STAT 528 Project 1

Matthew Brown

1. Find a PDE describing evolution of probability densities for the hydrodynamic limit of random walks with asymmetric jump probabilities [diffusion with drift].

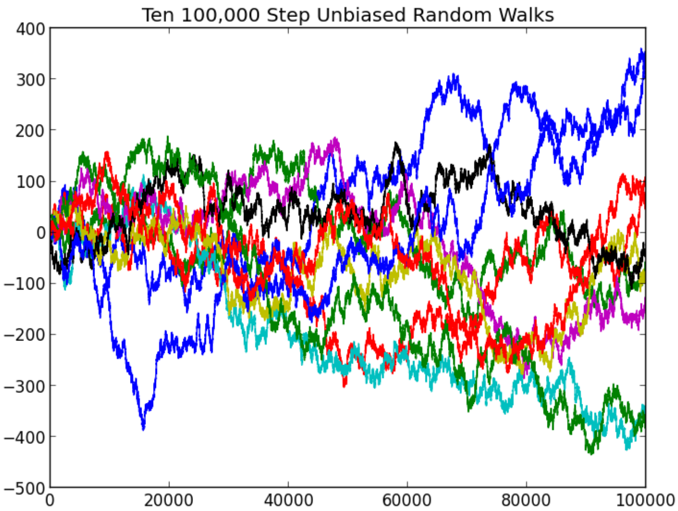
$$
\begin{aligned}
\mathrm{u}(t+\triangle t, x) &= p\mathrm{u}(t, x+\triangle x) + q\mathrm{u}(t, x-\triangle x) \\
\mathrm{u}(t+\triangle t, x) - \mathrm{u}(t, x) &= p\mathrm{u}(t, x+\triangle x) - \frac{1}{2}\mathrm{u}(t,x) + q\mathrm{u}(t, x-\triangle x) - \frac{1}{2}\mathrm{u}(t,x) \\
\mathrm{u}(t+\triangle t, x) - \mathrm{u}(t, x) &= \frac{1}{2}\left((\mathrm{u}(t, x+\triangle x) - \mathrm{u}(t,x)) - (\mathrm{u}(t, x-\triangle x) + \mathrm{u}(t,x))\right) \\
&\quad + \left(\left(p - \frac{1}{2}\right) u(t, x+\triangle x\right) + \left(\left(q - \frac{1}{2}\right) u(t, x-\triangle x\right) \\
\mathrm{u}(t+\triangle t, x) - \mathrm{u}(t, x) &= \frac{1}{2}\left((\mathrm{u}(t, x+\triangle x) - \mathrm{u}(t,x)) - (\mathrm{u}(t, x-\triangle x) + \mathrm{u}(t,x))\right) \\
&\quad + \left(\left(\frac{p-q}{2}\right) u(t, x+\triangle x\right) - \left(\left(\frac{p-q}{2}\right) u(t, x-\triangle x\right)
\end{aligned}
$$

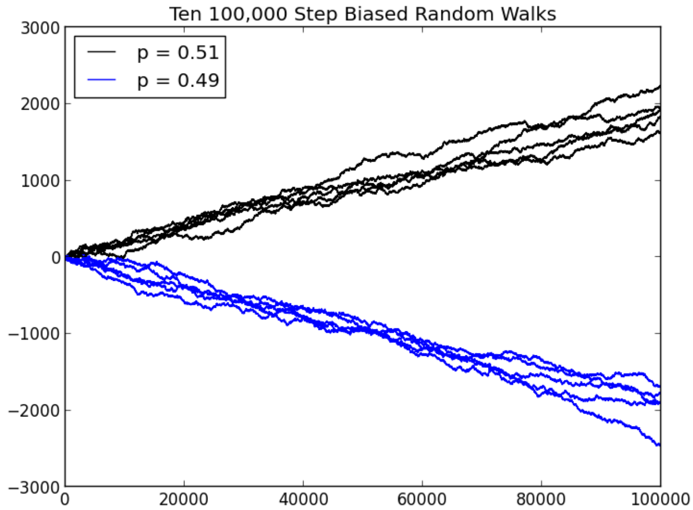
$$
\begin{aligned}
\frac{\mathrm{u}(t+\triangle t, x) - \mathrm{u}(t, x)}{\triangle t} &= \frac{1}{2}\frac{\frac{\mathrm{u}(t, x+\triangle x) - \mathrm{u}(t,x)}{\triangle x} - \frac{\mathrm{u}(t, x-\triangle x) + \mathrm{u}(t,x)}{\triangle x}}{\triangle x} \\
&\quad + \left(\frac{p-q}{2\,\triangle x}\right)\frac{(u(t,x)+\triangle x) - (u(t,x)-\triangle x)}{\triangle x} \\
\frac{\delta \mathrm{u}}{\delta t} &= \frac{1}{2}\frac{\delta^2 \mathrm{u}}{\delta x^2} + \left(\frac{p-q}{2\,\triangle x}\right)\frac{\delta \mathrm{u}}{\delta x} \\
\frac{\delta \mathrm{u}}{\delta t} &= D\frac{\delta^2 \mathrm{u}}{\delta x^2} + A\frac{\delta \mathrm{u}}{\delta x}
\end{aligned}
$$

Fourier Transform:

$$
\begin{aligned}
\mathrm{F}_x\left(\frac{\delta \mathrm{u}}{\delta t}\right) &= \mathrm{F}_x\left(D\frac{\delta^2 \mathrm{u}}{\delta x^2}\right) + \mathrm{F}_x\left(A\frac{\delta \mathrm{u}}{\delta x}\right) \\
\frac{\delta}{dt}\mathrm{F}_x(\mathrm{u}(t,\cdot)(\xi)) &= \frac{D}{2}(i\xi)^2\mathrm{F}_x(\mathrm{u}(t,\cdot)(\xi)) + A(i\xi)\mathrm{F}_x(\mathrm{u}(t,\cdot)(\xi)) \\
\frac{\delta}{dt}\rho(t,\xi) &= \frac{D}{2}(i\xi)^2\rho(t,\xi) + A(i\xi)\rho(t,\xi) \\
\frac{\delta}{dt}\rho(t,\xi) &= -\left(\frac{D\xi^2}{2} - A(i\xi)\right)\rho(t,\xi) \\
\rho(t,\xi) &= Ce^{-\frac{1}{2}(D\xi^2 - iA\xi)t} \\
u(t,x) &= \frac{1}{\sqrt{2\pi t D}}e^{-\frac{1}{2tD}(x-At)^2}
\end{aligned}
$$

2. Simulate 10 trajectories (on the same plot) of the Brownian motion, and BM with drift.





3. Calculate the following probabilities:

   a) $P[-1 < B(1) < +1, -1 < B(2) < +1, -1 < B(3) < +1]$

$$\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} \frac{e^{-\frac{x_1^2}{2}}}{\sqrt{2\pi}} \frac{e^{-\frac{(x_2-x_1)^2}{2(2-1)}}}{\sqrt{2\pi(2-1)}} \frac{e^{-\frac{(x_3-x_2)^2}{2(3-2)}}}{\sqrt{2\pi(3-2)}}\, dx_1\, dx_2\, dx_3 = 0.26021$$

   b) $P[-1 < B(1) < +1, -2 < B(2) < +2, -3 < B(3) < +3]$

$$\frac{1}{(2\pi)^{3/2}} \int_{-3}^{3}\int_{-2}^{2}\int_{-1}^{1} e^{-\frac{1}{2}x_1^2} e^{-\frac{1}{2}(x_2-x_1)^2} e^{-\frac{1}{2}(x_3-x_2)^2}\, dx_1\, dx_2\, dx_3 = 0.61303$$

   Do the same for BM with drift 1.

   a) $P[-1 < B_d(1) < +1, -1 < B_d(2) < +1, -1 < B_d(3) < +1]$

$$\frac{1}{(2\pi)^{3/2}} \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} e^{-\frac{1}{2}(x_1-1)^2} e^{-\frac{1}{2}(x_2-x_1-2)^2} e^{-\frac{1}{2}(x_3-x_2-3)^2}\, dx_1\, dx_2\, dx_3 = 0.06724$$

   b) $P[-1 < B_d(1) < +1, -2 < B_d(2) < +2, -3 < B_d(3) < +3]$

$$\frac{1}{(2\pi)^{3/2}} \int_{-3}^{3}\int_{-2}^{2}\int_{-1}^{1} e^{-\frac{1}{2}(x_1-1)^2} e^{-\frac{1}{2}(x_2-x_1-2)^2} e^{-\frac{1}{2}(x_3-x_2-3)^2}\, dx_1\, dx_2\, dx_3 = 0.29003$$

4. Calculate multivariate Brownian motion in 3 dimensions

$$f_{(t_1,\ldots,t_n)}(\vec{x}) = \frac{1}{(2\pi)^{n/2} \mid det(\Sigma) \mid^{1/2}} e^{-\frac{1}{2}\vec{x}^T \Sigma^{-1} \vec{x}}$$

$$\Sigma = \begin{pmatrix} t_1 & t_1 & t_1 \\ t_1 & t_2 & t_2 \\ t_1 & t_2 & t_3 \end{pmatrix}$$

$$\Sigma^{-1} = \begin{pmatrix} \frac{-t_2}{t_1^2 - t_1 t_2} & \frac{1}{t_1 - t_2} & 0 \\ \frac{1}{t_1 - t_2} & \frac{t_3 - t_1}{(t_1 - t_2)(t_2 - t_3)} & \frac{1}{t_2 - t_3} \\ 0 & \frac{1}{t_2 - t_3} & \frac{1}{t_3 - t_2} \end{pmatrix}$$

$$\vec{x}^T \Sigma^{-1} \vec{x} = \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \begin{pmatrix} \frac{-t_2}{t_1^2 - t_1 t_2} & \frac{1}{t_1 - t_2} & 0 \\ \frac{1}{t_1 - t_2} & \frac{t_3 - t_1}{(t_1 - t_2)(t_2 - t_3)} & \frac{1}{t_2 - t_3} \\ 0 & \frac{1}{t_2 - t_3} & \frac{1}{t_3 - t_2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\begin{aligned} \vec{x}^T \Sigma^{-1} \vec{x} = & x_1 \left(\frac{x_2}{t_1 - t_2} - \frac{t_2 x_1}{t_1^2 - t_1 t_2} \right) + x_2 \left(\frac{x_1}{t_1 - t_2} + \frac{x_2 (t_3 - t_1)}{(t_1 - t_2)(t_2 - t_3)} + \frac{x_3}{t_2 - t_3} \right) \\ & + x_3 \left(\frac{x_2}{t_2 - t_3} + \frac{x_3}{t_3 - t_2} \right) \end{aligned}$$

$$\det \Sigma = t_1 (t_1 - t_2)(t_2 - t_3)$$

$$f_{(t_1,t_2,t_3)}(\vec{x}) = \frac{e^{-\frac{1}{2}\left(x_1 \left(\frac{x_2}{t_1 - t_2} - \frac{t_2 x_1}{t_1^2 - t_1 t_2} \right) + x_2 \left(\frac{x_1}{t_1 - t_2} + \frac{x_2 (t_3 - t_1)}{(t_1 - t_2)(t_2 - t_3)} + \frac{x_3}{t_2 - t_3} \right) + x_3 \left(\frac{x_2}{t_2 - t_3} + \frac{x_3}{t_3 - t_2} \right) \right)}}{(2\pi)^{3/2} \mid t_1 (t_1 - t_2)(t_2 - t_3) \mid^{1/2}}$$

$$f_{(t_1,t_2,t_3)}(\vec{x}) = \frac{e^{-\frac{1}{2} \frac{x_1^2 (t_2 - t_1)(t_3 - t_2) + (x_2 - x_1)^2 t_1 (t_3 - t_2) + (x_3 - x_2)^2 t_1 (t_2 - t_1)}{t_1 (t_2 - t_1)(t_3 - t_2)}}}{(2\pi)^{3/2} \mid t_1 (t_1 - t_2)(t_2 - t_3) \mid^{1/2}}$$

$$f_{(t_1,t_2,t_3)}(\vec{x}) = \frac{e^{-\frac{1}{2}\left(\frac{x_1^2}{t_1} + \frac{(x_2 - x_1)^2}{t_2 - t_1} + \frac{(x_3 - x_2)^2}{t_3 - t_2} \right)}}{(2\pi)^{3/2} \mid t_1 (t_1 - t_2)(t_2 - t_3) \mid^{1/2}}$$

$$f_{(t_1,t_2,t_3)}(\vec{x}) = \frac{e^{-\frac{x_1^2}{2t_1}}}{\sqrt{2\pi t_1}} \frac{e^{-\frac{(x_2 - x_1)^2}{2(t_2 - t_1)}}}{\sqrt{2\pi (t_2 - t_1)}} \frac{e^{-\frac{(x_3 - x_2)^2}{2(t_3 - t_2)}}}{\sqrt{2\pi (t_3 - t_2)}}$$