7.)

(a.) $$L(x \mid \beta_0, \beta_1, \dots \beta_p) = \prod_{i=1}^{n} \left(\frac{1}{\sqrt{2\pi\sigma^2}} \right) \exp\left(\frac{-\left(y_i - \beta_0 - \sum_{j=1}^{p} x_{ij}\beta_j\right)^2}{2\sigma^2} \right)$$

(b) Let $\vec{\beta} = (\beta_0, \beta_1, \dots \beta_p)$

$$p(\beta \mid x) \propto p(\beta) \cdot L(x \mid \beta)$$

$$\propto \left(\frac{1}{2b}\right) \exp\left(\frac{-|\beta|}{b}\right) (2\pi\sigma^2)^{-n/2} \cdot \exp\left(\frac{-\sum (y_i - \sum \vec{\beta}\vec{x})^2}{2\sigma^2} \right)$$

$$\propto \left(\frac{1}{2b}\right) (2\pi\sigma^2)^{-n/2} \exp\left(\frac{-|\beta|}{b} - \frac{\sum (y_i - \vec{\beta}\vec{x})^2}{2\sigma^2} \right)$$

~~(c) Strategy: to show mode = most probable value, we'll differentiate wrt $\vec{\beta}$ and find value of $\vec{\beta}$ that has slope $= 0$, and then show it is max.~~

~~$X \sim n \times p$~~

~~$\beta \sim p \times 1$~~

$$\cancel{\ln(\text{post}) = \ln\left(\frac{1}{2b}\right) + \ln\left((2\pi\sigma^2)^{-n/2}\right) - \frac{|\beta|}{b} - \frac{\sum (y_i - \sum \vec{x}\vec{\beta})^2}{2\sigma^2}}$$

$$\cancel{\frac{\partial \ln(\text{post})}{\partial \beta} = -2 \sum (y_i - \sum \vec{x}\vec{\beta})\, x}$$

(c) Strategy: ① log,

$$\ln(\text{post}) = \ln\left(\frac{1}{2b}\right) + \ln\left((2\pi\sigma^2)^{-n/2}\right) - \frac{|\beta|}{b} - \frac{\sum (y_i - \sum x\beta)^2}{2\sigma^2}$$

② Note that minimizing $\ln(\text{post})$ is equiv to minimizing:
(since first 2 terms don't impact)

$$\arg\min_{\vec{\beta}} \left(\frac{1}{2\sigma^2}\right) \left(-\frac{\sum_{j=1}^{p} |\beta| \cdot (2\sigma^2)}{b} - \sum (y_i \sum x\beta)^2 \right)$$

$\downarrow$ BACK $\downarrow$

(c)- continued.

Let $\lambda = \frac{2\sigma^2}{b}$.

constant, so doesn't effect soln

Thus,

$$\arg\min_{\hat{\beta}} \left(\frac{-1}{2\sigma^2}\right)\left((y_i - \sum x_i \beta_i)^2 + \lambda \sum_{j=1}^{p} |\beta_j|\right)$$

Gives same optimization problem as LASSO.

Thus, when prior distribution is Laplace, the mode of the posterior dist. is given by the LASSO soln.

(d.)

Again, let $\vec{\beta} = (\beta_0, \beta_1, \ldots \beta_p)$

$$p(\vec{\beta} \mid x) = \frac{p(\vec{\beta}) \cdot L(x \mid \vec{\beta})}{\int p(\vec{\beta}) \cdot L(x \mid \vec{\beta})} \propto p(\vec{\beta}) L(x \mid \beta)$$

$$\propto (2\pi\sigma^2)^{-n/2} \exp\left(\frac{-\sum (y_i - \sum x_i \beta_i)^2}{2\sigma^2}\right) \cdot \sum_{j=1}^{p} \left(\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(\vec{\beta}_j)^2}{2c}}\right)$$

$$\star \quad \propto (2\pi\sigma^2)^{-n/2} (2\pi\sigma^2)^{-p/2} \exp\left(\frac{-\sum_i (y_i - \sum x\beta)^2}{2\sigma^2} - \frac{\sum_j \beta_j^2}{2c}\right)$$

(e.) Mode: Since first 2 constants don't affect the soln, equivalent to solving: (once we log)

$$\ln(p(\vec{\beta} \mid x)) = -\frac{\sum_i (y_i - \sum x\beta)^2}{2\sigma^2} - \frac{\sum \beta_j^2}{2c}$$

equiv to:

$$\arg\min_{\vec{\beta}} \left(\frac{1}{2\sigma^2}\right)\left(-\sum_i (y_i - \sum_j x_{ij}\beta)^2 - \lambda \sum_j \beta_j^2\right) =$$

Ridge soln, when $\lambda = \frac{\sigma^2}{c}$.

Also, since post is Gaussian $\Rightarrow$ Symmetric, we know mean = mode.