Boston University

Department of Electrical and Computer Engineering

EC505 STOCHASTIC PROCESSES

Problem Set No. 3 Solutions

Fall 2016

Issued: Monday, Sept. 26, 2016 **Due:** Wednesday, Oct. 5 2016

Problem 3.1 Problem 2.8, 2.9, 2.11

Solution: See solutions to problem set 2.

Problem 3.2

(a) Let $X_1(t)$ be a random telegraph wave. Specifically, let $N(t)$ be a Poisson counting process with

$$\Pr\left[N(t)=k\right]=\frac{(\lambda t)^k e^{-\lambda t}}{k!}\ .$$

Let $X_1(0)=+1$ with probability 1/2 and $X_1(0)=-1$ with probability 1/2, assume that $X_1(0)$ is independent of $N(t)$, and define

$$X_1(t)=\begin{cases} X_1(0) & \text{if } N(t) \text{ is even} \\ -X_1(0) & \text{if } N(t) \text{ is odd}\end{cases}$$

Sketch a typical sample function of $X_1(t)$. Find $m_{X_1}(t)$, $K_{X_1X_1}(t,s)$, $p_{X_1(t)}(x)$, and $p_{X_1(t)|X_1(s)}(x_t|x_s)$. You may find the following sums useful:

$$\sum_{k=0}^{\infty}\frac{\alpha^k}{k!}=e^{\alpha},\qquad \sum_{\substack{k=0\\ k\text{ even}}}^{\infty}\frac{\alpha^k}{k!}=\cosh(\alpha),\qquad \sum_{\substack{k=0\\ k\text{ odd}}}^{\infty}\frac{\alpha^k}{k!}=\sinh(\alpha).$$

(b) Let $X_2(t)$ be a Gaussian random process with

$$m_{X_2}(t)=0,\qquad \text{and}\qquad K_{X_2X_2}(t,s)=e^{-2\lambda|t-s|}$$

Find $p_{X_2(t)}(x)$, and $p_{X_2(t)|X_2(s)}(x_t|x_s)$. Sketch a typical sample function of $x_2(t)$. Show that $X_2(t)$ is not an independent increments process.

Note: The processes in (a) and (b) have the same second order properties, yet have very different sample paths, first order densities, etc.

Solution:

(a) A typical sample function of $X_1(t)$ is given below:

Now lets find $p_{X_1(t)}(x)$. Note that $X_1(t) = (-1)^{N(t)} X_1(0)$ and $X_1(t)$ only takes on two values, $+1$ and -1, so we just need to find the probability of these two possibilities (we really only need to find one). Now note that there are two ways that we could get $X_1(t) = +1$, thus:

$$\begin{aligned}
\Pr(X_1(t) = +1) &= \Pr[(X_1(0) = +1) \text{ and } (N(t) \text{ even})] + \Pr[(X_1(0) = -1) \text{ and } (N(t) \text{ odd})] \\
&= \Pr[X_1(0) = +1] \Pr[N(t) \text{ even}] + \Pr[X_1(0) = -1] \Pr[N(t) \text{ odd}] \\
&= \frac{1}{2} \left(\Pr[N(t) \text{ even}] + \Pr[N(t) \text{ odd}] \right) = \frac{1}{2}
\end{aligned}$$

since $(\Pr[N(t) \text{ even}] + \Pr[N(t) \text{ odd}]) = 1$, as there are only two possibilities. We thus immediately have that $\Pr(X_1(t) = -1) = 1 - \frac{1}{2} = \frac{1}{2}$ as well, so that:

$$p_{X_1(t)}(x) = \frac{1}{2} \left[\delta(x+1) + \delta(x-1) \right]$$

Next lets find the conditional density $p_{X_1(t)|X_1(s)}(x_t|x_s)$. The conditional density has 4 possibilities, which we enumerate below:

$$\begin{aligned}
p_{X_1(t)|X_1(s)}(+1|+1) &= \Pr\left[N(t) - N(s) \text{ even}\right] \\
p_{X_1(t)|X_1(s)}(-1|-1) &= \Pr\left[N(t) - N(s) \text{ even}\right] \\
p_{X_1(t)|X_1(s)}(+1|-1) &= \Pr\left[N(t) - N(s) \text{ odd}\right] \\
p_{X_1(t)|X_1(s)}(-1|+1) &= \Pr\left[N(t) - N(s) \text{ odd}\right]
\end{aligned}$$

We can thus see that

$$p_{X_1(t)|X_1(s)}(x_t|x_s) = \delta(x_t - x_s) \Pr\left[N(t) - N(s) \text{ even}\right] + \delta(x_t + x_s) \Pr\left[N(t) - N(s) \text{ odd}\right]$$

Now the process $N(t)$ is Poisson so recall:

$$\Pr\left[|N(t) - N(s)| = k\right] = \frac{[\lambda(|t-s|)]^k}{k!} e^{-\lambda(|t-s|)} \qquad k = 1, 2, ...$$

Thus:

$$\begin{aligned}
\Pr\left[N(t) - N(s) \text{ even}\right] &= \sum_{\substack{k=0 \\ k \text{ even}}}^{\infty} \frac{[\lambda(|t-s|)]^k}{k!} e^{-\lambda(|t-s|)} = e^{-\lambda(|t-s|)} \cosh(\lambda|t-s|) \\
\Pr\left[N(t) - N(s) \text{ odd}\right] &= \sum_{\substack{k=0 \\ k \text{ odd}}}^{\infty} \frac{[\lambda(|t-s|)]^k}{k!} e^{-\lambda(|t-s|)} = e^{-\lambda(|t-s|)} \sinh(\lambda|t-s|)
\end{aligned}$$

So that:

$$p_{X_1(t)|X_1(s)}(x_t|x_s) = e^{-\lambda(|t-s|)} \left[\delta(x_t - x_s) \cosh(\lambda|t-s|) + \delta(x_t + x_s) \sinh(\lambda|t-s|) \right]$$

Now lets find the moments. First, its easy to see from the first order density that:

$$m_{X_1}(t) = E[X_1(t)] = .5 - .5 = 0.$$

Now to find $K_{X_1X_1}(t,s)$ we calculate:

$$\begin{aligned}
E[X_1(t), X_1(s)] &= E\left[(-1)^{N(t)} X_1(0) (-1)^{N(s)} X_1(0)\right] = E[X_1(0)^2] E\left[(-1)^{N(t)} (-1)^{-N(s)}\right] \\
&= E\left[(-1)^{N(t)} (-1)^{-N(s)}\right] = E\left[(-1)^{N(t)-N(s)}\right] \\
&= +1 \Pr\left[N(t) - N(s) \text{ even}\right] - 1 \Pr\left[N(t) - N(s) \text{ odd}\right]
\end{aligned}$$

We have already found the quantities $\Pr[N(t)-N(s) \text{ even}]$ and $\Pr[N(t)-N(s) \text{ odd}]$ above, thus:

$$\begin{aligned} K_{X_1X_1}(t,s) &= E[X_1(t)X_1(s)] \\ &= +1\Pr[N(t)-N(s) \text{ even}] - 1\Pr[N(t)-N(s) \text{ odd}] \\ &= (+1)e^{-\lambda(|t-s|)}\cosh(\lambda|t-s|) + (-1)e^{-\lambda(|t-s|)}\sinh(\lambda|t-s|) \\ &= e^{-2\lambda(|t-s|)} \end{aligned}$$

where the last equality follows from the properties of cosh and sinh (recall, $\cosh(z) = .5(e^z + e^{-z})$ and $\sinh(z) = .5(e^z - e^{-z})$).

Another way to find $K_{X_1X_1}(t,s) = R_{X_1X_1}(t,s)$ is to first find the joint density $p_{X_1(t),X_1(s)}(x_t,x_s)$, which we may write as $p_{X_1(t),X_1(s)}(x_t,x_s) = p_{X_1(t)|X_1(s)}(x_t|x_s)p_{X_1(s)}(x_s)$. We already have both the marginal and the conditional density, so the joint density is given by:

$$\begin{aligned} p_{X_1(t),X_1(s)}(x_t,x_s) &= p_{X_1(t)|X_1(s)}(x_t|x_s)p_{X_1(s)}(x_s) \\ &= \frac{1}{2}e^{-\lambda(|t-s|)}\left[\delta(x_t-x_s)\cosh(\lambda|t-s|) + \delta(x_t+x_s)\sinh(\lambda|t-s|)\right] \qquad x_t,x_s = \pm 1 \end{aligned}$$

Finally we have:

$$\begin{aligned} &K_{X_1X_1}(t,s) = E[X_1(t)X_1(s)] \\ &= (+1)\left[\Pr(x_t=1,x_s=1) + \Pr(x_t=-1,x_s=-1)\right] + (-1)\left[\Pr(x_t=1,x_s=-1) + \Pr(x_t=-1,x_s=1)\right] \\ &= (+1)e^{-\lambda(|t-s|)}\cosh(\lambda|t-s|) + (-1)e^{-\lambda(|t-s|)}\sinh(\lambda|t-s|) \\ &= e^{-2\lambda(|t-s|)} \end{aligned}$$

where the last equality again follows from the properties of cosh and sinh.

(b) A typical sample function of $X_1(t)$ is given below, note that the "coherence" of the sample is on the order of $1/\lambda$:

Since the process is a Gaussian random process, we know that $p_{X_2(t)}(x) \sim N(m_{X_2}(t), K_{X_2X_2}(t,t))$. Now:

$$\begin{aligned} m_{X_2}(t) &= 0; \quad K_{X_2X_2}(t,t) = 1 \\ \Longrightarrow & \quad p_{X_2(t)}(x) \sim N(0,1) \end{aligned}$$

Lets find $p_{X_2(t)|X_2(s)}(x_t|x_s)$. Now $p_{X_2(t)|X_2(s)}(x_t|x_s)$ is Gaussian since $X_2(t)$ and $X_2(s)$ are jointly Gaussian (since the process is Gaussian). Thus the conditional density is completely determined by the conditional mean and the conditional variance. Using our results for the conditional mean and variance of jointly Gaussian random variables these quantities are given by:

$$\begin{aligned} E[X_2(t)|X_2(s)=x_s] &= m_{X_2}(t) + \frac{K_{X_2X_2}(t,s)}{K_{X_2X_2}(s,s)}\left[x_s - m_{X_2}(s)\right] = x_s e^{-2\lambda|t-s|} \\ \sigma^2_{X_t|X_s} &= K_{X_2X_2}(t,t) - \frac{K^2_{X_2X_2}(t,s)}{K_{X_2X_2}(s,s)} = 1 - \frac{e^{-4\lambda|t-s|}}{1} = 1 - e^{-4\lambda|t-s|} \end{aligned}$$

So we have that:

$$p_{X_2(t)|X_2(s)}(x_t|x_s) \sim N\left(x_s e^{-2\lambda|t-s|},\ 1 - e^{-4\lambda|t-s|}\right)$$

To show that $X_2(t)$ is not an IIP process, we can simply note that it is WSS. Now recall that an IIP process cannot be WSS, so this cannot be a IIP process! More directly, note that if the increments are independent, then they must at least be uncorrelated. Now it is straightforward to compute the correlation:

$$E\left\{[X_2(t) - X_2(s)]\, X_2(s)\right\} = K_{X_2X_2}(t,s) - K_{X_2X_2}(s,s) = e^{-2\lambda|t-s|} - 1 \neq 0$$

so $X_2(t) - X_2(s)$ and $X_2(s)$ are correlated and thus cannot be independent.

Problem 3.3

Let $N(t)$ be a Poisson counting process on $t \geq 0$ with rate λ. Let $\{Y_i\}$ be a collection of statistically independent, identically-distributed random variables with mean and variance

$$E[Y_i] = m_Y \qquad \text{and} \qquad \text{var}[Y_i] = \sigma_Y^2$$

respectively. Assume that the $\{Y_i\}$ are statistically independent of the counting process $N(t)$ and define a new random process $Y(t)$ on $t \geq 0$ via

$$y(t) = \begin{cases} 0 & N(t) = 0 \\ \displaystyle\sum_{i=1}^{N(t)} Y_i & N(t) > 0 \end{cases}$$

(a) Sketch a typical sample function on $N(t)$ and the associated typical sample function of $Y(t)$.

(b) Use the smoothing property of expectations (condition on $N(t)$ in the inner average) to find $E[Y(t)]$ and $E[Y^2(t)]$ for $t \geq 0$.

(c) Prove that $Y(t)$ is an independent increment process on $t \geq 0$ and use this fact to find the covariance function $K_{YY}(t,s)$ for $t, s \geq 0$.

Solution:

(a) The sample functions are shown in the figure for a case where $m_Y = 0$. Note that the size of the jumps in $Y(t)$ for this case should be not much more than σ_Y^2

(b) First lets find $E[Y(t)] = E\{E[Y(t)|N(t)]\}$. Now

$$E[Y(t)|N(t)=k] = E\left[\sum_{i=1}^{k} Y_i \middle| N(t)=k\right] = km_Y$$

since the Y_i are iid and independent of $N(t)$. Thus

$$E[Y(t)] = E\left[N(t)m_Y\right] = \sum_{k=0}^{\infty} km_Y \Pr(N(t)=k) = m_Y E[N(t)] = m_Y \lambda t$$

This is a pleasing result, as we can see that the average value of $Y(t)$ is given by (average height/jump)$\times$(average # of jumps in time t). Similarly we can compute the correlation using iterated expectation $E[Y^2(t)] = E\left\{E[Y^2(t)|N(t)=k]\right\}$. Now

$$\begin{aligned} E[Y^2(t)|N(t)=k] &= E\left[\left(\sum_{i=1}^{k} Y_i\right)^2 \middle| N(t)=k\right] = \mathrm{Var}\left(\sum_{i=1}^{k} Y_i\right) + \left[E\left(\sum_{i=1}^{k} Y_i\right)\right]^2 \\ &= \left(\sum_{i=1}^{k} \mathrm{Var}(Y_i)\right) + \left[\sum_{i=1}^{k} E(Y_i)\right]^2 = k\sigma_Y^2 + k^2 m_Y^2 \end{aligned}$$

where we have used the facts that $N(t)$ is independent of the Y_i and the Y_i are iid and that $E(x^2) = \mathrm{Var}(x) + m_X^2$. Now:

$$E[Y^2(t)] = E\left[N(t)\sigma_Y^2 + N^2(t)m_Y^2\right] = E[N(t)]\sigma_Y^2 + E[N^2(t)]m_Y^2 = \lambda t\sigma_Y^2 + \left[\lambda t + \lambda^2 t^2\right] m_Y^2$$

where in the last equality we have simply used the mean and correlation of the underlying Poisson process. Note that

$$\begin{aligned} \mathrm{Var}[Y(t)] &= E[Y^2(t)] - E[Y(t)]^2 = \sigma^2\lambda t + m_Y^2 \lambda t = \lambda(\sigma_Y^2 + m_Y^2)t \\ &= (\text{Avg \# jumps in } t \text{ sec})(\text{Var of height/jump}) + (\text{Var in \# jumps in } t \text{ sec})(\text{Avg height/jump}) \end{aligned}$$

(c) For $s < t$, $N(t) \geq N(s)$ thus:

$$Y(t) - Y(s) = \sum_{i=1}^{N(t)} Y_i - \sum_{i=1}^{N(s)} Y_i = \begin{cases} 0 & \text{if } N(t) = N(s) \\ \displaystyle\sum_{i=N(s)+1}^{N(t)} Y_i & \text{else} \end{cases}$$

Consider $Y(s)$:

$$Y(s) = \sum_{i=1}^{N(s)} Y_i$$

Now, note that the increment $Y(t) - Y(s)$ is the sum of $N(s) - N(t)$ mutually independent random variables (the Y_i's). Further $Y(s)$ is the sum of $N(s)$ mutually independent random variables, and since the interval $[t, s]$ doesn't overlap the interval $[s, 0]$ the Y_i's composing $Y(s)$ are independent of the Y_i's composing $Y(t) - Y(s)$. Also, the increment $N(s) - N(t)$ is independent of the increment $N(s)$ since it is a Poisson process and these values are independent of the Y_i's composing $Y(t) - Y(s)$ and $Y(s)$. Thus the increments are independent. Note that if we lose any of these pieces then the increments are not independent – e.g. if somehow $N(s) - N(t)$ was dependent on $Y(s)$.

Now to find $K_{YY}(t,s)$, recall that we said in class that for an arbitrary IIP process $X(t)$, $K_{XX}(t,s) = \text{Var}[X(\min(t,s))]$. To see this note for any IIP process, suppose $t \geq s$:

$$\begin{aligned}
\text{Cov}\left[X(t), X(s)\right] &= \text{Cov}\left\{[X(t) - X(s) + X(s)], X(s)\right\} = \text{Cov}\left\{[X(t) - X(s)], X(s)\right\} + \text{Cov}\left\{X(s), X(s)\right\} \\
&= \text{Var}(X(s))
\end{aligned}$$

where in the last equality term we have used the independence of the increments. Similarly for $s < t$, $\text{Cov}\left[X(t), X(s)\right] = \text{Var}(X(t))$, thus $K_{XX}(t,s) = \text{Var}[X(\min(t,s))]$. Now applying this result to the present case we obtain:

$$K_{YY}(t,s) = \text{Var}[Y(\min(t,s))] = \lambda\left(\sigma_Y^2 + m_Y^2\right)\min(t,s)$$

where we have used the expression for $\text{Var}(Y(t))$ found in part b).

Problem 3.4

Suppose we want to estimate the unknown value of a constant signal by observing and processing a noisy version of the signal for T seconds. Let $X(t) = c + N(t)$, where c is the unknown constant signal value (nonrandom) and $N(t)$ is a zero-mean, stationary bandlimited Gaussian random process with a psd $S_{NN}(f) = N_0$ for $|f| < B$ and zero elsewhere. Assume further that $B >> \frac{1}{T}$. The estimate of c will be the time averaged value:

$$\widehat{c} = \frac{1}{T}\int_0^T X(t)\,dt$$

(a) Show that $E[\widehat{c}] = c$. Such an estimate is termed *unbiased*.

(b) Show that $\text{Var}[\widehat{c}] \approx \frac{N_0}{T}$.

(c) Find the value of T such that $\Pr\left\{|\widehat{c} - c| < 0.1c\right\} \geq 0.999$. Express your answer in terms of c and N_0.

Useful facts:

$$\frac{1}{T^2}\int_0^T\int_0^T g(t_1 - t_2)\,dt_1\,dt_2 = \frac{1}{T}\int_{-T}^T \left[1 - \frac{|\tau|}{T}\right] g(\tau)\,d\tau = \int_{-\infty}^{\infty}\left[\frac{\sin^2(\pi f T)}{(\pi f T)^2}\right] G(f)\,df, \qquad \int_{-\infty}^{\infty}\frac{\sin^2 z}{z^2}\,dz = \pi$$

Solution:

(a) To find the mean:

$$E\left\{\widehat{c}\right\} = E\left\{\frac{1}{T}\int_0^T X(t)\,dt\right\} = E\left\{\frac{1}{T}\int_0^T (c + N(t))\,dt\right\} = \frac{1}{T}\int_0^T (E[c] + E[N(t)])\,dt = c + 0 = c$$

(b) We need to find σ^2:

$$\begin{aligned}
\sigma^2 &= E[(\widehat{c} - c)^2] = E\left\{\frac{1}{T}\int_0^T N(t_1)\,dt_1\,\frac{1}{T}\int_0^T N(t_2)\,dt_2\right\} = \frac{1}{T^2}\int_0^T\int_0^T E\left[N(t_1)N(t_2)\right]\,dt_1\,dt_2 \\
&= \frac{1}{T^2}\int_0^T\int_0^T R_{NN}(t_1 - t_2)\,dt_1\,dt_2 = \frac{1}{T}\int_{-T}^T\left[1 - \frac{|\tau|}{T}\right] R_{NN}(\tau)\,d\tau
\end{aligned}$$

where we have used the formulas given in the problem. Now substituting the inverse PSD expression for $R_{NN}(\tau)$ we obtain:

$$\begin{aligned}
\sigma^2 &= \frac{1}{T}\int_{-T}^T\left[1 - \frac{|\tau|}{T}\right]\left[\int_{-\infty}^{\infty} S_{NN}(f)e^{j2\pi f\tau}\,df\right]d\tau = \int_{-\infty}^{\infty} S_{NN}(f)\left[\frac{1}{T}\int_{-T}^T\left[1 - \frac{|\tau|}{T}\right]e^{j2\pi f\tau}\,d\tau\right]df \\
&= \int_{-\infty}^{\infty} S_{NN}(f)\left[\frac{1}{T}\int_{-T}^T\left[1 - \frac{|\tau|}{T}\right]e^{-j2\pi f\tau}\,d\tau\right]df = \int_{-\infty}^{\infty} S_{NN}(f)\left[\frac{\sin^2(\pi f T)}{(\pi f T)^2}\right]df \\
&= N_0\int_{-B}^{B}\frac{\sin^2(\pi f T)}{(\pi f T)^2}df
\end{aligned}$$

where in the second line we have first used the fact that the sign of the exponent does not change the integral and then the fact that the inner integral is just the transform of a triangular function, given e.g. in a Table of transforms. Now we can make use of the assumption that $B >> \frac{1}{T}$. Let $z = \pi f T$, then

$$\sigma^2 = N_0 \int_{-B}^{B} \frac{\sin^2(\pi f T)}{(\pi f T)^2} df = N_0 \int_{-\pi BT}^{\pi BT} \frac{\sin^2 z}{z^2} \frac{1}{\pi T}\, dz \approx N_0 \int_{-\infty}^{\infty} \frac{\sin^2 z}{z^2} \frac{1}{\pi T}\, dz = \frac{N_0}{\pi T}\pi = \frac{N_0}{T}$$

(c) First note that since the process is a Gaussian process, and the estimate is a linear function of the process, the estimate $\widehat{c}$ itself will be a Gaussian random variable. Since c is a constant, the error $\widehat{c} - c$ will also be a Gaussian random variable:

$$\widehat{c} - c = \frac{1}{T} \int_0^T N(t)\, dt \sim N(m, \sigma^2)$$

From part (a) we know that $m = E(\widehat{c} - c) = 0$ and from part (b) we know that $\sigma^2 = N_0/T$. Therefore $(\widehat{c} - c) \sim N(0, N_0/T)$. Now:

$$\Pr\left[|\widehat{c} - c| < \frac{c}{10}\right] = \int_{-c/10}^{c/10} N(x; 0, N_0/T)\, dx = Q\left(\frac{-c/10}{\sqrt{N_0/T}}\right) - Q\left(\frac{c/10}{\sqrt{N_0/T}}\right) = 1 - 2Q\left(\frac{c/10}{\sqrt{N_0/T}}\right)$$

where $Q(\alpha)$ is the probability $x \geq \alpha$ for the standard Gaussian as tabulated in tables. Now we want to choose T so that:

$$1 - 2Q\left(\frac{c/10}{\sqrt{N_0/T}}\right) \geq 0.999$$

From the table we find that we need:

$$\frac{c/10}{\sqrt{N_0/T}} \geq 3.28 \Longrightarrow T \geq \frac{1076}{c^2} N_0$$

Problem 3.5

Let $X(t)$ be a Gaussian random process, wide-sense stationary, with mean $E[X(t)] = 1$ and power spectral density function $S_{XX}(f)$ given by

$$S_{XX}(f) = \frac{2}{1 + (2\pi f)^2} + \delta(f)$$

(a) Compute the autocovariance function $K_{XX}(\tau)$.

Solution:

(a) To find $K_{XX}(\tau) = R_{XX}(\tau) - m_X^2$ we first need to find $R_{XX}(\tau)$, the inverse fourier transform of $S_{XX}(f)$. Now from the tables we find

$$R_{XX}(\tau) = e^{-|\tau|} + 1$$

From this we can find $K_{XX}(\tau) = R_{XX}(\tau) - 1 = e^{-|\tau|}$

(b) First note that:

$$\int_{-\infty}^{\infty} |K_{XX}(\tau)|\, d\tau = \int_{-\infty}^{\infty} |e^{-|\tau|}|\, d\tau = 2\int_0^{\infty} e^{-\tau}\, d\tau = 2 < \infty$$

so that the autocovariance function is absolutely integrable. Since the process is a Gaussian random process, this means that it is totally ergodic, and thus obviously ergodic in autocorrelation.

Problem 3.6 Consider a Poisson process $N(t)$, with rate 1. Define a new process $X(t) = N(t+2) - N(t) - 2$.

(a) Find the mean and autocorrelation of $X(t)$.

(b) Is $X(t)$ wide-sense stationary? Explain.

(c) If the process is wide-sense stationary, what is its power spectral density $S_{XX}(f)$?

Solution:

(a) First recall that for the Poisson process with rate 1 we have $E[X(t)] = t$ and $R_{NN}(t,s) = ts + \min(t,s)$. Now the mean of $X(t)$ is obtained as

$$E[X(t)] = E[N(t+2)] - E[N(t)] - 2 = t + 2 - t - 2 = 0$$

The autocorrelation is given by

$$\begin{aligned} R_{XX}(t,s) &= E[(N(t+2) - N(t) - 2)(N(s+2) - N(s) - 2)] \\ &= E\left[(N(t+2) - N(t))(N(s+2) - N(s))\right] - 2E\left[N(s+2) - N(s)\right] - 2E\left[N(t+2) - N(t)\right] + 4 \\ &= E\left[(N(t+2) - N(t))(N(s+2) - N(s))\right] - 4 \end{aligned}$$

Now note that if $|t-s| > 2$ then the increments $N(t+2) - N(t)$, and $N(s+2) - N(s)$ do not overlap, are independent, and so the value of $R_{XX}(t,s) = 0$ for this case. When this is not the case, i.e. when $|t-s| < 2$:

$$\begin{aligned} R_{XX}(t,s) &= E\left[(N(t+2) - N(t))(N(s+2) - N(s))\right] - 4 \\ &= E[N(t+2)N(s+2)] - E[N(t+2)N(s)] - E[N(t)N(s+2)] + E[N(t)N(s)] - 4 \\ &= (t+2)(s+2) + \min(t,s) + 2 - (t+2)s - \min(t+2,s) - t(s+2) - \min(t,s+2) + ts + \min(t,s) - 4 \\ &= 2 + 2\min(t,s) - s - t = 2 - |t-s| \end{aligned}$$

Thus, putting the pieces together we get:

$$R_{XX}(t,s) = \begin{cases} 0 & \text{if } |t-s| > 2 \\ 2 - |t-s| & \text{otherwise} \end{cases}$$

(b) Yes the process is WSS, since it has constant mean and autocorrelation depending on the time difference only.

(c) We need to take the transform of the triangular function represented by $R_{XX}(\tau)$. This is actually in the transform table in the notes and yields:

$$S_{XX}(f) = 4\frac{\sin^2(2\pi f)}{(2\pi f)^2}$$

Problem 3.7 Let $N_1(t)$ and $N_2(t)$ be two independent Poisson processes with rates $\lambda_1 = 1$ and $\lambda_2 = 2$ respectively.

1. Find the probability of no arrivals in $(3,5]$ for $N_1(t)$ and $N_2(t)$.

2. Find the probability that there are two arrivals in $(0,2]$ and three arrivals in $(1,4]$.

3. Find the probability that the second arrival in $N_1(t)$ occurs before the third arrival in $N_2(t)$. Hint: Think of $N_1(t)$ and $N_2(t)$ as two processes obtained from splitting a Poisson process $N(t)$ with rate $\lambda_1 + \lambda_2$.

Solution:

1. For $N_1(t)$ let Y be the random variables associated with number of arrivals in (3,5]. Y is a Poisson random variable with rate 2. So $P(Y=0)=e^{-2}$ and for $N_2(t)$ the corresponding probability is e^{-4}.

2. Let X, Y, Z be the numbers of arrivals in $(0,1]$, $(1,2]$, $(2,4]$ respectively. Then X, Y, Z are independent Poisson random variables with rate λ, λ and 2λ respectively. Let A be the event that there are two arrivals in $(0,2]$ and three arrivals in $(1,4]$. We can use the law of total probability to obtain $P(A)$. In particular,

$$P(A)=P(X+Y=2,Y+Z=3)=\sum_{k=0}^{\infty}P(X+Y=2,Y+Z=3\mid Y=k)P(Y=k)$$

This leads to:

$$\begin{aligned}P(A)=&P(X=2,Z=3\mid Y=0)P(Y=0)+P(X=1,Z=2\mid Y=1)P(Y=1)\\&+P(X=0,Z=1|Y=2)P(Y=2)\\&=P(X=2,Z=3)P(Y=0)+P(X=1,Z=2)P(Y=1)+P(X=0,Z=1)P(Y=2)\\&=P(X=2)P(Z=3)P(Y=0)+P(X=1)P(Z=2)P(Y=1)+P(X=0)P(Z=1)P(Y=2)\\&=(e^{\lambda}\lambda^2/2)(e^{2\lambda}(2\lambda)^3/6)(e^{\lambda})+(\lambda e^{\lambda})(e^{2\lambda}(2\lambda)^2/2)(\lambda e^{\lambda})+(e^{\lambda})(e^{2\lambda}(2\lambda))(e^{\lambda}\lambda^2/2)\end{aligned}$$

3. Let $N(t)$ be a Poisson process with rate $=1+2=3$. We split $N(t)$ into two processes $N_1(t)$ and $N_2(t)$ in the following way. For each arrival, a coin with $P(H)=1/3$ is tossed. If the coin lands heads up, the arrival is sent to the first process ($N_1(t)$), otherwise it is sent to the second process. The coin tosses are independent of each other and are independent of $N(t)$. Then $N_1(t)$ is a Poisson process with rate $p=1$; $N_2(t)$ is a Poisson process with rate $(1p)=2$; $N_1(t)$ and $N_2(t)$ are independent. Thus, $N_1(t)$ and $N_2(t)$ have the same probabilistic properties as the ones stated in the problem. We can now restate the probability that the second arrival in $N_1(t)$ occurs before the third arrival in $N_2(t)$ as the probability of observing at least two heads in four coin tosses, which is $\sum_{k=2}^{4}\binom{4}{k}(1/3)^k(2/3)^{4-k}$.

Problem 3.8 For each of the autocorrelation functions below, find the power spectral density function.

(a) $e^{-a|\tau|}$

(b) $\dfrac{\sin(1000\tau)}{1000\tau}$

(c) $\dfrac{1}{4}e^{-|\tau|}\left[\cos|\tau|+\sin|\tau|\right]$

(d) $e^{-.01f_0^2\tau^2}$

(e) $\cos(1000\tau)$

Solution: Note that all transforms are in cycles per second.

(a) $\frac{2a}{a^2+(2\pi f)^2}$

(b) $\frac{\pi}{1000}\left[u(2\pi f+1000)-u(2\pi f-1000)\right]$

(c) This one takes a bit of work. Lets break it into pieces and find the transform of $e^{-|\tau|}\cos|\tau|$ first. First we can rewrite this as:

$$e^{-|\tau|}\cos|\tau| \quad = \quad \frac{1}{2}e^{-|\tau|}\left[e^{j|\tau|}+e^{-j|\tau|}\right]=\frac{1}{2}\left[e^{-\frac{|\tau|}{1/(1-j)}}+e^{-\frac{|\tau|}{1/(1+j)}}\right]$$

Now looking in the tables we find that the transform of the terms is given by:

$$\begin{aligned}
&\mathcal{F}\left\{e^{-|\tau|}\cos|\tau|\right\} \\
&= \frac{1}{2}\left[\frac{2/(1-j)}{1+\frac{(2\pi f)^2}{(1-j)^2}}+\frac{2/(1+j)}{1+\frac{(2\pi f)^2}{(1+j)^2}}\right] = \frac{1-j}{(1-j)^2+(2\pi f)^2}+\frac{1+j}{(1+j)^2+(2\pi f)^2} \\
&= \frac{1-j}{-2j+(2\pi f)^2}+\frac{1+j}{+2j+(2\pi f)^2} = \frac{4+2(2\pi f)^2}{4+(2\pi f)^4}
\end{aligned}$$

By a similar argument we can show that:

$$\mathcal{F}\left\{e^{-|\tau|}\sin|\tau|\right\} = \mathcal{F}\left\{\frac{1}{2j}e^{-|\tau|}\left[e^{j|\tau|}-e^{-j|\tau|}\right]\right\} = \frac{4-2(2\pi f)^2}{4+(2\pi f)^4}$$

Combining and dividing by 4 gives for the overall transform:

$$\mathcal{F}\left\{\frac{1}{4}e^{-|\tau|}\left[\cos|\tau|+\sin|\tau|\right]\right\} = \frac{2}{4+(2\pi f)^4}$$

(d) $\frac{10\sqrt{\pi}}{f_0}e^{\frac{-100\pi^2 f^2}{f_0^2}}$

(e) $\frac{1}{2}\left[\delta(2\pi f-1000)+\delta(2\pi f+1000)\right]$

Problem 3.9 Let $W(t)$ be a standard Brownian motion.

1. Find $\text{Prob}(W(1)+W(2)>2)$.
2. Find the conditional PDF of $W(s)$ given $W(t)=a$ for $0\le t\le s$.
3. Let $X(t)=\exp(W(t))$ for $0\le t<\infty$. Compute mean, variance and covariance functions for this process.

Solution:

1. $X=W(1)+W(2)$ is a Gaussian random variable with mean zero and variance 5. Therefore, $P(X>2)=0.186$.
2. Again $W(s)\mid W(t)$ is a Gaussian because conditioning preserves Gaussianity. We note that,

$$E\left[W(s)\mid W(t)=a\right]=\frac{s}{t}a,\ \text{Var}(W(s)\mid W(t)=a)=s(1-s/t)$$

   This implies that $W(s)\mid W(t)=a\sim N(\frac{s}{t}a, s(1-s/t))$.

3. $W(t)\sim N(0,t)$ is Gaussian random variable. Therefore, $E(\exp(W(t))$ is just the moment generating function (MGF) evaluated at $s=1$. From your notes we know that for a Gaussian random variable, $Y\sim N(\mu,\sigma^2)$ the MGF is given by,

$$E[\exp(sY)]=\exp\left\{s\mu+\sigma^2 s^2/2\right\}$$

   Therefore, $E[\exp(W(t)]=\exp(t/2)$. $\text{Var}(X(t))=E[X^2(t)]-(E[X(t)])^2=\exp(2t)-\exp(t)$. Similarly, we can also compute the covariance,

$$\text{Cov}(X(s),X(t))=E[X(s)X(t)]-E[X(s)]E[X(t)]=E[(X(t)-X(s))X(s)]+E[X^2(s)]-E[X(s)]E[X(t)]$$

   We now invoke the IIP property for Brownian motion to conclude that $X(t)-X(s)$ is independent of $X(s)$ (note that $s<t$). Substituting we obtain

$$\text{Cov}(X(s),X(t))=\exp((3s+t)/2)-\exp((s+t)/2)$$

Computer Problems

Problem 3.10 Special Random Processes

In this project we will construct and explore some special random processes discussed in class.

(a) **The Poisson Process:**

We will write a function `poisson.m` to construct sample paths of a Poisson random process by following the steps outlined in the class. A Poisson process is often used to model e.g. arrivals, lines, queues, etc., such as found in banks, computer and network packet traffic, telephone hold queues, etc. The function call for our program will be the following: `[T,NT]=poisson(Na,lam);`, where `Na` will be the desired number of arrivals to generate, `lam` will be the arrival rate of the process, `T` will be the generated vector of the arrival times, and `NT` will be the corresponding number of arrivals that have occured up to the times in `T`. Each step below will form a line of the program:

(i) The first step is to generate a vector `tau` of `Na` independent, exponentially distributed, interarrival times. This can be done using the supplied function `randexp.m` as follows:

```
tau = randexp(Na,1,lam);
```

Note that `randexp.m` generates exponentially distributed random variables using the method developed in Problem Set #1.

(ii) Now given these interarrival times, we generate the corresponding vector of waiting times `T` for each arrival as the cumulative sum of the interarrival times:

```
T = cumsum(tau);
```

(iii) Finally, we can generate the corresponding vector containing the number of arrivals at each time `NT` by simply counting up (since each arrival increments the count by 1):

```
NT = [1:length(T)]';
```

Add these steps together to create your program and generate a few sample paths of the Poisson process for various choices of the rate parameter `lam`. Plot them using `stairs.m`. Note that we are generating the process by finding the *times* at which a particular event (a jump) takes place, not by finding the values the process takes at certain times, as we usually do. Comment on the difficulty this will cause in estimating ensemble averages and the like.

One the web stie you will find the function `poissrp.m`, which *does* generate sample path values of Poisson random processes at fixed sets of times. Using the supplied function, generate multiple Poisson process sample paths and estimate and plot the mean and autocovariance function of the process. Compare to the theoretical functions of these quantities discussed in class.

(b) **A Random Telegraph Wave:**

Recall that (one definition) of a random telegraph wave is $X(t) = Z(-1)^{N(t)}$ where $N(t)$ is a Poisson random process and $Z = \pm 1$ with equal probability. Use the output of `poissrp.m` to write a function `telerp.m` which generates sample paths of a random telegraph wave. Again, estimate and plot the mean process and autocovariance function.

(c) **The Wiener Process:**

We will write a function `wiener.m` to construct a sample of a Wiener-Levy random process on the interval [0,1] as the limit of a discrete-time random walk following the steps outlined in the class. The function call will be the following: `[X,t] = wiener(T,alpha);`, where `alpha` will be the ratio of the step size squared to time interval s^2/T, `T` will be the time step, `Z` will be the vector process values at the times in the vector `t`. Again, each step below will form a line of the program:

(i) First generate the set of time points we will use based on the input information:

```
t = 0:T:1;
```

(ii) Next, set the step size `s` of the discrete random walk. For convergence, we required that this step size scale with the time interval as $s^2/T = \alpha$:

```
s = sqrt(alpha*T);
```

(iii) Generate a random vector of positive and negative jumps of size `s` by scaling Bernoulli trials (which we generate by rounding uniform random variables to ± 1):.

```
z = round(rand(length(t),1)); % Bernoulli Trials via rounded uniform RVs
jumps =  s*( sign(z - .5) );
```

(iv) Finally, we generate the Wiener process value at each time as the cumulative sum of the jumps:

```
X = cumsum(jumps);
```

Add these steps together to create your program and generate a few sample paths of the Wiener process for various choices of the parameter `alpha`. You can plot your sample paths via `stairs(t,X)`. As the sampling interval `T` is made smaller, the discrete time random walk should converge to the Wiener process. Start with `T=.1` and progressively decrease it, generating and plotting sample paths along the way. Does the process appear to converge? Plot a sample path corresponding to a small value of `T`. Zoom in on a portion of your plot in MATLAB by typing `zoom on` and then clicking where you want to zoom. The Wiener process is often called "self-similar" – can you explain why from this experiment?

On the web site you will find the function `wienerp.m` which will generate multiple sample paths of our approximate Weiner process at arbitrary sampling points. Generate multiple sample paths and estimate and plot the mean and autocovariance functions of the process. Compare to the theoretical functions of these quantities discussed in class.

Solution:

(a) **The Poisson Process :**

The code for the function `poisson.m` is given below :

```
function [T,NT] = poisson(Na,lam)
                                % [T,NT] = poisson(Na,lam)
%
% N   : Number of arrivals to  generate
% lam : Arrival rate. OPTIONAL. Default l=1.
%
% T   : Vector of arrival times
% NT  : Number of arrivals upto time T

if max(size(Na))>1
  error('Na must be a scalar number of arrivals to generate')
end;

if nargin<2
  l = 1;
end;

% Generate Na exponential interarrival
% Tau contains the set of exponential interarrival times for each experiment
% -- one per row
Tau = randexp(Na,1,lam);
```

```
% Generate Waiting times from interarrival times
T = cumsum(Tau')';

% Generate number of arrivals at each time, NT, by counting up from
% 1 to total number of arrivals
NT = [1:length(T)]';
```

Using the above function, several sample paths of the Poisson process were generated for different values of the rate parameter `lam`. These plots are shown in Figure 1(a). Since the function `poisson.m` generates the *times* at which the event occurs and not the actual process value at samples of time we cannot estimate the mean (or covariance) of the process simply by the mean (covariance) of several sample path vectors. Instead we need to generate the process values at fixed times in order to do this. This is done by the supplied routine `poissonrp.m`. Figure 1(b) shows several such sample paths of the Poisson random process generated for the same rate parameter of $\lambda = 1$. The empirical mean and covariance of the process, calculated using 1000 sample paths, are shown in Figure 1(c) and Figure 1(d) respectively. We observe that the empirical mean of the process is close to it's theoretical value which is given by λt. The covariance too matches the theoretical value of $K_{XX}(t_1, t_2) = \lambda \min(t_1, t_2)$.

(a) (b)

(c) (d)

Figure 1: Poisson Process : (a) Sample paths for different values of λ, (b) Sample paths for $\lambda = 1$, (c) Sample path and empirical mean for $\lambda = 1$, (d) Empirical autocovariance of the process.

(b) **A Random Telegraph Wave**

The `MATLAB` code for the function `telerp.m` is given below:

```
function [X,t] = telerp(N,t,l)
% [X,t] = telerp(N,t,l)
%
% N : Number of sample paths to generate
% t : Vector of time points at which to generate samples
% l : Arrival rate. OPTIONAL. Default l=1.
%
% X : Matrix of process sample paths. Each row is a sample path, each column
%     is a different time point.
%
% X(t) = Random Telegraph Wave with rate l
%
%               N(t)
%  X(t) = Z (-1)
%
%  where N(t) is a Poisson process with rate l and Z = +/- 1 with equal
%  probability.

% W. C. Karl 1/20/98

if min(size(t)) > 1
  error('t must be a vector of time point')
end;

if max(size(N))>1
  error('N must be a scalar number of sample paths to generate')
end;

if (round(t)-t)~= 0
   error('t must be a vector of integers');
end

if nargin<3
  l = 1;
end;

% Generate sample paths of Poisson random process P(t)
[P,t] = poissrp(N,t,l);

% Generate vector of random variable Z (= +1/-1 with equal probability)
Z = sign(randn(N,1));

% Generate sample paths of Random telegraph wave
X = (Z*ones(1,length(t))).*(-1).^P;
```

A few sample paths generated by this function are shown in Figure 2(a). The mean and autocovariance of the process were empirically calculated using 1000 sample paths, and are plotted in Figure 2(b) and Figure 2(c) respectively.

(c) **Wiener Process**

The code for the function `wiener.m` to construct a sample path of the Wiener-Levy random process is

(a)

(b) (c)

Figure 2: Random Telegraph Wave : (a) Sample paths of the process, (b) Sample path and empirical mean, (d) Empirical autocovariance of the process.

given below :

```
function [X,t] = wiener(T,alpha)
% [X,t] = wiener(T,alpha)
%
% T     : Time step of approximation
% alpha : Variance parameter of resulting process. OPTIONAL. Default: alpha=1
%
% X     : Approximation to Wiener-Levy process on the interval t=[0,1].
%         [X]_k = X(kT) with X(t) the approximation of the process.
% t     : Vector of corresponding time points at which X(t) is being
%         sampled. i.e. t(k) = kT.
%
% Generates discrete approximation to the Wiener-Levy process or Brownian
% motion. Recall this process is defined as the limit of a discrete time
% random walk. In particular, let
%
%          inf                            / +s; Prob = .5
% X_T(t) = sum W[k] u(t-kT), where W[k] = |
%          k=0                            \ -s; Prob = .5
```

```
%
% is a sequence of i.i.d. scaled Bernoulli trials. Then the Wiener-Levy
% process is obtained as:
%
%     lim    X_T(t)
%     T->0
%  s^2/T=alpha
%
% This program recreates that process over a fixed interval [0,1]. Recall
% that m_x=0, Var[X(t)]=alpha t

% W. C. Karl 2/5/97

if max(size(T))>1
  error('T must be a scalar')
end;
if T<=0 | T>1
  error('T is outside of the allowed range')
end;
if nargin<2
  alpha = 1;
end;

% Generate time axis and N=number of points
t = 0:T:1;
N = length(t);

% For convergence the step size of the discrete time random walk must scale
% with alpha and T so that s^2/T = alpha. This assures that.
s = sqrt(alpha*T);

% Now generate a sequence of Bernoulli trials and scale it to +-s
z = rand(length(t),1) > .5;    % Bernoulli Trials
jumps =  s*( sign(z - .5) );

% Now generate the Wiener process at cumulative sum of values in jumps
X = cumsum(jumps);
```

Figure 3(a) shows a few sample paths generated by this function for different values of `alpha`. We observe that by reducing `alpha` we are reducing the step-size `s` of the discrete random walk used to approximate the Wiener process. Figure 3(b) illustrates the effect of reducing the sampling interval `T`. By zooming onto a sample path we observe that the Wiener process looks similar at different time-scales and hence is called "self-similar". As seen in Figure 3(c) and 3(d) the Wiener process has zero mean and autocovariance $K_{XX}(t_1, t_2) = \alpha \min(t_1, t_2)$.

(a) (b)

(c) (d)

Figure 3: Wiener Process : (a) Sample paths for different values of α, (b) Sample paths for different values of T, (c) Sample path and empirical mean calculated using 1000 sample paths, (d) Empirical autocovariance of the process.