Name: KEY

MA 578 — Bayesian Statistics
Fall 2019

Midterm Exam

Thursday 10/17/19, 12:30–1:45 PM

11

1. Extreme value distributions such as the *Weibull* distribution are often used to model times to events in survival and reliability analyses, and have thus found many applications in clinical studies and engineering. Suppose you observe (data) times $y_i \mid \theta \overset{\text{iid}}{\sim} \text{Weibull}(\theta^{1/k}, k)$, for $i = 1, \dots, n$, with a *known* shape parameter k. The density of a single observation is

$$\mathbb{P}(y_i \mid \theta) = \frac{k y_i^{k-1}}{\theta} \exp\left\{ -\frac{y_i^k}{\theta} \right\}.$$

3 (a) Write the full likelihood for the whole data. What is the conjugate prior for θ?

2 (b) Derive the posterior distribution for θ. How do you update the parameters in the prior?

3 (c) Show that the posterior mode θ^* can be written as a linear combination of the MLE $\widehat{\theta}$ for θ and the prior mode. Moreover, show that $\theta^* \to \widehat{\theta}$ as the sample size grows.

3 (d) Knowing that $\mathbb{E}[y_i^k \mid \theta] = \theta$, find Jeffreys prior. Does it correspond to a specific parameterization of your conjugate prior?

(A) $\mathbb{P}(y \mid \theta) = \dfrac{k^n \prod_i y_i^{k-1}}{\theta^n} \exp\left\{-\dfrac{\sum_i y_i^k}{\theta}\right\} \propto \theta^{-n} \exp\left\{-\dfrac{\sum y_i^k}{\theta}\right\}$ (1) (1)

$\mathbb{P}(\theta) \propto \theta^{-\alpha} \exp\{-\beta/\theta\} \sim F(\alpha, \beta)$ (1) or $\mathbb{P}(\theta) \propto \theta^{-\alpha-1} \exp\{-\beta/\theta\}$, $\theta \sim \text{Inv-Gamma}(\alpha, \beta)$

(B) $\mathbb{P}(\theta \mid y) \propto \mathbb{P}(y \mid \theta)\mathbb{P}(\theta) \propto \theta^{-(n+\alpha)-1} \exp\left\{-\dfrac{\sum_i y_i^k + \beta}{\theta}\right\}$ (1) $\quad \alpha \to \alpha + n$, $\beta \to \beta + \sum_i y_i^k$ (1)

$\Rightarrow \theta \mid y \sim \text{Inv-Gamma}(\alpha + n, \beta + \sum_i y_i^k)$

(C) $\theta^* = \dfrac{\beta + \sum_i y_i^k}{\alpha + n + 1}$ (1) $= \underbrace{\dfrac{\sum_i y_i^k}{n}}_{\hat\theta} \cdot \dfrac{n}{\alpha + n + 1}$ (1) $+ \underbrace{\dfrac{\beta}{\alpha + 1}}_{\theta_0^*} \dfrac{\alpha + 1}{\alpha + n + 1} \xrightarrow[n \to \infty]{} \hat\theta$ (1)

(D) $\ell(\theta) = -n \log \theta - \dfrac{\sum y_i^k}{\theta}$ (1) $\Rightarrow \ell'(\theta) = -\dfrac{n}{\theta} + \dfrac{\sum y_i^k}{\theta^2}$

$\ell''(\theta) = \dfrac{n}{\theta^2} - 2\dfrac{\sum y_i^k}{\theta^3}$ (1) $\Rightarrow \mathbb{E}[-\ell''(\theta)] = -\dfrac{n}{\theta^2} + \dfrac{2n\theta}{\theta^3} = \dfrac{n}{\theta^2}$

$\mathbb{P}_J(\theta) \propto I(\theta)^{1/2} \propto \theta^{-1} \sim \theta \sim F(1, 0)$ or (1) $\Rightarrow \theta \sim \text{Inv-Gamma}(0, 0)$

Name: KEY

2. To test if your friend knows classical music, you setup the following game: you play a random piece of classical music and ask your friend to guess the author; if she guesses correctly, you play a new piece, and so on, until she makes a mistake. Assume that your friend has the same probability θ of correctly guessing the author of a piece of classical music and that each guess is independent. Thus, if in the i-th round of the game she got y_i correct guesses (and, of course, one wrong guess at the end), the likelihood is $\mathbb{P}(y_i \mid \theta) = \theta^{y_i}(1-\theta)$. You repeated this game for $n = 5$ rounds and, overall, your friend got $\sum_{i=1}^{n} y_i = 20$ correct guesses.

3 (a) Assume that $\theta \sim \mathsf{Beta}(1/2, 0)$ and derive the posterior distribution for θ, writing the parameters as a function of the data.

3 (b) Use an asymptotic approximation to provide an approximate 95% posterior interval for θ. Based on this interval, do you have evidence that your friend knows classical music or is she just randomly guessing?

3 (c) What is the (posterior) probability that if your friend plays one more round of the game that she makes no right guesses?

3+ (d) (*Optional*) Show that $\mathsf{Beta}(1/2, 0)$ is Jeffreys prior. Hint: $\mathbb{E}[y_i \mid \theta] = \theta/(1-\theta)$.

(A) $\mathbb{P}(y|\theta) = \theta^{\sum y_i}(1-\theta)^n$ (1)

$\mathbb{P}(\theta|y) \propto \mathbb{P}(y|\theta)\mathbb{P}(\theta) \propto \theta^{\sum y_i}(1-\theta)^n \theta^{-1/2}(1-\theta)^{-1}$ (1)

$\Rightarrow \theta^{\sum y_i + 1/2 - 1}(1-\theta)^{n-1} \Rightarrow \boxed{\theta|y \sim \mathrm{Beta}(\textstyle\sum_i y_i + 1/2, n)}$ (1)

(B) SINCE $\theta \sim \mathrm{Beta}(\alpha,\beta) \approx N\left(\theta^*, \frac{\theta^*(1-\theta^*)}{\alpha+\beta-2}\right)$, $\theta^* = \frac{\alpha-1}{\alpha+\beta-2}$, (1)

$\theta|y \approx N\left(\theta^*, \frac{\theta^*(1-\theta^*)}{n+\sum_i y_i + 1/2 - 2}\right)$, $\theta^* = \frac{\sum_i y_i + 1/2 - 1}{n+\sum_i y_i + 1/2 - 2} = \frac{20+1/2-1}{5+20+1/2-2} = \frac{19.5}{23.5} = 0.817$ (1) (1)

$\frac{\theta^*(1-\theta^*)}{5+20+1/2-2} = 0.0064$ THUS 95% POST. INT. $0.817 \pm 2\sqrt{0.0064}$

$0.817 \pm 0.16 = (0.657, 0.977)$

(C) $\mathbb{P}(\tilde{y}|y) = \int_0^1 \theta^{\tilde{y}}(1-\theta) \frac{1}{B(\sum_i y_i + 1/2, n)} \theta^{\sum y_i + 1/2 - 1}(1-\theta)^{n-1}\, d\theta$ (1)

$= \frac{B(\sum_i y_i + 1/2 + \tilde{y}, n+1)}{B(\sum_i y_i + 1/2, n)}$ (1)

$\mathbb{P}(\tilde{y}=0|y) = \frac{B(\sum y_i + 1/2, n+1)}{B(\sum_i y_i + 1/2, n)} = \frac{\frac{\cancel{\Gamma(\sum y_i + 1/2)}\Gamma(n+1)}{\Gamma(\sum y_i + 1/2 + n+1)}}{\frac{\cancel{\Gamma(\sum y_i + 1/2)}\Gamma(n)}{\Gamma(\sum y_i + 1/2 + n)}} = \frac{n}{\sum y_i + 1/2 + n}$ (1)

$= \frac{5}{20+\frac{1}{2}+5} \approx 0.2$