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CS 421, SP15
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Lab #3

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§4.3 4. Prove the following languages are not regular.

§4.3 4a. $L = \{a^n b^l a^k : k \geq n + l\}$

Assume L is regular and therefore pumpable. By the Pumping Lemma, we know there exists $m \in \mathbb{N}$ and $w \in L$ such that $|w| \geq m$. We choose $w = a^m b^m a^{2m}$, which is clearly in L with $|w| \geq m$. By the Pumping Lemma, we know we need a decomposition $w = xyz$ with a prefix xy such that $|xy| \leq m$ and $|y| \geq 1$. It is easy to see that we must have $xy = a^p$ for some $p \in [1, m]$. Now consider $i = 2$: $w = xy^2z = a^{m+p}b^m a^{2m}$. For w to be in L, we must have $m + p + m \leq 2m \Rightarrow p \leq 0$. This, however, cannot be true since $p \in [1, m]$. We see that $w \notin L$. Thus, we have a contradiction in the Pumping Lemma. Since L is not pumpable, it cannot be regular.

#QED

§4.3 4b. $L = \{a^n b^l a^k : k \neq n + l\}$

Assume L is regular and therefore pumpable. We have the properties of complementation and intersection for regular languages. Let us say $L_1 = \overline{L} \cap L(a^*b^*a^*)$. Note that $L(a^*b^*a^*)$ is regular because $a^*b^*a^*$ is a regular expression. It is easy to see that $L_1 = \{a^n b^l a^k : k = n+l\}$. Since we have assumed L is regular, L_1 must be regular. Let us apply the Pumping Lemma to L_1. By the Pumping Lemma, we know there exists $m \in \mathbb{N}$ and $w \in L$ such that $|w| \geq m$. We choose $w = a^m b^m a^{2m}$, which is clearly in L_1 with $|w| \geq m$. By the Pumping Lemma, we must have a decomposition $w = xyz$ with $|xy| \leq m$ and $|y| \geq 1$. This forces $xy = a^p$ for some $p \in [1, m]$. We could now consider $i = 0$ and $i = 1$, but these are unhelpful. Consider instead $i = 2$: $w = xy^2z = a^{m+p}b^m a^{2m}$. For w to be in L, we need $m+p+m = 2m \Rightarrow p = 0$. However, this is impossible since $p \in [1, m]$. Thus, we have a contradiction in the Pumping Lemma for L_1. Since L_1 is not pumpable and therefore not regular, we have contradicted our assumption that L is regular. Therefore, L is not a regular language.

#QED

§4.3 4c. $L = \{a^n b^l a^k : n = l \text{ or } l \neq k\}$

Assume L is regular and therefore pumpable. By the Pumping Lemma, we know there exists $m \in \mathbb{N}$ and $w \in L$ such that $|w| \geq m$. We choose $w = a^m b^m a^m$, which is clearly in L with $|w| \geq m$. By the Pumping Lemma, we need a decomposition $w = xyz$ with $|xy| \leq m$ and $|y| \geq 1$. We see that $xy = a^p$ for some $p \in [1, m]$. Now consider $i = 0$: $w = xy^0z = a^{m-p}b^m a^m$. For w to be in L, we need $m - p = m \Rightarrow p = 0$ or $m \neq m$. It is clear to see that neither condition is true. Thus, we have a contradiction in the Pumping Lemma. Since L is not pumpable, L cannot be regular.

#QED

§4.3 4d. $L = \{a^n b^l : n \leq l\}$

Assume L is regular and therefore pumpable. By the Pumping Lemma, we know there exists $m \in \mathbb{N}$ and $w \in L$ such that $|w| \geq m$. We choose $w = a^m b^{m+1}$, which is clearly in L with $|w| \geq m$. By the Pumping Lemma, we need a decomposition $w = xyz$ with $|xy| \leq m$ and $|y| \geq 1$. It is easy to see that we must have $xy = a^p$ for some $p \in [1, m]$. Consider $i = 2$: $w = xy^2z = a^{m+p}b^{m+1}$. For w to be in L, we must have $m + p \leq m + 1 \Rightarrow p \leq 1$, which is impossible. Thus, we have a contradiction in the Pumping Lemma. Since L cannot be pumped, it cannot be a regular language.

#QED

§4.3 4e. $L = \{w : n_a(w) \neq n_b(w)\}$

Assume L is regular and therefore pumpable. We have the property of regular languages that $L_1 = \overline{L}$ is regular. We have, then, $L_1 = \{w : n_a(w) = n_b(w)\}$ is regular, which implies L_1 is pumpable. By the Pumping Lemma, we know there exists $m \in \mathbb{N}$ and $w \in L$ such that $|w| \geq m$. We choose $w = a^m b^m$, which is clearly in L_1 with $|w| \geq m$. By the Pumping Lemma, we need a decomposition $w = xyz$ with $|xy| \leq m$ and $|y| \geq 1$. We see that we must have $xy = a^p$ for some $p \in [1, m]$. Now consider $i = 0$: $w = xy^0z = a^{m-p}b^m$. For w to be in L_1, we must have $m - p = m \Rightarrow p = 0$, which is impossible. Thus, we have a contradiction in the Pumping Lemma, which implies L_1 is not regular. Recall that $L_1 = \overline{L}$. Since L_1 is not regular, we must conclude that L is also not regular.

#QED

§4.3 4f. $L = \{ww : w \in \{a, b\}^*\}$

Assume L is regular and therefore pumpable. By the Pumping Lemma, we know there exists $m \in \mathbb{N}$ and $v \in L$ such that $|v| \geq m$. We choose $v = a^m b^m a^m b^m$, which is clearly in L with $|v| \geq m$. By the Pumping Lemma, we need a decomposition $v = xyz$ with $|xy| \leq m$ and $|y| \geq 1$. Clearly we cannot have $xy = a^m b^m$ since $|a^m b^m| = 2m > m$. We are forced, then, to have $xy = a^p$ for some $p \in [1, m]$. Now consider $i = 0$: $v = xy^0z = a^{m-p}b^m a^m b^m$. For v to be in L, we need $m - p = m \Rightarrow p = 0$, which is impossible. Thus, we have a contradiction in the Pumping Lemma, which implies L cannot be regular.

#QED

§4.3 4g. $L = \{www w^R : w \in \{a, b\}^*\}$

Assume L is regular and therefore pumpable. By the Pumping Lemma, we know there exists $m \in \mathbb{N}$ and $v \in L$ such that $|v| \geq m$. We choose $v = wwww^R$ where $w = a^m b$. We see $v \in L$ and $|v| \geq m$. We also need a decomposition $v = xyz$ with $|xy| \leq m$ and $|y| \geq 1$. Since $|a^m b| > m$, we must consider the substring a^m. Thus, to satisfy the Pumping Lemma, we must have $xy = a^p$ for some $p \in [1, m]$. Now consider $i = 2$: $v = xy^2z = a^{m+p}ba^m ba^m bba^m$. We have, then, $v = w_1 www^R$, where $w_1 = a^{m+p}b$. For v to be in L, we need $w_1 = w$, which means we need $m + p = m \Rightarrow p = 0$. It is clear that $w_1 \neq w$. Thus, we see that $v \notin L$, which contradicts the Pumping Lemma. We have, then, that L is not pumpable and not regular.

#QED