Boston University

Department of Electrical and Computer Engineering

SC505 STOCHASTIC PROCESSES

Exam 2 Summary

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1. **Linear Systems and Random Processes:** $Y(t) = \int h(t,\tau)X(\tau)\,d\tau$
   - Complete characterization difficult $\Longrightarrow$ Use second order relationships
   - General Relations between 2nd order statistics:
     - Mean: $m_Y(t) = \int_{-\infty}^{\infty} h(t,\tau)m_X(\tau)\,d\tau$
     - Cross-correlation: $R_{YX}(t,s) = \int_{-\infty}^{\infty} h(t,\tau)R_{XX}(\tau,s)\,d\tau$
     - Output-correlation: $R_{YY}(t,s) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(t,\tau)R_{XX}(\tau,\sigma)h(s,\sigma)\,d\tau d\sigma$
2. **LTI Systems and Wide-Sense Stationary Processes:**
   - If $h(t)$ LTI and $X(t)$ WSS $\Longrightarrow$ $X(t)$,$Y(t)$ are JWSS
   - LTI Time-domain Relations between 2nd order statistics:
     - Mean: $m_Y = m_X \int_{-\infty}^{\infty} h(\tau)\,d\tau = H(0)m_X$
     - Cross-correlation: $R_{YX}(t) = \int_{-\infty}^{\infty} h(-\tau)R_{XX}(t-\tau)\,d\tau = h(-t) * R_{XX}(t)$
     - Output-correlation: $R_{YY}(t) = h(t) * R_{XX}(t) * h(-t)$
   - LTI Frequency-domain Relations:
     - Cross-PSD: $S_{YX}(j\omega) = H(-j\omega)S_{XX}(j\omega)$ or $S_{YX}(s) = H(-s)S_{XX}(s)$
     - Output-PSD: $S_{YY}(j\omega) = |H(j\omega)|^2 S_{UU}(j\omega)$ or $S_{YY}(s) = H(s)H(-s)S_{UU}(s)$
   - Shaping Filter: LTI system $H(s)$ driven by white noise.
   - Properties of PSD of $S_{YY}(s)$: <u>Quadrantal Symmetry of poles and zeros</u>.
   - Spectral Factorization: Can always write $S_{YY}(s) = G(s)G(-s)$ with $G(s)$ stable and causal. Yields shaping filter for given $S_{YY}(s)$.
3. **DT Linear Models:**
   - Autoregressive (AR). All pole model, IIR
     - $x(n) = \sum_{i=1}^{P} a_i x(n-i) + w(n)$, $R_{WW}(n) = \sigma^2\delta(n)$
     - $R_{XX}(m) = \sum_{i=1}^{P} a_i R_{XX}(m-i) + \sigma^2\delta(m)$
     - <u>Yule-Walker Equations</u>. Linear equations for coefficients a_i:

$$\begin{bmatrix} R_{XX}(1) \\ R_{XX}(2) \\ \vdots \\ R_{XX}(P) \end{bmatrix} = \begin{bmatrix} R_{XX}(0) & R_{XX}(1) & \cdots & R_{XX}(P-1) \\ R_{XX}(1) & R_{XX}(0) & & R_{XX}(P-2) \\ \vdots & & \ddots & \vdots \\ R_{XX}(P-1) & R_{XX}(P-2) & \cdots & R_{XX}(0) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_p \end{bmatrix}$$

- Moving Average (MA). All zero model, FIR
  - $x(n) = \sum_{k=1}^{Q} b_k w(n-k) + w(n)$, $R_{WW}(n) = \sigma^2 \delta(n)$
  - Nonlinear equations for coefficients b_i:

$$R_{XX}(m) = \begin{cases} \sigma^2 \left(\sum_{k=m}^{Q} b_k b_{k-m} \right) = \sigma^2 b(m) * b(-m) & m \le Q \\ 0 & m > Q \end{cases}$$

  - Correlation function is finite for MA
- Autoregressive Moving Average (ARMA). Both poles and zeros
  - $x(n) = \sum_{i=1}^{P} a_i x(n-i) + \sum_{k=1}^{Q} b_k w(n-k) + w(n)$, $R_{WW}(n) = \sigma^2 \delta(n)$
  - In general, equations for coefficients a_i, b_k are nonlinear:

$$R_{XX}(m) = \sum_{i=1}^{P} a_i R_{XX}(m-i) + \sum_{j=0}^{Q} b_j R_{XW}(m-j), \qquad b_0 = 1$$

  - For $m > Q$: $R_{XX}(m) = \sum_{i=1}^{P} R_{XX}(m-i)$.
  - Solution approach:
    (a) Solve linear equations for $m > Q$ for AR coefficients
    (b) Solve NL equations for MA coefficients: $S_{XX}(z)A(z)A(z^{-1}) = B(z)B(z^{-1})\sigma^2$

4. **Sampling of Random Processes:**

- Stochastic Nyquist Criterion Exists
- Result: If $S_{XX}(\omega) = 0$ for $|\omega| > W$ (i.e. is bandlimited) and $T_s < \pi/W$ then:

$$\lim_{N\to\infty} \left[2T_s \frac{W}{2\pi} \sum_{n=-N}^{N} X(nT_s) \frac{\sin W(t-nT_s)}{W(t-nT_s)} \right] \overset{mss}{=} X(t)$$

- Result: If $S_{XX}(\omega) = 0$ for $|\omega| > W$ and $T_s < \pi/W$ then:

$$\lim_{N\to\infty} 2T_s \frac{W}{2\pi} \sum_{n=-N}^{N} R_{XX}(nT_s) \frac{\sin W(\tau-nT_s)}{W(\tau-nT_s)} = R_{XX}(\tau)$$

5. **Detection/Hypothesis Testing:**

- Deterministic Decision Rule: Mapping of Observation space onto H_0, H_1.
- Bayes Risk Approach:
  - Priors $P_i = \Pr(H_i)$
  - Observation Model: $P(y|H_i)$
  - Costs: C_{ij} = Cost of deciding H_i when H_j true.
  - Choose decision rule to $\min E(\text{cost})$
  - Likelihood Ratio Test (LRT) minimizes $E(\text{cost})$:

$$\mathcal{L}(y) = \frac{P_{Y|H_1}(y|H_1)}{P_{Y|H_0}(y|H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} \frac{(C_{10}-C_{00})P_0}{(C_{01}-C_{11})P_1} = \eta$$

- Special Cases:
  - MPE: $C_{ij} = 1 - \delta_{ij} \Longrightarrow$ MAP decision rule

$$P_{H_1|Y}(H_1|y) \underset{H_0}{\overset{H_1}{\gtrless}} P_{H_0|Y}(H_0|y)$$

  - MPE and $P_0 = P_1 = 1/2 \Longrightarrow$ ML decision rule

$$P_{Y|H_1}(y|H_1) \underset{H_0}{\overset{H_1}{\gtrless}} P_{Y|H_0}(y|H_0)$$

  - Gaussian Problems
- Randomized Tests: Given Two LRTs (LRT_1 and LRT_2) use LRT_1 with probability p and LRT_2 with probability $1-p$. Has performance on line connecting two P_D,P_F pairs.

- Discrete Random Variables – know how to handle
- Sufficient Statistic: Function of the data that contains all information needed for test
- Performance:
  - $E[\text{Cost}] = \underbrace{C_{00}P_0 + C_{01}P_1}_{\text{Fixed Cost}} + \underbrace{(C_{10} - C_{00})P_0P_F + (C_{11} - C_{01})P_1P_D}_{\text{Fn of threshold } \eta}$
  - $\Pr(\text{error}) = \Pr[\text{choose } H_0,\ H_1 \text{ true}] + \Pr[\text{choose } H_1,\ H_0 \text{ true}] = (1 - P_D)P_1 + P_FP_0$
  - Both only depend on:

$$\begin{aligned} P_D &= \Pr(\text{Choose } H_1|H_1) = \int_{\{y|\text{say } H_1\}} P(y|H_1)\,dy = \int_{\mathcal{L}>\eta} P(\mathcal{L}|H_1)\,d\mathcal{L} \\ P_F &= \Pr(\text{Choose } H_1|H_0) = \int_{\{y|\text{say } H_1\}} P(y|H_0)\,dy = \int_{\mathcal{L}>\eta} P(\mathcal{L}|H_0)\,d\mathcal{L} \end{aligned}$$

  - Receiver Operating Characteristic: (ROC): Plot of P_D vs P_F as threshold is varied.
    - Know properties – Concave, etc
    - Discrete random variables: ROC consists of points
    - Role of randomized tests
- Minimax Tests: Minimize the maximum $E[\text{Cost}]$ as P_1 is varied. Minimax test satisfies $P_D = \left(\dfrac{C_{01} - C_{00}}{C_{01} - C_{11}}\right) - \left(\dfrac{C_{10} - C_{00}}{C_{01} - C_{11}}\right) P_F$
- Neyman-Pearson Tests: Maximize P_D subject to $P_F \leq \alpha$. Solution is a LRT for some threshold
- M-ary Bayes Hypothesis Tests:
  - Solution is:

$$\text{Choose } H_k \text{ if} \quad \sum_{j=0}^{M-1} C_{kj}P(H_j|y) \leq \sum_{j=0}^{M-1} C_{ij}P(H_j|y) \quad \forall i$$

  - Generates set of $M(M-1)/2$ unique comparisons defining decision regions:

$$\sum_{j=0}^{M-1} C_{kj}P_jP_{Y|H_j}(y|H_j) \underset{\text{Not } H_i}{\overset{\text{Not } H_k}{\gtrless}} \sum_{j=0}^{M-1} C_{ij}P_jP_{Y|H_j}(y|H_j) \quad \forall\, i,k \text{ pairs}$$

- Define $L_j(y) = \frac{P_{Y|H_j}(y|H_j)}{P_{Y|H_0}(y|H_0)}$, then test is:

$$\sum_{j=0}^{M-1} C_{kj} P_j L_j(y) \underset{\text{Not } H_i}{\overset{\text{Not } H_k}{\gtrless}} \sum_{j=0}^{M-1} C_{ij} P_j L_j(y) \qquad \forall\, i,k \text{ pairs}$$

<u>Linear Decision Boundaries in L_i space</u>

- Special Cases:
  - <u>MPE</u> Cost Assignment $C_{ij} = 1 - \delta_{ij} \implies$ <u>MAP decision rule</u>:

$$\text{Choose } H_k \text{ if} \quad P(H_k|y) \geq P(H_i|y) \quad \forall i$$

  - MPE and $P_i = 1/M \implies$ <u>ML decision rule</u>:

$$\text{Choose } H_k \text{ if} \quad P(y|H_k) \geq P(y|H_i) \quad \forall i$$

  - Minimum Distance Classifier
    - $P_{\underline{Y}|H_K}(\underline{y}|H_k) = N(\underline{y}; \underline{m}_k, I)$
    - ML Rule, $P_k = 1/M$
    - $\implies$ <u>Minimum Distance Classifier</u>

$$\text{Choose } H_k \text{ if} \quad ||\underline{y} - \underline{m}_k||^2 \leq ||\underline{y} - \underline{m}_i||^2 \quad \forall i$$

6. **Series Expansions, KLE, and Detection of Continuous Time Processes:**

- Series expansions of stochastic processes: $X(t) = \sum_{i=1}^{\infty} X_i \phi_i(t)$
- KLE:
  - Find good basis functions for stochastic processes
  - Want uncorrelated coefficients: $E[X_i X_j] = \lambda \delta_{ij}$
  - Basis given by solutions to KL equation:

$$\int_{T_0}^{T_1} R_{XX}(t,\tau)\phi_m(\tau)\, d\tau = \lambda_m \phi_m(t)$$

  - Eigendecomposition of $R_{XX}(t,\tau)$
  - Gives optimal <u>approximation</u> of $X(t)$.
  - White Noise: Every complete orthonormal basis (CON) is a KL basis
- Detection of CT waveforms
  - 1 Known signal in white noise:

$$\begin{aligned} H_0 &: \quad y(t) = w(t), \qquad R_{WW}(\tau) = \sigma^2 \delta(\tau) \\ H_1 &: \quad y(t) = s(t) + w(t) \end{aligned}$$

    - Choose $\phi_1(t) = s(t)/\sqrt{E}$ and remaining ϕ_i to form CON set
    - y_1 is a sufficient statistic for problem
    - Matched Filter: $y_1 = \int y(s)s(t)\, dt \underset{H_0}{\overset{H_1}{\gtrless}} \gamma$
    - Performance depends on signal energy, not structure.

- 2 Known signals in white noise:

$$
\begin{aligned}
H_0: &\quad y(t) = s_0(t) + w(t), \qquad R_{WW}(\tau) = \delta(\tau) \\
H_1: &\quad y(t) = s_1(t) + w(t)
\end{aligned}
$$

  * Approach 1) Let $y'(t) = y(t) - s_0(t)$ and apply previous results
  * Approach 2) Let subset of basis functions span signal subspace

- M Known signals in white noise:

$$
\begin{aligned}
H_0: &\quad y(t) = s_0(t) + w(t), \qquad R_{WW}(\tau) = \delta(\tau) \\
H_1: &\quad y(t) = s_1(t) + w(t) \\
H_2: &\quad y(t) = s_2(t) + w(t) \\
&\quad \vdots \\
H_M: &\quad y(t) = s_M(t) + w(t)
\end{aligned}
$$

  * Project onto signal subspace. Choose $\phi_1(t), \cdots, \phi_{M+1}(t)$ to span the space of the signals.

$$
\begin{array}{llllll}
H_0: & y_1 = s_{01} + w_1 & H_1: & y_1 = s_{11} + w_1 & H_M: & y_1 = s_{M1} + w_1 \\
 & y_2 = s_{02} + w_2 & & y_2 = s_{12} + w_2 & & y_2 = s_{M2} + w_2 \\
 & \vdots & & \vdots & \cdots & \vdots \\
 & y_M = s_{0M} + w_M & & y_M = s_{1M} + w_M & & y_M = s_{MM} + w_M \\
 & y_{M+1} = w_{M+1} & & y_{M+1} = w_{M+1} & & y_{M+1} = w_{M+1} \\
 & \vdots & & \vdots & & \vdots
\end{array}
$$

- Known signals in correlated noise:

$$
\begin{aligned}
H_0: &\quad y(t) = s_0(t) + w(t), \qquad R_{WW}(\tau) \neq \delta(\tau) \\
H_1: &\quad y(t) = s_1(t) + w(t)
\end{aligned}
$$

  * Choose $\phi_i(t)$ via KLE of noise $w(t)$. w_i uncorrelated, but need all coefficients in general.

$$
\begin{array}{llll}
H_0: & y_1 = w_1 & H_1: & y_1 = s_1 + w_1 \\
 & y_2 = w_2 & & y_2 = s_2 + w_2 \\
 & y_3 = w_3 & & y_3 = s_3 + w_3 \\
 & \vdots & & \vdots
\end{array}
$$

  * In practice, truncate after some number of terms

7. **Advice:**
   - Have basic results at your fingertips
   - Know the assumptions/conditions behind formulas that you use!
   - Perform sanity checks on answers – go back to basics if totally stuck (i.e. defining equation of expectation, variance etc)