Day 2 Notes

Class Goals

- Become more familiar with the Laplace Transform
- Understand how to use tables to compute L.T.
- Know that if two function have the same L.T. then they are basically equal
- Solve D.E. using L.T.
- Intuition about the Inverse Laplace Transform
- Initial Value Theorem for Laplace Transform
- Final Value Theorem for Laplace Transform

More with Laplace Transform Remember that the Laplace Transform is given by:

$$\mathcal{L}(f)(s) = F(s) = \int_0^\infty f(t)e^{-st}dt$$

What kind of functions have a Laplace Transform Not every function has a Laplace Transform. Only functions that allow the improper integral to converge will work for the

Definition A function $f(t)$ is defined for $t \geq 0$ is said to be of **exponential order** if for some constant a and some constant $M > 0$ the inequality

$$|f(t)| \leq Me^{at}$$

If a function is of **exponential order** then the function has a Laplace Transform.

One of the nice things about a Laplace transform is that a function that comes from an integral is always continuous even if the function that is being integrated isn't continuous. So if $f(t)$ has discontinuity $F(s)$ will not.

Dictionary for Laplace Transforms

Function $f(t)$	Laplace Transform $F(s)$	Comment
C	C/s	
t^n	$n!/s^{n+1}$	n is an integer
e^{at}	$1/(s-a)$	$s > a$
$t^n e^{at}$	$n!/(s-a)^{n+1}$	n is an integer
$\sin(bt)$	$b/(s^2+b^2)$	
$\cos(bt)$	$s/(s^2+b^2)$	
$e^{at}\sin(bt)$	$b/((s-a)^2+b^2)$	
$e^{at}\cos(bt)$	$(s-a)/((s-a)^2+b^2)$	

Laplace Transform of the derivatives

$$\mathcal{L}(f')(s) = sF(s) - f(0)$$

Proof: Use integration by parts.

$$\mathcal{L}(f'')(s) = s\mathcal{L}(f') - f'(0)$$
$$\mathcal{L}(f'')(s) = s(sF(s) - f(0)) - f'(0)$$
$$\mathcal{L}(f'')(s) = s^2F(s) - sf(0) - f'(0)$$

Proof: Use integration by parts.

Example Solve the following differential equation using the Laplace Transform:

$$u''(t) + 4u'(t) + 3u(t) = 0$$
$$u(0) = 2, u'(0) = 4$$

First Shifting Theorem

$$\mathcal{L}(e^{at}f(t)) = F(s-a)$$

Inverse Laplace Transform The main point here is that if $U_1(s) = U_2(s)$ then $u_1(t) = u_2(t)$ except for a finite number of points where u_i has a jump discontinuity.

Intuition about the Inverse Laplace Transform Most of the information about what functions are involved in the Laplace Transform comes from the denominator. If we have a function in the s-domain then we should look at the factors of the denominator to determine what functions are involved.

Initial Value Theorem for Laplace Transform If $\lim_{t\to 0^+}$ exists then $\lim_{t\to 0^+} f(t) = \lim_{s\to\infty} sF(s)$

Final Value Theorem for Laplace Transform If every pole of $F(s)$ is of the form $a + bi$ with $a < 0$ or if F has a pole at $s = 0$ then these poles are of multiplicity 1. Then $\lim_{t\to\infty}$ exists and $\lim_{t\to\infty} f(t) = \lim_{s\to 0^+} sF(s)$

Practice **Exercise 5.2.7:** Compute the Laplace transform of e^{it}, treating i as a constant by taking $a = i$ in Table 5.1. Compare the result to the Laplace transform of $\cos(t) + i\sin(t)$. Try simplifying the difference. Do the rules of Table 5.1 seem to work for complex exponents?