Boston University
Department of Electrical and Computer Engineering

SC505 STOCHASTIC PROCESSES

FINAL

December 18, 2002

NAME:________________________________

ID NUMBER:________________________________

Honor Code: This exam represents only my own work. I did not give or receive help on this exam.

Signature:________________________________ Date:__________

Instructions:

- This is a closed-book exam, but three $8\frac{1}{2}$" $\times$ 11" sheets of notes (both sides) are allowed.
- There are 4 problems on the exam, approximately equally weighted.
- The problems are not necessarily in order of difficulty.
- You should concisely indicate your reasoning and **show all relevant work** for each problem. Partial credit will be given for partial work. No credit will be given for no work.
- All work you want graded must go in this booklet. Use the blue books for scratch work.
- If it cannot be read, it cannot be graded.

(1) ________________

(2) ________________

(3) ________________

(4) ________________

Total: ________________

Problem #1:

Mark each statement as true or false. Give a brief justification for each answer (No credit will be given if there is no justification).

(a) In a Kalman filter, since we have more data as time increases, we must have $P(t+1 \mid t+1) \leq P(t \mid t)$.

(b) The Kalman filter is a recursive algorithm for computing the Bayes least square error estimate of $X(n)$ based on the window of data $Y(m)$, $n_0 \leq m \leq n$.

(c) If the Bayes least square estimate of random variable X based on Y is linear, then X and Y are jointly Gaussian.

(d) Maximum likelihood estimators of non-random parameters are always efficient estimators.

(e) If $X(t)$ is a Gaussian random process, then $Y(t) = |t|X(t)$ is a Gaussian random process.

Answer #1:

Answer #1 (Continued):

Problem #2:

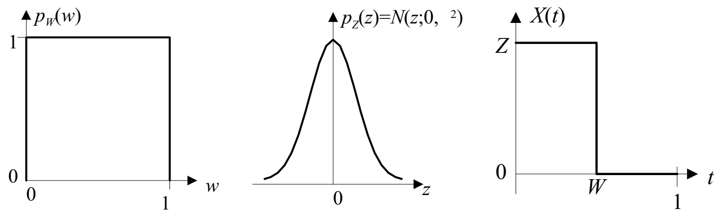
Let W and Z be independent random variables, with W uniformly distributed and Z Gaussian distributed, as illustrated in the figure below. The process $X(t)$ is defined on the interval $[0,1]$ in terms of the random variables W and Z as shown on the right in the figure below. The process $X(t)$ is known to be Markov.

(a) What is the linear least square error estimate $\widehat{X}_{LLSE}$ of $X(t_2)$ based on the observation $X(t_1) = x_1$ and the associated error variance λ_{LLSE}, if $t_2 > t_1$?

(b) What is the Bayes least square error (BLSE) estimate $\widehat{X}_{BLSE}$ of $X(t_2)$ based on the observation $X(t_1) = x_1$ and the associated error variance λ_{BLSE}, if $t_2 > t_1$?

(c) What is the Maximum A Posteriori (MAP) estimate $\widehat{X}_{MAP}$ of $X(1/2)$ based on the observation $X(1/3) = \pi$?

(d) Would either of the estimates in parts (b) or (c) change with the addition of observations $X(t_k)$ with t_k in the past of the current observation (i.e. $t_k < t_1$ in (b) or $t_k < 1/3$ in (c))? Explain.



$$X(t) = \begin{cases} Z & t < W \\ 0 & t \geq W \end{cases}$$

$$m_X(t) = 0$$

$$K_{XX}(t_1, t_2) = \sigma^2[1 - \max(t_1, t_2)]$$

$$p_{X(t)}(x) = N(x; 0, \sigma^2)(1-t) + \delta(x)t$$

$$p_{X(t_2)|X(t_1)}(x_2 \mid x_1) = \delta(x_2 - x_1)\tfrac{1-t_2}{1-t_1} + \delta(x_2)\tfrac{t_2-t_1}{1-t_1},$$
$$t_2 \geq t_1$$

Answer #2:

Answer #2 (Continued):

Problem #3:

We want to estimate the arrival rate α of customers at a ticket line. Suppose we make a set of N observations $\{Y_1, Y_2, \cdots Y_N\}$, where Y_i is the number of arrivals in the i-th of a set of N non-overlapping intervals, each of length T. Given α, we model the number of arrivals in non-overlapping intervals of length T as independent random variables with a Poisson density of mean αT:

$$\Pr[Y_i = k \mid \alpha] = \frac{(\alpha T)^k \, e^{-\alpha T}}{k!} \qquad \text{for } k = 0, 1, 2 \cdots$$

(a) Specify the joint density $p_{Y_1,\ldots,Y_N|\alpha}(y_1, \ldots, y_N \mid \alpha)$.

(b) Find an ML estimate $\widehat{\alpha}_{ML}(y_1, \cdots, y_N)$ of α.

(c) What is the Cramér-Rao bound on the error variance of any unbiased estimator of the parameter α based on $y_1, \cdots, y_N$?

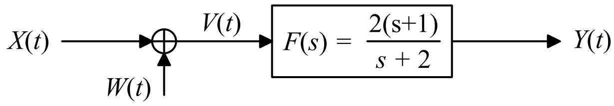
(d) What are the bias and error variance of $\widehat{\alpha}_{ML}(y_1, \cdots, y_N)$? Is $\widehat{\alpha}_{ML}(y_1, \cdots, y_N)$ efficient? Explain.

Answer #3:

Answer #3 (Continued):

Problem #4:

Consider the system shown in the figure below, where $X(t)$ and $W(t)$ are uncorrelated, zero-mean, wide-sense stationary random processes and $S_{WW}(s) = \dfrac{4 - s^2}{1 - s^2}$ and $S_{VV}(s) = \dfrac{9 - s^2}{1 - s^2}$.



(a) Provide expressions for $S_{XX}(s)$, $S_{YY}(s)$, and $S_{YX}(s)$.

(b) Find the noncausal linear filter $H_{nc}(s)$ that minimizes the mean square error in estimating $X(t)$ based on $Y(t)$ and the associated variance of the error λ_{nc}.

(c) Find the causal linear filter $H_c(s)$ that minimizes the mean square error in estimating $X(t)$ based on $Y(t)$ and the associated variance of the error λ_c.

Answer #4:

Answer #4 (Continued)