Boston University

Department of Electrical and Computer Engineering

EC505 STOCHASTIC PROCESSES

# Problem Set No. 4

Fall 2016

**Issued:** Monday, Sept. 26, 2016 **Due:** Wednesday, Oct. 5 2016

**Problem 4.1**

Let $X(t)$ be a Gaussian random process, wide-sense stationary, with mean $E[X(t)] = 1$ and power spectral density function $S_{XX}(f)$ given by:

$$S_{XX}(f) = \frac{2}{1 + (2\pi f)^2} + \delta(f)$$

(a) Compute the autocovariance function $K_{XX}(\tau)$.

(b) Is the process $X(t)$ ergodic in autocorrelation? Explain.

(c) Is the process $X(t)$ mean-square continuous? Is it mean-square differentiable? Explain.

(d) Define the process $Y(t) = \frac{d}{dt}X(t)$. Using generalized functions if necessary, find the mean and autocorrelation of the process $Y(t)$. Is the process $Y(t)$ wide-sense stationary? Explain.

**Problem 4.2**

Consider a Poisson process $N(t)$, with rate 1. Define a new process $X(t) = N(t + 2) - N(t) - 2$.

(a) Find the mean and autocorrelation of $X(t)$.

(b) Is $X(t)$ wide-sense stationary? Explain.

(c) If the process is wide-sense stationary, what is its power spectral density $S_{XX}(f)$?

(d) If the process is wide-sense stationary, is it ergodic in the mean?

**Problem 4.3** (Markov chain) Consider a 4 state Markov chain with the transition probability matrix

$$P = \begin{pmatrix} 0.1 & 0.2 & 0.3 & 0.4 \\ 0 & 0.5 & 0.2 & 0.3 \\ 0 & 0 & 0.3 & 0.7 \\ 0.1 & 0 & 0 & 0.9 \end{pmatrix}$$

1. Draw the state transition diagram, with the probabilities for the transitions.
2. Find the transient states and recurrent states.
3. Is the Markov chain irreducible? Explain.
4. Is the Markov chain aperiodic? Explain.
5. Find the steady state distribution of this Markov chain.

**Problem 4.4** (Markov Chain) Consider a 4 state Markov chain with the transition probability matrix (this is different from the previous problem...)

$$P = \begin{pmatrix} 0.1 & 0.2 & 0.3 & 0.4 \\ 0 & 0.5 & 0.2 & 0.3 \\ 0 & 0 & 0.3 & 0.7 \\ 0 & 0 & 0.1 & 0.9 \end{pmatrix}$$

1. Draw the state transition diagram, with the probabilities for the transitions.

2. Find the transient states and recurrent states.
3. Is the Markov chain irreducible? Explain.
4. Is the Markov chain aperiodic? Explain.
5. Find the steady state distribution of this Markov chain.

**Problem 4.5** (Markov Chain) Consider the following gambling problem. Suppose you go to a casino to play an unusual gambling game, where the probability of winning is 0.45. You start with \$3.00 and you want to double your stake. If you ever win enough to reach \$6.00, you quit and walk out. Similarly, if you lose your original stake and have no money left, you quit also.

1. The first betting scheme is simple: You bet 1.00 on every bet until you either go broke or you reach \$6.00. Draw a Markov chain, with transition probabilities, corresponding to this betting scheme. You should have 7 states, corresponding to the amount of money you have at any time.
2. Write the probability transition matrix $P$ for this Markov chain.
3. Which states are transient? Which states are recurrent? How many recurrent classes are there?
4. I'll assume you have access to a reasonable calculator or MATLAB. Enter the matrix $P$ into Matlab, and an initial column vector $\underline{p}(0)$ which is zero everywhere except for the 4th state (corresponding to \$3.00), where it is 1. You can find the steady state probability by evolving this Markov chain. Compute the probability mass function at time 20 by computing

$$\underline{p}(20) = (P^T)^{40}\underline{p}(0)$$

   What is the probability that by time 40, you go broke? What is the probability that, by time 40, you have left with \$6.00?
5. Let's repeat the above analysis for the following betting scheme. You start with \$1.00 bet. Every time you lose, if you can, you double your bet. Every time you win, you reduce your bet to \$1.00. If you can't double your bet, you bet everything you have left. If you ever reach \$6.00 or more, you walk away. Draw a Markov chain, with transition probabilities, corresponding to this betting scheme. Note: The states here are more complicated, as they involve the size of the bet and the amount remaining. Note also that the maximum bet size will be limited to \$2.00, and only when yoou have between \$2.00 and \$4.00 (work this out...) so you should get by with 10 states.
6. Number your states and write the probability transition matrix $P$ for this Markov chain.
7. Which states are transient? Which states are recurrent? How many recurrent classes are there?
8. I'll assume you have access to a reasonable calculator or MATLAB. Enter the matrix $P$ into Matlab, and an initial column vector $\underline{p}(0)$ which is zero everywhere except for the 4th state (corresponding to \$3.00), where it is 1. You can find the steady state probability by evolving this Markov chain. Compute the probability mass function at time 40 by computing

$$\underline{p}(20) = (P^T)^{40}\underline{p}(0)$$

   What is the probability that by time 40, you go broke? What is the probability that, by time 40, you have left with \$6.00?
9. Which betting scheme do you prefer? Why?

**Problem 4.6** Assume that the number of call arrivals between two locations has Poisson distribution with intensity $\lambda$. Also, assume that the holding times of the conversations are exponentially distributed with a mean of $1/\mu$. Calculate the average number of call arrivals for a period of a conversation.