2.6 Z-Transform (Probability Generating Function)

Consider this example: Poisson random variable X is the total number of UNL students arriving at the union in a unit of time, and Poisson random variable Y is the total number of UNL professors arriving at the union in a unit of time. Define a new random variable $U = X + Y$. Is U also a Poisson random variable, and what is the PMF of U?
This example can be easily analyzed using a powerful technique, called Z-transform (also called a probability generating function), which can be used to describe and analyze discrete random variables.

2.6.1 Definition

Consider a discrete random variable X which takes *non-negative integer* values. The Z-transform of X is defined by

$$G_X(z) = \sum_{k=0}^{\infty} P(X = k) z^k \qquad \text{with} \quad |z| \le 1$$

Example 1: Geometric Random Variables:

$$\begin{aligned}
G_X(z) &= \sum_{k=1}^{\infty} (1-p)^{k-1} p z^k \\
&= pz \sum_{k=1}^{\infty} \big((1-p)z\big)^{k-1} && \text{Replacing } k-1 \text{ with } i \\
&= pz \sum_{i=0}^{\infty} \big((1-p)z\big)^{i} && \text{Using } \sum_{i=0}^{n} r^i = \frac{1-r^{n+1}}{1-r} \\
&= pz \frac{1-\big((1-p)z\big)^{\infty}}{1-(1-p)z} && \text{Considering } r = (1-p)z < 1 \text{ and } \lim_{n\to\infty} r^n = 0 \\
&= \frac{pz}{1-(1-p)z}
\end{aligned}$$

Example 2: Poisson Random Variables:

$$\begin{aligned}
G_X(z) &= \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} e^{-\lambda} z^k \\
&= e^{-\lambda} \sum_{k=0}^{\infty} \frac{(\lambda z)^k}{k!} && \text{Using Taylor Series } \sum_{k=0}^{\infty} \frac{x^k}{k!} = e^x \\
&= e^{-\lambda} e^{\lambda z} \\
&= e^{\lambda(z-1)}
\end{aligned}$$

2.6.2 From Z-transform to PMF

If the Z-transform of a random variable is given, the PMF can be obtained as follows. This is why it is called a probability generating function.

$$\begin{aligned} P(X=0) &= G_X(0) \\ P(X=k) &= \frac{1}{k!}\frac{d^k G_X(z)}{dz^k}\bigg|_{z=0} \qquad k=1,2,3,\cdots \end{aligned}$$

Example 1: Geometric Random Variables:

$$\begin{aligned} P(X=0)=G_X(0) &= 0 \\ P(X=1)=G'_X(0) &= \frac{dG_X(z)}{dz}\bigg|_{z=0} \qquad \text{Using } \left(\frac{f}{g}\right)' = \frac{f'g-fg'}{g^2} \\ &= \frac{p(1-(1-p)z)-pz(-(1-p))}{(1-(1-p)z)^2}\bigg|_{z=0} \\ &= p \end{aligned}$$

Example 2: Poisson Random Variables:

$$\begin{aligned} P(X=0)=G_X(0) &= e^{-\lambda} \\ P(X=1)=G'_X(0) &= \frac{dG_X(z)}{dz}\bigg|_{z=0} \qquad \text{Using } \frac{df(z)}{dz} = \frac{df(u)}{du}\frac{du(z)}{dz} \text{ and } \frac{de^u}{du}=e^u \\ &= \lambda e^{\lambda(z-1)}\Big|_{z=0} \\ &= \lambda e^{-\lambda} \end{aligned}$$

2.6.3 From Z-Transform to Moments

The moments of a random variable can be calculated by

$$\begin{aligned} \overline{X} &= \frac{dG_X(z)}{dz}\bigg|_{z=1} \\ \overline{X^2} &= \frac{d^2G_X(z)}{dz^2}\bigg|_{z=1} + \overline{X} \end{aligned}$$

Example 1: Geometric Random Variables:

$$\begin{aligned} \overline{X} &= \frac{dG_X(z)}{dz}\bigg|_{z=1} \\ &= \frac{p(1-(1-p)z)-pz(-(1-p))}{(1-(1-p)z)^2}\bigg|_{z=1} \\ &= \frac{1}{p} \end{aligned}$$

Example 2: Poisson Random Variables:

$$\begin{aligned} \overline{X} &= \frac{dG_X(z)}{dz}\bigg|_{z=1} \\ &= \lambda e^{\lambda(z-1)}\Big|_{z=1} \\ &= \lambda \end{aligned}$$