# Approximate Tests of Correlation in Time-Series

By M. H. Quenouille

[Received August 20th, 1948]

## *Introduction*

In a recent investigation of the joint distribution of serial correlation coefficients (Quenouille, 1948a, 1948b), it has been pointed out that the algebraic forms suggest that multiple and partial correlation coefficients might provide approximate tests for serial correlation. For example, a Markoff scheme might be tested by calculating the partial correlation coefficient between terms $x_i$ and $x_{i+2}$ when $x_{i+1}$ has been eliminated, i.e. by $r_{13\cdot 2} = \dfrac{r_2 - r_1^2}{1 - r_1^2}$, where $r_1$ and $r_2$ are appropriate estimates of the first and second serial correlation coefficients. Partial correlation coefficients, such as $r_{14\cdot 23}$, $r_{15\cdot 234}$, . . . might also be used to test the scheme, and similarly to test schemes of a higher order. A further approximation of a similar type might be used to test the cross-correlations between time-series. Thus, for example, if $x_1, x_2, \ldots$ are observations in a Markoff scheme and $y_1, y_2, \ldots$ are any other set of observations, then the partial correlation coefficient between $x_i$ and $y_i$, $x_{i-1}$ eliminated, might be used to test the correlation between the two series, although more frequently we shall wish to test whether the error terms of the two series are correlated by eliminating $y_{i-1}$, $y_{i-2}$, . . . as well as $x_{i-1}$, while a general test might be made using a canonical analysis with the degrees of freedom appropriately adjusted. In the first part of this paper the former suggestion concerning serial correlations is investigated while the latter suggestion is discussed in the second part.

This former suggestion is strengthened by two other facts. Firstly it has been shown (Dixon, 1944; Rubin, 1945; Quenouille, 1948c) that the ordinary correlation coefficient might be used to test a serial correlation coefficient to a high order of approximation provided that 3 extra degrees of freedom are allowed. For example, if 20 observations are used, 19 pairs of observations are correlated, and the normal tests must be applied as if 22 pairs of independent observations were involved. Secondly, if a series of observations are connected in an autogressive scheme of extent $p$

$$x_{n+p} + a_1 x_{n+p-1} + \ldots + a_p x_n = \varepsilon_{n+p}$$

where the $\varepsilon_{n+p}$ are uncorrelated, then

$$x_{n-p} + a_1 x_{n-p+1} + \ldots + a_p x_n = \varepsilon'_{n-p}$$

where the $\varepsilon_i'$ are uncorrelated and $E(\varepsilon_{i+j}\, \varepsilon_i') = 0, j > p$.

This has two main consequences:

(*a*) The forms $R_s$ considered in a previous paper (Quenouille, 1947) and shown to be independent in large samples, are seen to represent large-sample approximations to the correlations between $\varepsilon_{i+j}$ and $\varepsilon_i'$ or to the partial correlation coefficients between $x_{i+j}$ and $x_{i-p}$ when the coefficients $a_1, a_2, \ldots a_p$ are known.

(*b*) The use of a partial correlation coefficient between $x_{i+j}$ and $x_{i-p}$ will involve an approximation of the same order of magnitude as that employed in the use of the ordinary correlation coefficient. It is worth noting that in large samples the partial correlation coefficient of order is also equal to the estimate of $a_p$ in an autogressive scheme of extent $p$ and might be found in this manner.

It is apparent that the use of a partial correlation coefficient will be justified if the sample is large enough, and the major problem is the validity of this approximation for small samples. When we are dealing with small samples three main problems are encountered:

(1) the validity of the large-sample approximation;

(2) the appropriate definitions for the serial correlation coefficients and the manner in which these coefficients are used in the partial correlation coefficients; and—

(3) the bias in the definition.

These three factors will all be important in the application of this test.

The bias will operate partly as a result of the correction for the mean, and partly as a result of the high correlation between terms such as $r_{12}$ and $r_{23}$ that are usually independent. Both forms of bias will be of order, but will in general depend upon the intermediate correlations and will thus be important in practice. For this reason artificial series have been used for which the true mean is known, and the partial correlation coefficients have been calculated both with and without correction for the mean. Also the form of the definition that is employed is likely to influence the result. Thus, for example, a set of observations $x_1, x_2, \ldots x_n$ may be used to calculate the first partial correlation coefficient by calculating the correlation between $s_1 = x_1, x_2, \ldots x_{n-2}$ and $s_3 = x_3, x_4, \ldots x_n$ when $s_2 = x_2, x_3, \ldots x_{n-1}$ has been eliminated, i.e. using

$$r_{13\cdot 2} = \frac{r_{13} - r_{12}\, r_{23}}{\sqrt{(1 - r^2_{12})\,(1 - r^2_{23})}},$$

where $r_{ij}$ is the estimated correlation between $s_i$ and $s_j$. However, it would seem profitable to pool $r_{12}$ and $r_{23}$ and to use an overall estimate of the mean. This may be done with definitions such as—

$$r_1 = \frac{(x_1 - \bar{x}_1)(x_2 - \bar{x}_1) + (x_2 - \bar{x}_1)(x_3 - \bar{x}_1) + \ldots + (x_{n-1} - \bar{x}_1)(x_n - \bar{x}_1)}{\frac{1}{2}(x_1 - \bar{x}_1)^2 + (x_2 - \bar{x}_1)^2 + (x_3 - \bar{x}_1)^2 + \ldots + \frac{1}{2}(x_n - \bar{x}_1)^2}$$

$$\bar{x}_1 = \frac{1}{n-1}\left(\tfrac{1}{2}x_1 + x_2 + \ldots + x_{n-1} + \tfrac{1}{2}x_n\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$r_2 = \frac{(x_1 - \bar{x}_2)(x_3 - \bar{x}_2) + (x_2 - \bar{x}_2)(x_4 - \bar{x}_2) + \ldots + (x_{n-2} - \bar{x}_2)(x_n - \bar{x}_2)}{\frac{1}{2}(x_1 - \bar{x}_2)^2 + \frac{1}{2}(x_2 - \bar{x}_2)^2 + (x_3 - \bar{x}_2)^2 + \ldots + \frac{1}{2}(x_n - \bar{x}_2)^2}$$

$$\bar{x}_2 = \frac{1}{n-2}\left(\tfrac{1}{2}x_1 + \tfrac{1}{2}x_2 + x_3 + \ldots + \tfrac{1}{2}x_n\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

in which the estimated variance in the denominator refers directly to the numerator. However, it would also seem desirable to make definitions such as these strictly comparable with regard to the terms that are involved. Thus, for example, if we were considering only the first and second serial correlation coefficients $r_2$ would be defined as above, but $r_1$ would be calculated from the definition—

$$r_1 = \frac{\frac{1}{2}(x_1 - \bar{x}_3)(x_2 - \bar{x}_3) + (x_2 - \bar{x}_3)(x_3 - \bar{x}_3) + \ldots + \frac{1}{2}(x_{n-1} - \bar{x}_3)(x_n - \bar{x}_3)}{\frac{1}{4}(x_1 - \bar{x}_3)^2 + \frac{3}{4}(x_2 - \bar{x}_3)^2 + (x_3 - \bar{x}_3)^2 + \ldots + \frac{1}{4}(x_n - \bar{x}_3)^2}$$

$$\bar{x}_3 = \frac{1}{n-2}\left(\tfrac{1}{4}x_1 + \tfrac{3}{4}x_2 + x_3 + \ldots + \tfrac{1}{4}x_n\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

It is seen that this definition does not differ from (1) except for small samples or for extreme cases.

This problem of deciding upon an appropriate definition disappears when $x_1, x_2, \ldots x_n$ are circularly related, and for this reason methods of obtaining a circular definition are worth mentioning. Three possible wavs suggest themselves:

(1) The use of the unadjusted observations, e.g. $x_1, x_2, \ldots x_n, x_1, x_2, \ldots$.

(2) The use of end adjustments so that the transition from $x_n$ to $x_1$ is less marked, e.g $\frac{1}{2}(x_n + x_1)/x_2, \ldots x_{n-1}, \frac{1}{2}(x_n + x_1) \ldots$

(3) The use of fitted constants to cover the transition, e.g. $x_1, x_2, \ldots x_n, a, b, c, x_1, x_2 \ldots$ where $a, b, c$ are constants determined from the series to preserve its continuity.

The last method of these three would seem to be preferable, but its application is complicated except when we are concerned with the appropriate definition of $r_1$ for a Markoff scheme. In this case $x_1, x_2, \ldots x_n, x_{n-1}, \ldots x_2, x_1, x_2, \ldots$ are circularly related and we arrive at the definition of $r_1$ given by (1). The second method is simpler in application, and it will be seen in a subsequent paper (Quenouille, 1948d) that it leads readily to a method of trend elimination. However, since we are concerned with small samples the definitions (1) (2) and (3) will be used in what follows.

### *Preliminary Investigation*

For an initial investigation thirty sets of twenty numbers were generated according to the following scheme,

$$x_n = \pm 1$$

$$p(x_n = 1 \mid x_{n-1} = 1) = p(x_n = -1 \mid x_{n-1} = -1) = 0{\cdot}8$$

$$p(x_n = 1 \mid x_{n-1} = -1) = p(x_n = -1 \mid x_{n-1} = 1) = 0{\cdot}2 \quad . \quad . \quad . \quad (4)$$

the first pair of each set being taken equal to the last pair of the preceding set for a reason to be seen later. It has been shown that this scheme approximates to a Markoff process with $\rho = 0{\cdot}6$.* The partial correlation coefficients were then calculated for each series: (*a*) with the mean assumed equal to zero using equations (2) and (3); (*b*) with the mean assumed equal to zero using equations (1) and (2); (*c*) with the correction applied for the mean using equations (1) and (2). The values obtained are given in Table 1. It is fairly clear that all methods are biased and that the major portion of the bias results from the correction for the mean, a second portion results from the term $r_1^2$ occurring in the definition of the partial correlation coefficient, while a third smaller portion arises from the fact that the definitions of $r_1$ and $r_2$ are not directly comparable. The expected variances for the three sets are approximately $1/18 = {\cdot}056$† and the observed variances are significantly less than this. It would appear from these figures that the bias leads to a reduction in the variance, and also that the effect of this reduction will prevent us from making more errors of the first kind. In order to investigate these points further the number of series was extended to sixty-four. The partial correlation coefficients for these series were worked out under definition (*a*), i.e. uncorrected for the mean using formulae (2) and (3), and the series were combined to form series of 38, 74, 146, . . . items. This could be done quite easily, since under the definition employed a serial correlation for a combined series was equal to the mean of the coefficients of the sub-series. The partial correlations for these series are shown in Table 2. The bias is diminished as the length of series is successively doubled, while the second moment is relatively increased. The bias is shown clearly to be inversely proportional to $n$, and the following method is suggested for the removal of such bias: if $r$ is a serial correlation coefficient or a partial correlation coefficient calculated from a series, and ${}_1r$ and ${}_2r$ are the corresponding coefficients from the first and second halves of the series, then $R = 2r - \frac{1}{2}({}_1r + {}_2r)$ will be free from bias to the order $1/n^2$. This method will tend to remove the bias due to the mean so that it can be usefully employed with

* Schemes such as this may be usefully employed to indicate approximate distributions in serial correlation. The variance and serial correlation coefficients are easily calculated since each term is equal to $\pm 1$, while schemes approximately to the general linear autoregressive process can also be set up.

† If we are calculating $r_{1p \cdot 23 \ldots}$ from a series of $n$ observations, then in effect $p$ sets of $n - p + 1$ observations are used. Since these sets are interdependent we add 3 to the effective number of observations, while $p - 2$ is subtracted for the partial correlation giving a variance of $1/(n - 2p + 4)$. If a correction is made for the mean this becomes $1/(n - 2p + 3)$.

TABLE 1.—*Values of* $r_{13 \cdot 2}$

| Series | (a) Using (2) and (3) uncorrected for mean | (b) Using (1) and (2) uncorrected for mean | (c) Using (1) and (2) corrected for mean |
|---|---|---|---|
| 1 | −0·1613 | −0·0445 | −0·0930 |
| 2 | −0·0588 | −0·1137 | −0·1565 |
| 3 | −0·1250 | −0·1799 | −0·1652 |
| 4 | −0·1076 | −0·1461 | −0·1461 |
| 5 | −0·2413 | −0·2447 | −0·1814 |
| 6 | −0·1782 | −0·1570 | −0·1435 |
| 7 | 0·0357 | −0·0027 | −0·2544 |
| 8 | −0·2000 | −0·2534 | −0·2579 |
| 9 | −0·0588 | −0·1137 | −0·1357 |
| 10 | 0·2000 | 0·1644 | 0·1072 |
| 11 | 0·0000 | −0·0285 | −0·0676 |
| 12 | 0·2500 | 0·2286 | 0·1993 |
| 13 | −0·2000 | −0·2534 | −0·3394 |
| 14 | −0·0588 | −0·1137 | −0·1564 |
| 15 | −0·1613 | −0·0445 | −0·1941 |
| 16 | −0·0907 | 0·1151 | −0·0974 |
| 17 | −0·1613 | −0·0445 | −0·0358 |
| 18 | 0·3454 | 0·3572 | −0·2551 |
| 19 | 0·2000 | 0·1644 | 0·1615 |
| 20 | 0·2500 | 0·2286 | 0·1395 |
| 21 | −0·2000 | −0·2534 | −0·2713 |
| 22 | 0·2000 | 0·1644 | 0·1525 |
| 23 | −0·2000 | −0·2534 | −0·2964 |
| 24 | −0·1250 | −0·1799 | −0·2600 |
| 25 | −0·1250 | −0·1799 | −0·1129 |
| 26 | −0·0588 | −0·1137 | −0·7860 |
| 27 | −0·2000 | −0·2534 | −0·4123 |
| 28 | 0·0836 | 0·1001 | −0·2445 |
| 29 | 0·1572 | 0·1644 | 0·1656 |
| 30 | −0·1250 | −0·1799 | −0·1916 |
| Mean | −0·0375 | −0·0489 | −0·1443 |
| $m_2'$ | 0·0287 | 0·0326 | 0·0605 |
| $m_2$ | 0·0154 | 0·0090 | 0·0411 |

TABLE 2.—*Values of* $r_{13 \cdot 2}$

| Series | Effective number of pairs of observations on which the coefficient is based: 18 | 36 | 72 | 144 | 288 | 576 | 1152 |
|---|---|---|---|---|---|---|---|
| 1 | −0·1613 | −0·1079 | .. | .. | .. | .. | .. |
| 2 | −0·0588 | .. | −0·0592 | .. | .. | .. | .. |
| 3 | −0·1250 | −0·0640 | .. | .. | .. | .. | .. |
| 4 | −0·1076 | .. | .. | −0·0868 | .. | .. | .. |
| 5 | −0·2413 | −0·1852 | .. | .. | .. | .. | .. |
| 6 | −0·1782 | .. | −0·1250 | .. | .. | .. | .. |
| 7 | 0·0357 | −0·0640 | .. | .. | .. | .. | .. |
| 8 | −0·2000 | .. | .. | .. | 0·0035 | .. | .. |
| 9 | −0·0588 | 0·1562 | .. | .. | .. | .. | .. |
| 10 | 0·2000 | .. | 0·1964 | .. | .. | .. | .. |
| 11 | 0·0000 | 0·1250 | .. | .. | .. | .. | .. |
| 12 | 0·2500 | .. | .. | 0·1000 | .. | .. | .. |
| 13 | −0·2000 | −0·1250 | .. | .. | .. | .. | .. |
| 14 | −0·0588 | .. | −0·1250 | .. | .. | .. | .. |

TABLE 2—(*Contd.*)

| Series | 18 | 36 | 72 | 114 | 288 | 576 | 1152 |
|---|---|---|---|---|---|---|---|
| 15 | −0·1613 | −0·1250 | . . | . . | . . | . . | . . |
| 16 | −0·0907 | . . | . . | . . | . . | 0·0492 | . . |
| 17 | −0·1613 | 0·1964 | . . | . . | . . | . . | . . |
| 18 | 0·3454 | . . | 0·2299 | . . | . . | . . | . . |
| 19 | 0·2000 | 0·2593 | . . | . . | . . | . . | . . |
| 20 | 0·2500 | . . | . . | 0·1133 | . . | . . | . . |
| 21 | −0·2000 | 0·0000 | . . | . . | . . | . . | . . |
| 22 | 0·2000 | . . | −0·0730 | . . | . . | . . | . . |
| 23 | −0·2000 | −0·1613 | . . | . . | . . | . . | . . |
| 24 | −0·1250 | . . | . . | . . | 0·0946 | . . | . . |
| 25 | −0·1250 | −0·0909 | . . | . . | . . | . . | . . |
| 26 | −0·0588 | . . | 0·0169 | . . | . . | . . | . . |
| 27 | −0·2000 | −0·0011 | . . | . . | . . | . . | . . |
| 28 | 0·0836 | . . | . . | 0·0686 | . . | . . | . . |
| 29 | 0·1572 | 0·1529 | . . | . . | . . | . . | . . |
| 30 | −0·1250 | . . | 0·1186 | . . | . . | . . | . . |
| 31 | −0·2000 | −0·0907 | . . | . . | . . | . . | . . |
| 32 | 0·0000 | . . | . . | . . | . . | . . | 0·0242 |
| 33 | −0·1613 | −0·1613 | . . | . . | . . | . . | . . |
| 34 | −0·1613 | . . | −0·0452 | . . | . . | . . | . . |
| 35 | 0·2000 | 0·0710 | . . | . . | . . | . . | . . |
| 36 | −0·1250 | . . | . . | −0·0592 | . . | . . | . . |
| 37 | −0·2000 | −0·2000 | . . | . . | . . | . . | . . |
| 38 | −0·2000 | . . | −0·0730 | . . | . . | . . | . . |
| 39 | −0·0588 | 0·0710 | . . | . . | . . | . . | . . |
| 40 | 0·0357 | . . | . . | . . | 0·0025 | . . | . . |
| 41 | −0·1250 | 0·0809 | . . | . . | . . | . . | . . |
| 42 | −0·1613 | . . | 0·0793 | . . | . . | . . | . . |
| 43 | −0·0836 | 0·0357 | . . | . . | . . | . . | . . |
| 44 | −0·0907 | . . | . . | 0·0186 | . . | . . | . . |
| 45 | −0·0588 | −0·1613 | . . | . . | . . | . . | . . |
| 46 | −0·2857 | . . | −0·0795 | . . | . . | . . | . . |
| 47 | −0·3333 | −0·0725 | . . | . . | . . | . . | . . |
| 48 | 0·1692 | . . | . . | . . | . . | −0·0008 | . . |
| 49 | 0·3032 | 0·2000 | . . | . . | . . | . . | . . |
| 50 | 0·1133 | . . | −0·0640 | . . | . . | . . | . . |
| 51 | −0·2857 | −0·2857 | . . | . . | . . | . . | . . |
| 52 | −0·2857 | . . | . . | 0·0069 | . . | . . | . . |
| 53 | 0·5000 | 0·3571 | . . | . . | . . | . . | . . |
| 54 | −0·1250 | . . | 0·0737 | . . | . . | . . | . . |
| 55 | −0·1250 | −0·2413 | . . | . . | . . | . . | . . |
| 56 | −0·3846 | . . | . . | . . | −0·0090 | . . | . . |
| 57 | −0·2000 | −0·1250 | . . | . . | . . | . . | . . |
| 58 | −0·0588 | . . | −0·1000 | . . | . . | . . | . . |
| 59 | 0·0357 | −0·1250 | . . | . . | . . | . . | . . |
| 60 | −0·2857 | . . | . . | −0·0671 | . . | . . | . . |
| 61 | 0·0000 | −0·0907 | . . | . . | . . | . . | . . |
| 62 | −0·2000 | . . | −0·0746 | . . | . . | . . | . . |
| 63 | −0·1250 | −0·0588 | . . | . . | . . | . . | . . |
| 64 | 0·0000 | . . | . . | . . | . . | . . | . . |
| Mean | −0·0637 | −0·0260 | −0·0065 | 0·0118 | 0·0229 | 0·0242 | . . |
| $m_2'$ | 0·0352 | 0·0238 | 0·0119 | 0·0054 | 0·0023 | 0·0012 | . . |
| $m_2$ | 0·0317 | 0·0239 | 0·0127 | 0·0060 | 0·0023 | 0·0012 | . . |

ordinary serial correlation coefficients. This approach also shows that instead of the usual definition of the partial correlation coefficient, i.e.

$$r_{1n\cdot 23} = \frac{\begin{vmatrix} r_{n-1} & r_1 & r_2 & \dots & r_{n-2} \\ r_{n-2} & 1 & r_1 & \dots & r_{n-3} \\ r_{n-3} & r_1 & 1 & \dots & r_{n-4} \\ & & \dots & & \\ r_1 & r_{n-3} & r_{n-4} & \dots & 1 \end{vmatrix}}{\begin{vmatrix} 1 & r_1 & r_2 & \dots & r_{n-2} \\ r_1 & 1 & r_1 & \dots & r_{n-3} \\ r_2 & r_1 & 1 & \dots & r_{n-4} \\ & & \dots & & \\ r_{n-2} & r_{n-3} & r_{n-4} & \dots & 1 \end{vmatrix}} \quad . \quad . \quad . \quad (4)$$

the form

$$r'_{1n\cdot 23} = \frac{\begin{vmatrix} {}_1r_{n-1} & {}_2r_1 & {}_2r_2 & \dots & {}_2r_{n-2} \\ {}_1r_{n-2} & 1 & {}_2r_1 & \dots & {}_2r_{n-3} \\ {}_1r_{n-3} & {}_1r_1 & 1 & \dots & {}_2r_{n-4} \\ & & \dots & & \\ {}_1r_1 & {}_1r_{n-3} & {}_1r_{n-4} & \dots & 1 \end{vmatrix} + \begin{vmatrix} {}_2r_{n-1} & {}_1r_1 & {}_1r_2 & \dots & {}_1r_{n-2} \\ {}_2r_{n-2} & 1 & {}_1r_1 & \dots & {}_1r_{n-3} \\ {}_2r_{n-3} & {}_2r_1 & 1 & \dots & {}_1r_{n-4} \\ & & \dots & & \\ {}_2r_1 & {}_2r_{n-3} & {}_2r_{n-4} & \dots & 1 \end{vmatrix}}{\begin{vmatrix} 1 & {}_2r_1 & {}_2r_2 & \dots & {}_2r_{n-2} \\ {}_2r_1 & 1 & {}_2r_1 & \dots & {}_2r_{n-3} \\ {}_2r_2 & {}_2r_1 & 1 & \dots & {}_2r_{n-4} \\ & & \dots & & \\ {}_2r_{n-2} & {}_2r_{n-3} & {}_2r_{n-4} & \dots & 1 \end{vmatrix} + \begin{vmatrix} 1 & {}_1r_1 & {}_1r_2 & \dots & {}_1r_{n-2} \\ {}_1r_1 & 1 & {}_1r_1 & \dots & {}_1r_{n-3} \\ {}_1r_2 & {}_1r_1 & 1 & \dots & {}_1r_{n-4} \\ & & \dots & & \\ {}_1r_{n-2} & {}_1r_{n-3} & {}_1r_{n-4} & \dots & 1 \end{vmatrix}} \quad . \quad . \quad . \quad (5)$$

might be used when the mean is known (see Appendix). For example, in this case $r'_{13\cdot 2} = \frac{{}_1r_2 + {}_2r_2 - 2_1\, r_{1\cdot 2}\, r_1}{2 - {}_1r_1^2 - {}_2r_1^2}$ can be used. The values of $R_{13\cdot 2}$ and $r'_{13\cdot 2}$ for the combined series of 38 items are given in Table 3. These mirror each other closely, and the means and second moments agree well with the theoretical values of zero and $1/36 = 0\cdot 0278$. The only significant value (series 53–54) arises from four successive changes of sign in series 53, which completely obscures the "Markoff" nature of the series, and which can be regarded as a significant event.

As a final step in this investigation, the values of $r_{13\cdot 2}$, $r'_{13\cdot 2}$ and $R_{13\cdot 2}$ were calculated for series 1–30 using definitions (2) and (3), correcting each observation for the mean. These values, which are given in Table 4, agree with the general conclusions already reached. The mean value $0\cdot 0415$ obtained for $R_{13\cdot 2}$ is in agreement with the value $0\cdot 0501$ obtained for $r_{13\cdot 2}$ when the whole series is used without correction for the mean. The values obtained are remarkably stable with the possible exception of series 17–18, in which the large bias is overcompensated in $R_{13\cdot 2}$, whose variance is consequently increased. It is, however, impossible to justify the use of these coefficients $r_{13\cdot 2}$, $r'_{13\cdot 2}$ and $R_{13\cdot 2}$ on the basis of this test series alone, and it seems advisable to extend the investigation to other experimental series.

*Extension of the Investigation.*

The first step in extending this investigation must be to investigate $r_{13\cdot 2}$ or $r_{14\cdot 23}$ on an artificial series of the type constructed by Kendall (1946). However, it must be noted that for Kendall's four series the theoretical values of the denominator in (4) are $0\cdot 160$, $0\cdot 059$, $0\cdot 141$ and $0\cdot 231$ respectively, so that rounding-off errors, bias and differences in the type of definition will be greatly magnified, and this situation will be worsened by the variation in the denominator. However,

TABLE 3.—*Values of* $r'_{13\cdot2}$ *and* $R_{13\cdot2}$

| Series | $r'_{13\cdot2}$ | $R_{13\cdot2}$ | Series | $r'_{13\cdot2}$ | $R_{13\cdot2}$ |
|---|---|---|---|---|---|
| 1–2 | −0·0941 | −0·1058 | 33–34 | −0·1613 | −0·1613 |
| 3–4 | −0·0206 | −0·0117 | 35–36 | 0·0779 | 0·1045 |
| 5–6 | −0·1716 | −0·1607 | 37–38 | −0·2000 | −0·2000 |
| 7–8 | −0·0594 | −0·0459 | 39–40 | 0·1370 | 0·1536 |
| 9–10 | 0·1935 | 0·2418 | 41–42 | 0·3368 | 0·3050 |
| 11–12 | 0·1250 | 0·1250 | 43–44 | 0·1658 | 0·1586 |
| 13–14 | −0·0968 | −0·1206 | 45–46 | −0·1097 | −0·1504 |
| 15–16 | −0·1180 | −0·1240 | 47–48 | −0·0716 | −0·0630 |
| 17–18 | 0·2465 | 0·3008 | 49–50 | 0·2067 | 0·1918 |
| 19–20 | 0·2857 | 0·2936 | 51–52 | −0·2857 | −0·2857 |
| 21–22 | 0·0000 | 0·0000 | 53–54 | 0·4616 | 0·5267 |
| 23–24 | −0·1559 | −0·1601 | 55–56 | −0·2062 | −0·2278 |
| 25–26 | −0·0815 | −0·0899 | 57–58 | −0·0969 | −0·1206 |
| 27–28 | 0·0263 | 0·0560 | 59–60 | −0·1250 | −0·1250 |
| 29–30 | 0·2623 | 0·2897 | 61–62 | 0·0000 | −0·0814 |
| 31–32 | 0·0000 | −0·0814 | 63–64 | 0·0000 | −0·0551 |
| | | | Mean | 0·0147 | 0·0118 |
| | | | $m_2'$ | 0·0317 | 0·0369 |
| | | | $m_2$ | 0·0325 | 0·0379 |

TABLE 4.—*Values of* $r_{13\cdot2}$, $r'_{13\cdot2}$ *and* $R_{13\cdot2}$

(*Corrected for mean*)

| Series | Effective number of observations on which the coefficient is based: 18 | 36 | 36 | 36 |
|---|---|---|---|---|
| | $r_{13\cdot2}$ | $r_{13\cdot2}$ | $r'_{13\cdot2}$ | $R_{13\cdot2}$ |
| 1 | −0·1868 | −0·1060 | −0·1168 | −0·0816 |
| 2 | −0·0740 | .. | .. | .. |
| 3 | −0·1326 | −0·0674 | −0·0310 | −0·0147 |
| 4 | −0·1076 | .. | .. | .. |
| 5 | −0·2403 | −0·1728 | −0·1746 | −0·1368 |
| 6 | −0·1772 | .. | .. | .. |
| 7 | −0·2500 | −0·0860 | −0·0920 | 0·0545 |
| 8 | −0·2030 | .. | .. | .. |
| 9 | −0·0667 | 0·1370 | 0·1939 | 0·2281 |
| 10 | 0·1585 | .. | .. | .. |
| 11 | −0·0267 | 0·1016 | 0·1016 | 0·1016 |
| 12 | 0·2300 | .. | .. | .. |
| 13 | −0·2621 | −0·1250 | −0·1255 | −0·0820 |
| 14 | −0·0740 | .. | .. | .. |
| 15 | −0·2616 | −0·2156 | −0·1935 | −0·2356 |
| 16 | −0·1296 | .. | .. | .. |
| 17 | −0·1566 | 0·1357 | 0·3584 | 0·4498 |
| 18 | −0·2003 | .. | .. | .. |
| 19 | 0·1980 | 0·2522 | 0·3230 | 0·3331 |
| 20 | +0·1447 | .. | .. | .. |
| 21 | −0·2125 | −0·0103 | −0·0103 | −0·0103 |
| 22 | 0·1919 | .. | .. | .. |
| 23 | −0·2308 | −0·2271 | −0·2401 | −0·2138 |
| 24 | −0·2500 | .. | .. | .. |
| 25 | −0·1609 | −0·0961 | −0·2336 | 0·0682 |
| 26 | −0·3600 | .. | .. | .. |
| 27 | −0·3175 | −0·2062 | −0·1634 | −0·1296 |
| 28 | −0·2482 | .. | .. | .. |
| 29 | 0·1571 | 0·1533 | 0·2620 | 0·2915 |
| 30 | −0·1268 | .. | .. | .. |
| Mean | −0·1125 | −0·0355 | −0·0095 | 0·0415 |
| $m'_2$ | 0·0395 | 0·0235 | 0·0313 | 0·0412 |
| $m_2$ | 0·0278 | 0·0239 | 0·0424 | 0·0424 |

to investigate the distributions under these conditions the values of the serial correlations coefficients for series of fourteen items calculated by Orcutt (1948) were used. The definition of $r_s$ employed in this case had been

$$r_s = \frac{n \sum_{1}^{n-s} (x_i - \bar{x})(x_{i+s} - \bar{x})}{(n - s) \sum_{1}^{n} (x_i - \bar{x})^2}$$

$$\bar{x} = \sum_{i=1}^{n} x_i / n$$

These coefficients were used to calculate $r_{13\cdot 2}$ and $r_{14\cdot 23}$ as shown in Table 5

(*a*) for Kendall's series 1, i.e. $x_{i+2} = 1\cdot 1\, x_{i+1} - 0\cdot 5 x_i + \varepsilon_{i+2}$;
(*b*) for Orcutt's series, i.e. $x_{i+1} = 0\cdot 9 x_i + \varepsilon_{i+1}$;
(*c*) for Tinbergen's economic series.

The latter set of series were included purely for comparative purposes; and as an index of the reliability of $r_{14\cdot 23}$ its denominator has also been included. The standard deviation of these values should be approximately $1/\sqrt{11} = 0\cdot 30$, and to simplify the interpretation of this table all values exceeding $0\cdot 60$ in absolute value have been marked with an asterisk. It can then be seen that the majority of such values may be attributed to the smallness and inaccuracy of the denominator. Even on these crude values it is possible to see the difference between Kendall's and Orcutt's series, and to judge the type of process generating them. When Tinbergen's series are examined, the number of "significant" values of $r_{13\cdot 2}$ are found to be less than the number for Kendall's series, but these values are nearly all negative, and by comparison with the other series an autoregressive scheme of the form $x_{n+2} = a x_{n+1} - 0\cdot 2 x_n + \varepsilon_{n+2}$ is suggested. However, since the series were short and the effect of the definition doubtful, this evidence is of rather a negative quality. To gauge the effect of these difficulties it was decided to employ the definitions

$$r_1 = \frac{\frac{1}{3}(x_1 - \bar{x}_1)(x_2 - \bar{x}_1) + \frac{2}{3}(x_2 - \bar{x}_1)(x_3 - \bar{x}_1) + \ldots + \frac{1}{3}(x_{n-1} - \bar{x}_1)(x_n - \bar{x}_1)}{\frac{1}{6}(x_1 - \bar{x}_1)^2 + \frac{1}{2}(x_2 - \bar{x}_1)^2 + \frac{5}{6}(x_3 - \bar{x}_1)^2 \quad + \ldots + \frac{1}{6}(x_n - \bar{x}_1)^2}$$

$$\bar{x}_1 = (\tfrac{1}{6} x_1 + \tfrac{1}{2} x_2 + \tfrac{5}{6} x_3 + x_4 + \ldots + \tfrac{1}{6} x_n)/(n - 3)$$

$$r_2 = \frac{\frac{1}{2}(x_1 - \bar{x}_2)(x_3 - \bar{x}_2) + (x_2 - \bar{x}_2)(x_4 - \bar{x}_2) + \ldots + \frac{1}{2}(x_{n-2} - \bar{x}_2)(x_n - \bar{x}_2)}{\frac{1}{4}(x_1 - \bar{x}_2)^2 + \frac{1}{2}(x_2 - \bar{x}_2)^2 + \frac{3}{4}(x_3 - \bar{x}_2)^2 \quad + \ldots + \frac{1}{4}(x_n - \bar{x}_2)^2}$$

$$\bar{x}_2 = (\tfrac{1}{4} x_1 + \tfrac{1}{2} x_2 + \tfrac{3}{4} x_3 + x_4 + \ldots + \tfrac{1}{4} x_n)/(n - 3)$$

$$r_3 = \frac{(x_1 - \bar{x}_3)(x_4 - \bar{x}_3) + (x_2 - \bar{x}_3)(x_5 - \bar{x}_3) + \ldots + (x_{n-3} - \bar{x}_3)(x_n - \bar{x}_3)}{\frac{1}{2}(x_1 - \bar{x}_3)^2 + \frac{1}{2}(x_2 - \bar{x}_3)^2 + \frac{1}{2}(x_3 - \bar{x}_3)^2 \quad + \ldots + \frac{1}{2}(x_n - \bar{x}_3)^2}$$

$$\bar{x}_3 = (\tfrac{1}{2} x_1 + \tfrac{1}{2} x_2 + \tfrac{1}{2} x_3 + x_4 + \ldots + \tfrac{1}{2} x_n)/(n - 3) \qquad \ldots \quad (6)$$

As usual, the sums of squares and products may be corrected for the mean afterwards if desired, while two subseries whose three end terms overlap can easily be combined to form a series of effectively double the lengths of the sub-series. Thus for comparison with the values of $r_{14\cdot 23}$ given in Table 5, Kendall's series 1 was split into 32 sub-series of 17 items comprising the 12th–28th, 26th–42nd, . . . 446th–462nd terms. Serial correlation coefficients corrected and uncorrected for the mean were calculated for each sub-series and for the sub-series taken in pairs. These values are given in Table 6. The bias due to the correction for the mean as observed by Orcutt is still marked, but it should also be noted that

$$2 \times 0\cdot7119 - 0\cdot6504 = 0\cdot7734$$
$$2 \times 0\cdot2645 - 0\cdot1452 = 0\cdot3838$$
$$-2 \times 0\cdot0527 + 0\cdot1760 = 0\cdot0706$$

which are the corrected mean values of $r_1$, $r_2$ and $r_3$ agree very well with the values 0·7627, 0·3722 and 0·0685 for the whole series. The next step was to calculate $r_{13\cdot2}$ and $r_{14\cdot23}$ giving the values shown in Table 7. These values only confirm the conclusions reached with the discrete series (*a*) that the uncorrected partial correlation coefficient are biased, (*b*) that the correction for the mean introduces a further bias, (*c*) that the bias tends to reduce the second moment.

TABLE 5.—*Values of* $r_{13\cdot2}$ *and* $r_{14\cdot23}$

| Sub-series | Kendall's series 1 | | | Orcutt's series | | |
|---|---|---|---|---|---|---|
| | $r_{13\cdot2}$ | $r_{14\cdot23}$ | Denominator | $r_{13\cdot2}$ | $r_{14\cdot23}$ | Denominator |
| 1 | −0·24 | −0·17 | 0·57 | 0·09 | −0·32 | 0·30 |
| 2 | −0·36 | −0·05 | 0·82 | −0·27 | −0·08 | 0·88 |
| 3 | −0·34 | −0·03 | 0·74 | 0·24 | −0·31 | 0·63 |
| 4 | −0·85* | −0·43 | 0·03 | 0·01 | −0·16 | 0·76 |
| 5 | −0·68* | −0·32 | 0·20 | −0·12 | 0·15 | 0·83 |
| 6 | −0·40 | −0·42 | 0·42 | −0·19 | −0·45 | 0·90 |
| 7 | −0·53 | 0·25 | 0·47 | −0·91* | 1·49* | 0·00 |
| 8 | −0·36 | −0·56 | 0·39 | −0·09 | 0·09 | 0·33 |
| 9 | −0·80* | 0·64* | 0·16 | 0·18 | −0·02 | 0·74 |
| 10 | −0·64* | −0·29 | 1·25 | 0·03 | −0·04 | 0·40 |
| 11 | −0·36 | −0·92* | 0·26 | −0·21 | −0·13 | 0·39 |
| 12 | −0·30 | −0·03 | 0·28 | −0·44 | 0·07 | 0·62 |
| 13 | −0·25 | 0·48 | 0·35 | −0·48 | 0·08 | 0·12 |
| 14 | −0·64* | −0·21 | 0·20 | −1·20* | −0·75* | −0·04 |
| 15 | −0·74* | −0·23 | 0·17 | −0·33 | −0·33 | 0·57 |
| 16 | −0·58 | −0·33 | 0·12 | −0·28 | 0·03 | 0·21 |
| 17 | −0·91* | 1·78 | 0·02 | −0·29 | −0·54 | 0·03 |
| 18 | −0·61* | −0·41 | 0·26 | 0·02 | −0·19 | 0·88 |
| 19 | −0·60* | −0·01 | 0·31 | 0·24 | −0·17 | 0·32 |
| 20 | −0·28 | −0·12 | 0·38 | 0·07 | −0·26 | 0·21 |
| 21 | −0·64* | −0·59 | 0·06 | 0·07 | 0·28 | 0·96 |
| 22 | −0·53 | 0·15 | 0·25 | 0·47 | −0·15 | 0·31 |
| 23 | −0·83* | 0·26 | 0·08 | 0·11 | 0·09 | 0·24 |
| 24 | −0·47 | −0·06 | 0·17 | 0·22 | −0·10 | 0·85 |
| 25 | −0·45 | −0·67* | 0·37 | −0·28 | 0·10 | 0·10 |
| 26 | −0·80* | 0·42 | 0·05 | −0·50 | −0·46 | 0·04 |
| 27 | −0·30 | −0·23 | 0·09 | −0·15 | −0·79* | 0·32 |
| 28 | −0·59* | −0·25 | 0·06 | 0·02 | −0·34 | 0·29 |
| 29 | −1·60* | 0·79* | 0·01 | −0·41 | 0·10 | 0·21 |
| 30 | −0·65* | 0·45 | 0·31 | 0·13 | −0·49 | 0·61 |
| 31 | −0·48 | −0·40 | 0·05 | −0·38 | −0·12 | 0·77 |
| 32 | −0·11 | 0·27 | 0·67 | −0·53 | −0·18 | 0·45 |
| 33 | −0·97* | 6·86* | 0·00 | −0·16 | −0·05 | 0·28 |
| 34 | −0·69* | 0·49 | 0·05 | −0·25 | −0·03 | 0·19 |
| 35 | .. | .. | .. | −0·28 | −0·33 | 0·85 |
| Mean | −0·576 | 0·180 | .. | −0·167 | −0·123 | .. |

* Absolute value 0·60 or over.

Tinbergen's economic series

| Series | $r_{13\cdot2}$ | $r_{14\cdot23}$ | Denominator | Series | $r_{12\cdot3}$ | $r_{1423}$ | Denominator |
|---|---|---|---|---|---|---|---|
| 1 | −0·72* | 0·35 | 0·03 | 27 | −0·42 | 0·03 | 0·42 |
| 2 | −0·60* | 0·24 | 0·07 | 28 | 0·16 | −0·10 | 0·39 |
| 3 | −0·03 | 0·09 | 0·24 | 29 | −0·13 | −0·02 | 0·44 |
| 4 | −0·03 | −0·41 | 0·45 | 30 | −0·08 | −0·13 | 0·43 |
| 5 | 0·03 | −0·04 | 0·99 | 31 | −0·30 | −0·47 | 0·42 |
| 6 | −0·31 | 0·40 | 0·03 | 32 | −0·35 | −0·15 | 0·54 |
| 7 | −0·26 | 0·00 | 0·72 | 33 | −0·59 | −0·48 | 0·08 |
| 8 | −0·23 | −0·22 | 0·30 | 34 | −0·32 | −0·13 | 0·29 |
| 9 | −0·13 | −0·24 | 0·26 | 35 | 0·16 | 0·22 | 0·90 |
| 10 | −0·33 | −0·04 | 0·09 | 36 | −0·09 | 0·07 | 0·53 |
| 11 | −0·34 | 0·07 | 0·85 | 37 | −0·37 | −0·10 | 0·66 |
| 12 | −0·35 | −0·79* | 0·04 | 38 | −0·51 | 0·22 | 0·28 |
| 13 | −0·31 | −0·18 | 0·63 | 39 | −0·21 | 0·00 | 0·48 |
| 14 | −0·59 | −0·35 | 0·02 | 40 | −0·20 | 0·04 | 0·63 |
| 15 | −0·31 | −0·04 | 0·17 | 41 | −0·29 | 0·07 | 0·57 |
| 16 | −0·20 | −0·29 | 0·17 | 42 | −0·40 | 0·23 | 0·47 |
| 17 | −0·25 | −0·52 | 0·76 | 43 | −0·23 | 0·23 | 0·61 |
| 18 | −0·30 | 0·06 | 0·69 | 44 | −0·34 | 0·19 | 0·85 |
| 19 | −0·37 | 0·15 | 0·43 | 45 | −0·44 | 0·37 | 0·07 |
| 20 | −0·84* | 0·32 | 0·09 | 46 | −0·41 | −0·33 | 0·03 |
| 21 | −0·21 | −0·36 | 0·04 | 47 | −0·43 | −0·35 | 0·08 |
| 22 | −0·42 | −0·05 | 0·29 | 48 | −0·33 | −0·54 | 0·07 |
| 23 | −0·05 | −0·24 | 0·36 | 49 | −0·37 | −0·24 | 0·27 |
| 24 | −0·80* | 0·02 | 0·01 | 50 | −0·41 | −0·14 | 0·30 |
| 25 | −0·10 | −0·87* | 0·89 | 51 | −0·51 | −0·26 | 0·12 |
| 26 | −0·38 | −0·66* | 0·40 | 52 | −0·17 | −0·17 | 0·50 |
| | | | | Mean | −0·308 | −0·106 | .. |

* Absolute value 0·60 or over.

The estimates $m_2$ of the second moment in this case agree fairly well with their expected values, which are 0·0769 and 0·0374 for the series uncorrected for the mean and 0·0833 and 0.0385 for the series corrected for the mean. These variances are reduced by a factor $1 - (0\cdot5)^2 = 0\cdot75$ for $r_{13\cdot2}$ uncorrected for the mean to give values of 0·0577 and 0·0278, which agree fairly well with the observed values. For comparison with Table 5 the values exceeding 1·96 times their standard deviation in absolute value have been indicated with an asterisk. It is clear that in spite of the bias the coefficients $r_{13\cdot2}$ and $r_{14\cdot23}$ indicate fairly well the nature of the series, although there are two significant values for $r_{14\cdot23}$ uncorrected for the mean, in the pairs of sub-series. To show that this was a chance effect and not the first signs of an effect which increased with size of series, the corresponding coefficients were calculated for larger groups of series, as shown in Table 8. None of these values is significant, and in fact the grouping of the series in fours has led to a drop in the variance of $r_{14\cdot23}$ to about two-thirds its expected value. It is interesting to note that the mean value of $r_{14\cdot23}$ obtained for the individual sub-series is not greatly biased, and that $m_2$ is consequently increased. The next step was to calculate $r'_{14\cdot23}$ and $R_{14\cdot23}$ for the pairs of series. For this the definition

$$r'_{14\cdot23} = \frac{\begin{vmatrix} {}_1r_3 & {}_2r_1 & {}_2r_2 \\ {}_1r_2 & 1 & {}_2r_1 \\ {}_1r_1 & {}_1r_1 & 1 \end{vmatrix} + \begin{vmatrix} {}_2r_3 & {}_1r_1 & {}_1r_2 \\ {}_2r_2 & 1 & {}_1r_1 \\ {}_2r_1 & {}_2r_1 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & {}_1r_1 & {}_1r_2 \\ {}_1r_1 & 1 & {}_1r_1 \\ {}_1r_2 & {}_1r_1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & {}_2r_1 & {}_2r_2 \\ {}_2r_1 & 1 & {}_2r_1 \\ {}_2r_2 & {}_2r_1 & 1 \end{vmatrix}} \quad . \quad . \quad . \quad (7)$$

TABLE 6.—*Serial Correlations for Sub-series of Kendall's series* 1

| Sub-series | Uncorrected for mean $r_1$ | $r_2$ | $r_3$ | Corrected for mean $r_1$ | $r_2$ | $r_3$ |
|---|---|---|---|---|---|---|
| 1 | 0·5906 | 0·0893 | 0·1347 | 0·4677 | −0·2138 | −0·2568 |
| 2 | 0·5968 | 0·3429 | 0·4468 | 0·2636 | −0·1957 | 0·0042 |
| 3 | 0·9273 | 0·7746 | 0·5732 | 0·7954 | 0·3764 | −0·1204 |
| 4 | 0·6777 | 0·1324 | −0·1822 | 0·6707 | 0·1150 | −0·2014 |
| 5 | 0·5969 | 0·0090 | −0·4637 | 0·5627 | −0·0711 | −0·5657 |
| 6 | 0·4586 | −0·3854 | −0·5608 | 0·4569 | −0·3935 | −0·5886 |
| 7 | 0·7412 | 0·2913 | −0·2465 | 0·7382 | 0·2850 | −0·2501 |
| 8 | 0·5439 | −0·2494 | −0·5483 | 0·5191 | −0·3224 | −0·6593 |
| 9 | 0·6282 | 0·0880 | −0·3785 | 0·6282 | 0·0877 | −0·3842 |
| 10 | 0·6292 | 0·2228 | −0·2433 | 0·5347 | 0·0305 | −0·5243 |
| 11 | 0·6317 | 0·2608 | 0·0382 | 0·6200 | 0·2393 | 0·0168 |
| 12 | 0·8942 | 0·7714 | 0·7895 | 0·6231 | 0·2088 | 0·3291 |
| 13 | 0·7579 | 0·2953 | −0·0608 | 0·6594 | 0·0096 | −0·4860 |
| 14 | 0·7332 | 0·2989 | −0·0368 | 0·5915 | −0·0608 | −0·5159 |
| 15 | 0·6810 | 0·0901 | −0·3754 | 0·6764 | 0·0756 | −0·4026 |
| 16 | 0·7126 | 0·1325 | −0·3305 | 0·7126 | 0·1325 | −0·3306 |
| 17 | 0·6597 | 0·0623 | −0·3985 | 0·6418 | 0·0069 | −0·5097 |
| 18 | 0·6641 | 0·0771 | −0·4609 | 0·6574 | 0·0584 | −0·4924 |
| 19 | 0·6076 | 0·0863 | −0·0395 | 0·5733 | −0·0022 | −0·1722 |
| 20 | 0·8014 | 0·4613 | 0·1162 | 0·7700 | 0·3778 | −0·0132 |
| 21 | 0·7915 | 0·4758 | 0·3010 | 0·7192 | 0·3079 | 0·1241 |
| 22 | 0·7257 | 0·1714 | −0·3044 | 0·6778 | 0·0278 | −0·5474 |
| 23 | 0·8278 | 0·4649 | 0·0952 | 0·7840 | 0·3338 | −0·1034 |
| 24 | 0·7653 | 0·5384 | 0·3494 | 0·5294 | 0·0733 | −0·3137 |
| 25 | 0·7406 | 0·2462 | −0·1592 | 0·7289 | 0·2134 | −0·2047 |
| 26 | 0·8185 | 0·6100 | 0·4828 | 0·8068 | 0·5843 | 0·4460 |
| 27 | 0·8814 | 0·6458 | 0·4376 | 0·8732 | 0·6212 | 0·3996 |
| 28 | 0·9138 | 0·7284 | 0·4846 | 0·9090 | 0·7128 | 0·4520 |
| 29 | 0·6755 | 0·2524 | 0·2054 | 0·5114 | −0·1488 | −0·3025 |
| 30 | 0·8718 | 0·6075 | 0·2803 | 0·8711 | 0·6058 | 0·2786 |
| 31 | 0·5041 | 0·2373 | 0·2937 | 0·3876 | 0·0423 | 0·0630 |
| 32 | 0·8528 | 0·5298 | 0·2005 | 0·8527 | 0·5296 | 0·2003 |
| Mean | 0·7157 | 0·2926 | 0·0137 | 0·6504 | 0·1452 | −0·1760 |
| 1–2 | 0·5938 | 0·2209 | 0·2996 | 0·3900 | −0·1886 | −0·1235 |
| 3–4 | 0·8458 | 0·5565 | 0·2870 | 0·7872 | 0·3947 | 0·0573 |
| 5–6 | 0·5388 | −0·1624 | −0·5101 | 0·5315 | −0·1767 | −0·5166 |
| 7–8 | 0·6770 | 0·1127 | −0·3508 | 0·6692 | 0·0923 | −0·3771 |
| 9–10 | 0·6286 | 0·1436 | −0·3265 | 0·6118 | 0·1032 | −0·3966 |
| 11–12 | 0·8285 | 0·6408 | 0·5848 | 0·7832 | 0·5474 | 0·4821 |
| 13–14 | 0·7457 | 0·2970 | −0·0498 | 0·7456 | 0·2969 | −0·0499 |
| 15–16 | 0·6995 | 0·1144 | −0·3512 | 0·6985 | 0·1110 | −0·3585 |
| 17–18 | 0·6625 | 0·0713 | −0·4333 | 0·6624 | 0·0706 | −0·4370 |
| 19–20 | 0·7381 | 0·3344 | 0·0583 | 0·7063 | 0·2514 | −0·0692 |
| 21–22 | 0·7481 | 0·2791 | −0·0802 | 0·7481 | 0·2789 | −0·0811 |
| 23–24 | 0·8044 | 0·4916 | 0·1790 | 0·8038 | 0·4902 | 0·1775 |
| 25–26 | 0·7696 | 0·3815 | 0·0784 | 0·7576 | 0·3497 | 0·0315 |
| 27–28 | 0·8972 | 0·6858 | 0·4602 | 0·8907 | 0·6659 | 0·4248 |
| 29–30 | 0·8139 | 0·5011 | 0·2568 | 0·7998 | 0·4624 | 0·1953 |
| 31–32 | 0·8076 | 0·4918 | 0·2127 | 0·8044 | 0·4830 | 0·1975 |
| Mean | 0·7374 | 0·3225 | 0·0197 | 0·7119 | 0·2645 | −0·0527 |

TABLE 7.—*Partial Serial Correlations for Sub-series of Kendall's Series* 1

| | Uncorrected for mean | | Corrected for mean | |
|---|---|---|---|---|
| | $r_{13\cdot 2}$ | $r_{14\cdot 23}$ | $r_{13\cdot 2}$ | $r_{14\cdot 23}$ |
| 1 | −0·3985 | 0·5409 | −0·5537 | 0·2907 |
| 2 | −0·0206 | 0·3888 | −0·2850 | 0·1703 |
| 3 | −0·6087* | −0·2025 | −0·6976* | −0·3912 |
| 4 | −0·6045* | 0·2434 | −0·6086* | 0·2388 |
| 5 | −0·5395 | −0·3286 | −0·5674 | −0·3965 |
| 6 | −0·7544* | 0·2798 | −0·7612* | 0·2278 |
| 7 | −0·5727* | −0·5336 | −0·5712 | −0·5186 |
| 8 | −0·7743* | 0·4025 | −0·8102 | 0·2558 |
| 9 | −0·5065 | −0·3190 | −0·5070 | −0·3307 |
| 10 | −0·2865 | −0·4389 | −0·3577 | −0·5704 |
| 11 | −0·2300 | −0·0336 | −0·2357 | −0·0351 |
| 12 | −0·1407 | 0·6540* | −0·2933 | 0·6146* |
| 13 | −0·6558* | 0·2707 | −0·7523* | −0·0041 |
| 14 | −0·5162 | 0·0276 | −0·6317* | −0·2139 |
| 15 | −0·6968* | −0·0181 | −0·7040* | −0·0495 |
| 16 | −0·7625* | 0·2254 | −0·7625* | 0·2249 |
| 17 | −0·6602* | −0·0978 | −0·6887* | −0·2431 |
| 18 | −0·6511* | −0·3511 | −0·6583* | −0·3832 |
| 19 | −0·4484 | 0·3116 | −0·4929 | 0·2208 |
| 20 | −0·5058 | −0·1320 | −0·5284 | −0·1737 |
| 21 | −0·4034 | 0·2934 | −0·4337 | 0·3024 |
| 22 | −0·7448* | 0·0742 | −0·7984* | −0·2047 |
| 23 | −0·7001* | 0·1275 | −0·7288* | 0·0862 |
| 24 | −0·1141 | −0·0546 | −0·2876 | −0·3202 |
| 25 | −0·6695* | 0·1293 | −0·6782* | −0·1131* |
| 26 | −0·1816 | 0·1300 | −0·1909 | 0·1148 |
| 27 | −0·5874* | 0·3542 | −0·5948* | 0·3512 |
| 28 | −0·6464* | −0·2144 | −0·6532* | −0·2550 |
| 29 | −0·3750 | 0·4800 | −0·5556 | 0·1960 |
| 30 | −0·6357* | −0·2224 | −0·6344* | −0·2169 |
| 31 | −0·0225 | 0·2451 | −0·1270 | 0·1121 |
| 32 | −0·7240* | 0·3007 | −0·7236* | 0·3000 |
| Mean | −0·5043 | 0·0791 | −0·5586 | −0·0152 |
| $m_2'$ | 0·3034 | 0·0943 | 0·3467 | 0·0874 |
| $m_2$ | 0·0506 | 0·0909 | 0·0358 | 0·0872 |
| 1–2 | −0·2034 | 0·4230* | −0·4018* | 0·1917 |
| 3–4 | −0·5582* | 0·1312 | −0·5916* | 0·1152 |
| 5–6 | −0·6379* | −0·0548 | −0·6400* | −0·0529 |
| 7–8 | −0·6381* | −0·1364 | −0·6439* | −0·1477 |
| 9–10 | −0·4159* | −0·3856* | −0·4333* | −0·4368* |
| 11–12 | −0·1455 | 0·3166 | −0·1707 | 0·3034 |
| 13–14 | −0·5836* | 0·1184 | −0·5833* | 0·1174 |
| 15–16 | −0·7341* | 0·0999 | −0·7360* | 0·0895 |
| 17–18 | −0·6552* | −0·2418 | −0·6560* | −0·2499 |
| 19–20 | −0·4622* | 0·1077 | −0·4938* | 0·0378 |
| 21–22 | −0·6371* | 0·2088 | −0·6376* | 0·2074 |
| 23–24 | −0·4405* | −0·1276 | −0·4405* | −0·1263 |
| 25–26 | −0·5169* | 0·0865 | −0·5264* | 0·0841 |
| 27–28 | −0·6110* | 0·1403 | −0·6167* | 0·1187 |
| 29–30 | −0·4779* | 0·1652 | −0·4920* | 0·1356 |
| 31–32 | −0·4613* | 0·0176 | −0·4648* | 0·0083 |
| Mean | −0·5112 | 0·0543 | −0·5330 | 0·0247 |
| $m_2'$ | 0·2853 | 0·0422 | 0·3015 | 0·0339 |
| $m_2$ | 0·0256 | 0·0418 | 0·0186 | 0·0355 |

*Absolute value exceeding 1·96 (standard deviation).

These values are given in Table 9. The estimated variances of $r'_{14\cdot 23}$ and $R_{14\cdot 23}$ are a little high (due to the two significant values), but they agree very well with the estimated variance of $r_{14\cdot 23}$. In this case, when no correction for the mean is applied the mean value of $R_{14\cdot 23}$ is not very accurate, but that this is an isolated occurrence can be seen from Table 10. Hence, to summarize, serial correlation may be tested using partial correlation coefficients (but these are biased to the order $1/n$)

(*a*) by correlations between the terms;
(*b*) by the correction for the mean.

The former difficulty may be overcome by a suitable modification $r'_{1n\cdot 23}$ . . . of the definition, while if the latter difficulty is also to be overcome a second form $R_{1n\cdot 23}$ . . . must be used.

### *Testing the Correlations between Series*

The difficulty in testing cross-correlations between series of observations arises when *both* series are serially correlated, since otherwise the ordinary correlation coefficient may be used. The effectiveness of the ordinary correlation and partial correlation coefficients in cases where the sets of observations are inter-dependent show that these might also be used for testing cross-correlations. We will adopt the notation $r(x_p y_q \mid x_{p-1} \ldots y_{q-1} \ldots)$ for the estimated partial correlation between $x_p$ and $y_q$ when $x_{p-1} \ldots y_{q-1} \ldots$ have been eliminated while $\rho(x_p\, y_q \mid x_{p-1} \ldots y_{q-1} \ldots)$ will be corresponding true correlation.

As a first step in testing the correlation between any series it will be necessary to test that at least one of the series can be represented autoregressively. For this the above tests can be applied to determine the order of autoregressive schemes which might be used to represent the series. It is obvious that a general moving average of infinite extent cannot be completely represented by an autoregressive scheme any more than a general function can be represented by a finite polynomial, but for most practical purposes the autoregressive scheme and finite polynomial will suffice. It is worth noting that in this analogy the moving average of finite extent corresponds to the reciprocal of a finite polynomial, and can therefore frequently be represented by an autoregressive scheme.

TABLE 8.—*Values of* $r_{14\cdot 23}$ *(uncorrected for the mean) for Groups of Sub-series of Kendall's Series* 1

| Series | | | | |
|---|---|---|---|---|
| 1–4 | 0·2147 | | | |
| | .. | 0·1178 | | |
| 5–8 | −0·0713 | | | |
| | .. | .. | 0·0635 | |
| 9–12 | −0·0236 | | | |
| | .. | 0·0024 | | |
| 13–16 | 0·0994 | | | |
| | .. | .. | .. | 0·0780 |
| 17–20 | −0·0387 | | | |
| | .. | 0·0150 | | |
| 21–24 | 0·0406 | | | |
| | .. | .. | 0·0797 | |
| 25–28 | 0·1386 | | | |
| | .. | 0·1064 | | |
| 29–32 | 0·0827 | | | |
| Mean | 0·0553 | 0·0604 | 0·0716 | 0·0780 |
| $m_2'$ | 0·0114 | 0·0064 | 0·0052 | 0·0061 |
| $m_2$ | 0·0095 | 0·0036 | 0·0001 | .. |

TABLE 9.—*Adjusted Partial Serial Correlations for Sub-series of Kendall's Series* 1

| | Uncorrected for mean | | Corrected for mean | |
|---|---|---|---|---|
| | $r'_{14\cdot23}$ | $R_{14\cdot23}$ | $r'_{14\cdot23}$ | $R_{14\cdot23}$ |
| 1–2 | 0·4145* | 0·3812* | 0·1076 | 0·1529 |
| 3–4 | 0·1166 | 0·2420 | 0·0293 | 0·3066 |
| 5–6 | −0·0123 | −0·0852 | −0·2002 | −0·0214 |
| 7–8 | 0·0095 | −0·2072 | −0·0450 | −0·1640 |
| 9–10 | −0·4034* | −0·3922* | −0·4766* | −0·4230* |
| 11–12 | 0·2309 | 0·3230 | 0·2822 | 0·3170 |
| 13–14 | 0·1182 | 0·0876 | −0·1404 | 0·3438 |
| 15–16 | 0·0830 | 0·0962 | 0·0665 | 0·0913 |
| 17–18 | −0·2252 | −0·2592 | −0·3145 | −0·1866 |
| 19–20 | 0·2000 | 0·1256 | 0·1270 | 0·0520 |
| 21–22 | 0·1363 | 0·2338 | 0·0631 | 0·3660 |
| 23–24 | −0·1455 | −0·2916 | −0·2479 | −0·1356 |
| 25–26 | 0·0066 | 0·0434 | −0·0091 | 0·0542 |
| 27–28 | 0·1139 | 0·2107 | 0·0969 | 0·1893 |
| 29–30 | 0·3027 | 0·2016 | 0·2857 | 0·2816 |
| 31–32 | 0·1694 | −0·2377 | 0·1760 | −0·1894 |
| Mean | 0·0697 | 0·0295 | −0·0125 | 0·0647 |
| $m_2'$ | 0·0429 | 0·0560 | 0·0430 | 0·0559 |
| $m_2$ | 0·0380 | 0·0588 | 0·0457 | 0·0552 |

TABLE 10.—*Values of* $R_{14\cdot23}$ (*uncorrected for the mean*) *for Groups of Sub-series of Kendall's Series* 1

| Series | | | | |
|---|---|---|---|---|
| 1–4 | 0·1523 | | | |
| | .. | 0·1639 | | |
| 5–8 | −0·0470 | | | |
| | .. | .. | 0·0669 | |
| 9–12 | −0·0127 | | | |
| | .. | −0·0331 | | |
| 13–16 | 0·0896 | | | |
| | .. | .. | .. | 0·0844 |
| 17–20 | −0·0104 | | | |
| | .. | 0·0290 | | |
| 21–24 | 0·0406 | | | |
| | .. | .. | 0·0987 | |
| 25–28 | 0·1638 | | | |
| | .. | 0·1022 | | |
| 29–32 | 0·0740 | | | |
| Mean | 0·0563 | 0·0655 | 0·0828 | 0·0844 |
| $m_2'$ | 0·0085 | 0·0098 | 0·0071 | 0·0071 |
| $m_2$ | 0·0060 | 0·0074 | 0·0005 | .. |

When the orders of the suitable schemes have been determined then the next step is to calculate partial correlation coefficients. This may be done by the usual definitions, except that serial correlations will be calculated from such definitions as (1), (2), (3) or (6) according to the order of the autoregressive scheme. For example, if $x$ is connected in a Markoff scheme, then

$$r(x_2\, y_2 \mid x_1) = \frac{r(x_2\, y_2) - r(x_1\, x_2)\, r(x_1\, y_2)}{\sqrt{\{1 - r^2(x_1\, x_2)\}\,\{1 - r^2(x_1\, y_2)\}}} \quad . \quad . \quad . \quad (8)$$

where $r(x_1\, x_2)$ is defined by (1), might be used. Again there will be two sources of bias:

(*a*) the high correlation between such high estimates as $r(x_2\, y_2)$ and $r(x_1\, y_1)$;
(*b*) the correction for the mean.

Neither of these will be as important as previously since they primarily affect serial correlations. For example, (8) will not be affected by bias (*a*), and will be affected by bias (*b*) only if $\rho(x_1 y_2) \neq 0$ (in which case $\rho(x_2 y_2) \neq 0$ and (8) tests whether this correlation arises partly or wholly indirectly). In any case it would seem likely that any bias might be removed by the use of coefficients $R(x_p y_q \mid x_{p-1} \ldots y_{q-1} \ldots)$ analogous to $R_{1n \cdot 23}$, and the major task is to determine the validity of this large-sample approximation when applied to small samples. For this purpose two artificial series similar to (4) were used, corresponding to the cases $\rho = 0 \cdot 6$ (series $x$) and $\rho = 0 \cdot 4$ (series $y$). From a formula given by Bartlett (1935), the variance of a cross-correlation between these series should be $\frac{1 + 0 \cdot 4 \times 0 \cdot 6}{1 - 0 \cdot 4 \times 0 \cdot 6} = 1 \cdot 63$ times the variance of the cross-correlation between two random series of the same length. These two series were each split into thirty-two sub-series, consisting of the 1st–20th, 20th–39th, 39th–58th . . . terms, and the values of $r(x_2 y_2)$, $r(x_2 y_2 \mid x_1)$, $r(x_2 y_2 \mid y_1)$, $r(x_2 y_2 \mid x_1 y_1)$, $r(x_2 y_1 \mid x_1)$, $r(x_1 y_2 \mid y_1)$ were worked out for each series. These values are given in Table 11, and those exceeding twice the standard deviation are, as usual, marked with an asterisk.

**TABLE 11.**—*Cross-correlations and Partial Correlations for Series of Twenty Items*

| Series | $r(x_2 y_2)$ | $r(x_2 y_2 \mid x_1)$ | $r(x_2 y_2 \mid y_1)$ | $r(x_2 y_2 \mid x_1 y_1)$ | $r(x_2 y_1 \mid x_1)$ | $r(x_1 y_2 \mid y_1)$ |
|---|---|---|---|---|---|---|
| 1 | −0·684* | −0·637* | −0·484* | −0·527* | −0·484* | −0·050 |
| 2 | −0·157 | 0·385 | −0·257 | 0·564* | −0·308 | −0·049* |
| 3 | 0·684* | 0·322 | 0·571 | 0·365 | −0·149 | 0·485 |
| 4 | 0·053 | −0·149 | 0·088 | −0·122 | −0·088 | 0·391 |
| 5 | 0·474* | 0·278 | 0·420 | 0·286 | −0·015 | 0·366 |
| 6 | 0·053 | 0·122 | 0·018 | 0·042 | 0·122 | −0·018 |
| 7 | 0·263 | −0·015 | 0·108 | −0·099 | 0·205 | 0·366 |
| 8 | 0·158 | 0·167 | 0·150 | 0·194 | 0·077 | 0·012 |
| 9 | −0·158 | −0·248 | 0·150 | −0·246 | −0·028 | 0·045 |
| 10 | −0·158 | 0·032 | −0·156 | −0·031 | −0·077 | −0·258 |
| 11 | 0·053 | −0·121 | 0·011 | −0·075 | −0·191 | −0·150 |
| 12 | −0·474* | −0·579* | −0·472* | −0·581* | −0·026 | 0·191 |
| 13 | −0·053 | −0·121 | −0·088 | −0·214 | 0·121 | 0·268 |
| 14 | 0·368 | 0·308 | 0·338 | 0·317 | −0·141 | 0·217 |
| 15 | −0·158 | 0·032 | −0·150 | −0·035 | −0·216 | −0·258 |
| 16 | −0·474* | −0·184 | −0·489 | −0·139 | −0·177 | −0·545* |
| 17 | 0·263 | 0·016 | 0·216 | 0·492 | 0·032 | 0·268 |
| 18 | −0·263 | −0·016 | −0·141 | −0·131 | −0·215 | −0·344 |
| 19 | 0·158 | 0·026 | 0·231 | −0·115 | −0·121 | 0·344 |
| 20 | 0·053 | 0·121 | 0·045 | 0·116 | 0·036 | −0·168 |
| 21 | −0·368 | −0·069 | 0·287 | −0·036 | −0·327 | −0·420 |
| 22 | −0·579 | −0·397 | −0·482* | −0·392 | −0·208 | −0·327 |
| 23 | −0·158 | 0·032 | −0·191 | 0·014 | 0·122 | −0·263 |
| 24 | −0·158 | −0·055 | −0·031 | −0·015 | −0·094 | −0·039 |
| 25 | −0·053 | −0·018 | −0·045 | −0·016 | −0·018 | −0·045 |
| 26 | −0·684* | −0·081 | −0·484* | 0·051 | −0·102 | −0·604* |
| 27 | −0·263 | −0·016 | −0·244 | −0·018 | 0·369 | −0·352 |
| 28 | 0·684* | 0·593* | 0·454* | 0·426 | 0·457* | 0·184 |
| 29 | −0·263 | −0·205 | −0·233 | −0·202 | −0·094 | −0·454* |
| 30 | −0·053 | −0·040 | −0·102 | −0·078 | 0·040 | −0·102 |
| 31 | −0·053 | 0·122 | −0·045 | −0·137 | −0·077 | 0·080 |
| 32 | 0·579 | −0·102 | 0·505* | 0·000 | −0·187 | 0·792* |
| Mean | −0·043 | −0·016 | −0·016 | −0·011 | −0·055 | −0·014 |
| $m'_2$ | 0·127 | 0·061 | 0·087 | 0·067 | 0·038 | 0·104 |
| $m_2$ | 0·129 | 0·063 | 0·090 | 0·069 | 0·036 | 0·107 |

The desired reduction in the variance has been obtained, and the observed variances agree fairly well with the theoretical values, which are $1 \cdot 63/19 = 0 \cdot 086$, $1/18 = 0 \cdot 056$ and $1/17 = 0 \cdot 059$, according to whether none, one or two variables are eliminated. The cases where only $y_1$ is eliminated are hardly satisfactory, and it would seem advisable to eliminate both variables. However, to ensure that this agreement was maintained and improved with larger series, these series were combined in pairs and the correlations recalculated. These values are given in Table 12. The agreement is again fairly good, but while the elimination of one variable leads to a general improvement, especially for $x_1$, it is obviously desirable to eliminate both $x_1$ and $y_1$. Thus it appears that a fairly good test of significance is provided by the partial correlation coefficients, and that this test improves as the length of series is increased

In this paper the prime object has been to demonstrate numerically that the problems occurring in time-series analysis are capable of treatment by the usual methods, possibly with slight modifications. Much still remains to be done, especially in the mathematical investigation of these approximations, but it is hoped that with the aid of such investigations as this the way will at least be partially cleared for further and more accurate analysis.

TABLE 12.—*Cross-correlations and Partial Correlations for Pairs of Series*

| Series | $r(x_2\, y_2)$ | $r(x_2\, y_2 \mid x_1)$ | $r(x_2\, y_2 \mid y_1)$ | $r(x_2\, y_2 \mid x_1\, y_1)$ | $r(x_2\, y_1 \mid x_1)$ | $r(x_1\, y_2 \mid y_1)$ |
|---|---|---|---|---|---|---|
| 1–2 | −0·421* | −0·425* | −0·229 | −0·324* | −0·348 | −0·014 |
| 3–4 | 0·368* | 0·055 | 0·310 | 0·074 | −0·031 | 0·420* |
| 5–6 | 0·263 | 0·172 | 0·213 | 0·185 | 0·033 | 0·111 |
| 7–8 | 0·210 | 0·059 | 0·146 | 0·043 | 0·059 | 0·185 |
| 9–10 | −0·158 | 0·055 | −0·151 | −0·053 | −0·055 | −0·151 |
| 11–12 | −0·210 | −0·226 | −0·213 | −0·216 | −0·085 | −0·034 |
| 13–14 | 0·158 | 0·033 | 0·098 | 0·010 | 0·145 | 0·171 |
| 15–16 | −0·316* | −0·070 | −0·316 | −0·078 | −0·070 | −0·376* |
| 17–18 | 0·000 | 0·000 | 0·043 | −0·100 | −0·122 | 0·043 |
| 19–20 | 0·105 | 0·072 | 0·113 | 0·094 | −0·042 | 0·072 |
| 21–22 | −0·474* | −0·233 | −0·381* | −0·218 | −0·268 | −0·381* |
| 23–24 | −0·158 | −0·004 | −0·126 | −0·012 | −0·016 | −0·161 |
| 25–26 | −0·368* | −0·028 | −0·274 | −0·028 | −0·012 | −0·314 |
| 27–28 | 0·210 | 0·221 | 0·044 | 0·071 | 0·344* | −0·026 |
| 29–30 | −0·158 | −0·024 | −0·129 | −0·029 | −0·059 | −0·243 |
| 31–32 | 0·263 | 0·095 | 0·217 | 0·118 | −0·032 | 0·411* |
| Mean | −0·043 | −0·016 | −0·040 | −0·029 | −0·035 | −0·018 |
| $m_2'$ | 0·072 | 0·025 | 0·044 | 0·018 | 0·023 | 0·058 |
| $m_2$ | 0·075 | 0·026 | 0·046 | 0·018 | 0·024 | 0·061 |
| Whole series | −0·043 | −0·042 | −0·026 | −0·035 | −0·026 | 0·001 |

## *References*

Bartlett, M. S. (1935), "Some aspects of the time-correlation problem in regard to tests of significance," *Suppl. J. Roy. Stat. Soc.*, **98**, 536.

Dixon, W. J. (1944), " Further contributions to the problem of serial correlation," *Annals of Mathematical Statistics*, **15**, 119–144.

Kendall, M. G. (1946), *Contributions to the Study of Oscillatory Time-Series.* Cambridge University Press.

Orcutt, G. H. (1948), "A Study of the autoregressive nature of the time series used for Tinbergen's model of the economic system of the United States 1919–1932," *Suppl. J. Roy. Stat. Soc.*, **10**, 1.

Quenouille, M. H. (1947), "A large sample test for the goodness of fit of autoregressive schemes," *J. Roy. Stat. Soc.*, **110**, 123–129.

—— (1948a), "The joint distribution of serial correlation coefficients," *Annals of Mathematical Statistics*, **19**.

—— (1948b), "Note on a certain class of functions occurring in serial correlations." *Unpublished paper.*

—— (1948c), "Some results in the testing of serial correlation coefficients," *Biometrika*, **35**.

—— (1948d), "On a method of trend elimination." *Biometrika* (in press).

Rubin, H. (1945), "On the distribution of the serial correlation coefficient," *Annals of Mathematical Statistics*, **16**, 211–215.

## APPENDIX

It is necessary to show that the forms $r'_{1n\cdot 23}$ . . . can be used instead of $r_{1n\cdot 23}$ when no correction is made for the mean.

The method can be demonstrated using $r_{13\cdot 2}$ and $r'_{13\cdot 2}$. We have

$$\delta r_{13\cdot 2} = \frac{1}{1 - r_1^2}\,[\delta r_2 - 2\,r_1(1 - r_{13\cdot 2})\,\delta r_1]$$

$$\delta r'_{13\cdot 2} = \frac{1}{2 - {}_1r_1^2 - {}_2r_1^2}\,[\delta_1 r_2 + \delta_2 r_2 - 2({}_2r_1 - r'_{13\cdot 2}\,{}_1r_1)\,\delta_1 r_1 - 2({}_1r_1 - r'_{13\cdot 2}\,{}_2r_1)\,\delta_2 r_1].$$

Hence

$$\begin{aligned} \text{var}\, r'_{13\cdot 2} &= \mathrm{E}(\delta r'_{13\cdot 2})^2 \\ &= \tfrac{1}{2}[\mathrm{E}(\delta_1 r_{13\cdot 2})^2 + \mathrm{E}(\delta_2 r_{13\cdot 2})^2] \\ &= \text{var}\, r_{13\cdot 2} \end{aligned}$$

A similar proof can be derived for $r'_{1n\cdot 23}$ . . . as given by (5) by differentiating the determinants column by column.