Volatility Skew Models

Fixed Income Derivatives

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Outline of the Lecture

- History of Volatility Skew
- Mechanism for creating fat tails
- General form of SDE creating volatility structure
- Some models with deterministic volatility structure
- Stochastic volatility models

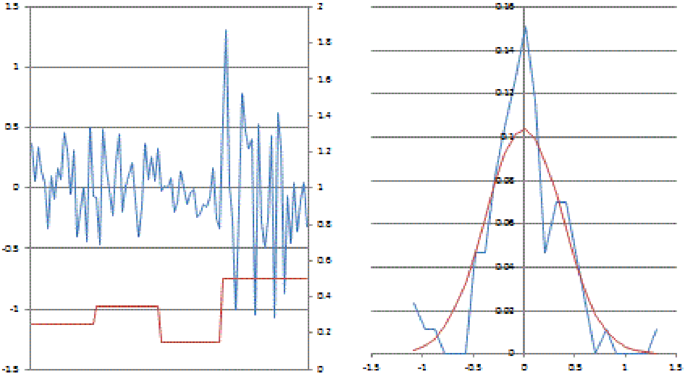
Interest Rate Modeling, Volume 1: Foundations and Vanilla Models, by Leiff B.G. Andersen and Vladimir V. Piterbarg, 1st edition, Atlantic Financial Press, 2010

Brief History of Volatility Skew

- Original Black-Scholes-Merton option pricing approach (1973) was based on double flat world paradigm:
  - Volatility is flat across the strike space;
  - Volatility is flat across time;
- Traders immediately rejected the "flat over time" assumption, but kept the "flat across strikes" assumption, which is impossible mathematically
- After the crash of 1987 the problem of the "flat across strikes" paradigm was realized
- However, the solution was only half step away from it
- 1990s the lesson repeated in OTC fixed income derivatives options world; it resulted in significant effort of combining volatility skew models with interest rates models
- Limited choice of volatility skew models for fixed income derivatives

Time-Variant Volatility Creates Fat Tails

Mechanism for Generating Heavy Tailed Distribution



General Assumption of Volatility Structure

Let $S(t)$ denote a forward Libor or swap rate and let $W(t)$ be a one-dimensional Brownian motion under measure P in which $S(\cdot)$ is a martingale.
$S(t)$ follows an SDE:

$$dS(t) = \lambda\varphi(S(t))\,dW(t)\,, \tag{1}$$

where λ is a positive constant and $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$ satisfies some general regularity conditions.
Both λ and φ can be time-dependent.
Compare 1 with the Black-Scholes type of SDE:

$$dS(t) = S(t)\,\lambda\frac{\varphi(S(t))}{S(t)}dW(t)\,,$$

which shows that expression $\lambda\frac{\varphi(S(t))}{S(t)}$ plays the role of implied Black-Scholes volatility dependent of the level of $S(t)$.

Quadratic Assumption

The role of φ is to allow for any reasonable marginal distribution of $S(T)$. If $\sigma(0, S(0); t, K)$ represents volatility smile at expiration t (assuming that information is available for the continuum $(t, K) \in [0, T] \times [0, \infty]$), $\varphi(t, K)$ can be uniquely implied by using some non-parametric expressions.

Assume

$$\varphi(S) = a + (S - S(0))^2,$$

$a > 0$.

Such φ implies U-shaped volatility smile when S is close to $S(0)$. But when S moves far away from $S(0)$ in either direction, volatility "smile" acquires undesirable feature: it is either strongly increasing or strongly decreasing for all strikes (instability of volatility structure).

CEV Model I

A class of volatility smiles with more realistic features can be produced by Constant Elasticity of Variance (CEV) Model.
The CEV model is characterized by the choice of φ:

$$\varphi(S) = S^p$$

In terms of Black-Scholes implied volatility CEV has volatility skew as function of level $S(t)$

$$\lambda S(t)^{p-1}.$$

CEV Model Properties

Proposition

Consider the SDE

$$dS(t) = \lambda S(t)^p \, dW(t) \tag{2}$$

where $p > 0$ is constant and $W(t)$ is a one-dimensional Brownian motion. The following holds:

1. *All solutions to (2) are non-explosive.*
2. *For $p \geq \frac{1}{2}$, the SDE (2) has a unique solution.*
3. *For $0 < p < 1$, $S = 0$ is an attainable boundary for (2); for $p \geq 1$, $S = 0$ is an unattainable boundary for (2).*
4. *For $0 < p \leq 1$, $S(t)$ in (2) is a martingale; for $p > 1$, $S(t)$ is a strict supermartingale.*

Possible States of CEV Model I

For $p \geq \frac{1}{2}$ the state $S = 0$ is absorbing for the solution to (2).
For $0 < p < \frac{1}{2}$ the solution is not unique. In order to make it unique and consistent with the case $p \geq \frac{1}{2}$ we set $S = 0$ to be an absorbing barrier. This is also the only boundary condition consistent with the absence of arbitrage.
For $p < 1$, the probability of hitting the absorbing state $S = 0$ before time horizon T can be calculated (see formulas (7.10) – (7.12) on page 281 of [1]). If this probability is substantial it is recommended to prevent absorption by using regularization.
In order to ensure that $S(t)$ is a martingale we normally prefer to avoid using $p > 1$ (due to the Property 4 of Proposition 1). This is not a significant restriction for interest rates applications, because upward sloping volatility skew, corresponding to the case $p > 1$, is not typical.

Call Option Pricing in CEV

Proposition

Consider the CEV model (2). Let $\chi^2_\nu(\gamma)$ be a non-central chi-square distributed variable with ν degrees of freedom and non-centrality parameter γ, and let

$$\Upsilon(x,\nu,\gamma) = \mathbb{P}\left\{\chi^2_\nu(\gamma) \le x\right\}$$

be the cumulative distribution function for $\chi^2_\nu(\gamma)$. Also define

$$a = \frac{K^{2(1-p)}}{(1-p)^2\lambda^2(T-t)}, b = |p-1|^{-1}, c = \frac{S^{2(1-p)}}{(1-p)^2\lambda^2(T-t)}.$$

Then for $0 < p < 1$ and an absorbing boundary at $S = 0$ we have, for $K > 0$

$$c_{CEV}(t,S;T,K) = S\left(1-\Upsilon(a,b+2,c)\right) - K\Upsilon(c,b,a).$$

Special Cases of Option Pricing with CEV I

See Proposition 7.2.6, comments and references in [1, pp. 282-283] for valuation of $c_{CEV}\left(t, S\left(t\right); T, K\right) = \mathbb{E}\left[\left(S\left(T\right) - K\right)^{+}\right]$.

- The result of the Proposition holds for all $p < 1$ including negative p.
- For $p > 1$ the result is complimentary:

$$c_{CEV}\left(t, S; T, K\right) = S\left(1 - \Upsilon\left(c, b, a\right)\right) - K\Upsilon\left(a, b+2, c\right).$$

- The special case $p = 1$ leads to the Black formula with volatility λ:

$$\begin{aligned} c_B\left(t, S; T, K, \lambda\right) &= S\Phi\left(d_+\right) - K\Phi\left(d_-\right), \\ d_\pm &= \frac{\ln\left(\frac{S}{K}\right) \pm \lambda^2 \frac{(T-t)}{2}}{\lambda\sqrt{T-t}}, \end{aligned}$$

where $\Phi\left(\cdot\right)$ is the standard Gaussian CDF.

Special Cases of Option Pricing with CEV II

- For $p = 0$, if we remove the assumption of an absorbing barrier at the origin, $S(t)$ is a Gaussian process. In this case we have Normal Black-Scholes with basis point (b.p., Normal) volatility λ:

$$\begin{aligned} c_N(t, S; T, K, \lambda) &= (S - K)\Phi(d) + \lambda\sqrt{T - t}\phi(d), \\ d &= \frac{S - K}{\lambda\sqrt{T - t}}, \end{aligned}$$

where $\Phi(\cdot), \phi(\cdot)$ are the standard Gaussian CDF and PDF, respectively.

Regularization I

Recall that the CEV process implies a positive probability of absorption at $S = 0$ in case $p < 1$.
It is possible to specify a regularized version of the CEV model:

$$\varphi(x) = x \min\left(\varepsilon^{p-1}, x^{p-1}\right), \varepsilon > 0, p < 1.$$

After such modification the process will not be able to reach the origin, but the cost of the fix is loss of closed-form pricing formulas.
Options have to be priced numerically.
Or we can use the non-regularized pricing formulas as approximation.

Regularization II

Proposition

For $p < 1$ and $\varepsilon > 0$, let

$$\begin{aligned} dx(t) &= \lambda x(t)^p \, dW(t), \\ dy(t) &= \lambda y(t) \min\left(\varepsilon^{p-1}, y(t)^{p-1}\right) dW(t), \end{aligned}$$

where $x(0) = y(0) > 0$ and $W(t)$ is a Brownian motion in measure $\mathbb{P}$.
For $p < \frac{1}{2}$, 0 is assumed to be an absorbing boundary x.
For some T and some constant K, we then have

$$\begin{aligned} \lim_{\varepsilon \to 0} \left[\mathbb{P}(x(T) < h) - \mathbb{P}(y(T) < h)\right] &= 0, \\ \lim_{\varepsilon \to 0} \left[\mathbb{E}\left[(x(T) - K)^+\right] - \mathbb{E}\left[(y(T) - K)^+\right]\right] &= 0. \end{aligned}$$

Displaced Diffusion Model I

Defining

$$\varphi(x) = (\alpha + x)^p$$

for some constant α moves the absorbing boundary from zero to $-\alpha$. Indeed, let $Z(t) = \alpha + S(t)$, then by Ito's lemma, for $Z(t)$ holds the CEV SDE

$$dZ(t) = \lambda Z(t)^p \, dW(t)$$

Displaced Diffusion Model II

Proposition

Pricing of the call option under the displaced CEV model

$$c_{DCEV}\left(t, S\left(t\right); T, K, \alpha\right) = \mathbb{E}\left[\left(S\left(T\right) - K\right)^{+}\right]$$

is reduced to pricing under the CEV model

$$c_{DCEV}\left(t, S; T, K, \alpha\right) = c_{CEV}\left(t, S + \alpha; T, K + \alpha\right),$$

where $S, K > -\alpha$.

The displaced model is often used in log-normal case $p = 1$.

Displaced Diffusion Model III

Log-Normal Case

Proposition

Consider the displaced log-normal process

$$dS(t) = \lambda(\beta + \zeta S(t))\, dW(t),$$

where $W(t)$ *is a Brownian motion in measure* $\mathbb{P}$, *and* $\lambda, \zeta \neq 0$. *Assuming* $S(t), K > -\frac{\beta}{\zeta}$,

$$\begin{aligned} c_{DLN}(t, S(t); T, K) &= \left(S(t) + \frac{\beta}{\zeta}\right)\Phi(d_+) \\ &\quad - \left(K + \frac{\beta}{\zeta}\right)\Phi(d_-), \\ d_\pm &= \frac{\ln\left(\frac{S(t)+\frac{\beta}{\zeta}}{K+\frac{\beta}{\zeta}}\right) \pm \frac{1}{2}\zeta^2\lambda^2(T-t)}{\zeta\lambda\sqrt{T-t}} \end{aligned}$$

Displaced Diffusion Model IV

It is convenient to rewrite the displaced log-normal model as

$$dS(t) = \sigma(bS(t) + (1-b)L)\,dW(t). \tag{3}$$

The parameter L is often set to near or at the initial value $S(0)$. The volatility parameter σ is similar to Black's volatility, it changes the level of the volatility smile, but not the slope. The other parameter b changes the slope, but not the level.
The displaced log-normal volatility model is also used as a first-order approximation for the general model (1).
By expanding $\varphi(\cdot)$ around at-the-money to the first order, obtain

$$dS(t) \approx \lambda\left(\varphi(S(0)) + \varphi'(S(0))(S(t) - S(0))\right) dW(t),$$

which has the form of (3) with

$$\sigma = \lambda\frac{\varphi(S(0))}{S(0)}, b = \varphi'(S(0))\frac{S(0)}{\varphi(S(0))}, L = S(0).$$

Displaced Diffusion Model V

Similarly, displaced log-normal model is a good approximation for CEV with

$$\sigma = \lambda S^{p-1}(0), b = p, L = S(0).$$

Possible issues with displaced model: the process $S(t)$ can become negative if $\beta > 0$; in the case of stochastic volatility model the asymptotic linear growth *of* $\varphi(x)$ can lead to unbounded second moments of $S(t)$.

Quadratic Polynomial Volatility Model

To allow for a more realistic behavior of the volatility skew we can use a special choice of φ, corresponding to the quadratic volatility model

$$\varphi(x) = \alpha + \beta x + \gamma x^2 \tag{4}$$

for constants α , β and γ (the latter parameter usually has a small value).
Such model can produce shapes of volatility skew which are not strictly monotonic.
The features of the quadratic volatility model depend on the configuration of the roots of the polynomial $\alpha + \beta x + \gamma x^2$.
If the roots l and u, $l < u$ satisfy

$$l \leq S(0) \leq u,$$

then $S(t) \in [l, u]$ and every time the process comes close to one of the bounds the volatility drops down towards zero.
Since such behavior is unrealistic we usually exclude this possibility.

Quadratic Polynomial Volatility: Roots on One Side I

If $l < u < S(0)$, consider process

$$dS(t) = \frac{(S(t) - u)(S(t) - l)}{u - l} dW(t). \tag{5}$$

For this process (see [1]) :

1. $S(t) \in (u, \infty)$, $S(t)$ does not explode under the measure $\mathbb{P}$.
2. The process $S(t)$ is a strict supermartingale under $\mathbb{P}$.
3. Suppose that $S(t)$ follows (5) in some measure $\mathbb{P}$ and assume that put-call parity holds. Then options prices are:

$$\begin{aligned} p(0, S(0); T, K) &= \mathbb{E}\left[(K - S(T))^{+}\right], \\ c(0, S(0); T, K) &= p(0, S(0); T, K) + S(0) - K \\ &> \mathbb{E}\left[(S(T) - K)^{+}\right]. \end{aligned}$$

4. Let

$$
\begin{aligned}
K_1 &= \frac{(K-u)(S(0)-l)}{u-l}, X_1 = \frac{(S(0)-u)(K-l)}{u-l}, \\
K_2 &= \frac{(K-l)(S(0)-l)}{u-l}, X_2 = \frac{(S(0)-u)(K-u)}{u-l}.
\end{aligned}
$$

Assuming $K > u$, the put price is

$$
\begin{aligned}
p(0, S(0); T, K) &= K_1 \Phi\left(-d_-^{(1)}\right) - X_2 \Phi\left(d_+^{(2)}\right) \\
&\quad - X_1 \Phi\left(-d_+^{(1)}\right) + K_2 \Phi\left(d_-^{(2)}\right),
\end{aligned}
$$

$$
d_\pm^{(i)} = \frac{\ln\left(\frac{X_i}{K_i}\right) \pm \frac{T}{2}}{\sqrt{T}},
$$

$$
c(0, S(0); T, K) = S(0) - K + p(0, S(0); T, K).
$$

Quadratic Polynomial Volatility: Roots on One Side III

Recall that the correct SDE (1) with φ given by (4) is not (5), but

$$\begin{aligned} dS(t) &= \lambda\gamma\left(S(t) - u\right)\left(S(t) - l\right) dW(t) \\ &= q\frac{\left(S(t) - u\right)\left(S(t) - l\right)}{u - l} dW(t), \\ q &= \lambda\gamma\left(u - l\right). \end{aligned}$$

Then the pricing formulas above can be used if we replace T with $q^2 T$.

Quadratic Polynomial Volatility Model: One Root

If there is only one root u of $\alpha + \beta x + \gamma x^2$, $u < S(0)$, consider the process

$$dS(t) = (S(t) - u)^2 \, dW(t),$$

which is a special case of the displaced CEV with the power equal to 2. The pricing formulas of the displaced CEV in this case are simplified to

$$\begin{aligned} p(0, S(0); T, K) &= (S(0) - u)(K - u)\sqrt{T} \\ &\quad \times \{d_+ \Phi(d_+) + \phi(d_+) - d_- \Phi(d_-) - \phi(d_-)\}, \end{aligned}$$

where $\phi(x)$ is the Gaussian density, and

$$d_\pm = \frac{1}{\sqrt{T}} \left(\pm \frac{1}{(S(0) - u)} - \frac{1}{(K - u)} \right)$$

As in the previous case, use the pricing formulas with T replaced by $\lambda^2 \gamma^2 T$ for the correct SDE

$$dS(t) = \lambda \gamma (S(t) - u)^2 \, dW(t).$$

Time-Dependent Diffusion Function I

Pricing swaptions and caplets does not require time-dependent volatility structure or term structure model.

But if we move to a more structured products interest rate models become necessary, their calibration to the market prices may require time-dependent volatility structure.

If (1) can be extended to a time-dependent case in the following separable form

$$dS(t) = \lambda(t)\,\varphi(S(t))\,dW(t), \tag{6}$$

then the generalized approach to pricing is obtained in the following proposition.

Time-Dependent Diffusion Function II

Proposition

Define

$$\tau(t) = \int_0^t \lambda^2(u)\,du,$$

and define $s(\cdot)$ *by* $S(t) = s(\tau(t))$, *with* $S(t)$ *following (6). Then*

$$ds(\tau) = \varphi(s(\tau))\,d\tilde{W}(\tau),\ s(0) = S(0),$$

where $\tilde{W}(\tau)$ *is a Brownian motion.*

The SDE in the Proposition is of the type (1) with $\lambda = 1$. Thus, all pricing formulas of the previous sections hold with the substitutions:

$$T \mapsto \tau(T),\ t \mapsto \tau(t),\ \lambda^2(T-t) \mapsto \int_t^T \lambda^2(u)\,du.$$

Time-Dependent Diffusion Function III

In more general case

$$dS(t) = \varphi(t, S(t))\, dW(t) = \lambda(t)\, g(t, S(t))\, dW(t),$$

where

$$X_0 = S(0),\, \lambda(t) = \varphi(t, X_0),\, g(t, x) = \frac{\varphi(t, x)}{\varphi(t, X_0)},$$

we may hope to find an approximation with a time-independent local volatility function which then can be interpreted as a time average.

Time-Dependent Diffusion Function IV

Example

The time-dependent displaced log-normal function

$$g\left(t, x\right) = b\left(t\right)\frac{x}{S\left(0\right)} + \left(1 - b\left(t\right)\right), t \in [0, T]$$

with the time-independent average function

$$\bar{g}\left(x\right) = \bar{b}\frac{x}{S\left(0\right)} + \left(1 - \bar{b}\right).$$

Time-Dependent Diffusion Function V

Example

The time-dependent CEV function

$$
\begin{aligned}
g(t,x) &= \left(\frac{x}{S(0)}\right)^{p(t)}, t \in [0, T] \\
\bar{g}(x) &= \left(\frac{x}{S(0)}\right)^{\bar{p}}.
\end{aligned}
$$

In both examples the averaged parameters $b(t)$ and $p(t)$ play the role of skew control.
Averaging weights are

$$
w_T(t) = \frac{v^2(t)\,\lambda^2(t)}{\int_0^T v^2(t)\,\lambda^2(t)\,dt}, v^2(t) = \int_0^t \lambda^2(s)\,ds.
$$

So that

$$\bar{b} = \int_0^T b(t)\, w_T(t)\, dt, \bar{p} = \int_0^T p(t)\, w_T(t)\, dt.$$

Note that in case of constant volatility $\lambda(t) \equiv \lambda$ the weights simplify to

$$\begin{aligned} v^2(t) &= \lambda^2 t, \\ w_T(t) &= \frac{t}{\int_0^T t dt} = \frac{t}{\frac{T^2}{2}}. \end{aligned}$$

This shows that the skew averaging weights increase with t.

Models with Stochastic Volatility I

Models with variable local volatility of the type (1) capture an important market effect: dependence of volatility on the level of underlying Libor rate.

But in these models local volatility is a deterministic function of the Libor rate.

Models with stochastic volatility produce more realistic shapes of volatility skew.

Let again $S(t)$ be the underlying forward Libor or swap rate; let also $W(t)$ and $Z(t)$ be two different Brownian motions under the same measure $\mathbb{P}$ under which $S(t)$ is a martingale. Assume that $W(t)$ and $Z(t)$ have constant cross-correlation ρ

$$\mathbb{E}\left[W(t)\,Z(t)\right] = \rho dt.$$

In real implementations ρ is usually put equal to zero.

Models with Stochastic Volatility II

Consider a family of stochastic volatility models

$$
\begin{aligned}
dS(t) &= \lambda\left(bS(t) + (1-b)L\right)\sqrt{z(t)}dW(t), \\
dz(t) &= \theta\left(z_0 - z(t)\right)dt + \eta\sqrt{z(t)}dZ(t), \\
z(0) &= z_0 = 1,
\end{aligned}
\tag{7}
$$

where parameters $b, \theta, \eta, \lambda > 0$ and $|\rho| < 1$.
In the special case $b = 1$, the model is well known as Heston stochastic volatility model.

Interpretation of the Parameters I

- Like in (1), parameter λ is responsible for the overall level of volatility, and b is responsible for the slope of the smile.
- The volatility of variance parameter η controls the curvature of the volatility smile.
- The mean reversion parameter of variance θ defines the force that pulls $z(t)$ towards z_0 whenever it deviates from it.
- When θ increases the long term variance of $z(t)$ decreases which reduces the effect of stochastic variance on the volatility skew for long enough expirations.
  θ controls the speed of decay of the volatility skew convexity.
- As before, parameter L is usually set equal or close to $S(0)$ This ensures interpretation of λ as relative (Black) volatility regardless of setting of b.
- The skew b is typically set as $b \in [0\%, 100\%]$.

Interpretation of the Parameters II

- The parameter η is usually expressed as annualized relative volatility of variance. Traders often interpret it as volatility of volatility.
- Volatility process is $\sqrt{z(t)}$, then by Ito's lemma

$$d\sqrt{z(t)} = O(dt) + \frac{\eta}{2} dZ(t),$$

so, if η is volatility of variance, then $\frac{\eta}{2}$ is approximately volatility of volatility.
- The parameter θ is expressed in percentage points per year. The inverse quantity θ^{-1} has units of years and shows the time over which volatility shocks dissipate.

Implied Volatility Structure

From Lemma 8.7.1 of [1, p. 345] derive the expression for implied Black's volatility $\sigma_B\left(t, S\left(t\right); T, K\right)$ where $t = 0$ is now, $S\left(t\right)$ is the initial underlying price, T is the expiration and K is the strike:

$$
\begin{aligned}
\sigma_B\left(0, S\left(0\right); T, K\right) &\approx \sigma_{ATM} + R\chi + \frac{1}{2}B\chi^2, \\
\sigma_{ATM} &\approx \lambda, \\
R &\approx \frac{\lambda}{2}\left(-\left(1-b\right) + \frac{\eta\rho}{2\lambda}\right), \\
B &\approx \lambda\left(\frac{\left(1-b\right)^2}{6} + \frac{\eta^2\left(2-5\rho^2\right)}{24\lambda^2}\right).
\end{aligned}
$$

By varying parameters of the model we can observe their effects on the volatility structure.

Some Basic Properties I

Proposition

The stationary distribution of $z(\cdot)$ in (7) is Gamma distribution with probability density

$$\pi(z) = \frac{z^{\alpha-1} e^{-\beta z}}{\Gamma(\alpha)\, \beta^{-\alpha}},$$

where

$$\alpha = \frac{2\theta z_0}{\eta^2}, \beta = \frac{2\theta}{\eta^2}.$$

Proposition

The process $S(\cdot)$ in (7) is a proper martingale.

Fourier Integration I

The most commonly used method of option pricing under stochastic volatility models is based on Fourier integration.

Theorem

[1, p. 324, Theorem 8.4.1] Let ξ be a random variable, define its moment generating function $\chi(u)$:

$$\chi(u) = \mathbb{E}\left[e^{u\xi}\right].$$

Then for $k \in \mathbb{R}$,

$$\mathbb{E}\left[\left(e^{\xi} - e^{k}\right)^{+}\right] = \chi(1) - \frac{e^{k}}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-k(\alpha+iw)}\chi(\alpha + iw)}{(\alpha + iw)(1 - \alpha - iw)} dw$$

for any $0 < \alpha < 1$ for which the right-hand side exists.

Fourier Integration II

Theorem

In the notations of the previous theorem

$$\mathbb{E}\left[(\xi - k)^+\right] = \left.\frac{d\chi(k)}{dk}\right|_{k=0} - k + \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{-k(-\alpha+iw)}\chi(-\alpha+iw)}{(-\alpha+iw)^2}dw$$

for any $\alpha > 0$ for which the right-hand side exists.

The moment-generating function for the model (7) is obtained in [1, pp. 320-322, Propositions 8.3.6, 8.3.7 and 8.3.8].
Combining it with the theorems above, obtain the following result.
Let $c_B(0, S'; T, K', \sigma)$ be the Black formula for underlying S', strike K', expiry T, volatility σ.

Fourier Integration III

Theorem

The price of a call option $c_{SV}(0, S; T, K)$ in the stochastic volatility model (7) is given by

$$
\begin{aligned}
c_{SV}(0, S; T, K) &= \frac{1}{b} c_B\left(0, S'; T, K', \lambda b\right) \\
&\quad - \frac{K'}{2\pi b} \int_{-\infty}^{\infty} \frac{e^{\left(\frac{1}{2} + iw\right) \ln\left(\frac{S'}{K'}\right)} q\left(\frac{1}{2} + iw\right)}{w^2 + \frac{1}{4}} dw
\end{aligned}
$$

$$
\begin{aligned}
S' &= bS + (1 - b) L, K' = bK + (1 - b) L, \\
q(u) &= \Psi_{\bar{z}}\left(\frac{1}{2} (\lambda b)^2 u (u - 1), u; T\right) - e^{\frac{1}{2}\lambda^2 b^2 z_0 T u (u-1)},
\end{aligned}
$$

with $\Psi_{\bar{z}}$ defined in [1, p. 322, Proposition 8.3.8].

SABR Model

SABR is a popular model which falls as a particular case in the general framework.
It is in fact a version of CEV with stochastic volatility.

$$\begin{aligned} dS(t) &= \lambda S^c(t)\sqrt{z(t)}dW(t), \\ dz(t) &= \frac{1}{4}\eta^2 z(t)\,dt + \eta z(t)\,dZ(t), \end{aligned}$$

where correlation between $dW(t)$ and $dZ(t)$ is ρ and $0 < c < 1$.
Stochastic volatility under SABR is a simple Ito drift-diffusion process.
This model has negative mean reversion parameter $\theta = -\frac{\eta^2}{4}$.
Under SABR $S(\cdot)$ cannot go negative, but might be absorbed at zero.
The model allows fairly accurate asymptotic expansions for European option prices.