Boston University

Department of Electrical and Computer Engineering

EC505 STOCHASTIC PROCESSES

Problem Set No. 2

Fall 2016

Issued: Wednesday, Sept. 14, 2016 **Due:** Monday, Sept. 26, 2016

Problem 2.1

Consider the random variables X and Y whose joint density function is given by $p_{X,Y}(x,y)$. For each of the possible choices of the joint density function given below determine i) whether X and Y are uncorrelated, ii) whether X and Y are independent, and iii) specify the corresponding covariance matrix:

$$\Sigma_{XY} = \begin{bmatrix} \sigma_{XX} & \sigma_{XY} \\ \sigma_{XY} & \sigma_{YY} \end{bmatrix}.$$

$$p_{X,Y}(x,y) = \begin{cases} 2 & \begin{cases} x, y \geq 0 \ \& \\ x + y \leq 1 \end{cases} \\ 0 & \text{otherwise} \end{cases}$$

(a)

$$p_{X,Y}(x,y) = \begin{cases} 1/2 & \begin{cases} |x+y| \leq 1 \ \& \\ |x-y| \leq 1 \end{cases} \\ 0 & \text{otherwise} \end{cases}$$

(b)

$$p_{X,Y}(x,y) = \begin{cases} 1 & 0 \leq x, y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

(c)

Problem 2.2

Given two random variables X and Y, knowledge of Y generally gives us information about the random variable X (and vice versa). Suppose we want to estimate X based on knowledge of Y (a topic we will study in much greater detail later in the course). In particular, suppose we want to estimate X as an affine function of Y:

$$\widehat{X} = \widehat{X}(Y) = aY + b,$$

where a and b are constants.

(a) Find expressions for a and b in terms of σ_{XX}, σ_{XY}, σ_{YY}, m_X, and m_Y so that the expected value of the square error between X and its estimate $\widehat{X}$ is minimized:

$$\min_{a,b} E\left[(\widehat{X} - X)^2\right]$$

(b) For each of the joint densities specified in Problem 2.1, what are the corresponding expressions for the values of a and b and the resulting estimator? In each case, graph the function $\widehat{X}(Y)$ on the plot of the joint density. What observations can you make? When does knowledge of Y *not* affect the affine estimate of X?

Problem 2.3

Consider the following 3×3 matrices:

$$A_1 = \begin{bmatrix} 10 & 3 & 1 \\ 2 & 5 & 0 \\ 1 & 0 & 2 \end{bmatrix} \qquad A_2 = \begin{bmatrix} 10 & 5 & 2 \\ 5 & 3 & 3 \\ 2 & 3 & 2 \end{bmatrix} \qquad A_3 = \begin{bmatrix} 10 & 5 & 2 \\ 5 & -3 & 3 \\ 2 & 3 & 2 \end{bmatrix} \qquad A_4 = \begin{bmatrix} 10 & 5 & 2 \\ -5 & 3 & 1 \\ -2 & -1 & 2 \end{bmatrix}$$

$$A_5 = \begin{bmatrix} 10 & -5 & 2 \\ -5 & 3 & -1 \\ 2 & -1 & 2 \end{bmatrix} \qquad A_6 = \begin{bmatrix} 10 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix} \qquad A_7 = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

(a) Which of the above matrices could be the covariance matrix of some random vector?

(b) Which of the above matrices could be the cross-covariance matrix of two random vectors?

(c) Which of the above matrices could be the covariance matrix of a random vector in which one component is a linear combination of the other two components?

(d) Which of the above matrices *could* be the covariance matrix of a vector with statistically independent components? Must a random vector with such a covariance matrix have statistically independent components?

Problem 2.4

Let $\underline{X} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$ denote a Gaussian random vector with mean $\underline{m}_X = \begin{bmatrix} 0 \\ a \end{bmatrix}$ and covariance $\Sigma_X = \begin{bmatrix} 4 & 2 \\ 2 & 2 \end{bmatrix}$ for some real a.

(a) Verify that Σ_X is a valid covariance matrix.

(b) What is the marginal probability density function $p_{X_1}(x_1)$ for X_1?

(c) Find the conditional density $p_{X_1|X_2}(x_1|x_2)$.

(d) Find a linear transformation T defining two new variables $\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = T \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$ such that Y_1 and Y_2 are uncorrelated and such that $TT^T = I$.

(e) Are Y_1 and Y_2 also statistically independent?

Problem 2.5

The joint density function for two two-dimensional random vectors $\underline{X}$ and $\underline{Y}$ is

$$p_{\underline{X},\underline{Y}}(\underline{x}, \underline{y}) = \begin{cases} x_1 x_2 + 3 y_1 y_2 & 0 \le x_1, x_2, y_1, y_2 \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Are $\underline{X}$ and $\underline{Y}$ statistically independent? Explain.

Problem 2.6

Define a random process $X(t)$ based on the outcome k of tossing a fair die independently *at each time* t, where t is discrete-valued. The definition of X is as follows:

$$X(t) = \begin{cases} -2 & \text{if } k = 1, \\ -1 & \text{if } k = 2, \\ 1 & \text{if } k = 3, \\ 2 & \text{if } k = 4, \\ t & \text{if } k = 5, \\ -t & \text{if } k = 6 \end{cases}$$

(a) Find the joint probability density function of $X(0), X(2)$.

(b) Find the marginal probabilty density functions of $X(0), X(2)$.

(c) Find $E[X(0)], E[X(2)], E[X(0)X(2)]$.

Problem 2.7 (Old Exam Problem) Let α and β be two statistically independent, identically distributed Gaussian random variables with means $E[\alpha] = E[\beta] = 0$ and variances $\sigma_\alpha^2 = \sigma_\beta^2 = 1$. Define the stochastic process $X(t) = \alpha \cos(t) + \beta \sin(t)$.

(a) Find the mean $m_X(t)$ and autocorrelation $R_{XX}(t_1, t_2)$ of the process $X(t)$.

(b) Is the process $X(t)$ wide-sense stationary? Explain.

(c) Is the process $X(t)$ a Gaussian random process? Explain.

Problem 2.8 (Old Exam Problem) Consider the random process

$$X(n) = \begin{cases} Z & n \text{ even} \\ -Z & n \text{ odd} \end{cases}$$

where $Z \sim N(m, 1)$. Be sure your answers are valid for all possible values of m and to give explanations for your answers.

(a) Find the mean and autocovariance functions of $X(n)$.

(b) Is $X(n)$ wide-sense stationary?

(c) Is $X(n)$ a Markov process?

(d) Is $X(n)$ strict-sense stationary?

(e) Is $X(n)$ a Gaussian random process?

(f) Is $X(n)$ an independent increments process?

Problem 2.9 (Old exam question) A random process $X(t)$ is defined as follows:

$$X(t) = \begin{cases} A & t < \Theta \\ B & t \geq \Theta \end{cases}$$

where A, B, and Θ are statistically independent unit-variance Gaussian random variables with means -1, 1, and 0 respectively, i.e., $A \sim N(-1, 1)$, $B \sim N(1, 1)$, $\Theta \sim N(0, 1)$.

(a) Sketch a typical sample function of $X(t)$.

(b) Find the first order density $p_{X(t)}(x)$, and the conditional density $p_{X(t_2)|X(t_1)}(x_2|x_1)$ for $t_2 > t_1$. Is $X(t)$ a Gaussian random process? Explain.

(c) Is $X(t)$ strict sense stationary? Explain.

(d) Is $X(t)$ a Markov process? Explain. (Hint: Consider $p_{X(t_2)|X(t_1),X(t_0)}(x_2|x_1, x_0)$ for different choices of $X(t_1), X(t_0)$).

(e) Is $X(t)$ an independent increments process? Explain.

If appropriate, you may express your answers to some parts of this problem in terms of the "Q" function defined below.

$$Q(t) = \frac{1}{\sqrt{2\pi}} \int_t^\infty e^{-\alpha^2/2} \, d\alpha$$

Problem 2.10 Let $X(t)$ be a Gaussian random process with mean $m_X(t)$, autocorrelation $R_X(t_1, t_2)$ and let $t_1 < t_2$. Find an expression for $E[X(t_2)|X(t_1)]$ in terms of m_X, R_X.

Problem 2.11 In the class we described how to construct a IID Bernoulli process from a random variable uniformly distributed random variable. Repeat the arguments here and verify the independence and identically distributed aspects for the first 5 time instants, X_1, X_2, X_3, X_4, X_5.

Computer Problems

Problem 2.12 Gaussian Random Vectors

The purpose of this problem is to learn how to generate samples of Gaussian random vectors with a given covariance structure and, conversly, given data, how to estimate the covariance matrix of a random vector. Random vectors also have a close relationship with random processes, particularly when we attempt to work with random processes in a computer. Thus we will also learn how to manipulate such vectors in MATLAB.

(a) Suppose $\underline{Z} = [Z_1, Z_2]^T$ is a vector of two independent Gaussian random variables Z_i, with $Z_i \sim N(0,1)$ and L is a 2×2 matrix. Define the (Gaussian) random vector $\underline{X} = [X_1, X_2]^T = L\underline{Z}$. What is the covariance matrix R_X of $\underline{X}$ in terms of L? If L is as given below in (1), what is the corresponding R_X?

$$L = \begin{bmatrix} 3/2 & 1/2 \\ 1/2 & 3/2 \end{bmatrix} \tag{1}$$

(b) We will use MATLAB to confirm part (a) experimentally. To this end we want to perform the following experiment:

  (i) Generate N independent samples $\underline{z}_i$ of the vector $\underline{Z}$ in MATLAB
  (ii) For each vector sample generated in (i), form the corresponding transformed sample $\underline{x}_i = L\underline{z}_i$,
  (iii) Estimate the covariance matrix by calculating: $\widehat{R}_x = \frac{1}{N}\sum_{i=1}^{N} \underline{x}_i \underline{x}_i^T - \widehat{m}_x \widehat{m}_x^T$ where $\widehat{m}_x = \frac{1}{N}\sum_{i=1}^{N} \underline{x}_i$.

Verify and implement this procedure for the L specified in (1) and compare your answers to what the theory says R_X should be. Use as large a value of N as is reasonable.

Note/Hint: With a bit of thought you can perform the necessary calculations efficiently in MATLAB with out having to resort to loops. For example, suppose we store each independent sample vector of $\underline{Z}$ as a row in the MATLAB matrix `Z`. Then calculating all the corresponding transformed vectors $\underline{X}$ given the matrix `L` is easily done in MATLAB via the operation: `X = (L*Z')'`, where the matrix `X` now stores the transformed sample vectors rowwise in the same way as `Z`. Further, if you read the help page on the MATLAB function `cov.m` you will see you are setup to find the covariance too.

We will use variants of this representational scheme for random vector and process experiments in MATLAB throughout the rest of the lab, so be sure you are comfortable with it.

(c) Use `pdf2d.m` to generate the empirical joint pdfs $p_{Z_1,Z_2}(z_1, z_2)$ and $p_{X_1,X_2}(x_1, x_2)$. What is the effect of applying the linear transformation L to $\underline{Z}$?

(d) You have now solved the "forward" problem of producing samples of a random vector given the *linear transformation* L. Use these insights to suggest a way to generate a Gaussian random vector $\underline{X}$ with a given, *desired covariance* structure R_X. Write a MATLAB function `covgen.m` based on your scheme, which takes as input a $P \times P$ desired covariance `R` and the desired number of sample vectors `N` and produces a $N \times P$ matrix `X`, whos rows are samples of the specified vector. Thus the function call would look like: `X = covgen(N,R)`. You may find the MATLAB functions `sqrtm.m`, which produces a "generic" matrix square root or `chol.m`, which factors a symmetric positive definite matrix, useful. Use your function when R_X is as given in (2) and verify it works as expected. You now have a way to generate vectors with arbitrary covariances!

$$R_X = \begin{bmatrix} 5/2 & 3/2 \\ 3/2 & 5/2 \end{bmatrix} \tag{2}$$

(e) In what ways would the random vector $\underline{X}$ and its covariance change if in the above development the elements of $\underline{Z}$ were independent, identically distributed, zero-mean, unit variance, *uniformly distributed* random variables rather than standard Gaussian random variables (i.e. if `Z = sqrt(12)*(rand(N,2)-.5)` were used instead of `Z = randn(N,2)`)? Try this and calculate both the covariance and the joint pdf.