International Reserve Management under Rollover Crises[*]

Mauricio Barbosa Alves
University of Minnesota

Javier Bianchi
Federal Reserve Bank of Minneapolis

César Sosa-Padilla
University of Notre Dame and NBER

March 2024

Abstract

This paper investigates how a government should manage international reserves when it faces the risk of a rollover crisis. We ask, should the government accumulate reserves or reduce debt to make itself less vulnerable? Our analysis shows that issuing debt to accumulate reserves may reduce the government's exposure to a rollover crisis and lead to a reduction in sovereign spreads. However, we find that the government should delay the accumulation of reserves until the point where increasing reserves makes it safe from a rollover crisis.

JEL classification: E4, E5, F32, F34, F41.
Keywords: International reserves, sovereign debt, rollover crises.

1 Introduction

Governments are frequently exposed to episodes where investors suddenly lose confidence in the government's ability to repay the debt. This sudden loss of confidence can be self-fulfilling and often results in governments being unable to roll over the debt and defaulting on their sovereign debt. Given the significant costs associated with rollover crises, understanding how governments can reduce their vulnerability to these crises is crucial for economic policy.

In this paper, we investigate whether governments should accumulate international reserves as a safeguard against rollover crises. A common argument is that reserves provide the government with liquid resources when it cannot roll over the debt. However, a government can also lower its vulnerability by reducing sovereign debt. To the extent that reserves earn a lower interest than the one paid on the debt, it is unclear whether building up a stock of reserves is an effective way to reduce the government's vulnerability to a rollover crisis.

We provide a simple environment to study the optimal management of international reserves for a government subject to the risk of rollover crises. Our paper establishes conditions under which issuing debt and accumulating reserves can be optimal. We show that if a government is heavily indebted, it should reduce debt gradually instead of accumulating reserves. Accumulating reserves becomes optimal only after debt has been substantially reduced. Furthermore, we show that it is possible that issuing debt to accumulate reserves can reduce sovereign spreads. This occurs when such an operation safeguards the government from a rollover crisis.

Our environment is a canonical model of rollover crises, following Cole and Kehoe (2000), expanded with reserve accumulation. The government starts with an initial stock of reserves and long-duration debt, receives a constant stream of income, and discounts the future at the same rate as external investors. To abstract from the insurance motive highlighted in Bianchi, Hatchondo and Martinez (2018), we assume that the only source of uncertainty is the possibility of a rollover crisis. When the government is in good credit standing, it decides how much debt to issue, how many reserves to accumulate, and whether to repay the coupons due or default on its obligations. Upon default, the government is permanently excluded from sovereign debt markets but can keep its reserves and continue to accumulate reserves in the future.

Our analysis begins by characterizing how the economy can be in one of three regions depending on the initial portfolio of debt and reserves. In the safe zone, the government repays the debt regardless of whether investors continue to roll over the debt, and thus it is not vulnerable to a run in equilibrium. In the default zone, the government finds it

optimal to default regardless of whether investors are willing to lend. In the *crisis zone*, the government's default decision depends on investors' beliefs. If investors refuse to roll over the debt, it becomes more costly for the government to repay, and the government defaults. On the other hand, if investors expect the government to repay, the government finds it optimal to repay. In the crisis region, the government is therefore vulnerable to self-fulfilling rollover crises (Cole and Kehoe, 2000).

The key question we tackle is the following: Suppose a government is in the crisis zone. What is the best strategy to reach the safe zone? Should the government accumulate reserves or reduce its debt?

Our analysis elucidates the conditions under which accumulating reserves is indeed optimal. A key consideration is that a higher stock of reserves raises the value of default, therefore raising the outside option for the government and making it potentially more vulnerable to a rollover crisis. On the other hand, higher reserves also raise the value of repayment. To the extent that only a fraction of the debt is due every period, a simultaneous increase in debt and reserves helps relax the budget constraint for the government in a situation where it is unable to borrow. Our analysis provides necessary and sufficient conditions for reserve accumulation to be optimal. Namely, it shows when the relaxation of the budget constraint is sufficiently large to compensate for the fact that reserves also raise value of default.

In addition, we show that it is not optimal for the government to accumulate reserves in the early phases of the transition to the safe zone. Rather, the government should increase the net foreign asset position exclusively through reductions in debt. Specifically, we find that the accumulation of reserves should be delayed until the point in which doing so guarantees that the government will be in the safe zone. Furthermore, we show that when the government issues debt to finance the accumulation of reserves, doing so lowers sovereign spreads.

Related literature.   Our paper belongs to a vast literature on international reserves. In particular, our paper is related to studies on the joint determination of reserves and defaultable sovereign debt, that build on the workhorse model of sovereign default (Eaton and Gersovitz, 1981; Aguiar and Gopinath, 2006; and Arellano, 2008).[1]

Alfaro and Kanczuk (2009) study reserve accumulation in the canonical sovereign default model with one-period debt and show that it is not optimal for the government to accumulate reserves. This is because reserves make default more attractive and therefore worsen debt sustainability. In a model with long-term debt, Bianchi, Hatchondo and Martinez (2018)

---

[1]See Aguiar and Amador (2014) and Aguiar, Chatterjee, Cole and Stangebye (2016) for a review of the literature on sovereign debt and default.

show that accumulating reserves provides hedging against negative income shocks that raise borrowing costs. In their model, when the government issues debt to accumulate reserves, it effectively reallocates resources from states with high bond prices and low marginal utility of consumption to states with low bond prices and high marginal utility of consumption. In that environment, the government trades off these hedging benefits against the costs of higher spreads from keeping larger gross debt positions.[2] Different from these studies, we consider the possibility of rollover crises, as in Cole and Kehoe (2000), and show that issuing debt to accumulate reserves can reduce the vulnerability and lead to a decrease in sovereign spreads. Hernandez (2018) provides numerical simulations in a model with fundamental risk and rollover risk. Our approach focuses exclusively on rollover risk and allows us to provide an analytical characterization of when issuing debt to accumulate reserves is optimal.[3]

Corsetti and Maeng (2023b) studies the joint accumulation of reserves and debt in a model of self-fulfilling debt crisis proposed by Aguiar, Chatterjee, Cole and Stangebye (2022). They show that accumulating reserves is desirable in that framework because it rules out an inefficient equilibrium with depressed bond prices.[4] We build instead on the canonical Cole and Kehoe (2000) model and show how issuing long-term debt and accumulating reserves can reduce vulnerability to a rollover crisis.

Aguiar and Amador (2014, 2024) and Bocola and Dovis (2019) study optimal debt maturity structure in a model with rollover crises. In these studies, a sufficiently long maturity can prevent a rollover crisis, but it has the cost of exacerbating debt dilution.[5] Our results on the optimal deleveraging are connected in particular to those in Aguiar and Amador (2014), who show that it is optimal for the government to initially remain passive in long-term debt bonds—actively reducing short-term debt—and to lengthen the maturity at the end of the process. Aguiar and Amador (2024) study maturity swaps and show how they can improve bond prices and welfare under rollover risk. Our paper takes as given the maturity structure and studies the accumulation of international reserves. An important distinction in the portfolio problem the government faces in our model is that reserves increase the outside

---

[2]Under a fixed exchange rate regime, Bianchi and Sosa-Padilla (2023) show that the hedging benefits lead to a macroeconomic stabilization motive for accumulating reserves.

[3]In his simulations, Hernandez (2018) finds negligible effects of changes in rollover risk on the accumulation of reserves, suggesting that the key motive for reserve accumulation in his model is related to fundamental risk instead of rollover risk. Our necessary condition for reserves to be optimal under rollover crisis does not appear to be satisfied in his calibration.

[4]See also Aguiar et al. (2022) for how the equilibrium with depressed bond prices can be avoided by issuing low or high amounts of debt relative to the good equilibrium level. Key for their mechanism is that the bond price schedule is a non-monotonic function of debt.

[5]The desirability properties of long-term debt are also highlighted in Cole and Kehoe (2000). See also Arellano and Ramanarayanan (2012) and Hatchondo, Martinez and Sosa-Padilla (2016) for studies of the maturity tradeoffs under fundamental risk.

option of defaulting, potentially reducing the sustainable debt level, whereas in their case, the outside option of defaulting is exogenous.

Conesa and Kehoe (2024) study a model of rollover crises where the government can commit to a tax rate in advance. They show that it is ex-ante optimal to set a high tax so that in the event of a rollover crisis, the government has sufficient tax revenue to repay the debt, thus deterring investors from running. However, this policy is suboptimal ex-post, as the government can borrow at the risk-free rate and would prefer to set a lower tax rate. They refer to this policy as "preemptive austerity." In our model, reserves also serve a preemptive role, but the use of reserves to service the debt is ex-post optimal and thus does not require commitment. Moreover, the accumulation of reserves allows the government to maintain a lower net foreign asset position and potentially higher consumption as it exits the crisis zone.[6]

Outline. The remainder of the paper is organized as follows. Section 2 introduces the environment. Section 3 analyzes the optimal government portfolio. Section 4 concludes. All proofs are in the Appendix.

2 Environment

Consider a small open economy with time indexed by $t = 0, 1....$ There is a single consumption good, which is freely tradable. The government in the small open economy receives a constant endowment y of the consumption good and trades long-term bonds, b, and one-period risk-free assets, a, with a continuum of external investors. Investors are risk-neutral and share the same discount factor, $\beta \in (0, 1)$, as the government.

2.1 Government Problem

The government's preferences are represented by

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t [u(c_t) - \phi d_t],$$

where c denotes consumption and $u(\cdot)$ is an increasing, twice continuously differentiable, and strictly concave function. We denote by d_t the default decision that takes the value of 1 if the government is in default and 0 otherwise.

The government enters every period with an initial portfolio of assets and outstanding bonds (a, b). We refer to the assets as "reserves" and assume without loss of generality that they have a one-period maturity.[7] The return on reserves is $1 + r = \beta^{-1}$.

The bonds the government issues are long-maturity bonds with geometrically decaying coupons, as in Hatchondo and Martinez (2009) and Chatterjee and Eyigungor (2012). A bond issued at any date t promises to pay $\left[\frac{\delta+r}{1+r}\right][1, (1-\delta), (1-\delta)^2, ...]$ in periods $t+1, t+2, t+3....$ Notice that the coupon payments are normalized so that the price of the bond equals $1/(1+r)$ if there is no default risk. The Macaulay duration of the bond is parameterized by $1/\delta$. To streamline notation, we will often use $\kappa \equiv \frac{\delta+r}{1+r}$ to denote the coupon per unit of bond.

If the government has not defaulted in the past, we can write the budget constraint of the government when it chooses to repay as follows:

$$c_t = y + a_t - \left(\frac{\delta+r}{1+r}\right)b_t - \frac{a_{t+1}}{1+r} + q_t[b_{t+1} - (1-\delta)b_t].$$

That is, the government collects its income and assets, consumes, pays the debt coupons, accummulates reserves, and issues bonds at a price schedule q_t. As we will see, in the Markov equilibrium, the bond price schedule will depend on the portfolio chosen by the government.

If the government defaults on its debt, it is excluded permanently from financial markets and faces a utility cost ϕ every period.[8] As occurs in practice, the government keeps its holdings of reserves and can continue trading. The budget constraint in case of default is as follows:

$$c_t = y + a_t - \frac{a_{t+1}}{1+r}.$$

Timing. The timing follows Cole and Kehoe (2000). We assume the government makes a repayment decision after borrowing and reserve accumulation have taken place. This differs from the Eaton-Gersovitz timing, in which the government first chooses to repay or default and then borrows. The fact that investors are atomistic will open the door to coordination failures where a *good equilibrium* where investors continue to roll over the bonds and the government repays coexists with a *bad equilibrium* where investors refuse to roll over the

---

[7]When reserves can be sold costlessly in a spot market, it is equivalent whether reserves mature every period or if they are long-term assets. The equivalence would break if there were shocks to the risk-free rate.

[8]Our results about the optimal portfolio are qualitatively the same if we impose an output cost of defaulting instead of a utility cost.

bonds and the government defaults.

We will use ζ to denote a sunspot variable that will determine the type of equilibrium. If $\zeta = 0$, investors will expect others to continue rolling over the bonds, while if $\zeta = 1$, investors will expect others to stop rolling over the bonds. The probability of $\zeta = 1$ is constant and denoted by λ. As we will see, whether a rollover crisis actually takes place will depend on the portfolio of the government.

Recursive problem. If the government has not defaulted in the past, it chooses whether to repay or default, and its value function is given by

$$V(a, b, \zeta) = \max \left\{ V_R(a, b, \zeta), V_D(a) \right\}, \tag{1}$$

where V_D and V_R represent, respectively, the values of default and repayment. We assume without loss of generality that if the government is indifferent between repaying and defaulting, it repays.

The value of the government under default is given by

$$V_D(a) = \max_{a' \geq 0} \left\{ u(c) - \phi + \beta V_D(a') \right\}, \tag{2}$$

subject to

$$c \leq y + a - \frac{a'}{1 + r}.$$

The value of repayment is given by

$$V_R(a, b, \zeta) = \max_{a' \geq 0, b'} \left\{ u(c) + \beta \mathbb{E} V(a', b', \zeta') \right\}, \tag{3}$$

subject to

$$c \leq y + a - \left(\frac{\delta + r}{1 + r} \right) b - \frac{a'}{1 + r} + q(a', b', s) \left(b' - (1 - \delta)b \right),$$

where we index the price schedule q by the initial state $s = (a, b, \zeta)$ to capture the possibility of multiple equilibria, as will become clear below. The expectation operator $\mathbb{E}$ is taken over the realization of the next period sunspot ζ'.

6

2.2 Markov Equilibrium

Before defining equilibrium, we present the no-arbitrage condition imposed by the presence of risk-neutral international investors. Investors' optimality implies that they equate the return from a risk-free asset to the expected return from government bonds. That is, the bond price must be given by

$$q(a', b', s) = \begin{cases} \frac{1}{1+r}\mathbb{E}\left[(1-d')\left(\frac{\delta+r}{1+r} + (1-\delta)q(a'', b'', s')\right)\right] & \text{if } d(s) = 0, \\ 0 & \text{if } d(s) = 1, \end{cases} \tag{4}$$

where $b'' = \hat{b}(a', b', \zeta')$, $a'' = \hat{a}(a', b', \zeta')$, and $d' = \hat{d}(a', b', \zeta')$ represent the policies the government is expected to follow in the next period. If the government is expected to default at the end of the current period, the bond price is zero. If the government is expected to repay in the current period, the bond price takes into account the coupon payments tomorrow and the secondary market price of the bond (which depends on future probabilities of default).

The Markov equilibrium in this economy is defined as follows:

Definition 1. A Markov equilibrium is defined by a set of policies for the government $\{c(\cdot), d(\cdot), a'(\cdot), b'(\cdot)\}$ and a bond price schedule $q(\cdot)$ such that

  i) the policies solve the government problem given the bond price schedule;

  ii) the price schedule satisfies (4) given the government policies.

2.3 Multiplicity of Equilibria

As in Cole and Kehoe (2000), the government can be subject to rollover crises where the government defaults on its debt because investors stop rolling over the bonds. To determine the states in which the government is vulnerable to a rollover crisis, we need to distinguish between two scenarios, one where investors continue rolling over the bonds and one where they do not.

Consider first a situation in which investors expect others to continue rolling over the

bonds. In this case, the government solves the following problem:

$$V_R^+(a,b) = \max_{a' \geq 0, b'} \left\{ u(c) + \beta \mathbb{E} V(a', b', \zeta') \right\},$$ (5)

subject to

$$c \leq y + a - \left(\frac{\delta + r}{1 + r} \right) b - \frac{a'}{1 + r} + \tilde{q}(a', b') \left(b' - (1 - \delta) b \right),$$

where $\tilde{q}$ denotes the "fundamental" bond price:

$$\tilde{q}(a', b') = \frac{1}{1 + r} \mathbb{E} \left\{ (1 - d') \left[\left(\frac{\delta + r}{1 + r} \right) + (1 - \delta), q(a'', b'', s') \right] \right\}.$$ (6)

Consider now a situation in which the government would like to issue new debt, but investors are unwilling to lend. In this case, the value of repayment for the government is given by

$$V_R^-(a,b) = \max_{a' \geq 0} \left\{ u(c) + \beta \mathbb{E} V(a', (1 - \delta) b, \zeta') \right\},$$ (7)

subject to

$$c \leq y + a - \left(\frac{\delta + r}{1 + r} \right) b - \frac{a'}{1 + r}.$$

An immediate implication is that $V_R^+(a,b) \geq V_R^-(a,b)$. That is, a government that can roll over the debt obtains at least the same value as a government that cannot. When $V_R^-(a,b) < V_D(a) \leq V_R^+(a,b)$, we thus have multiple equilibria. If the government faces a price $q = 0$, then the government defaults. If the government faces the fundamental price $\tilde{q}$, then the government repays.

The safe zone, the crisis zone, and the default zone. Given the value functions (2), (5), and (7), we can split the economy in three different zones depending on the initial portfolio (a, b):

$$\mathbf{S} = \left\{ (a, b) : V_D(a) \leq V_R^-(a, b) \right\},$$

$$\mathbf{D} = \left\{ (a, b) : V_D(a) > V_R^+(a, b) \right\},$$

$$\mathbf{C} = \left\{ (a, b) : V_R^-(a, b) < V_D(a) \leq V_R^+(a, b) \right\}.$$

S is the safe zone: the government is better off repaying, regardless of the sunspot realization. **D** is the default zone: the government is better off defaulting, regardless of the sunspot realization. **C** is the crisis zone: the government finds it optimal to repay if lenders are willing to lend, while it finds it optimal to default if the lenders are not willing to lend.

Default thresholds. Because the value functions of repayment are strictly decreasing in debt and strictly increasing in reserves, and continuous in both, by the intermediate value theorem, we can define thresholds $b^-(a)$ and $b^+(a)$ that determine respectively the boundary between **S** and **C** and the boundary between the **C** and **D**, for every a. That is, $b^-(a)$ is given by

$$V_R^-(a, b^-(a)) = V_D(a), \tag{8}$$

and $b^+(a)$ is given by

$$V_R^+(a, b^+(a)) = V_D(a). \tag{9}$$

Notice that these two thresholds are uniquely determined for any given a, and so we have constructed two functions that map an initial level of reserves to a level of debt that makes the government indifferent between repaying and defaulting depending on whether investors are willing to roll over the debt or not. A central aspect of our analysis will be how these thresholds vary with reserves and what this implies for the optimal portfolio.

2.4 Equilibrium Payoffs

Given the definition of **S**, **D**, and **C**, we can write the payoff for the government as

$$V(a, b, \zeta) = \begin{cases} V_R^+(a, b) & \text{if } (a, b) \in \mathbf{S}, \\ V_R^+(a, b) & \text{if } (a, b) \in \mathbf{C} \ \& \ \zeta \in \{0\}, \\ V_D(a) & \text{if } (a, b) \in \mathbf{C} \ \& \ \zeta \in \{1\}, \\ V_D(a) & \text{if } (a, b) \in \mathbf{D}. \end{cases} \tag{10}$$

When $(a, b) \in \mathbf{C}$, the equilibrium outcome is undetermined and the government's payoff depends on the sunspot. Notice that the equilibrium payoff for the government never takes the value V_R^-, as this is an off-equilibrium value. That is, although V_R^- does not appear explicitly in the equilibrium payoff (10), it is essential to determine which region the government is in.

We now analyze the values in each zone.

Safe zone. When $(a, b) \in \mathbf{S}$, given our assumption that $\beta(1 + r) = 1$, the government finds it optimal to remain in the safe zone and maintain constant consumption equal to its income plus the annuity value of the portfolio. Moreover, *the payoff for the government depends only on the net foreign asset position (NFA), defined as $a - b$, and not on the gross positions.*, We formalize this result in the lemma below.

Lemma 1. *In any Markov equilibrium, consider $(a, b) \in \mathbf{S}$. Then,*

$$V(a, b) = \frac{u(y + (1 - \beta)(a - b))}{1 - \beta} \equiv V^S(a - b), \tag{11}$$

$$c = y + (1 - \beta)(a - b),$$

and (a', b') satisfy $a' - b' = a - b$ and $(a', b') \in \mathbf{S}$.

Crucial for this result is that if the government stays in the safe zone, it can borrow at the risk-free rate. Moreover, notice that if the government were to choose a portfolio in the crisis zone, the bond price would fall, reflecting the positive probability of default, and the government would incur the expected costs of defaulting. Given that $\beta(1 + r) = 1$, remaining in the safe zone with a stationary level of consumption is strictly optimal.

Notice that while the choice for consumption and the NFA are unique in the safe zone, there are generically multiple portfolios that are optimal. That is, as long as the portfolio leaves unchanged the net foreign asset position and the government remains in the safe zone next period, the portfolio is (weakly) optimal. The value in the safe zone, which we denote by V^S, is therefore only a function of the net foreign asset position.

Default zone. Given our assumption that $\beta(1 + r) = 1$, under default, the government keeps the asset position and consumes its income plus the annuity value of the reserves. Thus, the value of default is given by

$$V_D(a) = \frac{u(y + (1 - \beta)a) - \phi}{1 - \beta}. \tag{12}$$

Crisis zone. Recall that the problem the government faces when investors are willing to roll over the bonds is given by (5). Using the results above, we can write the continuation

value for the government as follows:

$$
\mathbb{E}V(a', b', \zeta') = \begin{cases} V^S(a' - b') & \text{if } (a', b') \in \mathbf{S}, \\ (1 - \lambda)V_R^+(a', b') + \lambda V_D(a') & \text{if } (a', b') \in \mathbf{C}, \\ V_D(a') & \text{if } (a', b') \in \mathbf{D}. \end{cases} \tag{13}
$$

That is, if the government chooses a portfolio such in the safe zone in the next period, the continuation value is given by V^S. If the government chooses a portfolio in the crisis zone, then the government will obtain with probability λ the value V_D and with probability $1 - \lambda$, the value V_R^+. Finally, if the government chooses a portfolio in the default zone, its value is given by V_D. Given that $q = 0$ in that case, the latter will never be optimal, as the government can always raise consumption and the continuation value by borrowing a positive amount at a strictly positive price.

From Cole and Kehoe (2000), we know that if the government is in the crisis zone, it has incentives to deleverage until it reaches the safe zone. The idea is that as long as the government remains in the crisis zone, it pays a high interest rate to bondholders. While on expectation the government pays an actuarily fair price, the government is exposed to a costly default. To the extent that attempting to exit the crisis zone requires cutting consumption and that the utility function is strictly concave, exiting immediately may not be optimal. Therefore, depending on the sunspot realization, the government may be able to eventually reach the safe zone or it may default along the way. How fast the government attempts to exit depends on the perceived probability of facing the bad sunspot. In particular, the more likely is it that the government would face a rollover crisis, the higher the speed at which the government will try to exit. In addition, when the debt maturity is longer or when there is high curvature in the utility function, the government will try to exit at a lower speed.

Using that the safe zone is an absorbent state, as stated in Lemma 1, we can iterate on the bond price (6) to arrive at an expression for the bond price schedule faced by the government. In particular, consider a government with a portfolio $(a, b) \in \mathbf{C}$ and a policy function such that it would exit the crisis zone in $T(a, b)$ periods so long as $\{\zeta_t\}_{t=0}^{T-1}$. The bond price is then given by

$$
q(a, b) = \left(\frac{\delta + r}{1 + r}\right) \sum_{t=1}^{T(a,b)} (1 - \lambda)^t (1 - \delta)^{t-1} + \left[\frac{(1 - \lambda)(1 - \delta)}{1 + r}\right]^{T(a,b)}. \tag{14}
$$

The first term captures the expected bond coupon payments investors expect to receive, and the second term reflects the risk-free price of the bond once the government exits.

In the Cole and Kehoe model with one-period debt, consumption is constant over time as the government tries to exit the crisis zone, and once the government reaches the safe zone, consumption increases and then stays constant thereafter. We will show that a similar result applies here with long-term debt and reserves. The difference is that consumption may increase over time in the transition to the safe zone. Intuitively, the presence of long-term debt generates a debt dilution effect by which a strict reduction in consumption when the government is attempting to exit the crisis zone may improve the bond price by reducing the expected exit time, as suggested by (14). Proposition 1 formalizes that the exit toward the safe zone implies a non-decreasing path for consumption.[9]

Proposition 1 (Monotonically increasing consumption). *Consider an initial portfolio* $(a_0, b_0) \in \mathbf{C}$ *such that the government exit time is* T. *Then, if* $\zeta_t = 0$ *for all* $t \leq T - 1$, *we have* $c_{t+1} \geq c_t$ *for all* $t \leq T$.

Proof. In Appendix A.1. $\square$

As we will see in the numerical simulation, consumption will be constant between two periods for which a small change in debt would not affect the exit time while strictly increasing otherwise.

Solving for the boundaries. In general, solving for the two boundaries, $b^-(a)$ and $b^+(a)$, requires solving for a fixed point. Given a value for default, these two boundaries determine the bond price schedule, which in turn determines the value of V_R^+ and thus implies the boundary $b^+(a)$, as defined in (9). For the case of $b^-(a)$, the continuation value after repaying in the event of a run depends in general on $b^+(a)$. However, if a government that faces a run and repays the coupons due ceases to be vulnerable in the future, $b^-(a)$ can be computed directly from the following condition:

$$\max_{a' \geq 0} u\left(y - \left(\frac{\delta + r}{1 + r}\right) b^-(a) + a - \frac{a'}{1 + r}\right) + \beta V^S(a' - (1 - \delta)b^-(a)) = V^D(a), \quad (15)$$

where the continuation value on the left-hand side reflects the assumption that the government is safe going forward. Notice we can guarantee $(0, b^-(a)(1 - \delta)) \in \mathbf{S}$ for δ sufficiently close to one for any a while the portfolio with zero reserves $(0, b^-(0)(1 - \delta)) \in \mathbf{S}$.[10]

---

[9]In a continuous time model with outside option shocks and long-term bonds, Aguiar and Amador (2023) find that consumption is *strictly* increasing over time within the crisis zone. This is because the exit time is continuous in their model and strictly increasing in debt within the crisis zone.

[10]Intuitively, if δ is close to one, there is little debt left to make a government vulnerable after repaying the coupons due. Moreover, to see that $(0, b^-(0)(1 - \delta)) \in \mathbf{S}$, notice that if a government is indifferent between

We have the following results:

Lemma 2. *Consider an initial portfolio (a, b). We then have that*

(i) the solution to the left-hand side problem in (15) *is* $a'(a, b) = \max[0, a - \delta b]$;

(ii) if $a \geq \delta b$ and $(a - \delta b, (1 - \delta)b) \in \mathbf{S}$, then $V_R^-(a, b) = V_R^+(a, b)$.

Proof. In Appendix A.3 □

Result (i) says that when the government is safe tomorrow from a run, it will be at a corner with zero reserves if $a < \delta b$. The intuition is that when the government cannot rollover the bonds and starts with low reserves, it chooses the minimum reserves possible if it anticipates that it will be safe in the future. Notice that the solution to problem (7) is not in general the same as the one for (15). In particular, if the government remains vulnerable tomorrow after repaying during a run, it has more incentives to keep positive reserves.

Result (ii) says that if the government has a high level of reserves, it can achieve the unconstrained level of consumption even when investors refuse to roll over the debt. Notice that the result requires that the government is not vulnerable tomorrow after repaying the coupons due today. That is, it is possible that a government has enough reserves to pay the coupons due, achieve the same level of consumption than when the government can rollover the debt, keep positive reserves and have $V_R^-(a, b) < V_R^+(a, b)$. This is because the fall in reserves may make the government vulnerable in the future and therefore reduce the value from repaying today in the event of a run.

One interesting observation from (ii) is that reserves do not need to be large enough to pay for all coupons in order to be safe from a rollover crisis. That is, $\delta b < \left(\frac{\delta + r}{1 + r}\right) b$ for any $r > 0$. In fact, the government needs just enough reserves to repay the fraction of the debt that would allow the government to keep the NFA constant. The reason is that part of the endowment is used to pay the interest on the debt.[11]

3   Reserve Management under Rollover Crises

In this section turn to the main question we investigate in this paper: *How should the government choose its portfolio to manage the risk of a rollover crisis?*

---

repaying and defaulting with the portfolio $(0, b^-(0))$, it will find it strictly optimal to repay tomorrow given a portfolio with a lower amount of debt (i.e., $(0, (1 - \delta)b^-(0))$).

[11]To see this more clearly, suppose the government starts with the portfolio $(\delta b, b)$ and thus NFA $-(1 - \delta)b$. If the government cannot roll over the debt, $b' = (1 - \delta)b$. By setting $a' = 0$, the government keeps the same NFA and achieves the ideal level of consumption, which in this case is $c = y - (1 - \beta)(1 - \delta)b$.
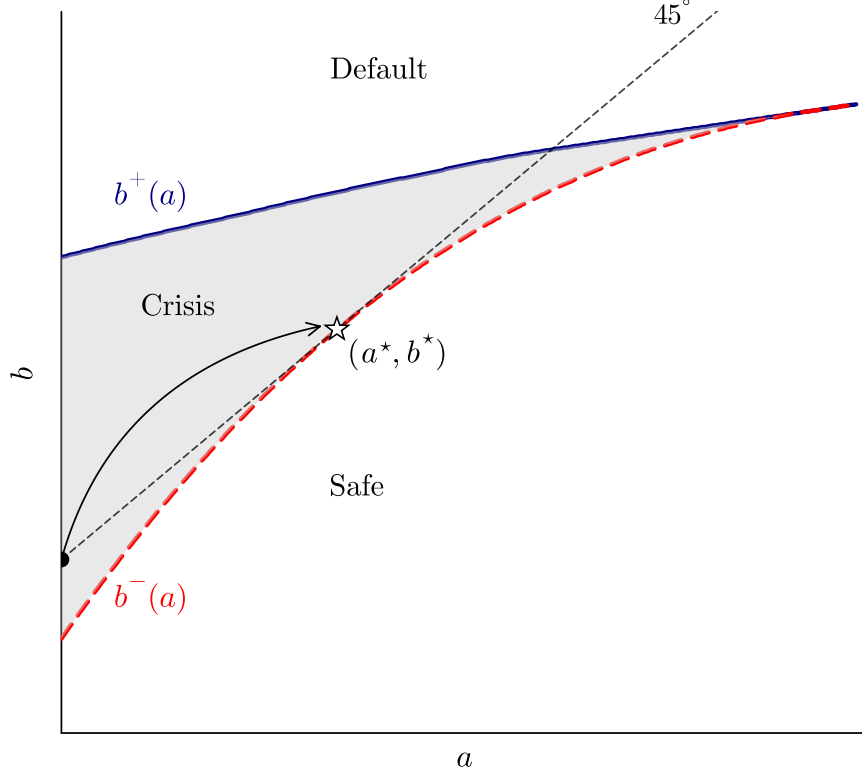
Figure 1: The three zones and the boundaries. The boundaries $b^-(a)$ and $b^+(a)$ are defined in (8) and in (9), respectively.

3.1 The Three Zones

We start the analysis by showing how the initial portfolio determines which zone the economy is in. Figure 1 presents the results.[12] The red-dashed line and the blue-solid line denote respectively $b^-(a)$ and $b^+(a)$. The former constitutes the boundary between the safe zone and the crisis zone. The latter constitutes the boundary between the crisis zone and the default zone. Recall that the government that starts with a portfolio $(a, b^-(a))$ is indifferent between defaulting and repaying when it cannot roll over the bonds, while a government that starts with a portfolio $(a, b^+(a))$ is indifferent between defaulting and repaying when it can roll over the bonds. Thus, we must have $b^+(a) \geq b^-(a)$, as observed in the figure. We can also see that the boundaries intersect for large values of a, a result in line with (ii) of Lemma 2.

Given any pair (a, b) at the $b^-(a)$ boundary, we can see that a lower level of debt implies that the government is in the safe zone, and a higher level of debt implies that the government is in the crisis zone. More interestingly, given any pair (a, b) at the $b^-(a)$ boundary, a higher level of reserves implies that the government is in the safe zone, while a lower level of reserves

---

[12]Throughout, we use calibrated parameter values listed in Table 1, to be described below.

implies the government is in the crisis zone. Likewise, we can see that given any pair (a, b) at the $b^+(a)$ boundary, a higher level of reserves implies that the government is in the crisis zone, while a lower level of reserves implies the government is in the default zone.

The figure also shows that for low a, the $b^-(a)$ boundary is strictly steeper than the $b^+(a)$ boundary. Intuitively, when reserves increase, the value function goes up more when the government cannot roll over the debt, because it is, in effect, borrowing constrained and its marginal utility is higher. Differentiating (8) and (9) with respect to reserves, we obtain

$$\frac{\partial b^-(a)}{\partial a} = \frac{\frac{\partial V_D(a)}{\partial a} - \frac{\partial V_R^-\left(a, b^-(a)\right)}{\partial a}}{\frac{\partial V_R^-\left(a, b^-(a)\right)}{\partial b}}, \qquad \frac{\partial b^+(a)}{\partial a} = \frac{\frac{\partial V_D(a)}{\partial a} - \frac{\partial V_R^+\left(a, b^+(a)\right)}{\partial a}}{\frac{\partial V_R^+\left(a, b^+(a)\right)}{\partial b}}. \tag{16}$$

As we show formally below, both expressions are positive because both the numerator and denominator are negative. For the case of b^-, the result follows because when the government cannot roll over the debt, making the debt payment forces an increase in the marginal utility of consumption. For the case where the government can roll over, the result follows because the government deleverages once in the crisis zone, as we showed in Proposition 1.

We summarize these results in the following proposition.

Proposition 2 (Monotonicity)**.** *In any Markov equilibrium, $b^-(\cdot)$ and $b^+(\cdot)$ are increasing in a, for all a. In addition, $\frac{\partial b^-(a)}{\partial a} \geq \frac{\partial b^+(a)}{\partial a}$ with strict inequality for a, such that $b^-(a) < b^+(a)$.*

Proof. In Appendix A.4 □

Although reserves help expand the safe zone, a government, in principle, could also reduce its vulnerability by reducing debt instead. The key issue is then whether increasing both debt and reserves leads to an increase in $V_R^-(a, b)$ that offsets the increase in $V_D(a)$. In particular, consider a government that is indifferent between repaying under a run and defaulting. That is, consider a portfolio that lies on the red curve in Figure 1. Does an increase in debt and reserves by one unit push the government into the crisis zone or the safe zone? We examine this question next.

3.2   A Joint Increase in Debt and Reserves

Each of the portfolios in the $b^-(a)$ boundary is associated with a different level of net foreign assets. It will be useful to examine the portfolio in the safe zone with the lowest NFA position. We refer to this portfolio as the "lowest-NFA safe portfolio", and we denote it by $(a^\star, b^\star)$.

15

Definition 2. The lowest-NFA safe portfolio $(a^\star, b^\star)$ is the portfolio in the safe zone with the lowest net foreign asset position.

Formally, we have that the lowest-NFA safe portfolio is given by

$$(a^\star, b^\star) = \operatorname*{argmin}_{a,b} \ a - b \tag{17}$$

$$\text{s.t.} \quad (a, b) \in \mathbf{S}.$$

Assuming a strictly interior solution for $a^\star$ and using Lemma 2, we obtain that the lowest-NFA safe portfolio satisfies $\frac{\partial b^-(a^\star)}{\partial a} = 1$. Graphically, this portfolio corresponds to the point highlighted in Figure 1 where the red line is tangent to the 45° line. It is then immediate that if the government starts from a point below this tangent line, it can move to the safe zone by increasing reserves and debt by the same amount (thus keeping the same NFA). The arrow in the figure illustrates how a government at the tangent line with zero initial reserves can jump to the safe zone by choosing $(a^\star, b^\star)$.

As we will see below, the lowest-NFA safe portfolio is a focal point. When a government is deep in the crisis zone (i.e., above the aforementioned tangent line), it needs to increase its net foreign asset position to reach the safe zone. The larger the required increase in the net foreign asset position, the higher the cost of exiting the crisis zone. Choosing the lowest-NFA safe portfolio will minimize the need for deleveraging.

A condition for $a^\star > 0$. The question is then, Under what conditions do we have that $a^\star > 0$? The proposition below provides a sufficient condition guaranteeing that the solution to (17) is strictly interior.

Proposition 3 (Positive reserves)**.** *Suppose that the crisis region boundary at zero reserves, $b^-(0)$, satisfies*

$$\beta(1-\delta)\left[u'\left(y - \left(\frac{\delta+r}{1+r}\right)b^-(0)\right) - u'\left(y - (1-\beta)(1-\delta)b^-(0)\right)\right] > u'(y). \tag{18}$$

Then, the lowest-NFA safe portfolio features an interior level of reserves. That is, $a^\star > 0$.

Proof. In Appendix A.5 □

The argument is that when (18) holds, the slope of $b^-(a)$ is larger than one at zero reserves. That is, starting from a point with zero reserves where the government is indifferent

between repaying and defaulting, an increase in reserves of one unit implies that there exists an increase in debt of more than one unit that keeps the government indifferent.

As condition (18) indicates, a key determinant for the lowest-NFA safe portfolio featuring strictly positive reserves is the curvature of the utility function. Intuitively, when the government cannot roll over the bonds, this implies a reduction in consumption. When the utility function is highly concave, this means that there is a large drop in utility, and thus a large drop in V_R^-. In this situation, a higher level of reserves has a particularly high marginal utility and can offset the fact that higher reserves also increase the value of default.

A necessary element for condition (18) to be satisfied is that debt maturity exceeds one period ($\delta < 1$). It is immediate to see that if $\delta = 1$, the condition cannot be satisfied. The proposition below further establishes that if $\delta = 1$, the lowest NFA-safe portfolio requires $a^\star = 0$. Intuitively, when $\delta = 1$, the value of repayment for the government is independent of the gross positions. On the other hand, because V^D increases with a. It thus follows that a one-unit increase in a and b must lower $V_R^-(a,b) - V_D(a)$. This implies that if a government is indifferent between repaying while facing a run and defaulting, an increase in debt and reserves will always push the economy into the crisis zone. That is, $a^\star = 0$ in the case with one-period debt. We formalize this result in the following Proposition.

Proposition 4 (No reserves with one-period debt). *Suppose that $\delta = 1$. Then, the lowest-NFA safe portfolio features $a^\star = 0$.*

Proof. In Appendix A.6. □

3.3 Exiting the Crisis Zone

We analyze now what the best strategy is for a government that is trying to exit the crisis zone: Should the government reduce its debt or increase reserves? If reserves are optimal, should the government slowly build up its stock of reserves?

Iso-T regions. We first examine how long it takes for the government to exit the crisis zone depending on the initial portfolio. Following the terminology of Aguiar and Amador (2013), we construct "Iso-T regions." As illustrated in Figure 2, the figure describes the initial portfolios (a, b) such that it takes T periods to exit the crisis zone. As we can see, when the gross positions are close to $(a^\star, b^\star)$, the government exists in one period, and as we move up and toward the left (that is, lowering the NFA), it takes more periods to exit.

Under the parameterization considered (which we explain in detail in section 3.4), we can see it takes at most three periods to exit.
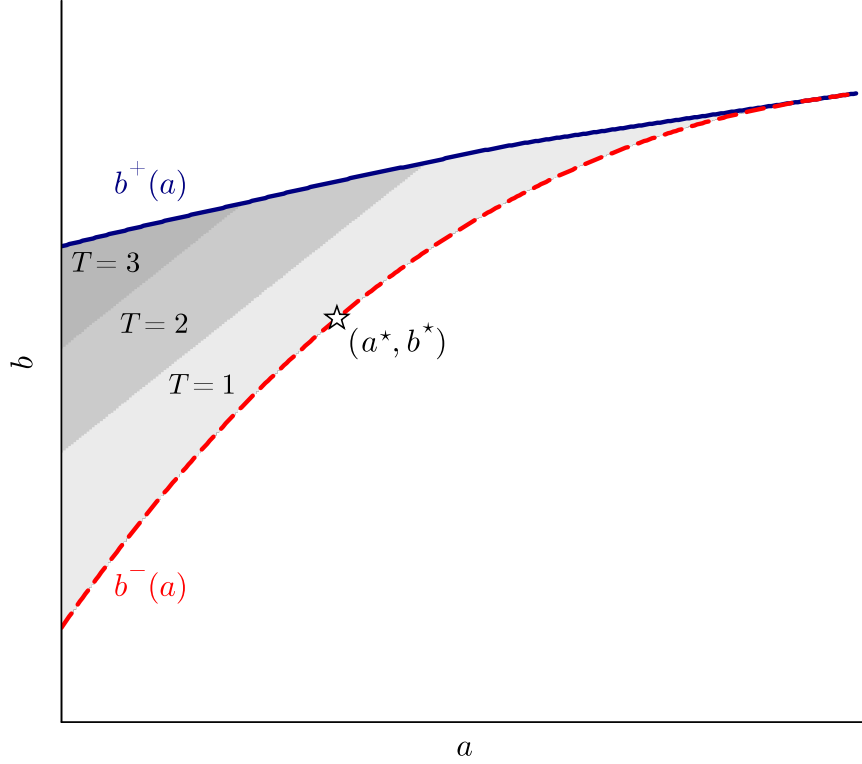
Figure 2: Iso-T Regions. The shaded areas describe how many periods it takes for the government to exit the crisis zone.

Optimal portfolio. The proposition provides a characterization of the optimal portfolio.

Proposition 5 (Optimal portfolio). *Consider an initial* $(a, b) \in \mathbf{C}$. *The optimal portfolio choice* $\{a'(a, b), b'(a, b)\}$ *that solves* (5) *satisfies the following conditions:*

(i) *Suppose that* $a - b < a^\star - b^\star$. *Then, if* $(a'(a, b), b'(a, b)) \in \mathbf{S}$ *we have that* $a'(a, b) = a^\star$ *and* $b'(a, b) = b^\star$.

(ii) *Suppose that* $(a'(a, b), b'(a, b)) \in \mathbf{C}$. *Then, we have that* $a'(a, b) = 0$.

Proof. In Appendix A.7. □

Part (i) considers a government that starts from an NFA lower than the lowest-NFA safe portfolio. The proposition shows that if the government chooses a portfolio that puts it in

the safe zone, then the government chooses the lowest-NFA safe portfolio $(a^\star, b^\star)$. The idea is that choosing $(a^\star, b^\star)$ is what allows the government to exit the crisis zone with the smallest cut in consumption. Portfolios in the safe zone with lower amounts of reserves would imply the government needs to reduce debt by more than the reduction in reserves (compared to the lowest-NFA safe portfolio). Portfolios in the safe zone with higher amounts of reserves would allow the government to borrow more but the increase in borrowing would be smaller than the increase in reserves (again, compared to the lowest-NFA safe portfolio).

Part (ii) considers a situation where the government chooses a portfolio that keeps it in the crisis zone in the next period. The proposition shows that it is not optimal to accumulate reserves in this case. To understand the intuition for this result consider an alternative strategy where the government issues more debt and accumulates reserves while keeping consumption constant. Notice that because the bond price is lower than $\frac{1}{1+r}$, the government's face value of debt increases by more than the reserves accumulated. Assume further that the number of periods it takes to exit does not change, thus implying the same bond price. Following this strategy implies that the government would have more resources available in the event of default but fewer resources available in the event of repayment. Moreover, because the government is deleveraging while it attempts to exit the crisis zone, this implies that the marginal utility is higher under repayment than under default.[13] Therefore, issuing debt to accumulate reserves in the crisis zone is strictly suboptimal as long as the government remains in the crisis zone.

Putting these two parts together establishes that *the optimal exit strategy is to delay the accumulation of reserves until the government is ready to exit the crisis zone.* We summarize this in the following corollary.

Corollary 1 (Optimal portfolio). *Consider a portfolio $(a, b) \in \mathbf{C}$ such that $a - b < a^\star - b^\star$ and the government exits after T periods for $T < \infty$, provided that $\{\zeta_t\}_{t=0}^{T-1} = 0$. Then, we have $a_{t+1} = 0$ for all $t < T$ and $a_{T+1} = a^\star$, $b_{T+1} = b^\star$.*

Proof. In Appendix A.8 □

We turn next to the case when when the government has an initial NFA higher than the lowest-NFA safe portfolio. The proposition below shows that the government exits in one period.

---

[13]A reader may wonder whether this result relies on the assumption that the cost of default is in terms of utility and not resources. We can show, however, that the same result holds under an output cost of defaulting.

Proposition 6 (Immediate exit). *Suppose that condition* (18) *holds. Consider an initial portfolio* $(a, b) \in \mathbf{C}$ *and suppose that* $a - b > a^\star - b^\star$. *Then, the government exists in one period. Moreover, any portfolio* (a', b') *such that* $(a', b') \in \mathbf{S}$ *and* $a - b = a' - b'$ *is optimal.*

Proof. In Appendix A.9 □

The logic for why the government exits in one period is that it is feasible for the government to choose a portfolio that takes it to the safe zone without having to reduce consumption relative to the unconstrained optimal. That is, exiting the crisis zone is not costly in this situation. Notice that in that case, there are a range of portfolios that are optimal. In particular, the portfolio $(a^\star, b + a^\star - a)$, achieves the optimal solution.

Illustration. In Figure 3 we show three simulations, one where the government exits in one period (panel [a]), one where it exists in two periods (panel [b]), and one where it exits in three periods (panel [c]). As is consistent with Proposition 5, we can see that when the government exits in two periods, it chooses zero reserves in the first period, and when the government exits in three periods, it chooses zero reserves in periods 1 and 2.

Panel (d) of Figure 3 shows all the possible portfolios chosen throughout the transition for the entire range of possible initial portfolios in the crisis zone such that $a - b < a^\star - b^\star$.[14] As one can see from the figure, the government never accumulates reserves unless doing so allows the government to exit the crisis zone. It is also interesting to note that there is a "hole" in the sense that there are a range of b' values that are never chosen by the government in equilibrium. This is because for those levels, the government is better off either borrowing slightly less and exiting in fewer periods (thus, obtaining a higher bond price) or borrowing slightly more and exiting in the same number of periods (thus, achieving a higher intertemporal smoothing of consumption).

---

[14]Initial portfolios such that $a - b > a^\star - b^\star$ have a correspondence for the chosen portfolios, as we will see below.

(a) One-Period Exit

(b) Two-Periods Exit

(c) Three-Periods Exit

(d) Chosen Portfolios
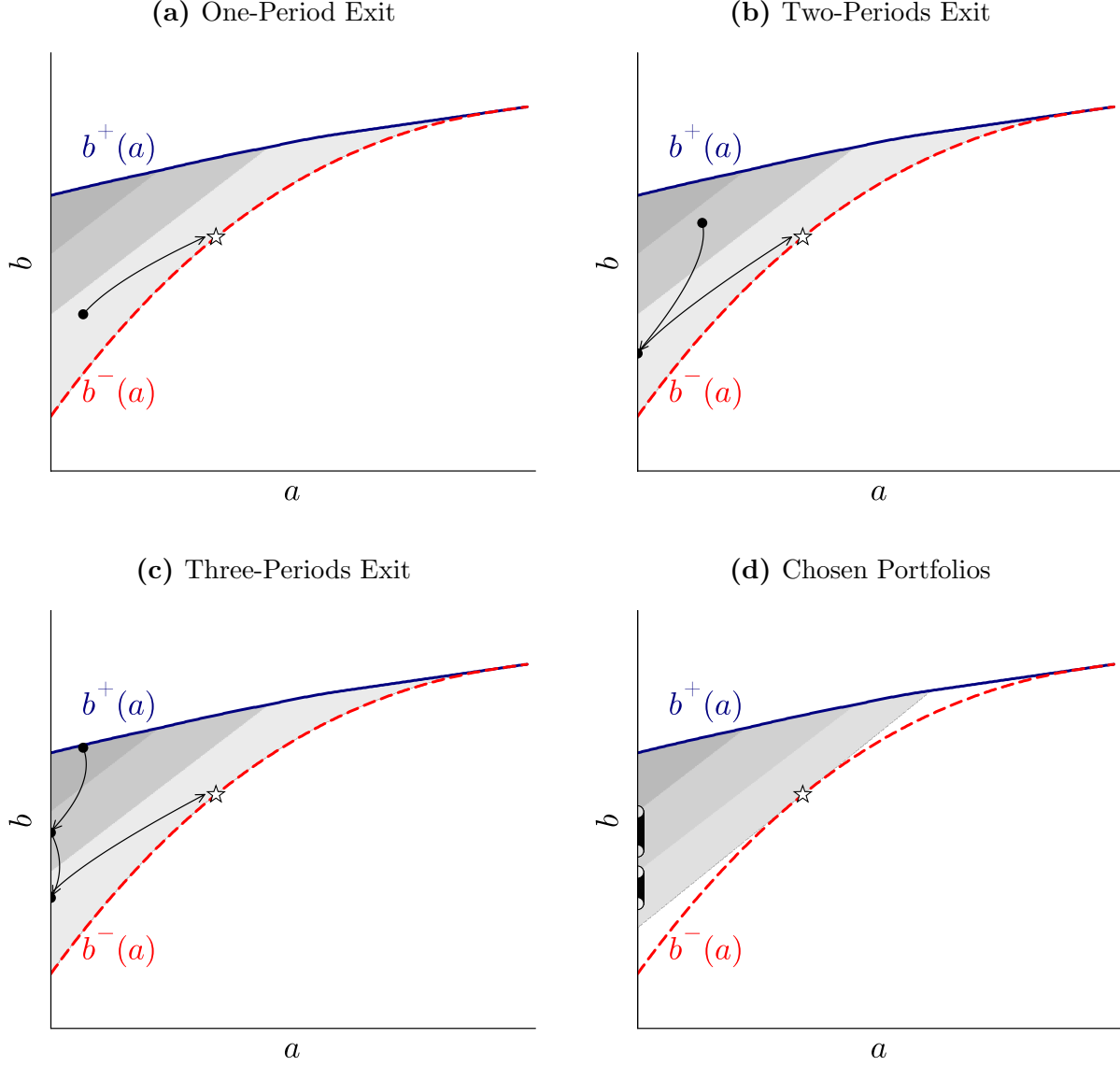
Figure 3: Periods to exit and portfolios. The figure presents examples where the government exits in one period (panel a), two periods (panel b), and three periods (panel c). Panel (d) shows all the possible portfolios the government chooses for any initial portfolio in the crisis zone such that $a - b < a^\star - b^\star$.

Policy correspondences. To further inspect the portfolio of the government, Figure 4 presents the policies for reserves, borrowing, NFA, and consumption in panels (a),(b),(c), and (d), respectively. These policies correspond to the solution of the government problem (5) (i.e., they are the policies conditional on repayment and having access to the bond market). The plots are presented for a range of initial values for b and for $a = 0$. The grey areas denote correspondences.

The figure highlights three vertical lines: the first vertical line denotes $b^-(0)$, the middle vertical line denotes $b^-(a^\star) - a^\star$ and the right vertical line denotes $b^+(0)$. We note that the middle vertical line corresponds to the portfolio with zero reserves that has the same NFA as the lowest NFA safe portfolio.[15]

Let us start from the left vertical line. For debt levels to the left of this line, the government is already in the safe zone and there is a range of portfolios that deliver the optimum solution, as illustrated in the shaded area in panels (a) and (b). In this region, NFA is kept constant (panel[c]) and consumption is given by $c = y - (1 - \beta)b$ (panel [d]). For debt levels between the left and the middle vertical line the government is initially in the crisis zone but jumps in one step to the safe zone keeping a constant NFA, as implied by Proposition 6. As in the previous case, portfolios are correspondences and the NFA is kept constant. This is because the government starts from a net foreign asset position which is higher than the lowest NFA safe portfolio.

For debt levels to the right of the middle vertical line, the government still exits in one period, but it now needs to increase the net foreign asset position to reach the safe zone. One can also see that as debt increases, this generates a sharper drop in consumption (panel [d]). Consistent with Propostion 5, the government finds it strictly optimal to choose $(a^\star, b^\star)$. Notably, the government increases the amount of debt to exit the crisis zone (panel [b]).

As we move further to the right, the government takes more than one period to exit. We can see that now the policy for b' is below the $45°$ line and that the government chooses zero reserves. That is, the government reduces its debt as it seeks to exit the crisis zone and delays the accumulation of reserves. The pattern in panels (c) and (d) for NFA and consumption have some common to Cole and Kehoe that features one-period debt and no reserve accumulation.[16] In particular, the saw-tooth pattern reflects that for certain debt levels, the government is indifferent between choosing a certain level of borrowing and a significantly higher one that delays the exit time by one period. At those points, consumption jumps upward as we increase the debt. Moreover, as debt is increased further, consumption falls continuously as the government smoothes consumption over the deleveraging phase. As we will see in more detail below, however, the possibility of accumulating reserves changes not only the gross positions but also consumption and the NFA. A more subtle difference with Cole and Kehoe (2000) is that the NFA also features flat portions before the discontinuity, which as we will see, it implies that consumption may be strictly increasing over time when the government is in the crisis zone.[17]

---

[15]In terms of Figure 1, this corresponds to the solid dot where the arrow originates.

[16]Figure A1 in the appendix presents the policies for b' and c for an economy without reserves.

[17]This is because of the debt dilution effect from long-term debt. When the government borrows more,
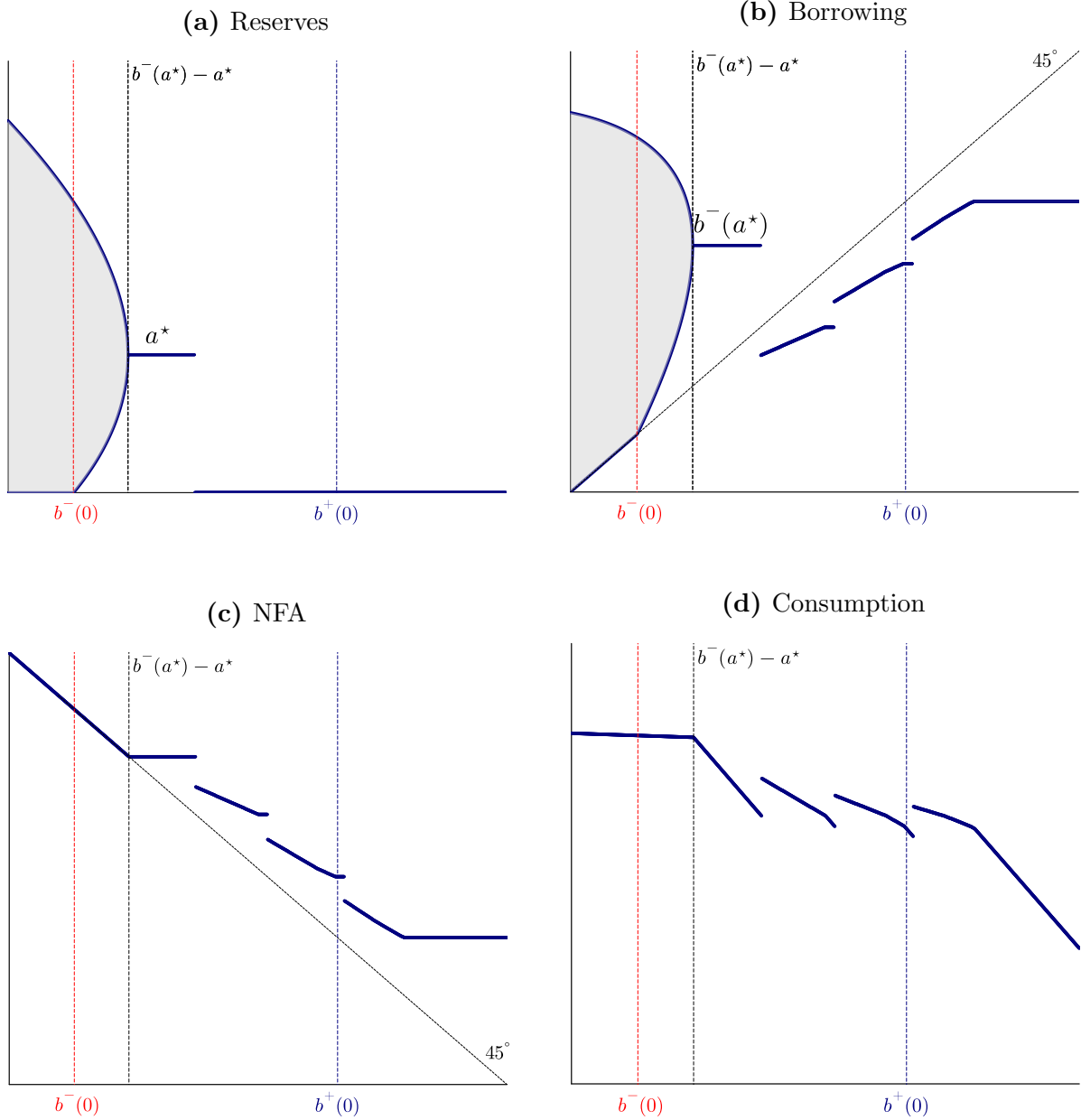
Figure 4: Policy correspondences. The plots are for an initial level of reserves $a = 0$ and a range of values of debt. The top panels denote $a'(0, b)$ and , $b'(0, b)$ and the bottom panels denote $a'(0, b) - b'(0, b)$, and $c(0, b)$. Light grey areas represent the portfolio indeterminacy in that region.

To conclude, we have characterized the optimal strategy for a government that seeks to

---

this may induce investors to anticipate that the government will take longer to exit and reduce the price at which they buy the bonds. To be able to obtain a better bond price, the government may stop itself from borrowing more even as initial debt increases. This gives rise to flat regions for the NFA and sharp drops in consumption.

exit the crisis zone. When the government is deep in the crisis zone, the optimal policy is to reduce the debt and keep zero reserves. As the government approaches the safe zone, it is optimal to increase borrowing and accumulate reserves.

3.4   Quantitative Results

In this section, we present a calibration of the model to assess the quantitative role of reserves. We use data for Italy to calibrate the model. Parameter values are presented in Table 1.[18]

Table 1: Parameter values

| Parameter	Value	Description	Reference
y	1	Endowment	Normalization
σ	2	Risk-aversion	Standard
r	3%	Risk-free rate	Standard
$1/\delta$	7	Maturity of debt	Italian Debt
$\underline{c}$	0.70	Consumption floor	Bocola and Dovis (2019)
β	$(1+r)^{-1}$	Discount factor	$\beta(1+r) = 1$
λ	0.5%	Sunspot probability	Baseline
ϕ	0.34	Default Cost	Debt-to-income =90%

A model period is one year and income is normalized to one. As in Conesa and Kehoe (2017) and Bocola and Dovis (2019), we assume the utility function takes the form

$$u(c) = \frac{(c - \underline{c})^{1-\sigma}}{1 - \sigma},$$ (19)

where $\underline{c}$ stands for the level of consumption that cannot be changed in the short run. Following Bocola and Dovis (2019), we set $\underline{c} = 0.70$, to a measure of non-discretionary spending for the Italian government and set $\bar{c} = 0.70$. We set $\sigma = 2$ and $r = 0.03$, which are common values in the literature.

We set the cost of default ϕ so that the midpoint between $b^-(0)$ and $b^+(0)$ equals 90% of GDP, the level of Italy's debt in the run-up to the sovereign debt crisis in 2012. Finally, we set $\lambda = 0.5\%$ as a baseline value. This value is important for the speed at which the economy exits the crisis zone but does not affect the lowest-NFA safe portfolio $(a^\star, b^\star)$.

[18]The algorithm is described in Appendix B.
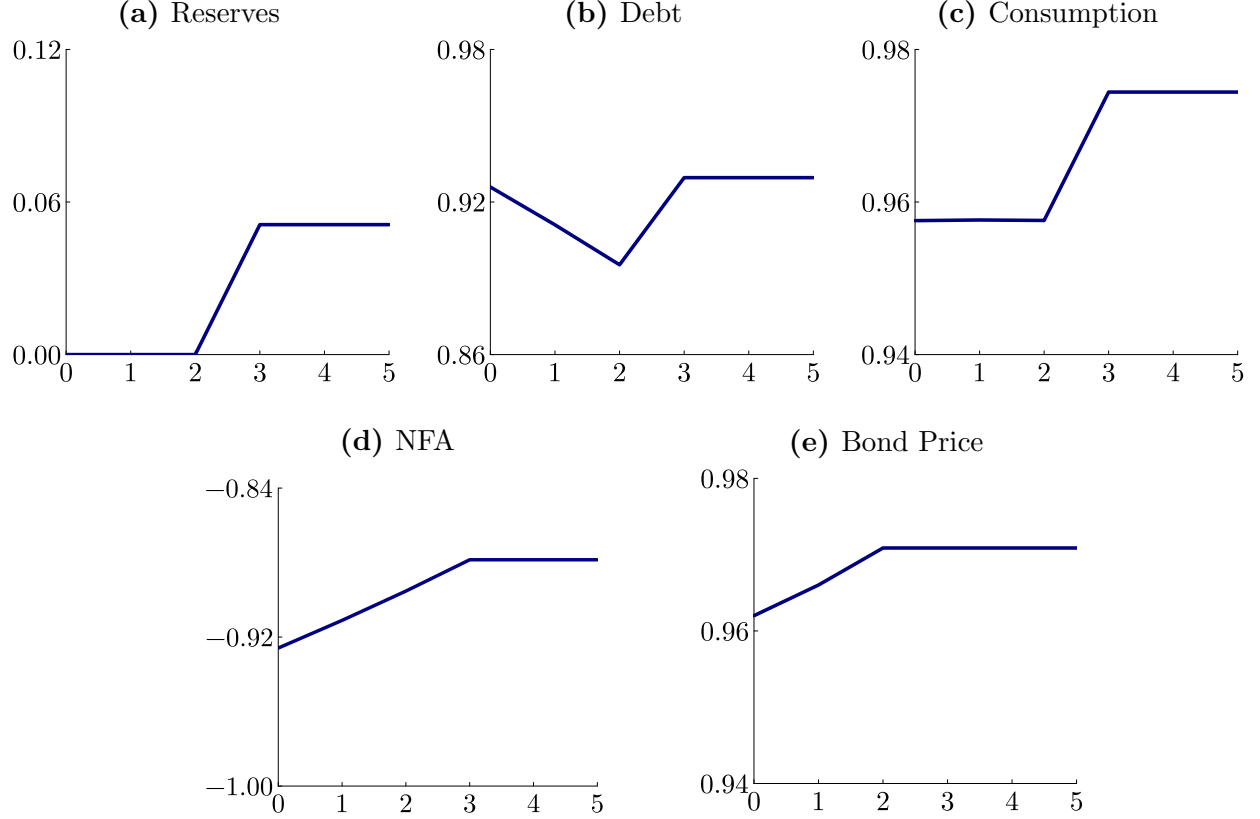
Figure 5: Deleveraging dynamics. The government is assumed to start in the crisis zone with $b = 0.945$ and zero initial reserves.

Simulation results. We obtain that the lowest-NFA safe portfolio is $a^\star = 0.06, b^\star = 0.93$. Figure 5 shows a time series simulation for a government that starts in the crisis zone. Recall that we normalize income to one, so the values can be interpreted as a fraction of GDP. For the initial portfolio considered, the government exits in three periods, as can be observed in the figure. Notice that the net foreign asset position increases monotonically while the debt level falls for two periods and then increases in the exit period. In addition, consumption is also increasing over time. The bond price also increases, because as the government approaches the safe zone, the probability of a future default falls.

Sensitivity. In Figure 6, we examine how changes in the parameters for the debt duration and risk aversion alter the optimal portfolio for the government. Specifically, we vary δ and σ and recalibrate ϕ to match the same level of debt as in the baseline. The figure presents the value of $a^\star$.

Panel (a) shows that when the debt maturity becomes shorter, the government accumulates

a larger amount of reserves. A shorter maturity implies that a larger fraction of the debt becomes due each period.[19] Therefore, the government needs higher amount of reserves to be safe from a rollover crisis. For a maturity of 4 years, the stock of reserves can reach close to 30% of GDP. Panel (b) shows that for higher risk aversion, the government accumulates a larger amount of reserves. A higher risk aversion implies higher curvature in the utility function. Therefore, reserves provide a higher marginal value when the government faces a run.

(a) Maturity   **(b)** Risk aversion
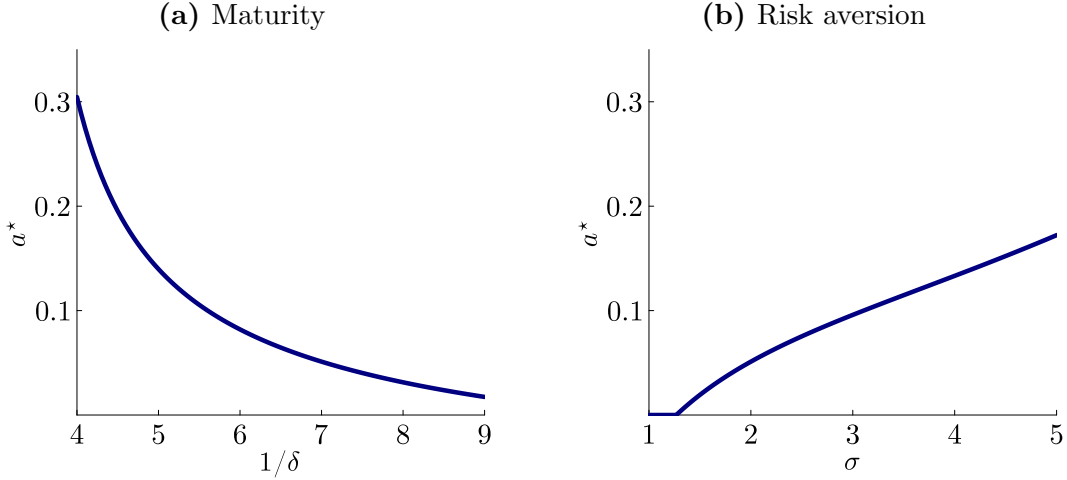
Figure 6: Sensitivity analysis. The figures shows the the level of reserves $a^\star$ for different parameters for δ and σ. In the simulations, the value of ϕ is recalibrated to match the same debt level $b^-(0)$ as in the baseline calibration.

Gains from reserves. We next conduct a counterfactual analysis where we compare our benchmark economy with an economy where the government never accumulates reserves under repayment. Namely, the government starts with portfolio (a, b) at some t, and from $t + 1$ onward, it is forced to set $a' = 0$.[20]

Exiting the crisis zone becomes more costly, as now the government's terminal point is $(0, b^-(0))$ instead of $(a^\star, b^\star)$. Because exiting is more costly, it follows that b^+ is lowered. Intuitively, if it is more costly to deleverage, less debt is sustainable even when the government is able to roll over the bonds. On the other hand, b^- remains the same. Figure 7 compares

---

[19]Recall, however, that in the limit when the maturity is one period there is no scope for reserve accumulation, as established in Proposition 4.

[20]We impose the restriction to accumulate reserves only under repayment, so this means that V^D remains the same as in the baseline.

the bond price schedules with and without reserves. One can see that the bond price with reserves is always above the economy without reserves, strictly so for some ranges of debt.
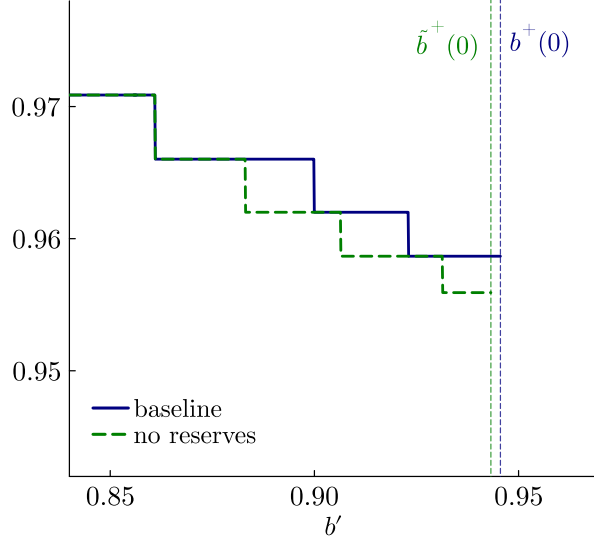
Figure 7: Bond prices schedules. The blue-solid line denotes the bond price schedule in the baseline model at zero reserves, $q(0, b')$. The green-dashed line denotes the bond price schedule in the economy without reserves, $q(b')$. The green vertical line $\tilde{b}^+(0)$ is given by $\tilde{V}_R^+(0, \tilde{b}^+(0)) = V^D(0)$ where $\tilde{V}_R^+$ corresponds to the value function when the government cannot accumulate reserves in the future.

In the absence of reserves, the government will either cut more consumption through the exit process or take longer to exit. In the appendix, we present these two different cases, starting from two different initial portfolios. Figure A2 shows how the government chooses lower consumption through $t = 0, 1, 2$ as it deleverages. Once it exits at $t = 3$, the government ends up with a higher net foreign asset position and thus a higher permanent level of consumption. A government that accumulates reserves is able to achieve superior consumption smoothing. Figure A3 presents the case where the government takes more time to exit. The figure now shows how the government chooses higher consumption through $t = 0, 1, 2$ as it chooses to postpone the terminal period for exiting from the crisis zone. Relative to the government without reserves, the government that accumulates reserves raises consumption at $t = 3$ because it is able to exit faster. As a result, it is able to reduce the expected default costs.

4 Conclusions

This paper studied optimal reserve management under the risk of a rollover crisis. We show that issuing debt to accumulate reserves can help the government repay in the event of a future run by investors, thus making it less vulnerable to rollover crises. We find, however, that when the government starts from a highly indebted position, it should delay the accumulation of reserves until doing so enables it to be safe from a rollover crisis.

Our findings are relevant for policy discussions on the adequate level of international reserves (IMF, 2015). Following a debt crisis, the IMF often prescribes countries to accumulate reserves to improve their liquidity position. Our results show, however, that accumulating reserves is not optimal at the beginning of a deleveraging process. Rather, the government should wait until the debt is sufficiently reduced so that the accumulation of reserves can make the government safe from a rollover crisis.

References

Aguiar, Mark and Gita Gopinath, "Defaultable Debt, Interest Rates and the Current Account," *Journal of International Economics*, 2006, *69* (1), 64–83.

Aguiar, Mark and Manuel Amador, "Take the Short Route: How to Repay and Restructure Sovereign Debt with Multiple Maturities," 2013. NBER Working Paper 19717.

Aguiar, Mark and Manuel Amador, "Sovereign Debt," in Gita Gopinath, Elhanan Helpman, and Kenneth Rogoff, eds., *Handbook of International Economics*, Vol. 4, North Holland, 2014, pp. 647–687.

Aguiar, Mark and Manuel Amador, *The Economics of Sovereign Debt and Default*, Princeton University Press, 2023.

Aguiar, Mark and Manuel Amador, "Sovereign swaps and sovereign default: Fundamental versus confidence risk," 2024. NBER Working Paper 19717.

Aguiar, Mark, Manuel Amador, Emmanuel Farhi, and Gita Gopinath, "Crisis and commitment: Inflation credibility and the vulnerability to sovereign debt crises," Technical Report, National Bureau of Economic Research 2013.

Aguiar, Mark, Manuel Amador, Emmanuel Farhi, and Gita Gopinath, "Coordination and crisis in monetary unions," *The Quarterly Journal of Economics*, 2015, *130* (4), 1727–1779.

Aguiar, Mark, Satyajit Chatterjee, Harold Cole, and Zachary Stangebye, "Quantitative Models of Sovereign Debt Crises," in John B. Taylor and Harald Uhlig, eds., *Handbook of Macroeconomics*, Vol. 2, Elsevier, 2016, pp. 1697–1755.

Aguiar, Mark, Satyajit Chatterjee, Harold L Cole, and Zachary Stangebye, "Self-Fulfilling Debt Crises, Revisited: The Art of the Desperate Deal," *Journal of Political Economy*, 2022, *130* (5), 1147–1183.

Alfaro, Laura and Fabio Kanczuk, "Optimal Reserve Management and Sovereign Debt," *Journal of International Economics*, 2009, *77* (1), 23–36.

Araujo, Aloisio, Marcia Leon, and Rafael Santos, "Welfare analysis of currency regimes with defaultable debts," *Journal of International Economics*, 2013, *89* (1), 143–153.

Arellano, Cristina, "Default Risk and Income Fluctuations in Emerging Economies," *American Economic Review*, 2008, *98* (3), 690–712.

Arellano, Cristina and Ananth Ramanarayanan, "Default and the Maturity Structure in Sovereign Bonds," *Journal of Political Economy*, 2012, *120* (2), 187–232.

Ayres, Joao, Gaston Navarro, Juan Pablo Nicolini, and Pedro Teles, "Sovereign default: The role of expectations," *Journal of Economic Theory*, 2018, *175*, 803–812.

Ayres, Joao, Gaston Navarro, Juan Pablo Nicolini, and Pedro Teles, "Self-fulfilling debt crises with long stagnations," 2023.

Bacchetta, Philippe, Elena Perazzi, and Eric Van Wincoop, "Self-fulfilling debt crises: What can monetary policy do?," *Journal of International Economics*, 2018, *110*, 119–134.

Bassetto, Marco and Carlo Galli, "Is inflation default? The role of information in debt crises," *American Economic Review*, 2019, *109* (10), 3556–3584.

Bianchi, Javier and César Sosa-Padilla, "Reserve Accumulation, Macroeconomic Stabilization, and Sovereign Risk," *The Review of Economic Studies*, 07 2023.

Bianchi, Javier and Jorge Mondragon, "Monetary independence and rollover crises," *The Quarterly Journal of Economics*, 2022, *137* (1), 435–491.

Bianchi, Javier, Juan Carlos Hatchondo, and Leonardo Martinez, "International reserves and rollover risk," *American Economic Review*, 2018, *108* (9), 2629–2670.

Bianchi, Javier, Pablo Ottonello, and Ignacio Presno, "Fiscal stimulus under sovereign risk," *Journal of Political Economy*, 2023, *131* (9), 2328–2369.

Bocola, Luigi and Alessandro Dovis, "Self-Fulfilling Debt Crises: A Quantitative Analysis," *American Economic Review*, 2019, *109* (12), 4343–4377.

Camous, Antoine and Russell Cooper, ""Whatever it takes" is all you need: Monetary policy and debt fragility," *American Economic Journal: Macroeconomics*, 2019, *11* (4), 38–81.

Chatterjee, Satyajit and Burcu Eyigungor, "Maturity, indebtedness, and default risk," *American Economic Review*, 2012, *102* (6), 2674–2699.

Cole, Harold L. and Timothy J. Kehoe, "Self-Fulfilling Debt Crises," *Review of Economic Studies*, 2000, *67* (1), 91–116.

Cole, Harold L, Daniel Neuhann, and Guillermo Ordonez, "Information Spillovers and Sovereign Debt: Theory Meets the Eurozone Crisis," 2022. Forthcoming, Review of Economic Studies.

Conesa, Juan Carlos and Timothy J Kehoe, "Gambling for Redemption and Self-Fulfilling Debt Crises," *Economic Theory*, 2017, *64* (4), 707–740.

Conesa, Juan Carlos and Timothy J Kehoe, "Preemptive Austerity with Rollover Risk," *Forthcoming, Journal of International Economcis*, 2024.

Corsetti, Giancarlo and Fred Seunghyun Maeng, "Debt crises, fast and slow," *Journal of the European Economic Association*, 2023.

Corsetti, Giancarlo and Luca Dedola, "The mystery of the printing press: Monetary policy and self-fulfilling debt crises," *Journal of the European Economic Association*, 2016, *14* (6), 1329–1371.

Corsetti, Giancarlo and Seung Hyun Maeng, "The Theory of Reserve Accumulation, Revisited," *Robert Schuman Centre for Advanced Studies Working Paper 2023/23*, 2023.

Eaton, Jonathan and Mark Gersovitz, "Debt with Potential Repudiation: Theoretical and Empirical Analysis," *Review of Economic Studies*, 1981, *48 (2)*, 289–309.

Hatchondo, Juan Carlos and Leonardo Martinez, "Long-Duration Bonds and Sovereign Defaults," *Journal of International Economics*, 2009, *79 (1)*, 117–125.

Hatchondo, Juan Carlos, Leonardo Martinez, and César Sosa-Padilla, "Debt dilution and sovereign default risk," *Journal of Political Economy*, 2016, *124 (5)*, 1383–1422.

Hernandez, Juan, "How International Reserves Reduce The Probability of Debt Crises," 2018. Inter-American Development Bank Discussion Paper IDB-DP-579.

IMF, "Assessing reserve adequacy," 2015. IMF Note.

Lorenzoni, Guido and Ivan Werning, "Slow moving debt crises," *American Economic Review*, 2019, *109* (9), 3229–3263.

A   Proofs

A.1   Proof of Lemma 1

Proof. Notice first that if $(a, b) \in \mathbf{S}$, we know that the portfolio $a' = a, b = b'$ and $c = y + (1 - \beta)(a - b)$ is feasible and delivers a value $\frac{1}{1-\beta} u(y + (1 - \beta)(a - b))$. We will argue that in any Markov equilibrium, this value puts an upper bound to the value of repayment. That is, $V^+(a, b) \leq \frac{1}{1-\beta} u(y + (1 - \beta)(a - b))$.

To see this, consider alternative candidate portfolios.

Suppose the government chooses another portfolio $(a', b') \in \mathbf{S}$, with $c \neq y + (1 - \beta)(a - b)$. By strict concavity of the government problem, we know this portfolio delivers strictly lower utility.

Suppose the government chooses $(a', b') \in \mathbf{C}$. Then,

$$
\max_{a',b'} u(y - \left(\frac{\delta + r}{1 + r}\right) b + a - \frac{a'}{1 + r} + q(a', b')(b' - b(1 - \delta))) + \beta[\lambda V^D(a') + (1 - \lambda)V_+^R(a', b')]
$$

$$
\leq \max_{a',b'} u(y - \left(\frac{\delta + r}{1 + r}\right) b + a - \frac{a'}{1 + r} + q(a', b')(b' - b(1 - \delta))) + \beta V_+^R(a', b')
$$

$$
< \max_{a',b'} u(y - b + a - \frac{a'}{1 + r} + \frac{1}{1 + r} b') + \beta V_+^R(a', b')
$$

$$
\leq \frac{1}{1 - \beta} u(y + (1 - \beta)(a - b)).
$$

The first inequality emerges because $V_+^R(a', b') \geq V_D$ in the crisis zone. The second inequality emerges because $q(a', b') < \frac{1}{1+r}$ for any $(a', b') \in \mathbf{C}$, given that $\lambda > 0$. The third inequality extends the argument forward.

Finally, if the government choosses $(a', b') \in \mathbf{D}$, the value is given by

$$
\max_{a',b'} u(y - \left(\frac{\delta + r}{1 + r}\right) b + a - \frac{a'}{1 + r}) + V^D(a') \leq \max_{a'} u\left(y - a - \frac{a'}{1 + r}\right) + V^D(a'). \quad \text{(A.1)}
$$

$$
= V^D(a)
$$

Given that $(a, b) \in \mathbf{S}$, it follows that $V_R^+(a, b) > V^D(a)$ and so the candidate portfolio achieves lower value.

Thus, it is not possible to find a portfolio for the government that gives strictly higher utility than the value from remaining safe and consuming $c = y + (1 - \beta)(a - b)$.

□

A.2 Proof of Proposition 1

Proof. Toward a contradiction, suppose that $c_t > c_{t+1}$ in the crisis zone is optimal. Consider a policy at t such that consumption $\tilde{c}_t = c_t - \Delta$ with $\Delta > 0$ and a reserve policy $\tilde{a} = a_{t+1}$. In addition, assume that the government keeps the same (a_{t+2}, b_{t+2}) as the original allocation in the event that $\zeta_{t+1} = 0$. Notice that this policy is feasible and implies that continuation values at $t+2$ remain the same as in the original allocation. Without loss of generality, suppose the alternative portfolio keeps the government in the crisis zone (i.e., $(\tilde{a}, \tilde{b}) \in \mathbf{C}$). We then have that $q(a_{t+1}, b_{t+1}) = q(\tilde{a}, \tilde{b})$, and from the budget constraint,

$$\tilde{b} = b_{t+1} - \frac{\Delta}{q(\tilde{a}, \tilde{b})}. \tag{A.2}$$

We will argue that the welfare goes up with the deviation. That is, given the same continuation value $t+2$, this means that the sum of utility from period t and expected utility from $t+1$ consumption is higher. That is, we want to show that

$$u(c_t - \Delta) + \beta(1-\lambda)u\left(y + a_t - \frac{a_{t+1}}{1+r} - \kappa\tilde{b} + q(a_{t+2}, b_{t+1})(b_{t+2} - (1-\delta)\left(b_{t+1} - \frac{\Delta}{q(\tilde{a}, \tilde{b})}\right))\right) >$$
$$u(c_t) + \beta(1-\lambda)u\left(y + a_t - \frac{a_{t+1}}{1+r} - \kappa b_{t+1} + q(a_{t+2}, b_{t+1})(b_{t+2} - (1-\delta)b_{t+1}))\right), \tag{A.3}$$

where $\kappa \equiv \frac{\delta+r}{1+r}$.

Using the equation for the bond price (6), we have that

$$u(c_t) + \beta(1-\lambda)u(y + a_t - \frac{a_{t+1}}{1+r} - \kappa b_{t+1} + q(a_{t+2}, b_{t+1})(b_{t+2} - (1-\delta)b_{t+1})) =$$
$$u(c_t) + \beta(1-\lambda)u(y + a_t - \frac{a_{t+1}}{1+r} - (\frac{1+r}{1-\lambda})b_{t+1} + q(a_{t+2}, b_{t+1})b_{t+2})$$
$$= u(c_t) + \beta(1-\lambda)u(c_{t+1}). \tag{A.4}$$

Similarly,

$$u(c_t - \Delta) + \beta(1-\lambda)u\left(y - \kappa\tilde{b} + q(a_{t+2}, b_{t+1})(b_{t+2} - (1-\delta)(b_{t+1} - \frac{\Delta}{q(a_{t+1}, \tilde{b})}))\right) =$$
$$u(c_t - \Delta) + \beta(1-\lambda)u\left(y - b_{t+1}\frac{1+r}{1-\lambda} + q(a_{t+2}, b_{t+1})b_{t+2} + \frac{\Delta}{(1-\lambda)R}\right). \tag{A.5}$$

33

Substituting (A.4)-(A.4) in (A.3), we arrive at

$$u(c_t - \Delta) + \beta(1 - \lambda)u\left(c_{t+1} + \frac{\Delta(1 + r)}{1 - \lambda}\right) > u(c_t) + \beta(1 - \lambda)u(c_{t+1}),$$

which holds if and only if

$$u(c_t - \Delta) - u(c_t) + \beta(1 - \lambda)\left[u\left(c_{t+1} + \frac{\Delta(1 + r)}{1 - \lambda}\right) - u(c_{t+1})\right] > 0$$

. For small Δ, it follows that we can improve utility, reaching a contradiction:

$$\lim_{\Delta \to 0} u(c_t - \Delta) - u(c_t) + \beta(1 - \lambda)\left[u\left(c_{t+1} + \Delta\frac{1 + r}{1 - \lambda}\right) - u(c_{t+1})\right] = -(u'(c_t) - u'(c_{t+1})) > 0.$$

$\square$

A.3 Proof of Lemma 2

Proof. Part (i):

Consider problem (15). A necessary and sufficient condition for optimality implies

$$u'\left(y + a - \frac{a'}{1 + r} - \left(\frac{\delta + r}{1 + r}\right)b\right) \geq \beta(1 + r)u'\left(y + (1 - \beta)(a' - (1 - \delta)b)\right), \qquad (A.6)$$

with equality if $a' > 0$. In the case of equality, we have

$$a - \frac{a'}{1 + r} - \left(\frac{\delta + r}{1 + r}\right)b = y + (1 - \beta)(a' - (1 - \delta)b),$$

which implies

$$a' = a - \delta b.$$

The solution is therefore

$$a' = \max[0, a - \delta b]. \qquad (A.7)$$

That is, if $a > \delta b$, then the unconstrained solution is feasible. Otherwise, $a' = 0$.

$\square$

Proof. Part (ii): Consider the problem when there is no run. In that case, $b' = (1 - \delta)b$, and $a' = a - \delta b$, and $c = y + (1 - \beta)(a - b)$ is feasible because $a \geq \delta b$. Moreover, the continuation

34

value is safe by the assumption that $(a - \delta b, (1 - \delta)b) \in \mathbf{S}$. Thus, the government can achieve the ideal stationary outcome. Consider now the problem under a run (7). Given that $a' = 0$ and $b' = b(1 - \delta)$ remains feasible, the government under a run can achieve the same value as the government that does not face a run. We thus obtain the result that $V_R^+ = V_R^-$, as we wanted to show. □

A.4   Proof of Proposition 2

We first prove the following lemma:

Lemma A.1. *For any* $(a, b) \in \mathbf{C}$, *we have that* $c_R^-(a, b) < c_R^+(a, b) < c_D(a)$.

Proof. By Proposition 1, consumption is increasing over time in the crisis zone. Moreover, $c_R^+(a, b^+(a)) > c_D$ for the government to be indifferent between repaying and defaulting at the portfolio $(a, b^+(a))$. Therefore it must be that $c_R^+(a, b) < c_D(a)$ for $(a, b) \in \mathbf{C}$.

The result that $c_R^-(a, b) < c_D(a)$ is as follows. Let $\bar{a}(a, b)$ be the solution of (15). If $(\bar{a}(a, b), (1 - \delta)b) \in \mathbf{S}$, the result that $c_R^-(a, b) < c_D(a)$ is immediate from replacing (A.7) in the government budget constraint.

If $(\bar{a}(a, b), (1 - \delta)b) \notin \mathbf{S}$, the government has a higher incentive to save in reserves and thus consumption is lower. To see this, first note that if $(a', b') \in \mathbf{C}$, we have

$$u' \left(y + a - \frac{a'}{1+r} - \left(\frac{\delta + r}{1+r} \right) b \right) \geq \beta(1+r)[\lambda u'(c_D(a')) + (1 - \lambda)u' \left(c^+(a', b(1 - \delta)) \right). \quad \text{(A.8)}$$

If $c_D(a) < c_R^-(a, b)$, we have $a' < a - (\delta + r)b$, and thus the end of period net foreign asset position is $a' - b' = a' - b(1 - \delta) < a - (\delta + r)b - b(1 - \delta) = a - (1 + r)b < a - b$, which in turn implies $a' < a$ and $c_D(a) > c_D(a')$.

Toward a contradiction, suppose that $c_D(a) < c_R^-(a, b)$. Using (A.8) and the fact that $c_R^+(a, b) < c_D(a)$, we arrive at

$$u' \left(c_D(a) \right) > u' \left(c_D(a') \right),$$

which implies $c_D(a) < c_D(a')$ and thus a contradiction.

□

We now prove Proposition 2.

Proof. We have that $V_D(a) = V_R^-(a, b^-(a))$. Differentiating with respect to a, we obtain

$$\frac{\partial V_D(a)}{\partial a} = \frac{\partial V_R^-(a, b^-(a))}{\partial a} + \frac{\partial V_R^-(a, b^-(a))}{\partial b} \frac{\partial b^-}{\partial a}.$$

Rearranging, we get

$$\frac{\partial b^-}{\partial a} = \frac{\frac{\partial V_D(a)}{\partial a} - \frac{\partial V_R^-(a, b^-(a))}{\partial a}}{\frac{\partial V_R^-(a, b^-(a))}{\partial b}}. \tag{A.9}$$

Proceeding analogously for $b^+(a)$, we have that

$$\frac{\partial b^+}{\partial a} = \frac{\frac{\partial V_D(a)}{\partial a} - \frac{\partial V_R^+(a, b^+(a))}{\partial a}}{\frac{\partial V_R^+(a, b^+(a))}{\partial b}}. \tag{A.10}$$

Applying the envelope condition in (5), (7), and (12), we obtain

$$\frac{\partial V_R^-(a, b)}{\partial a} = u'\left(c_R^-(a, b)\right), \tag{A.11}$$

$$\frac{\partial V_D(a)}{\partial a} = u'\left(c_D(a)\right), \tag{A.12}$$

$$\frac{\partial V_R^+(a, b)}{\partial a} = u'\left(c_R^+(a, b)\right). \tag{A.13}$$

From Lemma A.1, it then follows that $\frac{\partial V_R^-(a,b)}{\partial a} > \frac{\partial V_R^+(a,b)}{\partial a} > \frac{\partial V_D(a)}{\partial a}$ for any $(a, b) \in \mathbf{C}$. Using these inequalities in (A.9) and (A.10), we therefore have $\frac{\partial b^-}{\partial a} > 0$ and $\frac{\partial b^+}{\partial a} > 0$, implying that $b^-(a)$ and $b^+(a)$ are increasing in a. Moreover, using $\frac{\partial V_R^-(a,b)}{\partial a} > \frac{\partial V_R^+(a,b)}{\partial a}$ for $(a, b) \in \mathbf{C}$, the result that $\frac{\partial b^-}{\partial a} > \frac{\partial b^+}{\partial a} > 0$ for any a such that $b^-(a) < b^+(a)$ follows.

$\square$

A.5 Proof of Proposition 3

Proof. Toward a contradiction, suppose that the lowest-NFA safe portfolio $(a^\star, b^\star)$ is such that $a^\star = 0$. Consider an alternative portfolio $(\tilde{a}, b^-(\tilde{a}))$, with $\tilde{a} > 0$. By continuity of $\frac{\partial b^-(a)}{\partial a}$, we have

$$b^-(\tilde{a}) - b^-(0) = \int_0^{\tilde{a}} \frac{\partial b^-(a)}{\partial a} da > \int_0^{\tilde{a}} da = \tilde{a}. \tag{A.14}$$

The argument for the strict inequality comes from showing that $\frac{\partial b^-(a)}{a} > 1$ in the interval $[0, \tilde{a}]$. To prove this, we first note that the envelope condition in (7) is

$$\frac{\partial V_R^-(a,b)}{\partial b} = -u'\left(y + a - \left(\frac{\delta+r}{1+r}\right)b - \frac{a'}{1+r}\right)\left(\frac{\delta+r}{1+r}\right) - \beta(1-\delta)\mathbb{E}\frac{\partial V\left(a', b(1-\delta), \zeta'\right)}{\partial b}.$$

(A.15)

If we evaluate the above expression at $(0, b^-(0))$, then the government is repaying today, and it is straightforward to see that it will be safe tomorrow since it will have less debt and cannot have fewer reserves: $b' = (1-\delta)b$ and $a' \geq 0$. Moreover, we can show that a government that starts the period in the safe region will always choose to stay in the safe region. Therefore, the envelope condition above, evaluated at $(0, b^-(0))$ can be rewritten as:

$$\frac{\partial V_R^-(0, b^-(0))}{\partial b} = -u'\left(y - \left(\frac{\delta+r}{1+r}\right)b^-(0)\right)\left(\frac{\delta+r}{1+r}\right) - \beta(1-\delta)u\left(y - (1-\beta)(1-\delta)b^-(0)\right)$$

(A.16)

where we have used Lemma 2 to have $a' = 0$. We can then replace (A.16), together with (A.11), and (A.12) into (A.9) to obtain

$$\left.\frac{\partial b^-(a)}{\partial a}\right|_{a=0} = \frac{u'\left(y - \left(\frac{\delta+r}{1+r}\right)b^-(0)\right) - u'(y)}{u'\left(y - \left(\frac{\delta+r}{1+r}\right)b^-(0)\right)\left(\frac{\delta+r}{1+r}\right) + \beta(1-\delta)u\left(y - (1-\beta)(1-\delta)b^-(0)\right)}.$$

(A.17)

Condition (18) guarantees that $\left.\frac{\partial b^-(a)}{\partial a}\right|_{a=0} > 1$. By continuity, we then have that for a small $\tilde{a}$, it must be that $\frac{\partial b^-(\tilde{a})}{\partial a} > 1$.

Rearranging (A.14), we arrive at

$$\tilde{a} - b^-(\tilde{a}) < 0 - b^-(0),$$

which implies that we were able to find a portfolio $(\tilde{a}, b^-(\tilde{a})) \in \mathbf{S}$ with a lower net foreign asset position. We thus find a contradiction that the original portfolio with $a^\star = 0$ was part of a lowest-NFA safe portfolio. $\square$

A.6 Proof of Proposition 4

Proof. With $\delta = 1$, the value of repayment in a run becomes

$$V_R^-(a,b) = \max_{a'} u\left(y + a - b - \frac{a'}{1+r}\right) + \beta V^S(a'),$$

(A.18)

where we have used that the government is safe tomorrow after repaying all the debt. The solution for a' is given by (A.7). Under the assumption that $a < b$—and thus a negative net foreign asset position which we will verify below—we have $a' = 0$, and replacing (11) and (12) in (A.18), we arrive at

$$u(y + a - b) = \frac{u(y + a(1 - \beta)) - \phi}{1 - \beta} - \frac{\beta}{1 - \beta}u(y). \tag{A.19}$$

The lowest-NFA safe portfolio solves

$$\min_{a,b} \ a - b,$$

$$\text{subject to}$$

$$u(y + a - b) + \frac{\beta}{1 - \beta}u(y) \geq \frac{u(y + a(1 - \beta)) - \phi}{1 - \beta}.$$

Optimality implies that

$$1 \geq \lambda[u'(y + a - b) - u'(y + a(1 - \beta)) \quad \text{with equality if } a > 0,$$
$$1 = \lambda u'(y + a - b).$$

Combining these two conditions, we get

$$u'(y + a - b) \geq u'(y + a - b) - u'(y + a(1 - \beta)).$$

This condition holds with strict inequality, thus implying that $a^\star = 0$ and $b^\star$ is given by

$$u(y - b^\star) + \frac{\beta}{1 - \beta}u(y) \geq \frac{u(y) - \phi}{1 - \beta}.$$

Notice that $b^* > 0$; thus, the net foreign asset position is negative.

$\square$

A.7 Proof of Proposition 5

The proof has two parts:

Proof. **Part (i).**

Suppose the government chooses $(\tilde{a}, \tilde{b}) \in \mathbf{S}$ such that $(\tilde{a}, \tilde{b}) \neq (a^\star, b^\star)$. We can show that

the utility is then lower. To see this, we need to show that

$$u\left(y + a - b + \left(\frac{\tilde{b} - \tilde{a}}{1 + r}\right)\right) + \frac{\beta}{1 - \beta} u\left(y + (1 - \beta)(\tilde{a} - \tilde{b})\right)$$

$$< u\left(y + a - b + \left(\frac{b^\star - a^\star}{1 + r}\right)\right) + \frac{\beta}{1 - \beta} u\left(y + (1 - \beta)(a^\star - b^\star)\right). \quad (A.20)$$

Denote $\Delta \equiv \frac{b^\star - a^\star}{1 + r} - \frac{\tilde{b} - \tilde{a}}{1 + r}$, $c^\star \equiv y + a - b + \left(\frac{b^\star - a^\star}{1 + r}\right)$, and $\hat{c} \equiv y + (1 - \beta)(a^\star - b^\star)$. We know that by definition of $(a^\star, b^\star)$, $\Delta > 0$. Using these expressions, we can rewrite (A.20) as

$$u(c^\star - \Delta) + \frac{\beta}{1 - \beta} u\left(\hat{c} + \left(\frac{1 - \beta}{\beta}\right)\Delta\right) < u(c^\star) + \frac{\beta}{1 - \beta} u(\hat{c}).$$

Rearranging, we need to show that the following holds:

$$u(c^\star) - u(c^\star - \Delta) > \frac{\beta}{1 - \beta}\left[u\left(\hat{c} + \left(\frac{1 - \beta}{\beta}\right)\Delta\right) - u(\hat{c})\right]. \quad (A.21)$$

The result follows from an application of the mean-value theorem, the strict concavity of $u(\cdot)$, and the fact that $c^\star < \hat{c}$. Namely, there exists $x \in (c^\star - \Delta, c^\star)$, such that

$$u'(x)\Delta = u(c^\star) - u(c^\star - \Delta). \quad (A.22)$$

Similarly, there exists $z \in \left(\hat{c}, \hat{c} + \left(\frac{1 - \beta}{\beta}\right)\Delta\right)$, such that

$$u'(z)\Delta = \frac{u\left(\hat{c} + \left(\frac{1 - \beta}{\beta}\right)\Delta\right) - u(\hat{c})}{(1 - \beta)/\beta}. \quad (A.23)$$

Since $z > x$, $\Delta > 0$ and $u(\cdot)$ is strictly concave, we have that $u'(x) > u'(z)$. Using this strict inequality and rearranging (A.22) and (A.23), we obtain (A.21), as we wanted to show.

$\square$

Proof. **Part (ii).** Suppose, toward a contradiction, that $a' > 0$. Consider a portfolio such that

$$(\tilde{a}, \tilde{b}) = \left(a' - \Delta, b' - \frac{\Delta}{q(a', b')(1 + r)}\right) \quad (A.24)$$

and the government exits in the same number of steps. By construction, the portfolio delivers the same level of consumption and utility at time t. Moreover, assume that tomorrow the government keeps the same $t+1$ policy (a'', b'') under repayment and sets $c_D = y + (1 - \beta)(a' - \Delta)$

39

under default. This implies that the new portfolio delivers the same continuation value for $t+2$ in case of repayment.

We will show that the expected utility under the alternative policy is higher from $t+1$ onward, thus it is preferred to the policy with $a' > 0$. Let's define $\tilde{W}$ as follows

$$\tilde{W} = \frac{\lambda}{1-\beta} u\left(y + (1-\beta)(a' - \Delta)\right) +$$
$$(1-\lambda)u\left(y + (a' - \Delta) - \frac{a''}{1+r} - \kappa\left(b' - \frac{\Delta}{q(a',b')(1+r)}\right) + \right.$$
$$\left. q(a'',b'')\left[b'' - (1-\delta)\left(b' - \frac{\Delta}{q(a',b')(1+r)}\right)\right]\right). \quad \text{(A.25)}$$

From the expression above we see that the first term of $\tilde{W}$ is the lifetime expected utility under default, and the second term is the $t+1$ expected utility.

Noticing from (6) that $q(a',b')(1+r) = (1-\lambda)\left[\kappa + (1-\delta)q(a'',b'')\right]$, and rearranging terms, we obtain

$$\tilde{W} = \frac{\lambda}{1-\beta} u\left(y + (1-\beta)(a' - \Delta)\right) +$$
$$(1-\lambda)u\left(y + a' - \frac{a''}{1+r} - \kappa b' + q(a'',b'')\left[b'' - (1-\delta)b'\right] + \Delta\frac{\lambda}{1-\lambda}\right). \quad \text{(A.26)}$$

We want to show that

$$\tilde{W} > \frac{\lambda}{1-\beta} u\left(y + (1-\beta)a'\right) + (1-\lambda)u\left(y + a' - \frac{a''}{1+r} - \kappa b' + q(a'',b'')\left[b'' - (1-\delta)b'\right]\right),$$

where the right-hand side of the above inequality is an expression akin to $\tilde{W}$ but for the original portfolio with $a' > 0$. Denote $c_R^+(a',b') = y + a' - \frac{a''}{1+r} - \kappa b' + q(a'',b'')\left[b'' - (1-\delta)b'\right]$, and $c_D(a') = y + (1-\beta)a'$. Let's also define $\hat{\Delta} = \Delta\frac{\lambda}{1-\lambda}$. Plugging (A.26) into the last expression and using these expressions, we obtain that what we need to show is:

$$u\left(c_R^+(a',b') + \hat{\Delta}\right) - u\left(c_R^+(a',b')\right) > \frac{\lambda}{(1-\lambda)(1-\beta)}\left[u\left(c_D(a')\right) - u\left(c_D(a') - \frac{(1-\lambda)(1-\beta)}{\lambda}\hat{\Delta}\right)\right].$$

The result, as in the proof of part (i) of this proposition, follows from an application of the mean-value theorem. Namely, there exists $x \in (c_R^+(a',b'), c_R^+(a',b') + \hat{\Delta})$, such that

$$u'(x)\hat{\Delta} = u(c_R^+(a',b') + \hat{\Delta}) - u(c_R^+(a',b')). \quad \text{(A.27)}$$

40

Similarly, there exists $z \in \left(c_D(a') - \frac{(1-\lambda)(1-\beta)}{\lambda}\hat{\Delta}, c_D(a') \right)$, such that

$$u'(z)\hat{\Delta} = \frac{u\left(c_D(a')\right) - u\left(c_D(a') - \frac{(1-\lambda)(1-\beta)}{\lambda}\hat{\Delta}\right)}{\frac{(1-\lambda)(1-\beta)}{\lambda}}. \tag{A.28}$$

Since $c_D(a') > c_R^+(a', b')$ (from Lemma A.1), we can find a $\Delta > 0$ (and arbitrarily small) such that $z > x$. Then, from the strict concavity of the utility function, we have that $u'(x) > u'(z)$. Using this strict inequality and rearranging (A.27) and (A.28), we obtain the inequality we wanted to show.

$\square$

A.8 Corollary 1

Proof. Once the government reaches the safe zone, it stays at the safe zone. Thus, part (ii) of Proposition 3 indicates that the government chooses $a_{t+1} = 0$ for all $t < T$. Part (i) of of Proposition 3 indicates that the government exists by choosing $(a^\star, b^\star)$ $\square$

A.9 Proof of Proposition 6

Proof. If the government picks $(a^\star, b + a^\star - a)$, then the government is safe tomorrow. This follows from the fact that $b + a^\star - a < b^\star$ and that $b^-(a)$ is increasing in a, as shown in Proposition 2. In addition, notice that $c = y + (1 - \beta)(a - b)$ and the value of choosing this portfolio is given by $V^S(a - b)$. Given that $V_R^+(a, b) \leq V^S(a - b)$, it thus follows that the portfolio $(a^\star, b + a^\star - a)$, achieves the optimal solution.

$\square$

B Algorithm

1. Specify a grid for reserves and debt, which we denote respectively by A and B. We use 301 points in A and 1201 points in B.

2. For arbitrary value a, use $V_R(a, b^-(a)) = V_D(a)$ to find $b^-(a)$:

$$\max_{a' \geq 0} \left\{ u\left(y + a - \left(\frac{\delta + r}{1 + r}\right) b^-(a) - \frac{a'}{1 + r}\right) + \frac{\beta u(y + (1 - \beta)(a' - (1 - \delta)b^-(a)))}{1 - \beta} \right\} = $$
$$\frac{u(y + (1 - \beta)a) - \phi}{1 - \beta},$$

   employing a bisection scheme. Notice that this computation assumes that the government will be safe tomorrow after repaying the debt after the run. We then verify that this is the case in equilibrium.[21]

3. Compute $(a^\star, b^\star)$.

4. Compute $V_D(a)$ and $V_R^-(a, b)$ for each grid point.

5. Set an iterator counter $t = 1$.

6. For each $(a, b) \in A \times B$, start with an initial guess for $V_{R,t}^+(a, b) = \frac{u(y - (1 - \beta)b_{\max})}{1 - \beta}$ and $\tilde{q}_t(a, b) = \frac{1}{1 + r}$

7. For each $(a, b) \in A \times B$, go over all possible pairs $(\tilde{a}, \tilde{b}) \in A \times B$, searching for the optimal value under repayment with access to funding as follows:

   (a) If (a, b) satisfies $b \leqslant b^-(a)$, the government is safe under (a, b); i.e., $(a, b) \in \mathbf{S}$. Set $(a'_{t+1}, b'_{t+1}) = (a, b)$ and $V(a, b) = V^S(a - b)$.

   (b) If (a, b) satisfies $b > b^-(a)$, then run over all candidate pairs $(\tilde{a}, \tilde{b}) \in A \times B$ and store the maximizing values (a'_{t+1}, b'_{t+1}); that is,

   $$(a'_{t+1}, b'_{t+1}) \in \operatorname*{argmax}_{(\tilde{a}, \tilde{b}) \in A \times B} W_{R,t}^+(a, b, \tilde{a}, \tilde{b}),$$

   where

   $$W_{R,t}^+(a, b, \tilde{a}, \tilde{b}) \equiv u\left(y + a - \left(\frac{\delta + r}{1 + r}\right) b - \frac{\tilde{a}}{1 + r} + \tilde{q}_t(\tilde{a}, \tilde{b})\left(\tilde{b} - (1 - \delta)b\right)\right) + \beta \mathbb{E} V_t(\tilde{a}, \tilde{b}),$$

---

[21]It is also straightforward to extend the algorithm to consider parameterizations where the government can remain vulnerable tomorrow.

where we compute the expected continuation according to equation (13).

8. Store the value $V^+_{R,t+1}(a, b)$ as

$$V^+_{R,t+1}(a, b) \equiv W^+_{R,t}(a, b, a'_{t+1}, b'_{t+1}).$$

9. Given the updated policy functions, compute an update for the price schedule for each $(a', b') \in A \times B$ as follows:

$$\tilde{q}_{t+1}(a', b') = \begin{cases} \frac{1}{1+r} & \text{if } (a', b') \in \mathbf{S}, \\ 0 & \text{if } (a', b') \in \mathbf{D}, \\ \frac{1-\lambda}{1+r}\left(\frac{\delta+r}{1+r} + (1-\delta)\tilde{q}_t(a'', b'')\right) & \text{if } (a', b') \in \mathbf{C}. \end{cases}$$

where (a'', b'') is the optimal portfolio under state (a', b'); i.e., $a'' \equiv a'_{t+1}(a', b')$, and $b'' \equiv b'_{t+1}(a', b')$.

10. Check for convergence. Compute

$$\epsilon_q \equiv \|q_{t+1} - q_t\|_\infty,$$
$$\epsilon_{V_R^+} \equiv \|V^+_{R,t+1} - V^+_{R,t}\|_\infty,$$
$$\epsilon_a \equiv \|a'_{t+1} - a'_t\|_\infty,$$
$$\epsilon_b \equiv \|b'_{t+1} - b'_t\|_\infty.$$
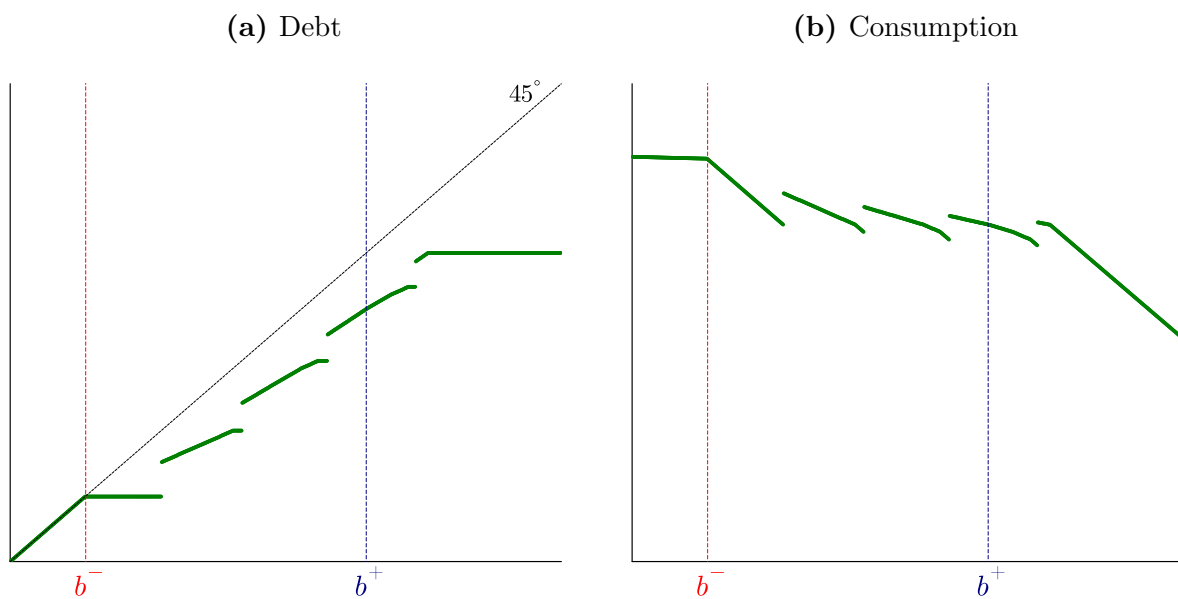
If $\max\{\epsilon_q, \epsilon_{V_R^+}, \epsilon_a, \epsilon_b\} < 10^{-8}$, stop. Otherwise, set

$$\tilde{q}_t(a, b) = \xi\tilde{q}_{t+1}(a, b) + (1 - \xi)\tilde{q}_t(a, b),$$

update the iterator counter $t = t + 1$ and go back to step (8). We set $\xi = 0.25$.

C   Additional Figures: Economy without Reserves

Figure A1: Policy without reserves.

(a) Debt

(b) Consumption

Note: The figure presents the policy functions for the economy when the government is restricted from accumulating reserves. Panel (a) shows the policy for debt, $b'(b)$. Panel (b) exhibits the policy for consumption, $c(b)$.

Figure A2: Lower consumption without reserves

(a) Reserves   **(b)** Debt   **(c)** Consumption

(d) NFA   **(e)** Bond Price
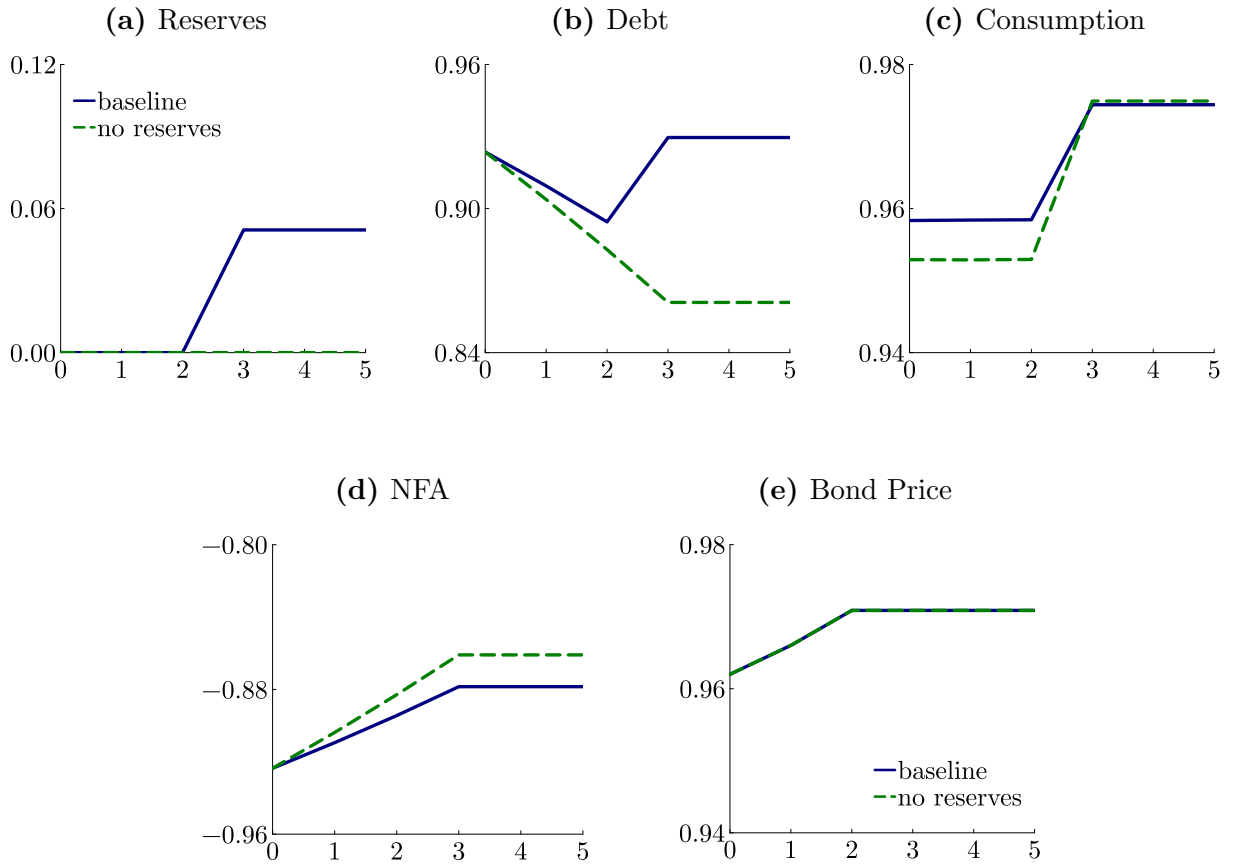
Note: The government is assumed to start in the crisis zone with $b = 0.924$ and zero initial reserves.

Figure A3: Longer to exit without reserves

(a) Reserves

(b) Debt

(c) Consumption

(d) NFA

(e) Bond Price
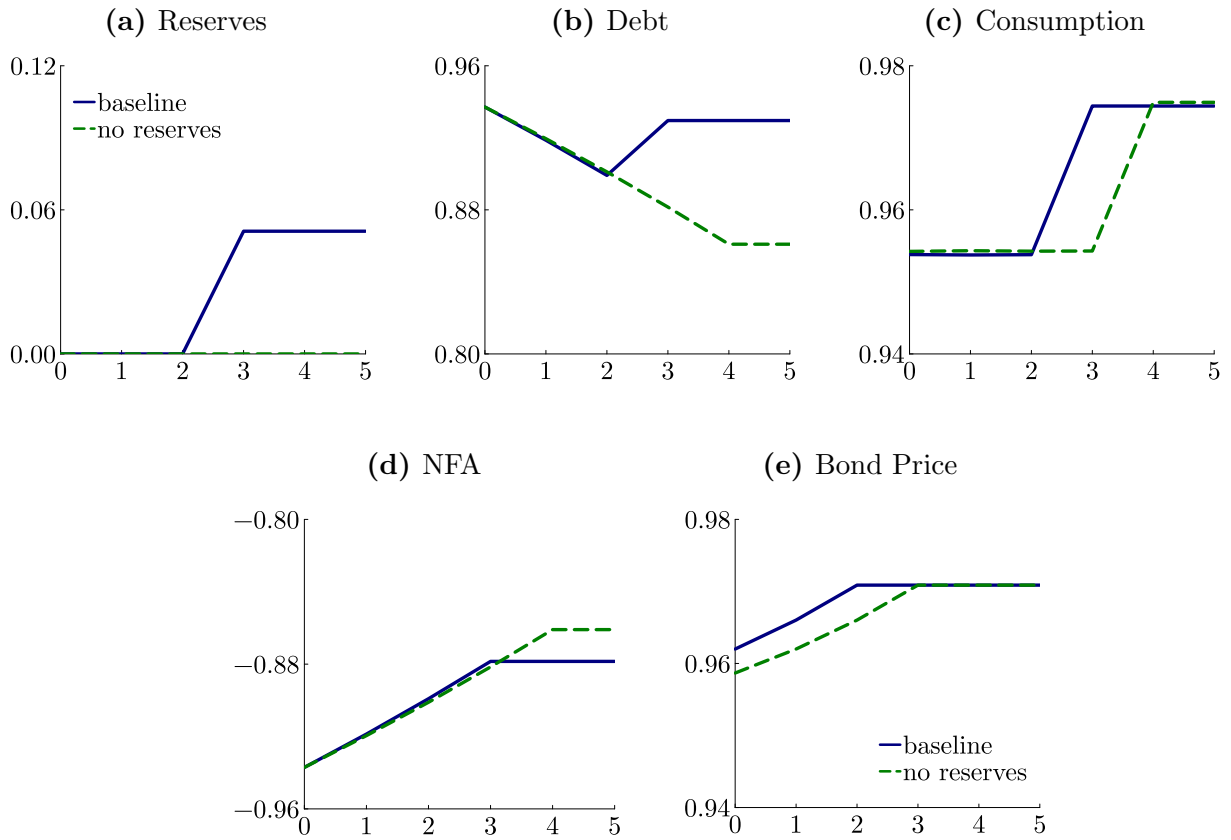
Note: The government is assumed to start in the crisis zone with $b = 0.937$ and zero initial reserves.