Lesson: Limits of Infinite Sequences

Part I: What Is a Limit?

- A **limit** describes the value that a sequence is approaching as n becomes very large.
- Notation:

$$\lim_{n \to \infty} a_n$$

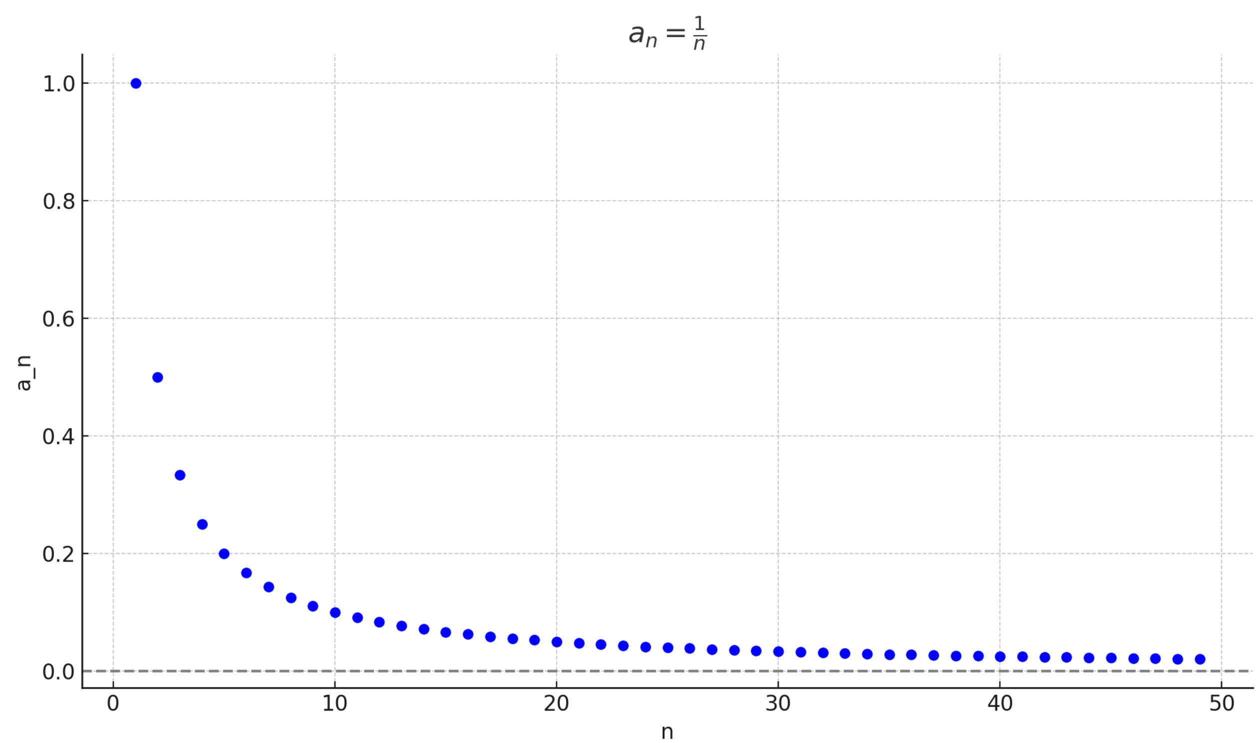
- This means: "What value is a_n getting closer and closer to as n gets larger?". This only has an answer if the values of a_n get arbitrarily close to a single number as n increases. We say that the sequence **converges**.
- When the values of a_n grow more and more positive without bound we say that the limit approaches ∞.
- When the values of a_n grow more and more negative without bound we say that the limit approaches $-\infty$.
- When the values are bounded but do not converge to a single value, then we say that "the limit does not exist!" (And plan to watch *Mean Girls* again).
- If the values do not approach a finite number—either because they grow without bound or oscillate—the sequence is said to diverge.

Graphical Interpretation

- When you graph the terms of a sequence as points (n, a_n), the **limit** is the horizontal value that the points approach as n increases.
- If the points level off and appear to settle near a specific y-value, that value is the limit.
- If the points grow without bound or bounce back and forth, the limit may **not** exist.

Example Graphs:

- $a_n = \frac{1}{n}$



Part II: Key Ideas to Know

1. If $k > 0$, then:

$$\lim_{n\to\infty} \frac{1}{n^k} = 0$$

2. If $-1 < r < 1$, then:

$$\lim_{n\to\infty} r^n = 0$$

3. You can **usually move limits inside functions**:

   - If $\lim_{n\to\infty} a_n = L$, then:

$$\lim_{n\to\infty} \sqrt{a_n} = \sqrt{L}, \quad \lim_{n\to\infty} \ln(a_n) = \ln(L)$$

Summary: How Sequences Shrink

- If the denominator grows faster than the numerator, the limit is usually 0.
- If you're raising a number between -1 and 1 to higher powers, it shrinks toward 0.
- You can often move limits inside functions like square roots or logarithms if the limit exists and the function is defined there.

Part III: Finding Limits of Sequences

1. $\lim_{n\to\infty} \frac{5}{n}$

2. $\lim_{n\to\infty} \frac{2n+1}{n^2+3}$

3. $\lim_{n\to\infty} \left(\frac{3}{4}\right)^n$

4. $\lim_{n\to\infty} \frac{n^2+5n}{n^2-3n+1}$

5. $\lim_{n\to\infty} \sqrt{\frac{1}{n}}$

6. $\lim_{n\to\infty} \left[3 + \frac{6}{n^3+4}\right]$

7. $\lim_{n\to\infty} \log_4 \left(\frac{16n+9}{n+1}\right)$

8. $\lim_{n\to\infty} \cos(e^{-n})$

Find the limit of these sequences or state that the limit does not exist

9. 1, 4, 9, 16, ...

10. 1/2, -1/4, 1/8, -1/16, ...

11. $t_n = \sin(\pi \cdot n)$

Part IV: When Limits Do Not Exist

A limit **does not exist** if: - The sequence grows without bound - The sequence oscillates

Examples:

12. $\lim_{n\to\infty} n$

13. $\lim_{n\to\infty} (-1)^n$

14. $\lim_{n\to\infty} \cos(n)$

15. $\lim_{n\to\infty} \frac{(-2)^n}{n}$

16. $\lim_{n\to\infty} \ln(n)$

Summary: When Limits Break Down

- A sequence does not have a limit if it:

- Grows without bound (positive or negative)
- Oscillates forever without settling down (like (-1)^n or cos(n))

- In those cases, we say “the limit does not exist” even if the values stay bounded.

Part V: Review – Multiple Choice and True/False

1. What is the limit of the sequence $a_n = \frac{5n+3}{n+2}$ as $n \to \infty$?

A) 5

B) 1

C) 0

D) The limit does not exist

2. True or False: If $a_n = (-1)^n$, then the sequence has a limit of 0.

3. Which of the following sequences has a limit of 0?

A) $a_n = \frac{1}{n^2}$

B) $a_n = 1 + \frac{1}{n}$

C) $a_n = (-2)^n$

D) $a_n = \cos(n)$

4. True or False: If $\lim_{n\to\infty} a_n = L$, then $\lim_{n\to\infty} \sqrt{a_n} = \sqrt{L}$, as long as $L \geq 0$.

Part VI: Reflection

- What does it mean for a sequence to have a limit?

- How can you recognize when a limit is zero?

- How can you tell when a limit does not exist?