Contents

1 Combinatorics

The combinatorics package houses functions related to combinatorics.

1.1 `divisorsInBC`

1.1.1 Usage

The `divisorsInBC(n, k, p)` function counts how many factors of p (where p is prime) there are in $\binom{n}{k}$.

1.1.2 Implementation

First, some notation. Let x be a positive integer. Define q_x, r_x and q'_x, r'_x (where we have $0 \leq r_x, r'_x < p^x$) as

$$n = p^x q_x + r_x,$$
$$k = p^x q'_x + r'_x.$$

Note that q_x, r_x, q'_x, r'_x all exist and are unique by the division algorithm.

Now, note that

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

To find the number of divisors of p in $\binom{n}{k}$, we will count the divisors of p in the numerator and in the denominator of the above fraction, then subtract.

Note that the desired difference is

$$\left(\lfloor\frac{n}{p}\rfloor + \lfloor\frac{n}{p^2}\rfloor + \cdots\right) - \left(\lfloor\frac{k}{p}\rfloor + \cdots\right) - \left(\lfloor\frac{n-k}{p} + \cdots\rfloor\right).$$

Note that, in the above sums, the $\cdots$ does not indicate (as it normally does) that the sum goes on forever; instead, it only indicates that the sum goes on until one term becomes 0 (as all subsequent terms will then be 0). This notation is choosen for succintness. It's important to distinguish that this is a finite sum, as soon we will do some algebra with it that would be harder to justify with infinite sums.

Note that

$$\left(\lfloor\frac{n}{p}\rfloor + \lfloor\frac{n}{p^2}\rfloor + \cdots\right) - \left(\lfloor\frac{k}{p}\rfloor + \cdots\right) - \left(\lfloor\frac{n-k}{p}\rfloor + \cdots\rfloor\right) = \left(\lfloor\frac{n}{p}\rfloor - \left(\lfloor\frac{k}{p}\rfloor + \lfloor\frac{n-k}{p}\rfloor\right)\right) + \cdots$$

Thus, it suffices to determine

$$\lfloor\frac{n}{p^x}\rfloor - \left(\lfloor\frac{k}{p^x}\rfloor + \lfloor\frac{n-k}{p^x}\rfloor\right)$$

for each integer $x > 0$ such that $n > p^x$.

Using the notation defined earlier, note that

$$\lfloor \frac{n}{p^x} \rfloor = q_x,$$

$$\lfloor \frac{k}{p^x} \rfloor = q'_x,$$

$$\lfloor \frac{n-k}{p^x} \rfloor = q_x - q'_x + \lfloor \frac{r_x - r'_x}{p^x} \rfloor.$$

Note that

$$-p^x < r_x - r'_x < p^x,$$

and thus

$$\lfloor \frac{r_x - r'_x}{p^x} \rfloor = 0 \text{ or } \lfloor \frac{r_x - r'_x}{p^x} \rfloor = -1.$$

Thus,

$$\lfloor \frac{n}{p^x} \rfloor - \left(\lfloor \frac{k}{p^x} \rfloor + \lfloor \frac{n-k}{p^x} \rfloor \right)$$

will be either 0 or 1, and it will be 1 if and only if $r_x < r'_x$.

Thus, the number of factors of p in $\binom{n}{k}$ is precisely the number of x such that $r_x < r'_x$. The function simply iterates over each x such that $p^x < n$, counting whenever we have $r_x < r'_x$.

1.1.3 Possible Updates

If I added a prime factorization function to this library, I could use that to extend this method to p that aren't prime.

It may be more efficient to first compute the largest x such that $p^x < n$, iterate backwards, then stop as soon as we reach some y such that $r_y < r'_y$, as all subsequent residues will follow in that pattern (this can be proven by looking at the base p representations of numbers involved).