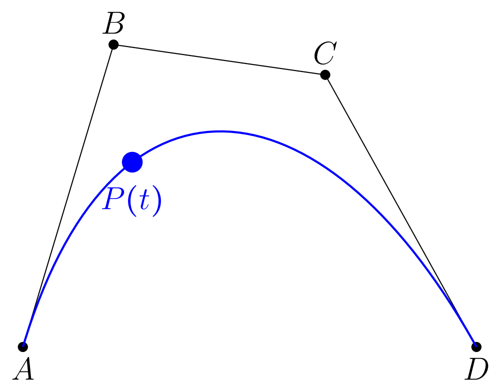
CS130 - Curves

Name: ______________________ SID: ______________



The following problems refer the figure above. In these problems, we will be deriving the cubic Bézier from some of its properties. This will help motivate the choices in the definition of the Bézier curve. Let A, B, C, and D be the control points for the curve, and let $p_a(t)$, $p_b(t)$, $p_c(t)$, and $p_d(t)$ be the blending functions, so that $P(t) = p_a(t)A + p_b(t)B + p_c(t)C + p_d(t)D$.

The cubic Bézier curve has the following properties:

1. Symmetry. Reversing the order of A, B, C, D results in the same curve, but with t replaced with $1 - t$.

2. Bézier curves end at two of the control points. $P(0) = A$ and $P(1) = D$.

3. The slope of the curve at the endpoints is given by the segment connecting the control points. $P'(0) = k(B - A)$ and $P'(1) = k(C - D)$ for some k.

4. Convex hull. The Bézier curve is inside the convex hull of the control points.

5. The blending functions form a *partition of unity*. That is, $p_a(t) + p_b(t) + p_c(t) + p_d(t) = 1$ for all t.

1. Bézier curves are *symmetrical*. Using property 1, express $p_a(t)$ and $p_b(t)$ as functions of $p_c(t)$ and $p_d(t)$.

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2. Since $p_c(t)$ and $p_d(t)$ are cubic polynomials, we can write them as

$$\begin{aligned} p_c(t) &= c_0 + c_1 t + c_2 t^2 + c_3 t^3 \\ p_d(t) &= d_0 + d_1 t + d_2 t^2 + d_3 t^3 \end{aligned}$$

for some coefficients $c_0, \ldots, c_3$ and $d_0, \ldots, d_3$. Use property 2 to solve for c_0, d_0, c_3, and d_3. Use these to eliminate those variables from $p_c(t)$ and $p_d(t)$.

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3. Use property 3 to solve for c_1, d_1, c_2, and d_2 as a function of k. Eliminate those variables from $p_c(t)$ and $p_d(t)$. These polynomials should now only depend on t and k.

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4. Write out $p_a(t)$ and $p_b(t)$ as functions of t and k.

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5. For Bézier curves, $k = 3$. This choice is made because of property 4. In particular, if $A = B$ (so that the convex hull is the triangle BCD), show that if $k > 3$ then $P(t)$ is outside triangle BCD for some $0 \le t \le 1$. (Hint: the blending functions serve as barycentric weights.)

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6. For the case $k = 3$, write out the blending functions. Show that the De Casteljau algorithm produces the same point as $P(t)$.

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7. A set is called *convex* if the segment connecting any two points in the set is itself entirely in the set. Use the De Casteljau algorithm to show that the Bézier curve is inside the convex hull of its control points.

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8. Let $\mathbf{x} \to \mathbf{M}\mathbf{x} + \mathbf{b}$ be an affine transformation, where $\mathbf{M}$ is a matrix and $\mathbf{b}$ is a vector. Show that transforming the control points A, B, C, and D is equivalent to transforming $P(t)$. That is, we can transform a Bézier curve by simply transforming its control points.

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9. The blending functions for the degree n Bézier curve are given by

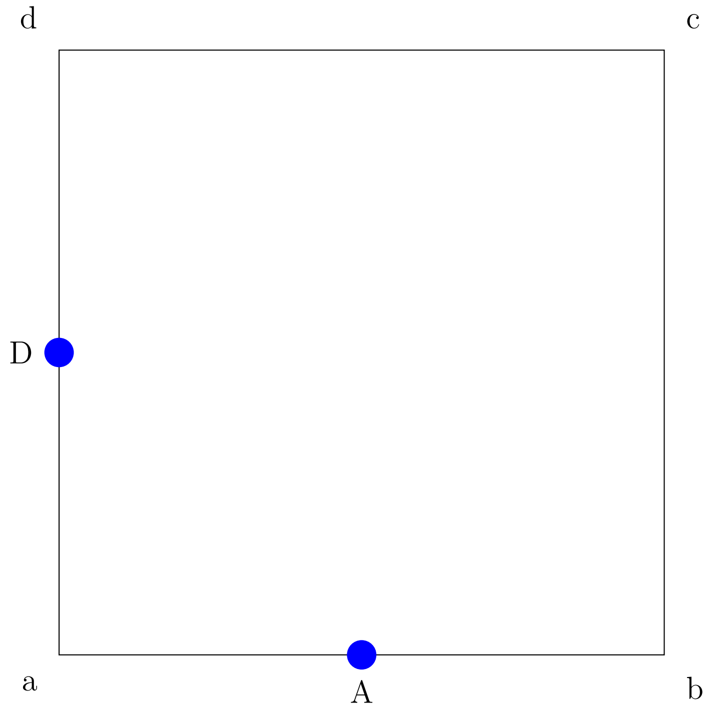
$$p_k(t) = \binom{n}{k} t^k (1-t)^{n-k}, \quad k = 0, \ldots, n$$

The binomial theorem states that

$$(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

Letting $x = t$ and $y = 1 - t$ in the binomial theorem, show that the Bézier blending functions satisfy property 5.

10.



Consider the figure above. In this exercise, you will be drawing new control points and sketching the resulting Bézier curves according to the instructions below. We will refer to the edges of the square *abcd* as e.g., edge *ab*, edge *bc*, etc.

1. Draw control points B and C so that the Bézier curve $p_1(t)$, with control points ABCD, satisfies $p_1'(0)$ parallel to edge *ab* and $p_1'(1)$ perpendicular to edge *ad*.

2. Sketch the Bézier curve $p_1(t)$.

3. Place a new control points E and F so that the Bézier curve $p_2(t)$, with control points DCEF, ends tangent to the edge *cd*.

4. Sketch the Bézier curve $p_2(t)$.

5. Add two control points G and H so that the Bézier curve $p_3(t)$, with control points by FGHA, is C^1 continuous with the Bézier curves $p_1(t)$ and $p_2(t)$.

6. Sketch the Bézier curve $p_3(t)$.

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