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Asymptotic Theory for the Principal Component Analysis of a Vector Random Function: Some Applications to Statistical Inference

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From the results of convergence by sampling in linear principal component analysis (of a random function in a separable Hilbert space), the limiting distribution is given for the principal values and the principal factors. These results can be explicitly written in the normal case. Some applications to statistical inference are investigated.

INTRODUCTION

The principal component analysis (PCA) of a finite set of real random variables (defined on a probability space $(\Omega', \mathcal{A}', P')$) or of statistical variables (in the particular case of Ω' with finite cardinality) is well known [1, 6] and its definition can be clarified in associating to it a "schema of duality" (see [6, 7] for instance). Such an analysis is often used to study and describe phenomena that can be summed up by random processes. Actually, the analysis is based on a sample and so it must be considered only as data description; in order to consider it as an approximation of the phenomenon description, the PCA of a process must be defined and then it must be shown that, when the sample size increases, the sequence of the obtained PCA converges (in a sense that will be stated more precisely) to the theoretical PCA of the process.

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Several studies have been performed to define factor analysis of processes; so the PCA definition has been extended to quantitative processes and is called the linear PCA [7, 10, 14]. However, for qualitative processes, such analysis cannot be done without suitable transformation ("scaling") and a non-linear PCA (that may be applied as well to the quantitative or the qualitiative case) has been defined [5, 7]. This question will not be entered upon here.

The sampling problem in PCA has been treated by Anderson [2], Hsu [13], and Krishnaiah and Lee [15] in the case of a finite set of real random variables of normal type and has been recently extended by Davis [9] and Fang and Krishnaiah [18] to the normal case; Dauxois and Pousse [7] and, in a different context, Deville [10] have extended it to the case of vector processes.

The PCA of a process is defined on infinite dimensional linear spaces; therefore the use of matrix theory as in [2, 9] is impossible. Hence, two difficulties appear: the first is due to the dimension and the second to the fact that the process may not be only a scalar one. We propose here an asymptotic study of the PCA of a vector random function; in Section 1 the definition of the PCA and the sampling problem are presented, in Section 2 the asymptotic theory for the PCA is developed and finally some applications in statistical inference are given in Section 3.

For more developments or details of some parts of this paper, the reader is referred to [7, 17].

1. Principal Component Analysis of a Vector Random Function

1.1. *Definition*

Let $(\Omega', \mathcal{O}\!\!\mathcal{I}')$ (respectively (T, ξ) be a mesurable space with the probability measure P' (resp. the bounded measure μ) and $X = (X_t)_{t \in T}$, also denoted by $(X(\cdot, t))_{t \in T}$, a vector random function (r.f.) mapping from $(\Omega', \mathcal{O}\!\!\mathcal{I}', P')$ into $(H, \mathscr{B}_H)$, where H is a separable Hilbert space and $\mathscr{B}_H$ its Borel field.

Let $L^2_H(P' \otimes \mu)$ denote the separable Hilbert space of the (equivalence classes of) measurable functions defined from $(\Omega' \times T, \mathcal{O}\!\!\mathcal{I}' \otimes \xi)$ to $(H, \mathscr{B}_H)$ and such that the squared norm is $P' \otimes \mu$-integrable. We assume that X belongs to $L^2_H(P' \otimes \mu)$ and, for all $t \in T$, the mean value $E(X_t) = \int_\Omega X(\omega', t)\, dP'(\omega')$ is zero.

We consider then the linear continuous operator Φ defined from $L^2_{\mathbb{R}}(\Omega', \mathcal{O}\!\!\mathcal{I}', P')$ (denoted by $L^2(P')$) to $L^2_H(T, \xi, \mu)$ (denoted by E) defined by

$$(\forall f \in L^2(P'))\ (\Phi f)(t) = \int_\Omega X(\omega', t) f(\omega')\, dP'(\omega') = E[X(\cdot, t) f],\ \mu\text{-a.e.}$$

Its adjoint Φ^* is such that

$$(\forall u \in E)\ (\Phi^* u)(\omega) = \int_T \langle X(\omega, t), u(t)\rangle_H \, d\mu(t) = \langle X(\omega, \cdot), u\rangle, \qquad P'\text{-a.e.}$$

($\langle , \rangle$ and $\| \, \|$ denote the inner product and the norm of E.)

The schema of duality, which is so associated to these operators (and to the r.f.), is

$$\begin{tikzcd}
E \arrow[d, shift left, "I"] & L^2(P') \arrow[l, "\Phi"'] \arrow[d, shift right, "W"'] \\
E \arrow[u, shift left, "V"] \arrow[r, "\Phi^*"] & L^2(P') \arrow[u, shift right, "I"']
\end{tikzcd}$$

where any Hilbert space and its dual are identified, I denotes the identity operator, V is $\Phi \circ \Phi^*$ and W is $\Phi^* \circ \Phi$; V and W are non-negative nuclear self-adjoint operators, so V belongs to $F = \sigma_2(E)$ the separable Hilbert space of Hilbert–Schmidt operators on E with the inner product defined by $\forall\ (T, U) \in F^2$, $\langle T, U\rangle_F = tr(TU^*)$. Since f mapping $x \in E$ to $(x, x) \in E \times E$ and g mapping $(x, y) \in E \times E$ to $x \otimes y \in F$ (where $x \otimes y$ is defined by $(\forall f \in E)$ $(x \otimes y)(f) = \langle x, f\rangle y)$ are continuous, $g \circ f(X) = X \otimes X$ is a random variable which maps $\omega' \in \Omega'$ into $X(\omega', \cdot) \otimes X(\omega', \cdot) \in F$; $X \otimes X$ is P'-integrable and V appears like $E(X \otimes X)$, which is an element of F.

The linear PCA of X is obtained by the spectral analysis of V, which gives the Schmidt's decomposition of V in F, $V = \sum_{i \in I} \lambda_i e_i \otimes e_i$, and the decomposition of X in $L^2_H(P' \otimes \mu)$, $X(\omega', t) = \sum_{i \in I} s_i e_i(t) f_i(\omega')$, where I is either a beginning section of $\mathbb{N}$ or $\mathbb{N}$ itself; $(\lambda_i)_{i \in I}$ is the decreasing complete sequence of non-null eigenvalues of V (or W), which are called the principal values (in such sequence λ_i is repeated in regard to its multiplicity order), and s_i is the squared root of λ_i; $(e_i)_{i \in I}$ (resp. $(f_i)_{i \in I}$) is a corresponding orthonormal sequence of V eigenvectors in E, called the principal factors (resp. of W eigenvectors in $L^2(P')$ called the principal components, such that $f_i = \phi^* e_i / \| \Phi^* e_i \|$). Furthermore, L will denote a subset of I such that $(\lambda_l)_{l \in L}$ is the strictly decreasing non-null V eigenvalue sequence.

In the case of $H = \mathbb{R}$ and T is the finite set $\{1, 2,..., p\}$, it is easily verified that the definition above is a generalization of the PCA of a finite set of real r.v.

When Ω' is $\{\omega_1, \ldots, \omega_n\}$ and P' the uniform probability measure on Ω', we may note that $L^2(P')$ is isomorphic to $\mathbb{R}^n$ with metric $(1/n) I_n$ (I_n is the identity matrix of n order), and the schema of duality, where $(\mathbb{R}^n)^*$ denotes the dual space of $\mathbb{R}^n$, becomes

$$\begin{tikzcd} E \arrow[d, shift left, "I"] & (\mathbb{R}^n)^* \arrow[l, "\Phi_n"'] \\ E \arrow[u, shift left, "V_n"] \arrow[r, "\Phi_n^*"] & \mathbb{R}^n \arrow[u, "\frac{1}{n} I_n"'] \end{tikzcd}$$

with

$$(\forall f \in (\mathbb{R}^n)^*) \qquad \Phi_n f = \frac{1}{n} \sum_{i=1}^{n} X(\omega_i', \cdot) f(\omega_i'),$$

$$(\forall u \in E) \qquad (\Phi_n^* u)(\omega_i') = \langle X(\omega_i', \cdot), u \rangle,$$

and then

$$\begin{aligned} (\forall g \in E) \qquad V_n g &= \frac{1}{n} \sum_{i=1}^{n} \langle X(\omega_i', \cdot)\, g \rangle X(\omega_i', \cdot) \\ &= \left(\frac{1}{n} \sum_{i=1}^{n} X(\omega_i', \cdot) \otimes X(\omega_i', \cdot) \right) (g). \end{aligned}$$

1.2. *The Sampling Problem*

In a concrete situation, when we are interested by a phenomenon described by a r.f., the data are based only on a sample of size n ($n \in \mathbb{N}^*$): then, the PCA is obtained by means of the n trajectories $X(\omega_i, \cdot)$ ($i = 1, 2, \ldots, n$). We recall here a model formalizing this sampling problem [8].

X may be considered as a measurable mapping from $(\Omega', \mathcal{A}', P')$ into $(E, \mathcal{B}_E)$, and let π_i denote the ith canonical projection from $(\Omega, \mathcal{A}, P) = (\Omega', \mathcal{A}', P')^{\otimes \mathbb{N}^*}$ into $(\Omega', \mathcal{A}', P')$ (i.e., $\forall \omega = (\omega_n)_{n \in \mathbb{N}^*} \in \Omega$, $\pi_i(\omega) = \omega_i$); then $(\pi_i)_{i=1,2,\ldots,n}$ is a sequence of identically distributed independent r.v. (the common distribution is P') and it follows that the random variables $X_i = X \circ \pi_i$ ($i = 1, 2, \ldots, n$) are independent, identically distributed as X (hence with common distribution P'_X), and $\forall \omega \in \Omega$ $X_i(\omega) = X(\omega_i)$.

The knowledge of the n trajectories $X(\omega_i, \cdot)$ implies consideration of the subset $\Omega_n'^{\omega}$ of Ω'^{ω} (the range of the finite sequence $(\omega_i)_{i=1,2,\ldots,n}$) and if X^n denotes the restriction of X on $\Omega_n'^{\omega}$, then $X_i(\omega) = X^n(\omega_i)$. Since neither P' nor P'_X are known, we consider on $\Omega_n'^{\omega}$ (provided with the σ-field of all sets in $\Omega_n'^{\omega}$) the probability measure $\alpha_n^{\omega} = n^{-1} \sum_{i=1}^{n} \delta_{\omega_i}$. Then, from the strong law of large numbers, the image of α_n^{ω} by X^n converges for almost all ω to P'_X.

This model corresponds (often unconsciously) to the current practice and leads to the consideration of, for all n, the schema of duality of the last part, where $\{\omega_1, \ldots, \omega_n\}$ has been substituted for $\Omega_n'^{\omega}$ provided with α_n^{ω}. The PCA is then obtained by the spectral analysis of the operator:

$$V_n(\omega) = n^{-1} \sum_{i=1}^{n} X^n(\omega_i, \cdot) \otimes X^n(\omega_i, \cdot) = n^{-1} \sum_{i=1}^{n} X_i(\omega) \otimes X_i(\omega).$$

The problem of convergence by sampling of the PCA is then translated in term of convergence of the sequence $(V_n)_{n \in \mathbb{N}^*}$, where each V_n is defined from $(\Omega, \mathcal{A}, P)$ into $(F, \mathcal{B}_F)$.

When the sampling problem is solved, another problem is raised in the case of the PCA of a r.f.; $(X_t)_{t \in T}$ is often known only on a discrete set of values of T. This question is not entered upon here but the interested reader is referred to [3, 7, 10].

1.3. *Convergence of the Sample Random Variables*

1.3.1. *Convergence of the Sequence* $(V_n)_{n \in \mathbb{N}^*}$

PROPOSITION 1. *$(V_n)_{n \in \mathbb{N}^*}$ is a sequence of integrable r.v. from $(\Omega, \mathcal{A}, P)$ into $(F, \mathcal{B}_F)$ which converges almost surely (a.s.) to V in F; the operators sequence $(V_n)_{n \in \mathbb{N}^*}$ converges uniformly a.s. to V.*

For each $i \in \{1, 2, \ldots, n\}$, $X_i \otimes X_i = (g \circ f)(X_i)$ is a measurable mapping into $(F, \mathcal{B}_F)$, integrable and with the same expectation V as $X \otimes X$. It results, from the strong law of large numbers in the separable Hilbert space F, that $V_n = n^{-1} \sum_{i=1}^{n} X_i \otimes X_i$ converges a.s. to V in F; let Ω_1 be the convergence set. Furthermore, as the (uniform) norm in $\mathcal{L}(E)$ is inferior to the F norm, the second part of the proposition is proved.

Then, we may use the results of [11, p. 1091; 16, p. 367]; they grant in particular the uniform convergence of the corresponding eigenmanifolds. In the case of an eigenvector e corresponding to a simple V eigenvalue λ, we can deduce, for all $\omega \in \Omega_1$, the existence of a sequence $(e_n(\omega))_{n \in \mathbb{N}^*}$ converging to e in E ($e_n(\omega)$ is, for each $n \in \mathbb{N}$, a $V_n(\omega)$ eigenvector corresponding to $\lambda_n(\omega)$, where $(\lambda_n(\omega))_{n\in\mathbb{N}^*}$ converges to λ in $\mathbb{R}$). It is important to notice that, for a V eigenvalue λ with multiplicity greater than 1, the orthonormal basis of the eigenmanifold corresponding to λ may be obtained by rotation and so we do not have such a convergence property for each corresponding eigenvector sequence.

1.3.2. *Principal Values*

As V in Section 1.1, V_n may be written $\sum_{i\in I}\lambda_i^n e_i^n \otimes e_i^n$ (with $\lambda_i^n = 0$ for $i \geqslant n+1$) and let

$$I_i = \{j \in I; \lambda_j = \lambda_i\}, \qquad |I_i| = k_i, \qquad I' = \{i \in I; I_i = \{i\}\}.$$

PROPOSITION 2. *For each n of $\mathbb{N}^*$, for all i of I, λ_i^n is a real r.v. defined on $(\Omega, \mathcal{A})$. When λ_i is of order k_i, there are, for all ω of Ω_1, k_i sequences $\{(\lambda_j^n(\omega)_{n\in\mathbb{N}^*}; j\in I_i\}$ converging to λ_i in $\mathbb{R}$.*

For each i of I, the function φ_i that maps $T \in F$ into $\lambda_i(T) \in \mathbb{R}$ is continuous since $|\lambda_i(T') - \lambda_i(T)| \leqslant \|T - T'\|_{\mathscr{L}(E)} \leqslant \|T - T'\|_F$. Now, as $\lambda_i^n = \varphi_i(V_n)$, Proposition 1 implies that λ_i^n is a real r.v. The second part of Proposition 2 comes directly from [11, p. 1091].

For all $(\varepsilon, \rho) \in (\mathbb{R}_+^*)^2$ such that $\varepsilon < \rho < \frac{1}{2}\min\{|\lambda_l - \lambda_i|; l \in I - I_i\}$ and for all $(\omega, j) \in \Omega_1 \times I_i$, there exists $n_1 \in N^*$ such that

$$n > n_1 \Rightarrow |\lambda_j^n(\omega) - \lambda_i| \leqslant \|V_n(\omega) - V\|_{\mathscr{L}(E)} \leqslant \|V_n(\omega) - V\|_F < \varepsilon,$$

and furthermore, if $D_{\rho,i}$ is the disk bounded by the circle $\Lambda_{\rho,i}$ of center λ_i and radius ρ, one has

$$D_{\rho,i} \cap \{\lambda_k^n(\omega)\}_{k\in I} = \{\lambda_j^n(\omega)\}_{j\in I_i}.$$

1.3.3. *Projection Operators*

Let E_j (resp. E_j^n) denote the eigenmanifold of V (resp. V_n) corresponding to the eigenvalue λ_j (resp. λ_j^n) and $P_j = \sum_{k\in I_j} e_k \otimes e_k$ (resp. $P_j^n = \sum_{k\in I_j} e_k^n \otimes e_k^n$) the orthogonal projection operator from E on E_j (resp. $\bigoplus_{k\in I_j} E_k^n$).

PROPOSITION 3. *For each j of I and each n of $\mathbb{N}^*$, P_j^n is a r. v. from $(\Omega, \mathcal{A}, P)$ into $(F, \mathscr{B}_F)$; for each ω of Ω_1, $(P_j^n(\omega))_{n\in\mathbb{N}^*}$ converges to P_j in F.*

Let $(T_n)_{n\in\mathbb{N}^*} \in F^{\mathbb{N}^*}$ be a sequence converging to T in F and Q_j^n (resp. Q_j) the analogous sequence for T_n (resp. T) of $P_j^n(\omega)$ (resp. P_j) for $V_n(\omega)$ (resp. V); the $\Lambda_{\rho,j}$ analogous term is denoted by $\Lambda'_{\rho,j}$ and the resolvent of T_n (resp. T) is denoted by R'_n (resp. R'). One has

$$\forall \varepsilon > 0 \ \exists n_2 \in \mathbb{N} \qquad (n > n_2) \Rightarrow \|T_n - T\|_F < \varepsilon.$$

Then (see [12, p. 15]) for $n > n_2$,

$$\|Q_j^n - Q_j\|_F = (2\pi)^{-1} \left\| \int_{\Lambda'_{\rho,j}} (R'_n(z) - R'(z))\, dz \right\|_F$$
$$\leqslant \rho \sup \left\{ \frac{\|T_n - T\|_F \, \|R'(z)\|_F^2}{1 - \|T_n - T\|_F \, \|R'(z)\|_F} : z \in \Lambda'_{\rho,j} \right\}$$

and $\varepsilon \leqslant 1/(2M'_{\rho,j})$ implies $\|Q_j^n - Q_j\|_F \leqslant 2\rho M'^2_{\rho,j}\varepsilon$, with $M'_{\rho,j} = \sup\{\|R'(z)\|_F^2 : z \in \Lambda'_{\rho,j}\}$. So, for all j of I, the mapping ψ_j defined by $T \in F \to \psi_j(T) = Q_j \in F$ is continuous and, as V_n is a r.v., $P_j^n = \psi_j(V_n)$ is also a r.v.

A similar proof based on $\{P_j^n(\omega)\}_{n\in\mathbb{N}^*}$ substituting $(Q_j^n)_{n\in\mathbb{N}^*}$ gives the second part of the proposition.

1.3.4. *Principal Factors*

We must only consider (normalized) eigenvector e_i (resp. e_i^n) of V (resp. V_n) coresponding to an eigenvalue λ_i (resp. λ_i^n) of multiplicity one (as λ_i is simple, λ_i^n is also simple for n large enough).

PROPOSITION 4. *For all i of I', e_i^n is a r.v. from $(\Omega, \mathcal{Cl}, P)$ into $(E, \mathscr{B}_E)$ and, for each ω of Ω_1, $(e_i^n(\omega))_{n\in\mathbb{N}^*}$ converges to e_i in E.*

With the same notations as in Section 1.3.3, let q_j (resp. q_j^n) be a normalized eigenvector of T (resp. T_n) corresponding to the eigenvalue λ_j (resp. λ_j^n). Let q_j^n be chosen as $Q_j^n q_j / \|Q_j^n q_j\|$; thus, one has

$$\|Q_j^n - Q_j\|_F^2 = 2(1 - \langle Q_j^n, Q_j\rangle_F) = 2(1 - \langle q_j^n \otimes q_j^n, q_j \otimes q_j\rangle_F)$$
$$= 2(1 - \langle q_j^n, q_j\rangle^2)$$
$$= 2(1 - \langle q_j^n, q_j\rangle)(1 + \langle q_j^n, q_j\rangle) = \|q_j^n - q_j\|^2 (1 + \langle q_j^n, q_j\rangle).$$

Since $\langle q_j^n, q_j\rangle$ is positive, then, for $\varepsilon > 0$ and $n > n_2$, we have

$$\|q_j^n - q_j\|^2 \leqslant \|Q_j^n - Q_j\|_F^2 \leqslant \varepsilon.$$

So, for all j of I', the mapping χ defined by $Q_j^n = q_j^n \otimes q_j^n \in F \mapsto (Q_j^n q_j / \|Q_j^n q_j\|) = q_j^n$ is continuous and $e_j^n = \chi(P_j^n)$ is a r.v. The second part of the proposition results from the end of the last chapter.

2. THE ASYMPTOTIC STUDY

Let U_n denote the r.v. $n^{1/2}(V_n - V)$ defined from $(\Omega, \mathcal{Cl}, P)$ into $(F, \mathscr{B}_F)$. We assume in this section that the X 4-th order moment is finite. So

$E[\|X \otimes X\|_F^2] = E[\|X\|^4] < +\infty$ and, with the central limit theorem in the separable Hilbert space F, one has

PROPOSITION 5. *$\{U_n\}_{n\in\mathbb{N}^*}$ converges in distribution to the gaussian random element of F with mean zero and covariance operator K the $X \otimes X$ one.*

When, for $(f, g) \in F^2$, $f \tilde{\otimes} g$ is the operator which maps $x \in F$ into $\langle x, f\rangle_F\, g$, then by definition $K = E[(X \otimes X - V) \tilde{\otimes} (X \otimes X - V)]$; K is positive, nuclear, and one has

$$K = E[(X \otimes X) \tilde{\otimes} (X \otimes X)] - E[V \tilde{\otimes} (X \otimes X)] \\ - E[(X \otimes X) \tilde{\otimes} V] + V \tilde{\otimes} V.$$

Since $E[V \tilde{\otimes} (X \otimes X)] = V \tilde{\otimes} E(X \otimes X) = V \tilde{\otimes} V = E[(X \otimes X) \tilde{\otimes} V]$, then $K = E[(X \otimes X) \tilde{\otimes} (X \otimes X)] - V \tilde{\otimes} V$. As $X(\omega', t) = \sum_{i\in I} s_i e_i(t) f_i(\omega')$ in $L_H^2(P' \otimes \mu)$ and $V = \sum_{i\in I} \lambda_i e_i \otimes e_i$ in F and since (the X 4th moment being finite and F being separable) the term by term integration is possible, K may be written

$$K = \sum_{(i,j,k,l)\in I^4} s_i s_j s_k s_l E[f_i f_j f_k f_l](e_i \otimes e_j) \tilde{\otimes} (e_k \otimes e_l) \\ - \sum_{(i,j)\in I^2} \lambda_i \lambda_j (e_i \otimes e_i) \tilde{\otimes} (e_j \otimes e_j).$$

Let U denote a gaussian random element of F with mean zero and covariance operator K; the law of U is denoted by $\mathcal{N}(0, K)$. If $\{\varepsilon_\alpha\}_{\alpha\in A}$ is an orthonormal family in F of K eigenvectors and $\{\mu_\alpha\}_{\alpha\in A}$ the complete decreasing corresponding eigenvalues sequence, then U may be written in F, $U = \sum_{\alpha\in A} (\mu_\alpha)^{1/2} \varepsilon_\alpha \xi_\alpha$ a.s., where $\{\xi_\alpha\}_{\alpha\in A}$ is a sequence of independent real standard normal variables (each ξ_α is $\mathcal{N}(0, 1)$).

When X is a gaussian r.f., the principal components $(f_i)_{i\in I}$ belong to the range of Φ^* in $L^2(P')$, the closure of which is the gaussian space generated by the family $(\langle u, X\rangle)_{u\in E}$. Therefore, in this particular case, they are jointly normal independent and each one of them is $\mathcal{N}(0, 1)$. It follows that $\forall (i, j, k, l) \in I^4$,

$$\begin{aligned} E(f_i) &= 0; \\ E(f_i f_j) &= 1 \quad \text{if} \quad i = j, \\ &= 0 \quad \text{if} \quad i \neq j; \end{aligned}$$

$$\begin{aligned} E[f_i f_j f_k f_l] &= 3 \quad \text{if} \quad i=j=k=l \\ &= 1 \quad \text{if} \quad i=j, k=l, i \neq k \\ &\qquad \text{or} \quad i=k, j=l, i \neq j \\ &\qquad \text{or} \quad i=l, j=k, i \neq j \\ &= 0 \quad \text{otherwise} \end{aligned}$$

and so $K = \sum_{i \neq j} s_i^2 s_j^2 [(e_i \otimes e_j) \tilde{\otimes} (e_i \otimes e_j) + (e_i \otimes e_j) \tilde{\otimes} (e_j \otimes e_i)] + 2 \sum_i s_i^4 (e_i \otimes e_i) \tilde{\otimes} (e_i \otimes e_i)$, that may be written $K = 2 \sum_{i<j} s_i^2 s_j^2 \varepsilon_{ij} \tilde{\otimes} \varepsilon_{ij} + 2 \sum_i s_i^4 \varepsilon_{ii} \tilde{\otimes} \varepsilon_{ii}$ with $\varepsilon_{ij} = 2^{-1/2}(e_i \otimes e_j + e_j \otimes e_i)$ for $i<j$ and $\varepsilon_{ii} = e_i \otimes e_i$. The family $\{\varepsilon_{ij}\}_{(i,j)I^2, i \leqslant j}$ is orthonormal in F and thus, U may be written

$$U = 2^{1/2} \sum_{i<j} s_i s_j \varepsilon_{ij} \xi_{ij} + 2^{1/2} \sum_i s_i^2 \varepsilon_{ii} \xi_{ii} \qquad \text{a.s.,}$$

the family $(\xi_{ij})_{(i,j) \in I^2, i \leqslant j}$ being constituted with independent $\mathcal{N}(0,1)$ distributed variables.

2.1. *Asymptotic Distribution of the Projection Operators*

Before the asymptotic joint distribution of $(n^{1/2}(P_j^n - P_j))_{j \in L}$ is considered, the marginal limiting distribution will be characterized.

2.1.1. *Asymptotic Distribution of* $n^{1/2}(P_j^n - P_j)$

One knows that, for $n > n_1$, $n^{1/2}(P_j^n - P_j) = -n^{1/2}(2i\pi)^{-1} \int_{\Lambda_{\rho,j}} [R_n(z) - R(z)]\, dz$. Further, from $\| V_n - V \|_F \leqslant \varepsilon < \rho$, it can be deduced that, for each z of $\Lambda_{\rho,j}$, $\| V_n - V \|_F \| R(z) \|_F < 1$ and then $R_n(z) = (V_n - zI)^{-1} = R(z)[I + \sum_{k=1}^{\infty} [(V - V_n) R(z)]^k]$. So, $R_n(z) - R(z) = R(z) \sum_{k=1}^{\infty} [(V - V_n) R(z)]^k = R(z)(V - V_n) R(z) \sum_{k=0}^{\infty} [(V - V_n) R(z)]^k$ and consequently

$$n^{1/2}(P_j^n - P_j) = +(2i\pi)^{-1} \int_{\Lambda_{\rho j}} R(z)\, U_n R(z)\, H_n(z)\, dz = \varphi_j^n(U_n),$$

with $H_n(z) = \sum_{k=0}^{\infty} [-n^{-1/2} U_n R(z)]^k$. Now, we consider F' the F closed subspace of the self-adjoint operators. As H_n is an analytic function of U_n, φ_j^n is a $\mathscr{B}_{F'}$-measurable mapping from F' into F'. We will verify then that φ_j^n satisfies the Rubin–Bellingsley theorem conditions [4, p. 34]. If $\{T_n\}_{n \in \mathbb{N}^*}$ is a sequence which converges to T in F' and $V'_n = V + n^{-1/2} T_n$, let $P_j'^n$ (resp. R'_n, H'_n, $\Lambda'_{\rho,j}$) denote the analogous term for V'_n of P_j^n (resp. R_n, H_n, $\Lambda_{\rho,j}$) for V_n. Then,

$$\begin{aligned} n^{1/2}[R'_n(z) - R(z)] = -[&R(z)(T_n - T)\, R(z)\, H'_n(z) + R(z)\, TR(z)(H'_n(z) - I) \\ &+ R(z)\, TR(z)] \end{aligned}$$

and so

$$M_n(z) = n^{1/2}[R_n'(z) - R(z)] + R(z)\,TR(z) = -[R(z)(T_n - T)\,R(z)\,H_n'(z) + R(z)\,TR(z)(H_n'(z) - I)].$$

One has

$$\left\|\varphi_j^n(T_n) - (2i\pi)^{-1}\int_{\Lambda'_{\rho j}} [R(z)\,TR(z)\,dz]\right\|_F = \left\|-(2i\pi)^{-1}\int_{\Lambda'_{\rho j}} M_n(z)\,dz\right\|_F \leqslant (2\pi)^{-1}\int_{\Lambda'_{\rho j}} \|M_n(z)\|_F\,dz.$$

Further,

$$\|M_n(z)\|_F \leqslant \|R(z)(T_n - T)\,R(z)\,H_n'(z)\|_F + \|R(z)\,TR(z)(H_n'(z) - I)\|_F \leqslant M_\rho^2\,\|T_n - T\|_F\,\|H_n'(z)\|_F + M_\rho^2\,\|T\|\,\|H_n'(z) - I\|_F$$

with $M_\rho = \sup\{\|R(z)\|_F,\ z \in \Lambda'_{\rho,j}\}$. Since, for each z of $\Lambda'_{\rho,j}$, $\|H_n'(z)\|_F \leqslant \sum_{k=0}^{\infty} [\|V_n' - V\|_F M_\rho]^k = [1 - \|V_n' - V\|_F M_\rho]^{-1} \leqslant 2$, when $\|V_n' - V\|_F \leqslant (2M_\rho)^{-1}$, and $\|H_n'(z) - I\|_F = \|(V - V_n')\,R(z)\sum_{k=0}^{\infty} [(V - V_n')\,R(z)]^k\|_F \leqslant 2M_\rho\,\|V_n' - V\|_F$, we obtain finally that $\lim_{n\to\infty}\|M_n(z)\|_F = 0$, which involves the convergence of $\varphi_j^n(T_n)$ to $\varphi_j(T) = +(2i\pi)^{-1}\int_{\Lambda'_{\rho j}} R(z)\,TR(z)\,dz$. φ_j is also $\mathscr{B}_{F'}$-measurable and the Rubin–Billingsley theorem [4] can be applied: it follows that $\varphi_j^n(U_n)$ converges in distribution to $\varphi_j(U)$. From the residue theorem, let us now derive $\varphi_j(T)$ explicitly: $\varphi_j(T) = +C_j$, the sum of residues of $R(z)\,TR(z)$ at the point λ_j. As $R(z) = (V - zI)^{-1} = \sum_{l\in I} (\lambda_l - z)^{-1} e_l \otimes e_l$, then $R(z)\,TR(z) = \sum_{(l,m)\in I^2} [(\lambda_l - z)(\lambda_m - z)]^{-1} (e_l \otimes e_l)\,T(e_m \otimes e_m)$ and so,

$$\varphi_j(T) = +C_j = \sum_{l\in I - I_j} (\lambda_j - \lambda_l)^{-1}[(e_l \otimes e_l)\,TP_j + P_jT(e_l \otimes e_l)].$$

Denoting by S_j the operator $\sum_{l\in I-I_j} (\lambda_j - \lambda_l)^{-1} e_l \otimes e_l$, we obtain

$$\varphi_j(T) = S_j TP_j + P_j TS_j.$$

Proposition 6. *For each j of I, $n^{1/2}(P_j^n - P_j)$ converges in distribution in F to $\varphi_j(U) = S_j UP_j + P_j US_j$, which is a gaussian random element with mean zero.*

The operator which maps $T \in F$ into $S_j TP_j + P_j TS_j \in F$ is linear and continuous and, since U is a gaussian element with mean zero, thus, $S_j UP_j + P_j US_j$ is also a gaussian element with mean zero.

The case of a gaussian random function. We know that $U = \sum_{h\in I^2} (\mu_h)^{1/2} \varepsilon_h \xi_h$ a.s. and furthermore $\varphi_j(U) = \sum_{h\in I^2} (\mu_h)^{1/2} \varphi_j(\varepsilon_h)\,\xi_h$ a.s.

Thus, replacing ε_h by its value, it is plain that, for $h=(k,l)$ or $h=(l,k)$ with $k<l$, $\varphi_j(\varepsilon_h)=2^{-1/2}(\lambda_j-\lambda_l)^{-1}[e_k \otimes e_l + e_l \otimes e_k]$ if $k\in I_j$ and $l\in I-I_j$, $\varphi_j(\varepsilon_h)$ is null otherwise, and, for $k\in I_j$ and $l\in I-I_j$, $\mu_h=2\lambda_j\lambda_l$. So, $\varphi_j(U)$ is expressed by

$$\varphi_j(U)=2^{1/2}\sum_{\substack{k\in I_j\\ l\in I-I_j\\ k<l}}(\lambda_j\lambda_l)^{1/2}(\lambda_j-\lambda_l)^{-1}\varepsilon_{kl}\xi_{kl}$$
$$+2^{1/2}\sum_{\substack{k\in I_j\\ l\in I-I_j\\ k>l}}(\lambda_j\lambda_l)^{1/2}(\lambda_j-\lambda_l)^{-1}\varepsilon_{kl}\xi_{kl},$$

where the ξ_{kl}, $(k,l)\in\{[I_j\times(I-I_j)]\cup[(I-I_j)\times I_j]\}$, $k<l$, are independent $\mathcal{N}(0,1)$ r.v.

2.1.2. *Joint Asymptotic Distribution of* $(n^{1/2}(P_j^n-P_j))_{j\in L}$

Let J be a p-tuple of L and φ^n the mapping from F' into the Hilbert sum $(F')^p$ defined by $\forall S\in F'$, $\varphi^n(S)=(\varphi_j^n(S))_{j\in J}$. Thus, $\varphi^n(U_n)=(n^{1/2}(P_j^n-P_j))_{j\in J}$ maps from $(\Omega,\mathcal{A},P)$ into $(F')^p$. For each sequence $(T_n)_{n\in\mathbb{N}^*}$ converging to T in F', the convergence of each component in F' involves the convergence of $\varphi^n(T_n)$ to $\varphi(T)=(\varphi_j(T))_{j\in J}$ in $(F')^p$. As φ is continuous, the Rubin–Billingsley theorem leads to

PROPOSITION 7. *For any p-tuple J,* $(n^{1/2}(P_j^n-J))_{j\in J}$ *converges in distribution in* $(F')^p$ *to* $\varphi(U)=(\varphi_j(U))_{j\in J}$, *which is a gaussian random element with mean zero.*

The case of a gaussian random function. The gaussian joint limiting distribution is characterized by its covariance operator K_J (which maps from F^p into F^p). The generating term of K_J is $K_{jj'}=E[\varphi_j(U)\ \tilde{\otimes}\ \varphi_{j'}(U)]$ (which maps from F to F). When $j=j'$, K_{jj} is the covariance operator of $\varphi_j(U)$ and it can be deduced, from Section 2.1.1,

$$K_{jj}=2\lambda_j\sum_{\substack{k\in I_j\\ l\in I-I_j}}\lambda_l(\lambda_l-\lambda_j)^{-2}\varepsilon_{kl}\ \tilde{\otimes}\ \varepsilon_{kl}.$$

Then, when $j\neq j'$, from the expressions of $\varphi_j(U)$ and $\varphi_{j'}(U)$ obtained in Section 2.1.1, it is easily verified that

$$K_{jj'}=2\lambda_j\lambda_{j'}(\lambda_j-\lambda_{j'})^{-1}\sum_{\substack{k\in I_j\\ l\in I_{j'}}}\varepsilon_{kl}\ \tilde{\otimes}\ \varepsilon_{kl}.$$

It should be noted that, for $j\neq j'$, the elements $n^{1/2}(P_j^n-P_j)$ and $n^{1/2}(P_{j'}^n-P_{j'})$ are not asymptotically independent ($K_{jj'}\neq 0$).

2.2. *Asymptotic Distribution of the Principal Values*

2.2.1. *Asymptotic Distribution of* $(n^{1/2}(\lambda_k^n - \lambda_j))_{k \in I_j}$

Let λ_j be a V eigenvalue and k_j its multiplicity ($k_j = \operatorname{card} I_j$). We consider the $\mathbb{R}^{k_j}$-valued r.v. $(n^{1/2}(\lambda_k^n - \lambda_j))_{k \in I_j}$, where $(\lambda_k^n)_{n \in \mathbb{N}^*}$ are k_j sequences of V_n eigenvalues converging a.s. to λ_j and we denote $Z_j^n(U_n) = n^{1/2}(P_j^n V_n P_j^n - \lambda_j P_j^n)$. For each k of I_j, e_k^n being the λ_j^n corresponding eigenvector of V_n, one has $Z_j^n(U_n)\, e_k^n = n^{1/2}(\lambda_k^n - \lambda_j)\, e_k^n$. So $n^{1/2}(\lambda_k^n - \lambda_j)$ is an eigenvalue of $Z_j^n(U_n)$; further, for n large enough, $Z_j^n(U_n)$ is of range order k_j. Finally, if Δ denotes the $\mathbb{R}^{k_j}$-valued function which maps k_j finite range-ordered operator of F into the $\mathbb{R}^{k_j}$ vector of its decreasing ordered eigenvalues, we obtain

$$\Delta[Z_j^n(U_n)] = (n^{1/2}(\lambda_k^n - \lambda_j))_{k \in I_j, k \nearrow}.$$

Let $(T_n)_{n \in \mathbb{N}^*}$ be a sequence of self-adjoint operators which converges to T in F. With $T_n = n^{1/2}(W_n - V)$ and denoting again $P_j'^n$ the analogous term for W_n of P_j^n for V_n, we have

$$\begin{aligned} Z_j^n(T_n) = n^{1/2}(P_j'^n W_n P_j'^n - \lambda_j P_j'^n) &= P_j'^n T_n P_j'^n \\ &\quad + n^{1/2}(P_j'^n - P_j)(I - P_j)(V - \lambda_j I)\, P_j'^n. \end{aligned}$$

$P_j'^n T_n P_j'^n$ converges to $P_j T P_j$ in F and, from Section 3.1.1, $n^{1/2}(P_j'^n - P_j)(I - P_j)$ converges to $\varphi_j(T)(I - P_j)$. As $(V - \lambda_j I)\, P_j'^n$ converges to $(V - \lambda_j I)\, P_j = 0$, so $Z_j^n(T_n)$ converges to $Z_j(T) = P_j T P_j$ in F and, from Rubin–Billingsley theorem, $Z_j^n(U_n)$ converges in distribution to $Z_j(U) = P_j U P_j$, which is a gaussian random element with mean zero in F. Further, as Δ is continuous, one has

PROPOSITION 8. *For each j of I, the asymptotic distribution of $(n^{1/2}(\lambda_k^n - \lambda_j))_{k \in I_j}$ in $\mathbb{R}^{k_j}$ is the joint distribution of the decreasing ordered eigenvalues of $P_j U P_j$, which is a gaussian random element with mean zero in F.*

The case of a gaussian r.f. In such case,

$$P_j U P_j = 2^{1/2} \lambda_j \left[\sum_{\substack{(k,l) \in I_j^2 \\ k < l}} \varepsilon_{kl} \xi_{kl} + \sum_{k \in I_j} \varepsilon_{kk} \xi_{kk} \right] \qquad \text{a.s.;}$$

$P_j U P_j$ is an operator of finite rank k_j and let B denote its random matrix in the basis $(e_k)_{k \in I_j}$; $B = (B_{kl})$ is symmetric, $B_{kk} = \langle P_j U P_j e_k, e_k \rangle = \lambda_j \xi_{kk}$ and, for $k < l$, $B_{kl} = \langle P_j U P_j e_k, e_k \rangle = 2^{1/2} \lambda_j \xi_{kl}$. The B_{kl}, $(k, l) \in I_j^2$, are gaussian independent elements and B is a gaussian element with mean zero. Its density with respect to the Lebesgue measure of $\mathbb{R}^{k_j(k_j+1)/2}$ is proportional to

$\exp[-\operatorname{tr} B^2/(4\lambda_j^2)]$. Thus, the joint distribution density of the decreasing ordered eigenvalues of $P_j U P_j$ (and of B) is given by [13].

$$f_{k_j}(t_1, ..., t_{k_j}) = C \exp\left[-\sum_{l=1}^{k_j} (t_l^2/4\lambda_j^2)\right] \prod_{l<l'} (t_l - t_{l'})$$

with

$$C^{-1} = 2^{k_j(k_j+3)/4} \prod_{l=1}^{k_j} \left[\Gamma\left(k_j + \frac{1-l}{2}\right) \lambda_j^{k_j(k_j+1)/2}\right].$$

For $j \in I'$, the limiting distribution of $n^{1/2}(\lambda_j^n - \lambda_j)$ is $\mathcal{N}(0, 2\lambda_j^2)$.

2.2.2. *The Joint Asymptotic Distribution of* $([n^{1/2}(\lambda_k^n - \lambda_j)]_{k\in I_j})_{j\in L}$

Let J denote again a p-tuple of L and consider now the mappings Z_n and Δ_p defined by

$$Z_n : T \in F \mapsto Z_n(T) = (Z_j^n(T))_{j\in J} \in F^p$$

and

$$\Delta_p : T' = (T_j')_{j\in J} \in F^p \mapsto (\Delta T_j')_{j\in J} \in \mathbb{R}^{\Sigma_{j\in J} k_j}.$$

It is easily verified that $(Z_n)_{n\in\mathbb{N}^*}$ satisfies the Rubin–Billingsley theorem conditions and so $Z_n(U_n)$ converges in distribution to $Z(U)$. Since Δ_p is continuous, $\Delta_p[Z_n(U_n)] = ([n^{1/2}(\lambda_k^n - \lambda_j)]_{k\in I_j})_{j\in J}$ converges in distribution to $\Delta_p[Z(U)]$ in $\mathbb{R}^{\Sigma^J k_j}$ and we get

PROPOSITION 9. *For each* J, $([n^{1/2}(\lambda_k^n - \lambda_j)]_{k\in I_j})_{j\in J}$ *converges in distribution to* $(\Delta[Z_j(U)])_{j\in J} = (\Delta[P_j U P_j])_{j\in J}$ *in* $\mathbb{R}^{\Sigma_J k_j}$.

The case of a gaussian r.f. The $P_j U P_j$ decomposition (in $L_F^2(\Omega, \mathcal{Q}, P)$) given in Section 2.2.1 shows that $P_j U P_j$ belongs to the gaussian space F_j ($j \in J$) generated by $\{\xi_{kl})_{(k,l)\in I_j^2, k<l}$. For each J of L, the sets I_j ($j \in J$) are disjoint and thus the F_j ($j \in J$) are independent; i.e., the random elements $(n^{1/2}(\lambda_k^n - \lambda_j))_{k\in I_j}$ and $(n^{1/2}(\lambda_{k'}^n - \lambda_{j'}))_{k\in I_{j'}}$, $j \neq j'$, $(j, j') \in J^2$ are asymptotically independent. So, the joint limiting distribution here is the product of the asymptotic distributions given in Section 2.2.1.

2.3. *Asymptotic Distribution of the Principal Factors*

We have seen that we must consider only a principal factor e_j corresponding to a principal value λ_j of multiplicity one (if not, we must only consider the whole eigenmanifold and then the projection operator P_j). Therefore, only indexes of I' will be considered in Section 2.3.

2.3.1. *Asymptotic Distribution of* $n^{1/2}(e_j^n - e_j)$

One may write $n^{1/2}(e_j^n - e_j) = P_j(n^{1/2}(e_j^n - e_j)) + (I - P_j)[n^{1/2}(e_j^n - e_j)]$. From the definition, $P_j(n^{1/2}(e_j^n - e_j)) = \langle n^{1/2}(e_j^n - e_j), e_j\rangle\, e_j = (n^{1/2}\langle e_j^n, e_j\rangle - 1)\, e_j$ and it is easily shown that $n^{1/2}(\langle e_j^n, e_j\rangle - 1) = \langle n^{1/2}(P_j^n - P_j), P_j\rangle_F \times (\langle e_j^n, e_j\rangle + 1)^{-1}$. Then, owing to the continuity of the inner product and Proposition 6, $\langle n^{1/2}(P_j^n - P_j), P_j\rangle_F$ converges in distribution to

$$\langle \varphi_j(U), P_j\rangle_F = \operatorname{tr}(\varphi_j(U)\, P_j) = \operatorname{tr}(P_j S_j U) = 0.$$

Thus, since $\langle e_j^n, e_j\rangle$ converges to 1, it can be deduced that, for each j of I', $P_j(n^{1/2}(e_j^n - e_j))$ converges in distribution to zero. Further, for n large enough, λ_j^n is a V_n eigenvalue with multiplicity one and $(I - P_j)[n^{1/2}(e_j^n - e_j)] = n^{1/2}(I - P_j)\, e_j^n = n^{1/2}(I - P_j)\, P_j^n e_j^n \times (\langle e_j^n, e_j\rangle)^{-1} = (\langle e_j^n, e_j\rangle)^{-1} \times (I - P_j)[n^{1/2}(P_j^n - P_j)](e_j)$. For each e of E, h_e, which maps $T \in F'$ into $Te \in E$, is continuous and, as $\varphi_j^n(U_n) = n^{1/2}(P_j^n - P_j)$ converges in distribution to $\varphi_j(U) = S_j U P_j + P_j U S_j$, the random element $h_{e_j}[\varphi_j^n(U_n)] = n^{1/2}(P_j^n - P_j)(e_j)$ converges in distribution to $\varphi_j(U)(e_j)$ in E. So $(I - P_j)[n^{1/2}(P_j^n - P_j)](e_j)$ converges in distribution to $(I - P_j)\, \varphi_j(U)(e_j) = S_j U(e_j)$. Finally, from the a.s. convergence of e_j^n to e_j, the convergence in distribution of $(I - P_j)[n^{1/2}(e_j^n - e_j)]$ and furthermore of $n^{1/2}(e_j^n - e_j)$ to $S_j U(e_j)$ is deduced and, as the mapping defined by $T \in F \to S_j T(e_j) \in E$ is linear and continuous and as U is gaussian with mean zero in F, we get

PROPOSITION 10. *For each j of I', $n^{1/2}(e_j^n - e_j)$ converges in distribution to the gaussian (with mean zero) vector $S_j U(e_j)$ of E.*

The case of a gaussian r.f. From the expressions of U and S_j in this case, we obtain

$$S_j U(e_j) = \sum_{\substack{k \in I \\ k > j}} (\lambda_j \lambda_k)^{1/2} (\lambda_j - \lambda_k)^{-1} e_k \xi_{jk} + \sum_{\substack{k \in I \\ k < j}} (\lambda_j \lambda_k)^{1/2} (\lambda_j - \lambda_k)^{-1} e_k \xi_{kj}.$$

So, $S_j U(e_j)$ is gaussian random element of E with mean zero and its covariance operator H_j is

$$H_j = \sum_{k \in I - \{j\}} \lambda_j \lambda_k (\lambda_j - \lambda_k)^{-2} e_k \otimes e_k.$$

2.3.2. *The Joint Asymptotic Distribution of* $(n^{1/2}(e_j^n - e_j))_{j\in I'}$

Let J be a p-tuple of elements of I'; the random element $(n^{1/2}(e_j^n - e_j))_{j\in J}$ is considered in $(E^p, \mathscr{B}_{E^p})$ and may be written

$$(n^{1/2}(e_j^n - e_j))_{j\in J} = (P_j[n^{1/2}(e_j^n - e_j)])_{j\in J} + ((I - P_j)[n^{1/2}(e_j^n - e_j)])_{j\in J}.$$

Since each $P_j[n^{1/2}(e_j^n - e_j)]$ converges to zero, we infer that $(P_j[n^{1/2}(e_j^n - e_j)])_{j\in J}$ converges in distribution to zero [4, p. 27]. Let us consider $e = (e_j)_{j\in J} \in E^p$, $\varphi^n(T) = ((\varphi_j^n(T))_{j\in J} \in (F')^p$, $I - P = (I - P_j)_{j\in J} \in (F')^p$, α_n which maps $(g_j)_{j\in J} \in E^p$ into $(\langle e_j^n, e_j\rangle^{-1} g_j)_{j\in J} \in E^p$, and generally each element $(T_j)_{j\in J} \in (F')^p$ as the mapping defined by $(\mu_j)_{j\in J} \in E^p \mapsto (T_j \mu_j)_{j\in J} \in E^p$. Hence, we can write

$$((I - P_j)[n^{1/2}(e_j^n - e_j)])_{j\in J} = \alpha_n (I - P)[\varphi^n(U_n)](e).$$

It is easily checked that the mapping $(T_j)_{j\in J} \in (F')^p \mapsto (T_j e_j)_{j\in J} \in E^p$ is linear and continuous. Then, from Proposition 7, $(\varphi^n(U_n))_{n\in \mathbb{N}^*}$ converges in distribution to $\varphi(U) = (\varphi_j(U))_{j\in J}$, which is gaussian with mean zero in $(F')^p$, and since $I - P$ is continuous and α_n converges a.s. to the identity in E^p, one has

Proposition 11. *For any p-tuple J of elements of* I', $(n^{1/2}(e_j^n - e_j))_{j\in J}$ *converges in distribution to* $(S_j U(e_j))_{j\in J}$ *which is gaussian with mean zero in* E^p.

The case of a gaussian r.f. In this case the covariance operator H_J of $(S_j U(e_j))_{j\in J}$ has the diagonal terms H_j $(j \in J)$ explicated in Section 2.3.1. The non-diagonal terms $H_{jj'}$ associated to $j \neq j'$ is

$$\begin{aligned} H_{jj'} &= E[S_j U(e_j) \otimes S_{j'} U(e_{j'})] \\ &= \sum_{k\in I-\{j\}} \sum_{k'\in I-\{j'\}} (\lambda_j \lambda_k)^{1/2} (\lambda_j - \lambda_k)^{-1} (\lambda_{j'} \lambda_{k'})^{1/2} (\lambda_{j'} - \lambda_{k'})^{-1} \\ &\quad e_k \otimes e_{k'} E[\xi_{ab} \xi_{cd}], \end{aligned}$$

where $a = \min\{k, j\}$, $b = \max\{k, j\}$, $c = \min\{k', j'\}$, $d = \max\{k', j'\}$. Since $\{\xi_{uv}\}_{(u,v)\in I^2, u<v}$ are independent $\mathscr{N}(0, 1)$, we obtain $H_{jj'} = -\lambda_j \lambda_{j'} (\lambda_j - \lambda_{j'})^{-2} e_{j'} \otimes e_j$. There is no asymptotic independence property.

3. Applications in Statistical Inference

3.1. *Statistical Structures*

In this section, three statistical structures are considered and denoted by $(E, \mathscr{B}_E, \mathscr{P}_j)^{\otimes \mathbb{N}^*}$ $(j = 1, 2, 3)$. $\mathscr{P}_1$ (resp $\mathscr{P}_2$; resp. $\mathscr{P}_3$) is the family of centered probability measures on E with finite second moment (resp. fourth moment; resp. gaussian). The statistics X_i $(i \in \mathbb{N}^*)$ are the coordinate mappings on these structures. Each family $\mathscr{P}_j$ may be written formally $\{P_\theta\}_{\theta \in \Theta}$ (with $P_\theta = \Theta$); for each θ of Θ, the covariance operator V of P_θ and each function of V are functions of θ; $\mathscr{P}_3$ may be written $\{P_V\}_{V \in \theta}$, where Θ is the space of the nuclear positive self-adjoint operators on F and V is the covariance operator of P_V. For each p of $[1, +\infty[$, let $\sigma_p(E)$ denote the separable Banach space of operators U on E such that $\|U\|_{\sigma_p}^p = \operatorname{tr}(UU^*)^{p/2}$ is finite (for $p = 2$, $\sigma_2(E) = F$ as in the other sections).

3.2. *Point Estimation*

Some properties which result from the last sections are indicated here without proof.

PROPOSITION 12. *On the structure* (1) (*resp.* (2)), *for each* $p \in [1, +\infty[$, V_n *is an estimator of* V *which is unbiased and converges a.s.* (*resp. a.s. and in quadratic mean*) *in* $\sigma_p(E)$.

PROPOSITION 13. *On the structure* (1), *for each* j *of* I, P_j^n *is an estimator of* P_j *which converges a.s. in* F, *and for each* i *of* I', e_i^n *is an estimator of* e_i *which converges a.s. in* E.

Let λ_j be a eigenvalue with multiplicity k_j and $\hat{\lambda}_j^n = (k_j)^{-1} \sum_{k \in I_j} \lambda_k^n$.

PROPOSITION 14. *On the structure* (1) (*resp.* (2)), *for each* j *of* I, $\hat{\lambda}_j^n$ *is an estimator of* λ_j *which converges a.s.* (*resp. a.s. and in quadratic mean*) *in* $\mathbb{R}$.

PROPOSITION 15. *On the structure* (1) (*resp.* (2)), *for each* $p \in [1, +\infty[$, $\|V_n\|_{\sigma_p}$ *is an estimator of* $\|V\|_{\sigma_p}$ *which converges a.s.* (*resp. a.s. and in quadratic mean*) *in* $\mathbb{R}$.

Thus, for $p = 1$, $\|V_n\|_{\sigma_1} = \sum_{l \in I} \lambda_l = \operatorname{tr} V_n$ is a consistent estimator of the total variance of X.

3.3. *Confidence Sets*

3.3.1. *For a Principal Value*

For briefness we are restricted here to the case of a principal value λ_j with multiplicity one ($j \in I'$) and we consider the third structure. We have seen that $[n^{1/2}(\lambda_j^n - \lambda_j)]/2^{1/2}\lambda_j$ is asymptotically $\mathcal{N}(0, 1)$ distributed. Thus, for $\alpha \in]0, 1[$ and θ_α defined by $\int_{-\theta_\alpha}^{\theta_\alpha} e^{-t^2/2}\, dt = (2\pi)^{1/2}(1-\alpha)$, the interval $]n^{1/2}\lambda_j^n/(n^{1/2} + 2^{1/2}\theta_\alpha), n^{1/2}\lambda_j^n/(n^{1/2} - 2^{1/2}\theta_\alpha)[$ is a confidence interval of λ_j with asymptotic confidence level $1-\alpha$. An immediate application of this result is to give a test (of asymptotic level α) of the null hypothesis $\{V \in \theta; \lambda_j = \lambda\}$ (where λ is given) against the alternative $\{V \in \theta; \lambda_j \neq \lambda\}$.

3.2.2. *For the Total Variance*

The "trace" operator which maps from the $\sigma_1(E)$ subspace of the self-adjoint operators into $\mathbb{R}$ is linear and continuous. Thus, $\operatorname{tr} V = \operatorname{tr} E(X \otimes X) = E(\operatorname{tr} X \otimes X)$. On the structure (2), the application of the central limit theorem to the sequence $(\operatorname{tr} X_i \otimes X_i)_{i \in \mathbb{N}^*}$ shows that $n^{1/2}(\|V_n\|_{\sigma_1} - \|V\|_{\sigma_1}) = n^{1/2}[n^{-1}\sum_{i=1}^n \operatorname{tr}(X_i \otimes X_i) - E[\operatorname{tr} X \otimes X]]$ converges in distribution to a gaussian random element with mean zero. If we are restricted to the structure (3), the variance of the limiting distribution is $2\|V\|_F^2$. Then, with the same notations as in Section 3.3.1,

$$]\|V_n\|_{\sigma_1} - (2/n)^{1/2}\theta_\alpha \|V_n\|_F, \|V_n\|_{\sigma_1} + (2/n)^{1/2}\theta_\alpha \|V_n\|_F[$$

is a confidence interval of the total variance of X (i.e., $\|V\|_{\sigma_1}$) with asymptotic confidence level $1-\alpha$. It should be noted that similar results can be obtained for $\|V\|_{\sigma_p}$ ($p \in [1, +\infty[$).

3.4. *Tests*

From the results of Section 3.3 several tests for the estimated parameters can be built. In this section, we are concerned with other types of tests and we give only two examples.

3.4.1. *Test for the Ratio of the Explained Variance*

Let $p \in \mathbb{N}^*$ and $\beta \in]0, 1[$. We consider the sub-structure of structure (3) obtained by restricting Θ to the subset Θ' of the elements with rank greater than p and with only simple eigenvalues. Let $H_0^{\beta,p} = \{V \in \Theta', A = \sum_{i=1}^p \lambda_i - \beta \sum_{i \in I} \lambda_i \geqslant 0\}$ and $H_1^{\beta,p} = \Theta' - H_0^{\beta,p}$. It can be shown that $A_n = \sum_{i=1}^p \lambda_i^n - \beta \sum_{i \in I} \lambda_i^n$ is a consistent estimator of A and, using Section 2.2.1, that $n^{1/2}(A_n - A)$ converges in distribution to a gaussian element with mean zero and variance $\gamma = 2(1-2\beta)\sum_{i=1}^p \lambda_i^2 + 2\beta^2 \sum_{i \in I} \lambda_i^2$. Since $\gamma_n = 2(1-2\beta)\sum_{i=1}^p (\lambda_i^n)^2 + 2\beta^2 \sum_{i \in I} (\lambda_i^n)^2$ is an a.s. converging estimator of γ, then $n^{1/2}\gamma_n^{-1/2}(A_n - A)$ is $\mathcal{N}(0, 1)$ asymptotically distributed. Let $\alpha \in]0, 1[$ and t_α defined by $\int_{t_\alpha}^{+\infty} e^{-x^2/2}\, dx = \alpha(2\pi)^{1/2}$; the deterministic test with critical region $[A_n < -t_\alpha n^{-1/2}\gamma_n^{1/2}]$ (which is included in $[n^{1/2}(A_n - A) < -t]$ under $H_0^{\beta,p}$) is a test of the null hypothesis $H_0^{\beta,p}$ against $H_1^{\beta,p}$ with asymptotic level inferior or equal to α.

3.4.2. *Test for a Principal Factor*

In this section $E = \mathbb{R}^q$ and V is regular; let $j \in I'$, and e_j^0 a given vector of E. On the structure (3) we want to test the null hypothesis $H_0^j = \{V \in \Theta; e_j = \varepsilon e_j^0\}$ (where $\varepsilon = \pm 1$) against $\bar{H}_0^j = \Theta - H_0^j$. For each $k = 1, 2,..., q$, let

$$\alpha_k = (\lambda_k \lambda_j)^{-1/2}(\lambda_k - \lambda_j); \quad \alpha_k^n = (\lambda_k^n \lambda_j^n)^{-1/2}(\lambda_k^n - \lambda_j^n); \quad y_j^n = n^{1/2}(e_j^n - e_j^0),$$

$$T_n = \sum_{k \neq j} \alpha_k^n e_k^n \otimes e_k^n \qquad \text{and} \qquad Z_j^n = \| T_n y_j^n \|_E^2.$$

The inverse operator V_n^{-1} of V_n exists for n large enough and converges a.s. to V^{-1}; thus, $T_n = \lambda_j^n V_n^{-1} + (\lambda_j^n)^{-1} V_n - 2I$ converges a.s. to the continuous operator $T = \lambda_j V^{-1} + \lambda_j^{-1} V - 2I = \sum_{k \neq j} \alpha_k e_k \otimes e_k$. Under H_0^j, y_j^n converges in distribution as y_j, which is $\mathcal{N}(0, \sum_{k \neq j} \alpha_k^{-2} e_k \otimes e_k)$, and so (see [4, p. 34]) $T_n y_j^n$ converges in distribution to Ty_j, which is $\mathcal{N}(0, I - e_j \otimes e_j)$, and then Z_j^n converges in distribution to $\| Ty_j \|^2$. Since $(\langle Ty_j, e_k \rangle)_{k \neq j}$ is a family of $\mathcal{N}(0, 1)$ independent variables and $\langle Ty_j, e_j \rangle$ is null a.s., then $\| Ty_j \|^2 = \sum_{k=1}^q \langle Ty_j, e_k \rangle^2$ has a χ^2 distribution with $p - 1$ degrees of freedom. Let $(\lambda_l)_{l \in L}$ be the strictly decreasing sequence of the V eigenvalues. Z_j^n may be decomposed

$$Z_j^n = n(A_n + B_n + C) \qquad \text{with} \qquad A_n = \sum_{l \neq j} \sum_{k \in I_l} [(\alpha_k^n)^2 - \alpha_l^2] \langle e_j^0, e_k^n \rangle^2.$$

$B_n = \sum_{l \neq j} \alpha_l^2 [\sum_{k \in I_l} \langle e_j^0, e_k^n \rangle^2 - \| P_l e_j^0 \|^2]$ and $C = \sum_{l \neq j} \alpha_l \| P_l e_j^0 \|^2$. Since $\forall l \in L \lim_{n \to \infty} P_l^n = P_l$ a.s., $\forall k \in I_l \lim_{n \to \infty} \alpha_k^n = \alpha_k$, and since C is null only under H_0^j, it can be deduced that, under $\bar{H}_0^j$, $\lim_{n \to \infty} Z_j^n = +\infty$ a.s.

So the deterministic test with critical region $[Z^n > \chi^2_{\alpha, p-1}]$ is a consistent test of H_0^j against $\bar{H}_0^j$ with asymptotic level $\alpha (\alpha \in]0, 1[)$.

Remark. The asymptotic results obtained upon the projection operators (in the case of a non-simple eigenvalue λ_j corresponding to the eigenmanifold E_j) allow one to build in a similar way a test (with asymptotic level α) of the null hypothesis $K_0^j = \{E_j = E_j^0\}$, where E_j^0 is a given subspace of E, against the alternative $\Theta - K_0^j$.

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