MA 578 — Bayesian Statistics

Homework 6

(Due: Tuesday, 11/26/19)

1. BDA problem 14.1. Also, conduct some posterior predictive checks to assess your model.
2. BDA problem 14.11, 14.12, and 14.13.
3. Consider the Bayesian linear model with semi-conjugate prior on the p coefficients β:

$$\begin{aligned} \mathbf{y} \mid \beta, \sigma^2 &\sim N(X\beta, \sigma^2 I_n) \\ \beta \mid \sigma^2 &\sim N(\beta_0, \sigma^2 \Sigma_0) \\ \sigma^2 &\sim \mathsf{Inv}\text{-}\chi^2(\nu, \tau^2). \end{aligned}$$

(a) Show that the joint posterior on β and σ^2 can be written as

$$\mathbb{P}(\beta, \sigma^2 \mid \mathbf{y}) \propto (\sigma^2)^{-\left(\frac{n+p+\nu}{2}+1\right)} \exp\left\{-\frac{1}{2\sigma^2}\left[\nu\tau^2 + \mathrm{RSS}(\widehat{\beta}) + (\widehat{\beta}-\beta_0)^\top \Sigma_0^{-1}(\widehat{\beta}-\beta_0)\right]\right\}$$
$$\times \exp\left\{-\frac{1}{2\sigma^2}(\beta-\widehat{\beta})^\top \Sigma_\beta^{-1}(\beta-\widehat{\beta})\right\},$$

where $\mathrm{RSS}(\beta) = (\mathbf{y}-X\beta)^\top(\mathbf{y}-X\beta)$, $\Sigma_\beta^{-1} = X^\top X + \Sigma_0^{-1}$ and $\widehat{\beta} = \Sigma_\beta(X^\top \mathbf{y} + \Sigma_0^{-1}\beta_0)$. (*Hint:* center with respect to the MLE β^* and note that $X^\top(\mathbf{y}-X\beta^*) = 0$.)

(b) Marginalize out β and show that

$$\sigma^2 \mid \mathbf{y} \sim \mathsf{Inv}\text{-}\chi^2\left(\nu+n, \frac{\nu\tau^2 + \mathrm{RSS}(\widehat{\beta}) + (\widehat{\beta}-\beta_0)^\top \Sigma_0^{-1}(\widehat{\beta}-\beta_0)}{\nu+n}\right).$$

(c) From now on, assume that $\beta_0 = 0$. Use a bit of linear algebra to show that

$$\mathrm{RSS}(\widehat{\beta}) + \widehat{\beta}^\top \Sigma_0^{-1}\widehat{\beta} = \mathbf{y}^\top(I_n - H)\mathbf{y},$$

where $H = X(X^\top X + \Sigma_0^{-1})^{-1}X^\top$ is the ("regularized") hat matrix, and so this term can be seen as a regularized RSS.

(d) $*$ Finally, marginalize out σ^2 to show that

$$\mathbb{P}(\mathbf{y}) \propto |I_n - H|^{1/2}\left(1 + \frac{\mathbf{y}^\top(I_n-H)\mathbf{y}}{\nu\tau^2}\right)^{-(n+\nu)/2},$$

that is, $\mathbf{y} \sim t_\nu(0, \tau^2(I_n-H)^{-1})$, a multivariate t distribution.