Parity Asymmetry in the CMB

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Outline

Intitial Calculations

Start by expanding the two correlation functions in terms of isotropic basis functions:

$$\begin{aligned}\zeta_{x_0}(\vec{r}_1, \vec{r}_2, \vec{r}_3) &= \sum_{\Lambda} \tilde{\zeta}_{\Lambda}(r_1, r_2, r_3)\mathcal{P}_{\Lambda}(\hat{r}_1, \hat{r}_2, \hat{r}_3) \\ \zeta_E(\vec{x}_0, \vec{x}_1, \vec{x}_2, \vec{x}_3) &= \sum_{\Lambda'} \tilde{\zeta}_{\Lambda'}(x_0, x_1, x_2, x_3)\mathcal{P}_{\Lambda'}(\hat{x}_0, \hat{x}_1, \hat{x}_2, \hat{x}_3)\end{aligned}$$

Where Λ is the set $\{l_1, l_2, l_3\}$ and Λ' is the set $\{l'_0, l'_1, l'_{01}, l'_2, l'_3\}$

Intitial Calculations

We can then set the two functions equal since they are correlation functions describing the same points

$$\begin{aligned} \zeta_{x_0}(\vec{r}_1, \vec{r}_2, \vec{r}_3) &= \zeta_E(\vec{x}_0, \vec{x}_1, \vec{x}_2, \vec{x}_3) \\ &\Downarrow \\ \sum_{\Lambda} \zeta_{\Lambda}(r_1, r_2, r_3)\mathcal{P}_{\Lambda}(\hat{r}_1, \hat{r}_2, \hat{r}_3) &= \sum_{\Lambda'} \zeta_{\Lambda'}(x_0, x_1, x_2, x_3)\mathcal{P}_{\Lambda'}(\hat{x}_0, \hat{x}_1, \hat{x}_2, \hat{x}_3) \end{aligned}$$

Intitial Calculations

We now multiply both sides by $\mathcal{P}^*_{\Lambda''}(\hat{x}_0, \hat{x}_1, \hat{x}_2, \hat{x}_3)$ and then integrate over all orientations $d\hat{x}_0 d\hat{x}_1 d\hat{x}_2 d\hat{x}_3 = d\hat{x}$

$$\int d\hat{x} \sum_{\Lambda} \zeta_{\Lambda}(r_1, r_2, r_3)\mathcal{P}_{\Lambda}(\hat{r}_1, \hat{r}_2, \hat{r}_3)P^*_{\Lambda''}(\hat{x}_0, \hat{x}_1, \hat{x}_2, \hat{x}_3) =$$

$$\int d\hat{x} \sum_{\Lambda'} \zeta_{\Lambda'}(x_0, x_1, x_2, x_3)\mathcal{P}_{\Lambda'}(\hat{x}_0, \hat{x}_1, \hat{x}_2, \hat{x}_3)P^*_{\Lambda''}(\hat{x}_0, \hat{x}_1, \hat{x}_2, \hat{x}_3)$$

Intitial Calculations

Using the orthogonality of isotropic basis functions which states

$$\int d\hat{R}\ \mathcal{P}_{\Lambda}(\hat{R})\mathcal{P}^*_{\Lambda'}(\hat{R}) = \delta_{\Lambda_1\Lambda'_1}\delta_{\Lambda_2\Lambda'_2}\dots$$

we can simplify the right hand side of the previous equation:

$$\int d\hat{x} \sum_{\Lambda'} \zeta_{\Lambda'}(x_0, x_1, x_2, x_3)\mathcal{P}_{\Lambda'}(\hat{x}_0, \hat{x}_1, \hat{x}_2, \hat{x}_3)P_{\Lambda''}(\hat{x}_0, \hat{x}_1, \hat{x}_2, \hat{x}_3)$$

$$= \zeta_{\Lambda''}(x_0, x_1, x_2, x_3)$$

Intitial Calculations

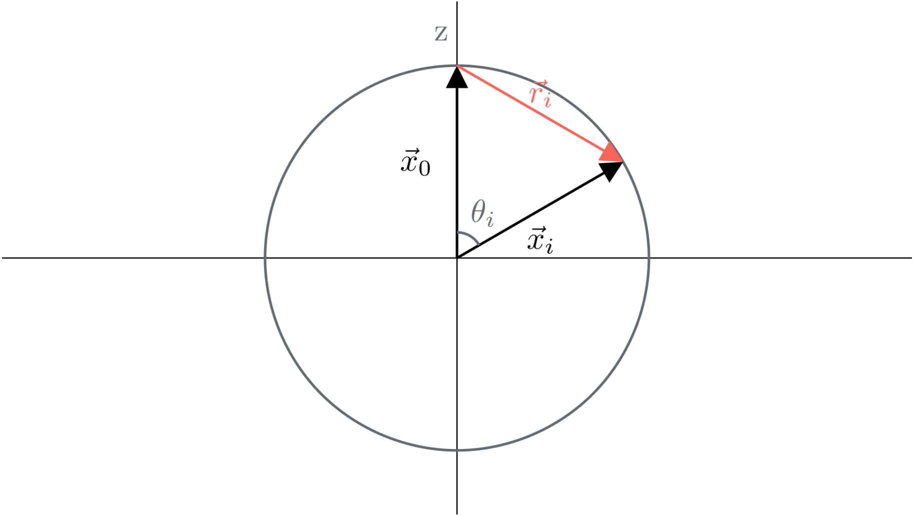
Plugging this back into the previous equation we get the key relation:

$$\boxed{\zeta_{\Lambda''}(x_0, x_1, x_2, x_3) = \int d\hat{x} \sum_{\Lambda} \zeta_{\Lambda}(r_1, r_2, r_3)\mathcal{P}_{\Lambda}(\hat{r}_1, \hat{r}_2, \hat{r}_3)P^*_{\Lambda''}(\hat{x}_0, \hat{x}_1, \hat{x}_2, \hat{x}_3)} \quad (1)$$

Solving the integral on the right will give an expression for the coefficients of the correlation function centered on earth in terms of the correlation function centered at x_0

Solving for $\vec{r}$ in terms of $\vec{x}$

Consider a great circle going through the poles at an angle ϕ_i to the x-axis where ϕ_i is the ϕ angle of vector $\vec{x}_i$ for $i = 1, 2, 3$

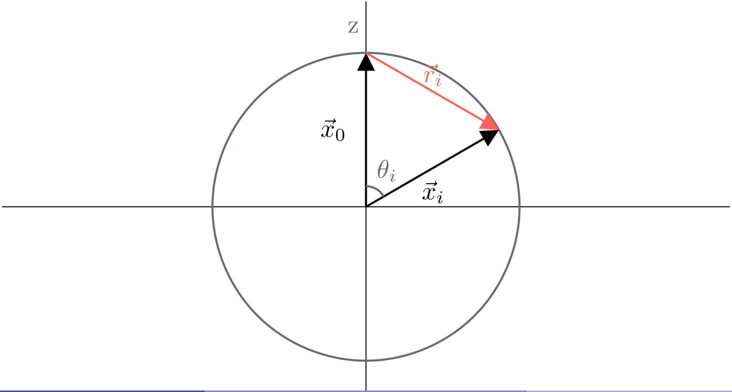


Solving for $\vec{r}$ in terms of $\vec{x}$

Using the law of Cosines and defining the distance to the cmb to be d^* we get:

$$|\vec{r}_i| = \sqrt{d^{*2} + d^{*2} - 2d^*d^*cos(\theta_i)}$$

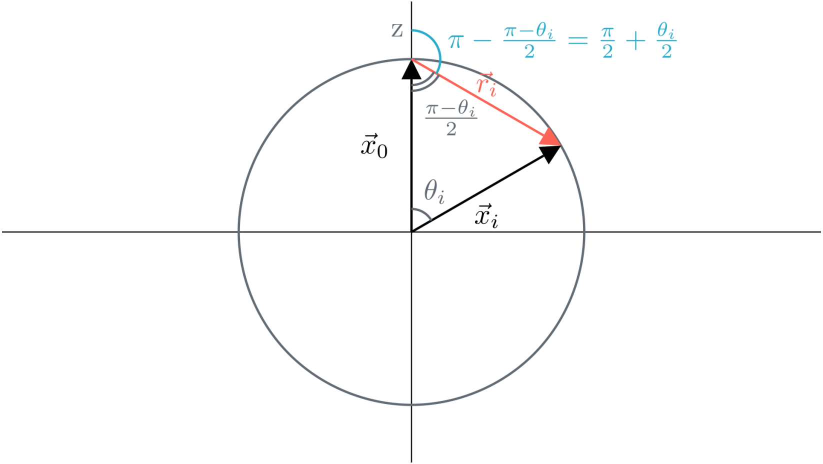
$$\boxed{r_i = \sqrt{2}d^*\sqrt{1 - cos(\theta_i)}}$$



Solving for $\vec{r}$ in terms of $\vec{x}$

Similarly, we can solve for the angular dependence of $\vec{r}_i$

$$\boxed{\theta_{ri} = \frac{\pi}{2} + \frac{\theta_i}{2};\ \phi_{ri} = \phi_i}$$



Solving for the coefficients

Plugging those back into equation 1 we get:

$$\zeta_{\Lambda''}(x_0, x_1, x_2, x_3) =$$

$$\int d\hat{x} \sum_{\Lambda} \zeta_{\Lambda}(\sqrt{2}d^* \sqrt{1 - cos(\theta_1)}, \sqrt{2}d^* \sqrt{1 - cos(\theta_2)}, \sqrt{2}d^* \sqrt{1 - cos(\theta_3)})$$

$$\times \mathcal{P}_{\Lambda}(\frac{\pi}{2} + \frac{\theta_1}{2}, \phi_1; \frac{\pi}{2} + \frac{\theta_2}{2}, \phi_2; \frac{\pi}{2} + \frac{\theta_3}{2}, \phi_3)$$

$$\times \mathcal{P}^*_{\Lambda''}(\theta_0, \phi_0; \theta_1, \phi_1; \theta_2, \phi_2; \theta_3, \phi_3)$$

Different approaches

- Directly solving the integral
- Rotating the spherical harmonics
- Using an addition formula
    - Associated Legendre Polynomials
    - Jacobi Polynomials

Directly solving the integral

We can expand the integral using the definitions of the isotropic basis functions:

$$\mathcal{P}_{l_1 l_2 l_3}(\hat{r}_1, \hat{r}_2, \hat{r}_3) = (-1)^{l_1+l_2+l_3} \sum_{m_1 m_1 m_3} \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \times Y_{l_1 m_1}(\hat{r}_1) Y_{l_2 m_2}(\hat{r}_2) Y_{l_3 m_3}(\hat{r}_3)$$

$$\mathcal{P}_{l_1 l_2 (l_{12}) l_3 l_4}(\hat{r}_1, \hat{r}_2, \hat{r}_3, \hat{r}_4) = (-1)^{l_1+l_2+l_3+l_4} \times \sum_{m_{12}} (-1)^{l_{12}-m_{12}} \sum_{m_1 m_2 m_3 m_4} \begin{pmatrix} l_1 & l_2 & l_{12} \\ m_1 & m_2 & -m_{12} \end{pmatrix} \begin{pmatrix} l_{12} & l_3 & l_4 \\ m_{12} & m_3 & -m_4 \end{pmatrix} \times Y_{l_1 m_1}(\hat{r}_1) Y_{l_2 m_2}(\hat{r}_2) Y_{l_3 m_3}(\hat{r}_3) Y_{l_4 m_4}(\hat{r}_4)$$

Directly solving the integral

Looking at just the angular dependent parts of the integral we have:

$$\int d\theta_0 d\phi_0 d\theta_1 d\phi_1 d\theta_2 d\phi_2 d\theta_3 d\phi_3 sin(\theta_0) sin(\theta_1) sin(\theta_2) sin(\theta_3)$$
$$\times\ Y_{l'_0 m'_0}(\theta_0, \phi_0) Y_{l'_1 m'_1}(\theta_1, \phi_1) Y_{l'_2 m'_2}(\theta_2, \phi_2) Y_{l'_3 m'_3}(\theta_3, \phi_3)$$
$$\times\ Y_{l_1 m_1}(\frac{\theta_1}{2} + \frac{\pi}{2}, \phi_1) Y_{l_2 m_2}(\frac{\theta_2}{2} + \frac{\pi}{2}, \phi_2) Y_{l_3 m_3}(\frac{\theta_3}{2} + \frac{\pi}{2}, \phi_3)$$

Directly solving the integral

We can further separate the integral into ϕ and θ dependent parts by using the following definition of sperical harmonics:

$$Y_l^m(\theta, \phi) = \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}P_l^m(cos\theta)e^{im\phi}$$

Solving the ϕ integral

Looking at just the ϕ dependent terms:

$$\int_0^{2\pi} d\phi_0 d\phi_1 d\phi_2 d\phi_3 e^{im_0'\phi_0} e^{im_1'\phi_1 + im_1\phi_1} e^{im_2'\phi_2 + im_2\phi_2} e^{im_3'\phi_3 + im_3\phi_3}$$

$$= \left[\frac{1}{im_0'} e^{im_0'\phi_0} \Big|_0^{2\pi} \right] \left[\frac{1}{i(m_j' + m_j)} e^{i(m_j'+m_j)\phi_j} \Big|_0^{2\pi} \right] \; for \; j = 1, 2, 3$$

Each of the above integrals will always be zero since all m's are integers unless $m_0' = 0$ and $m_j' + m_j = 0$ in which case each term becomes 2π so the total ϕ integral becomes:

$$16\pi^4 \delta^K_{m_0',0} \delta^K_{m_1,-m_1'} \delta^K_{m_2,-m_2'} \delta^K_{m_3,-m_3'}$$

Examining the 3j symbols

Our original expression had three 3j symbols:

$$\begin{pmatrix} l'_0 & l'_1 & l'_{01} \\ m'_0 & m'_1 & -m'_{01} \end{pmatrix} \begin{pmatrix} l'_{01} & l'_2 & l'_3 \\ m'_{01} & m'_2 & m'_3 \end{pmatrix} \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix}$$

If we now plug in the constraints given by the delta functions from the ϕ integral we get:

$$\begin{pmatrix} l'_0 & l'_1 & l'_{01} \\ 0 & -m_1 & -m'_{01} \end{pmatrix} \begin{pmatrix} l'_{01} & l'_2 & l'_3 \\ m'_{01} & -m_2 & -m_3 \end{pmatrix} \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix}$$

Since the bottom elements of the 3j symbol must sum to zero, this adds the additional constraint that $m'_{01} = -m_1$