A Book of Abstract Algebra: Solutions to Chapter 5

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Notes

A subgroup S is called a subgroup of a group G, if:

1. It is closed on the given operation, i.e. the operation $(\cdot)$ of two elements produces an element $\in S$.

2. It is closed under inverse, i.e. the inverse of each element of S is in S.

Also, each subgroup is a group as well, and therefore follows the three group laws:

1. Associativity

2. Identity

3. Inverse

The *identity*, e of the group is shared by the subgroup.

Trivial & Proper Subgroups

1. The one-element subset $\{e\}$ and the entire group G are the smallest and the largest subgroups of G and are called *trivial subgroups*.

2. All the other subgroups of G are called *proper subgroups*.

Cyclic Groups and Subgroups

If a group (or a subgroup) is generated by a single element, we call that group *Cyclic* and it is written as $\langle a \rangle$, where a is called the *generator* and is the single element which, along with the identity and a^{-1}, can define the entire group.

Defining Equations

A set of equations, involving only the generators and their inverses, is called a set of *defining equations*. These equations can completely define the operation table of the group.

Solutions

Set A

1. $G = \langle R, + \rangle, H = \{loga : a \in \mathbb{Q}, a > 0\}$

   - Addition:
     Let $a, b \in \mathbb{Q}$
     $log\ a + log\ b = log\ ab$
     $\because a, b \in \mathbb{Q},$
     $\therefore ab \in \mathbb{Q}, ab > 0,$
     $\Rightarrow log\ ab \in H$
   - Identity:
     The identity element would not change the value of $log\ a$ under addition. $log\ 1$ or 0 is the identity element, since:
     If $log\ a + log\ b = log\ a$, then $log\ b = 0$, and $b = 1$.
   - Inverse:

$$\begin{aligned} log\ a + log\ a^{-1} &= e \\ \Rightarrow log\ a &= -\ log\ a^{-1} \\ \Rightarrow loga &= log(\frac{1}{a^{-1}}) \\ \Rightarrow a &= \frac{1}{a^{-1}} \end{aligned}$$

     Since $a \in \mathbb{Q}$, $\frac{1}{a^{-1}} \in \mathbb{Q}$, $\therefore log\ a^{-1} \in H$