j} a^j b^{k-j} \right) (a+b) \\
&= \sum_{j=0}^{k} \binom{k}{j} a^{j+1} b^{k-j} + \sum_{j=0}^{k} \binom{k}{j} a^j b^{k-j+1} \\
&= \binom{k}{0} a^0 b^{k+1} + \binom{k}{k} a^{j+1} b^0 + \sum_{j=0}^{k-1} \binom{k}{j} a^{j+1} b^{k-j} + \sum_{j=1}^{k} \binom{k}{j} a^j b^{k-j+1} \\
&= \binom{k+1}{0} a^0 b^{k+1} + \binom{k+1}{k+1} a^{j+1} b^0 + \sum_{j=1}^{k} \left[\binom{k}{j-1} + \binom{k}{j} \right] a^j b^{k-j+1} \\
&= \sum_{j=0}^{k+1} \binom{k+1}{j} a^j b^{k-j+1}
\end{aligned}$$

Exercise 1.19 (Zorich 2.2.3)

Show that an inductive set is not bounded above.

Solution 1.18

Assume that a X is a nonempty inductive set that is bounded above. By definition, there exists a number $B \in \mathbb{R}$ such that $\max X < B$. Then, this means that there exists no numbers in $[B, B+1)$. Since X is inductive, this means that there cannot exist any elements of X in the interval $[B-1, B)$, and similarly for the interval $[B-2, B)$, and so on, meaning that if $x \in X$, then $x \notin [B-k, B-k+1)$ for all $k \in \mathbb{Z}$. By the Archimidean principle, this implies that $X = \emptyset$, contradicting our assumption.

Exercise 1.20 (Zorich 2.2.4)

Prove the following.

1. An inductive set is infinite (that is, equipollent with one of its subsets different from itself).
2. The set $E_n = \{x \in \mathbb{N} \mid x \leq n\}$ is finite.

Solution 1.19

Listed.

1. Assume that an inductive set X is finite $\implies$ X is bounded above (we can choose upper bound $B = \max X + 1$). But from 2.2.3, an inductive set cannot be bounded above, contradicting our assumption.
2. It is trivial that $E_1 = \{1\}$ is finite since $\text{card} E_1 = 1$. Now, if for some k, E_k is finite with cardinality e_k, then $\text{card} E_{k+1} = e_k + 1$, which implies finiteness.

Exercise 1.21 (Zorich 2.2.5)

Listed.

1. Let $m, n \in \mathbb{N}$ and $m > n$. Their greatest common divisor $\gcd(m, n) = d \in \mathbb{N}$ can be found in a finite number of steps using the following algorithm of Euclid involving successive divisions

with remainder.

$$\begin{aligned}
m &= q_1 n + r_1 \\
n &= q_2 r_1 + r_2 \\
r_1 &= q_3 r_2 + r_3 \\
\ldots &= \ldots \\
r_{k-2} &= q_k r_{k-1} + r_k \\
r_{k-1} &= q_{k+1} r_k + 0
\end{aligned}$$

Then $d = r_k$

2. If $d = \gcd(m, n)$, one can choose numbers $p, q \in \mathbb{Z}$ such that $pm + qn = d$.

Solution 1.20

Listed.

1.
2. Letting $n = r_0$, notice that the equations above satisfy for $i = 0, 1, \ldots$

$$r_i = q_{i+2} r_{i+1} + r_{i+2} \implies r_i - q_{i+2} r_{i+1} = r_{i+2} \tag{1}$$

Note that the second-to-last equation allows us to write r_k as a linear combination of r_{k-2} and r_{k-1}: $r_k = r_{k-2} - q_k r_{k-1}$. Now by applying (1), we can reduce the above to a linear combination of r_{k-3} and r_{k-2}.

$$\begin{aligned}
r_k &= r_{k-2} - q_k r_{k-1} \\
&= r_{k-2} - q_k (r_{k-3} - q_{k-1} r_{k-3}) \\
&= (1 + q_{k-1} q_k) r_{k-2} - q_k r_{k-3}
\end{aligned}$$

and repeatedly doing this allows us to reduce r_k to a linear combination $q_0 r_0 + q_1 r_1$. By the ring properties of $\mathbb{Z}$, the new linear coefficients are also in $\mathbb{Z}$. Reducing one last time using the first equation in the Euclidean algorithm gives

$$\begin{aligned}
r_k &= q_0 r_0 + q_1 r_1 \\
&= q_0 n + q_1 (m - q_1 n) \\
&= q_1 m + (q_0 - q_1) n \\
&= pm + qn
\end{aligned}$$

Exercise 1.22 (Zorich 2.2.9)

Show that if the natural number n is not of the form k^m, where $k, m \in \mathbb{N}$, then the equation $x^m = n$ has no rational roots.

Solution 1.21

Assume that there is a rational solution $x = p/q$, with $p, q \in \mathbb{N}$ of the equation. Then,

$$\left(\frac{p}{q}\right)^m = \frac{p^m}{q^m} = n \implies p^m = q^m n$$

By the fundamental theorem of arithmetic, the exponents of the prime factors of p^m must all be multiples of m, and so it must be so for the right hand side $\implies$ x must be of form $x = k^m$ for some k. This is a contradiction.

Exercise 1.23 (Zorich 2.2.12)

Knowing that $\frac{m}{n} \equiv m \cdot n^{-1}$ by definition, where $m \in \mathbb{Z}$ and $n \in \mathbb{N}$, derive the "rules" for addition, multiplication, and division of fractions, and also the condition for two fractions to be equal.

Solution 1.22

We can construct a $\mathbb{Q}$ as a quotient space $\mathbb{Z} \times \mathbb{N}/\sim$, where $\sim$ is an equivalent relation where

$$(q_1, p_1) \sim (q_2, p_2) \text{ iff } q_1 p_2 = p_1 q_2$$

which is the familiar equivalence relation from "simplifying" a fraction. We define addition and multiplication as the following

$$\begin{aligned}(a, b) + (c, d) &= (ad + bc, bd) \\ (a, b) \cdot (c, d) &= (ac, bd)\end{aligned}$$

which turns out to be algebraically closed in $\mathbb{Q}$. The additive identity is the equivalence class $0 = \{(0, c) \mid c \in \mathbb{N}\}$, and the multiplicative identity is the equivalence class $1 = \{(c, c) \mid c \in \mathbb{N}\}$. It is easy to check that $+$ is commutative, the additive inverse is $-(a, b) = (-a, b)$, and the multiplicative inverse is $(a, b)^{-1} = (b, a)$. We can subtract and divide these elements of $\mathbb{Q}$, called "fractions," as such:

$$\begin{aligned}(a, b) - (c, d) &= (a, b) + (-(c, d)) = (a, b) + (-c, d) = (ad - bc, bd) \\ (a, b) \div (c, d) &= (a, b) \cdot (c, d)^{-1} = (a, b) \cdot (d, c) = (ad, bc)\end{aligned}$$

Exercise 1.24 (Zorich 2.2.13)

Verify that the rational numbers $\mathbb{Q}$ satisfy all the axioms for real numbers except for the axiom of completeness.

Solution 1.23

From continuing the steps of 2.2.14, we can prove $\mathbb{Q}$ is an algebraic field (associativity, commutativity of addition and multiplication, along with distributive property).We can actually define the order relation $\leq_{\mathbb{Q}}$ in two ways:

1. $(a, b) \leq (c, d)$ iff $ad \leq_{\mathbb{Z}} bc$, where $\leq_{\mathbb{Z}}$ is the order relation on $\mathbb{Z}$ (which can be defined much more simply).
2. Recognizing that $\mathbb{Q} \subset \mathbb{R}$, we define the canonical injection map $i : \mathbb{Q} \longrightarrow \mathbb{R}$ and by abuse of language, endow the relation $\leq_{\mathbb{Q}}$ as the restriction of $\leq_{\mathbb{R}}$ onto $\mathbb{Q}$. That is, for $(a, b), (c, d) \in \mathbb{Q}$,

$$(a, b) \leq_{\mathbb{Q}} (c, d) \text{ iff } i(a, b) \leq_{\mathbb{R}} i(c, d)$$

The ordering for the 1st step can be checked for consistency.

1. $(a, b) \leq (a, b)$ since $ab \leq ab$ (true in $\mathbb{Z}$)
2. $(a, b) \leq (c, d), (c, d) \leq (a, b)$ means that $ad \leq bc$ and $bc \leq ad \implies ad = bc$ (true in $\mathbb{Z}$)
3. $(a, b) \leq (c, d) \leq (e, f)$ implies $ad \leq bc, cf \leq de$. Multiplying positive (important that $f > 0$!) to the first inequality gives $adf \leq bcf$, and multiplying positive b to the second gives $bcf \leq bde$, and by interpreting $\leq$ as the ordering defined on $\mathbb{Z}$, we use transitive property of $\leq_{\mathbb{Z}}$ to get $adf \leq bde \implies af \leq be \iff (a, b) \leq (e, f)$.
4. For any $(a, b), (c, d) \in \mathbb{Q}$, $(a, b) \leq (c, d)$ or $(a, b) \geq (c, d)$, which is equivalent to $ad \leq bc$ or $ad \geq bc$, which is true in $\mathbb{Z}$.

It is easy to prove $(a, b) \leq (c, d) \implies (a, b) + (p, q) \leq (c, d) + (p, q)$, and $0_{\mathbb{Q}} \leq (a, b), (c, d) \implies 0_{\mathbb{Q}} \leq (a, b) \cdot (c, d)$. However, $\mathbb{Q}$ is **not** complete. We prove this by showing that the subset $X = \{x \in$

$\mathbb{Q} \mid x^2 \leq 2\} \subset \mathbb{Q}$ does not satisfy the least upper bound property. Assume that there is a least upper bound $c \in \mathbb{Q}$. $c \neq \sqrt{2}$ (you should know how to prove irrationality of $\sqrt{2}$!), we have either $c > \sqrt{2}$ or $c < \sqrt{2}$.

1. Let $c < \sqrt{2} \iff c - \sqrt{2} > 0$. By the Archimedean principle, there exists a $k \in \mathbb{N}$ such that $0 < \frac{1}{k} < c - \sqrt{2}$. Then, $\frac{1}{k} \in \mathbb{Q}$ and $\mathbb{Q}$ is a field, so $c - \frac{1}{k} \in \mathbb{Q}$.
$$c - \frac{1}{k} < c - c + \sqrt{2} = \sqrt{2}$$
So c is not least and so it must be the case that $c < \sqrt{2}$.
2. Let $c < \sqrt{2} \iff \sqrt{2} - c > 0$. By the Archimedean principle, there exists a $k \in \mathbb{N}$ such that $0 < \frac{1}{k} < \sqrt{2} - c$. Then, $c + \frac{1}{k} \in \mathbb{Q}$ and
$$c + \frac{1}{k} < c + (\sqrt{2} - c) = c$$
So c is not an upper bound.

Note that given a well-defined $c = \sup X$ and in the case where $c < \sqrt{2}$, we have $2 - c^2 > 0$, so we can choose a well-defined δ satisfying (by Archimedean principle)
$$0 < \delta < \min\left\{1, \frac{2 - c^2}{2c + 1}\right\}$$
which gives us
$$\begin{aligned} (c + \delta)^2 &= c^2 + \delta(2c + \delta) \\ &< c^2 + \delta(2c + 1) && (\delta < 1) \\ &< c^2 + (2 - c^2) = 2 \end{aligned}$$
meaning that c is not an upper bound. Similarly for when $c > \sqrt{2}$.

Exercise 1.25 (Zorich 2.2.15)

Prove the equivalence of these two statements.

1. If X and Y are nonempty sets of $\mathbb{R}$ having the property that $x \leq y$ for every $x \in X, y \in Y$, then there exists $c \in \mathbb{R}$ such that $x \leq c \leq y$ for all $x \in X$ and $y \in Y$.
2. Every set $X \subset \mathbb{R}$ that is bounded above has a least upper bound.

Solution 1.24

Let S_1 be the first statement and S_2 the second.

1. ($S_2 \implies S_1$). Let $X \subset \mathbb{R}$ be a set that is bounded above, and Y is a set such that $x \leq y$ for all $x \in X, y \in Y$. Then, by LUB principle, there exists $c = \sup X \in \mathbb{R}$. Now, we claim that $c \leq y$ for all $y \in Y$. Assume it doesn't: then there exists $y' \in Y$ such that $y' < c$. But since we assumed $x \leq y$ for all $x \in X, y \in Y$, we have $x \leq y'$ for all $x \in X$, which means that y' is an upper bound of X. But $y' < c$, contradicting the given fact that c was the least upper bound.
2. ($S_1 \implies S_2$). Given a nonempty set $X \subset \mathbb{R}$, we wish to show the existence of $\sup X$. We are guaranteed the existence of nonempty set $Y \subset \mathbb{R}$ such that $x \leq y$ for all $x \in X, y \in Y$, which implies that X must be bounded above. Then, by S_1, there must exist a $c \in \mathbb{R}$ such that
$$x \leq c \leq y \text{ for all } x \in X, y \in Y$$
We claim that $c = \sup X$. It is an upper bound of X since $x \leq c$ for all $x \in X$. It is least since the set of all upper bounds of X is Y, and $c \leq y$ for all $y \in Y$.

Exercise 1.26 (Olmsted 1.15)

Prove **Dedekind's Theorem**: Let the real numbers be divided into two nonempty sets A and B such that (i) if $x \in A$ and if $y \in B$, then $x < y$ and (ii) if $x \in \mathbb{R}$ then either $x \in A$ or $x \in B$, then there exists a number c (which may belong to either A or B) such that any number less than c belongs to A and any number greater than c belongs to B.

Solution 1.25

This is really the same statement as Zorich 2.2.15.a, the original statement of completeness, but with the extra condition that the sets $A = X, B = Y$ must be disjoint.

Exercise 1.27 (Olmsted 1.7)

If x is an irrational number, under what conditions on the rational numbers a, b, c, d is $(ax+b)/(cx+d)$ rational?

Solution 1.26

Note that a trivial solution is $a = b = c = d = 1$ which gives 1. Since

$$\frac{ax + b}{cx + d} = \frac{acx + ad - ad + bc}{cx + d} = a + \frac{bc - ad}{cx + d}$$

for the above to be rational it is necessary that $1/(cx + d)$ is rational. But this cannot be the case, which leaves us with the condition that $bc = ad$.

Exercise 1.28 (Olmsted 1.8)

Prove that the system of integers satisfies the axiom of completeness.

Solution 1.27

Let $S \subset \mathbb{Z}$ be bounded from above. It must have a maximum element (justify?), call it c. Then we claim that $c \in \mathbb{Z}$ is the least upper bound. Being the maximum, it is an upper bound, and c is least since the next smallest element is $c - 1$, which is less than $c \in S$, and therefore cannot be an upper bound.

Exercise 1.29 (Zorich 2.2.16/Olmsted 1.16)

Prove the following.

1. If $A \subset B \subset \mathbb{R}$, then $\sup A \leq \sup B$ and $\inf A \geq \inf B$.
2. Let $\mathbb{R} \supset X \neq \emptyset$ and $R \supset Y \neq \emptyset$. If $x \leq y$ for all $x \in X, y \in Y$, then X is bounded above , Y is bounded below, and $\sup X \leq \inf Y$.
3. If the sets X, Y in (b), are such that $X \cup Y = \mathbb{R}$, then $\sup X = \inf Y$.
4. If X and Y are the sets defined in (c), then either X has a maximal element or Y as a minimal element.
5. Show that Dedekind's theorem is equivalent to the axiom of completeness.

Solution 1.28

Listed.

1. Let

$$A' = \{x \in \mathbb{R} \mid x \geq a \; \forall a \in A\}$$
$$B' = \{x \in \mathbb{R} \mid x \geq b \; \forall b \in B\}$$

   where we can easily verify that $B' \subset A'$. By definition, we get $\sup B = \min B'$ and $\sup A = \min A'$. But since $B' \subset A'$, for any $b' \in B'$, there exists an $a' \in A'$ such that $a' \leq b'$, which implies that $\sup B = \min B' \leq \min A' = \sup A$.

2. X is bounded above by any element of Y. Y is bounded below by any element of X. By the completion axiom, there exists a $c \in \mathbb{R}$ such that

$$x \leq c \leq y \text{ for all } x \in X, y \in Y$$

   Since c is an upper bound of X, $\sup X \leq c$ by definition, and since c is a lower bound of Y, $\inf Y \geq c$ by definition. Therefore, $\sup X \leq c \leq \inf Y$.

3. From completeness there exists a $c \in \mathbb{R}$ such that $x \leq c \leq y$ for all $x \in X, y \in Y$. Y is, by definition, the set of *all* upper bounds of X (i.e. *every* upper bound of X is in Y, unlike Y defined in 2.2.16.b). Since $c \leq y$ for all $y \in Y$, c is minimal and so $c = \sup X$. X is the set of all lower bounds of Y by definition, so $c \geq x$ for all $x \in X \implies c = \inf Y$. So, $\inf Y = c = \sup X$.
4. We know that there exists $c = \inf Y = \sup X$. Since $X \cup Y = \mathbb{R}$, c must be in at least X or Y. If $c \in X$, then $c = \sup X = \max X$, and if $c \in Y$, then $c = \inf Y = \min Y$.
5. This is the same statement as Zorich 2.2.15.a (an iff equivalence, not just one way implying).

Exercise 1.30 (Olmsted 1.13)

Let S be a nonempty set of numbers bounded above, and let x be the least upper bound of S. Prove that x has the two properties corresponding to an arbitrary positive number ϵ:

1. every element $s \in S$ satisfies the inequality $s < x + \epsilon$
2. at least one element $s \in S$ satisfies the inequality $s > x - \epsilon$

Solution 1.29

Listed.

1. x is an upper bound $\implies s \leq x$ for all $s \in S$, which implies that $s \leq x < x + \epsilon$.
2. By definition, $x - \epsilon$ cannot be an upper bound, so $x - \epsilon \geq s$ for all $s \in S$ is not true. Therefore, there must exist one $s \in S$ such that $s > x - \epsilon$.

Exercise 1.31 (Zorich 2.2.18)

Let $-A$ be the set of numbers of the form $-a$, where $a \in A \subset \mathbb{R}$. Show that $\sup(-A) = -\inf(A)$.

Solution 1.30

If A is unbounded below, then $-\inf A = \infty$ and $-A$ is unbounded above, implying that $\sup A = \infty$. Now assume that A is bounded below, then by completeness, it must have a greatest lower bound. Let us define the set $B = \{b \in \mathbb{R} \mid b \leq a \; \forall a \in A\}$. From 2.2.16.b, we have $b \leq \inf A \leq a$ for all $a \in A, b \in B$. Multiplying by -1 gives $-b \geq -\inf A \geq -a$ for all $a \in A, b \in B$, which is equivalent to saying

$$a \leq -\inf A \leq b \text{ for all } a \in -A, b \in -B$$

by definition of $-A, -B$. $-\inf A$ is clearly an upper bound of $-A$, and since

$$\begin{aligned} B &= \{b \in \mathbb{R} \,|\, b \leq a \; \forall a \in A\} \\ &= \{b \in \mathbb{R} \,|\, -b \geq -a \; \forall a \in A\} \\ &= \{b \in \mathbb{R} \,|\, -b \geq a \; \forall a \in -A\} \end{aligned}$$

implies that $-B = \{b \in \mathbb{R} \,|\, b \geq a \; \forall a \in -A\}$ is the set of all upper bounds of A. So, $-\inf A$ is the least upper bound of $-A$, i.e. $-\inf A = \sup(-A)$.

Exercise 1.32 (Zorich 2.2.21)

Show that the set $\mathbb{Q}(\sqrt{n})$ of numbers of the form $a + b\sqrt{n}$ where $a, b \in \mathbb{Q}$, n is a fixed natural number that is not the square of any integer, is an ordered set satisfying the principle of Archimedes but not the axiom of completeness.

Solution 1.31

The order on $\mathbb{Q}(\sqrt{n})$ can be embedded from the ordering on the reals by defining the canonical injection map $i : \mathbb{Q}(\sqrt{n}) \longrightarrow \mathbb{R}$ and defining for any $x, y \in \mathbb{Q}(\sqrt{n})$,

$$x \leq_{\mathbb{Q}(\sqrt{n})} y \iff i(x) \leq_{\mathbb{R}} i(y)$$

Now, let $h > 0$ be any fixed real number, and $x = (a, b) = a + b\sqrt{n}$. By the Archimidean principle, we can find a $k \in \mathbb{Z}$ such that

$$(k-1)h \leq x \leq kh \text{ for some } x \in \mathbb{Q}(\sqrt{n}) \subset \mathbb{R}$$

We now show that $\mathbb{Q}(\sqrt{n})$ is not complete since it doesn't satisfy the LUB property. Since there are infinite prime numbers in $\mathbb{N}$, choose a prime number p that is not a factor of n. Then, we are guaranteed that pn is not a perfect square, and can define the set

$$X = \{x \in \mathbb{Q}(\sqrt{n}) \,|\, x < \sqrt{pn}\} \subset \mathbb{Q}(\sqrt{n})$$

and assume that $c = c_1 + c_2\sqrt{n} = \sup X$ exists ($c_1, c_2 \in \mathbb{Q}$). Clearly, $c \neq \sqrt{pn} \notin \mathbb{Q}(\sqrt{n})$.

1. Assume $c < \sqrt{pn} \iff 0 < \sqrt{pn} - c \in \mathbb{R}$. By the Archimidean principle, there exists a $k \in \mathbb{N}$ such that $0 < \frac{1}{k} < \sqrt{pn} - c$. Then, we can verify that $c + \frac{1}{k} = (c_1 + \frac{1}{k}) + c_2\sqrt{n} \in \mathbb{Q}(\sqrt{n})$ and
$$c + \frac{1}{k} < c + \sqrt{pn} - c = \sqrt{pn} \implies c + \frac{1}{k} \in X$$
implies that c is not an upper bound. So we must turn to case 2.
2. Assume $c > \sqrt{pn} \iff c - \sqrt{pn} > 0$. By AP, there exists a $k \in \mathbb{N}$ such that $0 < \frac{1}{k} < c - \sqrt{pn}$. Then, we can verify that $c - \frac{1}{k} \in \mathbb{Q}(\sqrt{n})$ and
$$c - \frac{1}{k} > c - c + \sqrt{pn} = \sqrt{pn}$$
implies that $c - \frac{1}{k}$ is an upper bound of X, so c is not least.

Therefore, by contradiction, c does not exist.

Exercise 1.33 (Zorich 2.2.22)

Let $n \in \mathbb{N}$ and $n > 1$. In the set $E_n = \{0, 1, \ldots, n-1\}$, we define the sum and product of two elements as the remainders when the usual sum and product in $\mathbb{R}$ are divided by n. With these operations on it, the set E_n is denoted $\mathbb{Z}_n$.

1. Show that if n is not a prime number, then there are nonzero numbers $m, k \in \mathbb{Z}_n$ such that $m \cdot k = 0$, i.e. there exist nonzero zero divisors.
2. Show that if p is prime, then there are no zero divisors in $\mathbb{Z}_p$ and $\mathbb{Z}_p$ is a field.
3. Show that, no matter what the prime p, $\mathbb{Z}_p$ cannot be ordered in a way consistent with the arithmetic operations on it.

Solution 1.32

Listed.

1. n is composite implies that there exist $1 < m, k < n$ such that $n = mk$. These factors m, k are precisely the zero divisors of $\mathbb{Z}_n$ since $mk = n \equiv 0 \pmod{n}$.
2. With p prime, assume that there are nontrivial zero divisors $1 < m, k < p$ in $\mathbb{Z}_p$. Then, $mk \equiv 0 \pmod{n} \implies mk = lp$ for some $l \in \mathbb{N}$. But this implies that m or k must divide p, which is impossible since $1 < m, k < p$. Then prove field axioms.
3. For any field, we must have $0 \leq 1$, because if not, then
$$0 > 1 \implies 0 < 1^{-1} \cdot 1 = 1^{-1} \implies 0 \cdot 0 < 1^{-1} \cdot 1^{-1} = 1$$
So, $0 \leq 1$ implies that $0 \leq 1 \leq 2 \leq \ldots \leq p-1$. But
$$0 + 1 \leq (p-1) + 1 = 0$$
is false, so any ordering is impossible.

Exercise 1.34 (Zorich 2.2.23)

Show that if $\mathbb{R}$ and $\mathbb{R}'$ are two models of the set of real numbers and $f : \mathbb{R} \longrightarrow \mathbb{R}'$ (with $f \not\equiv 0'$) is a mapping such that $f(x+y) = f(x) + f(y)$ and $f(x \cdot y) = f(x) \cdot f(y)$ for any $x, y \in \mathbb{R}$. Prove that f is an order-preserving isomorphism.

Solution 1.33

Let $0, 0'$ be the additive identity of $\mathbb{R}, \mathbb{R}'$, respectively, and $1, 1'$ the multiplicative identity. We claim that $f(0) = 0'$ since

$$\begin{aligned} f(0) &= f(0+0) && \text{(definition of additive identity)} \\ &= f(0) + f(0) && \text{(homomorphism over } +\text{)} \end{aligned}$$

which implies that $f(0) + f(0) = f(0) = 0' + f(0)$. Since $f(0)$ lives in field $\mathbb{R}'$, its additive identity $-f(0)$ is well defined, and we get $f(0) = f(0) + f(0) + (-f(0)) = 0' + f(0) + (-f(0)) = 0'$. We also claim that $f(1) = 1'$ since

$$\begin{aligned} f(1) &= f(1 \cdot 1) && \text{(definition of multiplicative identity)} \\ &= f(1) \cdot f(1) && \text{(homomorphism over } \cdot\text{)} \end{aligned}$$

which implies that $f(1) \cdot f(1) = 1' \cdot f(1)$. Since $f(1)$ lives in field $\mathbb{R}'$, its multiplicative identity $f(1)^{-1}$ is well defined, and we get $f(1) = f(1) \cdot f(1) \cdot f(1)^{-1} = 1' \cdot f(1) \cdot f(1)^{-1} = 1'$. Now that we have proved mapping of identities, this implies the mapping of inverses.

$$\begin{aligned} 0' &= f(0) = f(x-x) = f(x) + f(-x) \implies f(-x) = -f(x) \\ 1' &= f(1) = f(x \cdot x^{-1}) = f(x) \cdot f(x^{-1}) \implies f(x^{-1}) = f(x)^{-1} \end{aligned}$$

With these conditions, we have proved that f is a homomorphism of fields. Now we prove that f is a bijection, but first, we claim that $f(x) = 0' \implies x = 0$. Assume that there exists a nonzero $x \in \mathbb{R}$ such that $f(x) = 0'$. Then, x^{-1} is well defined, and

$$\begin{aligned} f(x) \cdot f(x^{-1}) &= f(x) \cdot f(x)^{-1} = 0' \\ f(x) \cdot f(x^{-1}) &= f(x \cdot x^{-1}) = f(1) = 1' \end{aligned}$$

which implies that $0' = 1'$. So, $f(1) = 1' = 0'$, and so for all $k \in \mathbb{R}$, $f(k) = f(k \cdot 1) = f(k) \cdot f(1) = f(k) \cdot 0' = 0' \implies f \equiv 0'$, leading to a contradiction of the assumption that $f' \not\equiv 0'$.

1. (f injective). Assume f is not injective, i.e. there exists distinct $x_1, x_2 \in \mathbb{R}$ s.t. $f(x_1) = f(x_2)$. Then, using that fact $f(x) = 0 \implies x = 0$,

$$0 = f(x_1) - f(x_2) = f(x_1 - x_2) \implies x_1 - x_2 = 0 \implies x_1 = x_2$$

2. (f surjective). Let y be any nonzero element in $\mathbb{R}'$ (clearly if $y = 0'$ then its preimage is 0) and y^{-1} its multiplicative inverse. Assume there exists no $x \in \mathbb{R}$ satisfying $f(x) = y$, meaning that there exist no x satisfying

$$f(x) \cdot = y \cdot y^{-1} = 1'$$

But since f maps inverses to inverses, we can choose $x = (y^{-1})^{-1}$, which leads to

$$f(x) \cdot y^{-1} = ($$

Finally, we prove that f is order preserving. Assume that $x \leq y \iff 0 \leq y - x$, we wish to prove that

$$f(x) \leq f(y) \iff 0 \leq f(y) - f(x) = f(y - x)$$

Therefore, since this preservation of ordering is really the statement $0 \leq y - x \implies 0 \leq f(y - x)$, it suffices to prove that $0 \leq x \implies 0 \leq f(x)$. Now, assume that we have a x such that $f(x) < 0'$. Adding it with the equation $f(1) = 1'$ gives us

$$f(x + 1) < 1'$$

It is easy to prove that $0 \leq x \iff 0 \leq x^{-1}$. Now assume that $0 > f(x)$. **INCOMPLETE**

Exercise 1.35 (Density of Rationals in $\mathbb{R}$)

Prove that for any two distinct $a < b \in \mathbb{R}$, there exists an infinite number of rational numbers between a and b.

Solution 1.34

Since $a < b$, then $b - a > 0$ and by the Archimidean principle, there exists a $k \in \mathbb{N}$ such that

$$0 < \frac{1}{k} < b - a \implies 1 < kb - ka$$

which implies that the length of $[ka, kb)$ greater than 1. By the inductive property of $\mathbb{Z}$, there must be an integer $p \in [ka, kb)$. If there were not, then this would imply that $[ka+1, kb+1)$ and $[ka-1, kb-1)$

had no integers and repeating would mean that there were no integers in $\mathbb{R}$. Therefore,

$$ka \leq p < kb \implies a \leq \frac{p}{k} < b$$

for all $a, b \in \mathbb{R}$, with $p/k \in \mathbb{Q}$. If a is irrational we can replace the $\leq$ to $<$, leaving $a \leq \frac{p}{k} < b$, and if a is rational, we can construct another rational $a + \frac{1}{k} \in (a, b)$.

Exercise 1.36 (Nested Interval Lemma)

With the fact that $\mathbb{R}$ is complete, prove the following.

1. For a sequence of closed nested intervals $I_1 \supset I_2 \supset \ldots$ of $\mathbb{R}$, there exists a point $c \in \mathbb{R}$ belonging to all these intervals.
2. Furthermore, if the hypothesis also satisfies the fact that for any $\epsilon > 0$, there exists a $k \in \mathbb{N}$ such that $|I_k| < \epsilon$ (i.e. the length of the intervals decreases to 0), then the point c common to all sets is unique.

Solution 1.35

Listed.

1. Let $I_n = [a_n, b_n]$, with $a_n < b_n$ finite for all $n \in \mathbb{N}$. For all $n \in \mathbb{N}$, we have $I_n = [a_n, b_n]$ and can take the two subsets $X_n = (-\infty, a_n)$ and $Y_n = (b_n, \infty)$, where $x \leq y$ for every $x \in X_n, y \in Y_n$. We also have the fact that $\mathbb{R} = X_n \cup I_n \cup Y_n$. Since $\mathbb{R}$ is complete, there exists a c such that $x \leq c \leq y$ for all $x \in X, y \in Y$. But $x \leq c \iff c \notin X_n$ and $c \leq y \iff c \notin Y_n$, so for all $n \in \mathbb{N}$, c must be in I_n.
2. Since we have proved (a), it now suffices to prove uniqueness of c. Let there be two distinct points $c_1, c_2 \in \mathbb{R}$ belonging to these intervals. Without loss of generality, assume $c_1 - c_2 > 0$, and choose
$$\epsilon = \frac{c_1 - c_2}{3}$$
Then, there should exist a $k \in \mathbb{N}$ such that $|I_k| < \epsilon$. Since I_k must contain c_1, it must be a subset of $[c_1 - \epsilon, c_1 + \epsilon]$ (should be able to see why) and similarly for c_2.
$$I_k \subset \left[c_1 - \frac{c_1 - c_2}{3}, c_1 + \frac{c_1 - c_2}{3}\right] = \left[\frac{2c_1 + c_2}{3}, \frac{4c_1 - c_2}{3}\right] = L$$
$$I_k \subset \left[c_2 - \frac{c_1 - c_2}{3}, c_2 + \frac{c_1 - c_2}{3}\right] = \left[\frac{-c_1 + 4c_2}{3}, \frac{c_1 + 2c_2}{3}\right] = M$$
But since $c_1 > c_2 \implies \frac{c_1+2c_2}{3} < \frac{2c_1+c_2}{3}$, L and M are disjoint $\implies I_k$, as a subset of both, leaves us with $I_k = \emptyset$, contradicting that it is a closed interval.

Exercise 1.37 ()

Compactness of Closed Interval in $\mathbb{R}$ Prove that any system of open intervals covering (i.e. an open cover of) a closed interval contains a finite subsystem that covers the closed interval. Another way to state this is by saying that every closed interval of $\mathbb{R}$ is compact.

Solution 1.36

A closed interval with a finite open covering is trivially compact since any subcovering is also finite. We only need to deal with when a closed interval $I = [a, b]$ has an infinite open covering $\{U_\alpha\}_{\alpha \in A}$, which means that the set of indices A is infinite. Assume that there exists no finite covering of I.

Then, we divide I into two halves

$$I_1 = \left[a, \frac{a+b}{2}\right], \;\; I_2 = \left[\frac{a+b}{2}, b\right]$$

and define a subcovering for each of them. That is, we can define $A_1 \subset A$ and $A_2 \subset A$ such that $\{U_\alpha\}_{\alpha \in A_1} \subset \{U_\alpha\}_{\alpha \in A}$ is a covering of I_1 and $\{U_\alpha\}_{\alpha \in A_2} \subset \{U_\alpha\}_{\alpha \in A}$ is a covering of I_2. At least one of A_1 or A_2 must be infinite, since if they were both finite, then we can define a finite covering $\{U_\alpha\}_{\alpha \in A_1 \cup A_2}$ of I. Choose the interval with the infinite covering and repeat this procedure, which will result in a nested interval that decreases in length by a half.

$$I \supset I_1 \supset I_2 \supset \ldots$$

By the nested interval lemma, there exists a unique point c common to all these intervals. But since $c \in [a,b]$, the open cover $\{U\}$ should contain an open interval $(c-\delta_1, c+\delta_2)$ containing c. We wish to prove that this interval is a superset of some I_k in the sequence above, contradicting the fact that I_k has an infinite cover. Since the length of each I_i decreases arbitrarily (i.e. we can choose any $\epsilon > 0$ and find a I_k with length less than ϵ), we choose $\epsilon = \frac{1}{2}\min\{\delta_1, \delta_2\}$, and we should be able to find some I_k that is a subinterval of $[c-\epsilon, c+\epsilon]$, which itself is a subinterval of $(c-\delta_1, c+\delta_2)$.

$$I_k \subset \left[c - \frac{1}{2}\min\{\delta_1, \delta_2\}, c + \frac{1}{2}\min\{\delta_1, \delta_2\}\right] \subset (c-\delta_1, c+\delta_2)$$

Therefore, $(c-\delta_1, c+\delta_2)$ is a finite cover of I_k, contradicting the fact that all I_k's have infinite covers.

Exercise 1.38 ()

Bolzano-Weierstrass Theorem Prove that every bounded infinite set of real numbers has at least one limit point. (A limit point p of set X is a point such that every open neighborhood of p contains an infinite number of elements of X).

Solution 1.37

Let the set of points be denoted X, and let a be the lower bound and b be the upper bound. Then, $X \subset [a,b] = I$. Now divide $[a,b]$ into halves $[a, \frac{a+b}{2}] \cup [\frac{a+b}{2}, b]$. At least one of the halves must have an infinite number of points; choose the interval with infinite points as I_1 and doing this repeatedly gives the nested sequence

$$I \supset I_1 \supset I_2 \supset \ldots$$

By the nested interval lemma, there exists at least one point $c \in \mathbb{R}$ that is in all these intervals. Furthermore, since $|I_i| = \frac{1}{2^i}(b-a)$ decreases to 0, we can choose a $\epsilon > 0$ and find an interval I_k with $|I_k| < \epsilon$. We claim that c is a limit point of X. Given an ϵ, we wish to prove that there are an infinite number of points within the ϵ-neighborhood $(c-\epsilon, c+\epsilon)$ of c. Since we can find some I_k with $|I_k| < \epsilon$, we can see that

$$I_k \subset (c-\epsilon, c+\epsilon)$$

and therefore the ϵ-neighborhood of c contains I_k, which contains an infinite number of points in X.

Solution 1.38

We can construct another proof that is dependent on the compactness lemma. This construction will be useful for problem 2.3.4. Let X be a given subset of $\mathbb{R}$, and it follows from the definition of boundedness that X is contained in some closed interval $I \subset \mathbb{R}$. We show that at least one point of I is a limit point of X.

Assume that it is not. Then each point $x \in I$ would have a neighborhood $U(x)$ containing at most a finite number of points from X. The totality of such neighborhoods $\{U(x)\}$ constructed for the points $x \in I$ forms an open covering of X. Since I is closed, it is compact and therefore we can find a finite subcovering $\{U_i(x)\}_i$ of open intervals that cover I and therefore cover X. This open cover $\{U_i(x)\}_i$ of X is a finite union of sets that each contain at most a finite number of points from X, so the covering of X contains a finite number of points from X, a contradiction that X contains infinite points.

Exercise 1.39 (Zorich 2.3.1)

Show that

1. if I is any system of nested closed intervals, then

$$\sup\{a \in \mathbb{R} \mid [a, b] \in I\} = \alpha \leq \beta = \inf\{b \in \mathbb{R} \mid [a, b] \in I\}$$

   and

$$[\alpha, \beta] = \bigcap_{[a,b] \in I} [a, b]$$

2. if I is a system of nested open intervals (a, b), the intersection

$$\bigcap_{(a,b) \in I} (a, b)$$

   may happen to be empty.

Solution 1.39

Listed.

1. (May be tempted to say that $a_1 \leq a_2 \leq \ldots$, but this assumes that the indexing set I is countable). We claim that for any two intervals $[a_n, b_n]$ and $[a_m, b_m]$ in I,

$$a_n \leq b_m$$

   Assume that $a_n > b_m$. Then $b_n \geq a_n > b_m \geq a_m$ implies that $[a_n, b_n]$ and $[a_m, b_m]$ are disjoint, contradicting the fact that they are nested. Now given that X is the set of a_n's and Y is the set of b_n's, we have $x \leq y$ for all $x \in X, y \in Y$. So by 2.2.16.b, we have $\sup X \geq \inf Y$.
   To prove the second statement, we show that trying to "expand" the interval $[\alpha, \beta]$ will lead to a contradiction. Since α is the LUB, given any $\epsilon > 0$, there exists a $(a_l, b_l) \in X$ such that $\alpha - \epsilon < a_l < \alpha$, which implies that $[\alpha, \beta] \subset [a_l, \beta] \subset [\alpha - \epsilon, \beta]$. Assuming that this extended interval is the intersection, we should be able to choose any point in $[\alpha - \epsilon, \beta]$ and find that it is in every element of I. We choose a point in $[\alpha - \epsilon, a_l)$, which is not in the interval (a_l, b_l). We do the same for $\beta \mapsto \beta + \epsilon$. We also check that "shrinking" the interval $[\alpha, \beta] \mapsto [\alpha + \epsilon, \beta]$ is no good, since we can find an element in $[\alpha, \alpha + \epsilon)$ that is in every interval in I.
2. Take the system of nested open intervals

$$(0, 1) \supset (0, \frac{1}{2}) \supset (0, \frac{1}{3}) \ldots (0, \frac{1}{n}) \supset \ldots$$

   Take their infinite intersection, denote it S, and assume that some $\epsilon \in (0, 1)$ is in S. Since ϵ is a real number, by the Archimidean principle there exists a $k \in \mathbb{N}$ such that $\frac{1}{k} < \epsilon$. Therefore, $\epsilon \notin (0, \frac{1}{k}) \implies \epsilon \notin S$.

Exercise 1.40 (Zorich 2.3.2)

Show that
1. from a system of closed intervals covering a closed interval it is not always possible to choose a finite subsystem covering the interval.
2. from a system of open intervals covering a open interval it is not always possible to choose a finite subsystem covering the interval.
3. from a system of closed intervals covering a open interval it is not always possible to choose a finite subsystem covering the interval.

Solution 1.40

We show with the interval $(0,1)$ or $[0,1]$. Using linear transformations it is easy to generalize this to any other interval (a,b) or $[a,b]$.
1. Consider the infinite covering
$$[0,1] = \big[0, \frac{1}{2}\big] \cup \big[\frac{1}{2}, \frac{3}{4}\big] \cup \big[\frac{3}{4}, \frac{7}{8}\big] \cup \ldots$$
2. Consider the infinite covering
$$(0,1) = \big(0, \frac{1}{2}\big) \cup \big(\frac{1}{2}, \frac{3}{4}\big) \cup \big(\frac{3}{4}, \frac{7}{8}\big) \cup \ldots$$
3. Consider the infinite covering
$$(0,1) = \big[0, \frac{1}{2}\big] \cup \big[\frac{1}{2}, \frac{3}{4}\big] \cup \big[\frac{3}{4}, \frac{7}{8}\big] \cup \ldots$$

Exercise 1.41 (Zorich 2.3.3)

Show that if we only take the set $\mathbb{Q}$ of rational numbers instead of the complete set $\mathbb{R}$ of real numbers, with the definitions of closed, open, and neighborhood of a point $r \in \mathbb{Q}$ to mean respectively the corresponding subsets of $\mathbb{Q}$, then none of the three lemmas is true.

Solution 1.41

We prove only for the nested interval lemma. We choose the series of nested intervals
$$\left(\sqrt{2} - \frac{1}{n}, \sqrt{2} + \frac{1}{n}\right)$$
with $n \in \mathbb{N}$. Assume that there is a $r \in \mathbb{Q}$ such that
$$r \in \left(\sqrt{2} - \frac{1}{n}, \sqrt{2} + \frac{1}{n}\right) \text{ for all } n \in \mathbb{N}$$
which is equivalent to saying that $\big|r - \sqrt{2}\big| < \frac{1}{n}$ for all $n \in \mathbb{N}$. Clearly, $r \neq \sqrt{2}$, and by the Archimidean principle, there exists a $k \in \mathbb{N}$ such that
$$0 < \frac{1}{k} < |r - \sqrt{2}|$$
which contradicts the above.

Exercise 1.42 (Zorich 2.3.4)

Show that the three lemmas above are equivalent to the axiom of completeness.

Solution 1.42

Note that from the proofs, completeness implies nested interval lemma, which implies compactness of closed intervals, which implies the Bolzano-Weierstrass theorem. So, it is sufficient to prove that Bolzano-Weierstrass theorem implies completeness to determine equivalence. There are not a lot of direct proofs, so we prove that Weierstrass implies nested interval, which implies completeness.

1. (Weierstrass $\implies$ Nested) Assume that we have $\mathbb{R}$ with the Bolzano-Weierstrass theorem. Take the series of nested closed intervals

$$I = [a, b] \supset I_1 = [a_1, b_1] \supset I_2 = [a_2, b_2] \supset \ldots$$

We see that $a \leq a_i \leq b$, so the infinite sequence of monotonically nondecreasing values a_i is bounded. Therefore, it must have a limit point, which we will denote as c. We claim that $a_i \leq c$ for all a_i. Since if it were not, then $c < a_i$ for some i, and choosing $\epsilon = 0.5(a_i - c)$, the ϵ-neighborhood of c will not contain a_j for $j \geq i$ since

$$c < a_i \implies 0.5c < 0.5a_i \implies c + \epsilon = 0.5c + 0.5a_i < a_i < a_{i+1} < \ldots$$

. With similar reasoning, we can conclude that $b_i \geq c$ for all b_i. This implies that $a_i \leq c \leq b_i$ for all i which is equivalent to saying that $c \in [a_i, b_i] = I_i$ for all $i \in \mathbb{N}$.

2. (Nested $\implies$ LUB Principle) Let $X \subset \mathbb{R}$ be a set that is bounded above, with b_1 any upper bound. Since X is nonempty, there exists $a_1 \in X$ that is not an upper bound (otherwise, X would be a singleton set and it trivially has a least upper bound). Consider the well-defined interval $[a_0, b_0]$. Take the mean $m_0 = 0.5(a_0 + b_0)$, and if m_0 is an upper bound, set it to b_1 (with $a_1 = a_0$) and a_1 if else (with $b_1 = b_0$). Then, we have a sequence of nested intervals

$$[a_0, b_0] \supset [a_1, b_1] \supset [a_2, b_2] \supset \ldots$$

of decreasing lengths $|I_k| = \frac{1}{2^k}(b - a)$. All of them must contain a unique common point $c \in \mathbb{R}$ by the nested intervals lemma, which implies that

$$a_0 \leq a_1 \leq a_2 \leq \ldots \leq c \leq \ldots \leq b_2 \leq b_1 \leq b_0$$

I claim two things:

(a) c is an upper bound for X. Suppose it were not, then there exists some $x \in X$ such that $c < x$, and let the distance between them be $\epsilon = x - c > 0$. By AP, we can choose $k \in \mathbb{N}$ such that $\frac{1}{k} < \epsilon$. All the b_n are upper bounds of X, so we have $x \leq b_n$. Subtracting c on both sides gives

$$0 < x - c = \epsilon \leq b_n - c \leq |I_n| = \frac{1}{2^n}(b_0 - a_0)$$

where the last inequality follows from $c \in I_n = [a_n, b_n]$, so the maximum distance it can be from the endpoint b_n is $|I_n|$. The inequality above holds for all $n \in \mathbb{N}$, so increasing n arbitrarily should decrease $\frac{1}{2^n}(b_0 - a_0)$ past ϵ. To formalize this, we use the inequality

$$\frac{1}{2^n} < \frac{1}{n} \text{ for all } n \in \mathbb{N}$$

and so we have

$$\epsilon \leq b_n - c < \frac{1}{n}(b_0 - a_0)$$

We choose the natural number $n = \lceil \frac{2(b_0 - a_0)}{\epsilon} \rceil$, which does not satisfy the inequality above since

$$\epsilon < \frac{1}{n}(b_0 - a_0) = \frac{1}{\lceil 2(b_0 - a_0)/\epsilon \rceil}(b_0 - a_0) \leq \frac{\epsilon}{2(b_0 - a_0)}(b_0 - a_0) = \frac{\epsilon}{2}$$

This leads to a contradiction.

(b) We now prove that c is least. Assume that c is not least $\implies$ there exists an upper bound B such that $B < c$ and $x \leq B$ for all $x \in X$. **INCOMPLETE**

Exercise 1.43 (Zorich 2.4.1)

Show that the set of real numbers has the same cardinality as the points of the interval $(-1, 1)$.

Solution 1.43

We define the bijective map $\rho : (-1, 1) \longrightarrow \mathbb{R}$

$$p(x) = \begin{cases} 0 & \text{if } x = 0 \\ \frac{1}{x} & \text{if } x \neq 0 \end{cases}$$

Exercise 1.44 (Zorich 2.4.2)

Give an explicit one-to-one correspondence between

1. the points of two open intervals
2. the points of two closed intervals
3. the point of a closed interval and an open interval
4. the points of the closed interval $[0, 1]$ and $\mathbb{R}$

Solution 1.44

Listed.

1. $\rho : (a, b) \longrightarrow (c, d)$ defined
$$\rho(x) = \frac{d-c}{b-a}(x-a) + c$$
2. the extension of ρ defined on (a) to $[a, b]$
3. From (a) and (b), it suffices to prove a bijection from $(0, 1)$ to $[0, 1]$. We extract a countably infinite sequence from $(0, 1)$, say
$$x_1 = \frac{1}{3},\ x_2 = \frac{1}{4}, \ldots, x_i = \frac{1}{i+2}$$
Then, we define bijection $\rho : (0, 1) \longrightarrow [0, 1]$ as
$$\rho(x) = \begin{cases} x & \text{if } x \notin \{x_i\} \\ 0 & \text{if } x = x_1 = \frac{1}{2} \\ 1 & \text{if } x = x_2 = \frac{1}{3} \\ x_{i-2} & \text{if } x = x_i \text{ for } i > 2 \end{cases}$$
Colloquially, we extract a copy of $\mathbb{N}$ from $(0, 1)$ and use the bijection $\mathbb{N} \simeq \mathbb{N} \cup \{0, -1\}$ to "shift" the terms.
4. We compose the bijections $\rho_1 : [0, 1] \longrightarrow (0, 1)$ and $\rho_2 : (0, 1) \longrightarrow \mathbb{R}$.

Exercise 1.45 (Zorich 2.4.3)

Show that

1. every infinite set contains a countable subset

2. the set of even integers has the same cardinality as the set of all natural numbers
3. the union of an infinite set and an at most countable set has the same cardinality as the original infinite set.
4. the set of irrational numbers has the cardinality of the continuum
5. the set of transcendental numbers has the cardinality of the continuum

Solution 1.45

Listed.

1. Let A be an infinite set. By axiom of choice, choose $a_0 \in A$. Then, $A \setminus \{a_0\} \neq \emptyset$ since A is infinite. By induction, assume you have chosen $a_0, a_1, \ldots, a_k \in A$. Then, since A is infinite, $A \setminus \{a_0, a_1, \ldots, a_k\} \neq \emptyset$, so we can choose $a_{k+1} \in A \setminus \{a_0, \ldots, a_k\}$. Thus, we have constructed a countable subset $\{a_k\}_{k \in \mathbb{N}}$ of A.
2. Given the quotient ring $2\mathbb{Z}$, define the bijection $\rho : 2\mathbb{Z} \longrightarrow \mathbb{N}$ as
$$p(x) = \begin{cases} x+2 & \text{if } x \geq 0 \\ -x-1 & \text{if } x < 0 \end{cases}$$
3. From (a), we can extract a countable set from original set A, call it X. Since the product of countable sets is countable ($\mathbb{N} \cup \mathbb{N}$ is countable), we can define a bijection $\tilde{\rho} : X \longrightarrow X \cup B$. Therefore, we can define a bijection $\rho : A \longrightarrow A \cup B$ as
$$\rho(x) = \begin{cases} x & \text{if } x \in A \setminus X \\ \tilde{\rho}(x) & \text{if } x \in X \end{cases}$$
4. $\mathbb{Q}$ is countable and $\mathbb{R}$ is uncountable. So, $\mathbb{R} \setminus \mathbb{Q}$ must be uncountable since if it were countable, then the union of the rationals and irrationals, which is $\mathbb{R}$, would be countable.
5. It suffices to prove that the set of algebraic numbers (numbers that are possible roots of a polynomial with integer coefficients with leading coefficient nonzero) is countable, since we can apply (d) right after. The set of all kth degree polynomials with integer coefficients is isomorphic to $\mathbb{Z}^k$ through the map
$$a_k x^k + a_{k-1} x^{k-1} + \ldots + a_2 x^2 + a_1 x^1 + a_0 \mapsto (a^{k-1}, a^{k-2}, \ldots, a_1, a_0)$$
and the union of these countable sets (minus the 0 map)
$$P = \bigg(\bigcup_{k=1}^{\infty} \mathbb{Z}^k \bigg) \setminus \{0\} = \big(\mathbb{Z} \setminus \{0\}\big) \cup \mathbb{Z}^2 \cup \ldots$$
is countable. For any element in $\mathbb{Z}^k$, there are at most k real roots, and so we can define the set of roots of an element $z \in \mathbb{Z}^k \subset P$ as a j-tuple of algebraic numbers, which can have at most $j = k$ roots.
$$r(z) = \underbrace{(r_{1z}, r_{2z}, \ldots, r_{jz})}_{j \leq k}$$
Therefore, the union of all these j-tuples for all $z \in P$
$$\bigcup_{z \in P} r(z) = \bigcup_{k=1}^{\infty} \bigcup_{z \in \mathbb{Z}^k} r(z)$$
is a countable union of a countable union of finite sets, making it countable.

Exercise 1.46 (Zorich 2.4.4)

Show that
1. the set of increasing sequences of natural numbers has the same cardinality as the set of fractions of the form $0.\alpha_1\alpha_2\ldots$
2. the set of all subsets of countable set has cardinality of the continuum

Solution 1.46

Listed.
1. Given a sequence of increasing naturals $S = (n_1, n_2, \ldots)$, we can define a binary expansion $0.\alpha_1\alpha_2\ldots$ where $\alpha_i = 1$ if and only if $i \in \mathbb{N}$ is in S and $\alpha_i = 0$ if not. This is clearly a bijection.
2. The set of all segments of increasing natural is equipotent with $2^{\mathbb{N}}$, since the elements of each sequence define a subset of $\mathbb{N}$. Cantor's diagonalization argument proves that the set of infinite binary expansions is uncountable, and by (a), this proves that $2^{\mathbb{N}}$ is uncountable.

This is very interesting since $\mathbb{N} \simeq \mathbb{R}$, but $2^{\mathbb{N}} \simeq \mathbb{R}$, and the set of all infinite q-ary expansions is equipotent to $\mathbb{R}$ too.

Exercise 1.47 (Zorich 2.4.5)

Show that
1. the set $\mathcal{P}(X)$ of subsets of a set X has the same cardinality as the set of all functions $f : X \longrightarrow \{0, 1\}$.
2. for a finite set X of n elements, $\operatorname{card}\mathcal{P}(X) = 2^n$
3. one can write $\operatorname{card}\mathcal{P}(X) = 2^{\operatorname{card}X}$, which implies $\operatorname{card}\mathcal{P}(\mathbb{N}) = 2^{\operatorname{card}\mathbb{N}} = \operatorname{card}\mathbb{R}$
4. for any set X, $\operatorname{card}X < 2^{\operatorname{card}X}$

Solution 1.47

Listed.
1. An element $Y \in \mathcal{P}(X)$ is a subset of X by definition. Letting
$$f_Y(x) = \begin{cases} 0 & \text{if } x \notin Y \\ 1 & \text{if } x \in Y \end{cases}$$
we can construct the bijective map $Y \mapsto f_Y$.
2. We can prove this using the identity (which can be proved using induction)
$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$
3. Let $F(X; \{0, 1\})$ be the set of all binary valued functions from X to $\{0, 1\}$. From (a), $\operatorname{card}\mathcal{P}(X) \simeq F(X; \{0, 1\})$. Each binary-valued function f is determined by the assignment $f(x)$ for each $x \in X$. Since $f(x)$ has two possible values, the assignment of $f(x)$ for all $x \in X$ has $\{0, 1\}^{\operatorname{card}X}$ possible choices. This gives another bijection $F(X; \{0, 1\}) \simeq \{0, 1\}^{\operatorname{card}X}$, so
$$\mathcal{P}(X) \simeq \{0, 1\}^{\operatorname{card}X} \implies \operatorname{card}\mathcal{P}(x) = \operatorname{card}(\{0, 1\}^{\operatorname{card}X}) = 2^{\operatorname{card}X}$$
4. If X is finite, then letting $n = \operatorname{card}X$, we can simply prove $n < 2^n$ by induction (which we will not do here). If X is countable, then $\mathcal{P}(X)$ is uncountable (from 2.4.4) and so using (c),
$$\operatorname{card}X = \operatorname{card}\mathbb{N} < \operatorname{card}\mathbb{R} = \operatorname{card}\mathcal{P}(X) = 2^{\operatorname{card}X}$$
For uncountable sets (and for the two cases mentioned above), we can use Cantor's theorem, which states that $\operatorname{card}X < \operatorname{card}\mathcal{P}(X)$, and so using (c), we have $\operatorname{card}X < \operatorname{card}\mathcal{P}(X) = 2^{\operatorname{card}X}$.

Exercise 1.48 (Zorich 2.4.6)

Let $X_1, \ldots, X_m$ be a finite system of finite sets. Show that

$$\begin{aligned}
\text{card}\left(\bigcup_{i=1}^m X_i\right) &= \sum_{i_1} \text{card}\, X_{i_1} - \sum_{i_1 < i_2} \text{card}(X_{i_1} \cap X_{i_2}) + \ldots \\
&\sum_{i_1 < i_2 < i_3} \text{card}(X_{i_1} \cap X_{i_2} \cap X_{i_3}) - \ldots + (-1)^{m-1}\, \text{card}(X_1 \cap \ldots \cap X_m) \\
&= \sum_{k=1}^m \sum_{1 \leq i_1 \ldots i_k \leq m} (-1)^{k-1}\, \text{card}\left(\bigcap_{j=1}^k X_{i_j}\right)
\end{aligned}$$

Solution 1.48

Ignoring Russell's paradox (defining the universe set of all sets), we can use the commutative, associative, and distributive properties of $\cup, \cap$ on the algebra of sets. We prove using induction on m. For $m = 1$, we trivially have $\text{card}\, X_1 = \text{card}\, X_1$, and for $m = 2$, we claim

$$\text{card}(X_1 \cup X_2) = \text{card}(X_1) + \text{card}(X_2) - \text{card}(X_1 \cap X_2)$$

X_1 and $X_2 \setminus X_1$ are clearly exclusive sets by definition, with $X_1 \cup X_2 = X_1 \cup (X_2 \setminus X_1)$, so

$$\text{card}(X_1 \cup X_2) = \text{card}\big(X_1 \cup (X_2 \setminus X_1)\big) = \text{card}(X_1) + \text{card}(X_2 \setminus X_1) \tag{2}$$

By definition, the set $X_2 \setminus X_1$ and $X_1 \cap X_2$ are disjoint and satisfies $X_2 = (X_2 \setminus X_1) \cup (X_1 \cap X_2)$ (also by definition), so

$$\text{card}(X_2) = \text{card}(X_2 \setminus X_1) + \text{card}(X_1 \cap X_2) \tag{3}$$

and substituting (3) into (2) gives the claim for $m = 2$. Assuming that the claim is satisfied for some m, we have

$$\begin{aligned}
\text{card}\left(\bigcup_{i=1}^{m+1} X_i\right) &= \text{card}\left(\left[\bigcup_{i=1}^m X_i\right] \cup X_{m+1}\right) \\
&= \text{card}\left(\bigcup_{i=1}^k X_i\right) + \text{card}(X_{m+1}) - \text{card}\left(\left[\bigcup_{i=1}^m X_i\right] \cap X_{m+1}\right) && \text{(claim for } m = 2) \\
&= \text{card}\left(\bigcup_{i=1}^k X_i\right) + \text{card}(X_{m+1}) - \text{card}\left(\bigcup_{i=1}^m (X_i \cap X_{m+1})\right) && \text{(distributive prop.)} \\
&= \sum_{k=1}^m \sum_{1 \leq i_1 \ldots i_k \leq m} (-1)^{k-1}\, \text{card}\left(\bigcap_{j=1}^k X_{i_j}\right) + \text{card}(X_{m+1}) \\
&\quad - \sum_{k=1}^m \sum_{1 \leq i_1 \ldots i_k \leq m} (-1)^{k-1}\, \text{card}\left(\bigcap_{j=1}^k (X_{i_j} \cap X_{m+1})\right)
\end{aligned}$$

With a bit of thought, we can see that the kth term of the second summation contributes to adding another term to the $k{+}1$th summation term of the first. Therefore, we must try to shift the summation over by 1 index. Let us simplify this by taking the summations and extracting the first and last term, respectively. We have

$$\sum_{k=1}^m \sum_{1 \leq i_1 \ldots i_k \leq m} (-1)^{k-1} \operatorname{card}\left(\bigcap_{j=1}^k X_{i_j} \right) = \sum_{1 \leq i_1 \ldots i_k \leq m} \operatorname{card}(X_{i_1}) + \sum_{k=2}^m \sum_{1 \leq i_1 \ldots i_k \leq m} (-1)^{k-1} \operatorname{card}\left(\bigcap_{j=1}^k X_{i_j} \right)$$

and

$$\begin{aligned} \sum_{k=1}^m \sum_{1 \leq i_1 \ldots i_k \leq m} & (-1)^{k-1} \operatorname{card}\left(\bigcap_{j=1}^k (X_{i_j} \cap X_{m+1}) \right) \\ &= \sum_{k=1}^m \sum_{1 \leq i_1 \ldots i_k \leq m} (-1)^{k-1} \operatorname{card}\left(\left[\bigcap_{j=1}^k X_{i_j} \right] \cap X_{m+1} \right) \\ &= \sum_{k=1}^{m-1} \sum_{1 \leq i_1 \ldots i_k \leq m} (-1)^{k-1} \operatorname{card}\left(\bigcap_{j=1}^k (X_{i_j} \cap X_{m+1}) \right) + (-1)^{m-1} \operatorname{card}\left(\bigcap_{j=1}^{m+1} X_j \right) \\ &= \sum_{k=2}^{m} \sum_{1 \leq i_1 \ldots i_k \leq m} (-1)^{k-2} \operatorname{card}\left(\bigcap_{j=1}^{k-1} (X_{i_j} \cap X_{m+1}) \right) + (-1)^{m-1} \operatorname{card}\left(\bigcap_{j=1}^{m+1} X_j \right) \end{aligned}$$

So subtracting the summations gives

$$\begin{aligned} \sum_{k=1}^m \sum_{1 \leq i_1 \ldots i_k \leq m} & (-1)^{k-1} \operatorname{card}\left(\bigcap_{j=1}^k X_{i_j} \right) - \sum_{k=1}^m \sum_{1 \leq i_1 \ldots i_k \leq m} (-1)^{k-1} \operatorname{card}\left(\bigcap_{j=1}^k (X_{i_j} \cap X_{m+1}) \right) + |X_{m+1}| \\ &= \sum_{1 \leq i_1 \ldots i_k \leq m} \operatorname{card}(x_i) + \sum_{k=2}^m \sum_{1 \leq i_1 \ldots i_k \leq m} (-1)^{k-1} \operatorname{card}\left(\bigcap_{j=1}^k X_{i_j} \right) + \operatorname{card}(X_{m+1}) \\ &\quad + \sum_{k=2}^m \sum_{1 \leq i_1 \ldots i_k \leq m} (-1)^{k-1} \operatorname{card}\left(\bigcap_{j=1}^{k-1} (X_{i_j} \cap X_{m+1}) \right) + (-1)^m \operatorname{card}\left(\bigcap_{j=1}^{m+1} X_j \right) \\ &= \sum_{1 \leq i_1 \ldots i_k \leq m+1} \operatorname{card}(X_i) + \sum_{k=2}^m \sum_{1 \leq i_1 \ldots i_k \leq m} (-1)^{k-1} \left[\operatorname{card}\left(\bigcap_{j=1}^k X_{i_j} \right) \right. \\ &\quad \left. + \operatorname{card}\left(\left[\bigcap_{j=1}^{k-1} X_{i_j} \right] \cap X_{m+1} \right) \right] + (-1)^m \operatorname{card}\left(\bigcap_{j=1}^{m+1} X_j \right) \end{aligned}$$

and since the set of sequences of k terms bounded by $m+1$ (of form $1 \leq i_1 \ldots i_k \leq m+1$) is the set of sequences of k terms bounded by m (of form $1 \leq i_1 \ldots i_k \leq m$) unioned with the set of sequences of k terms with max element $m+1$ (of form $1 \leq i_1 \ldots i_k = m+1$), we have

$$\sum_{1 \leq i_1 \ldots i_k \leq m} (-1)^{k-1} \left[\operatorname{card}\left(\bigcap_{j=1}^k X_{i_j} \right) + \operatorname{card}\left(\left[\bigcap_{j=1}^{k-1} X_{i_j} \right] \cap X_{m+1} \right) \right] = \sum_{1 \leq i_1 \ldots i_k \leq m+1} \operatorname{card}\left(\bigcap_{j=1}^k X_{i_j} \right)$$

and therefore, substituting the above and observing that the independent terms are the first and last terms of the summation gives

$$\begin{aligned}\text{card}\left(\bigcup_{i=1}^{m+1} X_i\right) &= \sum_{1\leq i_1 \ldots i_k \leq m+1} \text{card}(X_i) + \sum_{k=2}^{m} \sum_{1\leq i_1 \ldots i_k \leq m+1} (-1)^{k-1} \text{card}\left(\bigcap_{j=1}^{k} X_{i_j}\right) \\ &\quad + \ldots + (-1)^m \text{card}\left(\bigcap_{j=1}^{m+1} X_j\right) \\ &= \sum_{k=1}^{m+1} \sum_{1\leq i_1 \ldots i_k \leq m+1} (-1)^{k-1} \text{card}\left(\bigcap_{j=1}^{k} X_{i_j}\right)\end{aligned}$$

Exercise 1.49 (Zorich 2.4.7)

On the closed interval $[0,1] \subset \mathbb{R}$, describe the sets of numbers $x \in [0,1]$ whose ternary representation $x = 0.\alpha_1\alpha_2\ldots$, $\alpha_i \in \{0,1,2\}$ has the property.

1. $\alpha_1 \neq 1$
2. $\alpha_1 \neq 1$ and $\alpha_2 \neq 1$
3. For all $i \in \mathbb{N}$, $\alpha_i \neq 1$ (the Cantor set)

Solution 1.49

Listed.

1. $[0, \frac{1}{3}) \cup [\frac{2}{3}, 1)$
2. $[0, \frac{1}{9}) \cup [\frac{2}{9}, \frac{3}{9}) \cup [\frac{6}{9}, \frac{7}{9}) \cup [\frac{8}{9}, 1)$
3. Made by recursively removing the middle third of every partitioned intervals.

Exercise 1.50 (Zorich 2.4.8)

Show that

1. the set of numbers $x \in [0,1]$ whose ternary representation does not contain 1 has the same cardinality as the set of all numbers whose binary representation has the form $0.\beta_1\beta_2\ldots$
2. the Cantor set has the same cardinality as the closed interval $[0,1]$

Solution 1.50

Listed.

1. We can define a bijection $0.\alpha_1\alpha_2\ldots \mapsto 0.\beta_1\beta_2\ldots$ as $\alpha_i = 0 \iff \beta_i = 0$ and $\alpha_i = 1 \iff \beta_i = 2$.
2. The map above defines a bijection between the Cantor set and the set of all infinite binary expansions in $[0,1]$, which is uncountable by Cantor's diagonalization theorem.

2 Euclidean Topology

With the construction of the real line and the real space, the extra properties of completeness, norm, and order (for the real line) allows us to restate these topological properties in terms of these "higher-order" properties. It also proves much more results than for general topological spaces. Therefore, the next few sections will focus on reiterating the topological properties of $\mathbb{R}$ and $\mathbb{R}^n$ (this can be done slightly more generally for metric spaces, but we talk about this in point-set topology). In this section, we will restate the notion of open sets, limit points, compactness, connectedness, and separability. Then we can continue in the next section sequences and their limits, and after that we describe continuity. Once this is done, we can focus constructing the derivative and integral, which are unique to Banach spaces.

2.1 Open Sets

It is well-known that the set of open-balls of a metric space (X, d) is indeed a topology, which we prove in point-set topology. Once we prove this, we have access to a whole suite of theorems on topological spaces that we can just apply to $\mathbb{R}^n$. We will restate many of these topological theorems for completeness but will not prove them. However, if any of these theorems use any other structure, such as order/metrics/norms/-completeness, we will have to prove them.

Definition 2.1 (Topology)

A

Theorem 2.1 (Euclidean Topology)

Let $\tau_{\mathbb{R}}$ (which we denote as τ) be the set of subsets S of $(\mathbb{R}^n, ||\cdot||)$ satisfying the property that if $x \in S$, then there exists an open ϵ-ball $B(x, \epsilon)$ s.t. $B \subset S$. τ is a topology of $\mathbb{R}^n$.

Proof. 2.1 ()

We prove the following three properties.

1. $\emptyset, \mathbb{R}^n$ are open.
2. For any collection $\{G_\alpha\}_\alpha$ of open sets, $\cup_\alpha G_\alpha$ is open.
3. For any finite collection $G_1, \ldots, G_n$ of open sets, $\cap_{i=1}^n G_i$ is open.

Listed.

1. Let $x \in \cup_\alpha G_\alpha$. Then, $x \in G_k$ for some k and since G_k is open, there exists a $B_\epsilon(x) \subset G_k \subset \cup_\alpha G_\alpha$, proving that $\cup_\alpha G_\alpha$ is open.
2. Let $x \in \cap_{i=1}^n G_i$. Then, $x \in G_i$ for every i, and so for each G_i, there exists an $\epsilon_i > 0$ s.t. $B_{\epsilon_i}(x) \subset G_i$. Since the set $\{e_i\}$ is finite, we can take
$$\epsilon = \min_i \{\epsilon_i\}$$
and see that $B_\epsilon(x) \subset G_i$ for all i, which implies that $B_\epsilon(x) \subset \cap_{i=1}^n G_i$. Since we have proved the existence of ϵ, $\cap_{i=1}^n G_i$ is open.

Definition 2.2 (Open Set)

An **open set** is an element of τ. An **open neighborhood**, or sometimes just the **neighborhood**, of $x \in \mathbb{R}^n$ is an open set U_x containing x. A **punctured neighborhood** is $U_x^\circ = U_x \setminus \{x\}$.

Theorem 2.2 (Equivalence to Open Ball Topology)

τ is equal to the topology τ' generated by the basis $\mathscr{B}$ of open balls
$$B(x, r) \coloneqq \{y \in \mathbb{R}^n \mid ||x - y|| < r\} \tag{90}$$

Proof. 2.2 ()

Let τ be the Euclidean topology and τ' be the open ball topology.

1. We show $\tau \subset \tau'$.

2. We show $\tau' \subset \tau$.

By defining the topology, we have automatically defined a bunch of topological objects and properties. For clarification, we will restate them.

Corollary 2.1 ()

An open ball is an open set.

Proof. 2.3 ()

Given $x \in B_r(p)$, we can imagine that x will always have some space between it and the boundary. We want to show that there exists some $\epsilon > 0$ s.t. $B_\epsilon(x) \subset B_r(p)$. That is, given any point $y \in B_\epsilon(x)$, we can show that $y \in B_r(p)$. Since $||x - p|| < r$, there exists some space $0 < r - ||x - p||$. There always exists a real number $0 < \epsilon < r - ||x - p||$, so given $y \in B_\epsilon(x)$, we can bound

$$||y - p|| = ||y - x + x - p|| \leq ||y - x|| + ||x - p|| \leq \epsilon + ||x - p|| < r \tag{91}$$

Example 2.1 ()

Here are some examples of sets which are open and not open.

1. $U = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \neq 1\}$ is open since for every point $x \in U$, we just need to find a radius $\epsilon > 0$ that is smaller than its distance to the unit circle.
2. $(a, b) \times (c, d) \subset \mathbb{R}^2$ is open since given a point x, we can take the minimum of its distance between the two sides of the rectangle and construct an open ball.
3. $S = \{(x, y) \in \mathbb{R}^2 : xy \neq 0\}$ is open since given a point $x \in S$, we can take the minimum of the distance between it and the x and y axes.
4. The set of all complex z such that $|z| \leq 1$ is not open since we cannot construct open balls at the boundary points that are fully contained in the set.
5. The set $S = \{1/n\}_{n \in \mathbb{N}}$ is not open since given any point $x = 1/n$, we can construct an open ball with radius $\epsilon < 1/(n + 1)$, which contains irrationals that are not in S.

Definition 2.3 (Interior Point)

A point $p \in S$ is an **interior point** if there exists an neighborhood N of p such that $N \subset S$.

An interior point means that we can always contain the point in S with some "breathing room." By definition an open set is a set where all of its points are interior points. A set is then said to be open if every point has this breathing room. This can be useful when defining differentiation at a point within an open set, since we can always find a neighborhood to take limits on.

Now that we have defined the Euclidean topology, we will prove that the features of topological objects can be reduced to features in $\mathbb{R}^n$.

Theorem 2.3 (Convexity)

An open ball is convex in a normed vector space.

Proof. 2.4 ()

The normed part is important here, as the properties of the metric is not sufficient. Given $B_r(p)$,

$x, y \in B_r(p)$ implies that $||x - p|| < r$ and $||y - p|| < r$. Therefore,

$$\begin{aligned} ||tx + (1-t)y - p|| &= ||tx - tp + (1-t)y - (1-t)p|| && (92) \\ &\leq t||x - p|| + (1-t)||y - p|| && (93) \\ &= tr + (1-t)r = r && (94) \end{aligned}$$

What happens if we weaken it to a metric?

2.2 Limit Points and Closure

Definition 2.4 (Limit Point)

A point $p \in \mathbb{R}^n$ is a **limit point** of $S \subset \mathbb{R}^n$ if every punctured neighborhood of p has a nontrivial intersection with X.[a] The set of all limit points of S is denoted S'.

Theorem 2.4 ()

Let $A_1, \ldots, A_n$ be a finite collection of sets. Then

$$\bigcup_{i=1}^n A_i' = \bigg(\bigcup_{i=1}^n A_i \bigg)'$$

Proof. 2.5 ()

Let the LHS be W and the RHS be V. If $x \in W$, $x \in A_i'$ for some i, and so for all $\epsilon > 0$, there exists a $B_\epsilon^\circ(x)$ s.t.

$$B_\epsilon^\circ(x) \cap A_i \neq \emptyset \implies B_\epsilon^\circ(x) \cap \bigg(\bigcup_{i=1}^n A_i \bigg) \neq \emptyset$$

which means that $x \in V$. Now assume that $x \in V$. Then for all $\epsilon > 0$, there exists a $B_\epsilon^\circ(x)$ s.t.

$$B_\epsilon^\circ(x) \cap \bigg(\bigcup_{i=1}^n A_i \bigg) \neq \emptyset$$

which implies that $B_\epsilon^\circ(x) \cap A_i \neq \emptyset$ for some i, which means that $x \in A_i' \subset W$.

A closed set can be defined in many equivalent ways for arbitrary topological spaces. The more general proof is done in topology, but we still prove it in the context of analysis.

Definition 2.5 (Closed Set)

A **closed set** $S \in \mathbb{R}^n$ is a set that contains all of its limit points.

Theorem 2.5 (Alternative Definition of Closed Set)

A set S is closed iff S^c is open.

[a] The definition just means that if we take a point and draw smaller and smaller circles around it, the circle itself should still overlap with S, no matter how small it gets.

Proof. 2.6 ()

We prove both ways:

1. ($\rightarrow$) Given that S is closed, then let $x \in S^c$. x is not a limit point of S since if it were, then it would be in S, and so there exists a punctured open neighborhood $B_\epsilon^\circ(x)$ of x s.t. $S \cap B_\epsilon^\circ(x) = \emptyset$. Since $x \notin S$, we also have $S \cap B_\epsilon(x) = \emptyset$, which implies that $B_\epsilon(x) \subset S^c$. Since for every $x \in S^c$, there exists a $B_\epsilon(x) \subset S^c$, S^c is open.
2. ($\leftarrow$) For simplicity, it suffices to prove if S open, then S^c is closed. Given that S is open, we have for every $x \in S$, there exists $B_\epsilon(x) \subset S$, which implies that $B_\epsilon(x) \cap S^c = \emptyset$. Since there exists an $B_\epsilon(x)$ that does not contain points in S^c, x cannot be a limit point of S^c, and so there exists no limit points of S^c in S. Therefore, all limit points of S^c are in S^c, proving that S^c is closed.

Theorem 2.6 ()

We have the following topological properties:

1. For any collection $\{F_\alpha\}_\alpha$ of closed sets, $\cap_\alpha F_\alpha$ is closed.
2. For any finite collection $F_1, \ldots, F_n$ of open sets, $\cup_{i=1}^n F_i$ is closed.

Proof. 2.7 ()

Listed.

1. Let x be a limit point of $\cap_\alpha F_\alpha$, and we want to show that $x \in \cap_\alpha F_\alpha$. By definition of limit points, for every $\epsilon > 0$, we have
$$B_\epsilon(x) \cap \left(\bigcap_\alpha F_\alpha \right)$$
which means that $B_\epsilon(x) \cap F_\alpha \neq \emptyset$ for all α. This means that x is a limit point for every F_α, and since they are all closed, $x \in F_\alpha$ for all α, which implies that $x \in \cap_\alpha F_\alpha$.

We can intuitively see a few properties about this. First, a finite set S of points does not have any limit points, since if we draw small enough circles around a $p \in S$, then at some point the circle will not contain any more points (remember that we're talking about deleted neighborhoods). Following this, we can deduce that a limit point must always have an infinite number of points close to it, as in no matter how small the circle gets, there are always an infinite number of points contained within that circle. This also means that if p is a limit point, then we can construct a sequence of points in S that converges to p, since every open ball with smaller and smaller radii will still have points in S.

Theorem 2.7 ()

If p is a limit point of S, then every neighborhood of p contains infinitely many points of S. The converse is also true trivially.

Proof. 2.8 ()

Assume p is a limit point and that there exists a finite number of points within a deleted neighborhood $B_r^\circ(p)$. Then, we can enumerate them $p_1, p_2, \ldots, p_n$ by their distances to p, with

$$d(p_1, p) \leq d(p_2, p) \leq \ldots \leq d(p_n, p) \tag{95}$$

Since $p_1 \neq p$, we have $d(p_1, p) > 0$ and so, we can choose an $0 < \epsilon < d(p_1, p)$ s.t. $B_\epsilon^\circ(p)$ does not contain any of the p_i's. This neighborhood does not contain any elements of S and so p is not a limit point.

Corollary 2.2 ()

A finite set has no limit points.

Proof. 2.9 ()

If S is a finite set, then every neighborhood of every point p in $\mathbb{R}^n$ will have at most finite points, which, by the previous theorem, is not a limit point.

We show a very useful result that will make things much more convenient when proving the following theorems and exercises. This is quite intuitive, since it shows that the limit points of a finite union of sets is the same as the finite union of the limit points of each set. This is clearly not true for infinite unions:

1. Look at the countable set $\mathbb{Q} \subset \mathbb{R}$. Each $\{q\}' = \emptyset$, but $\mathbb{Q}' = \mathbb{R}$.
2. Look at the uncountable set $\mathbb{R}$. Each $\{x \in \mathbb{R}\}' = \emptyset$, but $\mathbb{R}' = \mathbb{R}$.

Now, we give two more definitions for convenience of deriving open and closed sets from any arbitrary set.

Definition 2.6 (Closure)

Given a set S, let the set of all limit points of S be denoted S'. The **closure** of S is the set $\overline{S} = S \cup S'$. It is the smallest closed set that contains S.

Definition 2.7 (Interior)

Given a set S, the **interior** of S is denoted S°, the set of all interior points of S. It is the largest open set that is within S.

Theorem 2.8 ()

Let E be a nonempty set of real numbers which is bounded above. Let $y = \sup E$. Then $y \in \overline{E}$. Hence $y \in E$ if E is closed.

Proof. 2.10 ()

Assume that y is not a limit point of E. Then, there exists some $\epsilon > 0$ s.t. $(y - \epsilon, y + \epsilon)$ does not intersect with E. This means that $y - \epsilon$ is an upper bound of E, and so y is not the supremum.

2.3 Compactness

Definition 2.8 (Open Cover)

An **open cover** of a set E in a metric space X is a collection $\{G_\alpha\}$ of open subsets of X such that $E \subset \cup_\alpha G_\alpha$.

Definition 2.9 (Compact Set)

A subset S of a metric space X is said to be **compact** if every open cover of S contains a finite subcover.

While openness behaves differently depending on its embedding space, compactness stays constant. Therefore, we don't have to worry about talking about which space a compact set is embedded in.

Theorem 2.9 ()

Suppose $K \subseteq Y \subseteq X$. Then K is compact relative to X if and only if K is compact relative to Y.

Proof. 2.11 ()

Theorem 2.10 ()

A finite union of compact sets is compact.

Proof. 2.12 ()

It suffices to prove for two sets A, B by induction. Take an arbitrary cover $\mathscr{L}$ of $A \cup B$. Then $\mathscr{L}$ is a cover of A, so it has a finite subcover $\mathscr{F} \subset \mathscr{L}$. It is also a cover of B, so it has a finite subcover $\mathscr{G} \subset \mathscr{L}$. Therefore, $\mathscr{F} \cup \mathscr{G} \subset \mathscr{L}$ is a cover of $A \cup B$, and since it is the union of finite covers, it is finite.

Theorem 2.11 ()

Compact subsets of metric spaces are closed.

Proof. 2.13 ()

We would like to show that if A is compact in X, then A^c is open. What we would like to do is if we have some $x \in A^c$, then we must prove that there exists some open set $B_\epsilon(x)$ that is disjoint with A. For every point $a \in A$, we can construct an open balls $V_a = B_{d(x,a)/2}(a)$ and $U_a = B_{d(x,a)/2}(x)$. We know that if $y \in B_{d(x,a)/2}(a)$, then assuming $y \in B_{d(x,a)/2}(x)$ will give

$$d(x,a) \leq d(x,y) + d(y,a) < \frac{d(x,a)}{2} + \frac{d(x,a)}{2} = d(x,a) \tag{96}$$

which is absurd. Since $\{V_a\}_{a \in A}$ forms an open covering of A, then by compactness we can take a finite subcover $V_{a_1}, \ldots, V_{a_n}$, along with the respective neighborhoods of x $U_{a_1}, \ldots, U_{a_n}$. Since we have established

$$V_{a_i} \cap U_{a_i} = \emptyset \implies \bigcap_{i=1}^n V_{a_i} \cap \left(\bigcup_{i=1}^n U_{a_i} \right) = \emptyset \tag{97}$$

and since $\cap_{i=1}^n V_{a_i}$ is open (as it is the intersection of open sets) and disjoint from an open cover of A and hence from A, we have proved that A^c is open, and so A is closed.

The general notion of compactness[6] for topological spaces is not needed for analysis. Rather, we make use of the following theorem which allows us to focus on the compactness of subsets in Euclidean spaces $\mathbb{R}^n$.

[6] According to Terry Tao, a compact set is "small," in the sense that it is easy to deal with. While this may sound counterintuitive at first, since $[0, 1]$ is considered compact while $(0, 1)$, a subset of $[0, 1]$, is considered noncompact. More generally, a set that is compact may be large in area and complicated, but the fact that it is compact means we can interact with it in a finite way using open sets, the building blocks of topology. That finite collection of open sets makes it possible to account for all the points in a set in a finite way. This is easily noticed, since functions defined over compact sets have more controlled behavior than those defined over noncompact sets. Similarly, classifying noncompact spaces are more difficult and less satisfying.

Theorem 2.12 (Heine-Borel)

Let $E \subset \mathbb{R}^k$. The following are equivalent.

1. E is closed and bounded
2. E is compact.
3. Every infinite subset of E has a limit point in E.

Example 2.2 ()

An open set in $\mathbb{R}^2$ is not compact. Take the open rectangle $R = (0, 1)^2 \subset \mathbb{R}^2$. There exists an infinite cover of R

$$R = \bigcup_{n=0}^{\infty} (0, 1) \times \left(0, \frac{2^{n+1} - 1}{2^{n+1}}\right)$$

that does not have a finite subcover.

Theorem 2.13 ()

Closed subsets of compact sets are compact.

Proof. 2.14 ()

Theorem 2.14 ()

If F is closed and K is compact, then $F \cap K$ is compact.

Clearly, the limit point of an open set is its boundary points. Note that a sequence of points can also have a limit point.

Theorem 2.15 (Bolzano-Weierstrass Theorem)

Every bounded infinite sequence in $\mathbb{R}^n$ has an accumulation point. That is, there exists a point $p \in \mathbb{R}^n$ such that every open neighborhood U_p contains an infinite subset of the sequence.

Proof. 2.15 ()

The fact that the infinite sequence is bounded means that there exists some closed subset $I \in \mathbb{R}^n$ that contains all point of the sequence. By definition I is compact, so by the Heine-Borel theorem, every cover of I has a finite subcover.

Now, assume that there exists an infinite sequence in I that is not convergent, i.e. has no limit point. Then, each point $x_i \in I$ would have a neighborhood $U(x_i)$ containing at most a finite number of points in the sequence. We can define I such that the union of the neighborhoods is a cover of I. That is,

$$I \subset \bigcup_{i=1}^{\infty} U(x_i)$$

However, since every $U(x_i)$ contains at most a finite number of points, we must have an infinite open neighborhoods to cover $I \implies$ we cannot have a finite subcover. This contradicts the fact that I is compact.

2.4 Connectedness

Definition 2.10 (Separate, Connected Sets)

Two subsets A and B of a metric space X are said to be **separated** if both $A \cap \overline{B}$ and $\overline{A} \cap B$ are empty, i.e. if no point of A lies in the closure of B and no point of B lies in the closure of A.

Example 2.3 ()

It is clear that separate sets imply disjointness. However, this is not true for the other way around.
1. $(0, 1)$ and $[1, 2)$ are disjoint but not separate.
2. The rationals and irrationals are disjoint, but not separate.

Theorem 2.16 ()

A subset E of the real line $\mathbb{R}$ is connected if and only if it has the following property: if $x \in E, y \in E$ and $x < z < y$, then $z \in E$.

Proof. 2.16 ()

2.5 Separability

2.6 Perfect Sets

Definition 2.11 (Perfect Sets)

A set P is perfect if it is closed and all of its points are limit points of P. In other words, the limit points of P and P itself coincide.

$$P' = P \tag{98}$$

Theorem 2.17 ()

Let P be a nonempty perfect set in $\mathbb{R}^k$. Then P is uncountable.

2.7 Exercises

Exercise 2.1 (Rudin 2.1)

Prove that the empty set is a subset of every set.

Solution 2.1

It must suffice that if $x \in \emptyset$, then $x \in A$ for any arbitrary set A. This is vacuously true, since the initial condition is never met.

Exercise 2.2 ()

Show that the empty function $f : \emptyset \to X$, where X is an arbitrary set, is always injective. If $X = \emptyset$, then f is bijective.

Solution 2.2

Given distinct $x, y \in \emptyset$, $f(x) \neq f(y)$ is vacuously true, but if $X \neq \emptyset$, then there exists a $w \in X$ with no preimage. If $X = \emptyset$, then the statement for all $w \in X$, there exists an $x \in \emptyset$ s.t. $f(x) = w$ is vacuously true.

Exercise 2.3 (Rudin 2.2)

A complex number z is said to be algebraic if there are integers $a_0, a_1, \ldots, a_n$, not all zero, such that

$$a_0 z^n + a_1 z^{n-1} + \ldots + a_{n-1} z + a_n = 0.$$

Prove that the set of all algebraic complex numbers is countable. Hint: For every positive integer N there are only finitely many equations with

$$n + |a_0| + |a_1| + \ldots + |a_n| = N$$

Solution 2.3

Consider all polynomials s.t. $n + \sum_{i=0}^{n} |a_i| = N$. There is only a finite number of them, and each polynomial has at most n distinct complex roots. So this set is finite, an unioning over all $N \in \mathbb{N}$ gives an at most countable set of roots.

Exercise 2.4 (Rudin 2.3)

Prove there exists real numbers which are not algebraic.

Solution 2.4

From the previous exercise, if there were no no real numbers which are not algebraic, then every real number is algebraic. This contradicts the fact that the set of all complex numbers is countable.

Exercise 2.5 (Rudin 2.4)

Is the set of all irrational real numbers countable?

Solution 2.5

No. Assume that it is countable. We have $\mathbb{Q}$ countable. Then, by assumption, we must have $\mathbb{R} = \mathbb{Q} \cup \mathbb{Q}^c$ be the union of countable sets, which must be countable, contradicting the fact that it is uncountable.

Exercise 2.6 (Rudin 2.5)

Construct a bounded set of real numbers which exactly 3 limit points.

Solution 2.6

We can construct the union of 3 sequences that converge onto the limit points $0, 1, 2$.

$$\left\{\frac{1}{n}\right\}_{n \in \mathbb{N}} \cup \left\{\frac{1}{n} + 1\right\}_{n \in \mathbb{N}} \cup \left\{\frac{1}{n} + 2\right\}_{n \in \mathbb{N}}$$

Exercise 2.7 ()

Prove that the union of the limit points of sets is equal to the limit points of the union of the sets.

$$\bigcup_{k=1}^{m} A_k' = \left(\bigcup_{k=1}^{m} A_k\right)'$$

Exercise 2.8 (Rudin 2.6)

Let E' be the set of all limit points of a set E. Prove that E' is closed. Prove that E and $\overline{E}$ have the same limit points. (Recall that $\overline{E} = E \cup E'$). Do E and E' always have the same limit points?

Solution 2.7

Listed.

1. Let x be a limit point of E'. Then, for every $\epsilon > 0$, $U = B_\epsilon(x) \cap E' \neq \emptyset$. Take a $y \in U$. Since $y \in B_\epsilon(x)$, which is open, we can construct an open ball $B_\delta(y) \subset B_\epsilon(x)$. Since $y \in E'$, $B_\delta(y)$ must contain elements of E, which means that $B_\epsilon(x)$ must also contain elements of E, and so x is a limit point of $E \implies x \in E'$ and E' is closed.
2. To prove that $E' \subset \overline{E}'$, we know that if $x \in E'$, then for every $\epsilon > 0$, there exists a $B_\epsilon^\circ(x)$ that has a nontrivial intersection with E which means that it has a nontrivial intersection with $E \cup E'$. To prove that $\overline{E}' \subset E'$, we know that if $y \in \overline{E}'$, then for every $\delta > 0$ there exists a $B_\delta(x)$ that has a nontrivial intersection with $\overline{E}$. If $B_\delta(x)$ intersects E then we are done. If $B_\delta(x)$ intersects E', then we can find a $y \in E' \cap B_\delta(x)$. Since $B_\delta(x)$ is open, we can construct $B_\varepsilon(y) \subset B_\delta(x)$ and since $y \in E'$, we know that $B_\varepsilon(y)$ contains an element of E, which means that $B_\delta(x)$ contains an element of E. Therefore, $E' = \overline{E}'$.
3. No. Consider the set $E = \{1/n\}_{n\in\mathbb{N}}$. $E' = \{0\}$, but $E'' = \emptyset$.

Exercise 2.9 (Rudin 2.7)

Let $A_1, A_2, \ldots$ be subsets of a metric space.

1. If $B_n = \cup_{i=1}^n A_i$, prove that $\bar{B}_n = \cup_{i=1}^n \bar{A}_i$ for $n = 1, 2, 3, \ldots$
2. If $B = \cup_{i=1}^\infty A_i$, prove that $\bar{B} \supset \cup_{i=1}^\infty \bar{A}_i$.

Solution 2.8

Listed.

1. We will prove that $\overline{B_n} \subseteq \cup_{i=1}^n \overline{A_i}$ and $\cup_{i=1}^n \overline{A_i} \subseteq \overline{B_n}$. If $x \in B_n$, then $x \in \cup_{i=1}^n A_i$. Therefore, assume that $x \in B_n'$. Then for every $\epsilon > 0$, there exists a $B_\epsilon^\circ(x)$ s.t.

$$B_\epsilon^\circ(x) \cap B_n \neq \emptyset \implies B_\epsilon^\circ(x) \cap \left(\bigcup_{i=1}^{n} A_n\right) \neq \emptyset$$

This means that there exists some $i = i(\epsilon)$, a function of ϵ, s.t. $B_\epsilon^\circ(x) \cap A_i \neq \emptyset$. However, this i may change if we unfix ϵ. We have so far proved that just for one $\epsilon > 0$ there exists an i. Now if we take a sequence of $\epsilon = 1, \frac{1}{2}, \frac{1}{3}, \ldots$, we have a sequence of $i(\epsilon)$'s living in $\{1, \ldots, n\}$. By the pigeonhole principle, there must be at least one i that is hit infinitely many times, and so we can choose this i, that works for all $\epsilon > 0 \implies x \in A_i' \subseteq \cup_{i=1}^n \overline{A_i}$. If $x \in \cup_{i=1}^n \overline{A_i}$, then there exists an $\overline{A_i}$ s.t. $x \in \overline{A_i}$. If $x \in A_i$, then we are done. If $x \in A_i'$, then for every $\epsilon > 0$, there exists a $B_\epsilon^\circ(x)$ s.t.

$$B_\epsilon^\circ(x) \cap A_i \neq \emptyset \implies B_\epsilon^\circ(x) \cap \left(\bigcup_{i=1}^n A_i \right) \neq \emptyset$$

and so $x \in B_n' \subset \overline{B_n}$.

2. $x \in \cup_{i=1}^\infty \overline{A_i} \implies x \in \overline{A_i}$ for some i. If $x \in A_i$, then $x \in B$ and we are done. If $x \in A_i'$, then for every $\epsilon > 0$ there exists $B_\epsilon(x)$ s.t.

$$B_\epsilon^\circ(x) \cap A_i \neq \emptyset \implies B_\epsilon^\circ(x) \cap \left(\bigcup_{i=1}^\infty A_i \right) \neq \emptyset$$

and so $B_\epsilon^\circ(x) \cap B \neq \emptyset \implies x \in B' \subset \overline{B}$.

Exercise 2.10 (Rudin 2.8)

Is every point of every open set $E \subset \mathbb{R}^2$ a limit point of E? Answer the same question for closed sets in $\mathbb{R}^2$.

Solution 2.9

Yes for open. Given any $x \in U$ open, there always exists an $\epsilon > 0$ s.t.

$$B_\epsilon^\circ(x) \subset B_\epsilon(x) \subset U$$

and so $B_\epsilon^\circ(x)$ has a nontrivial intersection with U. If U is closed, then no. Note that for closed U, we have that every limit point is in U, but not every point in U is a limit point. Consider the isolated point $U = \{x\}$. x is not a limit point of U.

Exercise 2.11 (Rudin 2.9)

Let E° denote the set of all interior points of E in X. Prove the following:

(a) E° is always open.
(b) E is open if and only if $E^\circ = E$.
(c) If $G \subseteq E$ and G is open, then $G \subset E^\circ$.
(d) Prove that the complement of E° is the closure of the complement of E.
(e) Do E and $\bar{E}$ always have the same interiors?
(f) Do E and E° always have the same closures?

Solution 2.10

Listed.

1. We assume that E° is not open (this does not mean that E° is necessarily closed!). That is, there exists an $x \in E^\circ$ s.t. we can't construct an open ball $B_\epsilon(x) \subseteq E^\circ$. Since $x \in E^\circ \subset E$, by definition of an interior point we can construct a $B_\epsilon(x) \subset E$. But from our assumption $B_\epsilon(x) \not\subset E^\circ$. We choose a $y \in B_\epsilon(x) \setminus E^\circ$. Since $B_\epsilon(x)$ is open, there exists a $\delta > 0$ s.t.

$$B_\delta(y) \subset B_\epsilon(x) \subset E$$

But the fact that we can construct an open ball around y means that $y \in E^\circ$, leading to a contradiction.

2. If E is open, then by definition $E \subset E^\circ$. Now $E^\circ \subset E$ holds for all sets since E° must be composed of points from E. If $E = E^\circ$, then for every $x \in E$, $x \in E^\circ$, so by definition there exists an $\epsilon > 0$ s.t. $B_\epsilon(x) \subset E$, which means that E is open.

3. Let $x \in G$ open. Then there exists an $\epsilon > 0$ s.t. $B_\epsilon(x) \subset G$, and so $B_\epsilon(x) \subset E$. Since we can always construct an open ball around x contained within E, $x \in E^\circ$ and $G \subset E^\circ$.
4. ($(E^\circ)^c \subset \overline{E^c}$) If $x \in (E^\circ)^c$, then there exists no $\epsilon > 0$ s.t. $B_\epsilon(x) \subset E$. Then, for any $\epsilon > 0$, $B_\epsilon(x) \not\subset E \implies B_\epsilon(x) \cap E^c \neq \emptyset \implies x \in E^c \subset \overline{E^c}$. ($\overline{E^c} \subset (E^\circ)^c$) If $x \in \overline{E^c}$, then $x \in E^c$ or $x \in E^{c\prime}$. If $x \in E^c$, note $E^\circ \subset E \implies (E^\circ)^c \supset E^c \implies x \in (E^\circ)^c$. If $x \in E^{c\prime}$, then for all $\epsilon > 0$ $B_\epsilon(x) \cap E^c \neq \emptyset \implies B_\epsilon(x) \not\subset E \implies x \in E^\circ$.
5. No. Consider the rationals $\mathbb{Q} \subset \mathbb{R}$. $\mathbb{Q}^\circ = \emptyset$ but $\overline{\mathbb{Q}}^\circ = \mathbb{R}^\circ = \mathbb{R}$. It is true and straightforward to prove that $E^\circ \subset \overline{E}^\circ$. Let $x \in E^\circ$. Then there exists an $\epsilon > 0$ s.t. $B_\epsilon(x) \subset E \implies B_\epsilon(x) \subset \overline{E} \implies x \in \overline{E}^\circ$.
6. No. Consider $\mathbb{Q} \subset \mathbb{R}$. Then $\overline{\mathbb{Q}} = \mathbb{R}$ and $\overline{\mathbb{Q}^\circ} = \overline{\emptyset} = \emptyset$.

Exercise 2.12 (Rudin 2.10)

Let X be an infinite set. For $p \in X$ and $q \in X$, define

$$d(p, q) = \begin{cases} 1 & \text{if } p \neq q \\ 0 & \text{if } p = q \end{cases}$$

Prove that this is a metric. Which subsets of the resulting metric space are open? Which are closed? Which are compact?

Solution 2.11

This is a metric since clearly it satisfies symmetry and the fact that $d(p, p) = 0$. The triangle inequality

$$d(p, r) \leq d(p, q) + d(q, r)$$

is trivially satisfied if $p = r$, and if $p \neq r$, then either $p \neq q$ or $q \neq r$, and so the RHS ≥ 1. An open ϵ-ball around $x \in X$ is either X, when $\epsilon > 1$, or $\{x\}$ when $\epsilon \leq 1$. Therefore

Exercise 2.13 (Rudin 2.11)

For $x \in \mathbb{R}$ and $y \in \mathbb{R}$, define

$$\begin{aligned} d_1(x, y) &= (x - y)^2 \\ d_2(x, y) &= \sqrt{|x - y|} \\ d_3(x, y) &= |x^2 - y^2| \\ d_4(x, y) &= |x - 2y| \\ d_5(x, y) &= \frac{|x - y|}{1 + |x - y|} \end{aligned}$$

Determine, for each of these, whether it is a metric or not.

Solution 2.12

Listed. Positive semidefiniteness and symmetry are easy to check.

1. The triangle inequality gives

$$\begin{aligned} d_1(x, z) \leq d_1(x, y) + d_1(y, z) &\iff (x - z)^2 \leq (x - y)^2 + (y - z)^2 \\ &\iff 0 \leq (x - y)(y - z) \end{aligned}$$

which is not satisfied if $x < y < z$, so this is not a valid metric.

2. The triangle inequality gives $\sqrt{|x-z|} \leq \sqrt{|x-y|} + \sqrt{|y-z|}$, and since both sides are positive this inequality is equivalent to squaring both sides to get
$$|x-z| \leq |x-y| + |y-z| + 2\sqrt{|x-y||y-z|}$$
which is true since $|x-z| \leq |x-y| + |y-z|$ of the Euclidean distance satisfies the triangle inequality and $0 \leq \sqrt{|x-y||y-z|}$.
3. This does not satisfy triangle inequality, as taking $0, 1, 2$ gives
$$d_3(0,2) = 4 > 1 + 1 = d_3(0,1) + d_3(1,2)$$
4. This does not satisfy symmetry.
5. For simplicity, let us set $A = |x-y|, B = |y-z|, C = |x-z|$. Then, we get
$$\frac{C}{1+C} \leq \frac{A}{1+A} + \frac{B}{1+B} \iff C \leq A + B + 2AB + ABC$$

where $C \leq A + B$ is true by triangle inequality of Euclidean distance, $0 \leq AB$, and $0 \leq ABC$. Intuitively, we want a metric that doesn't "blow up" the distance between x and y. More precisely, we want a valid metric $d(x,y)$ to be $O(|x-y|)$. Having something like a quadratic growth rate $(x-y)^2$ will blow the distance $d(x,z)$ up too much overpowering the individual $d(x,y) + d(y,z)$.

Exercise 2.14 (Rudin 2.12)

Let $K \subset \mathbb{R}$ consist of 0 and the numbers $1/n$ for $n = 1, 2, 3, \ldots$. Prove that K is compact directly from the definition (without using the Heine-Borel theorem).

Proof. 2.17 ()

Every open cover of K must have an open set G s.t. $0 \in G$. Since G is open, there exists an open neighborhood $B_\epsilon(0) \subset G$ around 0. By the Archimidean principle, there exists an $N \in \mathbb{N}$ s.t.
$$\epsilon N > 1 \implies \epsilon > \frac{1}{N}$$
and so, $B_\epsilon(0)$ contains all points $\{1/n\}$ for $n > N$. For the rest of the points $1, 1/2, \ldots, 1/N$, we can simply construct a finite cover over each of them, hence getting a finite cover.

Exercise 2.15 (Rudin 2.13)

Construct a compact set of real numbers whose limit points form a countable set.

Solution 2.13

Consider the set
$$E = \left\{ \left(\frac{1}{10}\right)^n + \left(\frac{1}{10}\right)^{n+k} : n \in \{0\} \cup \mathbb{N}, k \in \mathbb{N} \right\} \cup \{0\}$$
This is clearly bounded by 0 and 1.1. Let us represent the elements of this set by (n, k). We can show that
$$(n_1, k_1) > (n_2, k_2)$$
if $n_1 < n_2$ or $n_1 = n_2$ and $k_1 < k_2$. Therefore, to prove closedness, we must prove that every limit point is a point in E. We can do this by proving that a point not in E cannot be a limit point.

Choose any $x \notin E$. Then, due to the ordering, we can see that there exists a (n, k) s.t.

$$A = \left(\frac{1}{10}\right)^n + \left(\frac{1}{10}\right)^{n+k} < k < \left(\frac{1}{10}\right)^n + \left(\frac{1}{10}\right)^{n+k+1} = B$$

and so we can take $\epsilon = \min\{k - A, B - k\}$ and show that $B_\epsilon(x)$ does not contain A nor B, and so has an empty intersection with E. Therefore, it cannot be a limit point of E and is closed. Since E is bounded and closed in $\mathbb{R}$, it is compact. Its limit points contain $1, 0.1, 0.01, \ldots, 0$ (simply by fixing n and letting $k \to \infty$, and so E' is infinite. We have just shown that since E is closed, $E' \subset E$. But E is countable, so E' is countable.

Exercise 2.16 (Rudin 2.14)

Given an example of an open cover of the segment $(0, 1)$ which has no finite subcover.

Solution 2.14

Consider

$$(0, 1/2) \cup \left(\bigcup_{i=1}^\infty \left[1 - \frac{1}{2^i}, 1 - \frac{1}{2^{i+1}}\right) \right)$$

Exercise 2.17 (Rudin 2.15)

Exercise 2.18 (Rudin 2.16)

Regard $\mathbb{Q}$, the set of all rational numbers, as a metric space, with $d(p, q) = |p - q|$. Let E be the set of all $p \in \mathbb{Q}$ s.t. $2 < p^2 < 3$. Show that E is closed and bounded in $\mathbb{Q}$ but is not compact. Is E open in $\mathbb{Q}$?

Solution 2.15

E is clearly bounded by 0 and 2 since $0^2 < 2 < p^2 < 3 < 2^2$. It is closed and we can show this by showing that E^c is open. Let $x \in E^c$. Then, $x^2 < 2$ or $x^2 > 3$.

1. $x^2 < 2 \iff -\sqrt{2} < x < \sqrt{2}$. Now let $\epsilon = \min\{\sqrt{2} - x, x + \sqrt{2}\} > 0$. Then by the Archimidean property there exists a $n \in \mathbb{N}$ s.t. $0 < \frac{1}{n} < \epsilon$. Therefore, the image of $B_{1/n}(x) \subset \mathbb{Q}$ will map onto $(0, 2)$.
2. $x^2 > 3 \iff x > \sqrt{3}$ or $x < -\sqrt{3}$. If $x > \sqrt{3}$, then by AP there exists a $n \in \mathbb{N}$ s.t. $x - \frac{1}{n} > \sqrt{3} \implies (x - \frac{1}{n})^2 > 3$. If $x < -\sqrt{3}$, then by AP there exist $n \in \mathbb{N}$ s.t. $x + \frac{1}{n} < -\sqrt{3} \implies (x + \frac{1}{n})^2 > 3$. Either way, the image of $B_{1/n}(x)$ will map within E^c.

It is not compact because E is not closed in $\mathbb{R}$. The limit points of E in $\mathbb{R}$ is $[\sqrt{2}, \sqrt{3}] \cup [-\sqrt{3}, -\sqrt{2}]$, which contains irrationals and is clearly not a subset of E. Since it is not closed in $\mathbb{R}$, it is not compact in $\mathbb{R}$, and it is not compact in $\mathbb{Q} \subset \mathbb{R}$. It is open because

$$E = \big((\sqrt{2}, \sqrt{3}) \cup (-\sqrt{3}, \sqrt{2})\big) \cup \mathbb{Q} \subset \mathbb{R}$$

which is the union of open $(\sqrt{2}, \sqrt{3}) \cup (-\sqrt{3}, \sqrt{2})\big)$ and subset $\mathbb{Q} \subset \mathbb{R}$, and so it is open.

Exercise 2.19 (Rudin 2.17)

Let E be the set of all $x \in [0,1]$ whose decimal expansion consists of only the digits 4 and 7. Is E countable? Is E dense in $[0,1]$? Is E compact? Is E perfect?

Exercise 2.20 (Rudin 2.18)

Is there a nonempty perfect set in $\mathbb{R}$ which contains no rational number?

Exercise 2.21 (Rudin 2.19)

Listed.

1. If A and B are disjoint closed sets in some metric space X, prove that they are separated.
2. Prove the same for disjoint open sets.
3. Fix $p \in X$, $\delta > 0$, define A to be the set of all $q \in X$ for which $d(p,q) < \delta$. Define B similarly, with $>$ in place of $<$. Prove that A and B are separated.
4. Prove that every connected metric space with at least two points is uncountable.

Solution 2.16

Listed.

1. This is trivial with the fact that the closure of the closure of A is the closure of A.
2. Let A, B be open. We wish to show that if $x \in A'$, then $x \notin B$. Assume $x \in B$. Then there exists $\epsilon > 0$ s.t. $B_\epsilon(x) \subset B$. But $B \cap A = \emptyset \implies B_\epsilon(x) \cap A = \emptyset$ and so $x \notin A'$, which is a contradiction.
3. Clearly, $A \cap B = \emptyset$. Not let $x \in A \implies$ there exists $\epsilon > 0$ s.t. $B_\epsilon(x) \subset A \implies B_\epsilon(x) \cap B = \emptyset \implies x \in B'$. We can prove similarly to show that $x \in B \implies x \notin A'$.
4. Assume X is countable (proof is very similar for finite). Then, we can enumerate a $X = \{x_i\}_{i=1}^\infty$. We wish to show that X can be decomposed into the union of an open ball and the interior of its complement as shown in (3). We fix $p \in X$. Then, we take the set $D = \{d(p,x)\}_{x \neq p} \subset \mathbb{R}$. Since D is a countable subset of $\mathbb{R}$, there must exist some $\alpha > 0$ s.t. $\alpha \notin D$. This α partitions the distances into two sets, and we can define

$$X = \{q \in X \mid d(p,q) < \alpha\} \cup \{q \in X \mid d(p,q) > \alpha\}$$

and by (3), these two sets are separated, which means that X is not connected, leading to a contradiction.

Exercise 2.22 (Rudin 2.20)

Are closures and interiors of connected sets always connected? Look at subsets of $\mathbb{R}^2$.

Solution 2.17

The interiors are not always connected. Consider the two closed balls $\overline{B_1((1,0))}$ and $\overline{B_1((-1,0))}$ as subsets of $\mathbb{R}^2$. They are connected but their interiors, which are the two open balls, are not connected. As for closures, they are always connected. Let W be connected. Then for any partition $A \cup B = W$, $\overline{A} \cap B \neq \emptyset$ WLOG. Consider $\overline{W} = W \cup W'$ and take any partition $\overline{W} = C \cup D$. Then, label $A = C \cap W, A^* = C \cap W', B = D \cap W, B^* = D \cap W'$. This implies that $C = A \cup A^*, D = B \cup B^*$, and $A \cup B = W$ (which is connected). Then, we can show that

$$\begin{aligned}\overline{C} \cap D &= (\overline{A \cup A^*} \cap D) = (\overline{A} \cup \overline{A^*}) \cap D = (\overline{A} \cap D) \cup (\overline{A^*} \cap D) \\ &= (\overline{A} \cap B) \cup (\overline{A} \cap B^*) \cup (\overline{A^*} \cap D)\end{aligned}$$

which cannot be empty since by connectedness of W, $\overline{A} \cap B \neq \emptyset$. Therefore, $\overline{W}$ is connected.

Exercise 2.23 (Rudin 2.21)

Let A and B be separated subsets of some $\mathbb{R}^k$. Suppose $a \in A, b \in B$ and define

$$p(t) = (1-t)a + tb$$

for $t \in \mathbb{R}$. Put $A_0 = p^{-1}(A), B_0 = p^{-1}(B)$.

1. Prove that A_0 and B_0 are separated subsets of $\mathbb{R}$.
2. Prove that there exists a $t_0 \in (0,1)$ s.t. $p(t_0) \notin A \cup B$.
3. Prove that every convex subset of $\mathbb{R}^k$ is connected.

Exercise 2.24 (Rudin 2.22)

A metric space is called *separable* if it contains a countable dense subset. Show that $\mathbb{R}^k$ is separable.

Proof. 2.18 ()

Consider the set $\mathbb{Q}^k \subset \mathbb{R}^k$. It is a finite Cartesian product (and hence, a countable union) of countable $\mathbb{Q}$, and so it is countable. $\mathbb{Q}^k$ is dense in $\mathbb{R}^k$ since given any $x \in \mathbb{R}^k$, we claim x is a limit point of $\mathbb{Q}^k$. Given any $\epsilon > 0$, we can construct $B^\circ_\epsilon(x)$. For each coordinate x_i, by density of rationals in $\mathbb{R}$ we can choose a $q_i \in \mathbb{Q}$ s.t. $0 < d(x_i, q_i) < \epsilon/k$. Then, using triangle inequality, we can take the distances between each coordinate changed from x_i to q_i. Let q^k be the vector x with the components $x_1, \ldots, x_k$ changed to $q_1, \ldots, q_k$, respectively.

$$d(x, q) = d(x, q^1) + d(q^1, q^2) + \ldots + d(q^{k-1}, q_k) < \frac{\epsilon}{k} + \ldots + \frac{\epsilon}{k} = \epsilon$$

and so $q \in B^\circ_\epsilon(x)$. Hence the intersection of $\mathbb{Q}^k$ and $B^\circ_\epsilon(x)$ for any $\epsilon > 0$ is nontrivial, so x is a limit point of $\mathbb{Q}^k$.

Exercise 2.25 (Rudin 2.23)

A collection $\{V_\alpha\}$ of open subsets of X is said to be a *base* for X if the following is true: For every $x \in X$ and every open set $G \subset X$ such that $x \in G$, we have $x \in V_\alpha \subset G$ for some α. In other words, every open set in X is the union of a subcollection of $\{V_\alpha\}$. Prove that every separable metric space has a countable base.

Solution 2.18

Since X is separable it contains a countable dense subset, call it S. Then for every $x \in S$, we can look at the set of all open balls with center x and rational radii, call it $\mathcal{B}$. Then $\mathcal{B}$ is countable. Now consider an open set U. By definition, for every $x \in U$, there exists an $\epsilon > 0$ s.t. $B_\epsilon(x) \subset U$. By AP, we can find a $n \in \mathbb{N}$ s.t. $0 < \frac{1}{n} < \epsilon$, and therefore we can find an open ball $B \in \mathcal{B}$ s.t. $B(x) \subset U$. We claim that

$$W := \bigcup_{x \in U} B(x) = U$$

If $x \in U$, then by construction it is contained in $B(x) \subset \cup_{x \in U} B(x)$, and so $U \subset W$. If $x \in W$, then it is in $B(x)$, which is fully contained in U and so $W \subset U$. Therefore every open set can be constructed by a countable union of open balls in countable $\mathcal{B}$.

Exercise 2.26 (Rudin 2.24)

Let X be a metric space in which every infinite subset has a limit point. Prove that X is separable.

Solution 2.19

We fix $\delta > 0$. Choose $x_1 \in X$. Then choose $x_2 \in X$ s.t. $d(x_1, x_2) \geq \delta$, and keep doing this until we choose $x_{j+1} \in X$ s.t. $d(x_{j+1}, x_i) \geq \delta$ for all $i \in 1, \ldots, j$.

1. We claim that this must stop after a finite number of steps. Assume it doesn't. Then by assumption $V = \{x_i\}_{i=1}^\infty$ should have a limit point in X, denote it x. Choose $\frac{\delta}{2} > 0$. Then, $B^\circ_{\delta/2}(x) \cap V \neq \emptyset$. This intersection can only have one point since if it had two x', x'', then since both are in $B_{\delta/2}(x)$, then

$$d(x', x'') \leq d(x', x) + d(x, x'') \leq \frac{\delta}{2} + \frac{\delta}{2} = \delta$$

and since they are both in V, then $d(x', x'') \geq \delta$, which is a contradiction. Since there is a finite number of points in $B_{\delta/2}(x)$ of V, x cannot be a limit point. So this must terminate at some finite $J < \infty$.
2. Denote $W = \{x_i\}_{i=1}^J$. Then, $\mathscr{B}_\delta = \{B_\delta(x) \mid x \in W\}$ must cover X, since if it didn't, there would exist a $y \in X$ s.t. $d(y, x) \geq \delta$ for all $x \in W$, and we can add another element in W.
3. Consider $\delta = 1, 1/2, 1/3, \ldots$ and construct the same cover

$$\mathscr{B}_k = \{B_{1/k}(x_{ki}) \mid i = 1, \ldots, J_k\}$$

which is finite. Therefore, $\mathscr{B} = \cup_{k=1}^\infty \mathscr{B}_k$ must be countable.
4. We claim that countable $\{x_{ki}\}_{k,i}$ is dense. Consider any $x \in X$. For every $\epsilon > 0$, we can find an arbitrarily large $n \in \mathbb{N}$ s.t. $0 < \frac{1}{n} < \epsilon$. Since $\mathscr{B}_n$ is an open cover, there must exist some x_{ni} s.t. $x \in B_{1/n}(x_{ni})$, which by symmetry implies that $x_{ni} \in B_{1/n}(x) \subset B_\epsilon(x)$. Therefore, there always exists an x_{ni} in every $B_\epsilon(x)$, and so $B_\epsilon(x) \cap \{x_{ki}\} \neq \emptyset \implies x$ is a limit point of $\{x_{ki}\}$ and so it is dense.

Exercise 2.27 (Rudin 2.25)

Prove that every compact metric space K has a countable base, and that K is therefore separable.

Solution 2.20

For every $n \in \mathbb{N}$, let us consider an open covering $\mathscr{F}_n := \{B_{1/n}(x_n) \mid x_n \in K\}$. Since K is compact, it has a finite subcovering

$$\mathscr{G}_n := \{B_{1/n}(x_{ni}) \mid i = 1, \ldots, k(n)\}$$

Now consider the union $\mathscr{G} = \cup_{i=1}^n \mathscr{G}_n$, which is countable. We claim that $\mathscr{G}$ is a base. Consider any open set U. Then for every $x \in U$, we want to show that x is contained in a $B_{1/n}(x_{ni}) \subset U$. Since U is open , there exists a $\epsilon > 0$ s.t. $B_\epsilon(x) \subset U$. Now by AP, there exists a $n \in \mathbb{N}$ s.t. $0 < \frac{1}{n} < \frac{\epsilon}{2}$. Therefore $B_{1/n}(x) \subset B_\epsilon(x)$. Since $\mathscr{G}$ is an open covering, there must exist some $B_{1/n}(x_{ni})$ that contains x. Now we wish to show that $B_{1/n}(x_{ni})$ is fully contained in U. Let $y \in B_{1/n}(x_{ni})$. Then, by triangle inequality,

$$d(y, x) = d(y, x_{ni}) + d(x_{ni}, x) < \frac{1}{n} + \frac{1}{n} < \epsilon$$

and therefore $x \in B_{1/n}(x_{ni}) \subset B_\epsilon(x)$. Therefore, for every $x \in U$, we can construct an open ball of $\mathscr{G}$ containing x and contained in U, proving that this is a base.
We claim that the set of all $\mathscr{P} = \{x_{ni}\}_{n,i}$ forms a countable dense subset. This is clearly countable since $\mathscr{G}$ is countable. We must prove that the closure of $\mathscr{P} = K$. Let $x \in K$. Given any $\epsilon > 0$, we wish to show that $B_\epsilon(x) \cap \mathscr{P} \neq \emptyset$. Since $B_\epsilon(x)$ is open, it can be covered by a subcollection of $\mathscr{G}$, and so their centers must be in $B_\epsilon(x)$, proving that $B_\epsilon(x) \cap \mathscr{P} \neq \emptyset$. Therefore, x is a limit point of $\mathscr{P}$.

Exercise 2.28 (Rudin 2.26)

Exercise 2.29 (Rudin 2.27)

Exercise 2.30 (Rudin 2.28)

Exercise 2.31 (Rudin 2.29)

Prove that every open set in $\mathbb{R}$ is the union of an at most countable collection of disjoint segments.

Solution 2.21

Let $U \subset \mathbb{R}$ be open. Then for all $x \in U$ there exists $\epsilon > 0$ s.t. $(x - \epsilon, x + \epsilon) \subset U$. Now since $\mathbb{R}$ is separable (by exercise Rudin 2.22), it has a countable dense subset $\mathbb{Q}$. Consider all segments of rational radius and rational centers

$$\mathscr{B} = \{(q - p, q + p) \subset \mathbb{R} \mid q, p \in \mathbb{Q}\} \tag{99}$$

This is clearly countable. We claim that every open U can be expressed as the union of a subset of $\mathscr{B}$. Now by AP, there exists $n \in \mathbb{N}$ s.t. $0 < \frac{1}{n} < \frac{\epsilon}{2}$, so for all $x \in U$, there exists $n \in \mathbb{N}$ s.t. $(x - \frac{1}{n}, x + \frac{1}{n}) \subset U$. Now since $\mathbb{Q}$ is dense in $\mathbb{R}$, $x \in \mathbb{Q}' \implies (x - \frac{1}{n}, x + \frac{1}{n}) \cap \mathbb{Q} \neq \emptyset$. Say r is in this intersection. Then, by symmetry of metric, $x \in (r - \frac{1}{n}, r + \frac{1}{n})$. Therefore, for all $x \in U$, we have found an open ball in $\mathscr{B}$ that contains x. Now, we must show that this actually is fully contained in U. This is easy, since if $y \in B_{1/n}(r)$, then

$$d(y, x) \leq d(y, r) + d(r, x) \leq \frac{1}{n} + \frac{1}{n} < \epsilon \tag{100}$$

and so $B_{1/n}(r)$ is complete contained in the ϵ-ball around x, which is a subset of U. So for all $x \in U$, we found an open set $U_x \in \mathscr{B}$ covering x and fully contained in U, which means that $\cup_{x \in U} U_x = U$. Now for some intervals $B_1, B_2 \in \mathscr{B}$, if $B_1 \cap B_2 \neq \emptyset$, take their union, which is another segment, and keep doing this until $B_i \cap B_j \neq \emptyset$ for all i, j. The cardinality of this new pruned set will be less than or equal to $\mathscr{B}$, which is countable, and so this must be at most countable.

3 Sequences

We have already defined sequences, but we'll add to our vocabulary to describe and classify sequences and series. To make this chapter a bit more self-contained, we redefine sequences and series.

Definition 3.1 (Sequence)

Definition 3.2 (Series)

Definition 3.3 (Constant Sequence)

Let X be any set.

1. $\{a_i\}$ is a **constant sequence** if $a_i = A$ for all i
2. $\{a_i\}$ is an **ultimately constant sequence** if $a_i = A$ for all $i > N$ for some $N \in \mathbb{N}$. If $A = 0$, then $\{a_i\}$ is **finary**.

Depending on the structure of X, we can further classify series.

Definition 3.4 (Bounded Sequences)

Let $(V, ||\cdot||)$ be a normed vector space. $\{a_i\}$ is **bounded** if there exists M such that $|x_n| < M$ for all $n \in \mathbb{N}$.

Definition 3.5 (Monotonic Sequences)

Let X be an ordered set. $\{x_n\}$ is

1. **increasing** if $x_{n+1} > x_n$ for all n.
2. **decreasing** if $x_{n+1} < x_n$ for all n.
3. **nondecreasing** if $x_{n+1} \geq x_n$ for all n.
4. **nonincreasing** if $x_{n+1} \leq x_n$ for all n.

Sequences of these types are called **monotonic**.

3.1 Convergent Sequences

Definition 3.6 (Limit of a Sequence)

A number $A \in \mathbb{R}$ is called the **limit of the sequence** $\{x_n\}$, written

$$\lim_{n\to\infty} x_n = A,$$

if for every neighborhood U_A there exists an index N such that

$$x_n \in U_A \text{ for all } n > N$$

Equivalently, A is the limit of $\{x_n\}$ if for every $\epsilon > 0$, there exists an index N such that

$$|x_n - A| < \epsilon \text{ for all } n > N$$

If A is the limit of $\{x_n\}$, then we say that $\{x_n\}$ **converges** to A. If the limit of $\{x_n\}$ is not well defined or finite, then we say that $\{x_n\}$ is **divergent**.

Theorem 3.1 (Properties of Limits)

Given that $\{x_n\}, \{y_n\}$ are numerical sequences with $y_n \neq 0$ for all n, and let

$$\lim_{n\to\infty} x_n = A, \qquad \lim_{n\to\infty} y_n = B \neq 0$$

then,

$$\begin{aligned}
&\lim_{n\to\infty}(x_n + y_n) = A + B \\
&\lim_{n\to\infty}(cx_n) = cA \\
&\lim_{n\to\infty}(x_n \cdot y_n) = A \cdot B \\
&\lim_{n\to\infty}\frac{x_n}{y_n} = \frac{A}{B}
\end{aligned}$$

It immediately follows that the set of all convergent sequences in $\mathbb{R}^{\mathbb{N}}$ is a subspace of $\mathbb{R}^{\mathbb{N}}$.

Proof. 3.1 ()

Assume that

$$\lim_{n\to\infty} x_n = A \text{ and } \lim_{n\to\infty} y_n = B \neq 0$$

This means that for every $\epsilon > 0$, there exists $N_1, N_2 \in \mathbb{N}$ such that

$$\begin{aligned}
|x_n - A| < \epsilon \text{ for all } n > N_1 \\
|y_n - B| < \epsilon \text{ for all } n > N_2
\end{aligned}$$

Therefore, for a given ϵ, we wish to prove that there exists a N such that for all $n > N$,

$$\begin{aligned}
&1.|(x_n + y_n) - (A + B)| < \epsilon \\
&2.|cx_n - cA| < \epsilon \\
&3.|(x_n y_n) - (AB)| < \epsilon \\
&4.\left|\frac{x_n}{y_n} - \frac{A}{B}\right| < \epsilon
\end{aligned}$$

1. By the triangle inequality, we can see that

$$|(x_n + y_n) - (A + B)| = |x_n - A| + |y_n - B|$$

Since we can choose the error between x_n and A for $n > N_1$, and y_n and B for $n > N_2$ as small as we want, we set it to $\epsilon/2$. Then, we have

$$|(x_n + y_n) - (A + B)| = |x_n - A| + |y_n - B| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

for all $n > N = \max\{N_1, N_2\}$. Therefore, for a given ϵ, there exists an N such that

$$|(x_n + y_n) - (A + B)| < \epsilon \text{ for all } n > N$$

2. This proof is easy. For a given ϵ, we choose the error to be $\frac{\epsilon}{c}$.

$$|x_n - A| < \frac{\epsilon}{c} \text{ for all } n > N_1$$

Then, there exists natural number N_1 such that

$$|cx_n - cA| < c|x_n - A| = c\frac{\epsilon}{c} = \epsilon \text{ for all } n > N_1$$

3. We first observe that since the limit of $\{y_n\}$ exists, it must be bounded by a value, say B. That is,
$$|y_n| < Y \text{ for all } n \in \mathbb{N}$$
Then, we see that
$$\begin{aligned} |x_n y_n - AB| &= |(x_n y_n - A y_n) + (A y_n - AB)| \\ &< |x_n y_n - A y_n| + |A y_n - AB| \\ &= |y_n||x_n - A| + |A||y_n - B| \end{aligned}$$
Suppose $\epsilon > 0$ is given. Then, we can set the error bounds freely; there exists $N_1, N_2 \in \mathbb{N}$ such that
$$\begin{aligned} |x_n - A| &< \frac{\epsilon}{2Y} \text{ for all } n > N_1 \\ |y_n - B| &< \frac{\epsilon}{2|A|} \text{ for all } n > N_2 \end{aligned}$$
Then, we can see that
$$|x_n y_n - AB| \leq |y_n||x_n - A| + |A||y_n - B| < Y \cdot \frac{\epsilon}{2Y} + |A| \frac{\epsilon}{2|A|} = \epsilon$$
for all $n > N = \max\{N_1, N_2\}$.
4. We use the estimate
$$\left| \frac{A}{B} - \frac{x_n}{y_n} \right| = \frac{|x_n||y_n - B| + |y_n||x_n - A|}{y_n^2} \cdot \frac{1}{1 - \delta(y_n)}, \quad \delta(y_n) = \frac{|y_n - B|}{|y_n|}$$
For a given $\epsilon > 0$, we find natural numbers N_1, N_2 such that
$$\begin{aligned} |x_n - A| &< \min\left\{1, \frac{\epsilon|B|}{8}\right\} \text{ for all } n > N_1 \\ |y_n - B| &< \min\left\{\frac{|B|}{4}, \frac{\epsilon B^2}{16(|A| + 1)}\right\} \text{ for all } n > N_2 \end{aligned}$$
From this we can deduce that
$$|x_n| = |x_n - A + A| < |x_n - A| + |A| < |A| + 1$$
and
$$\begin{aligned} & |B| = |y_n + B - y_n| < |y_n| + |B - y_n| \\ \implies & |y_n| > |B| - |y_n - B| > |B| - \frac{|B|}{4} > \frac{|B|}{2} \\ \implies & \frac{1}{|y_n|} < \frac{2}{|B|} \\ \implies & 0 < \delta(y_n) = \frac{|y_n - B|}{|y_n|} < \frac{|B|/4}{|B|/2} = \frac{1}{2} \\ \implies & 1 - \delta(y_n) > \frac{1}{2} \\ \implies & 0 < \frac{1}{1 - \delta(y_n)} < 2 \end{aligned}$$
So, we can substitute
$$\begin{aligned} |x_n| \cdot \frac{1}{y_n^2} \cdot |y_n - B| &< (|A| + 1) \cdot \frac{4}{B^2} \cdot \frac{\epsilon \cdot B^2}{16(|A| + 1)} = \frac{\epsilon}{4} \\ \left| \frac{1}{y_n} \right| \cdot |x_n - A| &< \frac{2}{|B|} \cdot \frac{\epsilon|B|}{8} = \frac{\epsilon}{4} \end{aligned}$$

into the final equation to get

$$\left|\frac{A}{B} - \frac{x_n}{y_n}\right| < \epsilon \text{ for all } n > N = \max\{N_1, N_2\}$$

Theorem 3.2 ()

Given convergent sequences $\{x_n\}$ and $\{y_n\}$, if

$$\lim_{n\to\infty} x_n < \lim_{n\to\infty} y_n$$

then there exists an index $N \in \mathbb{N}$ such that $x_n < y_n$ for all $n > N$.

Theorem 3.3 (Squeeze Theorem for Sequences)

Given sequences $\{x_n\}, \{y_n\}, \{z_n\}$ such that

$$x_n \leq y_n \leq z_n$$

for all $n > N$, if $\{x_n\}$ and $\{z_n\}$ both converge to the same limit, then the sequence $\{y_n\}$ also converges to that limit. That is,

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n = A \implies \lim_{n\to\infty} y_n = A$$

Note that so far, in order to show that a sequence is convergent, we must identify a real number first and then show using the ϵ-δ definition that it converges. This might be overkill in a case where we just want to prove that a sequence converges, but we don't care what it converges to.

Unsurprisingly, we use the fact that the sequence lives in the reals. We can determine convergence by using Cauchy-completeness, which gives us the "theorem" (though it is really a fact by construction).

Theorem 3.4 (Cauchy-Convergence Criterion)

A cauchy sequence in $\mathbb{R}$ converges.

The second is not so trivial. We take inspiration from the least-upper-bound property. We can't just say that every bounded sequence converges, since this is not true. However, if we have one more assumption, this is strong enough to guarantee convergence.

Lemma 3.1 (Convergence Criterion for Monotonic Sequences)

In order for a nondecreasing (nonincreasing) sequence to be convergent, it is necessary and sufficient that it is bounded above (or below).

3.2 Divergent Sequences

Note that while a convergent sequence can be visualized quite easily by the Cauchy convergence criterion, there are many way in which a sequence can be divergent.

1. Increasing/decreasing indefinitely
2. Oscillating between two constant values
3. Oscillating between a value tending to $+\infty$ and a value tending to $-\infty$
4. Many other classes of divergence

Definition 3.7 (Sequence Tending to Infinity)

The sequence $\{x_n\}$ **tends to positive infinity** if for each number c there exists $N \in \mathbb{N}$ such that $x_n > c$ for all $n > N$. It is denoted

$$x_n \to +\infty \text{ or } \lim_{n\to\infty} x_n = +\infty$$

We define sequences that **tend to negative infinity** similarly. And $\{x_n\}$ **tends to infinity** if for each c there exists $N \in \mathbb{N}$ such that $|x_n| > c$ for all $n > N$, which is written

$$x_n \to \infty$$

Note that

$$x_n \to +\infty \text{ or } x_n \to -\infty \implies x_n \to \infty$$

but the converse is not necessarily true. The simple example is the sequence $x_n = (-1)^n n$. Also, it is important to know that a sequence may be unbounded and yet not tend to $+\infty$, $-\infty$, or ∞.

Example 3.1 (Unbounded Sequence that Doesn't tend to ∞)

The sequence $x_n = n^{(-1)^n}$ is divergent yet does not tend to positive infinity, negative infinity, nor infinity.

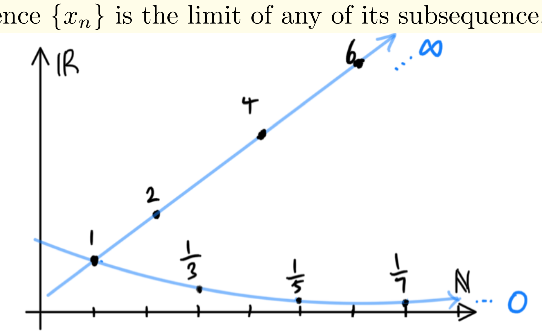
3.3 Subsequences

Definition 3.8 (Subsequences)

A **subsequence** of $\{a_n\}$ is a sequence $\{a_{\gamma_k}\}$, where $\{\gamma_k\}$ is a strictly increasing infinite subset of $\mathbb{N}$.

Definition 3.9 (Partial Limits)

The **partial limit** of a sequence $\{x_n\}$ is the limit of any of its subsequence.



Two (out of the many) partial limits of the sequence above is $+\infty$ and 0.

Theorem 3.5 (Bolzano-Weierstrass Theorem)

Every bounded sequence in $\mathbb{R}^n$ contains a convergent subsequence.

Proof. 3.2 ()

It suffices to prove that there exists a monotonic sequence within a bounded sequence $\{x_n\}$.

Corollary 3.1 ()

From each sequence of real numbers there exists either a convergent subsequence or a subsequence tending to infinity.

Example 3.2 ()

We claim that

$$\lim_{n\to\infty} \frac{n}{q^n} = 0 \text{ if } q > 1$$

Proof. 3.3 ()

Since $x_n = \frac{n}{q^n} \implies x_{n+1} = \frac{n+1}{nq} x_n$ for $n \in \mathbb{N}$. Since

$$\lim_{n\to\infty} \frac{n+1}{nq} = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)\frac{1}{q} = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right) \cdot \lim_{n\to\infty} \frac{1}{q} = 1 \cdot \frac{1}{q} = \frac{1}{q} < 1$$

there exists an index N such that $\frac{n+1}{nq} < 1$ for $n > N$. Thus, we have

$$x_n > x_{n+1} = x_n \cdot \frac{n+1}{nq} \text{ for } n > N$$

which means that the sequence will be monotonically decreasing from index N on. The terms of the sequence

$$x_{N+1} > x_{N+2} > x_{N+3} > \ldots$$

are positive (bounded below) and are monotonically decreasing, so it must have a limit.
Finding the actual limit is easy. Let $x = \lim_{n\to\infty} x_n$. It follows from the relation $x_{n+1} = \frac{n+1}{nq} x_n$ that

$$x = \lim_{n\to\infty} (x_{n+1}) = \lim_{n\to\infty} \left(\frac{n+1}{nq} x_n\right) = \lim_{n\to\infty} \frac{n+1}{nq} \cdot \lim_{n\to\infty} x_n = \frac{1}{q} x$$

which implies that $\left(1 - \frac{1}{q}\right) = 0 \implies x = 0$.

Example 3.3 ()

We claim that

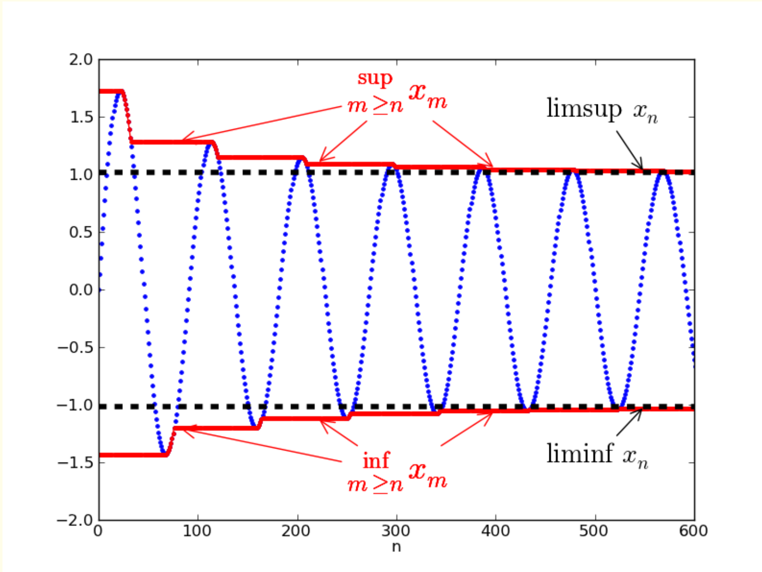
$$\lim_{n\to\infty} \sqrt[n]{n} = 1 \tag{101}$$

Definition 3.10 (Inferior, Superior Limits)

The **inferior limit** and **superior limit** of a sequence $\{x_k\}$ are defined as follows, and they can be shown to be the smallest and largest partial limits of the sequence. That is,

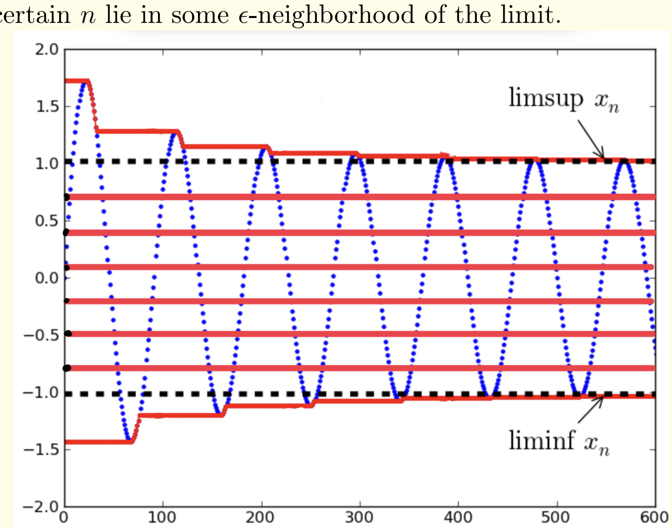
$$\begin{aligned} \varliminf_{k\to\infty} x_k &\equiv \lim_{n\to\infty} \inf_{k\geq n} x_k = \min\{\lim_{r\to\infty} y_r\} \\ \varlimsup_{k\to\infty} x_k &\equiv \lim_{n\to\infty} \sup_{k\geq n} x_k = \max\{\lim_{r\to\infty} y_r\} \end{aligned}$$

where $\{y_r\}$ is any subsequence of $\{x_n\}$. Despite the definition, it isn't too difficult to visualize this. For example, take a look at the superior and inferior limits of the divergent sequence below.



In order to find the superior limit, we first look the whole sequence in $\mathbb{N}$ and find the supremum. We now "decrease" our domain from $\mathbb{N}$ to $\{2, 3, \ldots\}$, then $\{3, 4, \ldots\}$, then $\{4, 5, \ldots\}$ and so on, continuing to label the supremum of the sequence. The limit of this sequence of supremums is the superior limit. Informally, the superior limit tells us what the supremum of the "end terms" of $\{x_n\}$ will be, and similarly for the inferior limit.

The second property of superior and inferior limits is that they represent the greatest and least possible partial limit of a sequence. For example, the six red lines marked in the middle (along with infinitely many others) are viable partial limits because one can choose a subsequence such that all of its points after a certain n lie in some ϵ-neighborhood of the limit.



Therefore, the superior and inferior limits represent some sort of "bound" on the sequence in the long run. That is, on the long run, the terms of the sequence $\{x_n\}$ cannot be greater than its superior limit and cannot be less than its inferior limit. With this interpretation, the following theorem should be clear.

Theorem 3.6 ()

A sequence has a limit or tends to $\pm\infty$ if and only if its inferior and superior limits are the same.

Corollary 3.2 ()

A sequence converges if and only if every subsequence of it converges.

3.4 Convergence Tests for Real Series

Definition 3.11 (Series over $\mathbb{R}$)

Given a sequence of real numbers $\{a_n\}$, the **series** of $\{a_n\}$ is defined

$$s = \sum_{k=1}^{\infty} a_k \tag{102}$$

The series can be interpreted as the sequence of partial sums $\{s_n\}$, where

$$s_n = \sum_{k=1}^{n} a_k \tag{103}$$

is the **nth partial term of the series**. Therefore, we can interpret the sum of the series s as the limit of $\{s_n\}$.

$$\lim_{n\to\infty} s_n = s \tag{104}$$

If the sequence $\{s_n\}$ converges, the series is **convergent** and **divergent** otherwise.

Since the convergence of a series is equivalent to convergence of its sequence of partial sums, applying the Cauchy convergence criterion to the sequence $\{s_n\}$ leads to the following theorem.

Theorem 3.7 (Cauchy Convergence Criterion for Series)

The series $a_1 + \ldots + a_n + \ldots$ converges if and only if for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $m \geq n > N$,

$$|a_n + \ldots + a_m| < \epsilon \tag{105}$$

Definition 3.12 (Cauchy Product of Real Series)

Corollary 3.3 (nth Term Test)

A necessary (but not sufficient) condition for convergence of the series $a_1 + \ldots a_n + \ldots$ is that the terms tend to 0 as $n \to \infty$. That is, it is necessary that

$$\lim_{n\to\infty} a_n = 0 \tag{106}$$

Proof. 3.4 ()

It suffices to set $m = n$ in the Cauchy convergence criterion. This would mean that for every $\epsilon > 0$ there exists a $N \in \mathbb{N}$ such that

$$|a_n| = |a_n - 0| < \epsilon \text{ for all } n > N \tag{107}$$

which, by definition, means that $\{a_n\}$ converges to 0.

Example 3.4 (Geometric Series)

The series
$$1 + q + q^2 + \ldots + q^n + \ldots \tag{108}$$
is called the **geometric series**.
Since $|q^n| = |q|^n$, we have $|q^n| \geq 1$ when $|q| \geq 1$. So, if $|q| \geq 1$, the terms q^n does not converge to 0 and the Cauchy convergence criterion is not met.
Now, suppose $|q| < 1$. Then,
$$s_n = 1 + q + \ldots + q^{n-1} = \frac{1 - q^n}{1 - q} \tag{109}$$
which implies that
$$\lim_{n \to \infty} s_n = \frac{1}{1 - q} \tag{110}$$
since $\lim_{n\to\infty} q^n = 0$ if $|q| < 1$. This, the series converges to if and only if $|q| < 1$, and its sum is $\frac{1}{1-q}$.

Example 3.5 (Harmonic Series)

The series
$$1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} + \ldots \tag{111}$$
is called the **harmonic series**, since each term from the second on is the harmonic mean of the two terms on either side of it. Clearly,
$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0 \tag{112}$$
but the sequence of partial sums s_n diverges, and thus the harmonic series diverges.

Definition 3.13 (Absolute Convergence)

The series $\sum_{n=1}^\infty a_n$ is **absolutely convergent** if the series
$$\sum_{n=1}^\infty |a_n|$$
converges. Clearly, every absolutely convergent series because
$$\left| \sum_{n=1}^\infty a_n \right| \leq \sum_{n=1}^\infty |a_n|$$

Theorem 3.8 (Direct Comparison Test)

Let $\sum_{n=1}^\infty a_n$ and $\sum_{n=1}^\infty b_n$ be 2 series with nonnegative terms. If there exists an index $N \in \mathbb{N}$ such that $a_n \leq b_n$ for all $n > N$, then
$$\begin{aligned} \sum_{n=1}^\infty b_n \text{ convergent } &\implies \sum_{n=1}^\infty a_n \text{ convergent} \\ \sum_{n=1}^\infty a_n \text{ divergent } &\implies \sum_{n=1}^\infty b_n \text{ divergent} \end{aligned}$$

Theorem 3.9 (Limit Comparison Test)

Suppose the limit

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \alpha$$

exists for the series $\sum_{n=1}^\infty a_n$. Then,

$$\begin{aligned} \alpha < 1 &\implies \sum_{n=1}^\infty a_n \text{ converges absolutely} \\ \alpha > 1 &\implies \sum_{n=1}^\infty a_n \text{ diverges} \\ \alpha = 1 &\implies \text{ Inconclusive} \end{aligned}$$

Theorem 3.10 (Root Test)

Let $\sum_{n=1}^\infty$ be a given series and

$$\alpha = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$$

Then, the following are true

$$\begin{aligned} \alpha < 1 &\implies \sum_{n=1}^\infty \text{ converges absolutely} \\ \alpha > 1 &\implies \sum_{n=1}^\infty \text{ diverges} \\ \alpha = 1 &\implies \text{ Inconclusive} \end{aligned}$$

Theorem 3.11 (Weierstrass M-test for Absolute Convergence)

Let $\sum_{n=1}^\infty$ and $\sum_{n=1}^\infty b_n$ be series. Suppose there exists an index $N \in \mathbb{N}$ such that $|a_n| \leq b_n$ for all $n > N$. Then,

$$\sum_{n=1}^\infty b_n \text{ converges } \implies \sum_{n=1}^\infty a_n \text{ converges absolutely}$$

The following theorem, while obvious, has interesting consequences.

Theorem 3.12 (Cauchy)

If $a_1 \geq a_2 \geq \ldots \geq 0$, the series $\sum_{n=1}^\infty a_n$ converges if and only if the series

$$\sum_{k=0}^\infty 2^k a_{2^k} = a_1 + 2a_2 + 4a_4 + 8a_8 + \ldots$$

converges.

Proof. 3.5 ()

Letting $A_k = a_1 + a_2 + \ldots + a_k$ and $S_n = a_1 + 2a_2 + \ldots + 2^n a_{2^n}$, it is clear that by adding up the inequalities

$$\begin{aligned}
&a_2 \leq a_2 \leq a_1 \\
&2a_4 \leq a_3 + a_4 \leq 2a_2 \\
&4a_8 \leq a_5 + a_6 + a_7 + a_8 \leq 4a_4 \\
&\ldots \\
&2^n a_{2^{n+1}} \leq a_{2^n+1} + \ldots + a_{2^{n+1}} \leq 2^n a_{2^n},
\end{aligned}$$

we get

$$\frac{1}{2}(S_{n+1} - a_1) \leq A_{2^{n+1}} - a_1 \leq S_n$$

Since the sequences $\{A_k\}$ and $\{S_k\}$ are nondecreasing, and hence from the inequalities we can conclude that they are either both bounded above (which means that they are both convergent since it is a bounded, nondecreasing series) or both unbounded above (which means that they are both divergent since they are nondecreasing and unbounded).

Corollary 3.4 (p-series Test)

The series

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

converges for $p > 1$ and diverges for $p \leq 1$.

Proof. 3.6 ()

Suppose $p \geq 0$. By the previous theorem, the series converges or diverges simultaneously with the series

$$\sum_{k=0}^{\infty} 2^k \frac{1}{(2^k)^p} = \sum_{k=0}^{\infty} (2^{1-p})^k$$

which is really just a geometric series. A necessary and sufficient condition for the convergence of this series is that $2^{1-p} < 1$, that is, $p > 1$.
Now suppose $p \leq 0$. The series is then clearly divergent since all of the terms are larger than 1.

3.5 Exercises

Exercise 3.1 ()

Prove that convergence of $\{x_n\}$ implies convergence of $\{|x_n|\}$. Is the converse true?

Solution 3.1

If $\{x_n\}$ converges to x, then for all $\epsilon > 0$, there exists a $N \in \mathbb{N}$ s.t. $|x_n - x| < \epsilon$ if $n > N$. We use the inequality $\big||x_n| - |x|\big| \leq |x_n - x|$ to show that then for every $\epsilon > 0$ there exists a $N \in \mathbb{N}$ s.t.

$$\big||x_n| - |x|\big| \leq |x_n - x| \leq \epsilon$$

and so $\{|x_n|\}$ converges to $|x|$.

Exercise 3.2 ()

Calculate

$$\lim_{n\to\infty} \sqrt{n^2+n} - n$$

Solution 3.2

We can compute

$$\lim_{n\to\infty} \sqrt{n^2+n} - n = \lim_{n\to\infty} (\sqrt{n^2+n} - n) \cdot \frac{\sqrt{n^2+n}+n}{\sqrt{n^2+n}+n} = \lim_{n\to\infty} \frac{n}{\sqrt{n^2+n}+n}$$

where

$$A_n = \frac{n}{\sqrt{n^2+2n+1}+n} \leq \frac{n}{2n+1} \leq \frac{n}{\sqrt{n^2+n}+n} \leq \frac{n}{\sqrt{n^2}+n} = \frac{n}{2n} = \frac{1}{2} = C_n$$

C_n is ultimately constant. It suffices to prove that A_n limits to $\frac{1}{2}$ by showing that

$$\frac{n}{2n+1} = \frac{n/n}{(2n+1)/n} = \frac{1}{2+\frac{1}{n}}$$

where $\{\frac{1}{n}\}$ is infinitesimal.

Exercise 3.3 (Rudin 3.3)

If $s_1 = \sqrt{2}$ and

$$s_{n+1} = \sqrt{2+\sqrt{s_n}}$$

for $n = 1, 2, \ldots$, prove that $\{s_n\}$ converges and that $s_n < 2$ for $n = 1, 2, \ldots$.

Solution 3.3

We can show that $s_n < 2$ by induction. $s_1 = \sqrt{2} < 2$, so the base case is proved. Now, given that $s_n < 2$, $\sqrt{s_n} < 2 \implies 2 + \sqrt{s_n} < 2 + \sqrt{2} < 4 \implies s_{n+1} = \sqrt{2+\sqrt{s_n}} < 2$ and we are done.

Exercise 3.4 (Rudin 3.4)

Exercise 3.5 (Rudin 3.5)

Exercise 3.6 (Rudin 3.6)

Exercise 3.7 (Rudin 3.7)

Exercise 3.8 (Rudin 3.8)

Exercise 3.9 (Rudin 3.9)

Exercise 3.10 (Rudin 3.10)

Exercise 3.11 (Rudin 3.11)