Parity Asymmetry in the CMB

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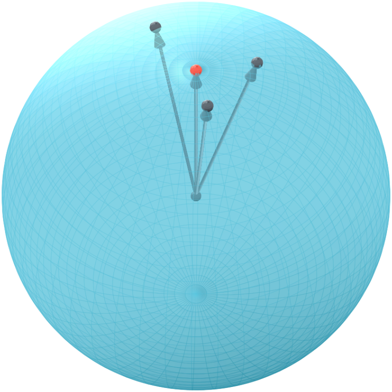
Outline

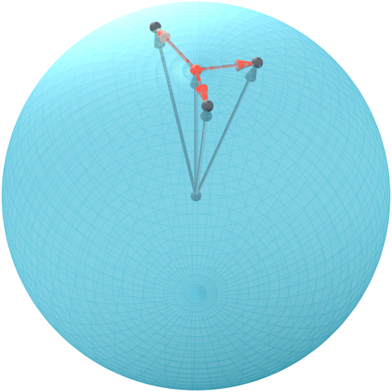
Overview of the problem

The goal is to determine whether or not parity asymmetry can be detected using the Cosmic Microwave Background. To do so we are modelling the 4-point correlation function as a function of 4 points on a sphere, defined by vectors $\vec{x}_0, \vec{x}_1, \vec{x}_2, \vec{x}_3$



Overview of the problem

We then define a set of vectors $\vec{r}_1, \vec{r}_2, \vec{r}_3$ where $\vec{r}_i = \vec{x}_i - \vec{x}_0$



Overview of the probelm

If we then take two correlation functions:

$$\zeta_E(\vec{x}_0, \vec{x}_1, \vec{x}_2, \vec{x}_3)$$

which is the correlation function of the vectors from the Earth the the CMB and

$$\zeta_{x_0}(\vec{r}_1, \vec{r}_2, \vec{r}_3)$$

which is the correlation function centered on point x_0, then our goal is to write one as a function of the other in order to help constrain the possible parity of both

Intitial Calculations

Start by expanding the two correlation functions in terms of isotropic basis functions:

$$\begin{aligned}
\zeta_{x_0}(\vec{r}_1, \vec{r}_2, \vec{r}_3) &= \sum_{\Lambda} \zeta_{\Lambda}(r_1, r_2, r_3)\mathcal{P}_{\Lambda}(\hat{r}_1, \hat{r}_2, \hat{r}_3) \\
\zeta_E(\vec{x}_0, \vec{x}_1, \vec{x}_2, \vec{x}_3) &= \sum_{\Lambda'} \zeta_{\Lambda'}(x_0, x_1, x_2, x_3)\mathcal{P}_{\Lambda'}(\hat{x}_0, \hat{x}_1, \hat{x}_2, \hat{x}_3)
\end{aligned}$$

Where Λ is the set $\{l_1, l_2, l_3\}$ and Λ' is the set $\{l'_0, l'_1, l'_{01}, l'_2, l'_3\}$

Intitial Calculations

We can then set the two functions equal since they are correlation functions describing the same points

$$
\begin{aligned}
\zeta_{x_0}(\vec{r}_1, \vec{r}_2, \vec{r}_3) &= \zeta_E(\vec{x}_0, \vec{x}_1, \vec{x}_2, \vec{x}_3) \\
&\Downarrow \\
\sum_{\Lambda} \zeta_\Lambda(r_1, r_2, r_3)\mathcal{P}_\Lambda(\hat{r}_1, \hat{r}_2, \hat{r}_3) &= \sum_{\Lambda'} \zeta_{\Lambda'}(x_0, x_1, x_2, x_3)\mathcal{P}_{\Lambda'}(\hat{x}_0, \hat{x}_1, \hat{x}_2, \hat{x}_3)
\end{aligned}
$$

Intitial Calculations

We now multiply both sides by $\mathcal{P}^*_{\Lambda''}(\hat{x}_0, \hat{x}_1, \hat{x}_2, \hat{x}_3)$ and then integrate over all orientations $d\hat{x}_0 d\hat{x}_1 d\hat{x}_2 d\hat{x}_3 = d\hat{x}$

$$\int d\hat{x} \sum_{\Lambda} \zeta_{\Lambda}(r_1, r_2, r_3)\mathcal{P}_{\Lambda}(\hat{r}_1, \hat{r}_2, \hat{r}_3)P^*_{\Lambda''}(\hat{x}_0, \hat{x}_1, \hat{x}_2, \hat{x}_3) =$$

$$\int d\hat{x} \sum_{\Lambda'} \zeta_{\Lambda'}(x_0, x_1, x_2, x_3)\mathcal{P}_{\Lambda'}(\hat{x}_0, \hat{x}_1, \hat{x}_2, \hat{x}_3)P^*_{\Lambda''}(\hat{x}_0, \hat{x}_1, \hat{x}_2, \hat{x}_3)$$

Intitial Calculations

Using the orthogonality of isotropic basis functions which states

$$\int d\hat{R}\ \mathcal{P}_{\Lambda}(\hat{R})\mathcal{P}^{*}_{\Lambda'}(\hat{R}) = \delta_{\Lambda_1\Lambda'_1}\delta_{\Lambda_2\Lambda'_2}\dots$$

we can simplify the right hand side of the previous equation:

$$\int d\hat{x} \sum_{\Lambda'} \zeta_{\Lambda'}(x_0, x_1, x_2, x_3)\mathcal{P}_{\Lambda'}(\hat{x}_0, \hat{x}_1, \hat{x}_2, \hat{x}_3)P_{\Lambda''}(\hat{x}_0, \hat{x}_1, \hat{x}_2, \hat{x}_3) = \zeta_{\Lambda''}(x_0, x_1, x_2, x_3)$$

Intitial Calculations

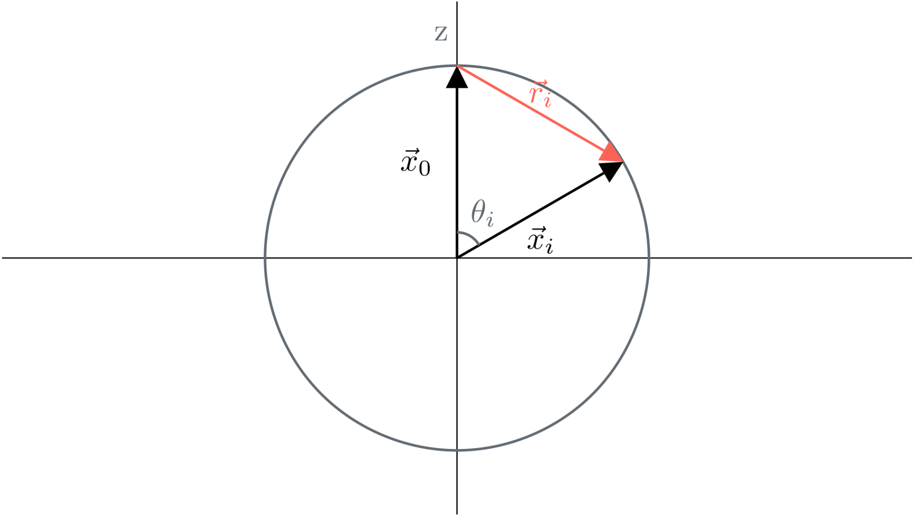
Plugging this back into the previous equation we get the key relation:

$$\boxed{\zeta_{\Lambda''}(x_0, x_1, x_2, x_3) = \int d\hat{x} \sum_{\Lambda} \zeta_{\Lambda}(r_1, r_2, r_3)\mathcal{P}_{\Lambda}(\hat{r}_1, \hat{r}_2, \hat{r}_3)P^{*}_{\Lambda''}(\hat{x}_0, \hat{x}_1, \hat{x}_2, \hat{x}_3)} \tag{1}$$

Solving the integral on the right will give an expression for the coefficients of the correlation function centered on earth in terms of the correlation function centered at x_0

Solving for $\vec{r}$ in terms of $\vec{x}$

Consider a great circle going through the poles at an angle ϕ_i to the x-axis where ϕ_i is the ϕ angle of vector $\vec{x}_i$ for $i = 1, 2, 3$

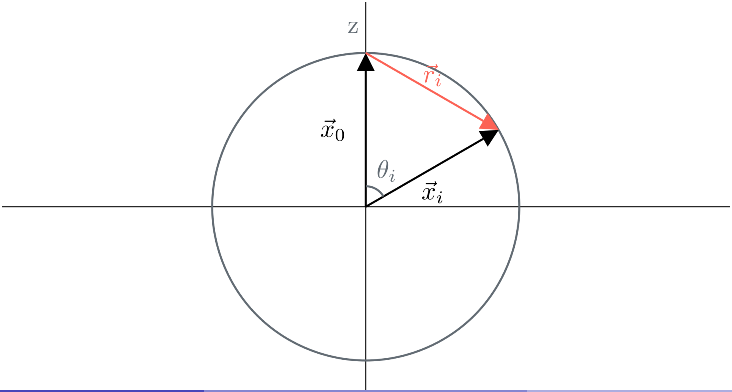


Solving for $\vec{r}$ in terms of $\vec{x}$

Using the law of Cosines and defining the distance to the cmb to be d^* we get:

$$|\vec{r}_i| = \sqrt{d^{*2} + d^{*2} - 2d^*d^*cos(\theta_i)}$$

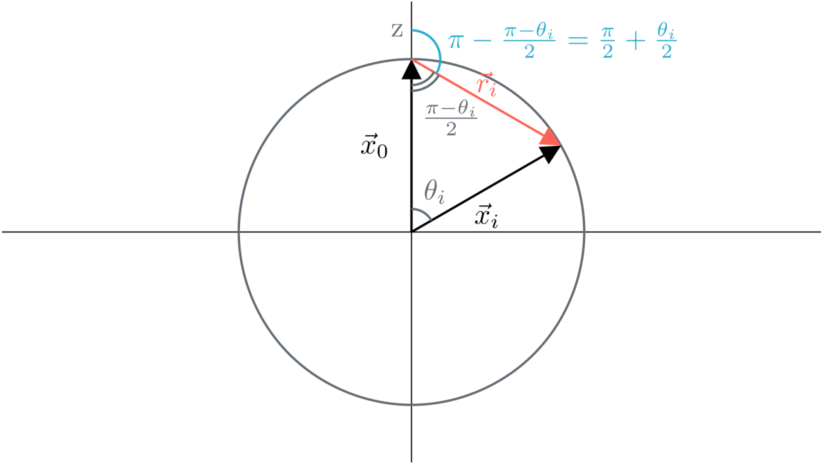
$$\boxed{r_i = \sqrt{2}d^*\sqrt{1 - cos(\theta_i)}}$$



Solving for $\vec{r}$ in terms of $\vec{x}$

Similarly, we can solve for the angular dependence of $\vec{r}_i$

$$\boxed{\theta_{ri} = \frac{\pi}{2} + \frac{\theta_i}{2};\ \phi_{ri} = \phi_i}$$



Solving for the coefficients

Plugging those back into equation 1 we get:

$$\zeta_{\Lambda''}(x_0, x_1, x_2, x_3) =$$

$$\int d\hat{x} \sum_{\Lambda} \zeta_{\Lambda}(\sqrt{2}d^* \sqrt{1 - cos(\theta_1)}, \sqrt{2}d^* \sqrt{1 - cos(\theta_2)}, \sqrt{2}d^* \sqrt{1 - cos(\theta_3)})$$

$$\times \mathcal{P}_{\Lambda}(\frac{\pi}{2} + \frac{\theta_1}{2}, \phi_1; \frac{\pi}{2} + \frac{\theta_2}{2}, \phi_2; \frac{\pi}{2} + \frac{\theta_3}{2}, \phi_3)$$

$$\times \mathcal{P}^*_{\Lambda''}(\theta_0, \phi_0; \theta_1, \phi_1; \theta_2, \phi_2; \theta_3, \phi_3)$$

Different approaches

- Directly solving the integral
- Rotating the spherical harmonics
- Using an addition formula
  - Associated Legendre Polynomials
  - Jacobi Polynomials

Directly solving the integral

We can expand the integral using the definitions of the isotropic basis functions:

$$\mathcal{P}_{l_1 l_2 l_3}(\hat{r}_1, \hat{r}_2, \hat{r}_3) = (-1)^{l_1+l_2+l_3} \sum_{m_1 m_1 m_3} \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \times Y_{l_1 m_1}(\hat{r}_1) Y_{l_2 m_2}(\hat{r}_2) Y_{l_3 m_3}(\hat{r}_3)$$

$$\mathcal{P}_{l_1 l_2 (l_{12}) l_3 l_4}(\hat{r}_1, \hat{r}_2, \hat{r}_3, \hat{r}_4) = (-1)^{l_1+l_2+l_3+l_4} \times \sum_{m_{12}} (-1)^{l_{12}-m_{12}} \sum_{m_1 m_2 m_3 m_4} \begin{pmatrix} l_1 & l_2 & l_{12} \\ m_1 & m_2 & -m_{12} \end{pmatrix} \begin{pmatrix} l_{12} & l_3 & l_4 \\ m_{12} & m_3 & -m_4 \end{pmatrix} \times Y_{l_1 m_1}(\hat{r}_1) Y_{l_2 m_2}(\hat{r}_2) Y_{l_3 m_3}(\hat{r}_3) Y_{l_4 m_4}(\hat{r}_4)$$

Directly solving the integral

Looking at just the angular dependent parts of the integral we have:

$$\int d\theta_0 d\phi_0 d\theta_1 d\phi_1 d\theta_2 d\phi_2 d\theta_3 d\phi_3 sin(\theta_0) sin(\theta_1) sin(\theta_2) sin(\theta_3)$$
$$\times \zeta_\Lambda(\sqrt{2}d^*\sqrt{1-cos(\theta_1)}, \sqrt{2}d^*\sqrt{1-cos(\theta_2)}, \sqrt{2}d^*\sqrt{1-cos(\theta_3)})$$
$$\times Y_{l_0' m_0'}(\theta_0,\phi_0) Y_{l_1' m_1'}(\theta_1,\phi_1) Y_{l_2' m_2'}(\theta_2,\phi_2) Y_{l_3' m_3'}(\theta_3,\phi_3)$$
$$\times Y_{l_1 m_1}(\frac{\theta_1}{2}+\frac{\pi}{2},\phi_1) Y_{l_2 m_2}(\frac{\theta_2}{2}+\frac{\pi}{2},\phi_2) Y_{l_3 m_3}(\frac{\theta_3}{2}+\frac{\pi}{2},\phi_3)$$

Directly solving the integral

We can further separate the integral into ϕ and θ dependent parts by using the following definition of sperical harmonics:

$$Y_l^m(\theta, \phi) = \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}} P_l^m(cos\theta)e^{im\phi}$$

Solving the ϕ integral

Looking at just the ϕ dependent terms:

$$\int_0^{2\pi} d\phi_0 d\phi_1 d\phi_2 d\phi_3 e^{im'_0\phi_0} e^{im'_1\phi_1 + im_1\phi_1} e^{im'_2\phi_2 + im_2\phi_2} e^{im'_3\phi_3 + im_3\phi_3}$$

$$= \left[\frac{1}{im'_0} e^{im'_0\phi_0} \Big|_0^{2\pi} \right] \prod_j \left[\frac{1}{i(m'_j + m_j)} e^{i(m'_j + m_j)\phi_j} \Big|_0^{2\pi} \right] \; for\ j = 1, 2, 3$$

Each of the above integrals will always be zero since all m's are integers unless $m'_0 = 0$ and $m'_j + m_j = 0$ in which case each term becomes 2π so the total ϕ integral becomes:

$$16\pi^4 \delta^K_{m'_0,0} \delta^K_{m_1,-m'_1} \delta^K_{m_2,-m'_2} \delta^K_{m_3,-m'_3}$$

Examining the 3j symbols

Our original expression had three 3j symbols:

$$\begin{pmatrix} l'_0 & l'_1 & l'_{01} \\ m'_0 & m'_1 & -m'_{01} \end{pmatrix} \begin{pmatrix} l'_{01} & l'_2 & l'_3 \\ m'_{01} & m'_2 & m'_3 \end{pmatrix} \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix}$$

If we now plug in the constraints given by the delta functions from the ϕ integral we get:

$$\begin{pmatrix} l'_0 & l'_1 & l'_{01} \\ 0 & -m_1 & -m'_{01} \end{pmatrix} \begin{pmatrix} l'_{01} & l'_2 & l'_3 \\ m'_{01} & -m_2 & -m_3 \end{pmatrix} \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix}$$

Since the bottom elements of the 3j symbol must sum to zero, this adds the additional constraint that $m'_{01} = -m_1$

Solving the θ integral

Looking at the θ integral, we have:

$$\int_0^\pi d\theta_0 d\theta_1 d\theta_2 d\theta_3 sin(\theta_0) sin(\theta_1) sin(\theta_2) sin(\theta_3)$$
$$\times \zeta_\Lambda(\sqrt{2}d^*\sqrt{1-cos(\theta_1)}, \sqrt{2}d^*\sqrt{1-cos(\theta_2)}, \sqrt{2}d^*\sqrt{1-cos(\theta_3)})$$
$$\times P_{l_0'}^{m_0'}(cos\theta_0) P_{l_1'}^{m_1'}(cos\theta_1) P_{l_2'}^{m_2'}(cos\theta_2) P_{l_3'}^{m_3'}(cos\theta_3)$$
$$\times P_{l_1}^{m_1}(cos(\frac{\theta_1}{2}+\frac{\pi}{2})) P_{l_2}^{m_2}(cos(\frac{\theta_2}{2}+\frac{\pi}{2})) P_{l_3}^{m_3}(cos(\frac{\theta_3}{2}+\frac{\pi}{2}))$$

Solving the θ integral

Since ζ_Λ is an unknown function, we cannot integrate $\theta_1, \theta_2, \theta_3$ in this form. To deal with this we can express ζ_Λ as an inverse Fourier Transform leading to:

$$\zeta_\Lambda(\sqrt{2}d^*\sqrt{1 - cos(\theta_1)}, \sqrt{2}d^*\sqrt{1 - cos(\theta_2)}, \sqrt{2}d^*\sqrt{1 - cos(\theta_3)})$$
$$= \int dk_1 dk_2 dk_3 j_{l_1}(k_1 r_1) j_{l_2}(k_2 r_2) j_{l_3}(k_3 r_3) \tilde{\zeta}_\Lambda(k1, k2, k3)$$

Where j_l are the spherical bessel functions of the first kind and $r_i = \sqrt{2}d^*\sqrt{1 - cos\theta_i}$

Rotating spherical harmonics

Another approach to equation 1 is to think of $Y_l^m(\frac{\theta}{2} + \frac{\pi}{2})$ as a rotation of $Y_l^m(\theta)$ by an angle of $\frac{-\theta}{2} + \frac{\pi}{2}$
Using the theorem:

$$Y_l^m(\theta', \phi') = \sum_{m'=-l}^{l} D^l_{m'm}(R) Y_l^{m'}(\theta, \phi)$$

Where $D^l_{m'm}(R)$ is a Wigner D-matrix of some rotation R.

Rotating spherical harmonics

Using the theorem:

$$D^{l}_{ms}(\alpha, \beta, \gamma) = (-1)^{s}\sqrt{\frac{4\pi}{2l+1}}\ {}_{-s}Y_{l}^{*m}(\beta, \alpha)e^{-is\gamma}$$

we can rewrite the D-matrices as spin-weighted spherical harmonics. When doing this approach however, since we are rotating by $\beta = \frac{-\theta}{2} + \frac{\pi}{2}$ we still have an argument in terms of $\frac{\theta}{2}$

Addition theorems

A third approach was inspired by the spherical harmonic addition theorem:

$$P_l(cos(\gamma)) = \frac{4\pi}{2l+1} \sum_{m=-l}^{l} (-1)^m Y_l^m(\theta_1, \phi_1) Y_l^m(\theta_2, \phi_2)$$

where γ is the angle between the directions of the (θ_1, ϕ_1) and (θ_2, ϕ_2)
If we could find a similar theorem for associated Legendre polynomials, we may be able change the form of $P_l^m(cos(\frac{\theta/2}{2} + \frac{\pi}{2}))$ into different form.

Addition theorems

B. E. Johnson (1964) proposed an addition theorem for associated Legendre polynomials, however, the domain of his function only includes real values greater than 0, which is not necessarily true for $cos(\frac{\theta/2}{2} + \frac{\pi}{2})$ Similarly, Koornwinder (1973) gives an addition theorem for Jacobi polynomials, however, we don't believe associated Legendre polynomials are a special case of Jacobi polynomials

Further Work

- We assumed we can set $\hat{x}_0 = \hat{z}$, however, we then integrate over $\hat{x}_0$
  - Define a rotation from absolute to relative coordinates $R\hat{x}_0 = \hat{z}$
  - How does this impact the integral?
- How do we solve the θ integral?
- Can we derive more information from the 3j symbols?