Homework 2: Question 4

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September 2017

1 Question

Assume that $x = (x_1, x_2)$ is a two dimensional vector and we have a function K defined as $K(x, z) = x_1 * z_1 + x_1 * e^{z_2} + z_1 * e^{x_2} + e^{x_2+z_2}$.

Prove that K is a kernel.

2 Approach: Mercer's Condition

I will prove that this function satisfies Mercer's Condition, which states that $k(x, x')$ is only a kernel if

1. It is symmetric
2. The resulting symmetric matrix is positive semi-definite

3 Mercer's 1st Condition

Let $\mathbf{x} = (x_1, x_2)$ and $\mathbf{z} = (z_1, z_2)$ be two vectors. Then

$$K(x, z) = x_1 * z_1 + x_1 * e^{z_2} + z_1 * e^{x_2} + e^{x_2+z_2}. \tag{1}$$

and

$$K(z, x) = z_1 * x_1 + z_1 * e^{x_2} + x_1 * e^{z_2} + e^{z_2+x_2} \tag{2}$$

Therefore

$$K(x, z) = K(z, x)$$

Thus $K(x, z)$ is symmetrical, satisfying the first condition of Mercer's Condition

4 Mercer's 2nd Condition

Let $\mathbf{x} = (x_1, x_2)$ and $\mathbf{z} = (z_1, z_2)$ be two vectors

and

$$M = \begin{matrix} k(x,x) & k(z,x) \\ k(x,z) & k(z,z) \end{matrix}$$

Where tr_M stands for the trace of matrix M
and det_M stands for the determinant of matrix M

For $K(x,z)$ to be considered positive semi-definite, matrix M must satisfy the following conditions

$$y \begin{cases} tr_M \geq 0 \\ det_M \geq 0 \end{cases}$$

a) Finding tr_M, the trace of our matrix

$$tr_M = k(x,x) + k(z,z) \tag{3}$$

$$tr_M = (x_1 + 2x_1 e^{x_2}) + (z_1 + 2z_1 e^{z_2}) \tag{4}$$

$$tr_M = (x_1 + e^{x_2})^2 + (z_1 + e^{z_2})^2 \tag{5}$$

Therefore tr_M will always be greater than or equal to zero, which means M satisfies the first condition to be considered a positive semi-definite matrix

b) Finding det_M, the determinant of our matrix

$$det_M = k(x,x) * k(z,z) - k(x,z) * k(z,x) \tag{6}$$

given that $k(x,z)$ is symmetrical $k(x,z) * k(x,z) = k(x,z)^2$

$$det_M = k(x,x) * k(z,z) - k(x,z)^2 \tag{7}$$

since

$$k(x,x) * k(z,z) = (x_1 + 2x_1 e^{x_2}) + (z_1 + 2z_1 e^{z_2}) \tag{8}$$

$$k(x,x) * k(z,z) = 4x_1 z_1 e^{z_2 + x_2} + 2xe^{2z_2 + x_2} + 2z_1 e^{z_1 + 2x_2} + e^{2z_2 + 2x_2} + 2e^2 x^2 z_1 + e^{2z_2} x^2 + 2e_2^x x_1 z_1^2 + 2e^{2x_2} z^2 + x_1^2 z_1^2 \tag{9}$$

and

$$k(x,z)^2 = 4x_1 z_1 e^{z_2 + x_2} + 2xe^{2z_2 + x_2} + 2z_1 e^{z_1 + 2x_2} + e^{2z_2 + 2x_2} + 2e^2 x^2 z_1 + e^{2z_2} x^2 + 2e_2^x x_1 z_1^2 + 2e^{2x_2} z^2 + x_1^2 z_1^2 \tag{10}$$

then

$$k(x,x) * k(z,z) = k(x,z)^2 \tag{11}$$

and thus

$$det_M = k(x, x) * k(z, z) - k(x, z)^2 \tag{12}$$

results in

$$det_M = 0 \tag{13}$$

Therefore det_M is great than or equal to zero, which means M satisfies the second condition to be considered a positive semi-definite matrix

5 Conclusion

Given that function K, defined as $K(x, z) = x_1 * z_1 + x_1 * e^{z_2} + z_1 * e^{x_2} + e^{x_2+z_2}$, satisfies Mercer's Condition, we can consider it a kernel.