A Book of Abstract Algebra: Solutions to Chapter 5

Tushar Tyagi

May 31, 2016

Notes

A subgroup S is called a subgroup of a group G, if:

1. It is closed on the given operation, i.e. the operation $(\cdot)$ of two elements produces an element $\in S$.

2. It is closed under inverse, i.e. the inverse of each element of S is in S.

Also, each subgroup is a group as well, and therefore follows the three group laws:

1. Associativity

2. Identity

3. Inverse

The *identity*, e of the group is shared by the subgroup.

Trivial & Proper Subgroups

1. The one-element subset $\{e\}$ and the entire group G are the smallest and the largest subgroups of G and are called *trivial subgroups*.

2. All the other subgroups of G are called *proper subgroups*.

Cyclic Groups and Subgroups

If a group (or a subgroup) is generated by a single element, we call that group *Cyclic* and it is written as $\langle a \rangle$, where a is called the *generator* and is the single element which, along with the identity and a^{-1}, can define the entire group.

Defining Equations

A set of equations, involving only the generators and their inverses, is called a set of *defining equations*. These equations can completely define the operation table of the group.

Solutions

Set A

1. $G = \langle R, + \rangle, H = \{loga : a \in \mathbb{Q}, a > 0\}$

    - Addition:
      Let $a, b \in \mathbb{Q}$
      $log\ a + log\ b = log\ ab$
      $\because a, b \in \mathbb{Q},$
      $\therefore ab \in \mathbb{Q}, ab > 0,$
      $\Rightarrow log\ ab \in H$
    - Identity:
      The identity element would not change the value of $log\ a$ under addition. log 1 or 0 is the identity element, since:
      If $log\ a + log\ b = log\ a$, then $log\ b = 0$, and $b = 1$.
    - Inverse:

    $$\begin{aligned} log\ a + log\ a^{-1} &= e \\ \Rightarrow log\ a &= -\ log\ a^{-1} \\ \Rightarrow loga &= log(\frac{1}{a^{-1}}) \\ \Rightarrow a &= \frac{1}{a^{-1}} \end{aligned}$$

    Since $a \in \mathbb{Q}$, $\frac{1}{a^{-1}} \in \mathbb{Q}$, $\therefore log\ a^{-1} \in H$

2. $G = \langle R, + \rangle, H = \{loga : a \in \mathbb{Z}, a > 0\}$

   - Addition:

     Same reasoning as previous question.

   - Inverse:
     As calculated in the previous question, $a^{-1} = \frac{1}{a}$ Since $a \in \mathbb{Z}, a^{-1} \notin \mathbb{Z}$

3. $G = \langle R, + \rangle, H = \{x \in \mathbb{R} : tanx \in \mathbb{Q}\}$

   - Addition:
     Let $x, y \in \mathbb{R}$.
     $\therefore tan(x + y) = \frac{tanx+tany}{1-tanx \cdot tany}$
     If $x = y = 45°$, then $tan\ x = tan\ y = 1$, which makes the denominator undefined, and therefore addition is not defined for H.

4. $G = \langle R, \cdot \rangle, H = \{2^n 3^m, m, n \in \mathbb{Z}\}$

   - Multiplication:
     Let $n, m, n', m' \in \mathbb{Z}$.
     $2^n 3^m \cdot 2^{n'} 3^{m'} = 2^{n+n'} 3^{m+m'}$
     $\because n + n', m + m' \in \mathbb{Z}$
     $\therefore 2^{n+n'} 3^{m+m'} \in H$

   - Inverse:
     Since $2^n 3^m \in \mathbb{Z}$,
     The inverse is $\frac{1}{2^n 3^m} = 2^{-n} 3^{-m} \because -n, -m \in \mathbb{Z}, \therefore 2^{-n} 3^{-m} \in \mathbb{Z}$

5. $G = \langle R \times R, + \rangle, H = \{(x, y) : y = 2x\}$

   - Addition:
     $(x, y) + (x', y') = (x + x, y + y')$
     $\because x,\ x',\ y,\ y' \in \mathbb{R}$,
     $\therefore x + x',\ y + y' \in \mathbb{R}$
     $\therefore (y + y') = 2(x + x')$

- Inverse:
  $e = (0, 0)$ Inverse: $(-x, -y)$
  $\because y = 2x$
  $\Rightarrow -y = -2x$
  $\Rightarrow y' = 2x'$

6. $G = \langle R \times R, + \rangle, H = \{(x, y) : x^2 + y^2 > 0\}$

   - Addition
     $(x, y) + (x', y') = (x + x', y + y')$
     $\Rightarrow (x + x')^2 + (y + y')^2 > 0,$
     $\therefore$ Addition operation is defined for H.
   - Inverse:
     $\because e = (0, 0)$
     Inverse is $(-x, -y)$
     Let $x, y \in H$,
     $\therefore x^2 + y^2 > 0,$
     $\Rightarrow (-x)^2 + (-y)^2 > 0 \in H$
     So inverse is defined.

7. Let C and D be sets, with $C \subseteq D$. Prove that P_C is a subgroup of P_D

   In a way, $G = \langle P_D, + \rangle, H = \langle P_c, + \rangle$

   - Identity:
     (common to both P_C and P_D): $\{\phi\}$
   - Inverse:
     $A^{-1} = A$
     We have already proved this in Chapter 3, Exercise C.
   - Addition:
     Let $A, B \subseteq H$,
     $A + B = (A - B) \cup (B - A)$
     Since $A \subseteq P_C, (A - B) \subseteq P_C$.
     Similarly, $(B - A) \subseteq P_C$. So the operation of symmetric difference is closed on subgroup H.

Set B

1. $G = \langle \mathscr{F}(R), + \rangle$, $H = \{f \in \mathscr{F}(R) : f(x) = 0\ \forall x \in [0,1]\}$

    - Addition
      Let $f, g \in H$
      $f(x) = 0, g(x) = 0, \forall x \in [0,1]$
      $[f+g](x) = f(x) + g(x) = 0 + 0 = 0$
      $\therefore f + g \in H$
    - Inverse
      Let $f \in H$, then
      $[-f](x) = -f(x) = 0 = f(x) x \in [0,1]$

2. $G = \langle \mathscr{F}(R), + \rangle$, $H = \{f \in \mathscr{F}(R) : f(-x) = -f(x)\}$

    - Addition
      Let $f, g \in H$
      $[f+g](-x)$
      $= f(-x) + g(-x)$
      $= -f(x) - g(x)$
      $= -(f(x) + g(x))$
      $= -[f+g](x)$
      $\therefore f + g \in H$