

# Parity Asymmetry in the CMB

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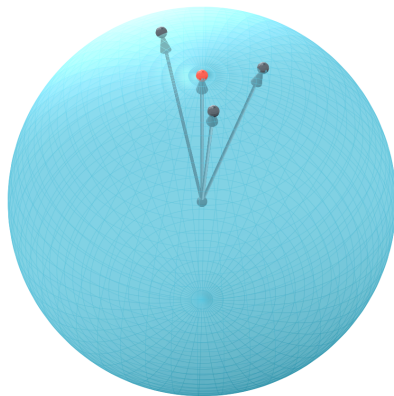
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# Outline

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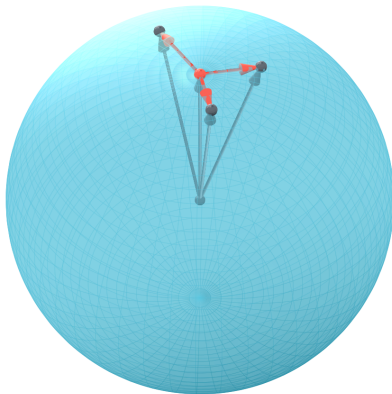
# Overview of the problem

The goal is to determine whether or not parity asymmetry can be detected using the Cosmic Microwave Background. To do so we are modelling the 4-point correlation function as a function of 4 points on a sphere, defined by vectors  $\vec{x}_0, \vec{x}_1, \vec{x}_2, \vec{x}_3$



# Overview of the problem

We then define a set of vectors  $\vec{r}_1, \vec{r}_2, \vec{r}_3$  where  $\vec{r}_i = \vec{x}_i - \vec{x}_0$



# Overview of the problem

If we then take two correlation functions:

$$\zeta_E(\vec{x}_0, \vec{x}_1, \vec{x}_2, \vec{x}_3)$$

which is the correlation function of the vectors from the Earth to the CMB and

$$\zeta_{x_0}(\vec{r}_1, \vec{r}_2, \vec{r}_3)$$

which is the correlation function centered on point  $x_0$ , then our goal is to write one as a function of the other in order to help constrain the possible parity of both

# Initial Calculations

Start by expanding the two correlation functions in terms of isotropic basis functions:

$$\begin{aligned}\zeta_{x_0}(\vec{r}_1, \vec{r}_2, \vec{r}_3) &= \sum_{\Lambda} \tilde{\zeta}_{\Lambda}(r_1, r_2, r_3) \mathcal{P}_{\Lambda}(\hat{r}_1, \hat{r}_2, \hat{r}_3) \\ \zeta_E(\vec{x}_0, \vec{x}_1, \vec{x}_2, \vec{x}_3) &= \sum_{\Lambda'} \tilde{\zeta}_{\Lambda'}(x_0, x_1, x_2, x_3) \mathcal{P}_{\Lambda'}(\hat{x}_0, \hat{x}_1, \hat{x}_2, \hat{x}_3)\end{aligned}$$

Where  $\Lambda$  is the set  $\{l_1, l_2, l_3\}$  and  $\Lambda'$  is the set  $\{l'_0, l'_1, l'_{01}, l'_2, l'_3\}$

# Initial Calculations

We can then set the two functions equal since they are correlation functions describing the same points

$$\begin{aligned}\zeta_{x_0}(\vec{r}_1, \vec{r}_2, \vec{r}_3) &= \zeta_E(\vec{x}_0, \vec{x}_1, \vec{x}_2, \vec{x}_3) \\ &\Downarrow \\ \sum_{\Lambda} \zeta_{\Lambda}(r_1, r_2, r_3) \mathcal{P}_{\Lambda}(\hat{r}_1, \hat{r}_2, \hat{r}_3) &= \sum_{\Lambda'} \zeta_{\Lambda'}(x_0, x_1, x_2, x_3) \mathcal{P}_{\Lambda'}(\hat{x}_0, \hat{x}_1, \hat{x}_2, \hat{x}_3)\end{aligned}$$

# Initial Calculations

We now multiply both sides by  $\mathcal{P}_{\Lambda''}^*(\hat{x}_0, \hat{x}_1, \hat{x}_2, \hat{x}_3)$  and then integrate over all orientations  $d\hat{x}_0 d\hat{x}_1 d\hat{x}_2 d\hat{x}_3 = d\hat{x}$

$$\int d\hat{x} \sum_{\Lambda} \zeta_{\Lambda}(r_1, r_2, r_3) \mathcal{P}_{\Lambda}(\hat{r}_1, \hat{r}_2, \hat{r}_3) P_{\Lambda''}^*(\hat{x}_0, \hat{x}_1, \hat{x}_2, \hat{x}_3) =$$
$$\int d\hat{x} \sum_{\Lambda'} \zeta_{\Lambda'}(x_0, x_1, x_2, x_3) \mathcal{P}_{\Lambda'}(\hat{x}_0, \hat{x}_1, \hat{x}_2, \hat{x}_3) P_{\Lambda''}^*(\hat{x}_0, \hat{x}_1, \hat{x}_2, \hat{x}_3)$$



# Initial Calculations

Using the orthogonality of isotropic basis functions which states

$$\int d\hat{R} \mathcal{P}_{\Lambda}(\hat{R}) \mathcal{P}_{\Lambda'}^*(\hat{R}) = \delta_{\Lambda_1 \Lambda'_1} \delta_{\Lambda_2 \Lambda'_2} \dots$$

we can simplify the right hand side of the previous equation:

$$\begin{aligned} \int d\hat{x} \sum_{\Lambda'} \zeta_{\Lambda'}(x_0, x_1, x_2, x_3) \mathcal{P}_{\Lambda'}(\hat{x}_0, \hat{x}_1, \hat{x}_2, \hat{x}_3) \mathcal{P}_{\Lambda''}(\hat{x}_0, \hat{x}_1, \hat{x}_2, \hat{x}_3) \\ = \zeta_{\Lambda''}(x_0, x_1, x_2, x_3) \end{aligned}$$

# Initial Calculations

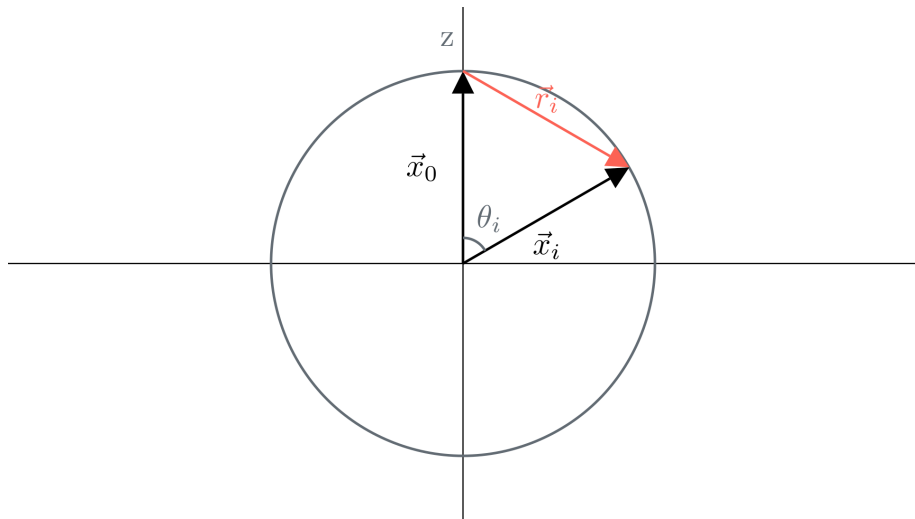
Plugging this back into the previous equation we get the key relation:

$$\zeta_{\Lambda''}(x_0, x_1, x_2, x_3) = \int d\hat{x} \sum_{\Lambda} \zeta_{\Lambda}(r_1, r_2, r_3) \mathcal{P}_{\Lambda}(\hat{r}_1, \hat{r}_2, \hat{r}_3) P_{\Lambda''}^*(\hat{x}_0, \hat{x}_1, \hat{x}_2, \hat{x}_3) \quad (1)$$

Solving the integral on the right will give an expression for the coefficients of the correlation function centered on earth in terms of the correlation function centered at  $x_0$

## Solving for $\vec{r}$ in terms of $\vec{x}$

Consider a great circle going through the poles at an angle  $\phi_i$  to the x-axis where  $\phi_i$  is the  $\phi$  angle of vector  $\vec{x}_i$  for  $i = 1, 2, 3$

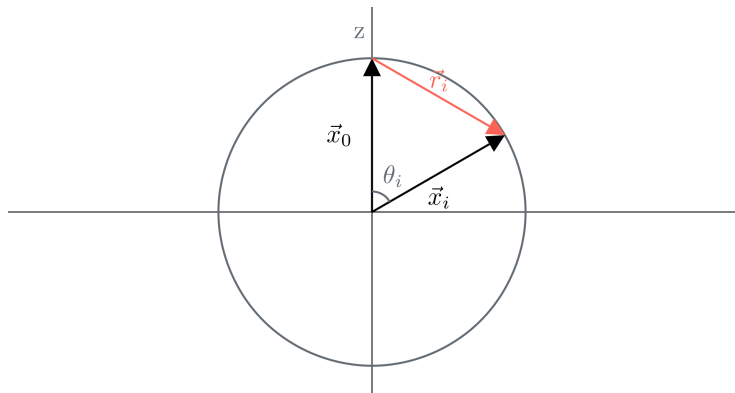


## Solving for $\vec{r}$ in terms of $\vec{x}$

Using the law of Cosines and defining the distance to the cmb to be  $d^*$  we get:

$$|\vec{r}_i| = \sqrt{d^{*2} + d^{*2} - 2d^*d^*\cos(\theta_i)}$$

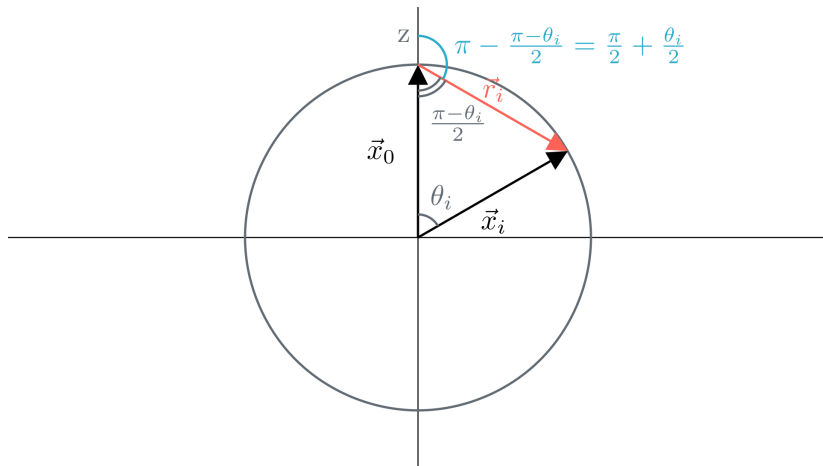
$$r_i = \sqrt{2}d^*\sqrt{1 - \cos(\theta_i)}$$



## Solving for $\vec{r}$ in terms of $\vec{x}$

Similarly, we can solve for the angular dependence of  $\vec{r}_i$

$$\theta_{ri} = \frac{\pi}{2} + \frac{\theta_i}{2}; \quad \phi_{ri} = \phi_i$$



# Solving for the coefficients

Plugging those back into equation 1 we get:

$$\begin{aligned}\zeta_{\Lambda''}(x_0, x_1, x_2, x_3) = & \int d\hat{x} \sum_{\Lambda} \zeta_{\Lambda}(\sqrt{2}d^* \sqrt{1 - \cos(\theta_1)}, \sqrt{2}d^* \sqrt{1 - \cos(\theta_2)}, \sqrt{2}d^* \sqrt{1 - \cos(\theta_3)}) \\ & \times \mathcal{P}_{\Lambda}(\frac{\pi}{2} + \frac{\theta_1}{2}, \phi_1; \frac{\pi}{2} + \frac{\theta_2}{2}, \phi_2; \frac{\pi}{2} + \frac{\theta_3}{2}, \phi_3) \\ & \times \mathcal{P}_{\Lambda''}^*(\theta_0, \phi_0; \theta_1, \phi_1; \theta_2, \phi_2; \theta_3, \phi_3)\end{aligned}$$

# Different approaches

- Directly solving the integral
- Rotating the spherical harmonics
- Using an addition formula
  - ▶ Associated Legendre Polynomials
  - ▶ Jacobi Polynomials

# Directly solving the integral

We can expand the integral using the definitions of the isotropic basis functions:

$$\mathcal{P}_{l_1 l_2 l_3}(\hat{r}_1, \hat{r}_2, \hat{r}_3) = (-1)^{l_1 + l_2 + l_3} \sum_{m_1 m_2 m_3} \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \\ \times Y_{l_1 m_1}(\hat{r}_1) Y_{l_2 m_2}(\hat{r}_2) Y_{l_3 m_3}(\hat{r}_3)$$

$$\mathcal{P}_{l_1 l_2(l_{12}) l_3 l_4}(\hat{r}_1, \hat{r}_2, \hat{r}_3, \hat{r}_4) = (-1)^{l_1 + l_2 + l_3 + l_4} \\ \times \sum_{m_{12}} (-1)^{l_{12} - m_{12}} \sum_{m_1 m_2 m_3 m_4} \begin{pmatrix} l_1 & l_2 & l_{12} \\ m_1 & m_2 & -m_{12} \end{pmatrix} \begin{pmatrix} l_{12} & l_3 & l_4 \\ m_{12} & m_3 & -m_4 \end{pmatrix} \\ \times Y_{l_1 m_1}(\hat{r}_1) Y_{l_2 m_2}(\hat{r}_2) Y_{l_3 m_3}(\hat{r}_3) Y_{l_4 m_4}(\hat{r}_4)$$



# Directly solving the integral

Looking at just the angular dependent parts of the integral we have:

$$\begin{aligned} & \int d\theta_0 d\phi_0 d\theta_1 d\phi_1 d\theta_2 d\phi_2 d\theta_3 d\phi_3 \sin(\theta_0) \sin(\theta_1) \sin(\theta_2) \sin(\theta_3) \\ & \times Y_{l'_0 m'_0}(\theta_0, \phi_0) Y_{l'_1 m'_1}(\theta_1, \phi_1) Y_{l'_2 m'_2}(\theta_2, \phi_2) Y_{l'_3 m'_3}(\theta_3, \phi_3) \\ & \times Y_{l_1 m_1}(\frac{\theta_1}{2} + \frac{\pi}{2}, \phi_1) Y_{l_2 m_2}(\frac{\theta_2}{2} + \frac{\pi}{2}, \phi_2) Y_{l_3 m_3}(\frac{\theta_3}{2} + \frac{\pi}{2}, \phi_3) \end{aligned}$$

# Directly solving the integral

We can further separate the integral into  $\phi$  and  $\theta$  dependent parts by using the following definition of spherical harmonics:

$$Y_l^m(\theta, \phi) = \sqrt{\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!}} P_l^m(\cos\theta) e^{im\phi}$$

# Solving the $\phi$ integral

Looking at just the  $\phi$  dependent terms:

$$\int_0^{2\pi} d\phi_0 d\phi_1 d\phi_2 d\phi_3 e^{im'_0\phi_0} e^{im'_1\phi_1+im_1\phi_1} e^{im'_2\phi_2+im_2\phi_2} e^{im'_3\phi_3+im_3\phi_3}$$
$$= \left[ \frac{1}{im'_0} e^{im'_0\phi_0} \right]_0^{2\pi} \left[ \frac{1}{i(m'_j+m_j)} e^{i(m'_j+m_j)\phi_j} \right]_0^{2\pi} \quad \text{for } j = 1, 2, 3$$

Each of the above integrals will always be zero since all  $m$ 's are integers unless  $m'_0 = 0$  and  $m'_j + m_j = 0$  in which case each term becomes  $2\pi$  so the total  $\phi$  integral becomes:

$$16\pi^4 \delta_{m'_0,0}^K \delta_{m_1,-m'_1}^K \delta_{m_2,-m'_2}^K \delta_{m_3,-m'_3}^K$$

# Examining the 3j symbols

Our original expression had three 3j symbols:

$$\begin{pmatrix} l'_0 & l'_1 & l'_{01} \\ m'_0 & m'_1 & -m'_{01} \end{pmatrix} \begin{pmatrix} l'_{01} & l'_2 & l'_3 \\ m'_{01} & m'_2 & m'_3 \end{pmatrix} \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix}$$

If we now plug in the constraints given by the delta functions from the  $\phi$  integral we get:

$$\begin{pmatrix} l'_0 & l'_1 & l'_{01} \\ 0 & -m_1 & -m'_{01} \end{pmatrix} \begin{pmatrix} l'_{01} & l'_2 & l'_3 \\ m'_{01} & -m_2 & -m_3 \end{pmatrix} \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix}$$

Since the bottom elements of the 3j symbol must sum to zero, this adds the additional constraint that  $m'_{01} = -m_1$

# Solving the $\theta$ integral

Looking at the  $\theta$  integral, we have:

$$\int_0^{2\pi} d\theta_0 d\theta_1 d\theta_2 d\theta_3 \sin(\theta_0) \sin(\theta_1) \sin(\theta_2) \sin(\theta_3)$$