Measure Theory

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Contents

1 Jordan Measure **5**
- 1.1 Jordan Measure . . . 6
- 1.2 Riemann Integration . . . 9

2 Lebesgue Measure **10**
- 2.1 Lebesgue Outer Measure . . . 11
- 2.2 Measurable Sets . . . 15
- 2.3 Measures . . . 22
- 2.4 Nonmeasurable Sets . . . 27
- 2.5 Exercises . . . 29

3 Lebesgue Integration **42**
- 3.1 Simple Functions . . . 42
- 3.2 Bounded Measurable Functions over Finite Measure . . . 44
- 3.3 Positive Measurable Functions . . . 47
- 3.4 Lebesgue Integral for Signed Functions . . . 52
- 3.5 Uniformly Integrable Families . . . 54
- 3.6 Riemann vs Lebesgue Integrability . . . 59
- 3.7 Exercises . . . 61

4 Convergence **74**
- 4.1 Exercises . . . 75

5 Differentiation **77**
- 5.1 Monotone Functions . . . 78
- 5.2 Functions of Bounded Variation . . . 83
- 5.3 Absolutely Continuous Functions . . . 86
- 5.4 Fundamental Theorem of Calculus . . . 87
- 5.5 Convex Functions . . . 90
- 5.6 Exercises . . . 93

6 Lp Spaces **94**
- 6.1 Nov 3 . . . 96
- 6.2 Exercises . . . 100

7 Measure **102**
- 7.1 Signed Measures . . . 106
- 7.2 Carathéodory Construction of Measurable Sets . . . 107
- 7.3 Premeasures . . . 109
- 7.4 Product Measure . . . 110
- 7.5 Stieltjes Construction of Measure . . . 111

8 Integration **114**
8.1 Radon-Nikodym Theorem 114
8.2 Fubini's and Tonelli's Theorem 115
8.3 Exercises 115

In math, we are first taught to solve simple equations like $x^2 - 2x + 4 = 0$ for a certain *number* x, but in real world applications, we must now solve for some *function* f satisfying an equation

$$\mathcal{L}(f) = 0 \tag{1}$$

where $\mathcal{L}$ is some operator on functions. This is usually difficult, and many times a solution does not exist. However, we can find approximate solutions, say

$$\begin{aligned} \mathcal{L}(f) &= 1/2 \\ \mathcal{L}(f) &= 1/4 \\ \mathcal{L}(f) &= 1/8 \\ \ldots &= \ldots \end{aligned}$$

and approximate the solution as

$$f = \lim_{n \to \infty} f_n \tag{2}$$

Given that this limit exists, we can usually define f pointwise using a point-wise limit

$$f(x) = \lim_{n \to \infty} f_n(x) \text{ for all } x \tag{3}$$

but the function in total is very ugly and not Riemann integrable. The classic non-Riemann integrable function is the

$$f(x) = \chi_{\mathbb{R} \setminus \mathbb{Q}}(x) := \begin{cases} 1 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \\ 0 & \text{if} x \in \mathbb{Q} \end{cases} \tag{4}$$

Since $\mathbb{Q}$ is countable, we can enumerate $\mathbb{Q} = \{q_n\}_{n=1}^{\infty}$ and define the sequence of functions

$$f_n = 1 - \chi_{\{q_j\}_{j=1}^n}(x) \tag{5}$$

that start off with the constant function 1 and then "removes" points in $\mathbb{Q}$, setting their image to 0. It is clear that since we are removing points, every function in the sequence has an integral (from 0 to 1) of 1, and therefore the integral of f should also be 1.

$$\int_0^1 f_n \, dx = 1 \implies \int_0^1 f \, dx = \int_0^1 \lim_{n \to \infty} f_n \, dx = \lim_{n \to \infty} \int_0^1 f_n \, dx \tag{6}$$

What is crucial for mathematicians to work with is the capability to take the limit from inside the integral to outside the integral. The problem is that f is not a Riemann integral function.

Definition 0.1 (Riemann Integrable Function)

Given a function $f : [0,1] \longrightarrow \mathbb{R}$, let us consider some partition of $[0,1]$ into intervals $P = \{I_0, I_1, \ldots, I_N\}$, then, for each $I \in P$, we can take the supremum $M_I = \sup_{x \in I} f(x)$ and infimum $m_I = \inf_{x \in I} f(x)$ and bound f by the upper and lower Riemann sums.

$$\sum_{I \in P} m_I |I| \leq \int_0^1 f \, dx \leq \sum_{I \in P} M_I |I| \tag{7}$$

where $|I|$ is the length of interval I. If we take *all* possible partitions, the bound should still hold.

$$m = \sup_P \Big\{ \sum_{I \in P} m_I |I| \Big\} \leq \int_0^1 f \, dx \leq \inf_P \Big\{ \sum_{I \in P} M_I |I| \Big\} = M \tag{8}$$

and if the lower bound is equal to the upper bound $m = M$, then the integral is this number and f is considered Riemann integrable.

Now since $\mathbb{Q}$ is dense in $\mathbb{R}$, for every interval I in every partition P will have $m_I = 0$ and $M_I = 1$ for the Riemann function, meaning that the lower bound will always be 0 and the upper bound will always be 1. So, $\int_0^1 \chi_{\mathbb{R}\setminus\mathbb{Q}}(x)$ can take on any value in $[0, 1]$, which isn't helpful. The fact that we can't integrate this really simple function is a problem. For nice functions, we can partition it so that the base of each Riemann rectangle is a nice interval, while the base of the Riemann function is an "interval with holes." The problem really boils down to measuring what the "length" of this set is. So the problem with the Riemann integral isn't the integral itself, but the fact that we can't give a meaningful size to the set $\mathbb{R} \setminus \mathbb{Q}$. Now mathematicians in the 19th century thought that as long as we solve this problem, we should be good to go, but Banach and Tarski proved that there exists sets that cannot be measured with their famous paradox, which says that you can take any set P, disassemble it into a finite set of pieces, and rearrange it (under isometry and translations) so that it has a different size than the original P. So, we have to exclude some sets that are not measurable. The collection of sets that we *can* measure is called the σ-algebra.

Exercise 0.1 (Tao 1.2.2)

Give an example of a sequence of uniformly bounded, Riemann integrable functions $f_n : [0, 1] \to \mathbf{R}$ for $n = 1, 2, \ldots$ that converge pointwise to a bounded function $f : [0, 1] \to \mathbf{R}$ that is *not* Riemann integrable. What happens if we replace pointwise convergence with uniform convergence?

Solution. The only non-Riemann integrable function that we know of is the Riemann function, so let's a try to construct such a limit with this. Enumerate the rationals $\mathbb{Q} = \{q_k\}_{k\in\mathbb{N}}$. Now, consider the function

$$f(x) = \begin{cases} 1 & \text{if } x \in \{q_1, \ldots, q_n\} \\ 0 & \text{if else} \end{cases} \tag{9}$$

Then, $f_n - > \chi_{\mathbb{Q}}$ pointwise.

1 Jordan Measure

We would like to generalize the concepts of size, which are specified as length/area/volume depending on the dimension of the space we live in. The most intuitive notion of size are those of segments, rectangles, and boxes, and these are the simplest forms of sets that we will work with.

Such that for "simple" sets A where we know what the area is, the outer measure of A should coincide with the area of A. Let's first start by defining what a "simple" set is.

Definition 1.1 (Box)

An **box** $E \subset \mathbb{R}^n$ is defined recursively as follows.

1. An **interval** $I \subset \mathbb{R}$ is one of the sets $(a, b), [a, b), (a, b], [a, b]$ for $a, b \in \mathbb{R}$.
2. For $n > 1$, an box $E \subset \mathbb{R}^n$ is $E = I_1 \times \ldots \times I_n$ for intervals $I_1, \ldots, I_k$.

Definition 1.2 (Size of a Box)

The **size** of a box $E = I_1 \times \ldots \times I_n \subset \mathbb{R}^n$ is defined as follows.

1. The **length** of an interval I is $\ell(I) \coloneqq b - a$.
2. The **size** of E is $|E| \coloneqq \prod_{i=1}^n (b_i - a_i)$.

Now we can combine these to get an elementary set.

Definition 1.3 (Elementary Set)

An **elementary set** is a set $E \subset \mathbb{R}^n$ that is a finite union of boxes.

We would like to have some nice properties of these elementary sets.

Lemma 1.1 (Boolean Closure of Elementary Sets)

Given two elementary sets $E, F \subset \mathbb{R}^n$,

1. $E \cup F$ is elementary.
2. $E \cap F$ is elementary.
3. $E \setminus F$ is elementary.

Proof.

Lemma 1.2 (Disjoint Finite Union of Elementary Sets)

Let $E \subset \mathbb{R}^n$ be an elementary set. Then, E can be expressed as a finite union of *disjoint* boxes.

Proof.

Definition 1.4 (Elementary Measure)

The **elementary measure** of an elementary set $E \subset \mathbb{R}^n$ is defined as the sum of the sizes of each box in a partition:

$$m(E) \coloneqq \sum_{i=1}^{k} |B_k| \tag{10}$$

We claim that this sum is invariant depending on the partition, and hence, well defined.

Proof.

This elementary measure clearly extends the notion of size, since

$$m(B) = s(B) \tag{11}$$

whenever B is elementary. Furthermore, we can deduce finite additivity and nonnegativity. These are really trivial but we state them as theorems to establish a pattern.

Lemma 1.3 (Fundamental Properties of Elementary Measure)

The elementary measure satisfies the following.

1. *Nonnegativity.* For any elementary set E, $m(E) \geq 0$.
2. *Finite Additivity* Given $E_1, \ldots, E_n$ are disjoint elementary sets,

$$m(E_1 \cup \ldots \cup E_k) = m(E_1) + \ldots + m(E_k) \tag{12}$$

3. *Monotonicity.* Given elementary sets $E \subset F$, we have

$$m(E) \leq m(F) \tag{13}$$

4. *Finite Subadditivity.* Let $E_1, \ldots, E_n$ be any elementary sets (not necessarily disjoint). Then

$$m(E_1 \cup \ldots \cup E_n) \leq m(E_1) + \ldots + m(E_n) \tag{14}$$

5. *Translation Invariance.* For $x \in \mathbb{R}^n$ and elementary set $E \subset \mathbb{R}^n$,

$$m(E) = m(x + E) \tag{15}$$

It turns out that these properties uniquely determine an elementary measure.

Theorem 1.4 (Uniqueness of Elementary Measure)

1.1 Jordan Measure

Now, we define the outer and inner measure, which are defined for *all* subsets of $\mathbb{R}^n$.

Definition 1.5 (Jordan Outer, Inner Measure)

Let $E \subset \mathbb{R}^n$.

1. The **Jordan inner measure** is defined

$$m_*(E) \coloneqq \sup_{A \subset E, A \text{ elementary}} m(A) \tag{16}$$

2. The **Jordan outer measure** is defined

$$m_*(E) \coloneqq \inf_{B \supset E, B \text{ elementary}} m(B) \tag{17}$$

Note that if E is unbounded, then there exists no elementary set that is a superset of E, and so the infimum of such a set is $+\infty$ conventionally.

This is where our first big leap in construction comes in. Before, we have defined elementary boxes, which are pretty much guaranteed to have a well-defined elementary measure. Here, we *begin* with a function on the power set of $\mathbb{R}^n$, and then we will filter the power set to those subsets that behave nicely.

Definition 1.6 (Jordan Measurable Set, Jordan Measure)

Let $E \subset \mathbb{R}^n$ be bounded.[a] If $m_*(E) = m^*(E)$, then E is said to be **Jordan-measurable**, and we define

$$m(E) := m_*(E) = m^*(E) \tag{18}$$

as the **Jordan measure** of E.

[a] Note that by convention, we don't consider unbounded sets to be Jordan measurable.

Note first of all that Jordan measure is a generalization of elementary measure, since if E is elementary, then we can set $A = E = B$ to achieve these bounds. Furthermore, by monotonicity, we can never get past them, and will always have

$$m(A) \leq m(E) \leq m(B) \tag{19}$$

where m is the elementary measure. So, we can overload the notation and just write m to denote elementary and Jordan measure. Second, note that the Jordan measure shares the same properties.

Lemma 1.5 (Boolean Closure of Jordan Measurable Sets)

Given two Jordan-measurable sets $E, F \subset \mathbb{R}^n$,

1. $E \cup F$ is elementary.
2. $E \cap F$ is elementary.
3. $E \setminus F$ is elementary.

Proof.

The properties of the Jordan measure parallel those of elementary measure.

Theorem 1.6 (Fundamental Properties of Jordan Measure)

The elementary measure satisfies the following.

1. *Nonnegativity.* For any elementary set E, $m(E) \geq 0$.
2. *Finite Additivity* Given $E_1, \ldots, E_n$ are disjoint elementary sets,
$$m(E_1 \cup \ldots \cup E_k) = m(E_1) + \ldots + m(E_k) \tag{20}$$
3. *Monotonicity.* Given elementary sets $E \subset F$, we have
$$m(E) \leq m(F) \tag{21}$$
4. *Finite Subadditivity.* Let $E_1, \ldots, E_n$ be any elementary sets (not necessarily disjoint). Then
$$m(E_1 \cup \ldots \cup E_n) \leq m(E_1) + \ldots + m(E_n) \tag{22}$$
5. *Translation Invariance.* For $x \in \mathbb{R}^n$ and elementary set $E \subset \mathbb{R}^n$,
$$m(E) = m(x + E) \tag{23}$$

Proof.

Jordan measurable sets are sets that are "almost" elementary, but a few sets already come to mind that are not Jordan measurable.

Example 1.1 (Rationals in Unit Interval)

$\mathbb{Q} \cap [0,1]$ is not Jordan measurable.

It may be hard to tell directly whether something is Jordan measurable. This is where the "Cauchy criterion" of Jordan measurable sets comes in.

Theorem 1.7 (Equivalent Notions)

E is Jordan measurable iff $\forall \epsilon > 0$, $\exists$ elementary sets $A \subset E \subset B$ s.t. $m(B \setminus A) < \epsilon$.

Proof.

Note how the previous lemma is very similar to this theorem on Riemann integrability.

Example 1.2 (Regions Under Graphs are Jordan Measurable)

Example 1.3 (Triangle is Jordan Measurable)

Example 1.4 (Compact Convex Polytopes are Jordan Measurable)

Example 1.5 (Open and Closed Balls in Euclidean Space are Jordan Measurable)

Example 1.6 (Subsets of Jordan Null Sets have 0 Jordan Measure)

Theorem 1.8 (Uniqueness of Jordan Measure)

Theorem 1.9 (Topological Approximations of Jordan Measurable Sets)

Let $E \subset \mathbb{R}^n$ be a bounded set. Then,
1. E and its closure $\overline{E}$ have the same Jordan outer measure.
2. E and its interior E° have the same Jordan outer measure.
3. E is Jordan measurable iff the topological boundary ∂E has Jordan outer measure 0.

Proof.

Example 1.7 (Bullet Riddled Square)

Show that both sets have a Jordan inner measure 0 and Jordan outer measure 1.

1. $[0,1]^2 \setminus \mathbb{Q}^2$.
2. $[0,1]^2 \cap \mathbb{Q}^2$.

Finally, a little teaser theorem.

Theorem 1.10 (Caratheodory Property)

Let $E \subset \mathbb{R}^n$ be a bounded set, and $F \subset \mathbb{R}^n$ be an elementary set. Show that

$$m^*(E) = m^*(E \cap F) + m^*(E \setminus F) \tag{24}$$

where m^* is the Jordan outer measure.

Proof.

1.2 Riemann Integration

Now, we connect the Riemann integral to the Jordan measure.

Theorem 1.11 (Jordan Measure with Riemann Integral)

If $E \subset [a, b]$ is Jordan measurable, then the indicator function $\mathbb{1}_E$ is Riemann integrable, and

$$\int_a^b \mathbb{1}_E \, dx = m(E) \tag{25}$$

2 Lebesgue Measure

We will see that

1. Borel $\implies$ measurable.
2. Countable $\implies$ measure 0 $\implies$ measurable.

But none of the reverse implications follow. For example, the Cantor set is measure 0 but uncountable. There is a subset of Cantor set that is measurable but not Borel. The Vitali set is not even measurable at all.

We have seen that there are some common sets that are not Jordan measurable, but a bigger problem is that countable unions and intersections aren't.

Example 2.1 (Countable Union/Intersection of Jordan Measurable Sets are Not J.M.)

We show a few counterexamples.

1. *Countable Union.* Take $\mathbb{N} = \{n\}_{n=1}^{\infty}$. Each point n has Jordan measure 0, but their union is unbounded so it isn't Jordan measurable.
2. *Bounded Countable Union.* Maybe we can fix this by considering bounded unions. But consider $E = \mathbb{Q} \cap [0, 1]$. By density of rationals,
$$m_*(E) = 0 \neq 1 = m^*(E) \tag{26}$$
3. *Intersection.* Consider the Cantor set C, which is bounded, but again
$$m_*(C) = 0 \neq 1 = m^*(C) \tag{27}$$

This motivates the definition of a σ-algebra.

Definition 2.1 (σ-Algebra)

A **σ-algebra** on a set X is a collection of subsets of X satisfying the following:

1. *Contains Empty and Full Set.* $\emptyset, X \in \mathcal{A}$.
2. *Closed Under Countable Unions.* $A_n \in \mathcal{A}$ for all $n \in \mathbb{N}$ implies $\bigcup_n A_n \in \mathcal{A}$.
3. *Closed Under Complements.* $A \in \mathcal{A} \implies A^c \in \mathcal{A}$.
4. *Closed Under Countable Intersections.*[a] $A_n \in \mathcal{A}$ for all $n \in \mathbb{N}$ implies $\bigcap_n A_n \in \mathcal{A}$.

[a] This is actually a consequence of the previous properties. We can utilize the fact that $\cap_{k=1}^{\infty} A_k = X \setminus \cup_{k=1}^{\infty} A_k^c$

In some applications, we can just ignore some of these sets as "pathological." But for Riemann integrability, we are integrating over Jordan measurable sets (closed and bounded intervals). It turns out that it is really important that we can *at least* guarantee that countable unions and intersections of Jordan measurable sets are measurable. If we don't, then not even uniform convergence—an extremely strong property—can preserve continuity, differentiability, and integrability of a sequence of functions.

A σ-algebra is similar to the topology τ of topological space. Both $\mathcal{A}$ and τ require $\emptyset$ and X to be in it. The three differences are that (i) τ does not allow compelmentation, (ii) τ allows any (even uncountable) union of sets (condition is strengthened), and (iii) τ allows only finite intersection of sets (condition is weakened). Now in order to construct σ-algebras, the following theorems are useful since they allow us to construct σ-algebras from other σ-algebras. It turns out that the intersection of σ-algebras is a σ-algebra, but not for unions.

Theorem 2.1

Let $\{\mathcal{A}_k\}$ be a family of σ-algebras of X. Then, $\cap \mathcal{A}_k$ is also a σ-algebra of X.

Proof. Clearly, $\emptyset, X$ is in $\cap \mathcal{A}_k$. To prove complementation,

$$A \in \bigcap \mathcal{A}_k \implies A \in \mathcal{A}_k \; \forall k \implies A^c \in A_k \; \forall k \implies A^c \in \bigcap \mathcal{A}_k \tag{28}$$

To prove countable union, let $\{A_j\}_{j \in J}$ be some countable family of subsets in $\cap \mathcal{A}_k$. Then,

$$A_j \in \bigcap \mathcal{A}_k \; \forall j \in J \implies A_j \in \mathcal{A}_k \; \forall k \forall j \implies \bigcup A_j \in \mathcal{A}_k \; \forall k \implies \bigcup A_j \in \bigcap \mathcal{A}_k \tag{29}$$

This allows us to easily prove the following theorem, which just establishes the existence of σ-algebras.

Theorem 2.2

Let $F \subset 2^X$ be a collection of subsets of X. Then there exists a unique smallest σ-algebra $\sigma(F)$ containing F. $\sigma(F)$ is called the σ-algebra **generated** by F.

Proof. Let us denote $\mathcal{M}$ as the set of all possible σ-algebras $\mathcal{B}$ of X. $\mathcal{M}$ is nonempty since it contains 2^X. Then, the intersection

$$\bigcap_{\mathcal{B} \in \mathcal{M}} \mathcal{B} \tag{30}$$

is the unique smallest σ-algebra.

2.1 Lebesgue Outer Measure

Therefore, we would like the collection of our measurable sets to be a σ-algebra. To do this, we tinker around with the definition of the Jordan measure. Note that by definition, the Jordan outer measure can be equivalently written as

$$m^*(S) \coloneqq \inf\{m(E) \mid S \subset E, E \text{ elementary}\} \tag{31}$$

$$= \inf \left\{ \sum_{i=1}^k |B_i| \;\middle|\; S \subset \bigcup_{i=1}^k B_i, B_i \text{ boxes}, k \in \mathbb{N} \right\} \tag{32}$$

Note that the *finite* number of boxes k are allowed to vary over all naturals. To define the Lebesgue measure, we change the finite to countable.

Definition 2.2 (Lebesgue Outer Measure)

Given any set $E \subset \mathbb{R}^n$, the **Lebesgue outer measure** is a map

$$m^* : 2^{\mathbb{R}^n} \to [0, +\infty], \qquad m^*(E) = \inf \left\{ \sum_{k=1}^\infty |B_k| \;\middle|\; E \subset \bigcup_{k=1}^\infty B_k \right\} \tag{33}$$

It's a hard definition, but a natural one, since we're taking all these boxes and trying to make them as snug as possible to define the outer measure of an arbitrary set. We first first check that this is indeed a generalization of Jordan measure, in the sense that if E is Jordan measurable, then its Lebesgue outer measure is the same as its Jordan measure.

Theorem 2.3 (Lebesgue Outer Measure Coincides with Interval Length)

m^* satisfies the property that for any interval $I \subset \mathbb{R}$, $m^*(I) = |I|$.

Proof. Let's first consider the case when I is closed. Let $I = [a, b]$. Then, we know from definition that

$$m^*(I) \coloneqq \inf \left\{ \sum_{k=1}^\infty |I_k| \;\middle|\; I \subset \bigcup_{k=1}^\infty I_k \right\} \tag{34}$$

where $I_k = [a_n, b_n]$. We wish to show that the above quantity equals $b - a$.

1. $m^*(I) \leq b - a$. This is pretty easy since we can just set the cover to consist of the single interval I, and since $m^*(I)$ must be the infimum of it, then we must have $m^*(I) \leq b - a$. A technicality is that we must strictly use countable covers, but in this case, we can just fix $\epsilon > 0$ and see

$$I_1 = [a, b], \qquad I_k = [b - \frac{\epsilon}{2^k}, b + \frac{\epsilon}{2^k}] \tag{35}$$

   In this case the sums of lengths of all $I_2, \ldots$ is $\epsilon/2 < \epsilon$, and so

$$I_k \leq b - a + \epsilon \qquad \forall \epsilon > 0 \tag{36}$$

2. Proving $m^*(I) \geq b - a$ is harder. In here, we use the "ϵ of room" trick. Take any $\epsilon > 0$. Then there exists a cover $\{I_k\}_k$ s.t.

$$m^*(I) = \epsilon \geq \sum_{k=1}^\infty |I_k| = \sum_{k=1}^\infty b_k - a_k \tag{37}$$

   All we wish to show that the RHS $\geq b - a$, but we can't really manipulate the infinite sum. This is where we use the fact that $[a, b]$ is compact[a], and so we can take a finite subcover $\{I_{k_j}\}_{j=1}^n$. Therefore,

$$m^*(I) + \epsilon \geq \sum_{j=1}^n b_{k_j} - a_{k_j} \tag{38}$$

   Now we can rearrange this: set the a_{k_j}'s to be increasing, and for simplicity let's reindex them to a_j, b_j. Then, it must be the case that $a_1 < a$.
   (a) Consider (a_1, b_1). If $b_1 > b$, we are done.
   (b) Otherwise, $b_1 \in (a_2, b_2)$. If $b_2 > b$, then

$$b_2 - a_2 + b_1 - a_1 \geq b_2 - a_1 > b - a \tag{39}$$

   (c) If not, then we keep going until we get to (a_n, b_n). If $b_n > b$, then

$$b_n - a_n + b_{n-1} - a_{n-1} + \ldots + b_1 - a_1 \geq b_n - a_1 > b - a \tag{40}$$

[a] since it's closed and bounded

The proof may be a bit unfamiliar since we have used two tricks.

1. *Geometric Sequence of ϵ Trick.* To account for countable collections, we set ϵ to be decreasing geometrically so that the series converges to ϵ.

$$\epsilon = \sum_{n=1}^\infty \frac{\epsilon}{2^n} \tag{41}$$

2. *ϵ of Room Trick.* This trick is used often when you need an opposite inequality that isn't as obvious,

and it can only be used for inequalities involving a supremum or infimum.

$$\inf\{S\} \geq x \tag{42}$$

By using the definition of sup/inf as the *least* upper/lower bound, we can scooch over by an ϵ to find an element in $s \in S$ that does satisfy the inequality

$$\inf\{S\} + \epsilon > s \geq x \tag{43}$$

and since ϵ was arbitrary, we are done.

It is clear that the Lebesgue outer measure is always less than the Jordan outer measure.

$$m^*(E) \leq m^{(J),*}(E) \tag{44}$$

When are these different?

Example 2.2 (Lebesgue Outer Measure Much Smaller than Jordan Outer Measure)

Consider $\mathbb{Q}$.

1. The Jordan outer measure is $+\infty$ since it is unbounded.
2. However, any countable set of $\mathbb{R}$ has Lebesgue outer measure 0. Just enumerate $E = \{x_1, \ldots\}$. Then, we set $I_k = \left(x_k - \frac{\epsilon}{2^k}, x_k + \frac{\epsilon}{2^k}\right)$. Then,

$$\sum_{k=1}^\infty \ell(I_k) = \epsilon \tag{45}$$

What about the inner measure? It turns out that we don't get much if we replace the finite to countable in the Jordan inner measure.

Lemma 2.4 (Axiomatic Properties of Lebesgue Outer Measure)

The Lebesgue outer measure satisfies the following.

1. *Null Empty Set.* $m^*(\emptyset) = 0$.
2. *Monotonicity.* Given sets $E \subset F \subset \mathbb{R}^n$, we have

$$m^*(E) \leq m^*(F) \tag{46}$$

3. *Countable Subadditivity.* For any countable collection of subsets $\{E_k\}$ of $\mathbb{R}^d$,

$$m^*\left(\bigcup_n E_n\right) \leq \sum_n m^*(E_n) \tag{47}$$

Proof. The first two properties are trivial. For the third, let's start by writing out the definition for the outer measure for each E_n.

$$m^*(E_n) \coloneqq \inf\left\{\sum_{k=1}^\infty |B_k^{(n)}| \;\middle|\; E_n \subset \bigcup_{k=1}^\infty B_k^{(n)}, B_k^{(n)} \text{ boxes}\right\} \tag{48}$$

Somehow, we want to sum these values over all n and prove that this is greater than the measure of the union. The first realization should be that for a fixed cover $\{B_k^{(n)}\}_k$ of E_n, the collection

$$\{B_k^{(n)}\}_{n,k} \text{ covers } \bigcup_n E_n \tag{49}$$

This gives us a clue as to comparing the collection of covers of each E_n, with the cover of $\cup E_n$. There is no straightforward way to do this,[a] so we want to try and *fix* these collections. We can do this with the ϵ of room trick. For each E_n, we can find a cover $\{B_k^{(n)}\}_k$ s.t.

$$m^*(E_n) + \frac{\epsilon}{2^n} \geq \sum_{k=1}^\infty |B_k^{(n)}| \tag{50}$$

Then, we can take the infinite sum.

$$\sum_{n=1}^\infty \left(m^*(E_n) + \frac{\epsilon}{2^n} \right) \geq \sum_{n=1}^\infty \sum_{k=1}^\infty |B_k^{(n)}| \geq m^* \left(\bigcup_n E_n \right) \tag{51}$$

where the final inequality follows from the fact that $\{B_k^{(n)}\}_{n,k}$ is a cover of $\cup_n E_n$, and so it must be greater than the infimum of all possible covers. All monotonic series converge in $[0, +\infty]$, so we can "take out" the ϵ term to get

$$\epsilon + \sum_{n=1}^\infty m^*(E_n) \geq m^* \left(\bigcup_n E_n \right) \tag{52}$$

which holds for all $\epsilon > 0$, and so we are done.

[a] On one side, we have the sum of a countable number of infimums of some sets, and on the other hand, we have the infimum of the unions of all of these sets.

Theorem 2.5 (Translation Invariance of Lebesgue Outer Measure)

m^* is translation invariant. That is, for any $E \subset \mathbb{R}^n$,

$$m^*(E) = m^*(E + a), \qquad E + a \coloneqq \{x + a \in \mathbb{R}^n \mid a \in E\} \tag{53}$$

Proof. This is straightforward and requires no tricks. Note that

$$\{B_k\} \text{ is a countable cover of } E \iff \{B_k + a\} \text{ is a countable cover of } E + a \tag{54}$$

It is also clear that

$$|B| = |B + a| \tag{55}$$

for any box $B \subset \mathbb{R}^n$, so the sets of sizes that we are taking the infimum over is exactly the same between the two.

The final property claims that we can always drop an outer-measure 0 set and it won't affect the outer measure of the original set. Therefore, when talking about measurability of intervals, we don't have to worry about endpoints, or even whether it is missing a countable number of elements from it!

Lemma 2.6 (Sets of Outer Measure 0 Doesn't Affect Outer Measure)

Suppose $m^*(E) = 0$ and A is any set.

1. Adding a set of outer measure 0 doesn't affect the outer measure.

$$m^*(A \cup E) = m^*(A) \tag{56}$$

2. Excising a set of outer measure 0 doesn't affect the outer measure.

$$m^*(A \setminus E) = m^*(A) \tag{57}$$

Proof. Listed.

1. We have

$$m^*(A \cup E) = \underbrace{m^*\big((A \cup E) \cap E\big)}_{=0} + m^* \underbrace{\big((A \cup E) \cap E^c\big)}_{\subset A} \leq m^*(A) \leq m^*(A) \tag{58}$$

   But $A \cup E \supset A$, so $m^*(A \cup E) = m^*(A)$.
2. Note that $m^*(A \setminus E) \leq m^*(A)$ by monotonicity. Also, by subadditivity

$$m^*(A) \leq m^*(A \setminus E) + \underbrace{m^*(E)}_{=0} = m^*(A \setminus E) \tag{59}$$

Lemma 2.7 (Finite Additivity for Outer Measure)

Let A and B be bounded sets for which there is an $\alpha > 0$ such that $|a - b| \geq \alpha$ for all $a \in A$, $b \in B$. Prove that $m^*(A \cup B) = m^*(A) + m^*(B)$.[a]

[a]This is a slightly weaker version of Tao Lemma 1.2.5.

Proof. The ideas is that if we have some cover $\{I_k\}$ of $A \cup B$, then either I_k only intersects A (and thus contributes to covering A only), only intersects B, or intersects both A and B (and thus also covers some non-essential interval of length α). Since A, B are bounded, $m^*(A), m^*(B) < +\infty$. We wish to show that

$$m^*(A) + m^*(B) \leq m^*(A \cup B) \iff m^*(A) + m^*(B) < m^*(A \cup B) + \epsilon \quad \forall \epsilon > 0 \tag{60}$$

By setting $\epsilon = \frac{\alpha}{2}$, we can force the covering to consist of intervals that can intersect only A and B, but not both. If such an interval I did, then it would contain a set of measure $\geq \alpha$, so

$$m(I) \geq \alpha \implies \sum_{k=1}^{\infty} \ell(I_k) \geq m(A \cup B) + \alpha \tag{61}$$

Therefore, we can partition all such covers to cover A and B, respectively, meaning

$$m^*(A \cup B) + \frac{\alpha}{2} > \sum \ell(I_k) + \sum \ell(I_k) \geq m^*(A) + m^*(B) \tag{62}$$

Therefore we can invoke this in special cases where one set may be compact and another may be closed.

2.2 Measurable Sets

The next step is to take the outer measure and define *Lebesgue measurable* sets. The problem is that—unlike the Jordan measure—we don't have the inner measure to work with. This turns out to be not much of a problem, since through *Littlewood's first principle*[1], we can define measurability as being well-approximated by an open set.

Definition 2.3 (Lebesgue Measurable Set)

A set $E \subset \mathbb{R}^d$ is **Lebesgue measurable** if for every set $A \subset \mathbb{R}^n$,

$$m^*(A) = m^*(A \cap E) + m^*(A \cap E^c) \tag{63}$$

[1]One of the major themes in measure theory, where we say that measurable sets are well-approximated by open and closed sets.

which is known as *Carathéodory's criterion*. Colloquially, no matter how nasty a subset A is, E should be nice enough that we can cut E into two pieces C and D.

Note that due to countable subadditivity, we are guaranteed to have $\leq$. Therefore, it suffices to prove only for $\geq$. The sets with which this inequality is strict is not measurable, and the measurable sets specifically satisfy equality for countable sets. So what kind of sets are measurable?

Example 2.3 (Outer Measure 0 Sets are Lebesgue Measurable)

For any outer measure m^* on X, $E \subset X$ with $m^*(E) = 0$ implies that E is m^*-measurable. Take any A. Then $(A \cap E) \subset E$ and $(A \cap E^c) \subset A$. So by monotonicity,

$$m^*(A \cap E) + m^*(A \cap E^c) \leq m^*(E) + m^*(A) = m^*(A) \tag{64}$$

and this by definition means that E is measurable.

We could continue with more examples, but our main priority is to show that this family of Lebesgue measurable sets is indeed a σ-algebra, and it covers much more than Jordan measurable sets. The path to prove that countable unions are measurable is a long one, and we'll a lot of lemmas.

Lemma 2.8 (Complements of Measurable Sets are Measurable)

If E is measurable, then so is E^c.

Proof. The definition is symmetric in both E and E^c.

Using unions and complements, we can prove that intersections and set differences of measurable sets are measurable.

Lemma 2.9 (Finite Intersections of Measurable Sets are Measurable)

If $E_1, \ldots, E_n$ is measurable, then so is $\cap_{k=1}^n E_k$.

Proof. By induction, it suffices to prove for $n = 2$. We have

$$E_1 \cap E_2 = (E_1^c \cup E_2^c)^c \tag{65}$$

Lemma 2.10 (Set Differences of Measurable Sets are Measurable)

If E_1, E_2 is measurable, then so is $E_1 \setminus E_2$.

Proof. We have

$$E_1 \setminus E_2 = E_1 \cap E_2^c \tag{66}$$

Lemma 2.11 (Excision Property)

If $E \subset \mathbb{R}^n$ is measurable with $m^*(E) < +\infty$, and $E \subset F$ for arbitrary set F, then

$$m^*(F \setminus E) = m^*(F) - m^*(E) \tag{67}$$

Proof. By measurability of E, we can see

$$m^*(F) = m^*(F \cap E) + m^*(F \cap E^c) \tag{68}$$
$$= m^*(E) + m^*(F \setminus E) \tag{69}$$

This excision property combined with the fact that outer measure 0 sets are always measurable gives us the property of *completeness*.[2] That is, given measurable sets $A \subset B \subset C$ with $m^*(A) = m^*(C)$, B must be measurable. This basically says that if you a set that is squeezed in between two measurable sets of equal measure, then the middle set will also be measurable.

Lemma 2.12 (Finite Unions are Measurable)

A finite union of measurable sets is measurable.

Proof. This proof is basically applying set theory laws, and there's not much more to that. It suffices to prove for E_1, E_2, and the rest follows by induction. Fix any A. Then

$$m^*(A) = m^*(A \cap E_1) + m^*(A \cap E_1^c) \tag{70}$$
$$= m^*(A \cap E_1) + m^*\big(A \cap E_1^c \cap E_2\big) + m^*\big((A \cap E_1^c) \cap E_2^c\big) \tag{71}$$

Now we apply the identity $(A \cap E_1^c) \cap E_2^c = A \cap (E_1 \cup E_2)^c$, so the third term can be changed

$$= m^*(A \cap E_1) + m^*\big((A \cap E_1^c) \cap E_2\big) + m^*\big(A \cap (E_1 \cup E_2)^c\big) \tag{72}$$

Then we apply the identity $(A \cap E_1) \cup (A \cap E_1^c \cap E_2) = A \cap (E_1 \cup E_2)$, so we can apply finite subadditivity on the first two terms to get

$$\geq m^*(A \cap (E_1 \cup E_2)) + m^*\big(A \cap (E_1 \cup E_2)^c\big) \tag{73}$$

which proves that $E_1 \cup E_2$ is measurable.

So we have proved that the set of all measurable sets is closed under finite unions. By definition it works for finite intersections. This makes it into an *algebra*, but we want to upgrade this to a σ-algebra by proving closure under *countable* unions. We first prove a lemma.

Lemma 2.13 (Finite Additivity of Outer Measure for Measurable Sets)

Suppose A is any set, $\{E_k\}_{k=1}^n$ disjoint and measurable.[a] Then,

$$m^*\bigg(A \cap \Big(\bigcup_{k=1}^n E_k\Big)\bigg) = \sum_{k=1}^n m^*(A \cap E_k) \tag{74}$$

In particular,

$$m^*\Big(\bigcup_{k=1}^n E_k\Big) = \sum_{k=1}^n m^*(E_k) \tag{75}$$

[a]Note that while outer measure isn't finitely additive for arbitrary sets, it is finitely additive for *measurable* sets.

[2]Nothing to do with completeness in the sense of real numbers or metric spaces.

Proof. It should be clear that we prove by induction, and intuitively, this disjointness should be essential for canceling out some measure terms. $n = 1$ is trivial. This takes a bit of fiddling around with where we should start, but if we just look at the LHS, we can try and use Carathéodory, by setting the arbitrary set to be $B = A \cap (\cup_k E_k)$ and writing $m^*(B) = m^*(B \cap E_n) + m^*(B \cap E_n^c)$.

$$m^*\bigg(A \cap \Big(\bigcup_{k=1}^n E_k\Big)\bigg) = m^*\Bigg(\bigg(A \cap \Big(\bigcup_{k=1}^n E_k\Big)\bigg) \cap E_n\Bigg) + m^*\Bigg(\bigg(A \cap \Big(\bigcup_{k=1}^n E_k\Big)\bigg) \cap E_n^c\Bigg) \tag{76}$$

$$= m^*(A \cap E_n) + m^*\bigg(A \cap \Big(\bigcup_{k=1}^{n-1} E_k\Big)\bigg) \tag{77}$$

$$= \sum_{k=1}^n m^*(A \cap E_k) \tag{78}$$

But note that by disjointness, we have

$$\bigg(A \cap \Big(\bigcup_{k=1}^n E_k\Big)\bigg) \cap E_n = A \cap E_n, \qquad \bigg(A \cap \Big(\bigcup_{k=1}^n E_k\Big)\bigg) \cap E_n^c = A \cap \Big(\bigcup_{k=1}^{n-1} E_k\Big) \tag{79}$$

by the induction hypothesis. ■ Here is a wrong proof that does an incorrect form of induction. I first assumed that we can just work with a family of two sets E_1, E_2, so I started deriving something like this:

$$m^*(A) = m^*(A \cap (E_1 \cup E_2)) + m^*(A \cap (E_1 \cup E_2)^c) \tag{80}$$

$$= m^*((A \cap E_1) \cup (A \cap E_2) + \underbrace{m^*((A \setminus E_1) \cap (A \setminus E_2))}_{\geq 0} \tag{81}$$

$$\geq m^*((A \cap E_1) \cup (A \cap E_2)) \tag{82}$$

Note that the E_k's being disjoint means that they are *pairwise* disjoint, and so it is *not* sufficient to prove for only E_1, E_2. So don't do this.

Theorem 2.14 (Countable Unions are Measurable)

Suppose $E_1, E_2, \ldots$ are a countable collection of measurable sets. Then, $E = \cup_{j=1}^\infty E_j$ is measurable.

Proof. They key is to look at disjoint sets. WLOG, one can assume E_j are disjoint. Indeed, we can define new sets

$$E_n' \coloneqq E_n \setminus \bigg(\bigcup_{j=1}^{n-1} E_j\bigg) \tag{83}$$

that are measurable, with $\cup E_n' = \cup E_n$. Now, fix any set A. Define sets $F_n = \cup_{j=1}^n E_j$. Then, $m^*(A) = m^*(A \cap F_n) + m^*(A \cap F_n^c)$. Then, $F_n^c \supset E^c \implies m^*(A \cap F_n^c) \geq m^*(A \cap E^c)$. Since we have established from the previous lemma that outer measure on disjoint measurable sets satisfies finite additivity, we can write

$$m^*(A \cap F_n) = m^*\bigg(\bigcup_{j=1}^n (A \cap E_j)\bigg) = \sum_{j=1}^n m^*(A \cap E_j) \tag{84}$$

Then,

$$m^*(A) \geq \sum_{j=1}^n m^*(A \cap E_j) + m^*(A \cap E^c) \tag{85}$$

for every n, therefore also with ∞. But by countable subadditivity of the outer measure,

$$\sum_{j=1}^{\infty} m^*(A \cap E_j) \geq m^*(A \cap E) \tag{86}$$

If follows that $m^*(A) \geq m^*(A \cap E) + m^*(A \cap E^c)$.

Proof. In class. We first prove this lemma, which is more general (arbitrary intersections than finite?). Suppose A is any set, E_j disjoint and measurable. Then,

$$\mu^*\bigg(A \cap \Big(\bigcup_{j=1}^n E_j\Big)\bigg) = \sum_{j=1}^n \mu^*(A \cap E_j) \tag{87}$$

By induction, $n = 1$ is true. Then,

$$\mu^*\bigg(A \cap \Big(\bigcup_{j=1}^n E_j\Big)\bigg) = \mu^*\Bigg(\bigg(A \cap \Big(\bigcup_{j=1}^n E_j\Big)\bigg) \cap E_n\Bigg) + \mu^*\Bigg(\bigg(A \cap \Big(\bigcup_{j=1}^n E_j\Big)\bigg) \cap E_n^c\Bigg) \tag{88}$$

$$= \mu^*(A \cap E_n) + \mu^*\bigg(A \cap \Big(\bigcup_{j=1}^{n-1} E_j\Big)\bigg) \tag{89}$$

$$= \sum_{j=1}^n \mu^*(A \cap E_j) \tag{90}$$

by the induction hypothesis.
They key is to look at disjoint sets. WLOG, one can assume E_j are disjoint. Indeed, we can define new sets

$$E_n' \coloneqq E_n \setminus \Big(\bigcup_{j=1}^{n-1} E_j\Big) \tag{91}$$

that are measurable, with $\cup E_n' = \cup E_n$. Now, fix any set A. Define sets $F_n = \cup_{j=1}^n E_j$. Then, $\mu^*(A) = \mu^*(A \cap F_n) + \mu^*(A \cap F_n^c)$. Then, $F_n^c \supset E^c \implies \mu^*(A \cap F_n^c) \geq \mu^*(A \cap E^c)$. Through the previous lemma, we have

$$\mu^*(A \cap F_n) = \mu^*\bigg(\bigcup_{j=1}^n (A \cap E_j)\bigg) = \sum_{j=1}^n \mu^*(A \cap E_j) \tag{92}$$

Then,

$$\mu^*(A) \geq \sum_{j=1}^n \mu^*(A \cap E_j) + \mu^*(A \cap E^c) \tag{93}$$

for every n, therefore also with ∞. But

$$\sum_{j=1}^{\infty} \mu^*(A \cap E_j) \geq \mu^*(A \cap E) \tag{94}$$

If follows that $\mu^*(A) \geq \mu^*(A \cap E) + \mu^*(A \cap E^c)$.

Corollary 2.15 (Measurable Sets form a σ-Algebra)

The set of all Lebesgue measurable sets of $\mathbb{R}^n$ form a σ-algebra.

Proof. Listed.
1. $\emptyset$ is measurable since it has outer measure 0.
2. $\mathbb{R}^n$ is measurable since for any set $A \subset \mathbb{R}^n$,

$$m^*(A) = m^*(A \cap \mathbb{R}^n) + m^*(A \cap (\mathbb{R}^n)^c) = m^*(A) \tag{95}$$

3. We proved closure under complements.
4. We just proved closure under countable union.

Now that we have established that Lebesgue measurable sets form a σ-algebra, let's give some examples.

Theorem 2.16 (Intervals are Measurable)

All intervals $I \subset \mathbb{R}$ are measurable.

Proof. It suffices to prove that every raya $(a, +\infty)$ is measurable, since every interval is a countable union/intersection of such rays. We simply prove using Carathéodory,[a] so we wish to show that for any set $A \subset \mathbb{R}$,

$$m^*(A) \geq m^*(A \cap (a, +\infty)) + m^*(A \cap (-\infty, a]) = m^*(A') + m^*(A'') \tag{96}$$

Again, the fact that we have to prove this nontrivial part of the inequality reminds us of using the ϵ of room trick. Let us $\{I_k\}_{k=1}^\infty$ is a countable cover of A s.t.

$$m^*(A) + \epsilon > \sum_{k=1}^\infty \ell(I_k) \tag{97}$$

We can take this cover and create two smaller covers, covering A' and A''.

$$\{I'_k \coloneqq I_k \cap A'\}_k, \qquad \{I''_k \coloneqq I_k \cap A''\}_k \tag{98}$$

Since these are valid covers, by definition it must be bounded below by the outer measures.

$$\sum_{k=1}^\infty \ell(I'_k) \geq m^*(A'), \qquad \sum_{k=1}^\infty \ell(I''_k) \geq m^*(A'') \tag{99}$$

Now all that remains is to connect the sums together. For each k, we have $\ell(I_k) = \ell(I'_k) + \ell(I''_k)$, and since both series converge in $[0, +\infty]$, we can indeed sum them up as limits to get

$$\sum_{k=1}^\infty \ell(I'_k) + \sum_{k=1}^\infty \ell(I''_k) = \sum_{k=1}^\infty \ell(I'_k) + \ell(I''_k) = \sum_{k=1}^\infty \ell(I_k) \tag{100}$$

Putting the bounds together gives

$$m^*(A') + m^*(A'') \leq \sum_{k=1}^\infty \ell(I'_k) + \sum_{k=1}^\infty \ell(I''_k) = \sum_{k=1}^\infty \ell(I_k) \leq m^*(A) + \epsilon \tag{101}$$

Since this is true for every $\epsilon > 0$, we are done.

[a] WLOG $a \notin A$ (since we can take the point out without affecting outer measure). TBD: Do we need this assumption really?

Note again that this ϵ of room trick is used so that we can *fix* some open cover that acts as a middle ground between the inequalities that we are trying to prove. Then as we let $\epsilon \to 0$, we are done.

Example 2.4 (Cantor Set is Measurable)

Let us define

$$C_0 = [0,1], \qquad C_1 = [0,1/3] \cup [2/3,1], \ldots \tag{102}$$

Basically, we take our every middle third of each subinterval. So C_k is the union of 2^k intervals of size 3^{-k}. Note that $C_k \subset C_{k-1}$. Now define the **Cantor set** as

$$C \coloneqq \bigcap_{k=1}^{\infty} C_k \tag{103}$$

The Cantor set is measurable since it is a countable intersection of closed sets, which are measurable.

Theorem 2.17 (Translations of Sets are Measurable)

If $E \subset \mathbb{R}^n$ is measurable, then for any $a \in \mathbb{R}^n$, $E + a \coloneqq \{x + a \in \mathbb{R}^n \mid x \in E\}$ is measurable.

Proof. We again use Carathéodory. Let $A \subset \mathbb{R}^n$ by any set. Then by translation invariance of outer measure, we have

$$\begin{aligned} m^*(A) &= m^*(A - a) && (104)\\ &= m^*((A-a) \cap E) + m^*((A-a) \cap E^c) && (105)\\ &= m^*(A \cap (E+a)) + m^*(A \cap (E+x)^c) && (106) \end{aligned}$$

and so $E + a$ is measurable.

Now we talk about how Lebesgue measurable sets interact with other common sets, such as open and closed sets in a topology.

Theorem 2.18 (Regularity Definitions of Lebesgue Measurable Set)

The following are equivalent in $\mathbb{R}^d$.

1. E is Lebesgue measurable.
2. *Outer Approximately Open.* $\forall \epsilon > 0$, $\exists$ open set $O \supset E$ s.t. $m(O \setminus E) \leq \epsilon$.
3. *Inner Approximately Closed.* $\forall \epsilon > 0$, $\exists$ closed set $F \subset E$ s.t. $m^*(E \setminus F) < \epsilon$.
4. *Outer Exactly G_δ.* $\exists$ a G_δ set G s.t. $E \subset G$ and $m^*(G \setminus E) = 0$.
5. *Inner Exactly F_σ.* $\exists$ a F_σ set F s.t. $F \subset E$ and $m^*(E \setminus F) = 0$.

Proof. Listed.

1. (2) $\implies$ (1). Then for every $k \in \mathbb{N}$, we can find $O_k \supset E$ s.t. $m^*(O_k \setminus E) \leq 1/k$. Define the G_δ set $G = \cap_{k=1}^{\infty} O_k$. Then, $(G \setminus E) \subset (O_k \setminus E)$ for all $k \implies m^*(G \setminus E) \leq 1/k$ for all k. Therefore $m^*(G \setminus E) = 0$, and $E = G \setminus (G \setminus E)$ is measurable.
2. (1) $\implies$ (2). Assume $m^*(E) < +\infty$. Find a cover $\{I_k\}_{k=1}^{\infty}$ s.t. $\sum_{k=1}^{\infty} \ell(I_k) \leq m^*(E) + \epsilon$. Call $O = \cup_k I_k$. Since E is measurable, $m^*(O \setminus E) = m^*(O) - m^*(E) \leq \sum_{k=1}^{\infty} \ell(I_k) - m^*(E) \leq \epsilon$
3. (1) $\iff$ (3). Straightforward with argument above.
4. (1) $\iff$ (4). Generally, we use the fact that E measurable iff E^c measurable. Find $O \supset E^c$ open, with $m^*(O \setminus E^c) \leq \epsilon$. Then $F = O^c$ is closed, $F \subset E$, and $m^*(E \setminus F) \leq \epsilon$.
5. (1) $\iff$ (5). Same argument as (1) $\iff$ (4).

Depending on the context, it is helpful to use one definition over another when proving measurability. Just remember that the Carathéodory definition is the most general, since it doesn't even assume a topology on a space, and that is the definition that we will use by default.

This next theorem is a different flavor of Littlewood's first principle. It tells us that we can use a finite union of intervals that "symmetrically" approximates measurable sets on the real line.

Theorem 2.19 (Measurable Sets can be Approximated by Borel Sets)

Suppose E is measurable, with $m^*(E) < +\infty$. For every $\epsilon > 0$, there exists a finite number of intervals $\{I_k\}_{k=1}^n$ s.t. if $O = \cup_{k=1}^n I_k$, then

$$m^*(O \setminus E) + m^*(E \setminus O) < \epsilon \tag{107}$$

Proof. In here, we use the outer approximately open definition of measurable sets. Since every open set can be written as a countable union of open intervals[a], we can find a collection of open intervals $\{I_k\}_{k=1}^\infty$ s.t.

$$E \subset U \coloneqq \bigcup_{k=1}^\infty I_k, \qquad m^*(U \setminus E) \leq \frac{\epsilon}{2} \tag{108}$$

The major thing to do now is to reduce the countable union into a finite union. Note that we can just take any subcollection of the I_k's, and we are guaranteed that their union O will satisfy

$$m^*(O \setminus E) \leq m(U \setminus E) \leq \frac{\epsilon}{2} \tag{109}$$

The problem is that we don't want to take too small of a collection so that the other difference is too big. To do this, we can just select the tail of the series: Find n s.t. $\sum_{k=n+1}^\infty \ell(I_k) \leq \epsilon/2$ where WLOG, I_k are disjoint. Define $O = \cup_{k=1}^n I_k$. Then, we have

$$m^*(O \setminus E) \leq m(U \setminus E) \leq \frac{\epsilon}{2} \tag{110}$$

$$m^*(E \setminus O) \leq m(U \setminus O) \leq \sum_{k=n+1}^\infty \ell(I_k) \leq \frac{\epsilon}{2} \tag{111}$$

[a] since $\mathbb{R}^n$ is second countable

This symmetry in difference induced me to use the inner approximately closed definition in addition. My idea was to just find a closed set F s.t. $F \subset E \subset U$, but there is no straightforward way of finding one finite collection of intervals O.

Example 2.5 (idk where to put this)

One should note that in particular, if E is m^*-measurable and A is any set disjoint from E, then we must have

$$\begin{aligned} m^*(A \cup E) &= m^*((A \cup E) \cap E) + m^*((A \cup E) \cap E^c) && (112) \\ &= m^*(E) + m^*(A) && (113) \end{aligned}$$

which solves a bit of the theorem on measures.

2.3 Measures

Now by restricting our outer measure to measurable sets, we get our measure.

Definition 2.4 (Lebesgue Measure)

The restriction the Lebesgue outer measure m^* to the set of all measurable sets $\mathcal{A}$, is called the **Lebesgue measure**

$$m = m^*\big|_{\mathcal{A}} \tag{114}$$

Note that for outer measures, they satisfy both countable subadditivity and finite additivity. With measures, we get the best of both worlds: countable subadditivity.

Lemma 2.20 (Axiomatic Properties of Lebesgue Measure)

The Lebesgue measure satisfies the following.

1. *Null Empty Set.* $m(\emptyset) = 0$.
2. *Countable Additivity.* For all countable collections $\{A_k\}_{k=1}^{\infty}$ of pairwise disjoint subsets $A_k \in \mathcal{A}$[a],

$$m\bigg(\bigsqcup_{k=1}^{\infty} A_k\bigg) = \sum_{k=1}^{\infty} m(A_k) \tag{115}$$

[a]Remember that we are allowed to take countable unions inside our σ-algebra, so this makes sense.

Proof. Listed.

1. *Null Empty Set.* Since this is true for outer measure m^*.
2. *Countable Additivity.* $m(\cup E_j) = \sum_j m(E_j)$. $\leq$ is trivial by countable subadditivity of the outer measure. For $\geq$, note that for every $n \in \mathbb{N}$,

$$m\bigg(\bigcup_{j=1}^{\infty} E_j\bigg) \geq m\bigg(\bigcup_{j=1}^{n} E_j\bigg) = \sum_{j=1}^{n} m(E_j) \tag{116}$$

where the inequality comes from monotonicity and the equality comes from finite additivity of the outer measure. Now take $n \to \infty$.

Unlike the outer measure, monotonicity is not an axiomatic property because of two independent reasons, both sufficient. First, the Lebesgue outer measure suffices. Second, it is actually a direct consequence of the two axiomatic properties.

Lemma 2.21 (Translation Invariance of Lebesgue Measure)

The Lebesgue measure is translation invariant.

Proof. We know that translations of Lebesgue measurable sets are also Lebesgue measurable, and the Lebesgue outer measure is translation invariant.

Now we provide some "continuity" properties of the Lebesgue measure.

Theorem 2.22 (Continuity From Below)

If $A_1 \subset A_2 \subset A_3 \subset \ldots$, then

$$m\bigg(\bigcup_{k=1}^{\infty} A_k\bigg) = \lim_{k\to\infty} m(A_k) \tag{117}$$

Proof. First, note that the limit on the RHS is defined, since $m(A_k)$ is nondecreasing and so must converge in $[0, +\infty]$. But why does this limit equal to the left hand side? The only property that makes sense to work with is countable additivity, so we should define the disjoint collection

$$B_1 = A_1, \quad B_k = A_k \setminus A_{k-1} \tag{118}$$

Then, it becomes straightforward

$$\begin{aligned} m\bigg(\bigcup_{k=1}^\infty A_k\bigg) &= m\bigg(\bigcup_{k=1}^\infty B_k\bigg) && \text{(Construction)} \\ &= \sum_{k=1}^\infty m(B_k) && \text{(Countable Additivity)} \\ &= \lim_{n\to\infty} \sum_{k=1}^n m(B_k) && \text{(Definition of Series)} \\ &= \lim_{n\to\infty} m\bigg(\bigcup_{k=1}^\infty B_k\bigg) && \text{(Finite Additivity)} \\ &= \lim_{n\to\infty} m(A_k) && (119) \end{aligned}$$

Proof. Old proof. We can see that

$$m\bigg(\bigcup_{k=1}^\infty A_k\bigg) = m(A_1) + \sum_{k=2}^\infty m(B_k) \tag{120}$$

$$= m(A_1) + \lim_{k\to\infty} \sum_{k=2}^\infty m(B_k) \tag{121}$$

$$= \lim_{k\to\infty} m(A_1 \cup B_2 \cup \ldots B_k) = \lim_{k\to\infty} m(A_k) \tag{122}$$

Now a similar theorem, but with a little twist to it.

Theorem 2.23 (Continuity from Above)

If $A_1 \supset A_2 \supset A_3 \supset \ldots$, then

$$m\bigg(\bigcap_{k=1}^\infty A_k\bigg) = \lim_{k\to\infty} m(A_k) \tag{123}$$

if $m(A_1) < \infty$.

Proof. First, note that the $m(A_1) < +\infty$ is a necessary condition, since if we take $A_k = [k, \infty)$ on the real number line, then we have $\cap_{k=1}^\infty A_k = \emptyset$, but the limit of the measure is ∞. We did not have this problem for continuity from below.

Well we can define $B_k = A_k \setminus A_{k+1}$ and write $\cap_{k=1}^\infty A_k = A_1 \setminus \cup_{k=1}^\infty B_k$, which means that

$$\begin{aligned}
m\bigg(\bigcap_{k=1}^\infty A_k\bigg) &= m\bigg(A_1 \setminus \bigcup_{k=1}^\infty B_k\bigg) && (124)\\
&= m(A_1) - m\bigg(\bigcup_{k=1}^\infty B_k\bigg) && \text{(Excision)}\\
&= m(A_1) - \sum_{k=1}^\infty m(B_k) && \text{(Countable Additivity)}\\
&= m(A_1) - \lim_{n\to\infty}\sum_{k=1}^n m(B_k) && \text{(Definition of Series)}\\
&= \lim_{n\to\infty}\bigg(m(A_1) - \sum_{k=1}^n m(B_k)\bigg) && (125)\\
&= \lim_{n\to\infty} m\bigg(A_1 \setminus \bigcup_{k=1}^n B_k\bigg) && \text{(Excision)}\\
&= \lim_{n\to\infty} m(A_n) && (126)
\end{aligned}$$

Now the first line uses the fact that if $A \subset B$, then $m(B \setminus A) + m(A) = m(B)$, and with the further assumption that $m(A) < \infty$, we can subtract on both sides like we do with regular arithmetic.

We will see two applications of continuity from above.

Example 2.6 (Cantor Set has Measure 0)

The Cantor set has measure 0. We can see that it is the intersection of all C_k's which are nested $C_k \supset C_{k+1}$ and $m(C_0) = m([0,1]) = 1$. Therefore, by continuity from above,

$$m\bigg(\bigcap_{k=1}^\infty C_k\bigg) = \lim_{k\to\infty} m(C_k) = \lim_{k\to\infty}\frac{2^k}{3^k} = 0 \tag{127}$$

It is also closed as an intersection of closed sets. It is also uncountable, since we can just do it using a tradic system and see that the Cantor set are all reals with infinite triadic representation of digits 0 and 2. Then create a bijection with binary represenation of reals. Here's a new way I learned. Suppose C is countable, so enumerate it: $c_1, c_2, \ldots$. Pick one interval I_1 in C_1 that doesn't contain c_1. Then, pick $I_2 \subset I_1 \cap C_2$ s.t. it doesn't contain c_2. Keep going, and we get

$$I_1 \supset I_2 \supset I_3 \supset \ldots \tag{128}$$

By nested intervals lemma, these are closed, bounded, and nested, which is nonempty. So we've found a point not in the Cantor set, contradicting the fact that we have enumerated it.

Colloquially, if I give you a bunch of sets so that their total measure if finite, then no point will be hit by an infinite number of the sets.

Lemma 2.24 (Borel-Cantelli)

Suppose $\{E_k\}_{k=1}^{\infty}$ are measurable, with $\sum_k m(E_k) < +\infty$. Then,

$$m\big(\limsup E_k\big) := m\bigg(\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} E_k\bigg) = 0 \tag{129}$$

That is, all $x \in \mathbb{R}$ belonging to an infinite number of E_k, has measure 0.

Proof. Let's consider the E_k's. How can I make a decreasing set of them? We can set

$$B_n := \bigcup_{k=n}^{\infty} E_k \tag{130}$$

As n increases, the tail grows smaller. Notice that $B_n \supset B_{k+1}$ and B_1 has finite measure since by countable subadditivity[a],

$$m(B_1) = m\bigg(\bigcup_{k=1}^{\infty} E_k\bigg) \leq \sum_{k=1}^{\infty} m(E_k) < +\infty \tag{131}$$

Therefore, we can derive

$$\begin{aligned}
m\bigg(\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} E_k\bigg) &= \lim_{n\to\infty}\bigg(\bigcup_{k=n}^{\infty} E_k\bigg) && \text{(Continuity from Above)} \\
&= \lim_{n\to\infty}\sum_{k=n}^{\infty} m(E_k) && \text{(Countable Additivity)} \\
&= 0 && (132)
\end{aligned}$$

because the tail of series converging to a finite value must tend to 0. Note that for the last step, we could have used countable subadditivity as well. ■ When proving this for the first time, it may not be clear whether to use continuity from above or below. I also tried invoking continuity from below by constructing the sets

$$A_n = \bigcap_{k=n}^{\infty} E_k \tag{133}$$

which is increasing since as n increases, I am intersecting less sets at the tail. However, if we invoke continuity of measure, we get

$$m\bigg(\bigcup_{n=1}^{\infty}\bigcap_{k=n}^{\infty} E_k\bigg) = m\bigg(\bigcup_{n=1}^{\infty} A_n\bigg) = \lim_{n\to\infty} m(A_n) = \lim_{n\to\infty} m\bigg(\bigcap_{k=n}^{\infty} E_k\bigg) \tag{134}$$

However, this is the liminf of these sets—i.e. the set of points that are missing in only a finite number of sets—and the limit on the RHS doesn't provide us with more information (other than that it's finite, but this is weaker than the union being finite).

[a]not countable additivity!

Example 2.7 (Practice Problem)

Let

$$E = \{x \in [0,1] : \Big|x - \frac{p}{q}\Big| < q^{-3} \text{ for infinitely many } p, q \in \mathbb{N}\} \tag{135}$$

We wish to show that this set has measure 0. Let

$$E_{p,q} = \{x \in [0,1] \colon \left|x - \frac{p}{q}\right| < q^{-3}\} \tag{136}$$

Then, letting $k \to (p,q)$ be some bijective map, we have

$$E = \bigcap_{n=1}^{\infty} \bigcup_{k \geq n} E_k \tag{137}$$

Well each

$$m(E_{p,q}) \leq 2q^{-3} \ldots \implies \sum_{p,q} m(E_{p,q}) < +\infty \tag{138}$$

So by Borel-Cantelli, $m(E) = 0$.

Definition 2.5 (Almost Everywhere)

Given a measure space $(X, \mathcal{A}, m)$, a subset $A \in \mathcal{A}$ is said to be a m-null set if $m(A) = 0$. If some property holds for all points $x \in X$ except on a null set, then we say that the property holds **almost everywhere**.

Example 2.8 (Rational Function)

The function $f(x) = \frac{1}{\sqrt{|x|}}$ is less than ∞ almost everywhere.

2.4 Nonmeasurable Sets

Lemma 2.25 (Quotienting over Countable Set Implies Lebesgue Measure 0)

Suppose E is measurable, bounded, and there exists a countably infinite, bounded set Λ s.t. $\{E+m\}_{m\in\Lambda}$ are disjoint. Then, $m(E) = 0$.

Proof. Consider $\cup_{m\in\Lambda}\{E + m\}$. It is bounded, so its measure is finite. Also, by countable additivity and translation invariance, we get

$$+\infty > m\bigg(\bigcup_{m\in\Lambda} \{E + m\}\bigg) = \sum_{m\in\Lambda} m(E + m) = \sum_{m\in\Lambda} m(E) \tag{139}$$

Since Λ is infinite, we must have $m(E) = 0$.

Recall that we say x is *rationally equivalent* to y if $x - y \in \mathbb{Q}$. This is an equivalence relation on $\mathbb{R}$, giving us a quotient set of an uncountable number of classes. A *choice set* for this equivalence relation is a set containing exactly one element from each class.[3] We can do this on any $E \subset \mathbb{R}$.

Lemma 2.26 (Properties of Choice Sets)

If C_E is any choice set on such E, then

1. $\forall x, y \in C_E$, if $x - y \notin \mathbb{Q}$, then for all $\Lambda \in \mathbb{Q}$, $\{m + C_E\}_{m\in\Lambda}$ are disjoint.

[3]This assumes axiom of choice.

2. $\forall x \in E$, there exists $y \in C_E$ s.t. $x - y \in \mathbb{Q}$.

Theorem 2.27 (Every Set of Positive Outer Measure Contains Nonmeasurable Set)

Any set $E \subset \mathbb{R}^n$ of positive outer measure contains a nonmeasurable set.

Proof. WLOG, let E be bounded.[a] Let C_E be any choice set. Suppose C_E is measurable. Let b be such that $E \subset [-b, b]$. Let $\Lambda = \mathbb{Q} \cap [-2b, 2b]$. Consider the disjoint family of sets $\{C_E + \lambda\}_{\lambda \in \Lambda}$. Also,

$$E \subset \bigcup_{\lambda \in \Lambda} \{C_E + \lambda\} \tag{140}$$

Indeed, $\forall x \in E$, there exists $y \in C_E$ s.t. $x - y \in \mathbb{Q}$ and in Λ by definition of Λ and $E \subset [-b, b]$. By the lemma, $m(C_E) = 0$. But also,

$$m^*(E) \leq \sum_{\lambda \in \Lambda} \underbrace{m(C_E + \lambda)}_{=m(C_E)} = 0 \tag{141}$$

which is a contradiction, so C_E is not measurable.

[a] Otherwise, just take a bounded subset.

Definition 2.6 (Cantor-Lebesgue Function, Devil's Staircase)

The **Cantor-Lebesgue function** $\phi : [0, 1] \to \mathbb{R}$ is defined as such.

1. Let us define $O_k = [0, 1] \setminus C_k$[a], which is an open set. So O_k consists of $2^k - 1$ open intervals I_j (j indexed from left to right). For O_k, we define $\phi(x) = j/2^k$, where j is the number of the interval I_j, indexed left to right. This defines ϕ on $O = \cup_{k=1}^{\infty} O_k = [0, 1] \setminus C$.
2. On C, let us define $\phi(x) \coloneqq \inf_{y \geq x, y \in O} \phi(x)$.

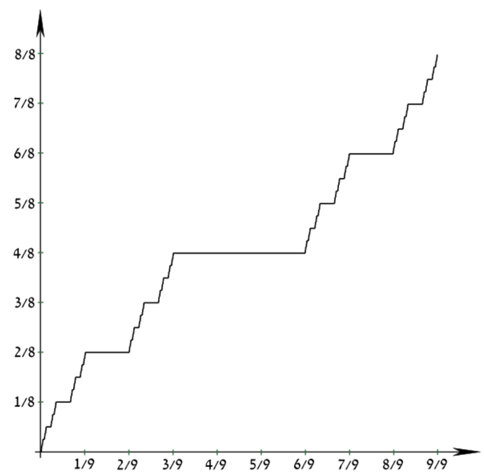


Figure 1: Plot

[a] C_k defined as before when constructing the Cantor set.

Theorem 2.28 (Properties of Cantor-Lebesgue Function)

ϕ is a nondecreasing, continuous function s.t. $\phi'(x) = 0$ for all $x \in O$ and $m(O) = 1$.

Proof. Listed.
1. *Increasing.* ϕ is increasing on each O_k, and so on O. Then, it is also increasing on C by definition.
2. *Continuity.* If $x \in C$, it lies between 2 intervals of O_k for any k. The difference in function values between 2 neighboring intervals of O_k is 2^{-k}, so ϕ is continuous.
3. *Derivative.* The derivative is 0 because it is constant around an interval.

Theorem 2.29 (Pathological Strictly Increasing Devil's Staircase)

Define $\psi(x) = \phi(x) + x$. Then,
1. ψ is continuous and strictly increasing.
2. ψ maps C into a set of positive measure.
3. ψ maps some subset of C into a nonmeasurable set.

Proof. Listed.
1. Continuity is from sum of continuous functions, and strictly increasing since ϕ is nondecreasing and x is strictly increasing.
2. We know that $\psi([0,1]) = [0,2]$ and $m(\psi(O)) = 1$, where for each interval $I_j \subset O$, $m(\psi(I_j)) = \ell(I_j)$. Therefore, $m(\psi(C)) = 1$.
3. Since ψ is strictly increasing, there exists a continuous inverse ψ^{-1}. Find $Z \subset \psi(C)$ that is nonmeasurable, which we can do from the previous theorem. Then, there exists $E \subset C$ s.t. $\psi(E) = Z$, and E is not Borel since if it were, then Z would be Borel, too.

2.5 Exercises

Exercise 2.1 (Royden 2.1)

Let m be a set function defined for all sets in a σ-algebra $\mathcal{A}$ with values in $[0, \infty]$. Assume m is countably additive over countable disjoint collections of sets in $\mathcal{A}$. Prove that if A and B are two sets in $\mathcal{A}$ with $A \subseteq B$, then $m(A) \leq m(B)$. This property is called *monotonicity*.

Solution. This is simply a manipulation of definition.

$$m(B) = m((B \cap A) \cup (B \setminus A)) = m(B \cap A) + m(B \setminus A) = m(A) + m(B \setminus A) \geq m(A) \tag{142}$$

■ Therefore, by upgrading countable subadditivity to countable additivity, we get the monotonicity property for free.

Exercise 2.2 (Royden 2.2)

Let m be a set function defined for all sets in a σ-algebra $\mathcal{A}$ with values in $[0, \infty]$. Assume m is countably additive over countable disjoint collections of sets in $\mathcal{A}$. Prove that if there is a set A in the collection $\mathcal{A}$ for which $m(A) < \infty$, then $m(\emptyset) = 0$.

Solution. By countable additivity,

$$m(A) = m\left(A \cup \bigcup_{i=1}^{\infty} \emptyset\right) = m(A) + \sum_{i=1}^{\infty} m(\emptyset) < +\infty \tag{143}$$

In order for the series to be finite, $m(\emptyset) = 0$. ■ Therefore, the finite subadditivity condition is not needed in most cases where there exists some set of finite measure.

Exercise 2.3 (Royden 2.3)

Let m be a set function defined for all sets in a σ-algebra $\mathcal{A}$ with values in $[0, \infty]$. Assume m is countably additive over countable disjoint collections of sets in $\mathcal{A}$. Let $\{E_k\}_{k=1}^{\infty}$ be a countable collection of sets in $\mathcal{A}$. Prove that $m(\bigcup_{k=1}^{\infty} E_k) \leq \sum_{k=1}^{\infty} m(E_k)$.

Solution. The idea is to manually make the E_k's disjoint. Let us set

$$E_1' = E_1, \quad E_2' = E_2 \setminus E_1, \quad \ldots, E_k' = E_k \setminus \bigcup_{i=1}^{k-1} E_i \tag{144}$$

Then, E_k's are pairwise disjoint, so

$$m\left(\bigcup_{k=1}^{\infty} E_k\right) = m\left(\bigcup_{k=1}^{\infty} E_k'\right) = \sum_{k=1}^{\infty} m(E_k') \leq \sum_{k=1}^{\infty} m(E_k) \tag{145}$$

Exercise 2.4 (Royden 2.4)

A set function c, defined on all subsets of $\mathbf{R}$, is defined as follows. Define $c(E)$ to be ∞ if E has infinitely many members and $c(E)$ to be equal to the number of elements in E if E is finite; define $c(\emptyset) = 0$. Show that c is a countably additive and translation invariant set function. This set function is called the *counting measure*.

Exercise 2.5 (Royden 2.5)

By using properties of outer measure, prove that the interval $[0, 1]$ is not countable.

Solution. If $[0, 1]$ was countable, its measure would be 0. ■

Exercise 2.6 (Royden 2.6)

Let A be the set of irrational numbers in the interval $[0, 1]$. Prove that $m^*(A) = 1$.

Solution. But we know that $m^*(\mathbb{Q}) = 0$ since rationals are countable, then by invoking excision, we get

$$m^*(A) = m^*([0, 1]) - m^*(\mathbb{Q}) = 1 - 0 = 1 \tag{146}$$

■ I tried proving this directly, but it turns out to be quite hard.[a]

[a] https://math.stackexchange.com/questions/3122008/direct-proof-that-the-irrationals-on-0-1-have-measure-1

Exercise 2.7 (Royden 2.7)

A set of real numbers is said to be a G_δ set provided it is the intersection of a countable collection of open sets. Show that for any bounded set E, there is a G_δ set G for which

$$E \subseteq G \text{ and } m^*(G) = m^*(E).$$

Solution. E is bounded so $m(E) < +\infty$. Let's write the definition

$$m(E) = \inf \left\{ \sum_{k=1}^\infty \ell(I_k) \colon A \subset \bigcup_k I_k \right\} \tag{147}$$

WLOG, we can let I_k be open, and we know that $\cup I_k$ is open as well. Perhaps we can construct a decreasing sequence of these open sets that converge onto $m(E)$, which is finite. For each $n \in \mathbb{N}$, we can find a cover $(I_{n,k})_k$ satisfying

$$m(E) \leq m\Bigg(\underbrace{\bigcup_k I_{n,k}}_{O_n} \Bigg) \leq m(E) + \frac{1}{n} \tag{148}$$

where the first inequality comes from monotonicity and the second comes from the ϵ of room trick. Now let

$$G = \bigcap_n O_n \tag{149}$$

■ Note that this applies to general sets and not just measurable ones, so for every set, we can always outer-approximate it (in terms of measure) with open sets and G_δ-sets.

Exercise 2.8 (Royden 2.8)

Let B be the set of rational numbers in the interval $[0, 1]$, and let $\{I_k\}_{k=1}^n$ be a finite collection of open intervals that covers B. Prove that $\sum_{k=1}^n m^*(I_k) \geq 1$.

Solution. The idea is to take advantage of the finiteness of the summation. WLOG, we can assume that $\{I_k = (a_k, b_k)\}$ is sorted in increasing order of a_k. Then, the following must be true.

1. $a_1 < 0$. If not, then $0 \not\in \cup I_k$.
2. $a_{k+1} \leq b_k$ for $k = 1, \ldots, n-1$. If not, then there exists some k s.t. $b_k < a_{k+1}$. By density of rationals, there exists some $q \in \mathbb{Q}$ s.t. $b_k < q < a_{k+1} \leq a_{k+2} \ldots$, and so $\{I_k\}$ cannot cover q.
3. $b_n > 1$.

Therefore, since this is a finite sum, we can rearrange

$$\sum_{k=1}^n m^*(I_k) = \sum_{k=1}^n b_k - a_k \tag{150}$$

$$= b_n \underbrace{-(a_n - b_{n-1}) - \ldots - (a_2 - b_1)}_{\geq 0} - a_1 \tag{151}$$

$$\geq b_n - a_1 \geq 1 \tag{152}$$

Exercise 2.9 (Royden 2.9)

Prove that if $m^*(A) = 0$, then $m^*(A \cup B) = m^*(B)$.

Solution. $\geq$ is already proven by monotonicity. For $\leq$, we can use finite subadditivity (which follows from countable subadditivity) to get

$$m^*(A \cup B) \leq m^*(A) + m^*(B) = m^*(B) \tag{153}$$

■ Therefore, adding or subtracting subsets of outer measure 0 does not affect the outer measure of any set. This will come in useful for making proofs more convenient.

Exercise 2.10 (Royden 2.10)

Let A and B be bounded sets for which there is an $\alpha > 0$ such that $|a - b| \geq \alpha$ for all $a \in A$, $b \in B$. Prove that $m^*(A \cup B) = m^*(A) + m^*(B)$.

Solution. Proven in Lemma 2.7

Exercise 2.11 (Royden 2.11)

Prove that if a σ-algebra of subsets of $\mathbf{R}$ contains intervals of the form (a, ∞), then it contains all intervals.

Solution. From countable union, it also contains all intervals of the form

$$[a, +\infty) = \bigcup_{x > a, x \in \mathbb{Q}} (x, +\infty) \tag{154}$$

Then, it also contains all intervals of the form $(-\infty, a)$, $(-\infty, a]$. By intersection, it contains all intervals of form $(a, b), [a, b), (a, b], [a, b]$.

Exercise 2.12 (Royden 2.12)

Show that every interval is a Borel set.

Solution. From the same logic as Exercise 2.11.

Exercise 2.13 (Royden 2.13)

Show that
1. the translate of an F_σ set is also F_σ,
2. the translate of a G_δ set is also G_δ, and
3. the translate of a set of measure zero also has measure zero.

Exercise 2.14 (Royden 2.14)

Show that if a set E has positive outer measure, then there is a bounded subset of E that also has positive outer measure.

Solution. The idea is to consider such a subset of E. If it has 0 outer measure, we can make it bigger, and eventually as we fill up E, it must be positive. However, we are strictly talking about outer measure, and E may not be measurable, so we can't use continuity of measure yet. Define

$$E_n = E \cap (-n, n) \tag{155}$$

Then $\cup_n E_n = E$ and E_n are increasing and so $m(E_n)$ must converge in the extended real number line.

1. If it converges to some finite $\alpha > 0$, then we can find a $N \in \mathbb{N}$ s.t. $n \geq N \implies m(E_n) > \alpha - \epsilon > 0$.
2. If it converges to $+\infty$, then we can find a $N \in \mathbb{N}$ s.t. $n \geq N \implies m(E_n) > K$ for all $K \in \mathbb{N}$.

Exercise 2.15 (Royden 2.15)

Show that if E has finite measure and $\epsilon > 0$, then E is the disjoint union of a finite number of measurable sets, each of which has measure at most ϵ.

Solution. By definition, we have

$$m^*(E) = \inf \left\{ \sum_{k=1}^{\infty} \ell(I_k) \right\} < +\infty \tag{156}$$

There are two cases.

1. If E is bounded, then $E \subset (-N, N)$. Choose M so large that $\frac{2N}{M} < \epsilon$, and divide E into the disjoint sets $E_m = [m, m+1) \cap E$. Each E_m—as the intersection of measurable sets—is measurable, and by monotonicity, we have

$$m^*(E \cap [m, m+1)) \leq m^*([m, m+1)) < \epsilon \tag{157}$$

2. If E is unbounded, then

$$m^*(E) = \inf \left\{ \sum_{k=1}^{\infty} \ell(I_k) \right\} = a \tag{158}$$

This is finite, so for some $\delta > 0$, we can find a cover $\{I_k\}_k$ s.t.

$$\sum_{k=1}^{\infty} \ell(I_k) < a + \delta \tag{159}$$

Since this series is finite, it converges and so the tail must tend to 0.

$$m^* \left(\bigcup_{k=n}^{\infty} I_k \right) \leq \sum_{k=n}^{\infty} \ell(I_k) < \epsilon \tag{160}$$

The rest of the intervals are bounded, so we can just use case 1.

Exercise 2.16 (Royden 2.16)

Complete the proof of Theorem 11 by showing that measurability is equivalent to (iii) and also equivalent to (iv).

Solution. We show the following.

1. Measurability $\to$ (iii). If E is measurable, then E^c is measurable. Then from (i), $\exists$ open $O \supset E^c$ s.t. $m^*(O \setminus E^c) < \epsilon$. But this implies that

$$m(O \setminus E^c) = m(O \cap E) = m(E \setminus O^c) < \epsilon \tag{161}$$

where O^c is closed.

2. (ii) $\to$ (iv). Let $G = \cap_{n=1}^\infty O_n$ be a G_δ-set. Then,

$$0 = m^*\bigg(\bigcap_{n=1}^\infty O_n \setminus E^c\bigg) = m^*\bigg(\bigg[\bigcap O_n\bigg] \cap E\bigg) \tag{162}$$

$$= m^*\bigg(E \setminus \bigg[\bigcap O_n\bigg]^c\bigg) \tag{163}$$

$$= m^*\bigg(E \setminus \bigg[\underbrace{\bigcup O_n^c}_{F_\sigma\text{-set}}\bigg]\bigg) \tag{164}$$

The converses of each statement can be proven almost identically. ■ I found out that the biggest leap was the identity (in blue), but the following diagram helps.

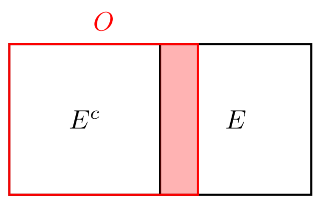


Figure 2: The shaded rectangle can be represented as the difference $O \setminus E^c$ or $E \setminus O^c$.

Exercise 2.17 (Royden 2.17)

Show that a set E is measurable if and only if for each $\epsilon > 0$, there is a closed set F and open set O for which $F \subseteq E \subseteq O$ and $m^*(O \setminus F) < \epsilon$.

Solution. We prove bidirectionally.

1. ($\to$). If E is measurable, then choose F, O s.t.

$$m^*(E \setminus F) < \frac{\epsilon}{2}, m^*(O \setminus E) < \frac{\epsilon}{2} \implies m^*(O \setminus F) \leq m^*(O \setminus E) + m^*(E \setminus F) < \epsilon \tag{165}$$

2. ($\leftarrow$).

Exercise 2.18 (Royden 2.18)

Let E have finite outer measure. Show that there is an F_σ set F and a G_δ set G such that

$$F \subseteq E \subseteq G \text{ and } m^*(F) = m^*(E) = m^*(G).$$

Exercise 2.19 (Royden 2.19)

Let E have finite outer measure. Show that if E is not measurable, then there is an open set O containing E that has finite outer measure and for which

$$m^*(O \setminus E) > m^*(O) - m^*(E).$$

Solution.

Exercise 2.20 (Royden 2.20)

(Lebesgue) Let E have finite outer measure. Show that E is measurable if and only if for each open, bounded interval (a,b),
$$b - a = m^*((a,b) \cap E) + m^*((a,b) \setminus E).$$

Solution. Since
$$m^*(E) = \inf \left\{ \sum_{k=1}^\infty \ell(I_k) \right\} < +\infty, \tag{166}$$
there exists a covering $\{I_k = (a_k, b_k)\}_k$ of E s.t.
$$\begin{aligned}
m^*(E) + \epsilon &> \sum_{k=1}^\infty (b_k - a_k) && (167)\\
&= \sum_{k=1}^\infty m^*((a_k,b_k) \cap E) + m^*((a_k,b_k) \setminus E) && (168)\\
&= \sum_{k=1}^\infty m^*((a_k,b_k) \cap E) + \sum_{k=1}^\infty m^*((a_k,b_k) \setminus E) && (169)\\
&\geq m^*\left(\bigcup_{k=1}^\infty (a_k,b_k) \cap E \right) + m^*\left(\bigcup_{k=1}^\infty (a_k,b_k) \setminus E \right) && (170)\\
&= m^*(E) + m^*(O \setminus E) && (171)
\end{aligned}$$
which implies $\epsilon > m^*(O \setminus E)$, which is equivalent to E being measurable.

Exercise 2.21 (Royden 2.21)

Use property (ii) of Theorem 11 as the primitive definition of a measurable set and prove that the union of two measurable sets is measurable. Then do the same for property (iv).

Exercise 2.22 (Royden 2.22)

For any set A, define $m^{**}(A) \in [0, \infty]$ by
$$m^{**}(A) = \inf\{m^*(O) \mid O \supseteq A, O \text{ open}\}.$$
How is this set function m^{**} related to outer measure m^*?

Solution. We claim that they are equal.

1. By monotonicity, $A \subset O \implies m^*(A) \leq m^*(O)$, and taking the infimum over O gives $m^*(A) \leq m^{**}(A)$.
2. Note that every open set can be represented as a countable pairwise-disjoint union of open intervals, so by countable subadditivity,
$$m^*(O) = m^*\left(\bigcup_{k=1}^\infty (a_k, b_k) \right) \leq \sum_{k=1}^\infty \ell((a_k,b_k)) \implies m^{**}(A) \leq m^*(A) \tag{172}$$

Exercise 2.23 (Royden 2.23)

For any set A, define $m^{***}(A) \in [0, \infty]$ by

$$m^{***}(A) = \sup\{m^*(F) \mid F \subseteq A, F \text{ closed}\}.$$

How is this set function m^{***} related to outer measure m^*?

Solution. By monotonicity,

$$F \subset A \implies m^*(F) \leq m^*(A) \implies m^{***}(A) \leq m^*(A) \tag{173}$$

Exercise 2.24 (Royden 2.24)

Show that if E_1 and E_2 are measurable, then

$$m(E_1 \cup E_2) + m(E_1 \cap E_2) = m(E_1) + m(E_2).$$

Solution. We know that $E_1 \cup E_2 = (E_1 \setminus E_2) \cup (E_2 \setminus E_1) \cup (E_1 \cap E_2)$, and so by finite additivity,

$$m(E_1 \cup E_2) = m(E_1 \setminus E_2) + m(E_2 \setminus E_1) + m(E_1 \cap E_2) \tag{174}$$

By adding $m(E_1 \cap E_2)$ and grouping, we get

$$m(E_1 \cup E_2) + m(E_1 \cap E_2) = \underbrace{m(E_1 \setminus E_2) + m(E_1 \cap E_2)}_{m(E_1)} + \underbrace{m(E_2 \setminus E_1) + m(E_1 \cap E_2)}_{m(E_2)} \tag{175}$$

Exercise 2.25 (Royden 2.25)

Show that the assumption that $m(B_1) < \infty$ is necessary in part (ii) of the theorem regarding continuity of measure.

Exercise 2.26 (Royden 2.26)

Let $\{E_k\}_{k=1}^\infty$ be a countable disjoint collection of measurable sets. Prove that for any set A,

$$m^*\left(A \cap \bigcup_{k=1}^\infty E_k\right) = \sum_{k=1}^\infty m^*(A \cap E_k).$$

Solution. We know that for all $n \in \mathbb{N}$,

$$\sum_{k=1}^n m^*(A \cap E_k) = m^*\left(A \cap \bigcup_{k=1}^n E_k\right) = m^*\Bigg(\underbrace{\bigcup_{k=1}^n (A \cap E_k)}_{B_n}\Bigg) \tag{176}$$

The B_n are an increasing sequence of sets, so by continuity from below,

$$m\left(\bigcup_{n=1}^\infty \bigcup_{k=1}^n (A \cap E_k)\right) = \lim_{n \to \infty} m\left(\bigcup_{k=1}^n (A \cap E_k)\right) \tag{177}$$

So by taking the limit as $n \to \infty$,

$$\sum_{k=1}^{\infty} m^*(A \cap E_k) = \lim_{n\to\infty} m^*\bigg(\bigcup_{k=1}^{n} (A \cap E_k)\bigg) \tag{178}$$

$$= m^*\bigg(\bigcup_{n=1}^{\infty} \bigcup_{k=1}^{n} (A \cap E_k)\bigg) \tag{179}$$

$$= m^*\bigg(\bigcup_{k=1}^{\infty} (A \cap E_k)\bigg) \tag{180}$$

Solution. We know from countable subadditivity that

$$m^*\bigg(A \cap \bigcup_{k=1}^{\infty} E_k\bigg) \leq \sum_{k=1}^{\infty} m^*(A \cap E_k) \tag{181}$$

$$\tag{182}$$

Now,

$$m^*\bigg(A \cap \bigcup_{k=1}^{\infty} E_k\bigg) \geq m^*\bigg(A \cap \bigcup_{k=1}^{n} E_k\bigg) = \sum_{k=1}^{n} m^*(A \cap E_k) \tag{183}$$

So by taking $n \to +\infty$, we are done.

Exercise 2.27 (Royden 2.27)

Let $\mathcal{M}'$ be any σ-algebra of subsets of $\mathbf{R}$ and m' a set function on $\mathcal{M}'$ which takes values in $[0, \infty]$, is countably additive, and such that $m'(\emptyset) = 0$.

1. Show that m' is finitely additive, monotone, countably monotone, and possesses the excision property.
2. Show that m' possesses the same continuity properties as Lebesgue measure.

Exercise 2.28 (Royden 2.28)

Show that continuity of measure together with finite additivity of measure implies countable additivity of measure.

Solution. This was basically my solution for Exercise 2.26. Let $\{E_k\}_{k=1}^{\infty}$ be a sequence of sets. Then, m being finitely additive implies that

$$m^*\bigg(\bigcup_{k=1}^{n} E_k\bigg) = \sum_{k=1}^{n} m(E_k) \tag{184}$$

But now define $B_n = \cup_{k=1}^n E_k$, which is increasing, and so by continuity of measure from below,

$$m\bigg(\bigcup_{k=1}^\infty E_k\bigg) = m\bigg(\bigcup_{n=1}^\infty B_n\bigg) \tag{185}$$
$$= \lim_{n\to\infty} m(B_n) \tag{186}$$
$$= \lim_{n\to\infty} m\bigg(\bigcup_{k=1}^n E_k\bigg) \tag{187}$$
$$= \lim_{n\to\infty} \sum_{k=1}^n m(E_k) \tag{188}$$
$$= \sum_{k=1}^\infty m(E_k) \tag{189}$$

Exercise 2.29 (Royden 2.29)

Listed.
1. Show that rational equivalence defines an equivalence relation on any set.
2. Explicitly find a choice set for the rational equivalence relation on $\mathbf{Q}$.
3. Define two numbers to be irrationally equivalent provided their difference is irrational. Is this an equivalence relation on $\mathbf{R}$? Is this an equivalence relation on $\mathbf{Q}$?

Exercise 2.30 (Royden 2.30)

Show that any choice set for the rational equivalence relation on a set of positive outer measure must be uncountably infinite.

Exercise 2.31 (Royden 2.31)

Justify the assertion in the proof of Vitali's Theorem that it suffices to consider the case that E is bounded.

Exercise 2.32 (Royden 2.32)

Does Lemma 2.25[a] remain true if Λ is allowed to be finite or to be uncountably infinite? Does it remain true if Λ is allowed to be unbounded?

[a] Originally Royden 4th Edition Lemma 11

Solution. Listed.
1. Finite? No. Let $E = (0, 1)$ and $\Lambda = \{0, 1, 2\}$. Therefore, the infinite part is essential since it makes things harder to overlap.
2. Uncountably infinite? Yes, since we can just find a countably infinite subcollection from it.
3. Λ unbounded? No. Let $E = (0, 1)$ and $\Lambda = \mathbb{Z}$. Then, $\{E + z\}_{z\in\mathbb{Z}}$ are disjoint but $m(E) = 1$. If it was unbounded, then we can just give as much space for $E + \lambda$ to roam around.

Exercise 2.33 (Royden 2.33)

Let E be a nonmeasurable set of finite outer measure. Show that there is a G_δ set G that contains E for which

$$m^*(E) = m^*(G), \text{ while } m^*(G \setminus E) > 0.$$

Solution. Note from Exercise 2.7 that we can outer-approximate *any* set with a G_δ-set. Therefore, choose such a set $G \supset E$ which satisfies

$$m^*(G) - m^*(E) = 0 \tag{190}$$

If E was measurable, then by excision property, $m^*(G \setminus E) = 0$, but E is not measurable, so we must have $m^*(G \setminus E) > 0$.

Exercise 2.34 (Royden 2.34)

Show that there is a continuous, strictly increasing function on the interval $[0, 1]$ that maps a set of positive measure onto a set of measure zero.

Exercise 2.35 (Royden 2.35)

Let f be an increasing function on the open interval I. For $x_0 \in I$ show that f is continuous at x_0 if and only if there are sequences $\{a_n\}$ and $\{b_n\}$ in I such that for each n, $a_n < x_0 < b_n$, and $\lim_{n\to\infty}[f(b_n) - f(a_n)] = 0$.

Exercise 2.36 (Royden 2.36)

Show that if f is any increasing function on $[0, 1]$ that agrees with the Cantor-Lebesgue function φ on the complement of the Cantor set, then $f = \varphi$ on all of $[0, 1]$.

Exercise 2.37 (Royden 2.37)

Let f be a continuous function defined on E. Is it true that $f^{-1}(A)$ is always measurable if A is measurable?

Exercise 2.38 (Royden 2.38)

Let the function $f : [a, b] \to \mathbf{R}$ be Lipschitz, that is, there is a constant $c \geq 0$ such that for all $u, v \in [a, b]$, $|f(u) - f(v)| \leq c|u - v|$. Show that f maps a set of measure zero onto a set of measure zero. Show that f maps an F_σ set onto an F_σ set. Conclude that f maps a measurable set to a measurable set.

Exercise 2.39 (Royden 2.39)

Let F be the subset of $[0, 1]$ constructed in the same manner as the Cantor set except that each of the intervals removed at the nth deletion stage has length $\alpha 3^{-n}$ with $0 < \alpha < 1$. Show that F is a closed set, $[0, 1] \setminus F$ dense in $[0, 1]$, and $m(F) = 1 - \alpha$. Such a set F is called a generalized Cantor set.

Exercise 2.40 (Royden 2.40)

Show that there is an open set of real numbers that, contrary to intuition, has a boundary of positive measure. (Hint: Consider the complement of the generalized Cantor set of the preceding problem.)

Exercise 2.41 (Royden 2.41)

A nonempty subset X of $\mathbf{R}$ is called perfect provided it is closed and each neighborhood of any point in X contains infinitely many points of X. Show that the Cantor set is perfect. (Hint: The endpoints of all of the subintervals occurring in the Cantor construction belong to C.)

Exercise 2.42 (Royden 2.42)

Prove that every perfect subset X of $\mathbf{R}$ is uncountable. (Hint: If X is countable, construct a descending sequence of bounded, closed subsets of X whose intersection is empty.)

Exercise 2.43 (Royden 2.43)

Use the preceding two problems to provide another proof of the uncountability of the Cantor set.

Exercise 2.44 (Royden 2.44)

A subset A of $\mathbf{R}$ is said to be *nowhere dense in* $\mathbf{R}$ provided that for every open set O has an open subset that is disjoint from A. Show that the Cantor set is nowhere dense in $\mathbf{R}$.

Exercise 2.45 (Royden 2.45)

Show that a strictly increasing function that is defined on an interval has a continuous inverse.

Exercise 2.46 (Royden 2.46)

Let f be a continuous function and B be a Borel set. Show that $f^{-1}(B)$ is a Borel set. (Hint: The collection of sets E for which $f^{-1}(E)$ is Borel is a σ-algebra containing the open sets.)

Exercise 2.47 (Royden 2.47)

Use the preceding two problems to show that a continuous strictly increasing function that is defined on an interval maps Borel sets to Borel sets.

Exercise 2.48 (Math 631 Fall 2025, Midterm Exercise 1)

Define a set $E \subset [0,1]$ to be all points such that in their decimal expansion the 10th digit (corresponding to 10^{-10} value) is equal to 4, and the 100th digit is equal to 5. Show that E is measurable and find $m(E)$.

Solution. There is just a finite set of real numbers where the ambiguity of the decimal expansion affects the definition of set E: for example, 0.0000000004000... = 0.00000000039999... - but finite number of points does not affect that measurability or Lebesgue measure of a set. We can adopt the convention that whenever we can use 4 on 10th place and 5 on the 100th place, we do. With this convention, the set of points with 4 on the 10th decimal place is a set of 10^9 disjoint closed intervals I_k of size 10^{-10}. These intervals correspond to the 10^9 choices of all possible digits in the larger position values. We further insist that 5 stands on the 100th place, then in each I_k there are 10^{89} closed intervals of size 10^{-100} where this is true. The number 10^{89} corresponds to all possible choices of digits at 89 positions from 11th to 99th. So, there is a total of

$$10^9 \times 10^{89} = 10^{98} \tag{191}$$

disjoint intervals of size 10^{-100}. This set is measurable as a finite union of intervals. The Lebesgue measure of such set is 0.01 by additivity of the Lebesgue measure.

Exercise 2.49 (Math 631 Fall 2025, Final Exam Exercise 3)

Let $E_k \subset (0,1)$ be measurable sets. Is it true that there exists a subsequence E_{k_j} such that $m(\cap_j E_{k_j}) > 0$ if (a) $\limsup m(E_k) = 1$, (b) $\liminf m(E_k) > 3/4$?

Solution. (a) Yes. Choose a subsequence k_j such that $\sum_{j=1}^{\infty}(1 - m(E_{k_j})) < 1$. This can be achieved, for example, if we require that $1 - m(E_{k_j}) < 2^{-j}$. This is equivalent to

$$\sum_{j=1}^{\infty} m(E_{k_j}^c) < 1. \tag{192}$$

This implies by countable subadditivity that $m(\cup_{j=1}^{\infty} E_{k_j}^c) < 1$, and since $\cap_j E_{k_j} = (\cup_{j=1}^{\infty} E_{k_j}^c)^c$, we get $m(\cap_j E_{k_j}) > 0$.
(b) No. Let E_k be all numbers without digits 1 and 2 at the kth position in their decimal expansion. Then $m(E_k) = 4/5$ for every k. On the other hand, for any k_j,

$$m(\cap_{j=1}^{J} E_{k_j}) = (4/5)^J \to 0 \tag{193}$$

as $J \to \infty$.

3 Lebesgue Integration

It turns out all the convergence theorems are dependent on Egorov.

Remember that Riemann integration is characterized by the approximation of step functions, which are the "building blocks" of Riemann integrable functions. To define the Lebesgue integral, we will start off with simple functions—which are a generalization of step functions. A function will be Lebesgue integrable if it can be approximated by these simple functions in some appropriate way. There are parallels between how we construct the Riemann and Lebesgue integral, namely that we can define the upper and lower integrals as the infimums and supremums of some set. However—while we were able to do this in "one shot" for the Riemann integral, the Lebesgue integral requires us to take intermediate steps. There are two ways that we can construct the Lebesgue integral.

1. We define the integral for simple functions ϕ over finite measure. Since simple functions are made up of a linear combination of indicators, which are themselves bounded, the finite measure locks in the property that $\int \phi$ will be bounded. This allows us to take the integral of simple functions that have real-valued coefficients—both positive and negative—but is inherently limiting as we can't define the integral over infinite measures. With this, we can define the integral of bounded measurable functions using the lower and upper integrals. This is similar to the construction of the Riemann integral. Furthermore, if we want to define the general integral, we have to now restrict what we have built up into the nonnegative case—a bit unnatural.

2. We first define the integral for *nonnegative* simple functions ϕ over any measure. We are sacrificing the ability to integrate negative simple functions early on, but this doesn't matter since we will split functions into a positive and negative part anyways. The true advantage of doing this is that we gain the ability to define integrals for infinite measures. This allows us to define for positive (possibly unbounded) measurable functions, and finally when we define the general Lebesgue integral, we deal with the $\infty - \infty$ problem by defining it only if at least one of the positive or negative integrals are finite.

I personally think the second way is superior, since in the end we can define integrals over infinite measure. Furthermore, since we are only working with positive functions, we only need to define the lower integral rather than checking to see if the lower and upper integrals coincide.

As we redefine these integrals over and over (sort of like method overriding), we really want to preserve three properties of the integral.

1. Linearity.
$$\int_E (\alpha f + \beta g) = \alpha \int_E f + \beta \int_E g \tag{194}$$

2. Monotonicity
$$f \leq g \implies \int_E f \leq \int_E g \tag{195}$$

3. Additivity.
$$A, B \text{ disjoint, measurable, and } E = A \cup B \implies \int_E f = \int_A f + \int_B f \tag{196}$$

3.1 Simple Functions

We will use ϕ, φ, ψ to denote simple functions.

Definition 3.1 (Lebesgue Integral of Simple Functions over Finite Measure)

Suppose $\phi = \sum_{k=1}^n a_k \chi_{E_k} : E \subset \mathbb{R} \to \mathbb{R}$ with $a_k \in \mathbb{R}$.

1. If $m(E) < +\infty$, then we can define the integral which is guaranteed to be a finite number.

$$\int_E \phi \coloneqq \sum_{k=1}^n a_k m(E_k) \tag{197}$$

2. If $a_k \geq 0$, we define the integral which lives in $\mathbb{R} \cup \{+\infty\}$.[a]

$$\int_E \phi \coloneqq \sum_{k=1}^n a_k m(E_k) \tag{198}$$

This is well defined for any representation of ϕ[b]

[a] As we have stated before, we could also define the Lebesgue integral of simple functions by letting a_k takes values in $\mathbb{R}$. But then, we might have a case where $E = A \sqcup B$, with $m(A) = +\infty, m(B) = +\infty$, and $\phi = \chi_A - 2\chi_B \implies \int_E \phi = \infty - \infty$. To prevent this from happening, some authors add the assumption that $m(E) < +\infty$, and I cover this case to make it as comprehensive as possible.

[b] We need this since the coefficients need not be unique. For example, we can write $1 \cdot \chi_{[0,1]} + 1 \cdot \chi_{[0.5,1]} = 1 \cdot \chi_{[0,0.5]} + 2 \cdot \chi_{[0.5,1]}$. If the E_i's are disjoint, then this decomposition is unique and is called the **standard representation** of ϕ.

Proof. It is clear that the two definitions coincide if $m(E) < +\infty$ and $a_k \geq 0$ is true; it is the same formula.

For bounded functions, we will temporarily focus on the first definition, and for general functions, we rely on the second definition.

Theorem 3.1 (Integral Properties for Simple Functions over Finite Measure)

Suppose $\phi, \psi : E \subset \mathbb{R} \to \mathbb{R}$ are simple with $m(E) < +\infty$. Then, the following properties hold.

1. Linearity. For $\alpha, \beta \in \mathbb{R}$,
$$\int_E (\alpha\phi + \beta\psi) = \alpha \int_E \phi + \beta \int_E \psi \tag{199}$$
2. Monotonicity.
$$\phi \leq \psi \implies \int_E \phi \leq \int_E \psi \tag{200}$$
3. Additivity. If A, B are disjoint, measurable, and $E = A \cup B$, then
$$\int_E \phi = \int_A \phi + \int_B \phi \tag{201}$$

Proof. Listed.

1. We can subdivide sets s.t. ϕ and ψ can be rewritten using the same finite family of sets A_k.
2. We can use (1) to rewrite
$$\int_E \psi - \int_E \phi = \int \underbrace{(\psi - \phi)}_{\geq 0 \forall x} \geq 0 \tag{202}$$
3. Trivial by definition of simple functions.

Example 3.1 (Step Function as Simple Function)

For $a, b \in \mathbb{R}$, with $a < b$, let $f : [a, b] \longrightarrow \mathbb{R}$ be a step function. That is, there exists a partition $a = x_0 < x_1 < \ldots < x_n = b$ and constants $c_1, c_2, \ldots, c_n \in \mathbb{R}$ s.t. $f(x) = c_i$ for all $x \in (x_{i-1}, x_i)$ and each $i = 1, \ldots, n$. Then, f is equal to the following simple function, taken over all open intervals and the points x_j at the boundary of each interval.

$$f = \sum_{i=1}^n c_i \chi_{(x_{i-1}, x_i)} + \sum_{j=0}^n f(x_j) \chi_{\{x_j\}} \tag{203}$$

If we ignore the behavior of f on the partition points x_j's, then f agrees almost everywhere with the simple function

$$\sum_{i=1}^n c_i \chi_{(x_{i-1}, x_i)} \tag{204}$$

3.2 Bounded Measurable Functions over Finite Measure

This isn't the most popular way to define Lebesgue integrability, but I'd like to compare it to Riemann integral. First, note that a Riemann integral

Definition 3.2 (Lower, Upper Lebesgue Integral)

Let $f : E \subset \mathbb{R} \to \mathbb{R}$ be bounded, with $m(E) < +\infty$. Then, the **upper and lower Lebesgue integrals** are defined

$$\overline{L}f \coloneqq \inf_{\phi} \bigg\{ \int \phi \;\bigg|\; f \leq \phi, \phi \text{ simple} \bigg\}, \qquad \underline{L}f \coloneqq \sup_{\phi} \bigg\{ \int \phi \;\bigg|\; \phi \leq f, \phi \text{ simple} \bigg\} \tag{205}$$

If the upper and lower Lebesgue integrals are equal, then f is said to be **Lebesgue integrable**.

This is exactly the same form as that of Riemann integration, notably that f must be bounded and we define integrability as the equivalence of the upper and lower integrals. The fact that $m(E)$ is finite is realized implicitly with partitions, though in this sense it's more similar to the Riemann-Stieltjes integral. From this, it's pretty easy to intuit that the Lebesgue integral agrees with the Riemann integral for step functions. Let $c_1, \ldots, c_n \in [0, \infty)$ and $a = x_0 < x_1 < \ldots < x_n = b$ be a partition. Let $f : [a, b] \longrightarrow [0, \infty]$ be a step function taking the value c_i on the interval (x_{i-1}, x_i) for $i = 1, \ldots, n$. Then the Riemann integral of f is simply

$$\int f(x)\, dx = \sum_{i=1}^n c_k |x_i - x_{i-1}| \tag{206}$$

The Lebesgue integral is

$$\int f\, d\mu = \sum_{i=1}^n c_i \mu((x_{i-1}, x_i)) + \sum_{j=0}^n f(x_j) \mu(\{x_j\}) = \sum_{i=1}^n c_k |x_i - x_{i-1}| \tag{207}$$

which agrees with the Riemann integral. In the Riemann integral, we write dx to indicate the variable that is being integrated over, but in the Lebesgue integral, we write $d\mu$, the measure which we are integrating over. We make this more rigorous in the following theorem.

Theorem 3.2 (Riemann Integrability Implies Lebesgue Integrability)

If f is Riemann integrable, then it is Lebesgue integrable.

Proof. Recall that f is Riemann integrable if

$$\sup_P L(P, f) = \inf_P U(P, f) \tag{208}$$

for a partition P. But for any $L(P, f)$ (or $U(P, f)$), there exist simple ϕ (or ψ) s.t.

$$\int_E \phi = L(P, f), \qquad \left(\int_E \psi = U(P, f) \right) \tag{209}$$

So,

$$\sup_P L(P, f) \leq \underline{L}f \leq \overline{L}f \leq \inf_P U(P, f) \tag{210}$$

where the first and third inequalities we just showed, and the middle inequality $\underline{L}f \leq \overline{L}f$ comes from monotonicity. So if the inf = sup, then $\underline{L}f = \overline{L}f$ has nowhere to go.

However, the converse is not true.

Example 3.2 (Lebesgue Integrable but Not Riemann Integrable)

Consider the simple function (consisting of one characteristic function) $\chi_{\mathbb{Q} \cap [0,1]}$. $\mathbb{Q} \cap [0, 1]$ is a Lebesgue measurable set of $\mathbb{R}$, and we have $\chi_{\mathbb{Q} \cap [0,1]} \geq 0$, so its Lebesgue integral is given by the above definition:

$$\int_{\mathbb{R}} \chi_{\mathbb{Q} \cap [0,1]} \, d\lambda = 1 \cdot \lambda(\mathbb{Q} \cap [0, 1]) = 0 \tag{211}$$

Remember that continuous functions are Riemann integrable. There is indeed an analogous result between measurable functions and Lebesgue-integrable functions.

Theorem 3.3 (Bounded Measurable Functions are Lebesgue Integrable)

Let $f : E \subset \mathbb{R} \to \mathbb{R}$ be measurable, bounded with $m(E) < +\infty$. Then, f is Lebesgue integrable.

Proof. We prove that $\forall n$, $\exists$ a simple ϕ_n, ψ_n s.t. $\phi_n \leq f \leq \psi_n$ and $\psi_n - \phi_n < 1/n$. Then,

$$\overline{L}f - \underline{L}f \leq \int \psi_n - \int \phi_n \leq \frac{1}{n} m(E) \tag{212}$$

So take $n \to \infty$.

To prove additivity, we'll need a specific lemma.

Lemma 3.4

Let $f : E \subset \mathbb{R} \to \mathbb{R}$ be measurable, bounded with $m(E) < +\infty$, and $A \subset E$ measurable. Then,

$$\int_E f \cdot \chi_A = \int_A f \tag{213}$$

Proof.

Theorem 3.5 (Integral Properties for Bounded Measurable Functions over Finite Measure)

Suppose $f, g : E \subset \mathbb{R} \to \mathbb{R}$ are bounded and measurable with $m(E) < +\infty$. Then, the following properties hold.

1. Linearity. For $\alpha, \beta \in \mathbb{R}$,
$$\int_E (\alpha f + \beta g) = \alpha \int_E f + \beta \int_E g \tag{214}$$
2. Monotonicity.
$$f \leq g \implies \int_E f \leq \int_E g \tag{215}$$
3. Additivity. If $A \subset E$ is measurable $B = A \setminus E$, then
$$\int_E f = \int_A f + \int_B f \tag{216}$$

Proof. Listed.

1. For scalar multiplication, we just use $\alpha\phi, \alpha\psi$ in the simple function approximation. Now for the sums of functions, we can see that
$$f \leq \psi_1, g \leq \psi_2 \implies f + g \leq \psi_1 + \psi_2 \implies \int (f+g) \leq \int f + \int g \tag{217}$$
$$f \geq \psi_1, g \geq \psi_2 \implies f + g \geq \psi_1 + \psi_2 \implies \int (f+g) \geq \int f + \int g \tag{218}$$
$$\tag{219}$$
and by taking the limit as the simple functions approach f, g, we get equality.
2.
3. We can define $f_1 = f \cdot \chi_A, f_2 = f \cdot \chi_B$. f_1, f_2 are measurable and bounded (as the product of measurable functions) implying that
$$\int_E f = \int_E (f_1 + f_2) = \int_E f_1 + \int_E f_2 = \int_E f \cdot \chi_A + \int_E f \cdot \chi_B = \int_A f + \int_B f \tag{220}$$
where the final equality follows from the lemma.

Corollary 3.6

Suppose $f : E \subset \mathbb{R} \to \mathbb{R}$ is bounded and measurable. Then
$$\left| \int_E f \right| \leq \int_E |f| \tag{221}$$

Proof. We know $-|f| \leq f \leq |f|$. By monotonicity, we have
$$\int -|f| \leq \int f \leq \int |f| \tag{222}$$

The next theorem is sort of like transferring convergence of functions to that of integrals. This results shouldn't be surprisingly since uniform convergence is so strong.

Theorem 3.7 (Uniform Convergence Implies Convergence of Integrals)

Suppose f_n are bounded, measurable, and $f_n \to f$ uniformly on E with $m(E) < +\infty$. Then,
$$\int_E f_n \to \int_E f \tag{223}$$

Proof. Since f is bounded, there exists $N \in \mathbb{N}$ s.t. $\|f_N - f\|_\infty \leq 1$. Then, by reverse triangle inequality (?), $\|f\|_\infty \leq \|f_N\|_\infty + 1$. Also, f is measurable as the limit of f_N. Given $\epsilon > 0$, find N_1 s.t. $\forall n \geq N_1$,

$$\|f_n - f\|_\infty \leq \frac{\epsilon}{m(E)} \implies \left| \int f - \int f_n \right| = \left| \int_E (f - f_n) \right| \leq \int_E |f - f_n| \leq \frac{\epsilon}{m(E)} \cdot m(E) = \epsilon \tag{224}$$

Naturally we might see if this results holds with weaker assumptions. Unfortunately, this is not true for pointwise convergence.

Example 3.3 (Integrals Don't Converge Under Pointwise Convergence)

Let $f_n(x) = n \cdot \chi_{[0,1/n]}(x)$, so $f_n(x) \to 0$ a.e. in $[0,1]$ but $\int f_n(x) = 1$ for all $n \in \mathbb{N}$.

But not all hope is lost. This is where the key theorems of measure theory comes in. The following theorem is more general, since uniform convergence implies uniformly bounded, but it isn't used in practice as much (according to Kiselev).

Theorem 3.8 (Bounded Convergence Theorem)

Suppose f_n are measurable, uniformly bounded.[a] Suppose $f_n \to f$ pointwise on E, with $m(E) < \infty$. Then,

$$\int_E f_n \to \int_E f \tag{225}$$

[a] $\|f_n\|_\infty \leq M < \infty$

Proof. The idea is to use Egorov to split the domain into F and $E \setminus F$. Over F, we can then use the fact that $f_n \to f$ uniformly, and over $E \setminus F$, we can control the non-uniformness with a small measure $m(E \setminus F)$.
f is bounded because f_n are uniformly bounded, and it is measurable since it's a limit of measurable functions. Fix $\epsilon > 0$. by Egoroff, $\exists$ closed $F \subset E$ s.t. $f_n \to f$ uniformly on F, and $m(E \setminus F) \leq \frac{\epsilon}{4M}$. Then,

$$\left| \int_E f - \int_E f_n \right| \leq \int_E |f - f_n| = \int_F |f - f_n| + \int_{E \setminus F} |f - f_n| \tag{226}$$

For the first term, $\exists N \in \mathbb{N}$ s.t. if $n \geq N$, then $|f_n(x) - f(x)| \leq \frac{\epsilon}{2m(F)} \leq \frac{\epsilon}{2}$ for all $x \in F$. For the second term, we can bound this by $2M \cdot \frac{\epsilon}{4M} \leq \frac{\epsilon}{2}$. Therefore, the whole expression $\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$.

However, this assumption is still too strong for convergence and is therefore not used much in practice unlike other theorems (e.g. monotone convergence theorem and dominated convergence theorem). One nice part is that it works for unbounded functions, which is good since working with bounded functions is not the most natural assumption to have in practice. This is why we will step back and reconstruct the Lebesgue integral using positive—and possibly unbounded—simple functions (1st definition).

3.3 Positive Measurable Functions

The next natural step to generalize Lebesgue integral is to look at unbounded functions and/or infinite measures. In order to do this, we must start off by stepping back into only nonnegative functions first.

Unlike Riemann integration and Lebesgue integration of signed bounded functions, which looks at both the supremum and infimum of integrals of simple functions, Lebesgue integration of positive only looks at the supremum, given that f is nonnegative, so for all these f, the Lebesgue integral always exists.

Definition 3.3 (Lebesgue Integral on Positive Measurable Functions)

Let $f : E \subset \mathbb{R} \to \mathbb{R}$ satisfy $f \geq 0$, with E measurable (but not necessarily that $m(E) < +\infty$). Then, the **Lebesgue integral** is defined

$$\int_E f \coloneqq \sup \left\{ \int_E h \;\middle|\; 0 \leq h \leq f, h \text{ measurable, bounded} \right\} \tag{227}$$

$$= \sup \left\{ \int_E \phi \;\middle|\; 0 \leq \phi \leq f, \phi \text{ simple} \right\} \tag{228}$$

A positive measurable function f is **Lebesgue integrable** if $\int f < +\infty$.

Proof. The only thing to show is that it suffices to check for simple functions only, so all we have to do is check the latter form.

Theorem 3.9 (Integral Properties for Nonnegative Measurable Functions)

Suppose nonnegative $f, g : E \subset \mathbb{R} \to \mathbb{R}$ are measurable. Then,

1. Linearity. For all $\alpha, \beta \geq 0$,[a]
$$\int_E (\alpha f + \beta g) = \alpha \int f + \beta \int g \tag{229}$$
2. Monotonicity.
$$f \leq g \implies \int f \leq \int g \tag{230}$$
3. Additivity. If A, B are disjoint with $E = A \cup B$, then
$$\int_E f = \int_A f + \int_B f \tag{231}$$

[a]Note that we have ≥ 0 since we are dealing with positive functions!

Proof. Listed.

1. We can easily show $\int \alpha f = \alpha \int f$ simply by multiplying the simple functions ϕ by α. For integrals of sums of functions, we show the following.
   (a) $\int f + \int g \leq \int (f + g)$. Since given simple ϕ_1, ϕ_2 with $\phi_1 \leq f, \phi_2 \leq g$, we have $\phi_1 + \phi_2 \leq f + g$, and so by taking the supremum, we have the bound.
   (b) $\int f + \int g \geq \int (f + g)$. Suppose h is simple s.t. $0 \leq h \leq f + g$. Define[a]
   $$\ell \coloneqq \min\{h, f\}, \qquad k \coloneqq h - \ell \tag{232}$$
   Note that ℓ, k are both measurable. Furthermore, ℓ is at most h (which is simple) and so is bounded, and k is also bounded since it is bounded h minus some nonnegative ℓ that is at most h. Therefore, we invoke our previous definition of the Lebesgue integral for bounded functions to get
   $$\int k + \int \ell = \int h \tag{233}$$
   Since $\ell \leq f$ and $k = h - \ell \leq g$, by monotonicity we have $\int k \leq \int g$ and $\int \ell \leq \int f$. Substituting this in gives
   $$\int h \leq \int f + \int g \implies \int (f + g) = \sup_h \int h \leq \int f + \int g \tag{234}$$
2. Direct.

3.

[a]Intuitively, we are trying to split the $f+g$ into ℓ and k, such that $\ell + k = h \leq f + g$.

Lemma 3.10 (Chebyshev)

Suppose $f : E \subset \mathbb{R} \to \mathbb{R}$, $f \geq 0$, and f is measurable. Then for all $\alpha > 0$,[a]

$$m(\{x \in E \mid f(x) \geq \alpha\}) \leq \frac{1}{\alpha} \int_E f \tag{235}$$

[a]In probability, this refers to a bound on the variance of a random variable, but in analysis, it seems to be a more general result of the bounds of form $\int |f|^p$.

Proof. Let us call the set on the LHS E_α. Then, E_α is measurable and define $\phi(x) = \alpha \chi_{E_\alpha}(x)$. Then, $\phi(x) \leq f(x)$, and so

$$\int f \geq \int \phi = \alpha m(E_\alpha) \tag{236}$$

Theorem 3.11 (Vanishing Integral iff a.e. Vanishing Nonnegative Function)

Suppose $f \geq 0$ on E. Then,

$$\int_E f = 0 \iff f = 0 \text{ a.e. on } E \tag{237}$$

Proof. We prove bidirectionally.

1. $(\rightarrow)$. By Chebyshev, we see that

$$m(\underbrace{\{x \in E \mid f(x) \geq 1/n\}}_{E_n}) \leq n \int f = 0 \tag{238}$$

for all $n \in \mathbb{N}$. But $\{x \in E \mid f(x) > 0\} = \cup_{n=1}^\infty E_n$. By countable additivity, $m(\{x \in E \mid f(x) > 0\}) = 0$.
2. $(\leftarrow)$. Since $f = 0$ a.e., any $0 \leq \phi \leq f$—ϕ simple—will satisfy $m(\{x \in E \mid \phi(x) = 0\}) = 0$, but it will never get off 0. Therefore, $\int_E \phi = 0$.

Theorem 3.12 (Integrable Functions can be Infinite on at Most 0 Measure Set)

If f is integrable on E, then $f(x) < +\infty$ a.e. on E.

Proof. By Chebyshev,

$$m(\{x \in E \mid f(x) \geq n\}) \leq \frac{1}{n} \int_E f \tag{239}$$

As $n \to \infty$, the LHS is bounded above by all $\epsilon > 0$.

$$m(\{x \in E \mid f(x) = \infty\}) = m\left(\bigcap_{n=1}^\infty E_n\right) \tag{240}$$

Lemma 3.13 (Fatou's Lemma)

Let $f_n \geq 0$ be measurable, and it converges to f. Then,

$$\int_E \liminf f_n \leq \liminf \int_E f_n, \qquad \int_E f \leq \liminf \int_E f_n \tag{241}$$

Proof. Here is a proof that doesn't require MCT. The idea is that we want to use bounded convergence theorem to do the work for us. The f_n's are all measurable already, but we are missing two things. First, the measure of the total space may not be finite. Second we don't have a uniformly bounded sequence of functions. We can solve both of these by appealing to the definition of the integral.
It suffices to show that if h is any bounded measurable function of finite support s.t. $0 \leq h \leq f$, then

$$\int_E h \leq \liminf_{n\to\infty} \int_E f_n \tag{242}$$

since taking the supremum w.r.t. h gives the result. The h being bounded gives us a start, so let it be bounded by M. As for the finite measure, we know that h has finite support, so define

$$E_0 = \{x \in E \mid h(x) \neq 0\} \tag{243}$$

and we have $m(E_0) < +\infty$. Now we must construct a sequence of uniformly bounded functions. Define the measurable functions

$$h_n \coloneqq \min\{h, f_n\} \tag{244}$$

Since $0 \leq h_n \leq M$ on E_0 and $h_n = 0$ on $E \setminus E_0$, we know that h_n is uniformly bounded on E_0. For each $x \in E$, we know that

$$h(x) \leq f(x), f_n(x) \to f(x) \implies h_n(x) \to h(x) \tag{245}$$

So we actually know what the (h_n) converges to. Therefore, we finally invoke BCT

$$\lim_{n\to\infty} \int_E h_n = \lim_{n\to\infty} \int_{E_0} h_n = \int_{E_0} h = \int_E h \tag{246}$$

Now we use the fact that $h_n \leq f_n$ on E to show that $\int_E h_n \leq \int_E f_n$, and by taking the liminf,

$$\int_E h = \lim_{n\to\infty} \int_E h_n = \liminf_{n\to\infty} \int_E h_n \leq \liminf_{n\to\infty} \int_E f_n \tag{247}$$

Proof. Define $g_k(x) = \inf_{j\geq k} f_j(x)$. Note that $g_k(x) \leq f_k(x)$ by definition, and $g_k(x)$ is increasing. Since by definition $\lim_{k\to\infty} g_k(x) = \liminf f_k(x)$, by MCT,

$$\lim_{k\to\infty} \int_E g_k = \int \liminf f_k \tag{248}$$

But since $\int g_k \leq \int f_k$ for all k,

$$\liminf \int f_k \geq \int \liminf f_k \tag{249}$$

Note that equality does not have to hold. For example, take $f_n(x) = \chi_{[n,n+1]}(x)$.

We know that the integral doesn't behave well under weak kinds of convergence, e.g. pointwise or a.e. The following theorem gives us a slightly stronger assumption.

Theorem 3.14 (Monotone Convergene Theorem (MCT))

Given a nondecreasing sequence of nonnegative measurable functions $f_1 \leq f_2 \leq f_3 \leq \ldots : E \subset \mathbb{R} \longrightarrow [0, +\infty]$, its limit $\lim_{n\to\infty} f_n$ always exists[a], is measurable, and

$$\int_E \lim_{k\to\infty} f_k = \lim_{k\to\infty} \int_E f_k \tag{250}$$

[a]since for every $x \in E$, $f_n(x)$ is a monotonic sequence in $[0, +\infty]$, which is guaranteed to converge.

Proof. By Fatou,

$$\int_E f \leq \liminf \int_E f_n \tag{251}$$

For each index n, $f_n \leq f$ a.e. on E, so by monotonicity of integration, we have

$$\int_E f_n \leq \int_E f \implies \limsup \int_E f_n \leq \int_E f \tag{252}$$

So combining the two inequalities give

$$\int_E f = \lim_{n\to\infty} \int f_n \tag{253}$$

Proof. Set the RHS $\lim_{n\to\infty} \int f_n = \alpha$ (could be infinity) and let $\lim_{n\to\infty} f_n = f$. $f(x) \geq f_n(x)$ for all x, n, so $\int f \geq \alpha$. Consider $0 \leq \phi \leq f$, ϕ simple. Take $0 < c < 1$[a] and define

$$E_n = \{x \mid f_n(x) \geq c\phi(x)\} \tag{254}$$

Then, the E_n are increasing and $\cup_{n=1}^\infty E_n = E$.[b] Observe that

$$\int_{E_n} f_n \geq c \int_{E_n} \phi \tag{255}$$

Suppose that $\phi(x) = \sum_{k=1}^M a_k \chi_{F_k}(x)$. Then,

$$\int_{E_n} \phi = \sum_{k=1}^M a_k m(F_k \cap E_n) \to \sum_{k=1}^M a_k m(F_k) = \int_E \phi \text{ as } n \to +\infty \tag{256}$$

Therefore, by taking the limit of 255 as $n \to \infty$, we get

$$\lim_{n\to\infty} \int_E f_n \geq c \int_E \phi \quad \forall c < 1 \implies \text{ also true for } c = 1 \tag{257}$$

Since $\phi \leq f$ is arbitrary, just take the supremum over all ϕ and we get $\lim_{n\to\infty} \int f_n \geq \int f$.

[a]We introduce the c because we can then claim that the union of E_n is E.
[b]This may not be true is $c = 1$.

Here is a variant of MCT.

Lemma 3.15 (Levi's Lemma)

Suppose $f_n \geq 0$ is increasing and $\int_E f_n \leq M$ for all n. Then,

1. $f(x) = \lim_{n\to\infty} f_n(x)$ is integrable, and

2. $\int f \leq M$.

Proof. By MCT, $\int_E f_n \to \int_E f$.

The huge problem with Riemann integrals is that this theorem doesn't hold, but it is the case for Lebesgue integration. In practice, the MCT is not as useful, but the following is.

3.4 Lebesgue Integral for Signed Functions

Definition 3.4 (Lebesgue Integral for Signed Functions)

Given $f : E \subset \mathbb{R} \to \mathbb{R} \cup \{\pm\infty\}$, let us write

$$f = f^+ - f^-, \qquad f^+ \coloneqq \max\{0, f\}, f^- \coloneqq \max\{-f, 0\} \tag{258}$$

f is **Lebesgue integrable** iff f^+ and f^- and we define the **Lebesgue integral** of f as

$$\int f \coloneqq \int f^+ - \int f^- \tag{259}$$

given that at least one of these integrals is finite. If one is infinite and the other is finite, then we can call it infinite. If we have *both* infinite integrals, then the integral doesn't exist.

If f^+, f^- are Lebesgue integrable, then it can only be $\pm\infty$ on a set of measure 0, so it generally won't affect anything. Also, there is a difference between the *existence* of the integral and a function *being* Riemann-integrable.

Since $|f| = f^+ + f^-$, f is also Lebesgue integrable if

$$\int |f| \, d\mu < \infty \tag{260}$$

since by triangle inequality, we have

$$\left| \int f \, d\mu \right| \leq \int |f| \, d\mu \tag{261}$$

Theorem 3.16

$f^\pm$ integrable iff $|f|$ integrable.

Proof. $|f| = f^+ + f^-$.

Theorem 3.17 (Integral Properties of Signed Measurable Functions)

Suppose $f, g : E \subset \mathbb{R} \to \mathbb{R}$ are measurable. Then,

1. Linearity. For all $\alpha, \beta \in \mathbb{R}$,
$$\int_E (\alpha f + \beta g) = \alpha \int f + \beta \int g \tag{262}$$
2. Monotonicity.
$$f \leq g \implies \int f \leq \int g \tag{263}$$

3. Additivity. If A, B are disjoint with $E = A \cup B$, then

$$\int_E f = \int_A f + \int_B f \tag{264}$$

Proof. Listed. As always, linearity is nontrivial.

1. For scalar multiplication, we divide into 2 cases.
   (a) $\alpha > 0$. Then, $(\alpha f)^+ = \alpha f^+$ and $(\alpha f)^- = \alpha f^-$, so

   $$\int (\alpha f) = \int \alpha f^+ - \int \alpha f^- = \alpha \int f \tag{265}$$

   (b) $\alpha < 0$. Then, $(\alpha f)^+ = -\alpha f^+$ and $(\alpha f)^- = \alpha f^+$, so

   $$\int (\alpha f) = \int -\alpha f^- + \int \alpha f^+ = \alpha \bigg(\int f^+ - \int f^- \bigg) = \alpha \int f \tag{266}$$

2. For addition, note that $|f + g| \leq |f| + |g|$, so it is integrable. So

   $$\int (f + g) = \int (f + g)^+ - \int (f + g)^- \tag{267}$$

   Now observe that

   $$(f + g)^+ - (f + g)^- = f + g = f^+ + g^+ - f^- - g^- \tag{268}$$
   $$(f + g)^+ + f^- + g^- = (f + g)^- + f^+ + g^+ \tag{269}$$

   Since it is nonnegative, our previous properties of linearity holds, and then we can just rearrange.
3.
4.

At this point, we have a general sense of the relationship between measurability and integrability. We know the following

1. If f is integrable, then it is necessary that it be measurable.[4] So measurability is a *must* for integrability.
2. If f is measurable plus bounded, then we get integrability. The boundedness basically locks in finiteness.

We actually generalize the boundedness now.

Theorem 3.18 (Integral Comparison Test)

Theorem 3.19 (Dominated Convergence Theorem)

Let (f_n) be a sequence of measurable functions on E. Suppose that there is a function g that is integrable over E and dominates (f_n) on E in the sense that

$$|f_n| \leq g \text{ on } E \quad \forall n \tag{270}$$

Then, if $f_n \to f$ pointwise a.e. on E, then f is integrable over E and

$$\lim_{n\to\infty} \int_E f_n = \int_E f \tag{271}$$

[4]This is unlike Riemann integration, where there may be discontinuous functions that are Riemann integrable.

Proof. Since f_n is dominated by integrable g, each f_n is integrable. Also, it must follow that f is also dominated by g, and so f is also integrable. We would like to use Fatou's lemma, but these f_n are not positive. This is easy, since we can define the sequence

$$(g - f_n)_n, \qquad g - f_n \to g - f \tag{272}$$

where each $g - f_n \geq g - |f_n| \geq 0$. Therefore, applying Fatou,

$$\int_E g - f = \int_E \liminf_{n\to\infty} (g - f_n) \leq \liminf_{n\to\infty} \int_E (g - f_n) \tag{273}$$

Expanding it gives

$$\int_E g - \int_E f \leq \int_E g - \liminf_{n\to\infty} \int_E f_n \implies \int_E f \geq \limsup_{n\to\infty} \int_E f_n \tag{274}$$

Now apply this same logic to the sequence $(g + f_n)_n$ and we get

$$\int_E f \leq \liminf_{n\to\infty} \int_E f_n \tag{275}$$

which completes the proof.

We can slightly generalize this by taking a sequence of functions g_n that dominates f_n.

Theorem 3.20 (Generalized Dominated Convergence Theorem)

Let measurable $f_n \to f$ a.e. Suppose there exists a sequence of *nonnegative* measurable (g_n) s.t. $g_n \to g$ a.e. and g_n dominates f_n in that

$$|f_n| \leq g_n \quad \forall n \in \mathbb{N} \tag{276}$$

Then,

$$\lim_{n\to\infty} \int_E g = \int_E g \implies \lim_{n\to\infty} \int_E f = \int_E f \tag{277}$$

Proof. It is the same proof, but by replacing the sequences with $(g_n - f_n)_n$ and $(g_n + f_n)_n$.

The dominated convergence theorem is used most heavily in analysis.

3.5 Uniformly Integrable Families

Lemma 3.21 (Decomposition of Finite Measure Set into δ Pieces)

Let E be measurable with $m(E) < +\infty$. Then $\forall \delta > 0$, E is the disjoint union of a finite collection of sets, each of which has measure less than δ.

Proof. If E is bounded, then say it is in $[-m, m]$, and divide it into subintervals each $< \delta$. If not, then by continuity of measure from above,

$$\lim_{n\to\infty} m(E \setminus [-n, n]) = m(\emptyset) = 0 \tag{278}$$

Therefore, we can choose $N \in \mathbb{N}$ s.t. $m(E \setminus [-N, N]) < \delta$, and for the rest of the pieces, just partition the bounded set $E \cap N$.

Great, so now we have established another condition for integrability, but we would like to do better. The following theorem gives an *almost* equivalent condition of integrability. This is a weird definition, so let's try to break it down. Colloquially, we are saying that by taking *any* $A \subset E$ with small measure, the integral is also guaranteed to be small. That is, there are no weird sets A that have small measure $m(A) < \delta$ but $\int_A f$ stays large.

Theorem 3.22 (Conditions for Integrability)

Let f be measurable on E.

1. If f is integrable over E, then $\forall \epsilon > 0$, $\exists \delta > 0$ s.t. if $\forall A \subset E$ is measurable with $m(A) < \delta$, then

$$\int_A |f| < \epsilon \tag{279}$$

2. If $m(E) < +\infty$, then the converse holds.

Proof. We prove bidirectionally.

1. $(\rightarrow)$. The idea is that first, if f is bounded, then the result is trivial since we can choose $\delta = \frac{\epsilon}{M}$, where M is bounded. If it isn't bounded, we would like to perhaps decompose it into a bounded part that we can control, plus an unbounded part, but with a small integral. To do this, we can by definition of the integral, choose a measurable bounded $0 \leq h \leq f$ s.t. its integral is arbitrarily close to f, say

$$\int_E f - \int_E h < \frac{\epsilon}{2} \tag{280}$$

Therefore, we have essentially bounded the integral of the unbounded portion of f. So, if there exists measurable $A \subset E$ s.t. $m(A) < \frac{\epsilon}{2M}$, then

$$\int_A f - \int_A h = \int_A f - h \leq \int_E f - h = \int_E f - \int_E h < \frac{\epsilon}{2} \tag{281}$$

Therefore, assuming that h is bounded by M, we can choose $\delta = \frac{\epsilon}{2M}$ to bound the integral of h, and finally get

$$\int_A f \leq \int_A h + \frac{\epsilon}{2} \leq \frac{\epsilon}{2M} \cdot M + \frac{\epsilon}{2} = \epsilon \tag{282}$$

[a]

2. $(\leftarrow)$. For $\epsilon = 1$, let us choose $\delta_0 > 0$ such that for all $A \subset E$ with $m(A) < \delta$, we have

$$\int_A |f| < 1 \tag{283}$$

Now using the lemma, let's find a finite partition of sets $\{E_k\}_{k=1}^n$, each with measure $< \delta_0$. Since $m(E_k) < \delta_0$, we have

$$\int_{E_k} f < 1 \implies \sum_{k=1}^n \int_{E_k} f < N \tag{284}$$

and by additivity of integration, we have $\int_E f < N$, and so if $0 \leq h \leq f$ is a measurable function of finite support, then $\int_E h < N$. Therefore, the supremum of all integrals of h must be finite, and so f is integrable.

[a] Otherwise, we want to choose a nice approximation f_ϵ s.t. $0 \leq f_\epsilon \leq |f|$, $\int_E |f| - \int_E f_\epsilon \leq \frac{\epsilon}{2}$ Maybe take $f_\epsilon = \min\{f, n\}$, which is bounded, and by MCT, we can prove it.

Now we state the family analogue of integrability.

Definition 3.5 (Uniformly Integrable)

A family $\mathscr{F}$ of measurable functions is **uniformly integrable (u.i.) over** E—also called **equi-integrable**—iff $\forall \epsilon > 0$, $\exists \delta > 0$ s.t.

$$A \subseteq E, m(A) < \delta \implies \int_A f < \epsilon \quad \forall f \in \mathscr{F} \tag{285}$$

Example 3.4 (Finite Family of Integrable Functions is UI)

A finite family of integrable functions $\{f_1, \ldots, f_n\}$ is always uniformly integrable, since we can take $\delta = \min_{1 \leq i \leq n}\{\delta_i\}$, which will satisfy the first theorem.

Example 3.5 (Dominated Family of Integrable Function is UI)

Fix some integrable g, and consider the uniformly integrable family

$$\mathscr{F} \coloneqq \{f \mid |f| \leq g\} \tag{286}$$

g integrable means that $\forall \epsilon > 0$, $\exists \delta > 0$ s.t.

$$A \subset E, m(A) < \delta \implies \int_A g < \epsilon \tag{287}$$

But $\int f \leq \int |f| < \int g$ for all $f \in \mathscr{F}$, and we are done.
There is a second proof I thought of, but this is incorrect. Here is goes: By DCT, f is integrable, so $\forall \epsilon > 0$, $\exists \delta > 0$ s.t.

$$A \subset E, m(A) < \delta \implies \int_A f < \epsilon \tag{288}$$

However, we still haven't removed the dependency on f, and since f may be infinite, we can't just take the minimum either. So this leads nowhere.

But not every function in a uniformly integrable family is integrable due to the extra awkward $m(E) < +\infty$ assumption.

Example 3.6 (Family of Integrable Functions May not be UI)

Even if all functions are integrable, the family may not be.

1. Consider
$$\{f_n \coloneqq n \cdot \chi_{[0,1/n]}\}_{n \in \mathbb{N}} \tag{289}$$
Take $\epsilon > 1/2$ over $E = [-1, 1]$.
2. Consider
$$\{f_n \coloneqq \min\{|x|^{-1}, n\}\}_n \tag{290}$$
which is not UI.

This is useful for showing convergence of integrals.

Lemma 3.23 (Limit of U.I. Functions is Integrable)

Assume $m(E) < +\infty$.[a] Let $\{f_n : E \to \mathbb{R}\}$ be u.i. over E and $f_n \to f$ a.e. on E. Then, f is integrable.

[a] This is to avoid the awkward assumption above.

Proof. It is similar to the proof for ϵ-δ criterion. Fix $\epsilon \geq 1$. By definition of u.i., we can find corresponding δ. Now split E into finite union of disjoint E_j. This is where we need the finite measure assumption from.

$$E = \bigsqcup_{j=1}^{N} E_j \qquad m(E_j) < \delta \tag{291}$$

Then, $\int_E |f_n| = \sum_{j=1}^N \int_{E_j} |f_n| \leq N$, so N is independent of n. By Fatou,

$$\int_E |f| \leq \liminf_{n\to\infty} \int_E |f_n| \leq N \tag{292}$$

The following convergence theorem gives the same conclusion in MCT or DCT, but under weaker assumptions, though now we must assue that the measure of the whole set is finite. This is in fact why Vitali is such a popular theorem in probability.

Theorem 3.24 (Vitali Convergence Theorem)

Assume $m(E) < +\infty$. Let $\{f_n : E \to \mathbb{R}\}$ be u.i. over E and $f_n \to f$ a.e. on E. Then,

$$\lim_{n\to\infty} \int_E f_n = \int_E f \tag{293}$$

Proof. Fix $\epsilon > 0$, and we wish to show that

$$\lim_{n\to\infty} \int_E |f_n - f| = 0 \tag{294}$$

Fix $\epsilon > 0$ (by definition of u.i.) s.t. if $m(A) < \delta$, $\int_A |f_n| < \frac{\epsilon}{3}$. By Fatou, $\int_A |f| < \frac{\epsilon}{3}$. Now by Egorov, since $m(E) < +\infty$, $f_n \to f$ a.e. We can find some set E_0 s.t. $m(E_0) < \delta$ and $f_n \to f$ uniformly on $E \setminus E_0$.
Choose N s.t. if $n > N$,

$$|f_n(x) - f(x)| < \frac{\epsilon}{3m(E)} \quad \forall x \in E \setminus E_0 \tag{295}$$

Then, if $n > N$,

$$\int_E |f_n - f| = \int_{E_0} |f_n - f| + \int_{E\setminus E_0} |f_n - f| \tag{296}$$

We know

1. For the first term, we know that the measure is small, so we use uniform integrability.

$$\int_{E_0} |f_n - f| \leq \int_{E_0} |f_n| + \int_{E_0} |f| \leq \frac{\epsilon}{3} + \frac{\epsilon}{3} \tag{297}$$

2. For the second term, we know that at each point, $|f_n - f|$ is bounded by $\frac{\epsilon}{3m(E)}$. We use uniform convergence to bound it.

$$\int_{E\setminus E_0} |f_n - f| \leq \frac{\epsilon}{3m(E)} \cdot m(E) \tag{298}$$

Note that the proof of the previous lemma tells us that f is integrable only if $\int |f_n| \leq C$ and $f_n \to f$ a.e. Equality does not hold under weaker conditions, since

$$f_n(x) = n\chi_{[0,1/n]}(x), f_n \to 0 \text{ a.e.} \tag{299}$$

but $\int f_n = 1$ and $\int f = 0$. So this is why we need integrability.

What is great about Vitali is that this conclusion is essentially "sharp." Let's make this more rigorous.

Theorem 3.25

If $m(E) < +\infty$, $h_n \geq 0$ is integrable, and $h_n \to 0$ a.e. on E, then

$$\lim_{n\to\infty} \int_E h_n = 0 \iff (h_n) \text{ is uniformly integrable over } E \tag{300}$$

Proof. For the backwards implication, this is Vitali. For the forward implication, fix $\epsilon \geq 0$. Choose $N \in \mathbb{N}$ large s.t.

$$\int_E h_n < \epsilon \quad \forall n \geq N \tag{301}$$

Note that $\{h_1, \ldots, h_n\}$ is a finite family, which we showed was u.i. So by definition $\exists \delta > 0$ s.t. if $m(A) < \delta$,

$$\int_A h_n < \epsilon \quad \forall n \in \{1, \ldots, N\} \tag{302}$$

But $\int_A h_n < \epsilon$ is also true for $n \geq N+1$. So essentially, we have combined a finite family with an infinite family to get one large u.i. family.

So we *need* the uniformly integrability condition for Vitali. We might ask if $m(E) < +\infty$ can be taken out, but in general it cannot. It is necessary because you can consider $f_n(x) = \chi_{[n,n+1]}(x)$ with $E = \mathbb{R}$. So a natural question is to ask: what should we add in order to replace $m(E) < +\infty$?

Lemma 3.26

Suppose f is integrable over set E. Then $\forall \epsilon > 0$, $\exists E_0 \subset E$ with $m(E_0) < +\infty$ s.t.

$$\int_{E \setminus E_0} |f| < \epsilon \tag{303}$$

Colloquially, the integral should be "concentrated" over a finite measure set, even if the space is infinite.

Proof. We again approximate by bounded functions. By definition of integrability, $\exists g$ bounded, with compact support (since integral is finite), s.t.

$$0 \leq g \leq |f|, \qquad \int_E |f| - \int_E g < \epsilon \tag{304}$$

Then, just take $E_0 = \text{supp}(g)$, and split the integral to write the inequality as

$$\int_{E\setminus E_0} |f| + \underbrace{\int_{E_0} |f| - \int_E g|}_{\geq 0 \text{ since } 0 \leq g \leq |f|} < \epsilon \implies \int_{E\setminus E_0} |f| < \epsilon \tag{305}$$

This lemma tells you that if you have even 1 integrable function, you can find that the integral is concentrated around a set of finite measure. But this is for 1 function only, and we want to do it for a *sequence* of functions.

Definition 3.6 (Tight)

A family $\mathscr{F}$ of measurable functions on E is **tight over** E if $\forall \epsilon > 0$, $\exists E_0 \subset E$ with $m(E_0) < +\infty$ s.t.

$$\int_{E \setminus E_0} |f| < \epsilon \quad \forall f \in \mathscr{F} \tag{306}$$

This is a more general condition since if $m(E) < +\infty$, then we can just take $E_0 = E$ and it is automatically tight. Therefore, we can replace the finite measure assumption with tightness.

Theorem 3.27 (Generalized Vitali Convergence Theorem)

Let (f_n) be a sequence of functions on E that is u.i. and tight, with $f_n \to f$ a.e. on E.

$$\lim_{n\to\infty} \int_E f_n = \int f, \qquad \lim_{n\to\infty} \int_E |f_n - f| = 0 \tag{307}$$

Proof. Fix $\epsilon > 0$. By tightness, $\exists E_0 \subset E$ with $m(E_0) < +\infty$ s.t.

$$\int_{E\setminus E_0} |f_n| < \frac{\epsilon}{4} \quad \forall n \in \mathbb{N} \tag{308}$$

By Fatou, $\int_{E\setminus E_0} |f| < \frac{\epsilon}{4}$. Now combine both inequalities and triangle inequality to get

$$\int_{E\setminus E_0} |f_n - f| < \int_{E\setminus E_0} (|f_n| + |f|) < \frac{\epsilon}{2} \tag{309}$$

Now $m(E_0) < +\infty$ and (f_n) is u.i. on E_0. By VCT, $\exists N \in \mathbb{N}$ s.t.

$$n \geq N \implies \int_{E_0} |f_n - f| < \frac{\epsilon}{2} \tag{310}$$

Now combine 309 and 310 to get

$$\int_E |f_n - f| = \int_{E\setminus E_0} |f_n - f| + \int_{E_0} |f_n - f| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \tag{311}$$

Note that the first term $\int_{E\setminus E_0} |f_n - f|$ has an infinite measure, though the integral itself is small. The second is over a finite measure set, which we can use VCT.

But this is not a complete generalization, since in VCT we don't assume f_n is integrable, but generalized VCT assumes uniform integrability.

3.6 Riemann vs Lebesgue Integrability

Now we present equivalent characterizations of Riemann and Lebesgue integrability.

Lemma 3.28

Suppose f is bounded, and there exists measurable sequences of functions ϕ_n, ψ_n s.t.

$$\psi_n(x) \leq f(x) \leq \psi_n(x) \quad \forall x \in E \tag{312}$$

and

$$\lim_{n\to+\infty}\int_E (\psi_n - \phi_n) = 0 \tag{313}$$

Then, there exists $\tilde{\phi}_n \to f$ and $\tilde{\psi}_n \to f$ a.e.

Proof. Define

$$\tilde{\phi}_n(x) = \max\{\phi_1(x), \ldots, \phi_n(x)\}, \quad \tilde{\psi}_n(x) = \min\{\psi_1(x), \ldots, \psi_n(x)\} \tag{314}$$

We still have $\tilde{\phi}_n(x) \leq f(x) \leq \tilde{\phi}_n(x)$ for all n and for all x. Also, $\tilde{\phi}_n(x)$ is increasing, $\tilde{\psi}_n(x)$ is decreasing. Now define

$$\phi^*(x) \coloneqq \lim_{n\to\infty} \tilde{\phi}_n(x), \qquad \psi^*(x) \coloneqq \lim_{n\to\infty} \tilde{\psi}_n(x) \tag{315}$$

Observe that

$$\int (\tilde{\psi}_n - \tilde{\phi}_n) \leq \int (\psi_n - \phi_n) \implies \int (\tilde{\psi}_n - \tilde{\phi}_n) \to 0 \text{ as } n \to \infty \tag{316}$$

Also,

$$\int (\underbrace{\psi^*(x) - \phi^*(x)}_{\geq 0})\, dx \leq \int (\tilde{\psi}^* - \tilde{\phi}^*) \tag{317}$$

for all n. Therefore,

$$\int (\psi^*(x) - \phi^*(x)) = 0 \implies \psi^*(x) = \phi^*(x) \text{ a.e.} \tag{318}$$

And so $f(x)$, which is sandwiched between ψ^* and ϕ^*, must be equal a.e. We didn't assume that f was measurable, but these ψ_n, ϕ_n is measurable by assumption.

Now, we can prove this master theorem.

Theorem 3.29 (Characterization of Lebesgue Integrability)

Let f be bounded on measurable set E of finite measure. Then f is Lebesgue integrable iff f is measurable.

Proof. The backward implication is true in general. We want to show that f is measurable. Recall that for bounded functions, we defined Lebesgue integrals with $\underline{L}f$ and $\overline{L}f$. Therefore, we can find simple ϕ_n, ψ_n s.t. $\phi_n \leq f \leq \psi_n$, and $\int \psi_n - \int \phi_n \leq 1/n$. Now we are exactly in the setting of the lemma, and so by the lemma, we can find measurable $\tilde{\psi}_n(x) \to f$ a.e. (in fact, $\tilde{\psi}$ will be simple). Since the limit of measurable functions is measurable, f is measurable.

This is a very reasonable criterion, and you can't really hope for more then Lebesgue measurability. This following theorem on Riemmann integrability is much more restrictive, while for above, measurable functions can be very wild.

Theorem 3.30 (Characterization of Riemann Integrability)

f is Riemann integrable on $[a, b]$ if the set of its discontinuities has measure 0.

Proof. Not stated. In book.

3.7 Exercises

Exercise 3.1 (Royden 4.1)

Show that, in the above Dirichlet function example, $\{f_n\}$ fails to converge to f uniformly on $[0,1]$.

Exercise 3.2 (Royden 4.2)

A partition P' of $[a,b]$ is called a refinement of a partition P provided each partition point of P is also a partition point of P'. For a bounded function f on $[a,b]$, show that under refinement lower Darboux sums increase and upper Darboux sums decrease.

Exercise 3.3 (Royden 4.3)

Use the preceding problem to show that for a bounded function on a closed, bounded interval, each lower Darboux sum is no greater than each upper Darboux sum. From this conclude that the lower Riemann integral is no greater than the upper Riemann integral.

Exercise 3.4 (Royden 4.4)

Suppose the bounded function f on $[a,b]$ is Riemann integrable over $[a,b]$. Show that there is a sequence $\{P_n\}$ of partitions of $[a,b]$ for which $\lim_{n\to\infty}[U(f,P_n) - L(f,P_n)] = 0$.

Exercise 3.5 (Royden 4.5)

Let f be a bounded function on $[a,b]$. Suppose there is a sequence $\{P_n\}$ of partitions of $[a,b]$ for which $\lim_{n\to\infty}[U(f,P_n) - L(f,P_n)] = 0$. Show that f is Riemann integrable over $[a,b]$.

Exercise 3.6 (Royden 4.6)

Use the preceding problem to show that since a continuous function f on a closed, bounded interval $[a,b]$ is uniformly continuous on $[a,b]$, it is Riemann integrable over $[a,b]$.

Exercise 3.7 (Royden 4.7)

Let f be an increasing real-valued function on $[0,1]$. For a natural number n, define P_n to be the partition of $[0,1]$ into n subintervals of length $1/n$. Show that $U(f,P_n) - L(f,P_n) \leq 1/n[f(1) - f(0)]$. Use Problem 5 to show that f is Riemann integrable over $[0,1]$.

Exercise 3.8 (Royden 4.8)

Let $\{f_n\}$ be a sequence of bounded functions that converges uniformly to f on the closed, bounded interval $[a,b]$. If each f_n is Riemann integrable over $[a,b]$, show that f also is Riemann integrable over $[a,b]$. Is it true that

$$\lim_{n\to\infty} \int_a^b f_n = \int_a^b f\,?$$

Exercise 3.9 (Royden 4.9)

Let E have measure zero. Show that if f is a bounded function on E, then f is measurable and $\int_E f = 0$.

Solution. For any measurable set S in the codomain of f,

$$m\big(f^{-1}(S)\big) \leq m(E) = 0 \implies f^{-1}(S) \text{ is measurable} \tag{319}$$

Also,

$$\int_E f \leq \int_E |f| \leq \int_E M = M \cdot m(E) = 0 \tag{320}$$

Exercise 3.10 (Royden 4.10)

Let f be a bounded measurable function on a set of finite measure E. For a measurable subset A of E, show that $\int_A f = \int_E f \cdot \chi_A$.

Exercise 3.11 (Royden 4.11)

Does the Bounded Convergence Theorem hold for the Riemann integral?

Solution. No, look at sequence of functions pointwise converging to Dirichlet function on $[0, 1]$. They are uniformly bounded by 1.

Exercise 3.12 (Royden 4.12)

Let f be a bounded measurable function on a set of finite measure E. Assume g is bounded and $f = g$ a.e. on E. Show that $\int_E f = \int_E g$.

Solution. Let $f = g$ on $F \subset E$. Then,

$$\int_E f = \int_F f + \underbrace{\int_{E \setminus F} f}_{=0} = \int_F f = \int_F g = \int_F g + \underbrace{\int_{E \setminus F} g}_{=0} = \int_E g \tag{321}$$

where the labeled integrals equal 0 from Exercise 3.9.

Exercise 3.13 (Royden 4.13)

Does the Bounded Convergence Theorem hold if $m(E) < \infty$ but we drop the assumption that the sequence $\{|f_n|\}$ is uniformly bounded on E?

Solution. No, consider the triangle function defined on $[0, 1/n]$ as $f(0) = 0$, $f(1/2n) = 2n$, $f(1/n) = 0$, and everything linearly interpolated.

Exercise 3.14 (Royden 4.14)

Show that Proposition 8 is a special case of the Bounded Convergence Theorem.

Solution. It suffices to prove that $f_n \to f$ uniformly implies f_n is uniformly bounded. $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ s.t.

$$n \geq N \implies |f(x) - f_n(x)| < \epsilon \quad \forall x \in E \tag{322}$$

So set $\epsilon > 1$ and then N^* s.t.

$$n \geq N^* |f - f_n| < 1 \implies |f_n| < |f| + 1 \tag{323}$$

So f_n is uniformly bounded by M_{N^*}. The rest of the functions are bounded by $M_1, \ldots, M_{N^*-1}$, so take the uniform bound

$$M = \max\{M_1, \ldots, M_{N^*}\} \tag{324}$$

Exercise 3.15 (Royden 4.15)

Verify the assertions in the last Remark of this section.

Exercise 3.16 (Royden 4.16)

Let f be a nonnegative bounded measurable function on a set of finite measure E. Assume $\int_E f = 0$. Show that $f = 0$ a.e. on E.

Solution. Assume $f \neq 0$ a.e. on E, so call this set $E_0 \subset E$. Then,

$$\int_E f = \int_{E_0} f + \int_{E \setminus E_0} f \geq \int_{E_0} f \tag{325}$$

Now we must prove $0 < \int_{E_0} f$. Define

$$E_n = \{x \in E \colon f(x) > 1/n\} \tag{326}$$

Then, E_n is ascending, and by continuity of measure from below,

$$m(E_0) = m\bigg(\bigcup_n E_n\bigg) = \lim_{n \to \infty} m(E_n) \tag{327}$$

So $\forall \delta > 0$, we can choose $N \in \mathbb{N}$ s.t.

$$m(E_0 \setminus E_N) < \delta, \quad f(x) > \frac{1}{N} \text{ on } E_N \tag{328}$$

which implies

$$\int_{E_0} f = \int_{E_N} f + \int_{E_0 \setminus E_N} f \geq \int_{E_N} f \geq \frac{1}{N} m(E_N) > \frac{1}{N}(m(E_0) - \delta) > 0 \tag{329}$$

Exercise 3.17 (Royden 4.17)

Let E be a set of measure zero and define $f \equiv \infty$ on E. Show that $\int_E f = 0$.

Solution. We know that

$$\int_E f = \sup \left\{ \int_E \phi \colon \phi \le h, \phi \text{ simple} \right\} \tag{330}$$

But for every ϕ, it is simple and therefore bounded by M_ϕ, and so

$$\int_E \phi \le \int_E M_\phi = M_\phi \cdot m(E) = 0 \tag{331}$$

So the supremum must be 0.

Exercise 3.18 (Royden 4.18)

Show that the integral of a bounded measurable function of finite support is properly defined.

Exercise 3.19 (Royden 4.19)

For a number α, define $f(x) = x^\alpha$ for $0 < x \le 1$, and $f(0) = 0$. Compute $\int_0^1 f$.

Exercise 3.20 (Royden 4.20)

Let $\{f_n\}$ be a sequence of nonnegative measurable functions that converges to f pointwise on E. Let $M \ge 0$ be such that $\int_E f_n \le M$ for all n. Show that $\int_E f \le M$. Verify that this property is equivalent to the statement of Fatou's Lemma.

Exercise 3.21 (Royden 4.21)

Let the function f be nonnegative and integrable over E and $\epsilon > 0$.

1. Show there is a simple function η on E that has finite support, $0 \le \eta \le f$ on E and $\int_E |f - \eta| < \epsilon$.
2. If E is a closed, bounded interval, show there is a step function h on E that has finite support and $\int_E |f - h| < \epsilon$.

Solution. Listed.

1. $f \ge 0$ on E, integrable and so measurable. By SAT, $\exists$ increasing sequence of simply functions $\phi_n \to f$. By MCT,
$$\lim_{n\to\infty} \int_E \phi_n = \int_E f \tag{332}$$
So for $\epsilon > 0$, choose $N \in \mathbb{N}$ s.t.
$$n \ge N \implies \left| \int_E f - \int_E \phi_n \right| < \epsilon \tag{333}$$
Set $\eta = \phi_N$. Then,
$$|f - \eta| = f - \eta \implies \left| \int_E f - \eta \right| = \int_E |f - \eta| < \epsilon \tag{334}$$
2. If E is closed and bounded, then $\forall \delta > 0$, by Lusin's there exists continuous g on closed set F s.t.
$$m(E \setminus F)\delta, \quad f = g \text{ on } F \tag{335}$$
f is continuous on F and so it Riemann integrable $\implies \exists$ step function ϕ s.t.
$$\left(\int_F f \right) - \frac{\epsilon}{2} < \int_F \phi < \int_F f \tag{336}$$

By EVT, f is bounded by M, so

$$\int_{E\setminus F} |f| < M \cdot m(E \setminus F) \to 0 \tag{337}$$

So, set $\phi = 0$ on $E \setminus F$ and

$$\int_E |f - \phi| = \underbrace{\int_F |f - \phi|}_{<\epsilon/2} + \underbrace{\int_{E\setminus F} |f|}_{<\epsilon/2} + \underbrace{\int_{E\setminus F} |\phi|}_{=0} \tag{338}$$

■ For the second part, I was thinking of Dini's.

Exercise 3.22 (Royden 4.22)

Let $\{f_n\}$ be a sequence of nonnegative measurable functions on $\mathbf{R}$ that converges pointwise on $\mathbf{R}$ to f and f be integrable over $\mathbf{R}$. Show that

$$\text{if } \int_{\mathbf{R}} f = \lim_{n\to\infty} \int_{\mathbf{R}} f_n, \text{ then } \int_E f = \lim_{n\to\infty} \int_E f_n \text{ for any measurable set } E.$$

Exercise 3.23 (Royden 4.23)

Let $\{a_n\}$ be a sequence of nonnegative real numbers. Define the function f on $E = [1, \infty)$ by setting $f(x) = a_n$ if $n \leq x < n + 1$. Show that $\int_E f = \sum_{n=1}^\infty a_n$.

Solution. Let $f_n = \sum_{k=1}^n a_k$. Then, f_n is increasing, and $f_n \to f$. By MCT,

$$\int_E f = \lim_{n\to\infty} \int_E f_n = \lim_{n\to\infty} \sum_{k=1}^n a_k \cdot m([k, k+1)) = \sum_{k=1}^\infty a_k \tag{339}$$

Exercise 3.24 (Royden 4.24)

Let f be a nonnegative measurable function on E.

1. Show there is an increasing sequence $\{\varphi_n\}$ of nonnegative simple functions on E, each of finite support, which converges pointwise on E to f.
2. Show that $\int_E f = \sup\{\int_E \varphi \mid \varphi \text{ simple, of finite support and } 0 \leq \varphi \leq f \text{ on } E\}$.

Solution. Listed.

1. By SAT, $\exists$ increasing sequence of ϕ_n simply on E s.t. $\phi_n \to f$. Since $f \geq 0$, we can assume $\phi_n = 0$ if negative, which is still simple. For finite support, let's restrict each ϕ_n to be on $E_n = E \cap [-n, n]$, and let $\phi_n = 0$ elsewhere. This still preserves nonnegativity and $\phi_n \to f$.
2. By monotonicity,
$$\forall n, \forall \phi_n \int_E f \geq \int_E \phi_n \implies \int_E f \geq \sup \int_E \phi \tag{340}$$
By MCT,
$$\int_E f = \lim_{n\to\infty} \int_E \phi_n = \lim_{n\to\infty} \sup_{k\geq n} \int_E \phi_k \leq \sup_n \int_E \phi_n \tag{341}$$
where the final inequality follows from the limsup being a monotonically decreasing sequence as

we are taking the supremum over a more restrictive class of functions.

■ . Fatou is not a bad choice for 2, but we see that it leads nowhere since the inequality is in the wrong direction.

$$\int_E f \leq \lim_{n\to\infty} \inf_{k\geq n} \int_E f_n \geq \inf_n \left\{ \int_E f_n \right\} \tag{342}$$

Exercise 3.25 (Royden 4.25)

Let $\{f_n\}$ be a sequence of nonnegative measurable functions on E that converges pointwise on E to f. Suppose $f_n \leq f$ on E for each n. Show that

$$\lim_{n\to\infty} \int_E f_n = \int_E f.$$

Solution. By monotonicity,

$$f \geq f_n \implies \int_E f \geq \int_E f_n \implies \int_E f \geq \limsup_n \int_E f_n \tag{343}$$

By Fatou,

$$\int_E f \leq \liminf_n \int_E f_n \tag{344}$$

By combining these inequalities we are done.

Exercise 3.26 (Royden 4.26)

Show that the Monotone Convergence Theorem may not hold for decreasing sequences of functions.

Exercise 3.27 (Royden 4.27)

Prove the following generalization of Fatou's Lemma: If $\{f_n\}$ is a sequence of nonnegative measurable functions on E, then

$$\int_E \liminf f_n \leq \liminf \int_E f_n.$$

Solution. Let $g_n = \int_{k\geq n} f_k$ which is measurable and increasing. By MCT,

$$\int_E \lim_{n\to\infty} \inf_{k\geq n} f_k = \int_E g = \lim_{n\to\infty} \int_E \inf_{k\geq n} f_k \tag{345}$$

But

$$\int_E \inf_{k\geq n} f_k = \sup \left\{ \int h \colon h \text{ measurable, bounded, finite support}, h \leq \int_{k\geq n} f_k \right\} \tag{346}$$

$$= \inf_{k\geq n} \sup \left\{ \int h \colon h \text{ measurable, bounded, finite support}, h \leq f_k \right\} \tag{347}$$

$$= \inf_{k\geq n} \int_E f_k \tag{348}$$

Exercise 3.28 (Royden 4.28)

Let f be integrable over E and C a measurable subset of E. Show that $\int_C f = \int_E f \cdot \chi_C$.

Exercise 3.29 (Royden 4.29)

For a measurable function f on $[1, \infty)$ which is bounded on bounded sets, define $a_n = \int_n^{n+1} f$ for each natural number n. Is it true that f is integrable over $[1, \infty)$ if and only if the series $\sum_{n=1}^{\infty} a_n$ converges? Is it true that f is integrable over $[1, \infty)$ if and only if the series $\sum_{n=1}^{\infty} a_n$ converges absolutely?

Exercise 3.30 (Royden 4.30)

Let g be a nonnegative integrable function over E and suppose $\{f_n\}$ is a sequence of measurable functions on E such that for each n, $|f_n| \leq g$ a.e. on E. Show that

$$\int_E \liminf f_n \leq \liminf \int_E f_n \leq \limsup \int_E f_n \leq \int_E \limsup f_n.$$

Exercise 3.31 (Royden 4.31)

Let f be a measurable function on E which can be expressed as $f = g + h$ on E, where g is finite and integrable over E and h is nonnegative on E. Define $\int_E f = \int_E g + \int_E h$. Show that this is properly defined in the sense that it is independent of the particular choice of finite integrable function g and nonnegative function h whose sum is f.

Exercise 3.32 (Royden 4.32)

Prove the General Lebesgue Dominated Convergence Theorem by following the proof of the Lebesgue Dominated Convergence Theorem, but replacing the sequences $\{g - f_n\}$ and $\{g + f_n\}$, respectively, by $\{g_n - f_n\}$ and $\{g_n + f_n\}$.

Solution. For all n, we have $|f_n| \leq g_n$ and $f_n \to f$, $g_n \to g$ a.e. on E. So, $|f| \leq g$ pointwise a.e. on E. Therefore, for all n we have

$$|f_n - f| \leq g_n + g \tag{349}$$

pointwise a.e. on E. Now apply Fatou to the nonnegative function $g_n + g - |f_n - f|$ to get

$$\liminf_{n\to\infty} \int_E (g_n + g - |f_n - f|) \geq \int_E \liminf_{n\to\infty} (g_n + g - |f_n - f|) = 2 \int_E g \tag{350}$$

The LHS equals

$$\liminf_{n\to\infty} \int_E (g_n + g - |f_n - f|) = 2 \int_E g - \limsup_{n\to\infty} \int_E |f_n - f| \tag{351}$$

since $\lim_{n\to\infty} \int_E g_n = \int_E g$. Substituting and applying the integral comparison test gives

$$0 \geq \limsup_{n\to\infty} \left| \int_E (f_n - f) \right| \geq \liminf_{n\to\infty} \left| \int_E f_n - f \right| \geq 0 \implies \lim_{n\to\infty} \left| \int_E f_n - f \right| = 0 \tag{352}$$

and by linearity, we can take the $\int_E f$ out.

Exercise 3.33 (Royden 4.33)

Let $\{f_n\}$ be a sequence of integrable functions on E for which $f_n \to f$ a.e. on E and f is integrable over E. Show that $\int_E |f - f_n| \to 0$ if and only if $\lim_{n\to\infty} \int_E |f_n| = \int_E |f|$. (Hint: Use the General Lebesgue Dominated Convergence Theorem.)

Solution. We prove bidirectionally.

1. $(\to)$. $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ s.t.

$$n \geq N \implies \int |f - f_n| < \epsilon \tag{353}$$

$$\implies \epsilon \int |f - f_n| \geq \int ||f| - |f_n|| \geq \left| \int |f| - \int |f_n| \right| \tag{354}$$

$$\implies \lim_{n\to\infty} \int_F |f_n| = \int |f| \tag{355}$$

2. $(\leftarrow)$. Suppose $\lim_{n\to\infty} \int_E |f_n| = \int_E |f|$. Then, for all $n \in \mathbb{N}$,

$$\lim_{n\to\infty} \int_E (|f_n| + |f|) = 2 \int_E |f| < \infty \tag{356}$$

Since $|f_n - f| \leq |f_n| + |f|$ on E for all n and $|f - f_n| \to 0$ a.e. on E, we have

$$\lim_{n\to\infty} \int_E |f - f_n| = 0$$

by the general Lebesgue DCT.

■ One wrong way I turned for the second part was that I use the wrong dominated bound $f_n \leq |f_n|$, so by GDCT,

$$\lim_{n\to\infty} \int f_n = f \tag{357}$$

But this doesn't lead anywhere.

Exercise 3.34 (Royden 4.34)

Let f be a nonnegative measurable function on $\mathbf{R}$. Show that

$$\lim_{n\to\infty} \int_{-n}^{n} f = \int_{\mathbf{R}} f.$$

Solution. We know

$$\int_{-n}^{n} f = \int_{\mathbb{R}} \underbrace{f \cdot \chi_{[-n,n]}}_{f_n} \tag{358}$$

See that $0 \leq f_n \leq f$, so by MCT,

$$\int_{\mathbb{R}} f = \lim_{n\to\infty} \int_{\mathbb{R}} f_n = \lim_{n\to\infty} \int_{\mathbb{R}} f \cdot \chi_{[-n,n]} = \lim_{n\to\infty} \int_{-n}^{n} f \tag{359}$$

Exercise 3.35 (Royden 4.35)

Let f be a real-valued function of two variables (x, y) that is defined on the square $Q = \{(x, y) \mid 0 \leq x \leq 1, 0 \leq y \leq 1\}$ and is a measurable function of x for each fixed value of y. Suppose for each fixed value of x, $\lim_{y\to 0} f(x, y) = f(x)$ and that for all y, we have $|f(x, y)| \leq g(x)$, where g is integrable over $[0, 1]$. Show that

$$\lim_{y\to 0} \int_0^1 f(x, y)dx = \int_0^1 f(x)dx.$$

Also show that if the function $f(x, y)$ is continuous in y for each x, then

$$h(y) = \int_0^1 f(x, y)dx$$

is a continuous function of y.

Exercise 3.36 (Royden 4.36)

Let f be a real-valued function of two variables (x, y) that is defined on the square $Q = \{(x, y) \mid 0 < x < 1, 0 < y < 1\}$ and is a measurable function of x for each fixed value of y. For each $(x, y) \in Q$ let the partial derivative $\partial f/\partial y$ exist. Suppose there is a function g that is integrable over $[0, 1]$ and such that

$$\left|\frac{\partial f}{\partial y}(x, y)\right| \leq g(x) \text{ for all } (x, y) \in Q.$$

Prove that

$$\frac{d}{dy}\left[\int_0^1 f(x, y)dx\right] = \int_0^1 \frac{\partial f}{\partial y}(x, y)dx \text{ for all } y \in [0, 1].$$

Exercise 3.37 (Royden 4.37)

Let f be a integrable function on E. Show that for each $\epsilon > 0$, there is a natural number N for which if $n \geq N$, then $\left|\int_{E_n} f\right| < \epsilon$ where $E_n = \{x \in E \mid |x| \geq n\}$.

Solution. Since f is integrable, $\int_E |f| < +\infty$. Note

$$E_n = \{x \in E \colon |x| \geq n\} \tag{360}$$

is decreasing, so by continuity of integrals,

$$\lim_{n\to\infty} \int_{E_n} f = \int_{\cap_{n=1}^\infty E_n} f \tag{361}$$

But $\cap_{n=1}^\infty E_n = \emptyset$, so this term is 0. By definition of the limit, $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ s.t.

$$n \geq N \implies \left|\int_{E_n} f\right| < \epsilon \tag{362}$$

■ . Note that

1. continuity of measure is not necessary here, since it is manifested by the continuity of the integral.
2. I was also thinking of using Chebyshev to bound the integral using the measure of E_n which gets smaller, but writing out

$$m(\{x \in E \colon |f(x)| \geq a\}) \leq \frac{1}{a}\int_E f \tag{363}$$

doesn't really lead anywhere since it is a lower bound on the integral.

Exercise 3.38 (Royden 4.38)

For each of the two functions f on $[1, \infty)$ defined below, show that $\lim_{n\to\infty} \int_1^n f$ exists while f is not integrable over $[1, \infty)$. Does this contradict the continuity of integration?

1. Define $f(x) = (-1)^n/n$, for $n \leq x < n + 1$.
2. Define $f(x) = (\sin x)/x$ for $1 \leq x < \infty$.

Exercise 3.39 (Royden 4.39)

Prove the theorem regarding the continuity of integration.

Exercise 3.40 (Royden 4.40)

Let f be integrable over $\mathbf{R}$. Show that the function F defined by

$$F(x) = \int_{-\infty}^{x} f \text{ for all } x \in \mathbf{R}$$

is properly defined and continuous. Is it necessarily Lipschitz?

Exercise 3.41 (Royden 4.41)

Show that Proposition 25 is false if $E = \mathbf{R}$.

Exercise 3.42 (Royden 4.42)

Show that Theorem 26 is false without the assumption that the h_n's are nonnegative.

Exercise 3.43 (Royden 4.43)

Let the sequences of functions $\{h_n\}$ and $\{g_n\}$ be uniformly integrable over E. Show that for any α and β, the sequence of linear combinations $\{\alpha f_n + \beta g_n\}$ also is uniformly integrable over E.

Exercise 3.44 (Royden 4.44)

Let f be integrable over $\mathbf{R}$ and $\epsilon > 0$. Establish the following three approximation properties.

1. There is a simple function η on $\mathbf{R}$ which has finite support and $\int_{\mathbf{R}} |f - \eta| < \epsilon$ (Hint: First verify this if f is nonnegative.]
2. There is a step function s on $\mathbf{R}$ which vanishes outside a closed, bounded interval and $\int_{\mathbf{R}} |f - s| < \epsilon$. (Hint: Apply part (i) and Problem 18 of Chapter 3.)
3. There is a continuous function g on $\mathbf{R}$ which vanishes outside a bounded set and $\int_{\mathbf{R}} |f - g| < \epsilon$.

Exercise 3.45 (Royden 4.45)

Let f be integrable over E. Define $\hat{f}$ to be the extension of f to all of $\mathbf{R}$ obtained by setting $\hat{f} \equiv 0$ outside of E. Show that $\hat{f}$ is integrable over $\mathbf{R}$ and $\int_E f = \int_{\mathbf{R}} \hat{f}$. Use this and part (i) and (iii) of the preceding problem to show that for $\epsilon > 0$, there is a simple function η on E and a continuous function

g on E for which $\int_E |f - \eta| < \epsilon$ and $\int_E |f - g| < \epsilon$.

Exercise 3.46 (Royden 4.46)

(Riemann-Lebesgue) Let f be integrable over $(-\infty, \infty)$. Show that

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f(x) \cos nx \, dx = 0.$$

(Hint: First show this for f is a step function that vanishes outside a closed, bounded interval and then use the approximation property (ii) of Problem 44.)

Exercise 3.47 (Royden 4.47)

Let f be integrable over $(-\infty, \infty)$.

1. Show that for each t,
$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{\infty} f(x+t)dx.$$
2. Let g be a bounded measurable function on $\mathbf{R}$. Show that
$$\lim_{t\to 0} \int_{-\infty}^{\infty} g(x) \cdot [f(x) - f(x+t)] = 0.$$

(Hint: First show this, using uniform continuity of f on $\mathbf{R}$, if f is continuous and vanishes outside a bounded set. Then use the approximation property (iii) of Problem 44.)

Exercise 3.48 (Royden 4.48)

Let f be integrable over E and g be a bounded measurable function on E. Show that $f \cdot g$ is integrable over E.

Exercise 3.49 (Royden 4.49)

Let f be integrable over $\mathbf{R}$. Show that the following four assertions are equivalent:

1. $f = 0$ a.e on $\mathbf{R}$.
2. $\int_{\mathbf{R}} fg = 0$ for every bounded measurable function g on $\mathbf{R}$.
3. $\int_A f = 0$ for every measurable set A.
4. $\int_O f = 0$ for every open set O.

Exercise 3.50 (Royden 4.50)

Let $\mathcal{F}$ be a family of functions, each of which is integrable over E. Show that $\mathcal{F}$ is uniformly integrable over E if and only if for each $\epsilon > 0$, there is a $\delta > 0$ such that for each $f \in \mathcal{F}$,

$$\text{if } A \subseteq E \text{ is measurable and } m(A) < \delta, \text{ then } \left| \int_A f \right| < \epsilon.$$

Exercise 3.51 (Royden 4.51)

Let $\mathcal{F}$ be a family of functions, each of which is integrable over E. Show that $\mathcal{F}$ is uniformly integrable over E if and only if for each $\epsilon > 0$, there is a $\delta > 0$ such that for all $f \in \mathcal{F}$,

$$\text{if } U \text{ is open and } m(E \cap U) < \delta, \text{ then } \int_{E\cap U} |f| < \epsilon.$$

Exercise 3.52 (Royden 4.52)

1. Let $\mathcal{F}$ be the family of functions f on $[0,1]$, each of which is integrable over $[0,1]$ and has $\int_0^1 |f| \leq 1$. Is $\mathcal{F}$ uniformly integrable over $[0,1]$?
2. Let $\mathcal{F}$ be the family of functions f on $[0,1]$, each of which is continuous on $[0,1]$ and has $|f| \leq 1$ on $[0,1]$. Is $\mathcal{F}$ uniformly integrable over $[0,1]$?
3. Let $\mathcal{F}$ be the family of functions f on $[0,1]$, each of which is integrable over $[0,1]$ and has $\int_a^b |f| \leq b - a$ for all $[a,b] \subseteq [0,1]$. Is $\mathcal{F}$ uniformly integrable over $[0,1]$?

Exercise 3.53 (Math 631 Fall 2025, Midterm Exercise 2)

Compute

$$\lim_{n\to\infty} \int_{\mathbb{R}} \frac{n \sin(x/n)}{x(1+x^2)} dx. \tag{364}$$

Solution. Observe that

$$\lim_{n\to\infty} \frac{n}{x} \sin \frac{x}{n} = 1 \tag{365}$$

for every $x \in \mathbb{R}$. Thus the limit above is equal to $1/(1+x^2)$ for every x. Moreover,

$$\left| \frac{n}{x} \sin \frac{x}{n} \right| \leq 1 \tag{366}$$

for all x, n since $|\sin y| \leq |y|$ for all y. Therefore, if we denote $f_n(x) = \frac{n \sin(x/n)}{x(1+x^2)}$, then we have $f_n(x) \to f(x) \equiv 1/(1+x^2)$ pointwise as $n \to \infty$, and $|f_n(x)| \leq f(x)$ for all x, n. But $f(x)$ is integrable. Recall that for a non-negative f,

$$\int_{\mathbb{R}} f dx = \sup_g \int g dx, \tag{367}$$

where sup is taken over all bounded g of finite support such that $0 \leq g \leq f$. We can take as test functions $f_R(x) = f(x)\chi_{[-R,R]}$ where χ_E denotes the characteristic function of the set E. Note that any test function $g \leq f_R$ for large enough R. Letting $R \to \infty$, we find that

$$\int f dx = \lim_{R\to\infty} \int_{-R}^{R} f dx \tag{368}$$

$$= \tan^{-1}(R) - \tan^{-1}(-R) = \pi. \tag{369}$$

Exercise 3.54 (Math 631 Fall 2025, Midterm Exercise 3)

Is $\{x^\alpha\}_{-1<\alpha<\infty}$ uniformly integrable on $[0,1]$?

Solution. No. Take $\epsilon = 1$. Suppose that there exists $\delta > 0$ such that if $m(E) < \delta$, then $\int_E |f| dx < 1$ for every f in the given family. Note that

$$\int_0^{\delta/2} x^\alpha dx = \frac{1}{\alpha+1}(\delta/2)^{\alpha+1}. \tag{370}$$

Once δ is fixed, we can always find α close enough to -1 so that

$$\frac{1}{\alpha+1}(\delta/2)^{\alpha+1} > 1. \tag{371}$$

Contradiction.

Exercise 3.55 (Math 631 Fall 2025, Final Exam Exercise 5)

Construct a function $f(x) \geq 0$ that is finite a.e., but f is not integrable on any subinterval of $[0, 1]$.

Solution. Consider $g(x) = \sum_k 2^{-k}|x - r_k|^{-1/2}$, where r_k is a sequence enumerating all rational numbers in $[0, 1]$. Then

$$\|g\|_{L^1} \leq 2 \sum_k 2^{-k} < \infty, \tag{372}$$

since each $|x - r_k|$ function is integrable and its L^1 norm over $[0, 1]$ does not exceed 2 by a simple estimate. Therefore, g is finite a.e. Now take $f(x) = g(x)^2$. On one hand, clearly $f(x)$ is finite a.e. since g is. On the other hand, for any k, we have

$$f(x) \geq 2^{-2k}|x - r_k|^{-1} \tag{373}$$

(simply ignore all other positive terms in the sum when squaring). This implies that f is not integrable on any subinterval of $[0, 1]$.

4 Convergence

Definition 4.1 (Convergence in Measure)

Let (f_n) be a sequence of measurable and finite[a] a.e. $f_n \to f$ **in measure** if for every $\eta > 0$,

$$\lim_{n\to\infty} m\big(\{x \mid |f_n(x) - f(x)| > \eta\}\big) = 0 \tag{374}$$

Colloquially, the set over which f_n and f differ too much is small.

[a] We add finite condition since we avoid dealing with $+\infty - \infty$.

So we have 3 types of convergence: uniform convergence, a.e. convergence, and now convergence in measure. Now we want to relate this convergence to the ones we already have.

Theorem 4.1

Suppose E is measurable, $m(E) < +\infty$, and $f_n \to f$ a.e. in E (assume f_n all measurable). Then , $f_n \to f$ in measure.

Proof. Observe that if $f_n \to f$ uniformly, then it converges in measure, because given some $\eta > 0$, $\exists N$ s.t.

$$\{x \mid |f_n(x) - f(x)| > \eta\} = \emptyset \tag{375}$$

by definition. It doesn't go to 0; it is 0. You can guess why we started with this, because now we can directly use Egorov's theorem. Fix any $\epsilon > 0$. Find $E_0 \subset E$ s.t. $m(E \setminus E_0) < \epsilon$, and $f_n \to f$ uniformly on E_0. It follows that for all $\eta > 0$,

$$m(\{x \mid |f_n(x) - f(x)| > \eta\}) \leq \epsilon \tag{376}$$

for all $n \geq N(\eta)$. Since this is true for every $\epsilon > 0$, so this implies

$$\lim_{n\to\infty} m(\{x \mid |f_n(x) - f(x)| > \eta\}) = 0 \tag{377}$$

for every $\eta > 0$.

A few remarks. First, if the measure of E is infinite, this need not be true. Consider $f_n(x) = \chi_{[n,n+1]}(x)$. Then, this converges to 0 pointwise, but it does not converge to 0 in measure. There is always a measure 1 set where f is 1. Where the proof breaks down is in Egorov's theorem, since it does not work when $m(E) = +\infty$.

The second remark is that the converse is not true. Consider $[0,1]$ and the sequence of functions

$$\chi_{[0,1/2]}, \chi_{[1/2,1]}, \chi_{[0,1/4]}, \chi_{[1/4,1/2]}, \chi_{[1/2,3/4]}, \ldots \tag{378}$$

Then $f_n \to 0$ in measure since the size shrinks at the rate of 2^{-n}. However, it doesn't converge a.e. since for any point $x \in [0,1]$, the function will be 1 eventually as we hit the subinterval containing x, like "waves." So $f_n(x)$ diverges for all $x \in [0,1]$. So indeed, convergence in measure is the weakest type of convergence.

Here is a sort-of converse.

Theorem 4.2 (Riesz)

Suppose $f_n \to f$ in measure. Then, there exists a subsequence $f_{n_k} \to f$ a.e.

Proof. For every k, find n_k s.t. for all $n \geq n_k$,

$$m(\underbrace{\{x \mid |f_n(x) - f(x)| > 1/k\}}_{E_k}) < 2^{-k} \tag{379}$$

Then,

$$\sum_{k=1}^{\infty} m(E_k) < +\infty \tag{380}$$

By Borel-Cantelli, the set of all x's that are in infinitely many E_k have measure 0. So, almost everywhere, x is only in a finite number of E_k. So for a.e., x, there exists $N(x)$ s.t. $x \notin E_k$ for all $k \geq N(x)$. This means

$$|f_{n_k}(x) - f(x)| < 1/k \tag{381}$$

for all $k \geq N(x)$. Therefore, $f_{n_k}(x) \to f(x)$ for a.e. x.

In the example above, we can just skip the functions that evaluate x to 1.

Practically, proving convergence in measure is still pretty good since we can pass in a subsequence that converges a.e. Here is a corollary.

Corollary 4.3

Let $f_n \geq 0$, integrable on E. Then,

$$\lim_{n \to +\infty} \int_E f_n \, dx = 0 \iff f_n \to 0 \text{ in measure} \tag{382}$$

f_n are tight and uniformly integrable.

Proof. We prove bidirectionally.

1. $(\to)$. Tight, uniformly integrable is true be definition. Also, $f_n \to 0$ in measure by Chebyshev.

$$m(\{x \mid f_n(x) > \eta\}) \leq \frac{1}{\eta} \int_E f_n \, dx \tag{383}$$

2. $(\leftarrow)$ For the opposite, we use the previous theorem. Find f_{n_k} s.t. that it converges to 0 a.e., and then use Vitali's convergence theorem.

In general, if $f_n \to 0$ in measure, it doesn't mean that the integral will go to 0 since you can take larger and larger bumps. So we need extra assumptions.

4.1 Exercises

Exercise 4.1 (Math 631 Fall 2025, Final Exam Exercise 1)

Let $f_n(x) = \frac{nx}{1+n^2x^4}$ defined on $E = (0, 1)$. Does f_n converge in measure? a.e.? uniformly? Does there exist integrable $g(x)$ such that $|f_n(x)| \leq g(x)$ for all n?

Solution. For each x, $f_n(x) \to 0$ as $n \to \infty$, so $f_n(x)$ converges a.e. and in measure since a.e. convergence implies convergence in measure. Observe that

$$f_n'(x) = \frac{n(1 + n^2x^4) - nx4n^2x^3}{(1 + n^2x^4)^2}. \tag{384}$$

The numerator is equal to zero if $n - 3n^3x^4 = 0$; the only root on $(0,1)$ is $x = 3^{-1/4}n^{-1/2}$. This suggests that we should look at $f_n(x)$ for $x \sim n^{-1/2}$ to get an idea of its maximal value. Indeed,

$$f_n(n^{-1/2}) = \frac{1}{2}n^{1/2}, \tag{385}$$

and so f_n does not converge to 0 uniformly (and can't converge to anything else due to pointwise convergence to zero). Also, given x close to zero, set $n = \lfloor \frac{1}{x^2} \rfloor$. Then

$$f_n(x) \geq \frac{1}{2}x\lfloor \frac{1}{x^2} \rfloor \geq \frac{1}{2x} + \frac{x}{2}(\lfloor \frac{1}{x^2} \rfloor - \frac{1}{x^2}) \geq \frac{1}{2x} - \frac{x}{2}. \tag{386}$$

So if x is sufficiently small, then $f_n(x) \geq \frac{1}{4x}$. This estimate implies that $g(x) \geq \frac{1}{4x}$ for all sufficiently small x, and so cannot be integrable.

5 Differentiation

Definition 5.1 (Divided Difference, Average Value)

Let f be integrable over closed and bounded $[a, b]$. Extend f to take value $f(b)$ on $(b, b+1]$. For $0 < h \leq 1$, we define

1. the **divided difference function** as
$$\operatorname{Diff}_h f := \frac{f(x+h) - f(x)}{h} \tag{387}$$
2. the **average value function** as
$$\operatorname{Av}_h f(x) := \frac{1}{h} \int_x^{x+h} f(x) \tag{388}$$

Note this important property, which we can think of as a finite version of the fundamental theorem of calculus.

Lemma 5.1 (Finite Version of Fundamental Theorem of Calculus)

We first establish that given integrable f over closed, bounded interval $[a, b]$, we extend f to equal $f(b)$ on $(b, b+1]$. Then for $0 < h \leq 1$, define

$$\operatorname{Diff}_h f := \frac{f(x+h) - f(x)}{h}, \qquad \operatorname{Av}_h f(x) := \frac{1}{h} \int_x^{x+h} f(x) \tag{389}$$

Then, for all $a \leq u < v \leq b$,
$$\int_u^v \operatorname{Diff}_h f = \operatorname{Av}_h f(v) - \operatorname{Av}_h f(u) \tag{390}$$

Proof. We see by employing a change of basis and the integral properties,

$$\int_a^b \operatorname{Diff}_h f = \int_u^v \frac{f(x+h) - f(x)}{h} \tag{391}$$
$$\frac{1}{h} \int_u^v f(x+h) - \frac{1}{h} \int_u^v f(x) \tag{392}$$
$$= \frac{1}{h} \int_{u+h}^{v+h} f(x) - \frac{1}{h} \int_u^v f(x) \tag{393}$$
$$= \frac{1}{h} \int_{u+h}^{v} f(x) + \frac{1}{h} \int_v^{v+h} f(x) - \frac{1}{h} \int_u^{u+h} f(x) - \frac{1}{h} \int_{u+h}^{v} f(x) \tag{394}$$
$$= \frac{1}{h} \int_v^{v+h} f(x) - \frac{1}{h} \int_u^{u+h} f(x) \tag{395}$$
$$= \operatorname{Av}_h f(v) - \operatorname{Av}_h f(u) \tag{396}$$

Now, we will establish differentiation and culminate in the fundamental theorem of calculus. We first focus on monotone functions, which has two nice properties. First, they can be discontinuous on at most a countable set, and second, they are differentiable a.e.![5] Therefore, the *difference* of two increasing functions in an open interval is also differentiable a.e. The class of functions of bounded variation are such functions that can be decomposed as this difference, and we call this the Jordan decomposition.

[5]Note that we gained differentiability by strengthening our assumption for a countable set to a measure 0 set.

Note that the finite version of the fundamental theorem of calculus applies for integrable functions. We would like a formula that looks more like

$$\int_a^b f'(x) = f(b) - f(a) \tag{397}$$

In order to do this, we first have to define the derivative.

Definition 5.2 (Dini Derivative)

For function $f : E \subset \mathbb{R} \to \mathbb{R}$ and an interior point $x \in E^\circ$, we define the **upper and lower Dini derivatives** as

$$\overline{D} f(x) \coloneqq \lim_{h \to 0} \sup_{0<|t|<h} \frac{f(x+t) - f(x)}{t}, \qquad \underline{D} f(x) \coloneqq \lim_{h \to 0} \inf_{0<|t|<h} \frac{f(x+t) - f(x)}{t} \tag{398}$$

If $\overline{D} f(x) = \underline{D} f(x) < +\infty$, then this value is the **Dini derivative** of f at x, denoted $f'(x)$, and we say that f is **differentiable** at x.

So given the punctured neighborhood $(x-h, x+h) \setminus \{x\}$, you can think of the upper derivative as the "maximum slope," and the lower derivative as the "minimum slope."[6]

Note that as h goes to 0, the first is nondecreasing and the second is nonincreasing, and clearly

$$\underline{D} f(x) \leq \overline{D} f(x) \tag{399}$$

Example 5.1 (Derivative of Riemann Function)

Consider the Riemann function $\chi_{\mathbb{Q}}$. Then

1. To compute the upper derivative, consider when x is a rational. Then no matter how small we set h, there will always be a rational and an irrational in $(x-h, x+h) \setminus \{x\}$.

$$\overline{D} \chi_{\mathbb{Q}} = \begin{cases} 0 & \text{if } x \in \mathbb{Q} \\ +\infty & \text{if } x \in \mathbb{Q}^c \end{cases} \tag{400}$$

2. To compute the lower derivative, consider when x is a rational. Then no matter how small we set h, there will always be a rational and an irrational in $(x-h, x+h) \setminus \{x\}$.

$$\overline{D} \chi_{\mathbb{Q}} = \begin{cases} -\infty & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{Q}^c \end{cases} \tag{401}$$

Great, so we have defined the derivative, and now we want to try and show something like

$$\lim_{h \to 0^+} \int_a^b \mathrm{Diff}_h\, f = \lim_{h \to 0^+} \mathrm{Av}_h\, f(b) - \mathrm{Av}_h\, f(a) \tag{402}$$

If f is continuous, then we can show that the RHS is $f(b) - f(a)$. If f is absolutely continuous, then we can show that the LHS becomes $\int_a^b f'$.

5.1 Monotone Functions

Monotone functions are a nice class of functions to study for differentiation and for constructing more general measures.

[6] Kiselev told me that while he always thinks of integrals as Lebesgue integrals, he often just thinks of the derivative as the limit rather than the limit of supremums/infimums.

Theorem 5.2 (Monotone Functions has At Most Countable Discontinuities)

Suppose f is monotone, increasing on $[a, b]$. Then, the set of discontinuities of f at most countable.

Proof. The idea is to show that the one-sided limits must exist by monotonicity, and one this is true, we can create intervals in the codomain for which there is a jump. Let x_k be any point of discontinuity. Note that

$$\lim_{x \to x_k^-} f(x) \coloneqq \sup\{f(x) \mid a < x < x_k\} \tag{403}$$

$$\lim_{x \to x_k^+} f(x) \coloneqq \inf\{f(x) \mid x_k < x < b\} \tag{404}$$

both exist by monotonicity, but since there is a discontinuity, we have

$$L_k^- = \lim_{x \to x_k^-} f(x) < \lim_{x \to x_k^+} f(x) = L_k^+ \tag{405}$$

Then, $L_k^+ - L_k^-$ is a jump of f at x_k. These intervals $[L_k^-, L_k^+]$ are disjoint due to monotonicity, and each interval contains a rational number. So there can only be at most countable intervals.[a]

[a]We can detour to a slight generalization for not necessarily monotone functions. Recall that a point x is a discontinuity of the first kind of $f(x)$ if both one-sided limits exist. Then, the set of discontinuities of the first kind is countable. Idea of the proof. Look at some jump discontinuity and record the jump $\eta > 0$. Then, find $\delta > 0$ s.t. if $0 < y - x < \delta$, then $\left|f(x) - \lim_{y \to x^+} f(y)\right| < \frac{\eta}{10}$. Then look at the rectangle on the graph associated with each jump. Because the limits exist, you can pick the rectangles so small that they are completely disjoint. Look at picture.

The following is a sort-of reverse statement.

Theorem 5.3 (Construction of Monotone Functions with Countable Discontinuities)

For any countable set $C \subset (a, b)$ (where the interval doesn't need to be bounded), there exists monotonically increasing f with a jump at each $x \in C$ and continuous at every $x \notin C$.

Proof. Let $x_1, x_2, \ldots$ be C, and define

$$f(x) = \sum_{x_k \leq x, x \in C} 2^{-k} \tag{406}$$

The sum is increasing and convergent (since it's dominated by geometric series). f also has a jump of 2^{-k} at every x_k.
Now we prove continuity. Suppose $x \notin C$. Take $N \in \mathbb{N}$. Find $\delta_N > 0$ s.t.

$$x_1, x_2, \ldots, x_N \notin (x - \delta_N, x + \delta_N) \tag{407}$$

which is possible since this is a finite set. The remaining sum can only add up to 2^{-N}, and so $f(x + \delta_N) - f(x - \delta_N) \leq 2^{-N}$.

Therefore, this is quite nice, since we can characterize the continuity of monotone functions. But we state even a stronger theorem on the differentiability of monotone functions.

Definition 5.3 (Vitali Covering)

A collection $\mathcal{F}$ of closed, bounded, nondegenerate intervals is said to cover a set E **in the sense of Vitali** if for each $x \in E$ and $\epsilon > 0$, there is an interval I in $\mathcal{F}$ that contains x and has $\ell(I) < \epsilon$.

In essence, I just think of the set of ϵ-balls around each $x \in E$ for $\epsilon = \frac{1}{n}$, and then just take the union of each covering for all $n \in \mathbb{N}$. Note that we don't assume that E is measurable. It is easy to see that a Vitali set can be uncountable (all subintervals of $[0, 1]$) and even countable (all subintervals with rational endpoints). Nevertheless, we can still select a finite set of intervals that almost covers E.

Lemma 5.4 (Vitali Covering Lemma)

Suppose $m^*(E) < +\infty$ and $\mathcal{F}$ covers E in Vitali sense. Then, $\forall \epsilon > 0$, $\exists$ a disjoint finite collection $I_1, I_2, \ldots, I_n$ of intervals from $\mathcal{F}$ s.t.

$$m^*\left(E \setminus \bigcup_{k=1}^{n} I_k\right) < \epsilon \tag{408}$$

Proof. This proof is very intuitive since given a set, take a bunch of big balls to cover it until you can't fit in a big ball without intersecting a smaller ball. Then, between the "gaps," choose a smaller ball to fill them in, and you can keep doing this (since it is a Vitali cover) until you can make the cover arbitrarily tight.

Since $m^*(E) < +\infty$, by definition $\exists$ open O s.t. $E \subset O$, $m(O) < +\infty$. WLOG we can assume that all intervals in $\mathcal{F}$ lie in O.[a]

Note two things.

1. If $I_1, I_2, \ldots, I_n$ are disjoint and belong to O, then $\sum_{k=1}^{n} \ell(I_k) < +\infty$ since it is less than the measure of O which is finite.
2. Second, if we have finite collection $\{I_k\}_{k=1}^{n} \in \mathcal{F}$, define

$$\mathcal{F}_n \coloneqq \{I \in \mathcal{F} \mid I \cap \bigcup_{k=1}^{n} I_k = \emptyset\} \tag{409}$$

   Then every $x \in E \setminus \bigcup_{k=1}^{n} I_k$ lies in some $I \in \mathcal{F}_n$.

The ideal is to define $I_1, \ldots, I_n \in \mathcal{F}$ s.t. they are disjiont and

$$E \setminus \bigcup_{k=1}^{n} I_k \subset \bigcup_{k=n+1}^{\infty} 5I_k \quad \forall n \tag{410}$$

where $5I$ means that we keep the center of the interval fixed and scale it up by 5 times. If we do that, then $\forall \epsilon > 0$, find n s.t. $\sum_{k=n+1}^{\infty} \ell(I_k) < \epsilon/5$. Take $I_1, \ldots, I_n$ as our intervals

$$m^*\left(E \setminus \bigcup_{k=1}^{n} I_k\right) \leq \sum_{k=n+1}^{\infty} \ell(5I_k) < \epsilon \tag{411}$$

So it remains to select these intervals $I_1, \ldots, I_n$. We will do this inductively.

1. I_1 be any interval in $\mathcal{F}$ s.t.

$$\ell(I_1) \geq \frac{1}{2} \sup_{I \in \mathcal{F}} \ell(I) \tag{412}$$

2. Once $I_1, \ldots, I_n$ have been selected, we select I_{n+1} from $\mathcal{F}_n$ s.t.

$$\ell(I_{n+1}) \geq \frac{1}{2} \sup_{I \in \mathcal{F}_n} \ell(I) \tag{413}$$

   So these intervals are clearly disjoint from the ones that we have selected earlier. So it remains to show 410. Suppose $x \in E \setminus \cup_{k=1}^{n} I_k$. Then, $\exists I \in \mathcal{F}_n$ s.t. $x \in I$.

Suppose $I \in \mathcal{F}_m$ for all $m \geq n$. This is impossible since by construction, $\ell(I_m) \geq \frac{1}{2}\ell(I)$. This contradicts $\sum \ell(I_m)$ is finite. Therefore, $\exists m$ s.t. $I \in \mathcal{F}_{m-1}$ but $I \notin \mathcal{F}_m$. This implies that $I \cap I_m \neq$ (while the intersection with the previous ones were empty). But then, $I \subset 5I_m$, since $\ell(I_m) \geq \frac{1}{2}\ell(I)$.[b]

[a]We can just discard any interval that it not Vitali in O and keep only those intervals in O such that it would still be in a Vitali cover. Indeed, we can discard all $I \subset \mathcal{F}$ s.t. $I \not\subset O$. Given $x \in E, x \in O$, so $d(x, O^c) > 0$ for all $\epsilon > 0$, $\exists I \in F$ s.t. $\ell(I) < \epsilon$, $x \in I, I \subset O$ (just take $\ell(I), < \min(\epsilon, d(x, O^c))$). So even remaining intervals cover E in Vitali sense.

[b]The 5 is needed since we have $1/2$. So we are taking the midpoint $3/4$ of the interval $[1/2, 1] \subset [0, 1]$, which should be blown up by 5.

Lemma 5.5 (MVT Inequality Generalization)

Let f be an increasing function on closed, bounded interval $[a, b]$. Then, for each $\alpha > 0$,

$$m^*(\{x \in (a, b) \colon \overline{D}f(x) \geq \alpha\}) \leq \frac{1}{\alpha}\big[f(b) - f(a)\big] \tag{414}$$

and[a]

$$m^*(\{x \in (a, b) \colon \overline{D}f(x) = \infty\}) = 0 \tag{415}$$

[a]This is sort of like Chebyshev's inequality applied to the derivative of f, if you think about it, assuming you know the fundamental theorem of calculus.

Proof. The general idea is to take this set and construct a Vitali cover for it. Then, using Vitali's covering lemma, we can reduce it to a finite disjoint union of intervals, which is better to work with.
Fix $\alpha > 0$, define $E_\alpha = \{x \mid \overline{D}f(x) \geq \alpha\}$. Take any $\alpha' < \alpha$, any $\epsilon > 0$. Consider all intervals $[c, d] \subset [a, b]$ s.t. $f(d) - f(c) > \alpha'(d - c)$. This collection covers E_α in Vitali sense. Since no matter how small h is, we can find t so that this ratio term is bigger than α'.
Now, we can use the covering lemma to find a finite disjoint collection $\{[c_k, d_k]\}_{k=1}^n$ s.t. $m^*(E \setminus \cup_{k=1}^n [c_k, d_k]) < \epsilon$. Then,

$$m^*(E) \leq \sum_{k=1}^n (d_k - c_k) + \epsilon \tag{416}$$

by subadditivity of outer measure. Using the inequality,

$$\leq \frac{1}{\alpha'} \sum_{k=1}^n \big(f(d_k) - f(c_k)\big) + \epsilon \tag{417}$$

But f is monotone, so

$$\leq \frac{1}{\alpha'}\big(f(b) - f(a)\big) + \epsilon \tag{418}$$

This is true for all $\alpha' < \alpha$ for all $\epsilon > 0$, proving the first claim. The second part follows since it is an intersection of all sets for $\alpha = n$ for all $n \in \mathbb{N}$, which go to 0.

Theorem 5.6 (Lebesgue's Theorem)

Let f be a monotone function over some bounded domain.

1. If f is monotone over (a, b), then it is differentiable a.e.[a]
2. If f is monotone over $[a, b]$, then f' is integrable over $[a, b]$, and

$$\int_a^b f' \leq f(b) - f(a)^b \tag{419}$$

[a]This is actually known as Lebesgue's theorem in the book, but the two results are used together too often for me to

separate them.

[b]Note that the integral $\int_a^b f'$ is independent of the values taken by f at the endpoints. On the other hand, the right-hand side of this equality holds for the extension of any increasing extension of f on the open bounded interval (a, b) to its closure $[a, b]$.

Proof. WLOG, (a, b) is bounded.[a] Consider the countable family of sets

$$E_{\alpha,\beta} = \{x \mid \overline{D}f(x) > \alpha > \beta > \underline{D}f(x), \alpha, \beta \in \mathbb{Q}\} \tag{420}$$

Note that if the derivatives aren't equal, we can always squeeze 2 rationals in, so

$$\{x \mid \overline{D}f(x) > \underline{D}f(x)\} \subset \bigcup_{\alpha,\beta\in\mathbb{Q}} E_{\alpha,\beta} \tag{421}$$

We want to prove that $m^*(E_{\alpha,\beta}) = 0 \quad \forall \alpha, \beta$. Let's find O open s.t. $E_{\alpha,\beta} \subset O$ and $m(O) < m^*(E) + \epsilon$, where we will denote $E = E_{\alpha,\beta}$.
Consider all intervals $[c, d] \subset O$ s.t. $f(d) - f(c) < \beta(d - c)$. Since we know $\underline{D}f(x) < \beta$, these intervals cover E in Vitali sense. So you find a disjoint subcollections $[c_k, d_k]$ for $k = 1, \ldots, n$ s.t.

$$m^*\bigg(E \setminus \bigcup_{k=1}^{n} [c_k, d_k]\bigg) < \epsilon \tag{422}$$

Observe that

$$\sum_{k=1}^{n} \big(f(d_k) - f(c_k)\big) < \beta \sum_{k=1}^{n} (d_k - c_k) \tag{423}$$

$$\leq \beta\big(m^*(E) + \epsilon\big) \tag{424}$$

On the other hand, we can apply the previous lemma to $E \cap [c_k, d_k]$ to get

$$m^*(E \cap [c_k, d_k]) \leq \frac{1}{\alpha}\big(f(d_k) - f(c_k)\big) \tag{425}$$

and so

$$m^*(E) \leq \frac{1}{\alpha} \sum_{k=1}^{n} \big(f(d_k) - f(c_k)\big) + \epsilon \tag{426}$$

$$\leq \frac{\beta}{\alpha}\big(m^*(E) + \epsilon\big) + \epsilon, \quad \forall \epsilon > 0 \tag{427}$$

So, $m^*(E) \leq \frac{\beta}{\alpha} m^*(E)$, where $\frac{\beta}{\alpha} < 1$. Therefore $m^*(E) = 0$.
For the second part, the idea is to try to approximate this with a sequence of finite differences using the identity above. Since f is increasing on $[a, b + 1]$, it is measurable, and $\mathrm{Diff}_{1/n} f(x)$ is also measurable. Since monotone functions are differentiable a.e., f is differentiable a.e., and so $f'(x) < +\infty$ a.e. Now construct the sequence of functions

$$f_n = \mathrm{Diff}_{1/n} f \tag{428}$$

Note that they are nonnegative due to monotonicity of f, and by definition of the derivative they converge pointwise to f. By Fatou,

$$\int_a^b f \leq \liminf_n \int_a^b f_n = \liminf_n \bigg\{ \int_a^b \mathrm{Diff}_{1/n} f \bigg\} \tag{429}$$

By Lemma 5.1, we can see that

$$\int_a^b \mathrm{Diff}_{1/n} f = \frac{1}{1/n} \int_b^{b+1/n} f - \frac{1}{1/n} \int_a^{a+1/n} f = f(b) - \int_a^{a+1/n} f \leq f(b) - f(a) \tag{430}$$

where the last inequality also follows by monotonicity. Therefore, take the limsup of both sides.

$$\limsup_n \left\{ \int_a^b \mathrm{Diff}_{1/n} f \right\} \leq f(b) - f(a) \tag{431}$$

and we are done. ■

[a] Otherwise, we can always split it into a countable union of bounded intervals.

Great, so we have established some condition of when f' can be integrated, and proven one side of the equality. But being differentiable doesn't imply that fundamental theorem of calculus holds. So integrating the derivative won't get you back these functions (think of step functions or the examples below). So we will have to specify a class of functions such that this holds.

Example 5.2 (Strict Inequality)

This inequality is strict for the Cantor Lebesgue function.

Example 5.3 (Continuous but not Monotone)

Note that we do need the monotonicity assumption since if it isn't, then we can't infer that f' is integrable over $[a, b]$. Consider the function

$$f(x) = \begin{cases} x^2 \sin(1/x^2) & \text{if } 0 < x \leq 1 \\ 0 & \text{if } x = 0 \end{cases} \tag{432}$$

Then, f' is not integrable over $[0, 1]$ since

$$f'(x) 2x \sin\left(\frac{1}{x^2}\right) - \frac{2}{x} \cos\left(\frac{1}{x^2}\right) \tag{433}$$

5.2 Functions of Bounded Variation

Definition 5.4 (Total Variation)

Let $f : [a, b] \to \mathbb{R}$ and $P = \{x_0, x_1, \ldots, x_k\}$ be a partition of $[a, b]$. The **variation** of f w.r.t. P is

$$V(f, P) \coloneqq \sum_{i=1}^k |f(x_i) - f(x_{i-1})| \tag{434}$$

The **total variation** of f is defined

$$\mathrm{TV}(f) = \sup_P \{V(f, P) \colon P \text{ partition } of [a, b]\} \tag{435}$$

Definition 5.5 (Bounded Variation)

A function $f : [a, b] \to \mathbb{R}$ is said to be of **bounded variation** if

$$\mathrm{TV}(f) < +\infty \tag{436}$$

Example 5.4 (Increasing Functions are of Bounded Variation)

Let $f : [a, b] \to \mathbb{R}$ be increasing. Then f is of bounded variation since with respect to any partition $P = \{x_0, \ldots, x_k\}$,

$$V(f, P) = \sum_{i=1}^{k} |f(x_i) - f(x_{i-1})| = \sum_{i=1}^{k} f(x_i) - f(x_{i-1}) = f(b) - f(a) \tag{437}$$

Example 5.5 (Lipschitz Functions are of Bounded Variation)

Let f be Lipschitz on $[a, b]$, where c is the Lipschitz constant satisfying

$$|f(u) - f(v)| \leq c|u - v| \text{ for all } u, v \in [a, b] \tag{438}$$

Then, f is of bounded variation since

$$V(f, P) = \sum_{i=1}^{k} |f(x_i) - f(x_{i-1})| \leq c \sum_{i=1}^{k} x_i - x_{i-1} = c(b - a) \tag{439}$$

Example 5.6 (Not Bounded Variation)

Define $f : [0, 1] \to \mathbb{R}$ as

$$f(x) = \begin{cases} x \cos(\pi/2x) & \text{if } 0 < x \leq 1 \\ 0 & \text{if } x = 0 \end{cases} \tag{440}$$

Then, consider the partition

$$P_n = \{a = 0, \frac{1}{2n}, \frac{1}{2n-1}, \frac{1}{2n-2}, \ldots, \frac{1}{3}, \frac{1}{2}, 1 = b\} \tag{441}$$

Then,

$$V(f, P_n) = 1 + \frac{1}{2} + \ldots + \frac{1}{n} \tag{442}$$

which diverges, and hence f is not of bounded variation.

Definition 5.6 (Total Variation Function)

For any $f : [a, b] \to \mathbb{R}$ that is of bounded variation, the **total variation function** is defined

$$x \to \mathrm{TV}(f_{[a,x]}) \tag{443}$$

Lemma 5.7 (Construction of Jordan Decomposition)

Let $f : [a, b] \to \mathbb{R}$ be of bounded variation. Then, f has the following explicit expression as the difference of two increasing functions on $[a, b]$.

$$f(x) = \big[f(x) + \mathrm{TV}(f_{[a,x]})\big] - \mathrm{TV}(f_{[a,x]}) \tag{444}$$

Any such decomposition is known as the **Jordan decomposition**.

Proof. Note that the total varGiation function is always real-valued and increasing on $[a, b]$. If P is a partition of $[a, b]$ and P' is a refinement by adjoining x to P, then we know that $V(f_{[a,b]}, P) = V(f_{[a,x]}, P_1) + V(f_{[x,b]}, P_2)$, where P_1, P_2 are partitions of $[a, x], [x, b]$ respectively.

$$\mathrm{TV}(f_{[a,b]}) = \mathrm{TV}(f_{[a,x]}) + \mathrm{TV}(f_{[x,b]}) \tag{445}$$

Therefore,

$$a \leq x < y \leq b \implies \mathrm{TV}(f_{[a,y]}) - \mathrm{TV}(f_{[a,x]}) = \mathrm{TV}(f_{[x,y]}) \geq 0 \tag{446}$$

So we proved increasing. Now if we take the crudest partition $P = \{x, y\}$ of x, y, then

$$f(x) - f(y) \leq |f(y) - f(x)| = V(f_{[x,y]}, P) \leq \mathrm{TV}(f_{[x,y]}) = \mathrm{TV}(f_{[a,y]}) - \mathrm{TV}(f_{[a,x]}) \tag{447}$$

So, for all such $a \leq x < y \leq b$,

$$f(u) + \mathrm{TV}(f_{[a,u]}) \leq f(y) + \mathrm{TV}(f_{[a,y]}) \tag{448}$$

It turns out that there is a converse, and so we can characterize this decomposition exactly with functions of total variation.

Theorem 5.8 (Jordan's Theorem)

A function $f : [a, b] \to \mathbb{R}$ is of bounded variation if and only if it is the difference of two increasing functions on $[a, b]$.

Proof. We already proved one direction. Now assume that $f = g - h$ for some g, h increasing. Then, for any partition $P = \{x_0, \ldots, x_k\}$ of $[a, b]$,

$$\begin{aligned}
V(f, P) &= \sum_{i=1}^{k} |f(x_i) - f(x_{i-1})| && (449)\\
&= \sum_{i=1}^{k} |[g(x_i) - g(x_{i-1})] + [h(x_{i-1}) - h(x_i)]| && (450)\\
&\leq \sum_{i=1}^{k} |g(x_i) - g(x_{i-1})| + \sum_{i=1}^{k} |h(x_{i-1}) - h(x_i)| && (451)\\
&= \sum_{i=1}^{k} [g(x_i) - g(x_{i-1})] + \sum_{i=1}^{k} [h(x_i) - h(x_{i-1})] && (452)\\
&= [g(b) - g(a)] + [h(b) - h(a)]. && (453)
\end{aligned}$$

Corollary 5.9

If $f : [a, b] \to \mathbb{R}$ is of bounded variation, then it is differentiable a.e. on (a, b), and f' is integrable over $[a, b]$.

Proof. Use the Jordan decomposition $f = g - h$ and use Lebesgue's theorem on each monotonic $g - h$. Now by linearity, the derivative and integral are preserved.

5.3 Absolutely Continuous Functions

Recall that uniformly continuous functions $f : E \to \mathbb{R}$ means that for $\epsilon > 0$, we can find a $\delta > 0$ that does not depend on $x \in E$. We now introduce a much stronger version.

Definition 5.7 (Absolutely Continuous Function)

A function $f : [a, b] \to \mathbb{R}$ is **absolutely** continuous if $\forall \epsilon > 0$, $\exists \delta > 0$ s.t. for every finite disjoint collection $\{(a_k, b_k)\}_{k=1}^n$ of open intervals in (a, b),

$$\sum_{k=1}^{n} [b_k - a_k] < \delta \implies \sum_{k=1}^{\infty} |f(b_k) = f(a_k)| < \epsilon \tag{454}$$

Note that if $n = 1$, then absolute continuity equals uniform continuity. The converse, however, isn't even true for monotonic functions.

Example 5.7 (Cantor Lebesgue)

The Cantor Lebesgue function is increasing and continuous on $[0, 1]$, but it is not absolutely continuous.

We now present the following classification.

$$\text{Lipschitz} \implies \text{AC} \implies \text{Bounded Variation} \tag{455}$$

Theorem 5.10 (Lipschitz Functions are AC)

Lipschitz functions are absolutely continuous.

Proof.

Example 5.8 (Absolutely Continuous but Not Lipschitz)

$f(x) = \sqrt{x}$ is AC over $[0, 1]$.

Theorem 5.11 (AC Functions are of Bounded Variation)

Let f be AC on $[a, b]$. Then, f is the difference of increasing AC functions, and so is of bounded variation.

Proof. By Jordan's theorem, finding such a Jordan decomposition implies that it is bounded variation.

Finally, what does it take for a continuous function to be AC? We present an equivalent condition.

Theorem 5.12 (Continuity + UI = AC)

Let $f : [a, b] \to \mathbb{R}$ be continuous. Then f is AC on $[a, b]$ if and only if the family $\{\mathrm{Diff}_h f\}_{0<h\leq 1}$ is uniformly integrable over $[a, b]$.

Proof.

5.4 Fundamental Theorem of Calculus

Theorem 5.13 (Second Fundamental Theorem of Calculus)

Suppose f is AC on closed and bounded $[a, b]$. Then f is differentiable a.e. on (a, b), and f' is integrable on $[a, b]$, and

$$\int_a^b f' = f(b) - f(a) \tag{456}$$

Proof. f is AC, so it is of bounded variation from Theorem 5.11. Therefore we can represent it as the difference $f = g - h$ of monotonic functions. By Lebesgue's Theorem 5.6, f' exists a.e. and is integrable. We know that since f is integrable, the finite version of the fundamental theorem of calculus holds

$$\int_a^b \mathrm{Diff}_h\, f = \mathrm{Av}_h\, f(b) - \mathrm{Av}_h\, f(a) \tag{457}$$

We want to take the limit as $h \to 0^+$, so we will consider the sequential limit as $n \to +\infty$ for $h = 1/n$. Since f is AC, it is continuous, so the RHS converges to $f(b) - f(a)$. As for the LHS, we know from Theorem 5.12 that f AC iff the family of divided differences are uniformly integrable, and so by Vitali Convergence Theorem,

$$\lim_{n\to\infty} \left\{ \int_a^b \mathrm{Diff}_{1/n}\, f \right\} = \int_a^b \lim_{n\to\infty} \mathrm{Diff}_{1/n}\, f = \int_a^b f' \tag{458}$$

where the last equality follows from the fact that the derivative exists and so $\mathrm{Diff}_{1/n}\, f \to f$.

This is a very nice result since it connects the derivative and the integral. But not that this is an implication in one direction.

Example 5.9

Show example of when the theorem holds but f is not AC.

However, we can make a stronger statement of the fundamental theorem of calculus to get an equivalent formulation of absolute continuity.

Theorem 5.14 (Fundamental Theorem of Calculus of Absolutely Continuous Functions)

f is AC on $[a, b]$ if and only if it is an **indefinite integral**, i.e. if it is of the form

$$f(x) = f(a) + \int_a^x g(y)\, dy \quad \forall x \in [a, b] \tag{459}$$

for some integrable g.

Proof. We prove bidirectionally.

1. $(\to)$. We can do
$$\int_a^x f = f(x) - f(a) \quad \forall x \in [a, b] \tag{460}$$
since if f is AC on $[a, b]$, then f is AC on $[a, x]$.
2. $(\leftarrow)$ We just need to prove that the antiderivative is absolutely continuous. Take $\{(a_k, b_k)\}_{k=1}^n$

disjoint. We estimate the total variation,

$$\sum_{k=1}^{n} |f(b_k) - f(a_k)| \tag{461}$$

and try to make this small if the measure of the unions of the intervals is small. Just using the definition that f is the antiderivative, the sum can be bounded by

$$\sum_{k=1}^{n} |f(b_k) - f(a_k)| \leq \sum_{k=1}^{n} \int_{a_k}^{b_k} |g| \, dx = \int_{\cup(a_k, b_k)} |g| \, dx \tag{462}$$

using the triangle inequality, and then using additivity. The rest is just ϵ-δ language. $\forall \epsilon > 0$, $\exists \delta > 0$ s.t.

$$m\Big(\bigcup (a_k, b_k)\Big) < \delta \implies \sum_{k=1}^{n} |f(b_k) - f(a_k)| < \epsilon \tag{463}$$

This is true since g is integrable, and whenever the measure of the region that you are integrating on is less than δ, your integral will be less than ϵ. So $\exists \delta > 0$ s.t. for all measurable E,

$$m(E) < \delta \implies \int_E |g| \, dx < \epsilon \tag{464}$$

This is just the definition of integrability. This implies that

$$\sum_{k=1}^{n} |a_k - b_k| < \delta \implies \sum_{k=1}^{n} |f(b_k) - f(a_k)| < \epsilon \tag{465}$$

Note that we are using the bounded

Lemma 5.15 (Vanishing Integral over All Subintervals Implies Vanishing Function)

Let f be integrable over $[a, b]$, with

$$\int_{x_1}^{x_2} f \, dx = 0 \quad \forall (x_1, x_2) \subset [a, b] \tag{466}$$

Then, $f = 0$ a.e. $[a, b]$.

Proof. Note that if we add the constraint that $f \geq 0$, then this is true. But the potential problem is that f might change signs, which may cancel out. So starting from the assumption, we know that for any open O,

$$\int_O f \, dx = 0 \quad \forall O \text{ open} \tag{467}$$

Since G_δ sets can be written as a decreasing sequence of open sets, by continuity of measure, we can write

$$\int_G f \, dx = 0 \quad \forall G \; G_\delta \tag{468}$$

Since any measurable set E can be written as $E = G \setminus E_0$ with $m(E_0) = 0$, we have

$$\int_E f \, dx = 0 \quad \forall E \text{ measurable} \tag{469}$$

So let $E^+ = \{x \in [a, b] \mid f(x) > 0\}$ and $E^- = \{x \in [a, b] \mid f(x) < 0\}$. So we have the first equalities

$$0 = \int_{E^+} f = \int_a^b f^+ \tag{470}$$

$$0 = \int_{E^-} f = -\int_a^b f^- \tag{471}$$

So this means that $f^+ = f^- = 0$ a.e.

Corollary 5.16 (First Fundamental Theorem of Calculus)

Let f be integrable over bounded, closed interval $[a, b]$. Then,

$$\frac{d}{dx}\left[\int_a^x f\,dt\right] = f(x) \quad \text{for a.e. } x \in (a, b) \tag{472}$$

So basically, the derivative of the antiderivative is the function itself.

Proof. Since $F(x) \coloneqq \int_a^x f$ is an indefinite integral, $F(x)$ is AC from the fundamental theorem of AC functions, so its derivative must exist a.e. with F' integrable. So we need to compare F' and f. For any $(x_1, x_2) \subset [a, b]$ by linearity, we have

$$\int_{x_1}^{x_2} [F' - f] = \int_{x_1}^{x_2} F' - \int_{x_1}^{x_2} f \tag{473}$$

The integral of the first term from the 2nd fundamental theorem is $F(x_2) - F(x_1)$, since it is AC. For the second term, we know that $F = \int_a^x f\,dt$, so can can split it

$$= F(x_2) - F(x_1) - \underbrace{\int_a^{x_2} f}_{F(x_2)} + \underbrace{\int_a^{x_1} f}_{F(x_1)} = 0 \tag{474}$$

So this is true for any open interval in $[a, b]$. Then by invoking the previous lemma, $f = F'$ a.e.

Another corollary is for monotone functions, and how we can determine whether they are AC or not.

Corollary 5.17 (AC of Monotone Functions)

Let f be monotone on $[a, b]$. Then f is AC if and only if

$$\int_a^b f'\,dx = f(b) - f(a) \tag{475}$$

If we have montone function, note that derivative should exist a.e., and the derivative is integrable for monotone functions. Note that in the previous corollary, we need to check for all $x \in [a, b]$, but in here, we only need to check at the endpoints a and b.

Proof. Bidirectional.

1. $(\rightarrow)$.

2. ($\leftarrow$). Let $x \in [a, b]$. We know from assumption—by rearranging the terms—that

$$0 = \int_a^b f' - \big(f(b) - f(a)\big) \tag{476}$$

But by additivity of the integral, we have

$$= \underbrace{\int_a^x f' - \big(f(x) - f(a)\big)}_{\leq 0} + \underbrace{\int_x^b f' - \big(f(b) - f(x)\big)}_{\leq 0} \tag{477}$$

But we know that WLOG, f is increasing. If f is increasing, then we know that both integrals should be positive, since the only type of discontinuities can be jump discontinuities.

$$\int_a^x f' - \big(f(x) - f(a)\big) + \int_x^b f' - \big(f(b) - f(x)\big) \tag{478}$$

This is the end of absolutely continuous functions.

5.5 Convex Functions

Definition 5.8 (Convex Function)

φ is convex on $(a, b) \subset \mathbb{R}$ if $\forall x_1, x_2 \in (a, b)$, $\forall \lambda \in [0, 1]$, the linear interpolation

$$\varphi(\lambda x_1 + (1 - \lambda) x_2) \leq \lambda \varphi(x_1) + (1 - \lambda) \varphi(x_2) \tag{479}$$

Note that if we have $x = \lambda x_1 + (1 - \lambda) x_2$, then $\lambda = \frac{x_2 - x}{x_2 - x_1}$, and so the definition can be rewritten as

$$\varphi(x) \leq \frac{x_2 - x}{x_2 - x_1} \varphi(x_1) + \frac{x - x_1}{x_2 - x_1} \varphi(x_2) \quad \forall x \in [x_1, x_2] \tag{480}$$

Note that the two fraction coefficients add up to 1. So we can write

$$\frac{x_2 - x}{x_2 - x_1} \varphi(x) + \frac{x - x_1}{x_2 - x_1} \varphi(x) \leq \frac{x_2 - x}{x_2 - x_1} \varphi(x_1) + \frac{x - x_1}{x_2 - x_1} \varphi(x_2) \quad \forall x \in (x_1, x_2) \tag{481}$$

and rearranging, we get

$$\frac{x_2 - x}{x_2 - x_1} \big(\varphi(x) - \varphi(x_1)\big) = \frac{x - x_1}{x_2 - x_1} \big(\varphi(x_2) - \varphi(x)\big) \tag{482}$$

Cancel out the common denominator to get

$$\frac{\varphi(x) - \varphi(x_1)}{x - x_1} \leq \frac{\varphi(x_2) - \varphi(x)}{x_2 - x} \qquad \forall x \in (x_1, x_2) \tag{483}$$

This is an if and only if derivation, so this is an equivalent.

Theorem 5.18

If φ is differentiable on (a, b) with φ' increaseing, then φ is convex.

Proof. Note that the LHS $= \varphi'(c_1)$, RHS $= \varphi'(c_2)$. Therefore,

$$\varphi'(c_1) \leq \varphi'(c_2) \tag{484}$$

Example 5.10

x^p for $p \geq 1$ is convex on $(0, \infty)$. Also, $e^{\alpha x}$ for $\alpha > 1$ is convex on $\mathbb{R}$.

Lemma 5.19 (Chorded Slope Lemma)

Let φ be convex on (a, b), with $x_1 < x < x_2$ belonging to a, b. Then,

$$\frac{\varphi(x) - \varphi(x_1)}{x - x_1} \leq \frac{\varphi(x_2) - \varphi(x_1)}{x_2 - x_1} \leq \frac{\varphi(x_2) - \varphi(x)}{x_2 - x} \tag{485}$$

Proof. Note that we can just write the second term as an interpolation of the first and third terms.

$$\frac{\varphi(x_2) - \varphi(x_1)}{x_2 - x_1} = \frac{\varphi(x) - \varphi(x_1)}{x - x_1} \frac{x - x_1}{x_2 - x_1} + \frac{\varphi(x_2) - \varphi(x)}{x_2 - x} \frac{x_2 - x}{x_2 - x_1} \tag{486}$$

Theorem 5.20

Let φ be convex on (a, b). Then φ has left and right derivatives at each point $x \in (a, b)$. Moreover, if $u < v$ on (a, b), then

$$\varphi'(u^-) \leq \varphi'(u^+) \leq \frac{\varphi(v) - \varphi(u)}{v - u} \leq \varphi'(v^-) \leq \varphi'(v^+) \tag{487}$$

Proof. From lemma, $\varphi(u^-)$ exists since $\frac{\varphi(u)-\varphi(w)}{u-w}$ is monotonically incraesing in w. It is also bounded from above by $\frac{\varphi(v)-\varphi(u)}{v-u}$. So $\varphi'(u^-)$ exists.

Corollary 5.21

If φ is convex on $[a, b]$, then φ is Lipschitz, i.e.

$$|\varphi(x) - \varphi(y)| \leq M(x - y) \quad \forall x, y \in [a, b], M = \max\{|\varphi'(a^+)|, |\varphi'(b^-)|\} \tag{488}$$

which hence implies that φ is AC, and hence differentiable a.e.

This is quite nice, because we don't make any assumption on regularity for convex functions. But this tells us that not only is it continuous, but also Lipschitz. Since every Lipschitz function is also absolutely continuous, then convex functions are also absolutely continuous, and hence differentiable a.e.

Theorem 5.22

Suppose φ is convex on $[a, b]$. Then, φ is differentiable except on at most a countable set, and φ' is increasing.

Proof. Consider $\varphi(x^-)$, $\varphi(x^+)$. They are increasing in x by the previous theorem. For monotone functions, we know that they can be discontinuous only on at most countable set. So, these functions $\varphi'(x^-), \varphi'(x^+)$ are continuous except on perhaps at a countable set.[a] Let us denote it as C. Let us consider $x_0 \in [a, b] \setminus C$, and take a sequence $x_n \to x_0$, with $x_n \geq x_0$. Then,

$$\varphi'(x_0^-) \leq \varphi'(x_0^+) \leq \frac{\varphi(x_n) - \varphi(x_0)}{x_n - x_0} \leq \varphi'(x_n^-) \tag{489}$$

where $\varphi'(x_n^-) \to \varphi'(x_0^-)$ as $n \to +\infty$, since $x_0 \in [a,b] \setminus C$. So, $\varphi'(x_0^-) = \varphi'(x_0^+) \implies \varphi$ is differentiable at x_0.

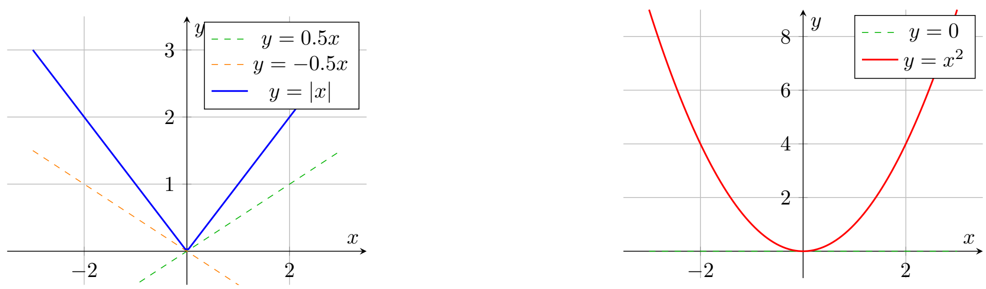
[a] Considering jumps.

Definition 5.9 (Supporting Line)

Let φ be a convex. A **supporting line** at $(x_0, \varphi(x_0))$ is a lower function

$$\ell(x) = a(x - x_0) + \varphi(x_0) \tag{490}$$

satisfying $\varphi(x) \geq \ell(x)$ for all x.



(a) Absolute value function with linear bounds (b) Quadratic function with horizontal bound

Figure 3: Basic mathematical functions with linear bounds through the origin

Note that there can be multiple supporting lines.

Theorem 5.23 (Jensen's Inequality)

suppose φ is convex on $\mathbb{R}$, with $f, \varphi \circ f$ integrable. Then,

$$\varphi\bigg(\int_0^1 f \, dx \bigg) \leq \int_0^1 \varphi(f(x)) \, dx \tag{491}$$

Proof. This first proof is good intuitively, but not a neat proof. This is like the definition of convexity, but in a "continuous appearance." Note that we can think of these as Riemann sums,

$$\varphi\bigg(\sum f_i \Delta x_i \bigg) \leq \sum_i \varphi_i \varphi(f_i) \Delta x_i \tag{492}$$

and then you can pass through the limit to get the integral version. But this is a good intuition.

Proof. The second proof is neater. Set $\alpha = \int_0^1 f \, dx$. Choose k between $\varphi'(\alpha^-), \varphi'(\alpha^+)$. Then, the line

$$\ell(x) := k(x - \alpha) + \varphi(\alpha) \tag{493}$$

is supporting for φ, so $\varphi(x) \geq \ell(x)$. Also, $\varphi(f(x)) \geq \ell(f(x))$. Integrate

$$\int_0^1 \varphi(f(x)) \, dx \geq \int_0^1 \Big(\underbrace{k(f(x) - \alpha)}_{=0} + \varphi(\alpha) \Big) \, dx = \varphi\bigg(\int_0^1 f \, dx \bigg) \tag{494}$$

On $[a,b]$, we have to normalize f,

$$\varphi\left(\frac{1}{b-a}\int_a^b f(x)\right) \leq \frac{1}{b-a}\int_a^b \varphi(f(x))\,dx \tag{495}$$

5.6 Exercises

Exercise 5.1 (Math 631 Fall 2025, Final Exam Exercise 2)

We say that $f \in C^\alpha[0,1]$, $0 < \alpha < 1$, if $\sup_{x\neq y, x,y\in[a,b]} \frac{|f(x)-f(y)|}{|x-y|^\alpha} < \infty$. Let f be absolutely continuous on $[0,1]$. Does it imply that $f \in C^\alpha[0,1]$ for some $\alpha > 0$?

Solution. No. Take $f(x)$ such that $f(0) = 0$ and $f'(x) = \frac{1}{x(\log x^{-1})^2}$. Then f' is integrable, and

$$f(x) = \int_0^x \frac{1}{y(\log y^{-1})^2}dy = \int_{\log x^{-1}}^\infty \frac{1}{z^2}dz = \frac{1}{\log x^{-1}} \tag{496}$$

is absolutely continuous. Since for every $\alpha > 0$, $\log x^{-1} \leq x^{-\alpha}$ for all sufficiently small x, f is not in $C^\alpha[0,1]$ for any $\alpha > 0$.

6 Lp Spaces

Suppose E is measurable, and f is measurable on E. Then, define

$$\|f\|_p := \left(\int_E |f|^p \right)^{1/p}, \qquad 1 \leq p < +\infty \tag{497}$$

This tells us more about the properties of the function, and how it may blow up or how its singularities might behave.

For $p = \infty$, we can define it as the maximum of the function (since by EVT). In general, we can define

$$\|f\|_\infty := \text{esssup}|f(x)| := \inf\{M \mid |f(x)| \leq M \text{ a.e. } x\} \tag{498}$$

The functions in L^p don't really satisfy the norm property that $\|f\| = 0 \iff f = 0$, so we can just consider equivalence classes of functions. For the subadditivity of the norm, checking for $p = 1$ is the easiest, and also to some extend $p = \infty$.

Definition 6.1 (Lp Space)

Given $1 \leq p \leq +\infty$, define $L^p(E)$ to be the vector space of all f measurable on E s.t. $\int_E |f|^p < +\infty$.

Proof. We first prove that it is a linear space. Since

$$|f + g|^p \leq 2^p(|f|^p + |g|^p) \tag{499}$$

so $f + g \in L^p$ if $f, g \in L^p$.

Theorem 6.1 (Young's Inequality)

Suppose $1 \leq p < +\infty$, with $q = \frac{p}{p-1}$, i.e. $\frac{1}{p} + \frac{1}{q} = 1$. Let $a, b \geq 0$. Then

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q} \tag{500}$$

Proof. Let the exponential function be $y = x^{p-1}$. Then, we get $x = y^{\frac{1}{p-1}}$. The proof is similar to if the function intersects the upper side of the rectangle.

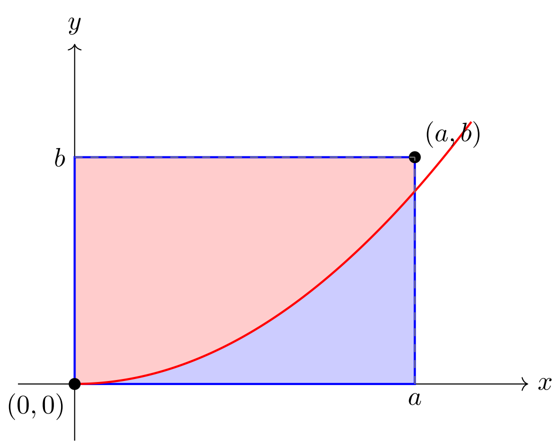


Figure 4: Rectangle with exponentially increasing curve and shaded regions

We can see that the blue region has area

$$\int_0^a x^{p-1}\,dx = \frac{a^p}{p} \tag{501}$$

We see that the red region has area A,

$$A \leq \int_0^b y^{\frac{1}{p-1}}\,dy = \frac{y^{\frac{1}{p-1}+1}}{\frac{1}{p-1}+1}\bigg|_0^b = \frac{b^q}{q} \tag{502}$$

Theorem 6.2 (Holder)

Let E be measurable, $1 \leq p \leq \infty$, $q = \frac{p}{p-1}$. Suppose $f \in L^p, g \in L^q$. Then, $f, g \in L^1(E)$, and

$$\int_E |fg|\,dx \leq \|f\|_p \|g\|_q \tag{503}$$

In fact, this is sharp. If $f \neq 0$, then (note that this is the dual vector of f)

$$f^*(x) \coloneqq \|f\|_p^{1-p} \mathrm{sgn}(f(x)) |f(x)|^{p-1} \in L^q \tag{504}$$

with

$$\|f^*\|_q = 1, \qquad \int f f^*\,dx = \|f\|_p \tag{505}$$

Proof. If $p = 1$,or $p = \infty$, then this can be proven by monotonicity. If $1 < p < \infty$, then assume that $\|f\|_p = \|g\|_q = 1$, since by linearity, we can just normalize them by multiplying them by a constant. Not it suffices to prove that the integral ≤ 1. Then, by Young's inequality,

$$|f(x)g(x)| \leq \frac{|f(x)|^p}{p} + \frac{|g(x)|^q}{q} \tag{506}$$

and by integrating over E, since the RHS is integrable (it is also bounded?), we get

$$\int_E |f(x)g(x)|\,dx \leq \frac{1}{p} + \frac{1}{q} = 1 \tag{507}$$

Finally,

$$\|f^*\|_q^q = \|f\|_p^{(1-p)q} \int |f(x)|^{(p-1)q}\,dx = \|f\|_p^{-p} \int |f(x)|^p\,dx = 1 \tag{508}$$

Corollary 6.3 (Cauchy-Schwartz)

If $p = q = 2$, then we get Cauchy Schwartz.

Theorem 6.4 (Minkowski)

If $f, g \in L^p$, then $f + g \in L^p$, and we get

$$\|f + g\|_p \leq \|f\|_p + \|g\|_q \tag{509}$$

This is the third condition for a norm. For $p = 1, +\infty$, this is immediate. Given $h \in L^p$, define

$$h^* \coloneqq \|f\|_p^{1-p} \mathrm{sgn}(h)\,|h|^{p-1} \tag{510}$$

Then, from the previous theorem,

$$\begin{aligned} \|f+g\|_p &= \int_E (f+g)(f+g)^* \, dx && (511) \\ &= \int_E f(f+g)^* \, dx + \int_E g(f+g)^* \, dx && (512) \\ &\leq \|f\|_p \underbrace{\|(f+g)^*\|_q}_{1} + \|g\|_p \underbrace{\|(f+g)^*\|_q}_{1} && (513) \end{aligned}$$

The following is an easy way to check that $\mathscr{F}$ is equi-integrable.

Corollary 6.5

let $\mathscr{F}$ be a family of functions s.t.

$$\int |f|^p \, dx < +\infty \forall f \in \mathscr{F} \tag{514}$$

Then $\mathscr{F}$ is equi-integrable.

Proof. Let A be ameasurable, $m(A) \leq \delta$. Then

$$\int_A |f| \, dx \leq \bigg(\int_A |f|^p \bigg)^{1/p} \bigg(\int_A 1^q \bigg)^{1/q} \leq M^{1/p} m(A) \leq m^{1/p} \delta^{1/q} \tag{515}$$

Given any $\epsilon > 0$, we can choose $\delta > 0$ s.t. $\int |f| \, dx \leq \epsilon$ if $m(A) < \delta$.

Corollary 6.6

Assume E has finite measure. Then, $L^{p_1} \subset L^{p_2}$ for any $1 \leq p_1 \leq p_2 \leq +\infty$. So a higher exponent is more restrictive in a finite measure set.

Proof. A simple application of Holder's inequality.

$$\int_E |f|^{p_1} \leq \bigg(\int_E |f|^{p_2} \, dx \bigg)^{p_1/p_2} x \tag{516}$$

In infinite measures, there are two ways that this can fail: singularities or tails. Either because it blows up, or it doesn't decay fast enough to be in L^p. $x^{-\alpha}$.

6.1 Nov 3

Example 6.1

Consider $f_m(x)$ in $L^p[0,1]$, with

$$f_m(x) = \chi_{I_k^{(n)}}(x) \cdot 2^{n/p}, \quad I_k^{(n)} = [(k-1)2^{-n}, k2^{-n}] \tag{517}$$

Note that $\|f_m\|_{L^p} = 1$, but $\|f_{m_1} - f_{m_2}\|_{L^p} \geq \frac{1}{2}$. So there are no limit points in f_m.

Definition 6.2 (Dual Vector)

Suppose X is a Banach space. Then $\mathcal{L} : X \to \mathbb{R}$ is called a **linear bounded functional** on X if

$$\mathcal{L}(\alpha f + \beta g) = \alpha \mathcal{L} f + \beta \mathcal{L} g, \quad \forall f, g \in X \tag{518}$$

and

$$|\mathcal{L}(f)| \leq \|\mathcal{L}\|_* \|f\|, \quad \forall f \in X \tag{519}$$

Whenever you want to prove that an integral is bounded, then you use Holder.

Example 6.2

Let $X = L^p(E)$. Then

$$\mathcal{L} f \coloneqq \int f g \, dx, \quad g \in L^q, q = \frac{p}{p-1} \tag{520}$$

is a linear bounded functional (by Holder).

L^∞ is not separable (as in there is no dense countable subset).

Theorem 6.7

Given a Banach space X, the space of all linear bounded functionals on X is a linear space with norm

$$\|\mathcal{L}\|_* = \sup_{f \in X, \|f\| \leq 1} |\mathcal{L}(f)| \tag{521}$$

This space is called **dual** to X, denoted X^*.

Proof. Just check that the sum, scalar multiplication is still a linear functional. Then for norm, just use triangle inequality.

We would like to prove that $(L^p)^* = L^q$, which we will do later. Now we'll define a different form of convergence. The regular pointwise convergence is known as strong convergence.

Definition 6.3 (Weak Convergence)

We say $f_n \rightharpoonup f$, i.e. **converges weakly** on X if

$$\mathcal{L}(f_n) \to \mathcal{L}(f), \quad \forall \mathcal{L} \in X^* \tag{522}$$

Example 6.3

Let $f_m(x)$ as before, and fix $g \in L^q[0,1]$. Then, by Holder,

$$\left| \int_0^1 \chi_{I_k^{(n)}}(x) g(x) \, dx \right| \leq \|f_m\|_{L^p} \cdot \|g \cdot \chi_{I_k^{(n)}}\|_{L^q} \to 0, \text{ as } m \to +\infty \tag{523}$$

since

$$\int_{I_k^{(n)}} |g|^p \, dx \to 0 \text{ as } n \to +\infty \tag{524}$$

by DCT, since $I_k^{(n)} = [(k-1)2^{-n}, k2^{-n}]$.

So if we know that $(L^p)^* = L^q$, we could conclude $f_m \rightharpoonup 0$.

Lemma 6.8

Suppose $\mathcal{L}_1, \mathcal{L}_2 \in X^*$. Suppose Y is a dense subset of X. If $\mathcal{L}_1 = \mathcal{L}_2$ on Y, then $\mathcal{L}_1 = \mathcal{L}_2$.

Proof. Take any $g \in X$. Find $f \in Y$ s.t. $\|f - g\|_X \leq \epsilon$. Then,

$$|\mathcal{L}_1 g - \mathcal{L}_2 g| \leq |\mathcal{L}_1 g - \mathcal{L}_1 f| + \underbrace{|\mathcal{L}_1 f - \mathcal{L}_2 f|}_{=0} + |\mathcal{L}_2 f - \mathcal{L}_2 g| \tag{525}$$

$$\leq (\|\mathcal{L}_1\|_* + \|\mathcal{L}_2\|_*)\epsilon \tag{526}$$

Lemma 6.9

Let $E \subset \mathbb{R}$ be measurable, $1 \leq p \leq +\infty$. Suppose $g \in L^1(E)$ and

$$\left| \int_E fg \, dx \right| \leq M \|f\|_p, \quad \forall f \in L^p \text{ simple} \tag{527}$$

Then, $g \in L^q, \|g\|_q \leq M$.

Proof. Consider $+\infty > p > 1$. Consider a sequence of simple φ_n s.t. $\varphi_n \to |g|$ a.e., with $0 \leq \varphi_n \leq |g|$. Then, it suffices to show that

$$\int |\varphi_n|^q \, dx \leq M, \quad \forall n \tag{528}$$

since we can just invoke Fatou's lemma. Define simple $f_n = \text{sgn}(g)|\varphi_n|^{q-1}$. Note $f_n \in L^p$ (simple functions). Then

$$\int_E |\varphi_n|^q \, dx \leq \int_E g \cdot f_n \, dx \leq M \|f_n\|_p, \tag{529}$$

where

$$\|f_n\|_p = \left(\int |\varphi_n|^{q-1)p} \, dx \right)^{1/p} = \|\varphi_n\|_q^{q/p} \tag{530}$$

So $\|\varphi_n\|_q^{q - \frac{q}{p}} = \|\varphi_n\|_q \leq M$. Since $q(1 - \frac{1}{p}) = q\frac{1}{q} = 1$.

Theorem 6.10

Let $[a, b]$ be a finite interval, with $1 \leq p < +\infty$. Suppose $\mathcal{L}$ is a linear bounded functional on $L^p([a, b])$. Then $\exists g \in L^q([a, b])$ s.t. $\mathcal{L}f = \int_a^b fg \, dx$.

Proof. Suppose $p > 1$. Define $\phi(x) = \mathcal{L}(\chi_{[0,x]})$. We claim that ϕ is AC. Take $[a_k, b_k]_{k=1}^n$ disjoint in $[a, b]$. We have

$$\sum_{k=1}^n |\phi(b_k) - \phi(a_k)| \tag{531}$$

Given $\epsilon > 0$, we want to show that $\exists \delta > 0$ s.t.

$$\sum_{k=1}^n |b_k - a_k| \leq \delta \implies \sum_{k=1}^n |\phi(b_k) - \phi(a_k)| < \epsilon \tag{532}$$

But

$$\sum_{k=1}^n |\phi(b_k) - \phi(a_k)| = \mathcal{L}\Bigg(\underbrace{\sum_{k=1}^n \operatorname{sgn}\big(\phi(b_k) - \phi(a_k)\big)\chi_{[a_k, b_k]}}_{f}\Bigg) \tag{533}$$

$$\leq \|\mathcal{L}\|_* \|f\|_p \tag{534}$$

$$= \|\mathcal{L}\|_* \bigg(\sum_{k=1}^n (b_k - a_k)\bigg)^{1/p} \tag{535}$$

Take $\delta = \big(\frac{\epsilon}{\|\mathcal{L}\|_*}\big)^p$

So now that we have proved that ϕ is AC, we are almost in a position to use the lemma we proved before. Then, $\exists g \in L^1$ s.t. $\phi(x) = \int_a^x g(t)\, dt$ by fundamental theorem of calculus. Given ay $\chi_{[c,d]}$, we have

$$\mathcal{L}(\chi_{c,d}) = \phi(d) - \phi(c) = \int_c^d g(t)\, dtjjA \tag{536}$$

So $\mathcal{L}f$ and $\int gf\, dx$ coincide on step functions, and step functions are a dense set in L^p. We have

$$\bigg| \int gf\, dx \bigg| \leq \|\mathcal{L}\|_* \|f\|_p \tag{537}$$

By the lemma, $g \in L^q$, with $\|g\|_q \leq \|\mathcal{L}\|_*$.

Note that this is *not true* on L^∞. Indeed $(L^1)^* = L^\infty$ (this is easier to prove), but $(L^\infty)^* \neq L^1$. But it is hard to construct a counterexample, and people use the Banach extension theorem to define such counterexamples.

Theorem 6.11 (Riesz Representation Theorem)

Let $E \subset \mathbb{R}$ be measurable, $1 \leq p < +\infty$. Suppose $\mathcal{L}$ is a bounded linear functional on L^p. Then $\exists g \in L^q$ s.t.

$$\mathcal{L}(f) = \int_E gf\, dx, \quad \|g\|_q = \|\mathcal{L}\|_* \tag{538}$$

The dual of continuous functions is Steljes measures.

Now let's talk about weak convergence. This gives you some sort of compactness of a unit ball in X^* with respect to the norm, which is basically sort of like weak convergence.

Theorem 6.12 (Helley)

Lt X be a Banach space, separable. Suppose $\mathcal{L}_n \in X^*$ satisfy $\|L_n\|_* \leq M$ for all n. Then, $\exists n_k$ s.t.

$$L_{n_k}(f) \to L(f), \quad \forall f \in X \tag{539}$$

Proof. Let $(f_n)_{n=1}^\infty$ be a countable dense subset in X. Then, $\{\mathcal{L}_n f_1\}_{n=1}^\infty$ is ? $\implies \exists$ subsequence $s_{1,m}$ s.t. $\mathcal{L}_{s_{1,m}} f_1 \to a_1$. Also, we can choose a $s_{2,m}$ subsequence of $s_{1,m}$ s.t. $\mathcal{L}_{s_2,m} f_2 \to a_2$, and so in $s_{l,m}$ s.t.

$$\mathcal{L}_{s_{l,m}} f_j \to a_j, \quad \forall 1 \leq j \leq l \tag{540}$$

Select $n_k = S_{k,k}$. Then, $\mathcal{L}_{n_k} f_j \to a_j$ for all j. Given any $g \in X$, consider

$$|\mathcal{L}_{n_{k_2}} g - \mathcal{L}_{n_{k_1}} g| \leq |\mathcal{L}_{n_{k_2}} g - \mathcal{L}_{n_{k_1}} f| + |\mathcal{L}_{n_{k_2}} f - \mathcal{L}_{n_{k_1}} f| + |\mathcal{L}_{n_{k_2}} f - \mathcal{L}_{n_{k_1}} g| \tag{541}$$

Fix $\epsilon > 0$, choose f s.t.

$$\|f - g\| \leq \frac{\epsilon}{3 \max\{\|f\|, \|g\|\}} \tag{542}$$

Then, k_1, k_2, large so middle term $\leq \frac{\epsilon}{3}$.

So this is what people mean by a unit ball in L^p is weakly compact.

6.2 Exercises

Exercise 6.1 (Math 631 Fall 2025, Final Exam Exercise 6)

Define c_0 to be a space of all sequences $x = x_1, x_2, ..., x_n, ...$ that converge to zero, with the norm $\|x\| = \sup_n |x_n|$. Prove that c_0 is a Banach space and find its dual.

Solution. Suppose that x^m is a Cauchy sequence in c_0. Then each component x_n^m is also a Cauchy sequence of real numbers, and converges to some y_n. It is not hard to see that y_n is a bounded sequence: take $\epsilon = 1$ and find N such that $\|x^m - x^N\|_\infty \leq 1$ if $m \geq N$. Then also $\|y - x^N\|_\infty \leq 1$, and so $y \in l^\infty$. Let us now show that $y \in c_0$. Fix $\epsilon > 0$. Find N such that $\|y - x^N\|_\infty \leq \epsilon/2$. Next, since $x^N \in c_0$, find M such that $|x_m^N| \leq \epsilon/2$ for all $m \geq M$. Then for every $m \geq M$,

$$|y_m| \leq |y_m - x_m^N| + |x_m^N| \leq \epsilon. \tag{543}$$

Hence $y \in c_0$. This proves c_0 is a Banach space.
Take any sequence a_n in l_1, then it defines a bounded linear functional A on c_0 defined by $A(x) = \sum_{n=1}^\infty a_n x_n$. Note that $\|A\| = \|a\|_{l^1}$: we can just take sequences x_n^m that take value 1 if $a_n > 0$ and -1 if $a_n < 0$ for $n = 1, ..., m$. Then $\|x^m\|_\infty = 1$, and

$$\sum_{n=1}^\infty a_n x_n^m \to \|a\|_{l^1} \tag{544}$$

as $m \to \infty$ by Lebesgue dominated convergence theorem. On the other hand, let A be any bounded linear functional on c_0. For every n, take $e^n \in c_0$ such that $e_j^n = \delta_{nj}$ (1 if $j = n$ and 0 if $j \neq n$). Set $a_n = A(e_n)$. By taking x^m before equal to ± 1 depending on the sign of a_n for the first m positions, we find that

$$A(x^m) = \sum_{n=1}^m |a_n| \leq \|A\|, \tag{545}$$

for every m. Therefore, $a_n \in l^1$ and then it is not hard to see $\|A\| = \|a\|_{l^1}$. Hence the dual space of c_0 can be identified with l^1.

Exercise 6.2 (Math 631 Fall 2025, Final Exam Exercise 7)

Let h be continuous function periodic on $\mathbb{R}$ with period one, and assume $\int_0^1 h(x)dx = 0$. Define $f_n(x) = h(nx)$. Show that for any finite interval $[a, b]$, $f_n \to 0$ in $L^p[a, b]$, $1 \leq p < \infty$.

Solution. Since step functions are dense in $L^p[a, b]$, it suffices to prove that for any $a \leq c < d \leq b$, $\int_c^d h(nx)dx \to 0$ as $n \to \infty$. But

$$\int_c^d h(nx)dx = n^{-1} \int_{nc}^{nd} h(y)dy \leq 2\|h\|_{L^\infty} n^{-1} \to 0 \tag{546}$$

as $n \to \infty$. Here we used that periodic continuous function must be bounded, and integrated over all full periods included into $[nc, nd]$ getting zero from these parts since h is mean zero.

7 Measure

Now we generalize to abstract measure spaces. Recall how we constructed the Lebesgue measure: (1) We took a set function ℓ that assigns lengths to all intervals in $\mathbb{R}$. (2) We have used this length to define the Lebesgue outer measure as an outer approximation of these intervals. (3) We then use the outer measure to define measurable sets with Caratheodory's criterion, which states that measurable sets should split any set nicely into measurable sets. (4) We verify that the collection of all measurable sets is a σ-algebra. (5) We define the Lebesgue measure as the restriction of the Lebesgue outer measure to the collection of mesaurable sets. This is called the *Caratheodory* construction of Lebesgue measure, and we can generalize this to an abstract space X under certain conditions.

First, we want all measurable sets to be a σ-algebra, which defines all the well-behaved sets.

Definition 7.1 (Measurable Space)

A **measurable space** is a tuple $(X, \mathcal{M})$ consisting of a set X with a σ-algebra of subsets of X. Elements of $\mathcal{M}$ are called **measurable sets**.

Definition 7.2 (Measure, Measure Space)

A **measure** μ on a measurable space $(X, \mathcal{M})$ is a set function $\mu : \mathcal{M} \to [0, +\infty]$ satisfying the following.

1. *Null empty set.* $\mu(\emptyset) = 0$.
2. *Countable Additivity.* For all countable collections $\{A_k\}_{k=1}^{\infty}$ of pairwise disjoint[a] subsets $A_k \subset 2^X$,

$$\mu\bigg(\bigsqcup_{k=1}^{\infty} A_k\bigg) = \sum_{k=1}^{\infty} \mu(A_k) \tag{547}$$

[a]Disjointness is clearly important since if it wasn't, then $\mu(A) = \mu(A \cup A) = 2\mu(A)$, which is absurd.

Example 7.1 (Measure Spaces)

The following are all valid measure spaces.

1. $(\mathbb{R}, \mathcal{L}, m)$, where $\mathcal{L}$ is the set of all Lebesgue-measurable sets.
2. $(\mathbb{R}, \mathcal{B}, m)$, where $\mathcal{B}$ is the set of all Borel-measurable sets.
3. $(X, 2^X, \eta)$, where $\eta(E)$ is the cardinality of the set. This is called the **counting measure**.
4. $(X, 2^X, \delta_{x_0})$ where $x_0 \in X$ and $\delta_{x_0}(E) = 1$ if $x_0 \in E$ and 0 if else. This is called the **Dirac measure**.

From just this definition, we can restore all of our familiar properties.

Theorem 7.1 (Axiomatic Properties of Measure)

Let $(X, \mathcal{M}, \mu)$ be a measure space.

1. *Finite Additivity.* For any finite disjoint collection $\{E_k\}_{k=1}^{n}$ of measurable sets,

$$\mu\bigg(\bigcup_{k=1}^{n} E_k\bigg) = \sum_{k=1}^{n} \mu(E_k) \tag{548}$$

2. *Monotonicity.* If $A \subset B$ are both measurable sets, then

$$\mu(A) \leq \mu(B) \tag{549}$$

3. *Excision.* If $A \subset B$ are both measurable sets and $\mu(A) < +\infty$, then

$$\mu(B \setminus A) = \mu(B) - \mu(A) \tag{550}$$

4. *Countable Monotonicity.* For any countable collection $\{E_k\}_{k=1}^\infty$ of measurable sets that covers a measurable set E,

$$\mu(E) \leq \sum_{k=1}^\infty \mu(E_k) \tag{551}$$

Proof. Listed.

1. *Finite Additivity.*
2. *Monotonicity.* Let $B \setminus A \coloneqq B \cap A^c$. Then, since A and $B \setminus A$ are disjoint, we have

$$\mu(B) = \mu\big(A \cup (B \setminus A)\big) = \mu(A) + \mu(B \setminus A) \geq \mu(A) \tag{552}$$

3. *Excision.*
4. *Countable Monotonicity.* We again try to divide this union into disjoint sets. Let $A_i' = A \cap A_i$, and let $B_1 = A_1'$ with

$$B_i = A_i \setminus \bigcup_{j=1}^{i-1} A_j' \tag{553}$$

Since B_i's are disjoint with $B_i \subset A_i$, we can use the first property to get

$$\mu(A) = \sum_{i=1}^\infty \mu(B_i) \leq \sum_{i=1}^\infty \mu(A_i) \tag{554}$$

Theorem 7.2 (Continuity of Measure)

Let $(X, \mathcal{M}, \mu)$ be a measure space.

1. *Continuity from Below.* If $\{A_k\}_{k=1}^\infty$ is an ascending sequence of measurable sets, then

$$\mu\bigg(\bigcup_{k=1}^\infty A_k\bigg) = \lim_{k \to \infty} A_k \tag{555}$$

2. *Continuity from Above.* If $\{B_k\}_{k=1}^\infty$ is a descending sequence of measurable sets and $\mu(B_1) < +\infty$, then

$$\mu\bigg(\bigcap_{k=1}^\infty B_k\bigg) = \lim_{k \to \infty} \mu(B_k) \tag{556}$$

Proof. Listeed.

1. *Continuity from Below.* With the fact that $\mu(A_k)$ must be nondecreasing, we can use real analysis and see that it is bounded by ∞, meaning that it must have a limit. But why does this limit equal to the left hand side? We can see that

$$\begin{aligned}
\mu\bigg(\bigcup_{k=1}^\infty A_k\bigg) &= \mu(A_1) + \sum_{k=2}^\infty \mu(B_k) && (557) \\
&= \mu(A_1) + \lim_{k \to \infty} \sum_{k=2}^\infty \mu(B_k) && (558) \\
&= \lim_{k \to \infty} \mu(A_1 \cup B_2 \cup \ldots B_k) = \lim_{k \to \infty} \mu(A_k) && (559)
\end{aligned}$$

where $B_k = A_k \setminus A_{k-1}$.

2. *Continuity from Above.* The $\mu(A_1) < \infty$ is a necessary condition, since if we take $A_k = [k, \infty)$ on the real number line, then we have $\cap_{k=1}^\infty A_k = \emptyset$, but the limit of the measure is ∞. Well we can define $B_k = A_k \setminus A_{k+1}$ and write $\cap_{k=1}^\infty A_k = A_1 \setminus \cup_{k=1}^\infty B_k$, which means that

$$\mu\bigg(\bigcap_{k=1}^\infty A_k\bigg) = \mu\bigg(A_1 \setminus \bigcup_{k=1}^\infty B_k\bigg) \tag{560}$$

$$= \mu(A_1) - \mu\bigg(\bigcup_{k=1}^\infty B_k\bigg) \tag{561}$$

$$= \mu(A_1) - \sum_{k=1}^\infty \mu(B_k) \tag{562}$$

$$= \mu(A_1) - \lim_{K\to\infty} \sum_{k=1}^K \mu(B_k) \tag{563}$$

$$= \lim_{K\to\infty} \bigg(\mu(A_1) - \sum_{k=1}^K \mu(B_k)\bigg) \tag{564}$$

$$= \lim_{K\to\infty} \mu\bigg(A_1 \setminus \bigcup_{k=1}^K B_k\bigg) = \lim_{K\to\infty} \mu(A_K) \tag{565}$$

Now the first line uses the fact that if $A \subset B$, then $\mu(B \setminus A) + \mu(A) = \mu(B)$, and with the further assumption that $\mu(A) < \infty$, we can subtract on both sides like we do with regular arithmetic.

Definition 7.3 (Almost Everywhere)

For a measure space $(X, \mathcal{M}, \mu)$ and a measurable subset E of X, we say that a property P holds **almost everywhere** on E if it holds for all $E \setminus E_0$ for some measurable subset E_0 where $\mu(E_0) = 0$.

Lemma 7.3 (Borel-Cantelli Lemma)

Let $(X, \mathcal{M}, \mu)$ be a measure space and $\{E_k\}_{k=1}^\infty$ be a countable collection of measurable sets for which $\sum_{k=1}^\infty \mu(E_k) < +\infty$. Then,

$$\mu\big(\limsup_k E_k\big) \coloneqq \mu\bigg(\bigcap_{n=1}^\infty \bigcup_{k\geq n} E_k\bigg) = 0 \tag{566}$$

That is, almost all $x \in X$ belong to at most a finite number of the E_k's.

Proof. By continuity of μ from above and countable monotonicity of μ,

$$\mu\bigg(\bigcup_{n=1}^\infty \bigg[\bigcap_{k\geq n} E_k\bigg]\bigg) = \lim_{n\to\infty} \mu\bigg(\bigcup_{k\geq n} E_k\bigg) \leq \lim_{n\to\infty} \sum_{k=n}^\infty \mu(E_k) = 0 \tag{567}$$

since the series converges.

Now here comes the new part.

Definition 7.4 (Finite, σ-Finite Measures and Measurable Sets)

Let $(X, \mathcal{M}, \mu)$ be a measure space.
1. The measure μ is **finite** if $\mu(X) < +\infty$.
2. The measure μ is **σ-finite** if X is the union of a countable collection of measurable sets, each of which has finite measure.

Let E be a measurable set.
1. E is of **finite measure** if $\mu(E) < +\infty$.
2. E is **σ-finite** if E is the union of a countable collection of measurable sets, each of which has finite measure.

Finiteness implies σ-finiteness.

Example 7.2

Listed.
1. The Lebesgue measure on $[0, 1]$ is a finite measure.
2. The Lebesgue measure on $\mathbb{R}$ is a σ-finite measure.
3. The counting measure on an uncountable set is not σ-finite.

Definition 7.5 (Complete Metric Spaces)

A measure space $(X, \mathcal{M}, \mu)$ is **complete** if $\mathcal{M}$ contains all subsets of sets of measure 0.

Example 7.3

Listed.
1. $(\mathbb{R}, \mathcal{L}, m)$ is complete.
2. $(\mathbb{R}, \mathcal{B}, m)$ is complete since we shows that the Cantor set (a Borel set of Lebesgue measure 0), contains a subset that is not Borel.

Theorem 7.4 (Every Measure Space can be Completed)

Let $(X, \mathcal{M}, \mu)$ be a measure space. Define $\mathcal{M}_0$ to be the collection of subsets E of X of the form $E = A \cup B$ where
1. $B \in \mathcal{M}$,
2. $A \subset C$ for some $C \in \mathcal{M}$ for which $\mu(C) = 0$.[a]

For such a set E, define $\mu_0(E) = \mu(B)$. Then, $\mathcal{M}_0$ is a σ-algebra that contains $\mathcal{M}$, μ_0 is a measure that extends μ, and $(X, \mathcal{M}_0, \mu_0)$ is a complete measure space.

[a] So we are basically splitting E into a measure 0 part C and everything else B.

Proof.

7.1 Signed Measures

Lemma 7.5 (Sums and Positive Multiples of Measures are Measures)

If μ_1 and μ_2 are two measures defined on the same measurable space $(X, \mathcal{M})$, then for $\alpha, \beta > 0$, the following is a measure as well.

$$\mu_3(E) = \alpha \cdot \mu_1(E) + \beta \cdot \mu_2(E) \tag{568}$$

Proof.

Note that we can't always define differences of such measures

$$\nu(E) = \mu_1(E) - \mu_2(E) \tag{569}$$

since ν may not always be nonnegative. Furthermore, it may not even be defined if $\mu_1(E) - \mu_2(E) = \infty - \infty$.

Definition 7.6 (Signed Measure)

A **signed measure** ν on the measurable space $(X, \mathcal{M})$ is a function $\nu : \mathcal{M} \to [-\infty, +\infty]$ satisfying

1. *Well-Defined.* ν assumes at most one of the values $+\infty, -\infty$.
2. *Null Empty Set.* $\nu(\emptyset) = 0$
3. *Countable Additivity.* For any countable collection $\{E_k\}_{k=1}^{\infty}$ of disjoint measurable sets,

$$\nu\bigg(\bigcup_{k=1}^{\infty} E_k\bigg) = \sum_{k=1}^{\infty} \nu(E_k) \tag{570}$$

where the series $\sum_k \nu(E_k)$ converges absolutely if $\nu\big(\cup_k E_k\big)$ is finite.

Definition 7.7 (Positive, Negative, Null Sets)

Let ν be a signed measure on $(X, \mathcal{M})$ and $A \in \mathcal{M}$.

1. A is **positive** w.r.t. ν if for all measurable $E \subset A$, $\nu(E) \geq 0$.
2. A is **negative** w.r.t. ν if for all measurable $E \subset B$, $\nu(E) \leq 0$.
3. A is **null** w.r.t. ν if for all measurable $E \subset B$, $\nu(E) = 0$.[a]

[a]By monotoncity of measure, a set is null if and only if it has measure 0.

Lemma 7.6 (Hahn's Lemma)

Let ν be a signed measure on $(X, \mathcal{M})$ and E a measurable set for which $0 < \nu(E) < +\infty$. Then, there is a measurable subset $A \subset E$ that is positive and of positive measure.

Theorem 7.7 (Hahn Decomposition Theorem)

Let ν be a signed measure on the measurable space $(X, \mathcal{M})$. Then, there is a positive set A and a negative set B, both with respect to ν, for which

$$X = A \sqcup B \tag{571}$$

That is, we can always decompose X into a positive and negative measure parts, and this is called the **Hahn decomposition** of X w.r.t. ν.[a]

[a]Note that this may not be unique, since if $A \cup B$ is a Hahn decomposition, then by excising a null set E from A and

adding to B, $(A \setminus E) \cup (B \cup E)$ is also a Hahn decomposition.

Therefore, the measure of $\nu^+(E) = \nu(E \cap A)$ and $\nu^-(E) = -\nu(E \cap B)$. This decomposition is nice since we know that the positive parts and the negative parts are "nicely separated." Let's formalize this notion.

Definition 7.8 (Mutually Singular Measures)

Two measures ν_1, ν_2 are said to be **mutually singular**, denoted $\nu_1 \perp \nu_2$, if $X = A \cup B$ with $\nu_1(A) = \nu_2(B) = 0$.

Theorem 7.8 (Jordan Decomposition Theorem)

Let ν be a signed measure on the measurable space $(X, \mathcal{M})$. Then, there is a unique pair of mutually singular measures ν^+, ν^- on $(X, \mathcal{M})$ for which

$$\nu = \nu^+ - \nu^- \tag{572}$$

called the **Jordan decomposition**, with ν^+, ν^- called the positive and negative parts of ν. Since ν assumes at most one of the values $\pm\infty$, either ν^+, ν^- must be finite.

Proof. We have already proven the first part as

$$\nu^+(E) = \nu(E \cap A), \quad \nu^-(E) = -\nu(E \cap B) \tag{573}$$

Example 7.4

Let $f : \mathbb{R} \to \mathbb{R}$ be a function that is Lebesgue integrable over $\mathbb{R}$, and define

$$\nu(E) \coloneqq \int_E f \, dm \tag{574}$$

Then, from the countable additivity of integration, ν is a signed measure on the measurable space $(\mathbb{R}, \mathcal{L})$. Define

$$A = \{x \in \mathbb{R} \mid f(x) \geq 0\}, \quad B = \{x \in \mathbb{R} \mid f(x) < 0\} \tag{575}$$

and

$$\nu^+(E) = \int_{A \cap E} f \, dm, \quad \nu^-(E) = -\int_{B \cap E} f \, dm \tag{576}$$

Then, A, B is a Hahn decomposition of $\mathbb{R}$ w.r.t. the signed measure ν. Moreover, $v = v^+ = v^-$ is a Jordan decomposition of ν.

7.2 Carathéodory Construction of Measurable Sets

Definition 7.9 (Outer Measure)

Given a space X, an **outer measure** is a function $\mu^* : 2^X \to [0, +\infty]$ satisfying either the two properties.

1. *Null Empty Set.* $\mu^*(\emptyset) = 0$.

2. *Countable Monotonicity.* For arbitrary subset $A, B_1, B_2, \ldots$,

$$A \subset \bigcup_{k=1}^{\infty} B_k \implies \mu(A) \leq \sum_{k=1}^{\infty} \mu(B_k) \tag{577}$$

Theorem 7.9 (Construction of Outer Measure)

Let $\mathcal{S}$ be a collection of subsets in X and $\mu : \mathcal{S} \to [0, +\infty]$ be a set functions. Define $\mu^*(\emptyset) = 0$ and

$$\mu^*(E) \coloneqq \inf \left\{ \sum_{k=1}^{\infty} \mu(E_k) \colon E \subset \bigcup_k E_k \right\} \tag{578}$$

where the infimum of an empty set is $+\infty$. Then, the set function $\mu^* : 2^X \to [0, +\infty]$ is an outer measure called the **outer measure induced by** μ.

Proof.

Definition 7.10 (Carathéodory's criterion)

Given outer measure μ^* on X, a set $E \subset X$ is called μ^***-measurable** if for every set $A \subset X$,

$$\mu^*(A \cap E) + \mu^*(A \cap E^c) = m^*(A) \tag{579}$$

Theorem 7.10 (Measurable Sets is a σ-Algebra)

Let μ^* be an outer measure on 2^X. Then, the collection $\mathcal{M}$ of sets that are measurable w.r.t. μ^*, also called μ^*-measurable, is a σ-algebra. If $\overline{\mu}$ is the restriction of μ^* to $\mathcal{M}$, then $(X, \mathcal{M}, \overline{\mu})$ is a complete measure space.

Proof. It is clear that complements are measurable by symmetricity of Carathéodory's criterion. The processes of proving this is identical to that of Lebesgue measure: first prove that finite union of measurable sets is measurable. Then show that for any $A \subset X$ and a finite disjoint collection $\{E_k\}_{k=1}^n$, we have

$$\mu^*\left(A \cap \left[\bigcup_{k=1}^{n} E_k\right]\right) = \sum_{k=1}^{n} \mu^*(A \cap E_k) \tag{580}$$

Finally, we prove that union of countable collection of measurable sets is measurable, where one direction is easy and the other is done by using the previous proposition and taking $n \to \infty$.

Definition 7.11 (Carathéodory Measure)

Let $\mathcal{S}$ be a collection of subsets of X, $\mu : \mathcal{S} \to [0, +\infty]$ a set function, and μ^* the outer measure induced by μ. The measure $\overline{\mu}$ that is the restriction of μ^* to the σ-algebra $\mathcal{M}$ of μ^*-measurable sets is called the **Carathéodory measure induced by** μ.

Recall the regularity properties of Lebesgue measurable sets. That is, $E \subset \mathbb{R}$ is measurable iff there exists a G_δ-set G s.t. $E \subset G$ and $m^*(G \setminus E) = 0$. The following is a generalization of this.

Theorem 7.11 (Regularity of μ^*-measurable Sets)

Let $\mu : \mathcal{S} \to [0, +\infty]$ be a set function defined on a collection $\mathcal{S}$ of subsets of a set X and $\overline{\mu} : \mathcal{M} \to [0, +\infty]$, the Carathéodory measure induced by μ. Let E be a subset of X for which $\mu^*(E) < +\infty$. Then, there is a subset $A \subset X$ for which

$$A \in S_{\sigma\delta}, \quad E \subset A, \quad \mu^*(E) = \mu^*(A) \tag{581}$$

Furthermore, if E and seach set in $\mathcal{S}$ is μ^*-measurable, then so is A, and

$$\overline{\mu}(A \setminus E) = 0 \tag{582}$$

Proof.

We can also generalize this further by introducing a increasing, continuous function $F : \mathbb{R} \to \mathbb{R}$ and defining the outer measure to be

$$\lambda^*(A) = \inf_{C_A} \sum_{j=1}^{\infty} \big(F(b_j) - F(a_j)\big) \tag{583}$$

7.3 Premeasures

Note that given a set function μ over $\mathcal{S}$, its Carathéodory extension $\overline{\mu}$ need not agree with μ for sets in $\mathcal{S}$. We would like to find adequate assumptions such that $\overline{\mu}$ is an extension of μ.

Definition 7.12 (Premeasure)

Let $\mathcal{S}$ be a collection of subsets of X and $\mu : \mathcal{S} \to [0, +\infty]$ a set function. Then, μ is called a **premeasure** if

1. μ is finitely additive,
2. μ is countably monotone,
3. if $\emptyset \in \mathcal{S}$, then $\mu(\emptyset) = 0$.

Theorem 7.12 (Premeasure Condition for Carathéodory Measure)

Let $\mathcal{S} \subset 2^X$ and $\mu : \mathcal{S} \to [0, +\infty]$ a set function. In order for the Carathéodory measure induced by μ be an extension of μ, it is necessary that μ is a premeasure.

Proof.

So being a premeasure is a necessary but not sufficient condition, but if we impose on $\mathcal{S}$ a finer set-theoretic structure, this necessary condition is also sufficient.

Definition 7.13 (Closure Under Relative Complements)

A collection $\mathcal{S} \subset 2^X$ is said to be closed w.r.t. the formation of relative complements provided

$$A, B \in \mathcal{S} \implies A \setminus B \in \mathcal{S} \tag{584}$$

Theorem 7.13 (Premeasure over Set Closured Under Relative Complements Induces Carathéodory Extension)

Let $\mu : \mathcal{S} \to [0, +\infty]$ be a premeasure on $\mathcal{S}$ that is closed w.r.t. the formation of relative complements. Then, the Carathéodory measure $\overline{\mu} : \mathcal{M} \to [0, +\infty]$ induced by μ is an extension of μ, called the **Carathéodory extension** of μ.

Proof.

However, a number of natural premeasures such as the premeasure length defined on the collection of bounded intervals of reals numbers, are defined on collections of sets that are not closed w.r.t. relative complements. So we consider alternate conditions for extending measures.

Definition 7.14 (Semiring)

A nonempty collection $\mathcal{S}$ of subsets of a set X is a **semiring** if

1. *Closure under finite intersections.*

$$A, B \in \mathcal{S} \implies A \cap B \in \mathcal{S} \tag{585}$$

2. *Disjoint decomposition of relative complements.*

$$A, B \in \mathcal{S} \implies A \setminus B = \bigsqcup_{k=1}^{n} C_k \tag{586}$$

for some collection $C_k \in \mathcal{S}$.

Theorem 7.14 (Carathéodory-Hahn Theorem)

Let $\mu : \mathcal{S} \to [0, +\infty]$ be a premeasures on a semiring $\mathcal{S}$ of subsets of X.

1. Then, the Carathéodory measure $\overline{\mu}$ induced by μ is an extension of μ.
2. Furthermore, if μ is σ-finite, then so is $\overline{\mu}$, and $\overline{\mu}$ is the unique measure on the σ-algebra of μ^*-measurable sets that extends μ.

7.4 Product Measure

Before, we saw how we can construct measures using Carathéodory construction. We will consider how to create product measures, which will be an extension of a set functions using the Carathéodory-Hahn theorem.

Definition 7.15 (Measurable Rectangle)

Let $(X, \mathcal{A}, \mu)$, $(Y, \mathcal{B}, \nu)$ be two measure spaces. Consider the product space $X \times Y$. If $A \in \mathcal{A}, B \in \mathcal{B}$, then the set $A \times B$ is called a **measurable rectangle**.

Lemma 7.15

Let $\{A_k \times B_k\}_{k=1}^{\infty}$ be a countable disjoint collection of measurable rectangles whose union is also a measurable rectangle $A \times B$. Then,

$$\mu(A) \times \nu(B) = \sum_{k=1}^{\infty} \mu(A_k) \times \nu(B_k) \tag{587}$$

Proof.

We want to set up the conditions to invoke the Carathéodory-Hahn theorem. This naturally leads to the following.

Theorem 7.16

Let $\mathcal{R}$ be the collection of measurable rectangles in $X \times Y$ and for a measurable rectangle $A \times B$, define

$$\lambda(A \times B) = \mu(A) \cdot \nu(B) \tag{588}$$

Then, $\mathcal{R}$ is a semiring and $\lambda : \mathcal{R} \to [0, +\infty]$ is a premeasure.

Proof.

Definition 7.16 (Product Measure)

Let $(X, \mathcal{A}, \mu)$, $(Y, \mathcal{B}, \nu)$ be two measure spaces, $\mathcal{R}$ the collection of measurable rectangles contained in $X \times Y$, and λ the premeasure defined on $\mathcal{R}$ by

$$\lambda(A \times B) = \mu(A) \cdot \nu(B) \tag{589}$$

for all $A \times B \in \mathcal{R}$. Then, the **product measure** $\lambda = \mu \times \nu$ is the Carathéodory extension of $\lambda : \mathcal{R} \to [0, +\infty]$ defined on the σ-algebra of $(\mu \times \nu)^*$-measurable subsets of $X \times Y$.

7.5 Stieltjes Construction of Measure

Let $\mathbb{R}^n$ be the continuum and $\mathcal{R}^n$ be the **Borel σ-algebra**, defined as the σ-algebra generated by the open sets of $\mathbb{R}^n$.

Example 7.5 (Stieltjes Measure Function)

Measures on $(\mathbb{R}, \mathcal{R})$ are defined by giving a **Stieltjes measure function** with the following properties:

1. F is nondecreasing
2. F is right continuous:
$$\lim_{y \downarrow x} F(y) = F(x) \tag{590}$$

Theorem 7.17

Associated with each Stieltjes measure function F there is a unique measure μ on $(\mathbb{R}, \mathcal{R})$ with

$$\mu((a, b]) = F(b) - F(a) \tag{591}$$

When $F(x) = x$, then the resulting measure is called the **Lebesgue measure**.

This is quite a hard proof, but we outline the construction of this measure on $\mathbb{R}$. First, we would like to define a "nice" set of half-open half-closed intervals, which we show is a semialgebra $\mathcal{S}$. We can easily define a measure μ on this semialgebra. We can extend this semialgebra to an algebra $\overline{\mathcal{S}}$, along with a proper extension $\overline{\mu}$ that is a unique measure on $\overline{\mathcal{S}}$.

Definition 7.17 (Semialgebra, Algebra)

A collection $\mathcal{S}$ of sets is said to be a **semialgebra** if
1. it is closed under intersection
2. If $S \in \mathcal{S}$, then S^c is a finite disjoint union of sets in $\mathcal{S}$

A collection $\mathcal{A}$ of subsets is said to be an **algebra** if
1. it is closed under union
2. it is closed under complementation
3. the first two imply that it is closed under intersection

We can see that a set that is a σ-algebra $\implies$ it is an algebra.

Here is an example of a semialgebra, which we will utilize in building a measure on $\mathbb{R}^n$.

Example 7.6

Let $\mathcal{S}_d$ be the empty set plus all sets of the form

$$(a_1, b_1] \times \ldots \times (a_d, b_d] \subset \mathbb{R}^d \tag{592}$$

where $-\infty \leq a_i < b_i \leq +\infty$. $\mathcal{S}_d$ is a semialgebra since

$$\left(\prod_i (a_i^1, b_i^1]\right) \cap \left(\prod_i (a_i^2, b_i^2]\right) = \prod_i (\max\{a_i^1, a_i^2\}, \min\{b_i^1, b_i^2\}] \tag{593}$$

and ...

Now, we show that we can extend this semialgebra to an algebra.

Lemma 7.18

If $\mathcal{S}$ is a semialgebra, then $\overline{\mathcal{S}} = \{\text{finite disjoint unions of sets in } \mathcal{S}\}$ is an algebra, called the algebra generated by $\mathcal{S}$.

Proof.

Example 7.7

Given $\mathbb{R}$ and its semialgebra $\mathcal{S}_1$, then $\overline{\mathcal{S}}_1$ consists of the empty set and all sets of the form

$$\bigcup_{i=1}^n (a_i, b_i] \text{ where } -\infty \leq a_i < b_i \leq +\infty \tag{594}$$

Now as for extending our measure function to $\overline{\mathcal{S}}$, we can simply use the properties. Note that since since an algebra is constructed from finite disjoint unions of a semialgebra, given that the finite collection $\{A_i\}_{i=1}^n$

all reside in $\mathcal{S}$ and are disjoint, then their disjoint union must be in $\overline{\mathcal{S}}$ and must be measurable, defined as

$$\overline{\mu}\bigg(\bigsqcup_{i=1}^{n} A_i\bigg) = \sum_{i=1}^{n} \mu(A_i) \tag{595}$$

Definition 7.18 (σ-finite measure)

Given a measure on an algebra $\mathcal{A}$, μ is said to be **$\boldsymbol{\sigma}$-finite** if there is a sequence of sets $A_1, A_2, \ldots \in \mathcal{A}$ s.t. $\mu(A_i) < \infty$ and $\cup_i A_i = \Omega$.

Theorem 7.19

Let $\mathcal{S}$ be a semialgebra and let μ defined on $\mathcal{S}$ have $\mu(\emptyset) = 0$. Suppose

1. if $S \in \mathcal{S}$ is a finite disjoint union of sets $\{S_i\}_{i=1}^n$, then

$$\mu(S) = \sum_{i=1}^{n} \mu(S_i) \tag{596}$$

2. f S is a countably infinite disjoint union of sets $\{S_j\}_{j=1}^{\infty}$, then

$$\mu(S) \leq \sum_{j=1}^{\infty} \mu(S_j) \tag{597}$$

Then, μ has a unique extension $\bar{\mu}$ that is a measure on $\overline{\mathcal{S}}$, the algebra generated by $\mathcal{S}$. If $\bar{\mu}$ is σ-finite, then there is a unique extension ν that is a measure on $\sigma(\mathcal{S})$ (the smallest σ-algebra containing $\mathcal{S}$).

8 Integration

8.1 Radon-Nikodym Theorem

Let $(X, \mathcal{M})$ be a measure space. For measure μ on $(X, \mathcal{M})$ and f a nonnegative function on X that is measurable w.r.t. $\mathcal{M}$, define the set function ν on $\mathcal{M}$ by

$$\nu(E) \coloneqq \int_E f \, d\mu \tag{598}$$

Then, by additivity of integration and by MCT, this is indeed a measure.

Definition 8.1 (Absolutely Continuous Measures)

On a given measurable space $(X, \mathcal{M})$, a measure ν is said to be **absolutely continuous** w.r.t. the measure μ if for all $E \in \mathcal{M}$,

$$\mu(E) = 0 \implies \nu(E) = 0 \tag{599}$$

So therefore, μ acts as an upper bound of ν over 0 measure sets.

Theorem 8.1

Let $(X, \mathcal{M}, \mu)$ be a measure space and ν a finite easure on the measurable space $(X, \mathcal{M})$. Then ν is absolutely continuous w.r.t. μ iff for each $\epsilon > 0$, $\exists \delta > 0$ s.t. for all $E \in \mathcal{M}$,

$$\mu(E) < \delta \implies \nu(E) < \epsilon \tag{600}$$

Theorem 8.2 (Radon-Nikodym Theorem)

Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space and ν a σ-finite measure defined on the measurable space $(X, \mathcal{M})$ that is absolutely continuous w.r.t. μ. Then there is a nonnegative function f on X that is measurable w.r.t. $\mathcal{M}$ for which

$$\nu(E) = \int_E f \, d\mu \tag{601}$$

for all $E \in \mathcal{M}$. Furthermore, f is unique in the sense that if g is any nonnegative measurable function on X that also has this property, then $g = f$ μ-a.e.

Proof.

Corollary 8.3

Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space and ν a finite signed measure on the measurable space $(X, \mathcal{M})$ that is AC w.r.t. μ, then there is a function f that is integrable over X w.r.t. μ, and

$$\nu(E) = \int_E f \, d\mu \tag{602}$$

for all $E \in \mathcal{M}$.

Theorem 8.4 (Lebesgue Decomposition Theorem)

Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space and ν a σ-finite measure on the measurable space $(X, \mathcal{M})$. Then there is a measure ν_0 on $\mathcal{M}$, singular w.r.t. μ, and a measure ν_1 on $\mathcal{M}$—AC w.r.t. μ—for which $\nu = \nu_0 + \nu_1$. The measures ν_0, ν_1 are unique.

Proof.

8.2 Fubini's and Tonelli's Theorem

Theorem 8.5 (Fubini's Theorem)

Let $(X, \mathcal{A}, \mu)$, $(Y, \mathcal{B}, \nu)$ be two measure spaces and ν be complete. Let f be integrable over $X \times Y$ w.r.t. to the product measure $\mu \times \nu$. Then for almost all $x \in X$, the x-section of f, $f(x, \cdot)$, is integrable over Y with respect to ν, and

$$\int_{X \times Y} f \, d(\mu \times \nu) = \int_X \left[\int_Y f(x, y) \, d\nu(y) \right] d\mu(x) \tag{603}$$

Theorem 8.6 (Tonelli's Theorem)

Let $(X, \mathcal{A}, \mu)$, $(Y, \mathcal{B}, \nu)$ be two measure spaces and ν be complete.Let f be a nonnegative $(\mu \times \nu)$-measurable function on $X \times Y$. Then,

1. For almost all $x \in X$, the x-section of f, $f(x, \cdot)$, is ν-measurable.
2. The function defined almost everywhere on x by

$$x \mapsto \int_Y f(x, y) \, d\nu(y) \tag{604}$$

   is μ-measurable.
3. Finally,

$$\int_{X \times Y} f \, d(\mu \times \nu) = \int_X \left[\int_Y f(x, y) \, d\nu(y) \right] d\mu(x) \tag{605}$$

8.3 Exercises

Exercise 8.1 (Math 631 Fall 2025, Final Exam Exercise 4)

Let $A = [-1, 1] \times [-1, 1]$. Let $f(x, y) = \frac{xy}{(x^2+y^2)^2}$ (when $x = 0$ or $y = 0$ we set $f(x, y) = 0$). Prove that the iterated integrals exist and are equal:

$$\int_{-1}^{1} (\int_{-1}^{1} f(x, y) dx) dy = \int_{-1}^{1} (\int_{-1}^{1} f(x, y) dy) dx, \tag{606}$$

but the double integral $\int_A f(x, y) dx dy$ does not exist.

Solution. Note that f is odd with respect to both x and y. Therefore

$$\int_{-1}^{1} f(x, y) dx = \int_{-1}^{1} f(x, y) dy = 0 \tag{607}$$

for all y and x respectively. Hence both iterated integrals are zero. Now passing to polar coordinates, we have that $|f(r, \theta)| = \frac{|\sin 2\theta|}{2r^2}$. Therefore, for every $\epsilon > 0$,

$$\int_A |f(x, y)| dx dy \geq \int_\epsilon^1 \int_0^{\pi/2} \frac{\sin 2\theta}{r^2} r d\theta dr \geq c \int_\epsilon^1 r^{-1} dr. \tag{608}$$

Since $\int_\epsilon^1 r^{-1} dr$ diverges as $\epsilon \to 0$, f is not integrable over A.