# Measure Theory

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## Contents

In math, we are first taught to solve simple equations like $x^2 - 2x + 4 = 0$ for a certain *number* $x$, but in real world applications, we must now solve for some *function* $f$ satisfying an equation

$$\mathcal{L}(f) = 0 \tag{1}$$

where $\mathcal{L}$ is some operator on functions. This is usually difficult, and many times a solution does not exist. However, we can find approximate solutions, say

$$\begin{aligned} \mathcal{L}(f) &= 1/2 \\ \mathcal{L}(f) &= 1/4 \\ \mathcal{L}(f) &= 1/8 \\ \ldots &= \ldots \end{aligned}$$

and approximate the solution as

$$f = \lim_{n \to \infty} f_n \tag{2}$$

Given that this limit exists, we can usually define $f$ pointwise using a point-wise limit

$$f(x) = \lim_{n \to \infty} f_n(x) \text{ for all } x \tag{3}$$

but the function in total is very ugly and not Riemann integrable. The classic non-Riemann integrable function is the

$$f(x) = \chi_{\mathbb{R} \setminus \mathbb{Q}}(x) := \begin{cases} 1 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \\ 0 & \text{if} x \in \mathbb{Q} \end{cases} \tag{4}$$

Since $\mathbb{Q}$ is countable, we can enumerate $\mathbb{Q} = \{q_n\}_{n=1}^\infty$ and define the sequence of functions

$$f_n = 1 - \chi_{\{q_j\}_{j=1}^n}(x) \tag{5}$$

that start off with the constant function 1 and then "removes" points in $\mathbb{Q}$, setting their image to 0. It is clear that since we are removing points, every function in the sequence has an integral (from 0 to 1) of 1, and therefore the integral of $f$ should also be 1.

$$\int_0^1 f_n \, dx = 1 \implies \int_0^1 f \, dx = \int_0^1 \lim_{n \to \infty} f_n \, dx = \lim_{n \to \infty} \int_0^1 f_n \, dx \tag{6}$$

What is crucial for mathematicians to work with is the capability to take the limit from inside the integral to outside the integral. The problem is that $f$ is not a Riemann integral function.

**Definition 0.1 (Riemann Integrable Function)**

Given a function $f : [0,1] \longrightarrow \mathbb{R}$, let us consider some partition of $[0,1]$ into intervals $P = \{I_0, I_1, \ldots, I_N\}$, then, for each $I \in P$, we can take the supremum $M_I = \sup_{x \in I} f(x)$ and infimum $m_I = \inf_{x \in I} f(x)$ and bound $f$ by the upper and lower Riemann sums.

$$\sum_{I \in P} m_I |I| \leq \int_0^1 f \, dx \leq \sum_{I \in P} M_I |I| \tag{7}$$

where $|I|$ is the length of interval $I$. If we take *all* possible partitions, the bound should still hold.

$$m = \sup_P \Big\{ \sum_{I \in P} m_I |I| \Big\} \leq \int_0^1 f \, dx \leq \inf_P \Big\{ \sum_{I \in P} M_I |I| \Big\} = M \tag{8}$$

and if the lower bound is equal to the upper bound $m = M$, then the integral is this number and $f$ is considered Riemann integrable.

Now since $\mathbb{Q}$ is dense in $\mathbb{R}$, for every interval $I$ in every partition $P$ will have $m_I = 0$ and $M_I = 1$ for the Riemann function, meaning that the lower bound will always be 0 and the upper bound will always be 1. So, $\int_0^1 \chi_{\mathbb{R}\setminus\mathbb{Q}}(x)$ can take on any value in $[0,1]$, which isn't helpful. The fact that we can't integrate this really simple function is a problem. For nice functions, we can partition it so that the base of each Riemann rectangle is a nice interval, while the base of the Riemann function is an "interval with holes." The problem really boils down to measuring what the "length" of this set is. So the problem with the Riemann integral isn't the integral itself, but the fact that we can't give a meaningful size to the set $\mathbb{R} \setminus \mathbb{Q}$. Now mathematicians in the 19th century thought that as long as we solve this problem, we should be good to go, but Banach and Tarski proved that there exists sets that cannot be measured with their famous paradox, which says that you can take any set $P$, disassemble it into a finite set of pieces, and rearrange it (under isometry and translations) so that it has a different size than the original $P$. So, we have to exclude some sets that are not measurable. The collection of sets that we *can* measure is called the $\sigma$-algebra.

**Exercise 0.1 (Tao 1.2.2)**

Give an example of a sequence of uniformly bounded, Riemann integrable functions $f_n : [0,1] \to \mathbf{R}$ for $n = 1, 2, \ldots$ that converge pointwise to a bounded function $f : [0,1] \to \mathbf{R}$ that is *not* Riemann integrable. What happens if we replace pointwise convergence with uniform convergence?

*Solution.* The only non-Riemann integrable function that we know of is the Riemann function, so let's a try to construct such a limit with this. Enumerate the rationals $\mathbb{Q} = \{q_k\}_{k \in \mathbb{N}}$. Now, consider the function

$$f(x) = \begin{cases} 1 & \text{if } x \in \{q_1, \ldots, q_n\} \\ 0 & \text{if else} \end{cases} \tag{9}$$

Then, $f_n -> \chi_{\mathbb{Q}}$ pointwise.

# 1 Jordan Measure

We would like to generalize the concepts of size, which are specified as length/area/volume depending on the dimension of the space we live in. The most intuitive notion of size are those of segments, rectangles, and boxes, and these are the simplest forms of sets that we will work with.

Such that for "simple" sets $A$ where we know what the area is, the outer measure of $A$ should coincide with the area of $A$. Let's first start by defining what a "simple" set is.

**Definition 1.1 (Box)**

An **box** $E \subset \mathbb{R}^n$ is defined recursively as follows.

1. An **interval** $I \subset \mathbb{R}$ is one of the sets $(a, b), [a, b), (a, b], [a, b]$ for $a, b \in \mathbb{R}$.
2. For $n > 1$, an box $E \subset \mathbb{R}^n$ is $E = I_1 \times \ldots \times I_n$ for intervals $I_1, \ldots, I_k$.

**Definition 1.2 (Size of a Box)**

The **size** of a box $E = I_1 \times \ldots \times I_n \subset \mathbb{R}^n$ is defined as follows.

1. The **length** of an interval $I$ is $\ell(I) \coloneqq b - a$.
2. The **size** of $E$ is $|E| \coloneqq \prod_{i=1}^n (b_i - a_i)$.

Now we can combine these to get an elementary set.

**Definition 1.3 (Elementary Set)**

An **elementary set** is a set $E \subset \mathbb{R}^n$ that is a finite union of boxes.

We would like to have some nice properties of these elementary sets.

**Lemma 1.1 (Boolean Closure of Elementary Sets)**

Given two elementary sets $E, F \subset \mathbb{R}^n$,

1. $E \cup F$ is elementary.
2. $E \cap F$ is elementary.
3. $E \setminus F$ is elementary.

*Proof.*

**Lemma 1.2 (Disjoint Finite Union of Elementary Sets)**

Let $E \subset \mathbb{R}^n$ be an elementary set. Then, $E$ can be expressed as a finite union of *disjoint* boxes.

*Proof.*

**Definition 1.4 (Elementary Measure)**

The **elementary measure** of an elementary set $E \subset \mathbb{R}^n$ is defined as the sum of the sizes of each box in a partition:

$$m(E) := \sum_{i=1}^{k} |B_k| \tag{10}$$

We claim that this sum is invariant depending on the partition, and hence, well defined.

*Proof.*

This elementary measure clearly extends the notion of size, since

$$m(B) = s(B) \tag{11}$$

whenever $B$ is elementary. Furthermore, we can deduce finite additivity and nonnegativity. These are really trivial but we state them as theorems to establish a pattern.

**Lemma 1.3 (Fundamental Properties of Elementary Measure)**

The elementary measure satisfies the following.

1. *Nonnegativity.* For any elementary set $E$, $m(E) \geq 0$.
2. *Finite Additivity* Given $E_1, \ldots, E_n$ are disjoint elementary sets,
$$m(E_1 \cup \ldots \cup E_k) = m(E_1) + \ldots + m(E_k) \tag{12}$$
3. *Monotonicity.* Given elementary sets $E \subset F$, we have
$$m(E) \leq m(F) \tag{13}$$
4. *Finite Subadditivity.* Let $E_1, \ldots, E_n$ be any elementary sets (not necessarily disjoint). Then
$$m(E_1 \cup \ldots \cup E_n) \leq m(E_1) + \ldots + m(E_n) \tag{14}$$
5. *Translation Invariance.* For $x \in \mathbb{R}^n$ and elementary set $E \subset \mathbb{R}^n$,
$$m(E) = m(x + E) \tag{15}$$

It turns out that these properties uniquely determine an elementary measure.

**Theorem 1.1 (Uniqueness of Elementary Measure)**

## 1.1 Jordan Measure

Now, we define the outer and inner measure, which are defined for *all* subsets of $\mathbb{R}^n$.

**Definition 1.5 (Jordan Outer, Inner Measure)**

Let $E \subset \mathbb{R}^n$.

1. The **Jordan inner measure** is defined
$$m_*(E) := \sup_{A \subset E, A \text{ elementary}} m(A) \tag{16}$$
2. The **Jordan outer measure** is defined
$$m_*(E) := \inf_{B \supset E, B \text{ elementary}} m(B) \tag{17}$$

Note that if $E$ is unbounded, then there exists no elementary set that is a superset of $E$, and so the infimum of such a set is $+\infty$ conventionally.

This is where our first big leap in construction comes in. Before, we have defined elementary boxes, which are pretty much guaranteed to have a well-defined elementary measure. Here, we *begin* with a function on the power set of $\mathbb{R}^n$, and then we will filter the power set to those subsets that behave nicely.

**Definition 1.6 (Jordan Measurable Set, Jordan Measure)**

Let $E \subset \mathbb{R}^n$ be bounded.[a] If $m_*(E) = m^*(E)$, then $E$ is said to be **Jordan-measurable**, and we define

$$m(E) := m_*(E) = m^*(E) \tag{18}$$

as the **Jordan measure** of $E$.

[a]Note that by convention, we don't consider unbounded sets to be Jordan measurable.

Note first of all that Jordan measure is a generalization of elementary measure, since if $E$ is elementary, then we can set $A = E = B$ to achieve these bounds. Furthermore, by monotonicity, we can never get past them, and will always have

$$m(A) \leq m(E) \leq m(B) \tag{19}$$

where $m$ is the elementary measure. So, we can overload the notation and just write $m$ to denote elementary and Jordan measure. Second, note that the Jordan measure shares the same properties.

**Lemma 1.4 (Boolean Closure of Jordan Measurable Sets)**

Given two Jordan-measurable sets $E, F \subset \mathbb{R}^n$,

1. $E \cup F$ is elementary.
2. $E \cap F$ is elementary.
3. $E \setminus F$ is elementary.

*Proof.*

The properties of the Jordan measure parallel those of elementary measure.

**Theorem 1.2 (Fundamental Properties of Jordan Measure)**

The elementary measure satisfies the following.

1. *Nonnegativity.* For any elementary set $E$, $m(E) \geq 0$.
2. *Finite Additivity* Given $E_1, \ldots, E_n$ are disjoint elementary sets,
$$m(E_1 \cup \ldots \cup E_k) = m(E_1) + \ldots + m(E_k) \tag{20}$$
3. *Monotonicity.* Given elementary sets $E \subset F$, we have
$$m(E) \leq m(F) \tag{21}$$
4. *Finite Subadditivity.* Let $E_1, \ldots, E_n$ be any elementary sets (not necessarily disjoint). Then
$$m(E_1 \cup \ldots \cup E_n) \leq m(E_1) + \ldots + m(E_n) \tag{22}$$
5. *Translation Invariance.* For $x \in \mathbb{R}^n$ and elementary set $E \subset \mathbb{R}^n$,
$$m(E) = m(x + E) \tag{23}$$

*Proof.*

Jordan measurable sets are sets that are "almost" elementary, but a few sets already come to mind that are not Jordan measurable.

**Example 1.1 (Rationals in Unit Interval)**

$\mathbb{Q} \cap [0, 1]$ is not Jordan measurable.

It may be hard to tell directly whether something is Jordan measurable. This is where the "Cauchy criterion" of Jordan measurable sets comes in.

**Theorem 1.3 (Equivalent Notions)**

$E$ is Jordan measurable iff $\forall \epsilon > 0$, $\exists$ elementary sets $A \subset E \subset B$ s.t. $m(B \setminus A) < \epsilon$.

*Proof.*

Note how the previous lemma is very similar to this theorem on Riemann integrability.

**Example 1.2 (Regions Under Graphs are Jordan Measurable)**

**Example 1.3 (Triangle is Jordan Measurable)**

**Example 1.4 (Compact Convex Polytopes are Jordan Measurable)**

**Example 1.5 (Open and Closed Balls in Euclidean Space are Jordan Measurable)**

**Example 1.6 (Subsets of Jordan Null Sets have 0 Jordan Measure)**

**Theorem 1.4 (Uniqueness of Jordan Measure)**

**Theorem 1.5 (Topological Approximations of Jordan Measurable Sets)**

Let $E \subset \mathbb{R}^n$ be a bounded set. Then,

1. $E$ and its closure $\overline{E}$ have the same Jordan outer measure.
2. $E$ and its interior $E^\circ$ have the same Jordan outer measure.
3. $E$ is Jordan measurable iff the topological boundary $\partial E$ has Jordan outer measure 0.

*Proof.*

**Example 1.7 (Bullet Riddled Square)**

Show that both sets have a Jordan inner measure 0 and Jordan outer measure 1.
1. $[0, 1]^2 \setminus \mathbb{Q}^2$.
2. $[0, 1]^2 \cap \mathbb{Q}^2$.

Finally, a little teaser theorem.

**Theorem 1.6 (Caratheodory Property)**

Let $E \subset \mathbb{R}^n$ be a bounded set, and $F \subset \mathbb{R}^n$ be an elementary set. Show that

$$m^*(E) = m^*(E \cap F) + m^*(E \setminus F) \tag{24}$$

where $m^*$ is the Jordan outer measure.

*Proof.*

## 1.2 Riemann Integration

Now, we connect the Riemann integral to the Jordan measure.

**Theorem 1.7 (Jordan Measure with Riemann Integral)**

If $E \subset [a, b]$ is Jordan measurable, then the indicator function $\mathbb{1}_E$ is Riemann integrable, and

$$\int_a^b \mathbb{1}_E \, dx = m(E) \tag{25}$$

# 2 Lebesgue Measure

We have seen that there are some common sets that are not Jordan measurable, but a bigger problem is that countable unions and intersections aren't.

**Example 2.1 (Countable Union/Intersection of Jordan Measurable Sets are Not J.M.)**

We show a few counterexamples.

1. *Countable Union.* Take $\mathbb{N} = \{n\}_{n=1}^\infty$. Each point $n$ has Jordan measure 0, but their union is unbounded so it isn't Jordan measurable.
2. *Bounded Countable Union.* Maybe we can fix this by considering bounded unions. But consider $E = \mathbb{Q} \cap [0,1]$. By density of rationals,
$$m_*(E) = 0 \neq 1 = m^*(E) \tag{26}$$
3. *Intersection.* Consider the Cantor set $C$, which is bounded, but again
$$m_*(C) = 0 \neq 1 = m^*(C) \tag{27}$$

This motivates the definition of a $\sigma$-algebra.

**Definition 2.1 ($\sigma$-Algebra)**

A ***σ*-algebra** on a set $X$ is a collection of subsets of $X$ satisfying the following:

1. *Contains Empty and Full Set.* $\emptyset, X \in \mathcal{A}$.
2. *Closed Under Countable Unions.* $A_n \in \mathcal{A}$ for all $n \in \mathbb{N}$ implies $\bigcup_n A_n \in \mathcal{A}$.
3. *Closed Under Complements.* $A \in \mathcal{A} \implies A^c \in \mathcal{A}$.
4. *Closed Under Countable Intersections.*[a] $A_n \in \mathcal{A}$ for all $n \in \mathbb{N}$ implies $\bigcap_n A_n \in \mathcal{A}$.

[a] This is actually a consequence of the previous properties. We can utilize the fact that $\cap_{k=1}^\infty A_k = X \setminus \cup_{k=1}^\infty A_k^c$

In some applications, we can just ignore some of these sets as "pathological." But for Riemann integrability, we are integrating over Jordan measurable sets. I turns out that it is really important that we can *at least* guarantee that countable unions and intersections of Jordan measurable sets are measurable. If we don't, then not even uniform convergence—an extremely strong property—can preserve continuity, differentiability, and integrability of a sequence of functions.

## 2.1 Lebesgue Outer Measure

Therefore, we would like the collection of our measurable sets to be a $\sigma$-algebra. To do this, we tinker around with the definition of the Jordan measure. Note that by definition, the Jordan outer measure can be equivalently written as

$$\begin{aligned} m^*(S) &\coloneqq \inf\{m(E) \mid S \subset E, E \text{ elementary}\} && (28) \\ &= \inf\left\{ \sum_{i=1}^k |B_i| \;\middle|\; S \subset \bigcup_{i=1}^k B_i, B_i \text{ boxes}, k \in \mathbb{N} \right\} && (29) \end{aligned}$$

Note that the *finite* number of boxes $k$ are allowed to vary over all naturals. To define the Lebesgue measure, we change the finite to countable.

**Definition 2.2 (Lebesgue Outer Measure)**

Given any set $E \subset \mathbb{R}^n$, the **Lebesgue outer measure** is a map

$$m^* : 2^{\mathbb{R}^n} \to [0, +\infty], \qquad m^*(E) = \inf \left\{ \sum_{k=1}^{\infty} |B_k| \;\middle|\; E \subset \bigcup_{k=1}^{\infty} B_k \right\} \tag{30}$$

It's a hard definition, but a natural one, since we're taking all these boxes and trying to make them as snug as possible to define the outer measure of an arbitrary set. We first first check that this is indeed a generalization of Jordan measure, in the sense that if $E$ is Jordan measurable, then its Lebesgue outer measure is the same as its Jordan measure.

**Theorem 2.1 (Lebesgue Outer Measure Coincides with Interval Length)**

$m^*$ satisfies the property that for any interval $I \subset \mathbb{R}$, $m^*(I) = |I|$.

*Proof.* Let's first consider the case when $I$ is closed. Let $I = [a, b]$. Then, we know from definition that

$$m^*(I) := \inf \left\{ \sum_{k=1}^{\infty} |I_k| \;\middle|\; I \subset \bigcup_{k=1}^{\infty} I_k \right\} \tag{31}$$

where $I_k = [a_n, b_n]$. We wish to show that the above quantity equals $b - a$.

1. $m^*(I) \leq b - a$. This is pretty easy since we can just set the cover to consist of the single interval $I$, and since $m^*(I)$ must be the infimum of it, then we must have $m^*(I) \leq b - a$. A technicality is that we must strictly use countable covers, but in this case, we can just fix $\epsilon > 0$ and see

$$I_1 = [a, b], \qquad I_k = [b - \frac{\epsilon}{2^k}, b + \frac{\epsilon}{2^k}] \tag{32}$$

In this case the sums of lengths of all $I_2, \ldots$ is $\epsilon/2 < \epsilon$, and so

$$I_k \leq b - a + \epsilon \qquad \forall \epsilon > 0 \tag{33}$$

2. Proving $m^*(I) \geq b - a$ is harder. In here, we use the "$\epsilon$ of room" trick. Take any $\epsilon > 0$. Then there exists a cover $\{I_k\}_k$ s.t.

$$m^*(I) = \epsilon \geq \sum_{k=1}^{\infty} |I_k| = \sum_{k=1}^{\infty} b_k - a_k \tag{34}$$

All we wish to show that the RHS $\geq b - a$, but we can't really manipulate the infinite sum. This is where we use the fact that $[a, b]$ is compact[a], and so we can take a finite subcover $\{I_{k_j}\}_{j=1}^n$. Therefore,

$$m^*(I) + \epsilon \geq \sum_{j=1}^{n} b_{k_j} - a_{k_j} \tag{35}$$

Now we can rearrange this: set the $a_{k_j}$'s to be increasing, and for simplicity let's reindex them to $a_j, b_j$. Then, it must be the case that $a_1 < a$.

(a) Consider $(a_1, b_1)$. If $b_1 > b$, we are done.

(b) Otherwise, $b_1 \in (a_2, b_2)$. If $b_2 > b$, then

$$b_2 - a_2 + b_1 - a_1 \geq b_2 - a_1 > b - a \tag{36}$$

(c) If not, then we keep going until we get to $(a_n, b_n)$. If $b_n > b$, then

$$b_n - a_n + b_{n-1} - a_{n-1} + \ldots + b_1 - a_1 \geq b_n - a_1 > b - a \tag{37}$$

[a] since it's closed and bounded

The proof may be a bit unfamiliar since we have used two tricks.

1. *Geometric Sequence of $\epsilon$ Trick.* To account for countable collections, we set $\epsilon$ to be decreasing geometrically so that the series converges to $\epsilon$.
$$\epsilon = \sum_{n=1}^{\infty} \frac{\epsilon}{2^n} \tag{38}$$
2. *$\epsilon$ of Room Trick.* This trick is used often when you need an opposite inequality that isn't as obvious, and it can only be used for inequalities involving a supremum or infimum.
$$\inf\{S\} \geq x \tag{39}$$
By using the definition of sup/inf as the *least* upper/lower bound, we can scooch over by an $\epsilon$ to find an element in $s \in S$ that does satisfy the inequality
$$\inf\{S\} + \epsilon > s \geq x \tag{40}$$
and since $\epsilon$ was arbitrary, we are done.

It is clear that the Lebesgue outer measure is always less than the Jordan outer measure.
$$m^*(E) \leq m^{(J),*}(E) \tag{41}$$
When are these different?

**Example 2.2 (Lebesgue Outer Measure Much Smaller than Jordan Outer Measure)**

Consider $\mathbb{Q}$.
1. The Jordan outer measure is $+\infty$ since it is unbounded.
2. However, any countable set of $\mathbb{R}$ has Lebesgue outer measure 0. Just enumerate $E = \{x_1, \ldots\}$. Then, we set $I_k = \left(x_k - \frac{\epsilon}{2^k}, x_k + \frac{\epsilon}{2^k}\right)$. Then,
$$\sum_{k=1}^{\infty} \ell(I_k) = \epsilon \tag{42}$$

What about the inner measure? It turns out that we don't get much if we replace the finite to countable in the Jordan inner measure.

**Lemma 2.1 (Axiomatic Properties of Lebesgue Outer Measure)**

The Lebesgue outer measure satisfies the following.
1. *Null Empty Set.* $m^*(\emptyset) = 0$.
2. *Monotonicity.* Given sets $E \subset F \subset \mathbb{R}^n$, we have
$$m^*(E) \leq m^*(F) \tag{43}$$
3. *Countable Subadditivity.* For any countable collection of subsets $\{E_k\}$ of $\mathbb{R}^d$,
$$m^*\left(\bigcup_n E_n\right) \leq \sum_n m^*(E_n) \tag{44}$$

*Proof.* The first two properties are trivial. For the third, let's start by writing out the definition for the outer measure for each $E_n$.

$$m^*(E_n) := \inf\left\{ \sum_{k=1}^\infty |B_k^{(n)}| \;\middle|\; E_n \subset \bigcup_{k=1}^\infty B_k^{(n)}, B_k^{(n)} \text{ boxes}\right\} \tag{45}$$

Somehow, we want to sum these values over all $n$ and prove that this is greater than the measure of the union. The first realization should be that for a fixed cover $\{B_k^{(n)}\}_k$ of $E_n$, the collection

$$\{B_k^{(n)}\}_{n,k} \text{ covers } \bigcup_n E_n \tag{46}$$

This gives us a clue as to comparing the collection of covers of each $E_n$, with the cover of $\cup E_n$. There is no straightforward way to do this,[a] so we want to try and *fix* these collections. We can do this with the $\epsilon$ of room trick. For each $E_n$, we can find a cover $\{B_k^{(n)}\}_k$ s.t.

$$m^*(E_n) + \frac{\epsilon}{2^n} \geq \sum_{k=1}^\infty |B_k^{(n)}| \tag{47}$$

Then, we can take the infinite sum.

$$\sum_{n=1}^\infty \left( m^*(E_n) + \frac{\epsilon}{2^n}\right) \geq \sum_{n=1}^\infty \sum_{k=1}^\infty |B_k^{(n)}| \geq m^*\left( \bigcup_n E_n \right) \tag{48}$$

where the final inequality follows from the fact that $\{B_k^{(n)}\}_{n,k}$ is a cover of $\cup_n E_n$, and so it must be greater than the infimum of all possible covers. All monotonic series converge in $[0, +\infty]$, so we can "take out" the $\epsilon$ term to get

$$\epsilon + \sum_{n=1}^\infty m^*(E_n) \geq m^*\left( \bigcup_n E_n \right) \tag{49}$$

which holds for all $\epsilon > 0$, and so we are done.

[a] On one side, we have the sum of a countable number of infimums of some sets, and on the other hand, we have the infimum of the unions of all of these sets.

**Theorem 2.2 (Translation Invariance of Lebesgue Outer Measure)**

$m^*$ is translation invariant. That is, for any $E \subset \mathbb{R}^n$,

$$m^*(E) = m^*(E + a), \qquad E + a := \{x + a \in \mathbb{R}^n \mid a \in E\} \tag{50}$$

*Proof.* This is straightforward and requires no tricks. Note that

$$\{B_k\} \text{ is a countable cover of } E \iff \{B_k + a\} \text{ is a countable cover of } E + a \tag{51}$$

It is also clear that

$$|B| = |B + a| \tag{52}$$

for any box $B \subset \mathbb{R}^n$, so the sets of sizes that we are taking the infimum over is exactly the same between the two.

The final property claims that we can always drop an outer-measure 0 set and it won't affect the outer measure of the original set. Therefore, when talking about measurability of intervals, we don't have to worry about endpoints, or even whether it is missing a countable number of elements from it!

**Lemma 2.2 (Sets of Measure 0 have no Effect)**

Suppose $m^*(E) = 0$ and $A$ is any set. Then, $m^*(A \cup E) = m^*(A)$.

*Proof.* We have

$$m^*(A \cup E) = \underbrace{m^*\big((A \cup E) \cap E\big)}_{=0} + m^* \underbrace{\big((A \cup E) \cap E^c\big)}_{\subset A} \leq m^*(A) \leq m^*(A) \tag{53}$$

But $A \cup E \supset A$, so $m^*(A \cup E) = m^*(A)$.

**Lemma 2.3 (Finite Additivity for Outer Measure)**

Let $A$ and $B$ be bounded sets for which there is an $\alpha > 0$ such that $|a - b| \geq \alpha$ for all $a \in A$, $b \in B$. Prove that $m^*(A \cup B) = m^*(A) + m^*(B)$.[a]

[a] This is a slightly weaker version of Tao Lemma 1.2.5.

*Proof.* From finite subadditivity

Therefore we can invoke this in special cases where one set may be compact and another may be closed.

## 2.2 Measurable Sets

The next step is to take the outer measure and define *Lebesgue measurable* sets. The problem is that—unlike the Jordan measure—we don't have the inner measure to work with. This turns out to be not much of a problem, since through *Littlewood's first principle*[1], we can define measurability as being well-approximated by an open set.

**Definition 2.3 (Measurable Set)**

A set $E \subset \mathbb{R}^d$ is **Lebesgue measurable** if it satisfies one of the equivalent properties.

1. *Carathéodory's criterion.*[a] For every set $A \subset \mathbb{R}^n$,

$$m^*(A) = m^*(A \cap E) + m^*(A \cap E^c) \tag{54}$$

   Note that due to countable subadditivity, we are guaranteed to have $\leq$. Therefore, it suffices to prove only for $\geq$. The sets with which this inequality is strict is not measurable, and the measurable sets specifically satisfy equality for countable sets.
2. *Outer Approximately Open.* $\forall \epsilon > 0$, $\exists$ open set $O \supset E$ s.t. $m(O \setminus E) \leq \epsilon$.
3. *Inner Approximately Closed.* $\forall \epsilon > 0$, $\exists$ closed set $F \subset E$ s.t. $m^*(E \setminus F) < \epsilon$.
4. *Outer Exactly $G_\delta$.* $\exists$ a $G_\delta$ set $G$ s.t. $E \subset G$ and $m^*(G \setminus E) = 0$.
5. *Inner Exactly $F_\sigma$.* $\exists$ a $F_\sigma$ set $F$ s.t. $F \subset E$ and $m^*(E \setminus F) = 0$.

[a] Colloquially, no matter how nasty a subset $A$ is, $E$ should be nice enough that we can cut $E$ into two pieces $C$ and $D$.

*Proof.* Listed.

1. (2) $\implies$ (1). Then for every $k \in \mathbb{N}$, we can find $O_k \supset E$ s.t. $m^*(O_k \setminus E) \leq 1/k$. Define the $G_\delta$ set $G = \cap_{k=1}^\infty O_k$. Then, $(G \setminus E) \subset (O_k \setminus E)$ for all $k \implies m^*(G \setminus E) \leq 1/k$ for all $k$. Therefore

[1] One of the major themes in measure theory, where we say that measurable sets are well-approximated by open and closed sets.

$m^*(G \setminus E) = 0$, and $E = G \setminus (G \setminus E)$ is measurable.

2. (1) $\implies$ (2). Assume $m^*(E) < +\infty$. Find a cover $\{I_k\}_{k=1}^{\infty}$ s.t. $\sum_{k=1}^{\infty} \ell(I_k) \leq m^*(E) + \epsilon$ . Call $O = \cup_k I_k$. Since $E$ is measurable, $m^*(O \setminus E) = m^*(O) - m^*(E) \leq \sum_{k=1}^{\infty} \ell(I_k) - m^*(E) \leq \epsilon$
3. (1) $\iff$ (3). Straightforward with argument above.
4. (1) $\iff$ (4). Generally, we use the fact that $E$ measurable iff $E^c$ measurable. Find $O \supset E^c$ open, with $m^*(O \setminus E^c) \leq \epsilon$. Then $F = O^c$ is closed, $F \subset E$, and $m^*(E \setminus F) \leq \epsilon$.
5. (1) $\iff$ (5). Same argument as (1) $\iff$ (4).

Depending on the context, it is helpful to use one definition over another when proving measurability. Just remember that the Carathéodory definition is the most general, since it doesn't even assume a topology on a space, and that is the definition that we will use by default. So what kind of sets are measurable?

**Example 2.3 (Outer Measure $0$ Sets are Lebesgue Measurable)**

For any outer measure $m^*$ on $X$, $E \subset X$ with $m^*(E) = 0$ implies that $E$ is $m^*$-measurable. Take any $A$. Then $(A \cap E) \subset E$ and $(A \cap E^c) \subset A$. So by monotonicity,

$$m^*(A \cap E) + m^*(A \cap E^c) \leq m^*(E) + m^*(A) = m^*(A) \tag{55}$$

and this by definition means that $E$ is measurable.

We could continue with more examples, but our main priority is to show that this family of Lebesgue measurable sets is indeed a $\sigma$-algebra, and it covers much more than Jordan measurable sets. The path to prove that countable unions are measurable is a long one, and we'll a lot of lemmas.

**Lemma 2.4 (Complements of Measurable Sets are Measurable)**

If $E$ is measurable, then so is $E^c$.

*Proof.* The definition is symmetric in both $E$ and $E^c$.

Using unions and complements, we can prove that intersections and set differences of measurable sets are measurable.

**Lemma 2.5 (Finite Intersections of Measurable Sets are Measurable)**

If $E_1, \ldots, E_n$ is measurable, then so is $\cap_{k=1}^n E_k$.

*Proof.* By induction, it suffices to prove for $n = 2$. We have

$$E_1 \cap E_2 = (E_1^c \cup E_2^c)^c \tag{56}$$

**Lemma 2.6 (Set Differences of Measurable Sets are Measurable)**

If $E_1, E_2$ is measurable, then so is $E_1 \setminus E_2$.

*Proof.* We have

$$E_1 \setminus E_2 = E_1 \cap E_2^c \tag{57}$$

**Lemma 2.7 (Excision Property)**

If $E \subset \mathbb{R}^n$ is measurable with $m^*(E) < +\infty$, and $E \subset F$ for arbitrary set $F$, then

$$m^*(F \setminus E) = m^*(F) - m^*(E) \tag{58}$$

*Proof.* By measurability of $E$, we can see

$$\begin{aligned} m^*(F) &= m^*(F \cap E) + m^*(F \cap E^c) & (59) \\ &= m^*(E) + m^*(F \setminus E) & (60) \end{aligned}$$

This excision property combined with the fact that outer measure 0 sets are always measurable gives us the property of *completeness*.[2] That is, given measurable sets $A \subset B \subset C$ with $m^*(A) = m^*(C)$, $B$ must be measurable. This basically says that if you a set that is squeezed in between two measurable sets of equal measure, then the middle set will also be measurable.

**Lemma 2.8 (Finite Unions are Measurable)**

A finite union of measurable sets is measurable.

*Proof.* This proof is basically applying set theory laws, and there's not much more to that. It suffices to prove for $E_1, E_2$, and the rest follows by induction. Fix any $A$. Then

$$\begin{aligned} m^*(A) &= m^*(A \cap E_1) + m^*(A \cap E_1^c) & (61) \\ &= m^*(A \cap E_1) + m^*\big(A \cap E_1^c \cap E_2\big) + m^*\big((A \cap E_1^c) \cap E_2^c\big) & (62) \end{aligned}$$

Now we apply the identity $(A \cap E_1^c) \cap E_2^c = A \cap (E_1 \cup E_2)^c$, so the third term can be changed

$$= m^*(A \cap E_1) + m^*\big((A \cap E_1^c) \cap E_2\big) + m^*\big(A \cap (E_1 \cup E_2)^c\big) \tag{63}$$

Then we apply the identity $(A \cap E_1) \cup (A \cap E_1^c \cap E_2) = A \cap (E_1 \cup E_2)$, so we can apply finite subadditivity on the first two terms to get

$$\geq m^*(A \cap (E_1 \cup E_2)) + m^*\big(A \cap (E_1 \cup E_2)^c\big) \tag{64}$$

which proves that $E_1 \cup E_2$ is measurable.

So we have proved that the set of all measurable sets is closed under finite unions. By definition it works for finite intersections. This makes it into an *algebra*, but we want to upgrade this to a $\sigma$-algebra by proving closure under *countable* unions. We first prove a lemma.

**Lemma 2.9 (Finite Additivity of Outer Measure on Disjoint Measurable Sets)**

Suppose $A$ is any set, $\{E_k\}_{k=1}^n$ disjoint and measurable. Then,

$$m^*\bigg(A \cap \Big(\bigcup_{k=1}^n E_k\Big)\bigg) = \sum_{k=1}^n m^*(A \cap E_k) \tag{65}$$

In particular,

$$m^*\Big(\bigcup_{k=1}^n E_k\Big) = \sum_{k=1}^n m^*(E_k) \tag{66}$$

[2] Nothing to do with completeness in the sense of real numbers or metric spaces.

*Proof.* It should be clear that we prove by induction, and intuitively, this disjointness should be essential for canceling out some measure terms. $n = 1$ is trivial. This takes a bit of fiddling around with where we should start, but if we just look at the LHS, we can try and use Carathéodory, by setting the arbitrary set to be $B = A \cap (\cup_k E_k)$ and writing $m^*(B) = m^*(B \cap E_n) + m^*(B \cap E_n^c)$.

$$\begin{aligned} m^*\bigg(A \cap \Big(\bigcup_{k=1}^n E_k\Big)\bigg) &= m^*\Bigg(\bigg(A \cap \Big(\bigcup_{k=1}^n E_k\Big)\bigg) \cap E_n\Bigg) + m^*\Bigg(\bigg(A \cap \Big(\bigcup_{k=1}^n E_k\Big)\bigg) \cap E_n^c\Bigg) && (67)\\ &= m^*(A \cap E_n) + m^*\bigg(A \cap \Big(\bigcup_{k=1}^{n-1} E_k\Big)\bigg) && (68)\\ &= \sum_{k=1}^n m^*(A \cap E_k) && (69) \end{aligned}$$

But note that by disjointness, we have

$$\bigg(A \cap \Big(\bigcup_{k=1}^n E_k\Big)\bigg) \cap E_n = A \cap E_n, \qquad \bigg(A \cap \Big(\bigcup_{k=1}^n E_k\Big)\bigg) \cap E_n^c = A \cap \Big(\bigcup_{k=1}^{n-1} E_k\Big) \tag{70}$$

by the induction hypothesis.

Here is a wrong proof that does an incorrect form of induction. I first assumed that we can just work with a family of two sets $E_1, E_2$, so I started deriving something like this:

$$\begin{aligned} m^*(A) &= m^*(A \cap (E_1 \cup E_2)) + m^*(A \cap (E_1 \cup E_2)^c) && (71)\\ &= m^*((A \cap E_1) \cup (A \cap E_2) + \underbrace{m^*((A \setminus E_1) \cap (A \setminus E_2))}_{\geq 0} && (72)\\ &\geq m^*((A \cap E_1) \cup (A \cap E_2)) && (73) \end{aligned}$$

Note that the $E_k$'s being disjoint means that they are *pairwise* disjoint, and so it is *not* sufficient to prove for only $E_1, E_2$. So don't do this.

**Theorem 2.3 (Countable Unions are Measurable)**

Suppose $E_1, E_2, \ldots$ are a countable collection of measurable sets. Then, $E = \cup_{j=1}^\infty E_j$ is measurable.

*Proof.* They key is to look at disjoint sets. WLOG, one can assume $E_j$ are disjoint. Indeed, we can define new sets

$$E_n' \coloneqq E_n \setminus \bigg(\bigcup_{j=1}^{n-1} E_j\bigg) \tag{74}$$

that are measurable, with $\cup E_n' = \cup E_n$. Now, fix any set $A$. Define sets $F_n = \cup_{j=1}^n E_j$. Then, $m^*(A) = m^*(A \cap F_n) + m^*(A \cap F_n^c)$. Then, $F_n^c \supset E^c \implies m^*(A \cap F_n^c) \geq m^*(A \cap E^c)$. Since we have established from the previous lemma that outer measure on disjoint measurable sets satisfies finite additivity, we can write

$$m^*(A \cap F_n) = m^*\bigg(\bigcup_{j=1}^n (A \cap E_j)\bigg) = \sum_{j=1}^n m^*(A \cap E_j) \tag{75}$$

Then,

$$m^*(A) \geq \sum_{j=1}^n m^*(A \cap E_j) + m^*(A \cap E^c) \tag{76}$$

for every $n$, therefore also with $\infty$. But by countable subadditivity of the outer measure,

$$\sum_{j=1}^{\infty} m^*(A \cap E_j) \geq m^*(A \cap E) \tag{77}$$

If follows that $m^*(A) \geq m^*(A \cap E) + m^*(A \cap E^c)$.

**Corollary 2.1 (Measurable Sets form a $\sigma$-Algebra)**

The set of all Lebesgue measurable sets of $\mathbb{R}^n$ form a $\sigma$-algebra.

*Proof.* Listed.

1. $\emptyset$ is measurable since it has outer measure 0.
2. $\mathbb{R}^n$ is measurable since for any set $A \subset \mathbb{R}^n$,

$$m^*(A) = m^*(A \cap \mathbb{R}^n) + m^*(A \cap (\mathbb{R}^n)^c) = m^*(A) \tag{78}$$

3. We proved closure under complements.
4. We just proved closure under countable union.

Now that we have established that Lebesgue measurable sets form a $\sigma$-algebra, let's give some examples.

**Theorem 2.4 (Rays are Measurable)**

Every interval $(a, +\infty)$ is measurable.

*Proof.* We simply prove using Carathéodory,[a] so we wish to show that for any set $A \subset \mathbb{R}$,

$$m^*(A) \geq m^*(A \cap (a, +\infty)) + m^*(A \cap (-\infty, a]) = m^*(A') + m^*(A'') \tag{79}$$

Again, the fact that we have to prove this nontrivial part of the inequality reminds us of using the $\epsilon$ of room trick. Let us $\{I_k\}_{k=1}^{\infty}$ is a countable cover of $A$ s.t.

$$m^*(A) + \epsilon > \sum_{k=1}^{\infty} \ell(I_k) \tag{80}$$

We can take this cover and create two smaller covers, covering $A'$ and $A''$.

$$\{I_k' \coloneqq I_k \cap A'\}_k, \qquad \{I_k'' \coloneqq I_k \cap A''\}_k \tag{81}$$

Since these are valid covers, by definition it must be bounded below by the outer measures.

$$\sum_{k=1}^{\infty} \ell(I_k') \geq m^*(A'), \qquad \sum_{k=1}^{\infty} \ell(I_k'') \geq m^*(A'') \tag{82}$$

Now all that remains is to connect the sums together. For each $k$, we have $\ell(I_k) = \ell(I_k') + \ell(I_k'')$, and since both series converge in $[0, +\infty]$, we can indeed sum them up as limits to get

$$\sum_{k=1}^{\infty} \ell(I_k') + \sum_{k=1}^{\infty} \ell(I_k'') = \sum_{k=1}^{\infty} \ell(I_k') + \ell(I_k'') = \sum_{k=1}^{\infty} \ell(I_k) \tag{83}$$

Putting the bounds together gives

$$m^*(A') + m^*(A'') \leq \sum_{k=1}^{\infty} \ell(I_k') + \sum_{k=1}^{\infty} \ell(I_k'') = \sum_{k=1}^{\infty} \ell(I_k) \leq m^*(A) + \epsilon \tag{84}$$

Since this is true for every $\epsilon > 0$, we are done.

[a] WLOG $a \notin A$ (since we can take the point out without affecting outer measure). TBD: Do we need this assumption really?

Note again that this $\epsilon$ of room trick is used so that we can *fix* some open cover that acts as a middle ground between the inequalities that we are trying to prove. Then as we let $\epsilon \to 0$, we are done.

**Corollary 2.2 (All Intervals are Measurable)**

All intervals $I \subset \mathbb{R}$ are measurable.

**Example 2.4 (Cantor Set is Measurable)**

Let us define

$$C_0 = [0, 1], \qquad C_1 = [0, 1/3] \cup [2/3, 1], \ldots \tag{85}$$

Basically, we take our every middle third of each subinterval. So $C_k$ is the union of $2^k$ intervals of size $3^{-k}$. Note that $C_k \subset C_{k-1}$. Now define the **Cantor set** as

$$C \coloneqq \bigcap_{k=1}^{\infty} C_k \tag{86}$$

The Cantor set is measurable since it is a countable intersection of closed sets, which are measurable.

**Theorem 2.5 (Translations of Sets are Measurable)**

If $E \subset \mathbb{R}^n$ is measurable, then for any $a \in \mathbb{R}^n$, $E + a \coloneqq \{x + a \in \mathbb{R}^n \mid x \in E\}$ is measurable.

*Proof.* We again use Carathéodory. Let $A \subset \mathbb{R}^n$ by any set. Then by translation invariance of outer measure, we have

$$\begin{aligned} m^*(A) &= m^*(A - a) && (87) \\ &= m^*((A - a) \cap E) + m^*((A - a) \cap E^c) && (88) \\ &= m^*(A \cap (E + a)) + m^*(A \cap (E + x)^c) && (89) \end{aligned}$$

and so $E + a$ is measurable.

This next theorem is a different flavor of Littlewood's first principle. It tells us that we can use a finite union of intervals that "symmetrically" approximates measurable sets on the real line.

**Theorem 2.6 (Finite Union of Intervals are Good Symmetric Approximations)**

Suppose $E$ is measurable, with $m^*(E) < +\infty$. Fix $\epsilon > 0$. Then there exists a finite number of intervals $\{I_k\}_{k=1}^n$ s.t. if $O = \cup_{k=1}^n I_k$, then

$$m^*(O \setminus E) + m^*(E \setminus O) < \epsilon \tag{90}$$

*Proof.* In here, we use the outer approximately open definition of measurable sets. Since every open set can be written as a countable union of open intervals[a], we can find a collection of open intervals

$\{I_k\}_{k=1}^\infty$ s.t.

$$E \subset U \coloneqq \bigcup_{k=1}^\infty I_k, \qquad m^*(U \setminus E) \leq \frac{\epsilon}{2} \tag{91}$$

The major thing to do now is to reduce the countable union into a finite union. Note that we can just take any subcollection of the $I_k$'s, and we are guaranteed that their union $O$ will satisfy

$$m^*(O \setminus E) \leq m(U \setminus E) \leq \frac{\epsilon}{2} \tag{92}$$

The problem is that we don't want to take too small of a collection so that the other difference is too big. To do this, we can just select the tail of the series: Find $n$ s.t. $\sum_{k=n+1}^\infty \ell(I_k) \leq \epsilon/2$ where WLOG, $I_k$ are disjoint. Define $O = \cup_{k=1}^n I_k$. Then, we have

$$m^*(O \setminus E) \leq m(U \setminus E) \leq \frac{\epsilon}{2} \tag{93}$$

$$m^*(E \setminus O) \leq m(U \setminus O) \leq \sum_{k=n+1}^\infty \ell(I_k) \leq \frac{\epsilon}{2} \tag{94}$$

[a]since $\mathbb{R}^n$ is second countable

This symmetry in difference induced me to use the inner approximately closed definition in addition. My idea was to just find a closed set $F$ s.t. $F \subset E \subset U$, but there is no straightforward way of finding one finite collection of intervals $O$.

**Example 2.5 (idk where to put this)**

One should note that in particular, if $E$ is $m^*$-measurable and $A$ is any set disjoint from $E$, then we must have

$$\begin{aligned} m^*(A \cup E) &= m^*((A \cup E) \cap E) + m^*((A \cup E) \cap E^c) && (95) \\ &= m^*(E) + m^*(A) && (96) \end{aligned}$$

which solves a bit of the theorem on measures.

## 2.3 Measures

Now by restricting our outer measure to measurable sets, we get our measure.

**Definition 2.4 (Lebesgue Measure)**

The restriction the Lebesgue outer measure $m^*$ to the set of all measurable sets $\mathcal{A}$, is called the **Lebesgue measure**

$$m = m^*\big|_{\mathcal{A}} \tag{97}$$

Note that for outer measures, they satisfy both countable subadditivity and finite additivity. With measures, we get the best of both worlds: countable subadditivity.

**Lemma 2.10 (Axiomatic Properties of Lebesgue Measure)**

The Lebesgue measure satisfies the following.

1. *Null Empty Set.* $m(\emptyset) = 0$.
2. *Countable Additivity.* For all countable collections $\{A_k\}_{k=1}^\infty$ of pairwise disjoint subsets $A_k \in \mathcal{A}$[a],

$$m\bigg(\bigsqcup_{k=1}^\infty A_k\bigg) = \sum_{k=1}^\infty m(A_k) \tag{98}$$

[a] Remember that we are allowed to take countable unions inside our $\sigma$-algebra, so this makes sense.

*Proof.* Listed.

1. *Null Empty Set.* Since this is true for outer measure $m^*$.
2. *Countable Additivity.* $m(\cup E_j) = \sum_j m(E_j)$. $\leq$ is trivial by countable subadditivity of the outer measure. For $\geq$, note that for every $n \in \mathbb{N}$,

$$m\bigg(\bigcup_{j=1}^\infty E_j\bigg) \geq m\bigg(\bigcup_{j=1}^n E_j\bigg) = \sum_{j=1}^n m(E_j) \tag{99}$$

   where the inequality comes from monotonicity and the equality comes from finite additivity of the outer measure. Now take $n \to \infty$.

Unlike the outer measure, monotonicity is not an axiomatic property because of two independent reasons, both sufficient. First, the Lebesgue outer measure suffices. Second, it is actually a direct consequence of the two axiomatic properties.

**Lemma 2.11 (Translation Invariance of Lebesgue Measure)**

The Lebesgue measure is translation invariant.

*Proof.* We know that translations of Lebesgue measurable sets are also Lebesgue measurable, and the Lebesgue outer measure is translation invariant.

Now we provide some "continuity" properties of the Lebesgue measure.

**Theorem 2.7 (Continuity From Below)**

If $A_1 \subset A_2 \subset A_3 \subset \ldots$, then

$$m\bigg(\bigcup_{k=1}^\infty A_k\bigg) = \lim_{k\to\infty} m(A_k) \tag{100}$$

*Proof.* First, note that the limit on the RHS is defined, since $m(A_k)$ is nondecreasing and so must converge in $[0, +\infty]$. But why does this limit equal to the left hand side? The only property that makes sense to work with is countable additivity, so we should define the disjoint collection

$$B_1 = A_1, \quad B_k = A_k \setminus A_{k-1} \tag{101}$$

Then, it becomes straightforward

$$\begin{aligned}
m\bigg(\bigcup_{k=1}^\infty A_k\bigg) &= m\bigg(\bigcup_{k=1}^\infty B_k\bigg) && \text{(Construction)} \\
&= \sum_{k=1}^\infty m(B_k) && \text{(Countable Additivity)} \\
&= \lim_{n\to\infty} \sum_{k=1}^n m(B_k) && \text{(Definition of Series)} \\
&= \lim_{n\to\infty} m\bigg(\bigcup_{k=1}^\infty B_k\bigg) && \text{(Finite Additivity)} \\
&= \lim_{n\to\infty} m(A_k) && (102)
\end{aligned}$$

*Proof.* Old proof. We can see that

$$\begin{aligned}
m\bigg(\bigcup_{k=1}^\infty A_k\bigg) &= m(A_1) + \sum_{k=2}^\infty m(B_k) && (103) \\
&= m(A_1) + \lim_{k\to\infty} \sum_{k=2}^\infty m(B_k) && (104) \\
&= \lim_{k\to\infty} m(A_1 \cup B_2 \cup \ldots B_k) = \lim_{k\to\infty} m(A_k) && (105)
\end{aligned}$$

Now a similar theorem, but with a little twist to it.

**Theorem 2.8 (Continuity from Above)**

If $A_1 \supset A_2 \supset A_3 \supset \ldots$, then

$$m\bigg(\bigcap_{k=1}^\infty A_k\bigg) = \lim_{k\to\infty} m(A_k) \tag{106}$$

if $m(A_1) < \infty$.

*Proof.* First, note that the $m(A_1) < +\infty$ is a necessary condition, since if we take $A_k = [k, \infty)$ on the real number line, then we have $\cap_{k=1}^\infty A_k = \emptyset$, but the limit of the measure is $\infty$. We did not have this problem for continuity from below.

Well we can define $B_k = A_k \setminus A_{k+1}$ and write $\cap_{k=1}^{\infty} A_k = A_1 \setminus \cup_{k=1}^{\infty} B_k$, which means that

$$\begin{aligned}
m\bigg(\bigcap_{k=1}^{\infty} A_k\bigg) &= m\bigg(A_1 \setminus \bigcup_{k=1}^{\infty} B_k\bigg) && (107) \\
&= m(A_1) - m\bigg(\bigcup_{k=1}^{\infty} B_k\bigg) && \text{(Excision)} \\
&= m(A_1) - \sum_{k=1}^{\infty} m(B_k) && \text{(Countable Additivity)} \\
&= m(A_1) - \lim_{n\to\infty} \sum_{k=1}^{n} m(B_k) && \text{(Definition of Series)} \\
&= \lim_{n\to\infty} \bigg(m(A_1) - \sum_{k=1}^{n} m(B_k)\bigg) && (108) \\
&= \lim_{n\to\infty} m\bigg(A_1 \setminus \bigcup_{k=1}^{n} B_k\bigg) && \text{(Excision)} \\
&= \lim_{n\to\infty} m(A_n) && (109)
\end{aligned}$$

Now the first line uses the fact that if $A \subset B$, then $m(B \setminus A) + m(A) = m(B)$, and with the further assumption that $m(A) < \infty$, we can subtract on both sides like we do with regular arithmetic.

We will see two applications of continuity from above.

**Example 2.6 (Cantor Set has Measure 0)**

The Cantor set has measure 0. We can see that it is the intersection of all $C_k$'s which are nested $C_k \supset C_{k+1}$ and $m(C_0) = m([0,1]) = 1$. Therefore, by continuity from above,

$$m\bigg(\bigcap_{k=1}^{\infty} C_k\bigg) = \lim_{k\to\infty} m(C_k) = \lim_{k\to\infty} \frac{2^k}{3^k} = 0 \tag{110}$$

It is also closed as an intersection of closed sets. It is also uncountable, since we can just do it using a tradic system and see that the Cantor set are all reals with infinite triadic representation of digits 0 and 2. Then create a bijection with binary represenation of reals. Here's a new way I learned. Suppose $C$ is countable, so enumerate it: $c_1, c_2, \ldots$. Pick one interval $I_1$ in $C_1$ that doesn't contain $c_1$. Then, pick $I_2 \subset I_1 \cap C_2$ s.t. it doesn't contain $c_2$. Keep going, and we get

$$I_1 \supset I_2 \supset I_3 \supset \ldots \tag{111}$$

By nested intervals lemma, these are closed, bounded, and nested, which is nonempty. So we've found a point not in the Cantor set, contradicting the fact that we have enumerated it.

**Lemma 2.12 (Borel-Cantelli)**

Suppose $\{E_k\}_{k=1}^{\infty}$ are measurable, with $\sum_k m(E_k) < +\infty$. Then,

$$m\big(\limsup E_k\big) := m\bigg(\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k\bigg) = 0 \tag{112}$$

That is, all $x \in \mathbb{R}$ belonging to an infinite number of $E_k$, has measure 0.

*Proof.* This is a direct application of continuity from above, where we can set

$$B_n \coloneqq \bigcup_{k=n}^{\infty} E_k \tag{113}$$

Notice that $B_n \supset B_{k+1}$ and $B_1$ has finite measure since by countable subadditivity[a],

$$m(B_1) = m\bigg(\bigcup_{k=1}^{\infty} E_k\bigg) \leq \sum_{k=1}^{\infty} m(E_k) < +\infty \tag{114}$$

Therefore, we can derive

$$\begin{aligned} m\bigg(\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} E_k\bigg) &= \lim_{n\to\infty} \bigg(\bigcup_{k=n}^{\infty} E_k\bigg) && \text{(Continuity from Above)} \\ &= \lim_{n\to\infty} \sum_{k=n}^{\infty} m(E_k) && \text{(Countable Additivity)} \\ &= 0 && (115) \end{aligned}$$

because the tail of series converging to a finite value must tend to 0. Note that for the last step, we could have used countable subadditivity as well.

[a] not countable additivity!

**Definition 2.5 (Almost Everywhere)**

Given a measure space $(X, \mathcal{A}, m)$, a subset $A \in \mathcal{A}$ is said to be a $m$-null set if $m(A) = 0$. If some property holds for all points $x \in X$ except on a null set, then we say that the property holds **almost everywhere**.

**Example 2.7 (Rational Function)**

The function $f(x) = \frac{1}{\sqrt{|x|}}$ is less than $\infty$ almost everywhere.

## 2.4 Nonmeasurable Sets

**Lemma 2.13 (Quotienting over Countable Set Implies Lebesgue Measure 0)**

Suppose $E$ is measurable, bounded, and there exists a countably infinite, bounded set $\Lambda$ s.t. $\{E+m\}_{m\in\Lambda}$ are disjoint. Then, $m(E) = 0$.

*Proof.* Consider $\cup_{m\in\Lambda}\{E+m\}$. It is bounded, so its measure is finite. Also, by countable addivitiy and translation invariance, we get

$$+\infty > m\bigg(\bigcup_{m\in\Lambda} \{E+m\}\bigg) = \sum_{m\in\Lambda} m(E+m) = \sum_{m\in\Lambda} m(E) \tag{116}$$

Since $\Lambda$ is infinite, we must have $m(E) = 0$.

Recall that we say $x$ is *rationally equivalent* to $y$ if $x - y \in \mathbb{Q}$. This is an equivalence relation on $\mathbb{R}$, giving us a quotient set of an uncountable number of classes. A *choice set* for this equivalence relation is a set containing exactly one element from each class.[3] We can do this on any $E \subset \mathbb{R}$.

**Lemma 2.14 (Properties of Choice Sets)**

If $C_E$ is any choice set on such $E$, then

1. $\forall x, y \in C_E$, if $x - y \notin \mathbb{Q}$, then for all $\Lambda \in \mathbb{Q}$, $\{m + C_E\}_{m \in \Lambda}$ are disjoint.
2. $\forall x \in E$, there exists $y \in C_E$ s.t. $x - y \in \mathbb{Q}$.

**Theorem 2.9 (Every Set of Positive Outer Measure Contains Nonmeasurable Set)**

Any set $E \subset \mathbb{R}^n$ of positive outer measure contains a nonmeasurable set.

*Proof.* WLOG, let $E$ be bounded.[a] Let $C_E$ be any choice set. Suppose $C_E$ is measurable. Let $b$ be such that $E \subset [-b, b]$. Let $\Lambda = \mathbb{Q} \cap [-2b, 2b]$. Consider the disjoint family of sets $\{C_E + \lambda\}_{\lambda \in \Lambda}$. Also,

$$E \subset \bigcup_{\lambda \in \Lambda} \{C_E + \lambda\} \tag{117}$$

Indeed, $\forall x \in E$, there exists $y \in C_E$ s.t. $x - y \in \mathbb{Q}$ and in $\Lambda$ by definition of $\Lambda$ and $E \subset [-b, b]$. By the lemma, $m(C_E) = 0$. But also,

$$m^*(E) \leq \sum_{\lambda \in \Lambda} \underbrace{m(C_E + \lambda)}_{= m(C_E)} = 0 \tag{118}$$

which is a contradiction, so $C_E$ is not measurable.

[a] Otherwise, just take a bounded subset.

**Definition 2.6 (Cantor-Lebesgue Function, Devil's Staircase)**

The **Cantor-Lebesgue function** $\phi : [0, 1] \to \mathbb{R}$ is defined as such.

1. Let us define $O_k = [0, 1] \setminus C_k{}^a$, which is an open set. So $O_k$ consists of $2^k - 1$ open intervals $I_j$ ($j$ indexed from left to right). For $O_k$, we define $\phi(x) = j/2^k$, where $j$ is the number of the interval $I_j$, indexed left to right. This defines $\phi$ on $O = \cup_{k=1}^{\infty} O_k = [0, 1] \setminus C$.
2. On $C$, let us define $\phi(x) \coloneqq \inf_{y \geq x, y \in O} \phi(x)$.

[3] This assumes axiom of choice.

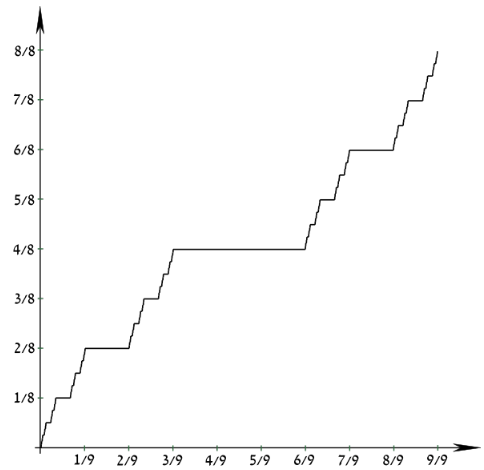


Figure 1: Plot

[a] $C_k$ defined as before when constructing the Cantor set.

**Theorem 2.10 (Properties of Cantor-Lebesgue Function)**

$\phi$ is a nondecreasing, continuous function s.t. $\phi'(x) = 0$ for all $x \in O$ and $m(O) = 1$.

*Proof.* Listed.

1. *Increasing.* $\phi$ is increasing on each $O_k$, and so on $O$. Then, it is also increasing on $C$ by definition.
2. *Continuity.* If $x \in C$, it lies between 2 intervals of $O_k$ for any $k$. The difference in function values between 2 neighboring intervals of $O_k$ is $2^{-k}$, so $\phi$ is continuous.
3. *Derivative.* The derivative is 0 because it is constant around an interval.

**Theorem 2.11 (Pathological Strictly Increasing Devil's Staircase)**

Define $\psi(x) = \phi(x) + x$. Then,

1. $\psi$ is continuous and strictly increasing.
2. $\psi$ maps $C$ into a set of positive measure.
3. $\psi$ maps some subset of $C$ into a nonmeasurable set.

*Proof.* Listed.

1. Continuity is from sum of continuous functions, and strictly increasing since $\phi$ is nondecreasing and $x$ is strictly increasing.
2. We know that $\psi([0, 1]) = [0, 2]$ and $m(\psi(O)) = 1$, where for each interval $I_j \subset O$, $m(\psi(I_j)) = \ell(I_j)$. Therefore, $m(\psi(C)) = 1$.
3. Since $\psi$ is strictly increasing, there exists a continuous inverse $\psi^{-1}$. Find $Z \subset \psi(C)$ that is nonmeasurable, which we can do from the previous theorem. Then, there exists $E \subset C$ s.t. $\psi(E) = Z$, and $E$ is not Borel since if it were, then $Z$ would be Borel, too.

## 2.5 Exercises

**Exercise 2.1 (Royden 2.1)**

Let $m$ be a set function defined for all sets in a $\sigma$-algebra $\mathcal{A}$ with values in $[0, \infty]$. Assume $m$ is countably additive over countable disjoint collections of sets in $\mathcal{A}$. Prove that if $A$ and $B$ are two sets in $\mathcal{A}$ with $A \subseteq B$, then $m(A) \leq m(B)$. This property is called *monotonicity*.

*Solution.* This is simply a manipulation of definition.

$$m(B) = m((B \cap A) \cup (B \setminus A)) = m(B \cap A) + m(B \setminus A) = m(A) + m(B \setminus A) \geq m(A) \tag{119}$$

■ Therefore, by upgrading countable subadditivity to countable additivity, we get the monotonicity property for free.

**Exercise 2.2 (Royden 2.2)**

Let $m$ be a set function defined for all sets in a $\sigma$-algebra $\mathcal{A}$ with values in $[0, \infty]$. Assume $m$ is countably additive over countable disjoint collections of sets in $\mathcal{A}$. Prove that if there is a set $A$ in the collection $\mathcal{A}$ for which $m(A) < \infty$, then $m(\emptyset) = 0$.

*Solution.* By countable additivity,

$$m(A) = m\left(A \cup \bigcup_{i=1}^{\infty} \emptyset\right) = m(A) + \sum_{i=1}^{\infty} m(\emptyset) < +\infty \tag{120}$$

In order for the series to be finite, $m(\emptyset) = 0$. ■ Therefore, the finite subadditivity condition is not needed in most cases where there exists some set of finite measure.

**Exercise 2.3 (Royden 2.3)**

Let $m$ be a set function defined for all sets in a $\sigma$-algebra $\mathcal{A}$ with values in $[0, \infty]$. Assume $m$ is countably additive over countable disjoint collections of sets in $\mathcal{A}$. Let $\{E_k\}_{k=1}^{\infty}$ be a countable collection of sets in $\mathcal{A}$. Prove that $m(\bigcup_{k=1}^{\infty} E_k) \leq \sum_{k=1}^{\infty} m(E_k)$.

*Solution.* The idea is to manually make the $E_k$'s disjoint. Let us set

$$E_1' = E_1, \quad E_2' = E_2 \setminus E_1, \quad \ldots, E_k' = E_k \setminus \bigcup_{i=1}^{k-1} E_i \tag{121}$$

Then, $E_k'$s are pairwise disjoint, so

$$m\left(\bigcup_{k=1}^{\infty} E_k\right) = m\left(\bigcup_{k=1}^{\infty} E_k'\right) = \sum_{k=1}^{\infty} m(E_k') \leq \sum_{k=1}^{\infty} m(E_k) \tag{122}$$

**Exercise 2.4 (Royden 2.4)**

A set function $c$, defined on all subsets of $\mathbf{R}$, is defined as follows. Define $c(E)$ to be $\infty$ if $E$ has infinitely many members and $c(E)$ to be equal to the number of elements in $E$ if $E$ is finite; define $c(\emptyset) = 0$. Show that $c$ is a countably additive and translation invariant set function. This set function is called the *counting measure.*

**Exercise 2.5 (Royden 2.5)**

By using properties of outer measure, prove that the interval $[0, 1]$ is not countable.

*Solution.* If $[0, 1]$ was countable, its measure would be 0. ■

**Exercise 2.6 (Royden 2.6)**

Let $A$ be the set of irrational numbers in the interval $[0, 1]$. Prove that $m^*(A) = 1$.

*Solution.* But we know that $m^*(\mathbb{Q}) = 0$ since rationals are countable, then by invoking excision, we get

$$m^*(A) = m^*([0, 1]) - m^*(\mathbb{Q}) = 1 - 0 = 1 \tag{123}$$

■ I tried proving this directly, but it turns out to be quite hard.[a]

[a] https://math.stackexchange.com/questions/3122008/direct-proof-that-the-irrationals-on-0-1-have-measure-1

**Exercise 2.7 (Royden 2.7)**

A set of real numbers is said to be a $G_\delta$ set provided it is the intersection of a countable collection of open sets. Show that for any bounded set $E$, there is a $G_\delta$ set $G$ for which

$$E \subseteq G \text{ and } m^*(G) = m^*(E).$$

*Solution.* $E$ is bounded so $m(E) < +\infty$. Let's write the definition

$$m(E) = \inf \left\{ \sum_{k=1}^\infty \ell(I_k) \colon A \subset \bigcup_k I_k \right\} \tag{124}$$

WLOG, we can let $I_k$ be open, and we know that $\cup I_k$ is open as well. Perhaps we can construct a decreasing sequence of these open sets that converge onto $m(E)$, which is finite. For each $n \in \mathbb{N}$, we can find a cover $(I_{n,k})_k$ satisfying

$$m(E) \leq m\underbrace{\left( \bigcup_k I_{n,k} \right)}_{O_n} \leq m(E) + \frac{1}{n} \tag{125}$$

where the first inequality comes from monotonicity and the second comes from the $\epsilon$ of room trick. Now let

$$G = \bigcap_n O_n \tag{126}$$

■ Note that this applies to general sets and not just measurable ones, so for every set, we can always outer-approximate it (in terms of measure) with open sets and $G_\delta$-sets.

**Exercise 2.8 (Royden 2.8)**

Let $B$ be the set of rational numbers in the interval $[0,1]$, and let $\{I_k\}_{k=1}^n$ be a finite collection of open intervals that covers $B$. Prove that $\sum_{k=1}^n m^*(I_k) \geq 1$.

*Solution.* The idea is to take advantage of the finiteness of the summation. WLOG, we can assume that $\{I_k = (a_k, b_k)\}$ is sorted in increasing order of $a_k$. Then, the following must be true.

1. $a_1 < 0$. If not, then $0 \notin \cup I_k$.
2. $a_{k+1} \leq b_k$ for $k = 1, \ldots, n-1$. If not, then there exists some $k$ s.t. $b_k < a_{k+1}$. By density of rationals, there exists some $q \in \mathbb{Q}$ s.t. $b_k < q < a_{k+1} \leq a_{k+2} \ldots$, and so $\{I_k\}$ cannot cover $q$.
3. $b_n > 1$.

Therefore, since this is a finite sum, we can rearrange

$$\sum_{k=1}^n m^*(I_k) = \sum_{k=1}^n b_k - a_k \tag{127}$$

$$= b_n \underbrace{-(a_n - b_{n-1}) - \ldots - (a_2 - b_1)}_{\geq 0} - a_1 \tag{128}$$

$$\geq b_n - a_1 \geq 1 \tag{129}$$

**Exercise 2.9 (Royden 2.9)**

Prove that if $m^*(A) = 0$, then $m^*(A \cup B) = m^*(B)$.

*Solution.* $\geq$ is already proven by monotonicity. For $\leq$, we can use finite subadditivity (which follows from countable subadditivity) to get

$$m^*(A \cup B) \leq m^*(A) + m^*(B) = m^*(B) \tag{130}$$

■ Therefore, adding or subtracting subsets of outer measure 0 does not affect the outer measure of any set. This will come in useful for making proofs more convenient.

**Exercise 2.10 (Royden 2.10)**

Let $A$ and $B$ be bounded sets for which there is an $\alpha > 0$ such that $|a - b| \geq \alpha$ for all $a \in A$, $b \in B$. Prove that $m^*(A \cup B) = m^*(A) + m^*(B)$.

*Solution.* Proven in Theorem 2.3

**Exercise 2.11 (Royden 2.11)**

Prove that if a $\sigma$-algebra of subsets of $\mathbf{R}$ contains intervals of the form $(a, \infty)$, then it contains all intervals.

*Solution.* From countable union, it also contains all intervals of the form

$$[a, +\infty) = \bigcup_{x > a, x \in \mathbb{Q}} (x, +\infty) \tag{131}$$

Then, it also contains all intervals of the form $(-\infty, a)$, $(-\infty, a]$. By intersection, it contains all intervals of form $(a, b), [a, b), (a, b], [a, b]$.

**Exercise 2.12 (Royden 2.12)**

Show that every interval is a Borel set.

*Solution.* From the same logic as Exercise 2.11.

**Exercise 2.13 (Royden 2.13)**

Show that (i) the translate of an $F_\sigma$ set is also $F_\sigma$, (ii) the translate of a $G_\delta$ set is also $G_\delta$, and (iii) the translate of a set of measure zero also has measure zero.

**Exercise 2.14 (Royden 2.14)**

Show that if a set $E$ has positive outer measure, then there is a bounded subset of $E$ that also has positive outer measure.

**Exercise 2.15 (Royden 2.15)**

Show that if $E$ has finite measure and $\epsilon > 0$, then $E$ is the disjoint union of a finite number of measurable sets, each of which has measure at most $\epsilon$.

**Exercise 2.16 (Royden 2.16)**

Complete the proof of Theorem 11 by showing that measurability is equivalent to (iii) and also equivalent to (iv).

**Exercise 2.17 (Royden 2.17)**

Show that a set $E$ is measurable if and only if for each $\epsilon > 0$, there is a closed set $F$ and open set $O$ for which $F \subseteq E \subseteq O$ and $m^*(O \setminus F) < \epsilon$.

**Exercise 2.18 (Royden 2.18)**

Let $E$ have finite outer measure. Show that there is an $F_\sigma$ set $F$ and a $G_\delta$ set $G$ such that

$$F \subseteq E \subseteq G \text{ and } m^*(F) = m^*(E) = m^*(G).$$

**Exercise 2.19 (Royden 2.19)**

Let $E$ have finite outer measure. Show that if $E$ is not measurable, then there is an open set $O$ containing $E$ that has finite outer measure and for which

$$m^*(O \setminus E) > m^*(O) - m^*(E).$$

**Exercise 2.20 (Royden 2.20)**

(Lebesgue) Let $E$ have finite outer measure. Show that $E$ is measurable if and only if for each open, bounded interval $(a, b)$,

$$b - a = m^*((a, b) \cap E) + m^*((a, b) \setminus E).$$

**Exercise 2.21 (Royden 2.21)**

Use property (ii) of Theorem 11 as the primitive definition of a measurable set and prove that the union of two measurable sets is measurable. Then do the same for property (iv).

**Exercise 2.22 (Royden 2.22)**

For any set $A$, define $m^{**}(A) \in [0, \infty]$ by

$$m^{**}(A) = \inf\{m^*(O) \mid O \supseteq A, O \text{ open}\}.$$

How is this set function $m^{**}$ related to outer measure $m^*$?

**Exercise 2.23 (Royden 2.23)**

For any set $A$, define $m^{***}(A) \in [0, \infty]$ by

$$m^{***}(A) = \sup\{m^*(F) \mid F \subseteq A, F \text{ closed}\}.$$

How is this set function $m^{***}$ related to outer measure $m^*$?

**Exercise 2.24 (Royden 2.24)**

Show that if $E_1$ and $E_2$ are measurable, then

$$m(E_1 \cup E_2) + m(E_1 \cap E_2) = m(E_1) + m(E_2).$$

**Exercise 2.25 (Royden 2.25)**

Show that the assumption that $m(B_1) < \infty$ is necessary in part (ii) of the theorem regarding continuity of measure.

**Exercise 2.26 (Royden 2.26)**

Let $\{E_k\}_{k=1}^{\infty}$ be a countable disjoint collection of measurable sets. Prove that for any set $A$,

$$m^*\left(A \cap \bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} m^*(A \cap E_k).$$

**Exercise 2.27 (Royden 2.27)**

Let $\mathcal{M}'$ be any $\sigma$-algebra of subsets of $\mathbf{R}$ and $m'$ a set function on $\mathcal{M}'$ which takes values in $[0, \infty]$, is countably additive, and such that $m'(\emptyset) = 0$.

1. Show that $m'$ is finitely additive, monotone, countably monotone, and possesses the excision property.
2. Show that $m'$ possesses the same continuity properties as Lebesgue measure.

**Exercise 2.28 (Royden 2.28)**

Show that continuity of measure together with finite additivity of measure implies countable additivity of measure.

**Exercise 2.29 (Royden 2.29)**

Listed.

1. Show that rational equivalence defines an equivalence relation on any set.
2. Explicitly find a choice set for the rational equivalence relation on $\mathbf{Q}$.
3. Define two numbers to be irrationally equivalent provided their difference is irrational. Is this an equivalence relation on $\mathbf{R}$? Is this an equivalence relation on $\mathbf{Q}$?

**Exercise 2.30 (Royden 2.30)**

Show that any choice set for the rational equivalence relation on a set of positive outer measure must be uncountably infinite.

**Exercise 2.31 (Royden 2.31)**

Justify the assertion in the proof of Vitali's Theorem that it suffices to consider the case that $E$ is bounded.

**Exercise 2.32 (Royden 2.32)**

Does Lemma 16 remain true if $\Lambda$ is allowed to be finite or to be uncountably infinite? Does it remain true if $\Lambda$ is allowed to be unbounded?

**Exercise 2.33 (Royden 2.33)**

Let $E$ be a nonmeasurable set of finite outer measure. Show that there is a $G_\delta$ set $G$ that contains $E$ for which

$$m^*(E) = m^*(G), \text{ while } m^*(G \setminus E) > 0.$$

*Solution.* Note from Exercise 2.7 that we can outer-approximate *any* set with a $G_\delta$-set. Therefore, choose such a set $G \supset E$ which satisfies

$$m^*(G) - m^*(E) = 0 \tag{132}$$

If $E$ was measurable, then by excision property, $m^*(G \setminus E) = 0$, but $E$ is not measurable, so we must have $m^*(G \setminus E) > 0$.

**Exercise 2.34 (Royden 2.34)**

Show that there is a continuous, strictly increasing function on the interval $[0, 1]$ that maps a set of positive measure onto a set of measure zero.

**Exercise 2.35 (Royden 2.35)**

Let $f$ be an increasing function on the open interval $I$. For $x_0 \in I$ show that $f$ is continuous at $x_0$ if and only if there are sequences $\{a_n\}$ and $\{b_n\}$ in $I$ such that for each $n$, $a_n < x_0 < b_n$, and $\lim_{n\to\infty}[f(b_n) - f(a_n)] = 0$.

**Exercise 2.36 (Royden 2.36)**

Show that if $f$ is any increasing function on $[0, 1]$ that agrees with the Cantor-Lebesgue function $\varphi$ on the complement of the Cantor set, then $f = \varphi$ on all of $[0, 1]$.

**Exercise 2.37 (Royden 2.37)**

Let $f$ be a continuous function defined on $E$. Is it true that $f^{-1}(A)$ is always measurable if $A$ is measurable?

**Exercise 2.38 (Royden 2.38)**

Let the function $f : [a, b] \to \mathbf{R}$ be Lipschitz, that is, there is a constant $c \geq 0$ such that for all $u, v \in [a, b]$, $|f(u) - f(v)| \leq c|u - v|$. Show that $f$ maps a set of measure zero onto a set of measure zero. Show that $f$ maps an $F_\sigma$ set onto an $F_\sigma$ set. Conclude that $f$ maps a measurable set to a measurable set.

**Exercise 2.39 (Royden 2.39)**

Let $F$ be the subset of $[0, 1]$ constructed in the same manner as the Cantor set except that each of the intervals removed at the $n$th deletion stage has length $\alpha 3^{-n}$ with $0 < \alpha < 1$. Show that $F$ is a closed set, $[0, 1] \setminus F$ dense in $[0, 1]$, and $m(F) = 1 - \alpha$. Such a set $F$ is called a generalized Cantor set.

**Exercise 2.40 (Royden 2.40)**

Show that there is an open set of real numbers that, contrary to intuition, has a boundary of positive measure. (Hint: Consider the complement of the generalized Cantor set of the preceding problem.)

**Exercise 2.41 (Royden 2.41)**

A nonempty subset $X$ of $\mathbf{R}$ is called perfect provided it is closed and each neighborhood of any point in $X$ contains infinitely many points of $X$. Show that the Cantor set is perfect. (Hint: The endpoints of all of the subintervals occurring in the Cantor construction belong to $C$.)

**Exercise 2.42 (Royden 2.42)**

Prove that every perfect subset $X$ of $\mathbf{R}$ is uncountable. (Hint: If $X$ is countable, construct a descending sequence of bounded, closed subsets of $X$ whose intersection is empty.)

**Exercise 2.43 (Royden 2.43)**

Use the preceding two problems to provide another proof of the uncountability of the Cantor set.

**Exercise 2.44 (Royden 2.44)**

A subset $A$ of $\mathbf{R}$ is said to be *nowhere dense in* $\mathbf{R}$ provided that for every open set $O$ has an open subset that is disjoint from $A$. Show that the Cantor set is nowhere dense in $\mathbf{R}$.

**Exercise 2.45 (Royden 2.45)**

Show that a strictly increasing function that is defined on an interval has a continuous inverse.

**Exercise 2.46 (Royden 2.46)**

Let $f$ be a continuous function and $B$ be a Borel set. Show that $f^{-1}(B)$ is a Borel set. (Hint: The collection of sets $E$ for which $f^{-1}(E)$ is Borel is a $\sigma$-algebra containing the open sets.)

**Exercise 2.47 (Royden 2.47)**

Use the preceding two problems to show that a continuous strictly increasing function that is defined on an interval maps Borel sets to Borel sets.