# Research Statement

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## Introduction

My research is in the area of geometric group theory whereby one studies a finitely generated infinite group $G$ by understanding the geometry of metric spaces on which $G$ can act geometrically. A *geometric action* of a finitely generated group $G$ on a metric space $X$ is one that is proper, cocompact and by isometries. With this setup, one can conclude certain algebraic properties of $G$ by understanding which metric spaces $G$ can and cannot act on in this way.

The focus of my work has been on groups which act geometrically on CAT(0) spaces. A CAT(0) space is a metric space which is non-positively curved in the sense of Alexandrov. We call $G$ a CAT(0) group if there exists a CAT(0) space $X$ on which $G$ acts geometrically. Familiar examples of CAT(0) spaces include $\mathbb{H}^n$, $\mathbb{R}^n$, metric trees, as well as finite dimensional cube complexes which satisfy Gromov's link condition. Groups acting on CAT(0) cube complexes have played a prominent role in the recent award winning work of Agol solving the Virtual Haken and Virtual Fibering Conjectures. More complicated CAT(0) spaces can be obtained by combining CAT(0) spaces using metric products and other natural constructions.

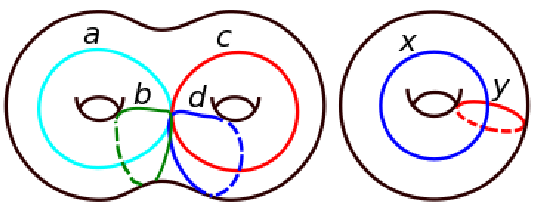


Figure 1: The fundamental groups of a genus 2 surface and a torus, shown with generating loops, are examples of CAT(0) groups.

Understanding Gromov hyperbolic groups has been a rich area of study over the past three decades. CAT(0) groups are a class of groups meant to be one attempt at generalizing the theory of hyperbolic groups to the setting of non-positive curvature. Thus it is natural to ask when properties of hyperbolic groups carry over to CAT(0) groups.

Specifically, my work has involved studying the connections between the group $G$ and the topology of a natural compactification of $X$ called the visual boundary of $X$. The *visual boundary* of a (complete, proper) CAT(0) space $X$, denoted $\partial X$, is this space of rays emanating from a fixed basepoint. There is a natural topology on $\overline{X} = X \cup \partial X$ which makes $\overline{X}$ compact. When $X$ is a Riemannian manifold, $\overline{X}$ is a closed ball and then $\partial X$ is a sphere. When $X$ is not from the manifold setting, the boundary can be much more exotic.

When $X$ admits a geometric group action by $G$, properties of the visual boundary can give strong algebraic information about the group. In particular, far-reaching generalizations of Stallings theorem about ends of groups use the topology of the boundary to detect splittings of a group as in work of Bowditch, Kapovich-Kleiner, Papasoglu-Swenson, and Mihalik-Ruane-Tschantz, and many more.

One question that has guided my work thus far is the following:

**Question 1.** Suppose $G$ is a group acting geometrically on a CAT(0) space $X$. What conditions on $G$ force $\partial X$ to be path connected?

One reason to care about path connectivity in $\partial X$ comes from the theory of hyperbolic groups. Combined work of Bestvina-Mess, Bowditch, and Swarup shows that all 1-ended hyperbolic groups have locally connected boundaries. Some natural examples of non-hyperbolic CAT(0) groups, such as $F_2 \times \mathbb{Z}$, show that this phenomenon does not generally occur in the CAT(0) setting. However, a boundary which is connected and locally connected is also globally path connected. In this sense, path connectivity is the 'next best thing' one can hope for in the boundary of a CAT(0) group. At the same time, there are known examples of CAT(0) groups which act on spaces with non-path connected visual boundaries. Therefore, my goal is to find conditions on $G$ which determine when $\partial X$ is path connected.

A second motivation for trying to answer Question 1 comes from the relationship between path connectivity in $\partial X$ and semistability. A proper 1-ended CAT(0) space $X$ is *semistable at infinity* if any two geodesic rays are properly homotopic. Being semistable at infinity is a quasi-isometry invariant, so showing a group is semistable at infinity comes down to finding an appropriate space on which this group acts. Geoghegan conjectured that all CAT(0) groups are semistable at infinity. Geoghegan also shows that given a CAT(0) space $X$, if $\partial X$ is path connected, then $X$ is semistable at infinity. Piecing all of this together, knowing that $G$ acts geometrically on a CAT(0) space with a path connected visual boundary shows that $G$ is semistable at infinity.

A third motivation has to do with understanding which topological properties of the boundary are group invariants. A theorem of Gromov says if $\Gamma$ is hyperbolic and acts geometrically on spaces $X$ and $Y$, then $\partial X$ and $\partial Y$ are homeomorphic. Croke-Kleiner show that this is false in general for CAT(0) groups. All known examples of CAT(0) groups with non-unique boundary have the property that all of their visual boundaries are all non-path connected. This leads to the following question:

**Question 2.** Suppose $G$ acts geometrically on CAT(0) spaces $X$ and $Y$ with $\partial X$ path connected. Are $\partial X$ and $\partial Y$ homeomorphic?

My research partially addresses Question 1 and provides a counterexample to Question 2.

# Previous Work

## CAT(0) groups with isolated flats

Part of my work has been answering Question 1 for a class of CAT(0) groups which exhibit more hyperbolicity than others: CAT(0) groups with isolated flats. These are CAT(0) groups which are hyperbolic relative to a collection of flat stabilizers, and therefore share many properties with hyperbolic groups. An example of such a group is the fundamental groups in Figure 3. Hruska-Ruane have categorized the CAT(0) groups with isolated flats which have locally connected boundaries. There are many examples, however, which do not have locally connected boundaries. My work has been on the general case where I fully answer Question 1 in this setting.

**Theorem 1.** *Let $G$ be a 1-ended CAT(0) group with isolated flats acting geometrically on a CAT(0) space $X$. Then $\partial X$ is path connected.*

The main approach I use to proving Theorem 1 is to decompose a CAT(0) space into more tractable pieces. Then, by showing the pieces fit together in a nice way, I conclude the resulting space will have a path connected boundary. Along the way, I prove a combination theorem for CAT(0) spaces related to path connectivity.

This approach is motivated by work of Croke-Kleiner. They study the fundamental group, $G$, of the object in Figure 2 acting geometrically on its universal cover $X$. The group $G$ splits as an amalgamated product $G = A *_C B$ and this group-theoretic structure imposes geometric structure on $X$. This structure is such that $X$ can be viewed as a union of closed, convex pieces called *blocks*. Each block admits a geometric group action by $A$ or $B$, and the way the blocks fit together to form $X$ mimics the way $G$ is made up of cosets of $A$ and $B$. Mooney later generalizes this idea, defining a general *block decomposition*. This is a way of decomposing a CAT(0) space $X$ into closed, convex blocks, and is independent of a group action on $X$.

By splitting up $X$ in this way, we can divide the kinds of points of $\partial X$ into two different sets. There are the points of $\partial X$ which are in the boundary of a block, and those which come from rays traveling through infinitely many blocks. We call the first kind *rational* and the second kind *irrational*. Rational points are easy to understand, since the idea of a block decomposition is to split up $X$ into pieces we understand well. Therefore, with a block decomposition questions about $\partial X$ can be reduced to understanding how the irrational points fit with the block boundaries. I prove sufficient conditions on the irrational rays which guarantees $\partial X$ will be path connected.

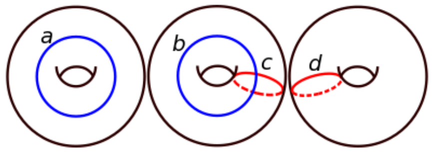


Figure 2: Three tori with the identifications $a = b$ and $c = d$.

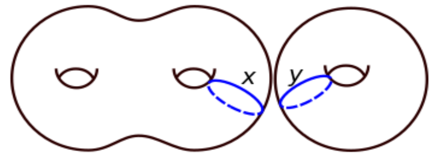


Figure 3: Torus and hyperbolic surface with the identification $x = y$.

**Theorem 2.** *Let $X$ be a 1-ended CAT(0) space with a block decomposition. Suppose the following conditions hold:*

1. *For each block $B$, $\partial B$ is path connected, and*
2. *no two rays go through the same infinite sequence of blocks.*

*Then $\partial X$ is path connected.*

This is a combination theorem for CAT(0) spaces with path connected boundaries. If $X$ is made up of pieces with path connected visual boundaries, plus a technical condition on the irrational points, then $\partial X$ will be path connected. In an ideal world, we would not require the second condition. However, Connor-Mihalik-Tschantz show the universal cover of Figure 2 is not path connected using two rays which travel through the same infinite sequence of blocks. In light of their result, Theorem 2 shows that avoiding this kind of pathology is sufficient for having path connectivity in $\partial X$.

The main application of Theorem 2 is in the proof of Theorem 1. After applying some results of Bowditch about splitting of relatively hyperbolic groups, a convex splitting result of Hruska-Ruane, and a gluing theorem of Bridson-Haefliger, the question of path connectivity for CAT(0) groups with isolated flats can be reduced to determining when Theorem 2 can be applied. With some work, one can show that a slight variation of Theorem 2 always applies in the isolated flats setting.

By applying a result of Geoghegan, the following is an immediate corollary of Theorem 1:

**Corollary 1.** *Let $G$ be a CAT(0) group with isolated flats. Then $G$ is semistable at infinity.*

This corollary has some overlap with work of other authors, but is ultimately distinct. Mihalik-Swenson show that if $G$ is relatively hyperbolic and has a trivial maximal peripheral splitting, then $G$ is semistable at infinity. Many CAT(0) groups with isolated flats, however, have a non-trivial maximal peripheral splitting. Additionally, Hruska-Ruane show that if $G$ is hyperbolic relative to polycyclic groups then $G$ is semistable at infinity, with the additional assumption that $G$ contains no non-central elements of order 2. While CAT(0) groups with isolated flats are hyperbolic relative to polycyclic groups, these groups can also have non-central elements of order 2.

## Right-angled Artin groups

Another portion of my work has been devoted to understanding right-angled Artin groups (RAAGs). These are groups which can be defined via a finite, simple graph. An Artin group with defining graph $\Gamma$ is denoted $A(\Gamma)$. A presentation for $A(\Gamma)$ has a generator for each vertex and the only relations are that two generators commute exactly when there is an edge between their corresponding vertices. Varying $\Gamma$ from a complete graph to a graph with no edges interpolates $A(\Gamma)$ between a free Abelian group and a free group.

There is a natural CAT(0) cube complex $S_\Gamma$ on which the RAAG $A(\Gamma)$ acts geometrically. Let $P_4$ be the graph of a path with 4 vertices, seen in Figure 4. Then the fundamental group in Figure 2 is $A(P_4)$ and when the curves $b$ and $c$ intersect at a right-angle the universal cover is $S_{P_4}$.

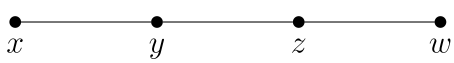


Figure 4: The graph $P_4$

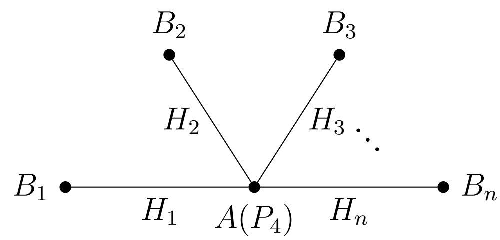


Figure 5: A graph of groups where each $B_i$ is a RAAG and $H_i$ is a proper parabolic subgroup of $A(P_4)$.

One major result of Croke-Kleiner is showing that $A(P_4)$ acts on two different CAT(0) spaces $X$ and $Y$ where $\partial X$ and $\partial Y$ are not homeomorphic. This is contrary to what happens in the hyperbolic setting. Later, Croke-Kleiner and Mooney provided more examples of CAT(0) groups which exhibit this phenomenon. In each of these examples, the resulting boundary was not path connected. These results lead to asking Question 2. In joint work with R. Kropholler, we show that the answer to this question is 'no,' as well as answer the high-dimensional extension of the question:

**Theorem 3.** *For each $n$, there is a group $G_n$ and CAT(0) spaces $X_n$ and $Y_n$ admitting geometric group actions by $G_n$ with the following properties:*

- *$\partial X_n$ and $\partial Y_n$ are $n$-connected*
- *$\partial X_n$ and $\partial Y_n$ are not homeomorphic*

A space is $n$-connected if the first $n$-homotopy groups are trivial. When $n = 0$, this is equivalent to the space being path connected. The groups we use to prove this are $G_n = A(P_4) \times Z^{n+1}$. The spaces $X_n$ and $Y_n$ are higher-dimensional analogues to those used by Croke-Kleiner to first show that $A(P_4)$ acts on spaces with non-homeomorphic boundaries.

A secondary point of interest around the study of $A(P_4)$ is understanding if $A(\Gamma)$ containing $A(P_4)$ as a subgroup forces $\partial S_\Gamma$ to not be path connected. Related to this is a conjecture of Mihalik:

**Conjecture** (Mihalik)**.** Let $\Gamma$ be a finite, simple, connected graph. $\partial S_\Gamma$ is path connected if and only if $\Gamma$ is a join.

Every finite, simple, connected graph is either a join or contains $P_4$ as an induced subgraph. It is known that whenever $\Gamma$ is a join, $\partial S$ will be path connected. Therefore, what is left to show is the other direction. R. Kropholler and I give more evidence for this conjecture.

**Theorem 4.** *Let $A_\Gamma$ be a RAAG admitting a graph of groups as in Figure 5 where $H_i$ is a proper parabolic subgroup of $A(P_4)$. Then $\partial S_\Gamma$ is not path connected.*

A *parabolic subgroup* is a subgroup generated by a subset of the vertices of $\Gamma$. An example of a RAAG which fits into this theorem is the RAAG on $C_n$, the cycle on $n$-vertices. When $n \geq 5$, then $C_n$ is not a join and $A(C_n)$ can be written as a graph of groups which fits into Theorem 4.

# Work in Progress

## Right-angled Coxeter Groups and Right-angled Artin Groups

There are two widely studied classes of groups which are defined via a graph: RAAGs and right-angled Coxeter groups (RACGs). Both are defined via a graph and interpolate between free-products and direct products. But one major difference between the two is that RAAGs are generated by infinite order elements while RACGs are generated by elements of order 2. Because these groups are defined in such similar ways, it is natural to ask how related are they algebraically. Davis-Januskiewicz show that if $A(\Gamma)$ is a RAAG then there is a graph $\Gamma'$ so that $W(\Gamma')$, the RACG with defining graph $\Gamma'$, contains $A(\Gamma)$ as a subgroup of finite index. I am working on going the other way. That is I want to understand when a RACG contain a RAAG of finite index.

## Applications of Theorem 2

I am currently working on finding more spaces that fit into the hypotheses of Theorem 2. In particular, finding examples where the block decomposition of the space does not come from the group theory. Potential examples are spaces which admit geometric group actions by groups which have Serre's property FA since these groups cannot split algebraically.

Another direction is to find algebraic conditions on $G$ which force the second condition of Theorem 2. The geometry of CAT(0) groups with isolated flats is what guaranteed all of those groups satisfy that condition, but there could be an algebraic reason as well. It might be possible to reduce checking this condition on rays to checking whether certain subgroups intersect. This is motivated by a phenomenon which occurs in $A(P_4)$ but does not happen in the fundamental group of Figure 3.

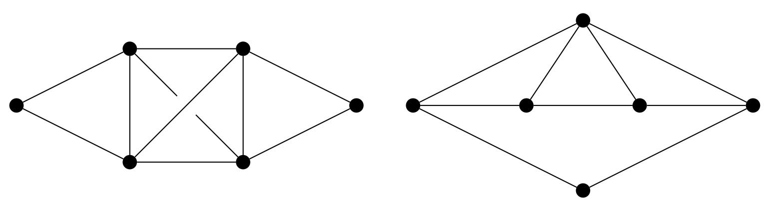

Figure 6: Two graphs for which we cannot apply Theorem 4.

## Extending Theorem 4

I would like to extend Theorem 4 to encompass all RAAGs which are not joins. Two specific graphs which are not covered by our theorem are seen in Figure 6. Both of these graphs contain multiple copies of $P_4$ as induced subgraphs, but their associated RAAGs do not split algebraically along parabolic subgroups of $A(P_4)$. In ongoing work with R. Kropholler, we are trying to extend our results to these graphs.