Boston University
Department of Electrical and Computer Engineering

EC505 STOCHASTIC PROCESSES

Problem Set No. 3

Fall 2016

Issued: Monday, Sept. 26, 2016 **Due:** Wednesday, Oct. 5 2016

Problem 3.1

Problem 2.8, 2.9, 2.11

Problem 3.2

(a) Let $X_1(t)$ be a random telegraph wave. Specifically, let $N(t)$ be a Poisson counting process with

$$\Pr[N(t) = k] = \frac{(\lambda t)^k e^{-\lambda t}}{k!} .$$

Let $X_1(0) = +1$ with probability 1/2 and $X_1(0) = -1$ with probability 1/2, assume that $X_1(0)$ is independent of $N(t)$ and define

$$X_1(t) = \begin{cases} X_1(0) & \text{if } N(t) \text{ is even} \\ -X_1(0) & \text{if } N(t) \text{ is odd} \end{cases}$$

Sketch a typical sample function of $X_1(t)$. Find $m_{X_1}(t)$, $K_{X_1X_1}(t,s)$, $p_{X_1(t)}(x)$, and $p_{X_1(t)|X_1(s)}(x_t|x_s)$. You may find the following sums useful:

$$\sum_{k=0}^{\infty} \frac{\alpha^k}{k!} = e^{\alpha}, \qquad \sum_{\substack{k=0 \\ k \text{ even}}}^{\infty} \frac{\alpha^k}{k!} = \cosh(\alpha), \qquad \sum_{\substack{k=0 \\ k \text{ odd}}}^{\infty} \frac{\alpha^k}{k!} = \sinh(\alpha).$$

(b) Let $X_2(t)$ be a Gaussian random process with

$$m_{X_2}(t) = 0, \qquad \text{and} \qquad K_{X_2X_2}(t,s) = e^{-2\lambda|t-s|}$$

Find $p_{X_2(t)}(x)$, and $p_{X_2(t)|X_2(s)}(x_t|x_s)$. Sketch a typical sample function of $x_2(t)$. Show that $X_2(t)$ is not an independent increments process.

Note: The processes in (a) and (b) have the same second order properties, yet have very different sample paths, first order densities, etc.

Problem 3.3

Let $N(t)$ be a Poisson counting process on $t \geq 0$ with rate λ. Let $\{Y_i\}$ be a collection of statistically independent, identically-distributed random variables with mean and variance

$$E[Y_i] = m_y \qquad \text{and} \qquad \text{var}[Y_i] = \sigma_y^2$$

respectively. Assume that the $\{Y_i\}$ are statistically independent of the counting process $N(t)$ and define a new random process $Y(t)$ on $t \geq 0$ via

$$y(t) = \begin{cases} 0 & N(t) = 0 \\ \displaystyle\sum_{i=1}^{N(t)} Y_i & N(t) > 0 \end{cases}$$

(a) Sketch a typical sample function on $N(t)$ and the associated typical sample function of $Y(t)$.

(b) Use smoothing property of expectations (condition on $N(t)$ in the inner average) to find $E[Y(t)]$ and $E[Y^2(t)]$ for $t \geq 0$.

(c) Prove that $Y(t)$ is an independent increment process on $t \geq 0$ and use this fact to find the covariance function $K_{YY}(t,s)$ for $t, s \geq 0$.

Problem 3.4

Suppose we want to estimate the unknown value of a constant signal by observing and processing a noisy version of the signal for T seconds. Let $X(t) = c + N(t)$, where c is the unknown constant signal value (nonrandom) and $N(t)$ is a zero-mean, stationary bandlimited Gaussian random process with a psd $S_{NN}(f) = N_0$ for $|f| < B$ and zero elsewhere. Assume further that $B >> \frac{1}{T}$. The estimate of c will be the time averaged value:

$$\widehat{c} = \frac{1}{T}\int_0^T X(t)\,dt$$

(a) Show that $E[\widehat{c}] = c$. Such an estimate is termed *unbiased.*

(b) Show that $\mathrm{Var}[\widehat{c}] \approx \frac{N_0}{T}$.

(c) Find the value of T such that $\Pr\{|\widehat{c} - c| < 0.1c\} \geq 0.999$. Express your answer in terms of c and N_0.

Useful facts:

$$\frac{1}{T^2}\int_0^T\int_0^T g(t_1 - t_2)\,dt_1\,dt_2 = \frac{1}{T}\int_{-T}^T \left[1 - \frac{|\tau|}{T}\right] g(\tau)\,d\tau = \int_{-\infty}^{\infty}\left[\frac{\sin^2(\pi f T)}{(\pi f T)^2}\right] G(f)\,df, \qquad \int_{-\infty}^{\infty}\frac{\sin^2 z}{z^2}\,dz = \pi$$

Problem 3.5

Let $X(t)$ be a Gaussian random process, wide-sense stationary, with mean $E[X(t)] = 1$ and power spectral density function $S_{XX}(f)$ given by:

$$S_{XX}(f) = \frac{2}{1 + (2\pi f)^2} + \delta(f)$$

(a) Compute the autocovariance function $K_{XX}(\tau)$.

(b) Is the process $X(t)$ ergodic in autocorrelation? Explain.

(c) Is the process $X(t)$ mean-square continuous? Is it mean-square differentiable? Explain.

(d) Define the process $Y(t) = \frac{d}{dt}X(t)$. Using generalized functions if necessary, find the mean and autocorrelation of the process $Y(t)$. Is the process $Y(t)$ wide-sense stationary? Explain.

Problem 3.6

Consider a Poisson process $N(t)$, with rate 1. Define a new process $X(t) = N(t+2) - N(t) - 2$.

(a) Find the mean and autocorrelation of $X(t)$.

(b) Is $X(t)$ wide-sense stationary? Explain.

(c) If the process is wide-sense stationary, what is its power spectral density $S_{XX}(f)$?

(d) If the process is wide-sense stationary, is it ergodic in the mean?

Problem 3.7 Let $N_1(t)$ and $N_2(t)$ be two independent Poisson processes with rates $\lambda_1 = 1$ and $\lambda_2 = 2$ respectively.

1. Find the probability of no arrivals in $(3, 5]$ for $N_1(t)$ and $N_2(t)$.

2. Find the probability that there are two arrivals in $(0, 2]$ and three arrivals in $(1, 4]$.

3. Find the probability that the second arrival in $N_1(t)$ occurs before the third arrival in $N_2(t)$. Hint: Think of $N_1(t)$ and $N_2(t)$ as two processes obtained from splitting a Poisson process $N(t)$ with rate $\lambda_1 + \lambda_2$.

Problem 3.8

For each of the autocorrelation functions below, find the power spectral density function.

(a) $e^{-a|\tau|}$

(b) $\dfrac{\sin(1000\tau)}{1000\tau}$

(c) $\dfrac{1}{4}e^{-|\tau|}\left[\cos|\tau| + \sin|\tau|\right]$

(d) $e^{-.01 f_0^2 \tau^2}$

(e) $\cos(1000\tau)$

Problem 3.9 Let $W(t)$ be a standard Brownian motion.

1. Find $\mathrm{Prob}(W(1) + W(2) > 2)$.
2. Find the conditional PDF of $W(s)$ given $W(t) = a$ for $0 \le t \le s$.
3. Let $X(t) = \exp(W(t))$ for $0 \le t < \infty$. Compute mean, variance and covariance functions for this process.

Computer Problems

Problem 3.10 Special Random Processes

In this project we will construct and explore some special random processes discussed in class.

(a) **The Poisson Process:**

We will write a function `poisson.m` to construct sample paths of a Poisson random process by following the steps outlined in the class. A Poisson process is often used to model e.g. arrivals, lines, queues, etc., such as found in banks, computer and network packet traffic, telephone hold queues, etc. The function call for our program will be the following: `[T,NT]=poisson(Na,lam);`, where `Na` will be the desired number of arrivals to generate, `lam` will be the arrival rate of the process, `T` will be the generated vector of the arrival times, and `NT` will be the corresponding number of arrivals that have occured up to the times in `T`. Each step below will form a line of the program:

(i) The first step is to generate a vector `tau` of `Na` independent, exponentially distributed, interarrival times. This can be done using the supplied function `randexp.m` as follows:

```
tau = randexp(Na,1,lam);
```

Note that `randexp.m` generates exponentially distributed random variables using the method developed in Problem Set #1.

(ii) Now given these interarrival times, we generate the corresponding vector of waiting times `T` for each arrival as the cumulative sum of the interarrival times:

```
T = cumsum(tau);
```

(iii) Finally, we can generate the corresponding vector containing the number of arrivals at each time `NT` by simply counting up (since each arrival increments the count by 1):

```
NT = [1:length(T)]';
```

Add these steps together to create your program and generate a few sample paths of the Poisson process for various choices of the rate parameter `lam`. Plot them using `stairs.m`. Note that we are generating the process by finding the *times* at which a particular event (a jump) takes place, not by finding the values the process takes at certain times, as we usually do. Comment on the difficulty this will cause in estimating ensemble averages and the like.

One the web stie you will find the function `poissrp.m`, which *does* generate sample path values of Poisson random processes at fixed sets of times. Using the supplied function, generate multiple Poisson process sample paths and estimate and plot the mean and autocovariance function of the process. Compare to the theoretical functions of these quantities discussed in class.

(b) **A Random Telegraph Wave:**

Recall that (one definition) of a random telegraph wave is $X(t) = Z(-1)^{N(t)}$ where $N(t)$ is a Poisson random process and $Z = \pm 1$ with equal probability. Use the output of `poissrp.m` to write a function `telerp.m` which generates sample paths of a random telegraph wave. Again, estimate and plot the mean process and autocovariance function.

(c) **The Wiener Process:**

We will write a function `wiener.m` to construct a sample of a Wiener-Levy random process on the interval [0,1] as the limit of a discrete-time random walk following the steps outlined in the class. The function call will be the following: `[X,t] = wiener(T,alpha);`, where `alpha` will be the ratio of the step size squared to time interval s^2/T, `T` will be the time step, `Z` will be the vector process values at the times in the vector `t`. Again, each step below will form a line of the program:

(i) First generate the set of time points we will use based on the input information:

```
t = 0:T:1;
```

(ii) Next, set the step size `s` of the discrete random walk. For convergence, we required that this step size scale with the time interval as $s^2/T = \alpha$:

```
s = sqrt(alpha*T);
```

(iii) Generate a random vector of positive and negative jumps of size `s` by scaling Bernoulli trials (which we generate by rounding uniform random variables to ± 1):.

```
z = round(rand(length(t),1)); % Bernoulli Trials via rounded uniform RVs
jumps =  s*( sign(z - .5) );
```

(iv) Finally, we generate the Wiener process value at each time as the cumulative sum of the jumps:

```
X = cumsum(jumps);
```

Add these steps together to create your program and generate a few sample paths of the Wiener process for various choices of the parameter `alpha`. You can plot your sample paths via `stairs(t,X)`. As the sampling interval `T` is made smaller, the discrete time random walk should converge to the Wiener process. Start with `T`=.1 and progressively decrease it, generating and plotting sample paths along the way. Does the process appear to converge? Plot a sample path corresponding to a small value of `T`. Zoom in on a portion of your plot in MATLAB by typing `zoom on` and then clicking where you want to zoom. The Wiener process is often called "self-similar" – can you explain why from this experiment?

On the web site you will find the function `wienerp.m` which will generate multiple sample paths of our approximate Weiner process at arbitrary sampling points. Generate multiple sample paths and estimate and plot the mean and autocovariance functions of the process. Compare to the theoretical functions of these quantities discussed in class.