

Chapter 17

Wide Sense Stationary Random Processes

17.1 Introduction

Having introduced the concept of a random process in the previous chapter, we now wish to explore an important subclass of stationary random processes. This is motivated by the very restrictive nature of the stationarity condition, which although mathematically expedient, is almost never satisfied in practice. A somewhat weaker type of stationarity is based on requiring the mean to be a constant in time and the covariance sequence to depend only on the separation in time between the two samples. We have already encountered these types of random processes in Examples 16.9–16.11. Such a random process is said to be stationary in the wide sense or *wide sense stationary* (WSS). It is also termed a *weakly stationary* random process to distinguish it from a stationary process, which is said to be *strictly stationary*. We will use the former terminology to refer to such a process as a WSS random process. In addition, as we will see in Chapter 19, if the random process is Gaussian, then wide sense stationarity implies stationarity. For this reason alone, it makes sense to explore WSS random processes since the use of Gaussian random processes for modeling is ubiquitous.

Once we have discussed the concept of a WSS random process, we will be able to define an extremely important measure of the WSS random process—the *power spectral density* (PSD). This function extends the idea of analyzing the behavior of a deterministic signal by decomposing it into a sum of sinusoids of different frequencies to that of a random process. The difference now is that the amplitudes and phases of the sinusoids will be random variables and so it will be convenient to quantify the *average power* of the various sinusoids. This description of a random phenomenon is important in nearly every scientific field that is concerned with the analysis of time series data such as systems control [Box and Jenkins 1970], signal processing [Schwartz and Shaw 1975], economics [Harvey 1989], geophysics [Robinson 1967],

vibration testing [McConnell 1995], financial analysis [Taylor 1986], and others. As an example, in Figure 17.1 the Wolfer sunspot data [Tong 1990] is shown, with the data points connected by straight lines for easier viewing. It measures the average number of sunspots visually observed through a telescope each year. The importance of the sunspot number is that as it increases, an increase in solar flares occurs. This has the effect of disrupting all radio communications as the solar flare particles reach the earth. Clearly from the data we see a periodic type property. The estimated PSD of this data set is shown in Figure 17.2. We see that the distribution of power versus frequency is highest at a frequency of about 0.09 cycles per year. This means that the random process exhibits a large periodic component with a period of about $1/0.09 \approx 11$ years per cycle, as is also evident from Figure 17.1. This is a powerful prediction tool and therefore is of great interest. How the PSD is actually estimated will be discussed in this chapter, but before doing so, we will need to lay some groundwork.

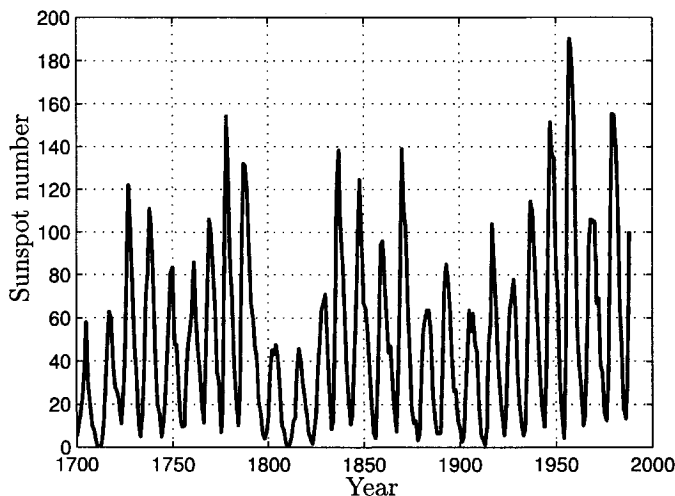


Figure 17.1: Annual number of sunspots – Wolfer sunspot data.

17.2 Summary

A less restrictive form of stationarity, termed wide sense stationarity, is defined by (17.4) and (17.5). The conditions require the mean to be the same for all n and the covariance sequence to depend only on the time difference between the samples. A random process that is stationary is also wide sense stationary as shown in Section 17.3. The autocorrelation sequence is defined by (17.9) with n being arbitrary. It is the covariance between two samples separated by k units for a zero mean WSS random process. Some of its properties are summarized by Properties 17.1–17.4. Under certain conditions the mean of a WSS random process can be found by using

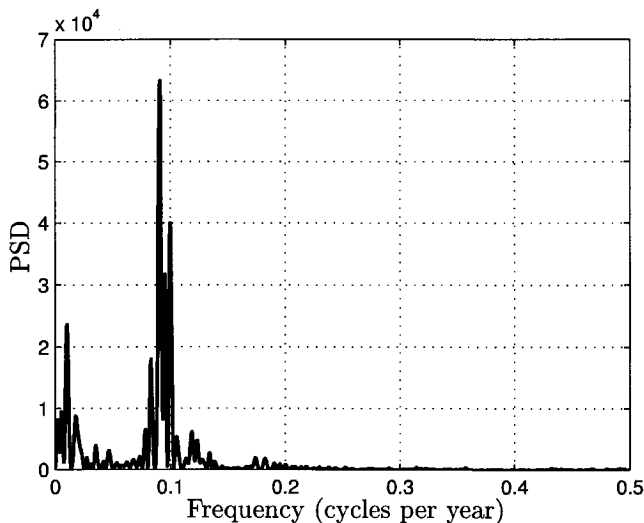


Figure 17.2: Estimated power spectral density for Wolfer sunspot data of Figure 17.1. The sample mean has been computed and removed from the data prior to estimation of the PSD.

the temporal average of (17.25). Such a process is said to be ergodic in the mean. For this to be true the variance of the temporal average given by (17.28) must converge to zero as the number of samples averaged becomes large. The power spectral density (PSD) of a WSS random process is defined by (17.30) and can be evaluated more simply using (17.34). The latter relationship says that the PSD is the Fourier transform of the autocorrelation sequence. It measures the amount of average power per unit frequency or the distribution of average power with frequency. Some of its properties are summarized in Properties 17.7–17.12. From a finite segment of a realization of the random process the autocorrelation sequence can be estimated using (17.43) and the PSD can be estimated by using the averaged periodogram estimate of (17.44) and (17.45). The analogous definitions for a continuous-time WSS random process are given in Section 17.8. Also, an important example is described that relates sampled continuous-time white Gaussian noise to discrete-time white Gaussian noise. Finally, an application of the use of PSDs to random vibration testing is given in Section 17.9.

17.3 Definition of WSS Random Process

Consider a discrete-time random process $X[n]$, which is defined for $-\infty < n < \infty$ with n an integer. Previously, we defined the mean and covariance sequences of