EE 214 PROBABILITY AND RANDOM PROCESSES IN ELECTRICAL ENGINEERING

Lecture 1B: Expected Value and Variance

STATISTICS OF RANDOM VARIABLES

- A random experiment is performed N times. Random variable X takes on values $x_1, x_2, \ldots, x_m$. x_1 occurs N_1 times, x_2 occurs N_2 times, etc.

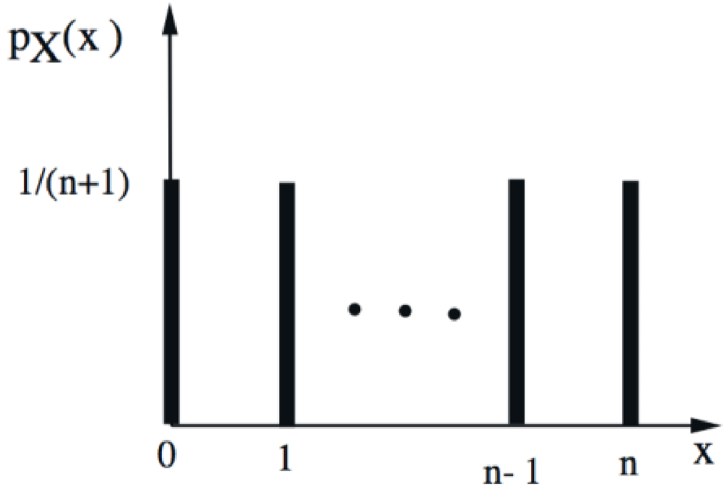
$$\overline{x} = \frac{N_1 x_1 + N_2 x_2 + \ldots + N_m x_m}{N_1 + N_2 + \ldots + N_m}$$

$$\overline{x} = \sum_i x_i \frac{N_i}{N}$$

$$\overline{x} = E[X] = \sum_i x_i p_X(x_i)$$

EXPECTATION

- A discrete RV with uniform PMF



$$E[X] = 0 \times \frac{1}{n+1} + 1 \times \frac{1}{n+1} + \cdots + n \times \frac{1}{n+1}$$

EXPECTATION

In general, the expectation value of a random variable g(X), denoted by E[g(X)], is given by:

discrete $$E[g(X)] = \sum_i g(x_i)p_X(x_i)$$

continuous $$E[g(X)] = \int_{-\infty}^{\infty} g(x)f_X(x)dx$$

PROPERTIES OF EXPECTATION

- $E[c] = c$
- $E[cg(x)] = cE[g(x)]$
- $E[g_1(x) + g_2(x)] = E[g_1(x)] + E[g_2(x)]$

DEFINITIONS

- Variance: $$\sigma_X{}^2 = E[(X - \bar{X})^2] = \overline{(X - \bar{X})^2}$$
- Standard Deviation $$\sigma_X = \sqrt{E[(X - \bar{X})^2]}$$
- Mean Square Value $$E[X^2] = \overline{X^2}$$

EXAMPLE

- Show that $\sigma_X^2 = E[X^2] - (E[X])^2$

$$\sigma_X{}^2 = E[(X - \overline{X})^2] = E[X^2 - 2X\overline{X} + \overline{X}^2]$$
$$\sigma_X{}^2 = E[X^2] - 2\overline{X}E[X] + \overline{X}^2$$
$$\sigma_X{}^2 = E[X^2] - 2\overline{X}\left(\overline{X}\right) + \overline{X}^2$$
$$\sigma_X{}^2 = E[X^2] - \overline{X}^2 = E[X^2] - E[X]^2$$

DEFINITIONS

- n^{th} moment $\quad m_n = E[X^n] = \overline{X^n}$

- n^{th} central moment $\quad \mu_n = E[(X - \overline{X})^n] = \overline{(X - \overline{X})^n}$

BERNOULLI RANDOM VARIABLE

- Probability mass function (PMF)

Outcome	X	$p_X(x)$
Failure	0	1 – p
Success	1	p

- Statistics

$$E[X] = \sum x p_X(x) = (0)(1-p) + (1)(p) = p$$

$$E[X^2] = \sum x^2 p_X(x) = (0)^2(1-p) + (1)^2(p) = p$$

$$\sigma_X{}^2 = p - p^2$$

BINOMIAL

- Number of successes on n trials of a Bernoulli experiment

$$p_X(x) = \binom{n}{x} p^x (1-p)^{n-x}$$

- Recall: Binomial theorem

$$\sum_{x=0}^{n} \binom{n}{x} p^x q^{n-x} = (p+q)^n$$

BINOMIAL

- mean

$$E[X] = np\sum_{y=0}^{n-1}\binom{n-1}{y}p^{y}(1-p)^{n-1-y} = np$$

- the mean square

$$E[X^2] = np(1-p) + n^2p^2$$

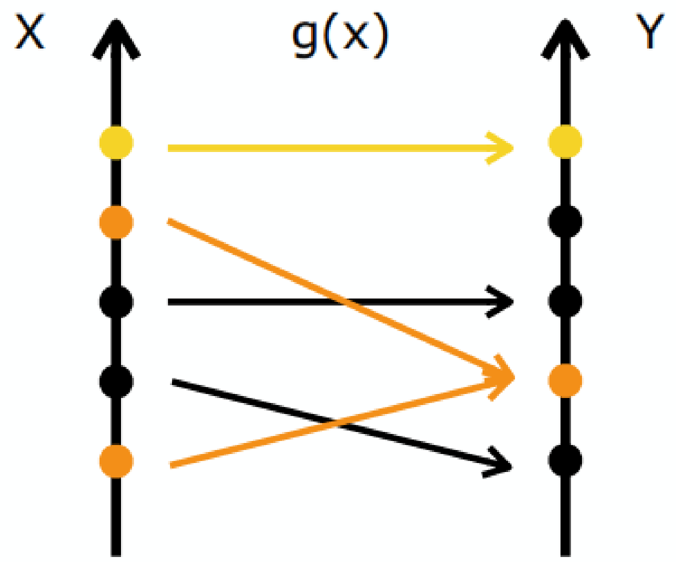
- The variance

$$\sigma_X{}^2 = (np(1-p) + n^2p^2) - (np)^2$$
$$\sigma_X{}^2 = np(1-p)$$

TRANSFORMATION OF RV

- Given 2 discrete RVs X and Y, and Y = g(X), how do we calculate $p_Y(y)$?

$$p_Y(y) = \mathbf{P}(g(X) = y) = \sum_{x:g(x)=y} p_X(x)$$



EXAMPLE

- Consider a discrete random variable X = {-1, 0, 1} with probabilities p_{-1}, p_0 and p_1. Determine the pmf of $Y = X^2$

$$p_X(x) = \begin{cases} p_{-1} & x = -1 \\ p_0 & x = 0 \\ p_1 & x = 1 \end{cases}$$

$$p_Y(y) = \begin{cases} p_{-1} & x = -1 \rightarrow y = 1 \\ p_0 & x = 0 \rightarrow y = 0 \\ p_1 & x = 1 \rightarrow y = 1 \end{cases}$$

Note: $p_{-1} + p_0 + p_1 = 1$

$$\boxed{p_Y(y) = \begin{cases} p_0 & y = 0 \\ p_{-1} + p_1 & y = 1 \end{cases}}$$