Boston University

Department of Electrical and Computer Engineering

EC505 STOCHASTIC PROCESSES

# Problem Set No. 4

Fall 2016

**Issued:** Monday, Oct., 10 2016 **Due:** Wednesday, Oct., 19 2016

**Problem 4.1**

Let $X(t)$ be a Gaussian random process, wide-sense stationary, with mean $E[X(t)] = 1$ and power spectral density function $S_{XX}(f)$ given by

$$S_{XX}(f) = \frac{2}{1 + (2\pi f)^2} + \delta(f)$$

(a) Compute the autocovariance function $K_{XX}(\tau)$.

(b) Is the process $X(t)$ ergodic in autocorrelation? Explain your answer.

(c) Is the process $X(t)$ mean-square continuous? Is it mean-square differentiable? Explain your answer.

(d) Define the process $Y(t) = \frac{d}{dt}X(t)$. Using generalized functions if necessary, find the mean and autocorrelation of the process $Y(t)$. Is the process $Y(t)$ wide-sense stationary? Explain

Solution:

(a) To find $K_{XX}(\tau) = R_{XX}(\tau) - m_X^2$ we first need to find $R_{XX}(\tau)$, the inverse fourier transform of $S_{XX}(f)$. Now from the tables we find

$$R_{XX}(\tau) = e^{-|\tau|} + 1$$

From this we can find $K_{XX}(\tau) = R_{XX}(\tau) - 1 = e^{-|\tau|}$

(b) First note that:

$$\int_{-\infty}^{\infty} |K_{XX}(\tau)| \, d\tau = \int_{-\infty}^{\infty} |e^{-|\tau|}| \, d\tau = 2\int_{0}^{\infty} e^{-\tau} \, d\tau = 2 < \infty$$

so that the autocovariance function is absolutely integrable. Since the process is a Gaussian random process, this means that it is totally ergodic, and thus obviously ergodic in autocorrelation.

(c) For mean-square continuity we must show that

$$\lim_{t \to t_0} E([X(t) - X(t_0)]^2) = 0$$

First, let's find $E([X(t) - X(t_0)]^2)$

$$E([X(t) - X(t_0)]^2) = 2R_{XX}(0) - 2R_{XX}(t - t_0) = 2 - 2e^{-|t-t_0|}$$

Taking the limit we find that this is in fact zero, so this process is mean-square continuous. More simply, note that $R_{XX}(\tau)$ is continuous at $\tau = 0$, so the process must be MSC.

With respect to mean-square differentiability, we may note that the process is not mean-square differentiable because its autocorrelation $R_{XX}(\tau)$ is not twice differentiable at $\tau = 0$.

(d) Despite the fact that the process $X(t)$ is not strictly differentiable we can still proceed with generalized functions to define the generalized derivative. In particular:

$$E[Y(t)] = E\left[\frac{d}{dt}X(t)\right] = \frac{d}{dt}1 = 0$$

This is not surprising since $X(t)$ is WSS. Now, since the mean is zero we have:

$$K_{YY}(t) = -\frac{d^2}{dt^2}K_{XX}(t) = -\frac{d^2}{dt^2}e^{-|t|}$$

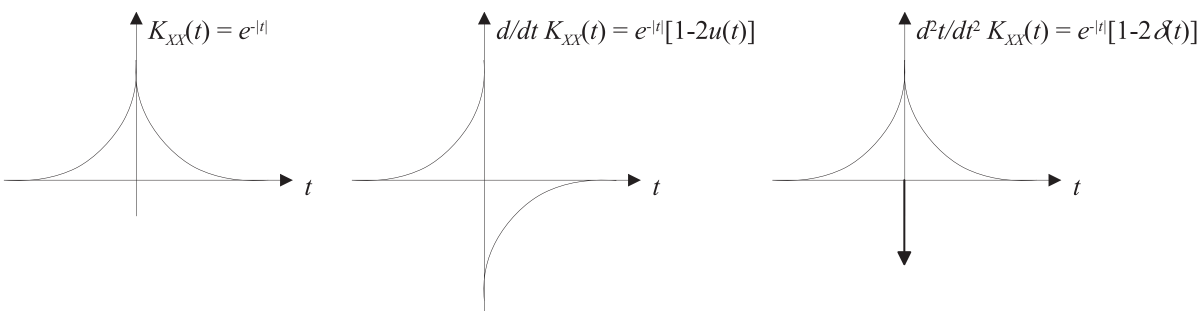
One way of getting this derivative is as follows. Note that:

$$\frac{d}{dt}e^{-|t|} = \begin{cases} -e^{-|t|} & \text{if } t > 0 \\ e^{-|t|} & \text{if } t < 0 \end{cases}$$

We can rewrite this as $\frac{d}{dt}e^{-|t|} = e^{-|t|}[1 - 2u(t)]$, where $u(t)$ is the unit step function. Then,

$$\frac{d^2}{dt^2}e^{-|t|} = e^{-|t|}[1 - 2u(t)]^2 - 2\delta(t)e^{-|t|} = e^{-|t|}[1 - 2\delta(t)]$$

These functions are shown in the Figure.




Substituting into the above formula for the second derivative into the formula yields:

$$K_{YY}(t) = -\frac{d^2}{dt^2}K_{XX}(t) = e^{-|t|}[2\delta(t) - 1] = 2\delta(t) - e^{-|t|}$$

Note the presence of the impulse in the solution, a reflection of the fact that the mean square derivative does not exist.

The process is of course wide-sense stationary since the mean is constant and the autocorrelation is a function of only the time difference.

**Problem 4.2**

Consider a Poisson process $N(t)$, with rate 1. Define a new process $X(t) = N(t + 2) - N(t) - 2$.

(a) Find the mean and autocorrelation of $X(t)$.

(b) Is $X(t)$ wide-sense stationary? Explain.

(c) If the process is wide-sense stationary, what is its power spectral density $S_{XX}(f)$?

(d) If the process is wide-sense stationary, is it ergodic in the mean?

Solution:

(a) First recall that for the Poisson process with rate 1 we have $E[X(t)] = t$ and $R_{NN}(t,s) = ts + \min(t,s)$. Now the mean of $X(t)$ is obtained as

$$E[X(t)] = E[N(t+2)] - E[N(t)] - 2 = t + 2 - t - 2 = 0$$

The autocorrelation is given by

$$\begin{aligned} R_{XX}(t,s) &= E[(N(t+2) - N(t) - 2)(N(s+2) - N(s) - 2)] \\ &= E\left[(N(t+2) - N(t))(N(s+2) - N(s))\right] - 2E\left[N(s+2) - N(s)\right] - 2E\left[N(t+2) - N(t)\right] + 4 \\ &= E\left[(N(t+2) - N(t))(N(s+2) - N(s))\right] - 4 \end{aligned}$$

Now note that if $|t-s| > 2$ then the increments $N(t+2) - N(t)$, and $N(s+2) - N(s)$ do not overlap, are independent, and so the value of $R_{XX}(t,s) = 0$ for this case. When this is not the case, i.e. when $|t-s| < 2$:

$$\begin{aligned} & R_{XX}(t,s) = E\left[(N(t+2) - N(t))(N(s+2) - N(s))\right] - 4 \\ &= E[N(t+2)N(s+2)] - E[N(t+2)N(s)] - E[N(t)N(s+2)] + E[N(t)N(s)] - 4 \\ &= (t+2)(s+2) + \min(t,s) + 2 - (t+2)s - \min(t+2,s) - t(s+2) - \min(t,s+2) + ts + \min(t,s) - 4 \\ &= 2 + 2\min(t,s) - s - t = 2 - |t-s| \end{aligned}$$

Thus, putting the pieces together we get:

$$R_{XX}(t,s) = \begin{cases} 0 & \text{if } |t-s| > 2 \\ 2 - |t-s| & \text{otherwise} \end{cases}$$

(b) Yes the process is WSS, since it has constant mean and autocorrelation depending on the time difference only.

(c) We need to take the transform of the triangular function represented by $R_{XX}(\tau)$. This is actually in the transform table in the notes and yields:

$$S_{XX}(f) = 4\frac{\sin^2(2\pi f)}{(2\pi f)^2}$$

(d) Yes. To be ergodic in the mean it is sufficient that the correlation function is absolutely integrable. Notice that $R_{XX}(\tau)$ is certainly absolutely integrable (it is only nonzero for a finite duration), therefore $X(t)$ is ergodic in the mean.

**Problem 4.3** Consider a 4 state Markov chain with the transition probability matrix

$$P = \begin{pmatrix} 0.1 & 0.2 & 0.3 & 0.4 \\ 0 & 0.5 & 0.2 & 0.3 \\ 0 & 0 & 0.3 & 0.7 \\ 0.1 & 0 & 0 & 0.9 \end{pmatrix}$$

1. Draw the state transition diagram, with the probabilities for the transitions.

   **Solution:** Imagine 4 circles, numbered 1 to 4. Arcs from 1 to 1, 1 to 2, 1 to 3, 1 to 4. Arcs from 2 to 2, 2 to 3, 2 to 4. Arcs from 3 to 3, 3 to 4. Arcs from 4 to 4, 4 to 1. Assign the appropriate probabilities from the matrix above.

2. Find the transient states and recurrent states.

   **Solution:** States 1, 2, 3,4 are recurrent, as every state can transition to 4, and from 4 to 1, and from 1 back to any state. Hence, every state has a cycle that connects back to it. There are no transient states.

3. Is the Markov chain irreducible? Explain.

   **Solution:** The Markov chain consists of all states being recurrent, and belonging to a single recurrent class. Hence it is irreducible.

4. Is the Markov chain aperiodic? Explain.

   **Solution:** There are self-cycles at one node, hence the period is 1, which makes it aperiodic.

5. Find the steady state distribution of this Markov chain.

   **Solution:** Doing a probability balance out of nodes to other nodes, one gets the following equations:

$$
\begin{aligned}
0.9\pi_1 &= 0.1\pi_4 (node\ 1) \\
0.2\pi_1 &= 0.5\pi_2 (node\ 2) \\
0.7\pi_3 &= 0.3\pi_1 + 0.2\pi_2 (node\ 3)
\end{aligned}
$$

   Expressing all the other probabilities in terms of $\pi_1$ yields

$$
\begin{aligned}
\pi_4 &= 9\pi_1 \\
\pi_2 &= \frac{2}{5}\pi_1
\end{aligned}
$$

$$\pi_3 = \frac{154}{35}\pi_1$$

   Since the sum of probabilities must equal to 1, we get

$$\frac{518}{35}\pi_1 = 1$$

   which implies

$$
\begin{aligned}
\pi_1 &= \frac{35}{518} \\
\pi_2 &= \frac{14}{518} \\
\pi_3 &= \frac{154}{518} \\
\pi_4 &= \frac{315}{518}
\end{aligned}
$$

**Problem 4.4** Consider a 4 state Markov chain with the transition probability matrix (this is different from the previous problem...)

$$P = \begin{pmatrix} 0.1 & 0.2 & 0.3 & 0.4 \\ 0 & 0.5 & 0.2 & 0.3 \\ 0 & 0 & 0.3 & 0.7 \\ 0 & 0 & 0.1 & 0.9 \end{pmatrix}$$

1. Draw the state transition diagram, with the probabilities for the transitions.

   **Solution:** It is the same as the previous graph, except that the arc from 4 to 1 gets changed to an arc from 4 to 3.

2. Find the transient states and recurrent states.

   **Solution:** States 1 and 2 are transient, as there is a path to state 3 but no path back from state 3 in these two states. States 3 and 4 are a single recurrent class.

3. Is the Markov chain irreducible? Explain.

   **Solution:** Not irreducible, as there are transient states.

4. Is the Markov chain aperiodic? Explain.

   **Solution:** It is aperiodic, as there are cycles of length 1 (self loop from 3 to 3).

5. Find the steady state distribution of this Markov chain.

   **Solution:** We only have to look at the recurrent states 3 and 4. By balance out of state 3, we have

   $$0.7\pi_3 = 0.1\pi_4$$

   so $\pi_4 = 7\pi_3$, so the steady state distribution is $\pi_4 = 7/8, \pi_3 = 1/8$.

**Problem 4.5** Consider the following gambling problem. Suppose you go to a casino to play an unusual gambling game, where the probability of winning is 0.45. You start with \$3.00 and you want to double your stake. If you ever win enough to reach \$6.00, you quit and walk out. Similarly, if you lose your original stake and have no money left, you quit also.

1. The first betting scheme is simple: You bet 1.00 on every bet until you either go broke or you reach \$6.00. Draw a Markov chain, with transition probabilities, corresponding to this betting scheme. You should have 7 states, corresponding to the amount of money you have at any time.

   **Solution:** This is a birth-death process. Start with state 1 corresponding to \$0, state 2 corresponding to \$1, etc, we get 7 states. The probability of going one step forward from states 2 to 6 is 0.45. The probability of going one state backwards from states 2 to 6 is 0.55. States 1 and 7 have self loops wwith probability 1.

2. Write the probability transition matrix $P$ for this Markov chain.

   $$P = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.55 & 0 & 0.45 & 0 & 0 & 0 & 0 \\ 0 & 0.55 & 0 & 0.45 & 0 & 0 & 0 \\ 0 & 0 & 0.55 & 0 & 0.45 & 0 & 0 \\ 0 & 0 & 0 & 0.55 & 0 & 0.45 & 0 \\ 0 & 0 & 0 & 0 & 0.55 & 0 & 0.45 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

3. Which states are transient? Which states are recurrent? How many recurrent classes are there?

   **Solution:** States 2 through 6 are transient. Two recurrent states: 1 and 7. Two recurrent classes (broke or double your money).

4. I'll assume you have access to a reasonable calculator or MATLAB. Enter the matrix $P$ into Matlab, and an initial column vector $\underline{p}(0)$ which is zero everywhere except for the 4th state (corresponding to \$3.00), where it is 1. You can find the steady state probability by evolving this Markov chain. Compute the probability mass function at time 20 by computing

   $$\underline{p}(40) = (P^T)^{40}\underline{p}(0)$$

   What is the probability that by time 40, you go broke? What is the probability that, by time 40, you have left with \$6.00?

   **Solution:** 0.3526 in my computer.

5. Let's repeat the above analysis for the following betting scheme. You start with \$1.00 bet. Every time you lose, if you can, you double your bet. Every time you win, you reduce your bet to \$1.00. If you can't double your bet, you bet everything you have left. If you ever reach \$6.00 or more, you walk away. Draw a Markov chain, with transition probabilities, corresponding to this betting scheme. Note: The states here are more complicated, as they involve the size of the bet and the amount remaining. Note also that the maximum bet size will be limited to \$2.00, and only when yoou have between \$2.00 and \$4.00 (work this out...) so you should get by with 10 states.

   **Solution:** Keep the 7 states you had before as money you have and the bid size 1. Then add the following states (numbered 8 to 10): State 8 is \$2.00 with bet size \$2, state 9 is \$3.00 with bet size \$2, and state 10 is \$4.00 with bet size \$2.

   Now, the main thing to capture is that, when you lose, you double the bet size. Hence, from state 6 (corresponding to \$5.00, bet size 1), if you lose, you go to state 10 (\$4 with bet size \$2) instead of state 5. From state 5, if you lose, you go to state 9 instead of state 4. From state 4, if you lose, you go to state 8 instead of state 3.

   We now have to add transitions out of states 8, 9, 10. From state 10, if you win, you go to state 7, and if you lose, you go to state 8. From state 9, if you win, you go to state 6, and if you lose, you go to state 2. From state 8, if you win, you go to state 5, and if you lose, you go to state 1.

6. Number your states and write the probability transition matrix $P$ for this Markov chain.

   **Solution:** Keep the numbers the same as above,

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.55 & 0 & 0.45 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.55 & 0 & 0.45 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.45 & 0 & 0 & 0.55 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.45 & 0 & 0 & 0.55 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.45 & 0 & 0 & 0.55 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0.55 & 0 & 0 & 0 & 0.45 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.55 & 0 & 0 & 0 & 0.45 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.45 & 0.55 & 0 & 0 \end{pmatrix}$$

7. Which states are transient? Which states are recurrent? How many recurrent classes are there?

   **Solution:** States 2-6 and 8-10 are transient. States 1 and 7 are recurrent. There are two recurrent classes.

8. I'll assume you have access to a reasonable calculator or MATLAB. Enter the matrix $P$ into Matlab, and an initial column vector $\underline{p}(0)$ which is zero everywhere except for the 4th state (corresponding to \$3.00), where it is 1. You can find the steady state probability by evolving this Markov chain. Compute the probability mass function at time 40 by computing

$$\underline{p}(40) = (P^T)^{40}\underline{p}(0)$$

   What is the probability that by time 40, you go broke? What is the probability that, by time 40, you have left with \$6.00?

   **Solution:** Probability of going broke is 0.6, and probability of going home with \$6.00 is 0.4.

9. Which betting scheme do you prefer? Why?

   **Solution:** Neither of the above! I would rather place a single bet for all the money. I have a probability of 0.45 of leaving with \$6.00 !!! When a game is not in your favor, the longer you play, the more likely you are to lose all your money.

**Problem 4.6** Assume that the number of call arrivals between two locations has Poisson distribution with intensity $\lambda$. Also, assume that the holding times of the conversations are exponentially distributed with a mean of $1/\mu$. Calculate the average number of call arrivals for a period of a conversation.

**Solution:** Let $N(t)$ be the random variable representing the number of calls in time $t$. We know that $N(t)$ is a PCP with rate $\lambda$. Let $X$ be the random variable representing the holding time of a call. We are interested in $E[N(X)]$. We can compute it using nested expectations:

$$E[N(X)] = E\left[E[N(X) \mid X]\right] = \int_0^\infty P_X(x)\lambda x dx = \int_0^\infty \mu \exp(-\mu x)\lambda x dx = \frac{\lambda}{\mu}$$