Boston University

Department of Electrical and Computer Engineering

EC505 STOCHASTIC PROCESSES

Exam 1 Summary

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1. **Probability and Random Variables:**

- Axiomatic definition of probability: Triple $(\Omega, \mathcal{F}, P)$, Ω = Set of outcomes, $\mathcal{F}$ = Set of events, and $P(\cdot)$ = Probability measure which satisfies axioms:
  (a) $P(\Omega) = 1$
  (b) $P(A) \geq 0, \forall\, A \in \mathcal{F}$.
  (c) $P(\cup A_i) = \sum P(A_i)$ if $A_i \cap A_j = \emptyset$, $i \neq j$.
- One Random Variable
  - Probability Distribution Function (PDF): $P_X(x) \equiv P(X \leq x)$
    * $P[x_1 < X \leq x_2] = P_X(x_2) - P_X(x_1)$
  - Probability Density Function (pdf): $p_X(x) = \dfrac{dP_X(x)}{dx}$
    * $P(A) = \int_A p_X(x)\, dx$
  - $E[g(X)] \equiv \int_{-\infty}^{\infty} g(x) p_X(x)\, dx$
  - Mean: $m_X = E[X]$
  - n-th moment: $E[X^n] = \int_{-\infty}^{\infty} x^n p_X(x)\, dx$
  - Variance: $\sigma_X^2 = E\left[(X - m_X)^2\right] = E[X^2] - (E[x])^2$
- Two Random Variables X, Y
  - Joint Distribution Function: $P_{X,Y}(x,y) \equiv P[(X \leq x) \cap (Y \leq y)]$
  - Joint Density Function: $p_{X,Y}(x,y) = \dfrac{\partial^2 P_{X,Y}(x,y)}{\partial x \partial y}$
  - Marginal pdf: $p_X(x) = \int_{-\infty}^{\infty} p_{X,Y}(x,y)\, dy$
  - Conditional Density: $p_{X|Y}(x \mid y) = \dfrac{p_{X,Y}(x,y)}{p_Y(y)} = \dfrac{p_{Y|X}(y|x) p_X(x)}{p_Y(y)}$
  - Bayes' Rule: $p_{X|Y}(x \mid y) = \dfrac{p_{Y|X}(y \mid x) p_X(x)}{p_Y(y)}$
  - X, Y <u>statistically independent</u> $\Leftrightarrow$ $p_{X,Y}(x,y) = p_X(x) p_Y(y)$
  - Expected value: $E[g(X,Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x,y) p_{X,Y}(x,y)\, dx\, dy$
  - Correlation: $E[XY]$
  - Covariance: $\sigma_{XY} = E[(X - m_X)(Y - m_Y)] = E[XY] - m_X m_Y = \mathrm{Cov}(X,Y)$
  - X, Y uncorrelated $\Leftrightarrow \sigma_{XY} = 0 \Leftrightarrow E[XY] = E[X]E[Y]$
  - X, Y orthogonal $\Leftrightarrow E[XY] = 0$
  - $(X,Y$ Independent$)$ $\underset{\not\Longleftarrow}{\Longrightarrow}$ $(X,Y$ Uncorrelated$)$,
  - Conditional Expectation (mean) of X given Y: $E[X \mid Y] = \int_{-\infty}^{\infty} x p_{X|Y}(x|y)\, dx$
- Random Vectors

- $\underline{X} = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_N \end{bmatrix}, \underline{Y} = \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_N \end{bmatrix}$
- Joint distribution function:
  $P_{\underline{X}}(\underline{x}) = P[(X_1 \le x_1), \ldots, (X_N \le x_N)]$ or $P_{\underline{X},\underline{Y}}(\underline{x},\underline{y}) = P[(X_1 \le x_1), \ldots, (X_N \le x_N), (Y_1 \le y_1), \ldots, (Y_N \le y_N)]$
- Joint density: $p_{\underline{X}}(\underline{x}) = \dfrac{\partial^N P_{\underline{X}}(\underline{x})}{\partial x_1 \ldots \partial x_N}$ or $p_{\underline{X},\underline{Y}}(\underline{x},\underline{y}) = \dfrac{\partial^{2N} P_{\underline{X},\underline{Y}}(\underline{x},\underline{y})}{\partial x_1 \ldots \partial x_N, \partial y_1 \ldots \partial y_N}$
- $\underline{X}$, $\underline{Y}$ independent if $p_{\underline{X},\underline{Y}}(\underline{x},\underline{y}) = p_{\underline{X}}(\underline{x}) p_{\underline{Y}}(\underline{y})$
- Conditional Density: $p_{\underline{X}|\underline{Y}}(\underline{x}|\underline{y}) = \dfrac{p_{\underline{X},\underline{Y}}(\underline{x},\underline{y})}{p_{\underline{Y}}(\underline{y})} = \dfrac{p_{\underline{Y}|\underline{X}}(\underline{y}|\underline{x}) p_{\underline{X}}(\underline{x})}{p_{\underline{Y}}(\underline{y})}$
- $E[\underline{g}(\underline{x})] = \begin{pmatrix} E[g_1(\underline{x})] \\ \vdots \\ E[g_N(\underline{x})] \end{pmatrix} = \int_{-\infty}^{\infty} \underline{g}(\underline{x}) f(\underline{x})\, d\underline{x}$
- Mean Vector: $E[\underline{X}] = m_{\underline{X}} = \begin{pmatrix} E[X_1] \\ \vdots \\ E[X_N] \end{pmatrix}$
- Covariance Matrix:
  $\mathrm{Cov}(\underline{X},\underline{X}) = \Lambda_{\underline{XX}} = \Sigma_{\underline{XX}} = E[(\underline{X} - m_X)(\underline{X} - m_X)^T] = E[\underline{XX}^T] - m_{\underline{X}} m_{\underline{X}}^T$
- Convariance matrix constraints: $\Sigma_{\underline{XX}}$ must be a symmetric, PSD matrix
- Cross-Covariance Matrix:
  $\mathrm{Cov}(\underline{X},\underline{Y}) = \Lambda_{\underline{XY}} = \Sigma_{\underline{XY}} = E[(\underline{X} - m_{\underline{X}})(\underline{Y} - m_{\underline{Y}})^T] = E[\underline{XY}^T] - m_{\underline{X}} m_{\underline{Y}}^T$
- Conditional Mean: $E[\underline{X}|\underline{Y}] = \int_{-\infty}^{\infty} \underline{x}\, f(\underline{x}|\underline{y}) d\underline{x}$
- Conditional Covariance: $\Sigma_{X|Y} = \int_{-\infty}^{\infty} (\underline{x} - E[\underline{x}|\underline{y}])(\underline{x} - E[\underline{x}|\underline{y}])^T f(\underline{x}|\underline{y})\, d\underline{x}$
- Uncorrelated: $\Sigma_{\underline{XY}} = 0, \Rightarrow E[\underline{XY}^T] = E[\underline{X}]E[\underline{Y}]^T$.
- Orthogonal: $E[\underline{XY}^T] = 0$.
- Gaussian Random Vectors: $\underline{a}^T \underline{X}$ is a Gaussian random variable for all $\underline{a}$.

2. **Characterization and Manipulation of Random Processes**

- Complete Characterization of Random Processes: In terms of N-th order probability distribution or density functions $p_{X(t_1),X(t_2),\cdots,X(t_N)}(x_1, x_2, \cdots, x_N)$ for all t_i, N.
- Joint pdfs
- Marginal pdfs
- Conditional pdfs
- Mean: $m_x(t) = E[X(t)]$
- Autocorrelation: $R_{XX}(t_1,t_2) = E[X(t_1)X(t_2)]$
- Autocovariance: $K_{XX}(t_1,t_2) = C_{XX}(t_1,t_2) = E[X(t_1)X(t_2)] - E[X(t_1)]E[X(t_2)]$
- Constraints on $R_{XX}(t_1,t_2)$, $K_{XX}(t_1,t_2)$: $\mathrm{Var}\left[\int a(t)X(t)\, dt\right] \ge 0$
- General Expectation $E[g(x)]$
- Conditional Expectation $E[g(x)|y]$
- Second order characterization: <u>Partial</u> characterization in terms of $m_x(t)$ and $K_{XX}(t_1,t_2)$.
- Special Types of Stochastic Processes

- Gaussian: $X(t)$ is Gaussian process $\Longleftrightarrow \sum_{i=1}^{N} a_i X(t_i)$ a Gaussian random variable for all a_i, t_i, N.
- Markov: $p_{X(t_N)|X(t_{N-1}),X(t_{N-2}),\cdots,X(t_1)}(X_N|X_{N-1},\cdots,X_1) = p_{X(t_N)|X(t_{N-1})}(X_N|X_{N-1})$, for all t_i with $t_i \geq t_{i-1}$
- IIP: $X(t_i) - X(t_{i-1})$ independent of $X(t_{i-1})$ for all $t_i \geq t_{i-1}$. IIP $\longrightarrow$ Markov. IIP $\longrightarrow K_{XX}(t,s) = \mathrm{Var}[\min(t,s)]$
- Strict Sense Stationary: $p_{X(t_1),X(t_2),\cdots,X(t_N)}(x_1,x_2,\cdots,x_N) = p_{X(t_1+\tau),X(t_2+\tau),\cdots,X(t_N+\tau)}(x_1,x_2,\cdots,x_N)$ for all τ, N.
- Wide Sense or weakly Stationary: $m_x(t) = m_x$, $K_{XX}(t_1,t_2) = K_{XX}(t_2 - t_1)$

- Special random processes and their means and variances: e.g.
  - Poisson Counting Process: $\Pr[N(t) = n] = \frac{(\lambda t)^n}{n!} e^{-\lambda t}$, $m_N(t) = \lambda t$, $K_{nn}(t,s) = \lambda \min(t,s)$. $N(t)$ is IIP.
  - Random Telegraph Wave
  - Random Walk
  - Wiener Process: $m_x(t) = 0$, $K_{XX}(t,s) = \alpha \min(t,s)$, IIP

3. **Convergence, Mean Square Calculus**

- Mean Square Convergence: $\lim_{n\to\infty} E\left[(X_n - X)^2\right] = 0$
- Cauchy Criterion for MSS Convergence: $\lim_{n\to\infty} E\left[(x_n - x_m)^2\right] \to 0$
- Mean Square Continuity: $\lim_{\varepsilon\to 0} X(t+\varepsilon) \stackrel{\text{mss}}{=} X(t)$. Mean Square Continuous if $R_{XX}(t_1,t_2)$ continuous.
- Mean Square Derivative: $\lim_{\varepsilon\to 0}\left(\dfrac{X(t+\varepsilon) - X(t)}{\varepsilon}\right) \stackrel{\text{mss}}{=} \dot{X}(t)$. Mean Square differentiable if $\frac{\partial^2}{\partial t_1 \partial t_2} R_{XX}(t_1,t_2)$ exists.
- Mean Square Integral: $\lim_{N\to\infty}\left(\sum_{i=1}^{N} X(s+i\Delta)\Delta\right) \stackrel{\text{mss}}{=} Y$. Mean square integrable if $\int_s^t \int_s^t R_{XX}(\sigma,\tau)\, d\sigma d\tau$ exists.

4. **Ergodicity**

- Idea: Time average = Ensemble average in MSS sense.
- Ergodic in the Mean: $\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} X(t)\, dt = \lim_{T\to\infty} \langle m_x \rangle_T \stackrel{\text{mss}}{=} m_x$
- Ergodic in Autocorrelation: $\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} X(t+\tau)X(t)\, dt = \lim_{T\to\infty} \langle R_{XX}(\tau)\rangle_T \stackrel{\text{mss}}{=} R_{XX}(\tau)$
- Completely Ergodic: $\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} g\{X(t)\}\, dt \stackrel{\text{mss}}{=} E[g\{X(t)\}]$

5. **Power Spectral Density**

- For $X(t)$ WSS
- Beware WSS Notation: $R_{XY}(t,t+\tau) \equiv R_{XY}(t+\tau-t) = R_{XY}(\tau)$ verses $R_{XY}(t,t+\tau) \equiv R_{XY}(t-t-\tau) = R_{XY}(-\tau)$.
- Power Spectral Density: $S_{XX}(\omega) = F[R_{XX}(\tau)] = \int_{-\infty}^{\infty} R_{XX}(\tau)e^{-j\omega\tau} d\tau$, $S_{XX}(f) = F[R_{XX}(\tau)] = \int_{-\infty}^{\infty} R_{XX}(\tau)e^{-j2\pi f\tau} d\tau$
- Inverse Power Spectral Density: $R_{XX}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)e^{j\omega\tau}\, d\omega = \int_{-\infty}^{\infty} S_{XX}(f)e^{j2\pi f\tau}\, df$
- Cross-Spectral Density: $S_{XY}(\omega) = F[R_{XY}(\tau)] = \int_{-\infty}^{\infty} R_{XY}(\tau)e^{-j\omega\tau} d\tau$, $S_{XY}(f) = F[R_{XY}(\tau)] = \int_{-\infty}^{\infty} R_{XY}(\tau)e^{-j2\pi f\tau} d\tau$

- Inverse Cross-Power Spectral Density: $R_{XY}(\tau) = \dfrac{1}{2\pi}\int_{-\infty}^{\infty} S_{XY}(\omega)e^{j\omega\tau}\,d\omega = \int_{-\infty}^{\infty} S_{XY}(f)e^{j2\pi f\tau}\,df$
- For sequences: $S_{XX}(f) = \sum_{n=-\infty}^{\infty} R_{XX}(n)e^{-j2\pi fn}$, $-1/2 < f < 1/2$. $S_{XX}(\omega) = \sum_{n=-\infty}^{\infty} R_{XX}(n)e^{-j\omega n}$, $-\pi < \omega < \pi$.
- $S_{XX}(f)$ average power at frequency f.
- Properties of $S_{XX}(f)$ and $S_{XY}(f)$:
  - $S_{XX}(\omega)$ real, non-negative.
  - Total average power in $X(t)$: $R_{XX}(0) = \int_{-\infty}^{\infty} S_{XX}(\omega)\,d\omega$
  - $X(t)$ Real $\longrightarrow S_{XX}(\omega) = S_{XX}(-\omega)$
  - $\alpha X(t) \longrightarrow \alpha^2 S_{XX}(\omega)$
  - $\dfrac{d}{dt}X(t) \longrightarrow \omega^2 S_{XX}(\omega)$
  - $X(t)e^{j\omega_0 t} \longrightarrow S_{XX}(\omega-\omega_0)$
  - $X(t)+b \longrightarrow S_{XX}(\omega)+2\pi|b|^2\delta(\omega)$.

6. **Advice:**

- Have basic results at your fingertips
- Know the assumptions/conditions behind formulas that you use!
- Perform sanity checks on answers – go back to basics if totally stuck (i.e. defining equation of expectation, variance etc)
- Don't forget to deal with the mean! (e.g. $R_{XX}(t)$ vs $K_{XX}(t)$, etc.)