The Delta-Method applied to Sobol' indices

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Abstract

We explore the use of the Delta-method in order to estimate the Sobol' indices.

Contents

1 Convergence in distribution

Definition 1. *(*Convergence in distribution*) Assume that $X_1, X_2, \ldots$ is a sequence of real-valued random variables with cumulative distribution functions $\{F_n\}_{n\geq 0}$. Assume that X is a real-valued random variable with cumulative distribution function $\{F\}_{n\geq 0}$. The sequence X_n converges in distribution to X if:*

$$\lim_{n\to\infty} F_n(X_n) = F(x).$$

for any $x \in \mathbb{R}$ at which F is continuous. In this case, we write:

$$X_n \xrightarrow{D} X.$$

The following theorem gives an example of such convergence.

Theorem 1. *(*Maximum of uniform random numbers*) Assume that $X_1, X_2, \ldots$ are independent random numbers such that $X_n \sim U(0,1)$. Let Y_n be the maximum:*

$$Y_n = \max_{1\leq i\leq n} X_i.$$

Therefore the sequence $n(1-Y_n)$ converges in distribution to an exponential random variable, i.e.:

$$n(1-Y_n) \xrightarrow{D} \mathcal{E}(1).$$

Proof. By definition, the exponential distribution with rate λ has the cumulative distribution function:

$$F(y) = 1 - \exp(-\lambda y),$$

for any real $y \geq 0$. With $\lambda = 1$ we must prove that:

$$F(y) = 1 - \exp(-y),$$

for any real $y \geq 0$. Let F_n be the cumulative distribution function of the random variable $n(1 - Y_n)$. By definition of the cumulative distribution function,

$$\begin{aligned} F_n(z) &= P(n(1 - Y_n) \leq z) \\ &= P(1 - Y_n \leq z/n) \\ &= P(1 - z/n \leq Y_n) \\ &= 1 - P(Y_n \leq 1 - z/n), \end{aligned}$$

for any $z \in \mathbb{R}$. However, the cumulative distribution function of the maximum is:

$$\begin{aligned} P(Y_n \leq y) &= P(X_1 \leq y, X_2 \leq y, ..., X_n \leq y) \\ &= P(X_1 \leq y)P(X_2 \leq y) \cdots P(X_n \leq y) \text{ (by independence)} \\ &= yy \cdots y \text{ (by definition of the C.D.F. of a uniform random variable)} \\ &= y^n, \end{aligned}$$

for any $y \in [0, 1]$. Therefore,

$$F_n(z) = 1 - (1 - z/n)^n$$

for any $z \in \mathbb{R}$. But

$$\lim_{n \to \infty} (1 + z/n)^n = \exp(z)$$

for any $z \in \mathbb{R}$. Therefore,

$$\lim_{n \to \infty} F_n(z) = 1 - \exp(-z)$$

for any $z \in \mathbb{R}$ which concludes the proof. □

2 Convergence in probability

Definition 2. *(*Convergence in probability*) Assume that $X_1, X_2, ...$ is a sequence of real-valued random variables. The sequence X_n converges in probability to X if, for any $\epsilon > 0$:*

$$\lim_{n \to \infty} P(|X_n - X| > \epsilon) = 0.$$

In this case, we write:

$$X_n \xrightarrow{P} X.$$

Theorem 2. *Assume that X and Y are random variables. Let $y \in \mathbb{R}$ and let $\epsilon > 0$. Therefore*

$$P(Y \leq y) \leq P(X \leq y + \epsilon) + P(|Y - X| > \epsilon)$$

Proof.

$$P(Y \leq y) = P(Y \leq y, X \leq y + \epsilon) + P(Y \leq y, X > y + \epsilon).$$

But $P(Y \leq y, X \leq y + \epsilon) \leq P(X \leq y + \epsilon)$, which implies :

$$\begin{aligned} P(Y \leq y) &\leq P(X \leq y + \epsilon) + P(Y \leq y, X > y + \epsilon) \\ &\leq P(X \leq y + \epsilon) + P(Y - X \leq y - X, y + \epsilon < X) \\ &\leq P(X \leq y + \epsilon) + P(Y - X \leq y - X, y - X < -\epsilon). \end{aligned}$$

Moreover, we have $P(Y - X \leq y - X, y - X < -\epsilon) \leq P(Y - X < -\epsilon)$ which implies :

$$P(Y \leq y) \leq P(X \leq y + \epsilon) + P(Y - X < -\epsilon).$$

By definition of a probability, we have $P(Y - X > \epsilon) \geq 0$. Therefore,

$$\begin{aligned} P(Y \leq y) &\leq P(X \leq y + \epsilon) + P(Y - X < -\epsilon) + P(Y - X > \epsilon) \\ &\leq P(X \leq y + \epsilon) + P(|Y - X| < \epsilon) \end{aligned}$$

which concludes the proof. □

Theorem 3. *(*Convergence in probability implies convergence in distribution*) Assume that $X_1, X_2, ...$ is a sequence of real-valued random variables. If*

$$X_n \xrightarrow{P} X$$

then

$$X_n \xrightarrow{D} X.$$

Proof. TODO □

3 Delta method

Theorem 4. *(*Slutky's theorem*) Let X_n and Y_n be two sequences of real random variables such that:*

$$X_n \xrightarrow{D} X \quad \text{and} \quad Y_n \xrightarrow{D} c,$$

where c is a non-random constant. Therefore,

$$X_n Y_n \xrightarrow{D} cX.$$

Proof. TODO □

Theorem 5. *(*Delta-method*) Assume that $X_1, X_2, ...$ is a sequence of real-valued random variables so that*

$$\sqrt{n}(X_n - \theta) \xrightarrow{D} \mathcal{N}(0, \sigma^2). \tag{1}$$

Assume that g is a real function. Let $\theta \in \mathbb{R}$. Suppose that $g \in C^1(\mathbb{R})$ and that $g'(\theta) \neq 0$. Therefore,

$$\sqrt{n}(g(X_n) - g(\theta)) \xrightarrow{D} \mathcal{N}(0, \sigma^2 g'(\theta)^2).$$

Proof. The Taylor expansion of g at the point θ implies that there exists a $\tilde{\theta}$ between X_n and θ such that:

$$g(X_n) = g(\theta) + g'(\tilde{\theta})(X_n - \theta) + O((X_n - \theta)^2), \quad X_n \to \theta.$$

This implies:

$$\sqrt{n}(g(X_n) - g(\theta)) = g'(\theta)\sqrt{n}(X_n - \theta) + O((X_n - \theta)^2), \quad X_n \to \theta.$$

Let Y_n and Z_n be the sequences of random variables defined by the equations:

$$Y_n = g'(\theta)\sqrt{n}(X_n - \theta) \text{ and } Z_n = O((X_n - \theta)^2).$$

The properties of the gaussian distribution and the equation 1 imply:

$$Y_n = g'(\theta)\sqrt{n}(X_n - \theta) \xrightarrow{D} \mathcal{N}(0, \sigma^2 g'(\theta)^2).$$

Furthermore,

$$Z_n = O((X_n - \theta)^2) \xrightarrow{D} 0.$$

In order to conclude the proof, we apply Slutky's theorem to Y_n and Z_n. □

4 References

Most of the results in this document can be found in [1].

References

[1] A. W. van der Vaart. *Asymptotic Statistics*. Cambridge Series in Statistical and Probabilistic Mathematics, 2000.