Chapter 5: Excercises

1.) $\min_{\alpha} \ var(\alpha X + (1-\alpha)\cdot Y)$

$$\frac{\partial var}{\partial \alpha} = \frac{\partial}{\partial \alpha}\left[var(\alpha X) + var((1-\alpha)\cdot Y) + 2cov(\alpha X, (1-\alpha)Y)\right]$$

$$= \frac{\partial}{\partial \alpha}\left[\alpha^2 \cdot var(X) + (1-\alpha)^2 var(Y) + 2\alpha(1-\alpha)\, cov(X,Y)\right]$$

$$\frac{\partial var}{\partial \alpha} = 2\alpha \cdot var(X) + 2(1-\alpha)(-1)var(Y) + 2cov(X,Y)\left[(1-\alpha) + \alpha(-1)\right]$$

$$0 = 2\alpha\, var(X) - 2(1-\alpha)var(Y) + 2cov(X,Y)(1-2\alpha)$$

$$0 = \underline{2\alpha\, var(X)} - \underline{2var(Y)} + \underline{2\alpha var(Y)} + \underline{2cov(X,Y)} - \underline{4\alpha cov(X,Y)}$$

$$\cancel{2}var(Y) - \cancel{2}cov(X,Y) = \cancel{2}\alpha\left[var(X) + var(Y) - 2cov(X,Y)\right]$$

$$\boxed{\alpha = \frac{var(Y) - cov(X,Y)}{\left[var(X) + var(Y) - 2cov(X,Y)\right]}}$$

Is it a minimum?

$$\frac{\partial^2 var}{\partial \alpha^2} = 2var(X) - 2(-1)var(Y) + 2cov(X,Y)(-2)$$

$$= 2var(X) + 2var(Y) - 4cov(X,Y)$$

$$= 2\cdot \underbrace{\left[var(X) + var(Y) - 2cov(X,Y)\right]}_{>0,\ \text{since if } X_2 = -X} > 0 \ \checkmark$$

$$var(X_2 + Y) = var(X_2) + var(Y) + 2cov(X_2, Y) \geq 0$$

$$= var(-X) + var(Y) + 2cov(-X, Y) \geq 0$$

$$= var(X) + var(Y) - 2cov(X, Y) \geq 0,$$