Collateral Agreements and Derivatives Pricing

Fixed Income Derivatives Workshop

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Disclaimer

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Outline

- Classical framework
  1. The role of risk-free interest rate
  2. Where does the risk-free interest rate come from
- Mechanics of Collateral
- Concept of CVA, DVA and FVA
- CSA and Perfect CSA
- Pricing under Perfect CSA
- Forward measure
- Effect on rates

The Mysterious Risk-Free Interest Rate

Traditional method of pricing derivative products is based on the following assumptions:

- Possibility of borrowing and lending at the same risk-free rate
- Under the risk-neutral measure all tradeable assets of the economy have expected return equal to the risk-free rate
- Under the real world measure expected return of an asset is related to the riskiness of this asset: the higher the risk – the higher the return. As a consequence, measures corresponding to different assets chosen as numeraires are defined by market prices of risk of the numeraire assets. In the risk neutral world defined by the numeraire asset all tradeable assets are expected to earn the risk-free rate

What is risk-free rate $r(t)$?

What is risk-free rate?

Candidates [1, Page 77]:

1. The safest asset: Treasury, but for tax reasons Treasury rate is too low;
2. The most liquid and convenient index: LIBOR, a short term borrowing rate between AA-rated financial institutions (banks); LIBOR curve is smooth and liquid. Several serious issues with Libor as a candidate for a risk-free rate have been realized in the last several years.
3. The shortest exposure: Overnight Indexed Swap (OIS) is more correct candidate, since collateral is typically in the form of cash and OIS rate is the return that it has; Collateralized agreement has no risk;

- Prior to 2007 market convention made LIBOR the risk-free rate.
- Basis spread (LIBOR-OIS) was small and stable, but after 2007 it widened out from 2 b.p. to 200 b.p.
- Collateralization and central clearing of swaps is becoming a standard (CSA). This makes OIS the best candidate in the post-crisis era.

New reality: Multiple rates

Prior to the crisis of 2007 the choice of risk-free rate was not important since all candidates were considered similar and had stable relationships with each other.

The reality of the post-crisis world:

- The choice of risk-free rate is important and far less obvious than we used to think
- Even with consensus regarding accepting OIS as risk-free rate, different levels and forms of collateralization result in simultaneous use of multiple rates

An OTC derivative trading desk may enter same contracts with CSA (secured by collateral) or not (unsecured).

In most cases collateral may be in the form of cash, but it also can be in the form of purchased assets.

The trading desk may use different sources of funding (funding accounts): for example, funding desk, collateral posted by counterparties, repo contracts on different assets.

Funding Rates

Let $r_C(t)$ be the short rate corresponding to collateral funding account, i.e. overnight rate paid on collateral by dealers under CSA.
Let $P_C(t, T)$, $0 \leq t \leq T < \infty$, be the corresponding discount factor.
According to HJM framework,

$$dP_C(t, T) = r_C(t) P_C(t, T) dt - \sigma_C(t, T) P_C(t, T) dW_C(t) \tag{1}$$

where $W_C(t)$ is a Brownian motion under the risk-neutral measure $\mathbb{P}$ and $\sigma_C(t, T)$ is a stochastic process of volatility.
Let $X(t)$ denote the price of the asset underlying the derivative and $r_R(t)$ is the short rate on funding secured by the asset (repo).
Let $r_F(t)$ be the short rate for unsecured funding.
Typically $r_C(t) \leq r_R(t) \leq r_F(t)$.
The spread $s_R(t) = r_R(t) - r_C(t)$ is often called lending fee, $s_F(t) = r_F(t) - r_C(t)$ can be interpreted as stochastic intensity of default by the bank.

Black-Scholes-Merton Framework

The underlying asset $X(t)$ follows the real world dynamics

$$dX(t) = \mu_X(t) X(t)\, dt + \sigma_X(t) X(t)\, dW(t)$$

Let $V(X,t)$ be a derivative security on the asset. Applying Ito's lemma to it obtain

$$\begin{aligned} dV(X,t) &= \mathfrak{D}(V(X,t))\, dt + \Delta(t)\, dX(t) && (2) \\ \mathfrak{D} &= \frac{\partial}{\partial t} + \frac{1}{2}\sigma_X^2(t) X^2(t) \frac{\partial^2}{\partial X^2(t)} \\ \Delta(t) &= \frac{\partial V(X,t)}{\partial X(t)}. \end{aligned}$$

Let $C(t)$ be the amount of cash in the collateral account held against the derivative at time t.

To replicate the derivative, we hold at time t $\Delta(t)$ units of the asset and $\gamma(t)$ cash

$$V(X,t) = \Pi(t) = \Delta(t) X(t) + \gamma(t). \qquad (3)$$

Funding Accounts

Let $B_\alpha(t)$ be a funding account, where α indicates the funding source. Assume the dynamics based on short rate r_α

$$\begin{aligned} dB_\alpha(t) &= r_\alpha(t)\, B_\alpha(t)\, dt,\, B_\alpha(0) = 1 \\ B_\alpha(t) &= \exp\left\{ \int_0^t r_\alpha(u)\, du \right\} \end{aligned}$$

The cash amount $\gamma(t)$ is split among several accounts:

- Amount $C(t)$ is in collateral account $B_C(t)$ with the short rate $r_C(t)$
- $B_F(t) = V(X,t) - C(t)$ is borrowed unsecured from the treasury desk at $r_F(t)$
- $B_R(t) = \Delta(t)\, X(t)$ is secured by the asset (repo) to finance hedge at $r_R(t)$
- The $\Delta(t)\, X(t)$ shares of the asset pay dividend rate $r_D(t)$ to $B_D(t)$

$$\begin{aligned} d\gamma(t) &= \left[r_C(t)\, C(t) + r_F(t)\,(V(X,t) - C(t)) \right. \\ &\quad \left. + (r_D(t) - r_R(t))\, \Delta(t)\, X(t) \right] dt \end{aligned} \tag{4}$$

Self-Financing Condition

From the self-financing equation (3) and Ito's lemma (2)

$$\begin{aligned} d\gamma(t) &= dV(X,t) - \Delta(t)\,dX(t) = (\mathfrak{D}V(X,t))\,dt \\ &= \left(\frac{\partial}{\partial t} + \frac{1}{2}\sigma_X^2(t)\,X^2(t)\,\frac{\partial^2}{\partial X^2(t)}\right) V(X,t)\,dt \end{aligned}$$

Combining with funding accounts split (4) obtain

$$\begin{aligned} &\left(\frac{\partial}{\partial t} + \frac{1}{2}\sigma_X^2(t)\,X^2\frac{\partial^2}{\partial X^2}\right) V \\ = \quad & r_C(t)\,C(t) + r_F(t)\,(V - C(t)) + (r_D(t) - r_R(t))\,\frac{\partial V}{\partial X}X \end{aligned}$$

or

$$\begin{aligned} &\frac{\partial V}{\partial t} + (r_R(t) - r_D(t))\,\frac{\partial V}{\partial X}X + \frac{1}{2}\sigma_X^2(t)\,X^2(t)\,\frac{\partial^2 V}{\partial X^2} \\ = \quad & r_F(t)\,V - (r_F(t) - r_C(t))\,C(t) \end{aligned} \tag{5}$$

Solution to Extended Black-Scholes-Merton Equation

Solution to (5) is given by [2]

$$V(X,t) = \mathbb{E}_t\left[e^{-\int_t^T r_F(u)du} V(X,T) + \int_t^T e^{-\int_t^u r_F(v)dv}\left(r_F(u) - r_C(u)\right) C(u)\,du\right] \tag{6}$$

Expectation here is taken with respect to the measure in which the asset grows at $r_R(t) - r_D(t)$:

$$dX(t) = \left(r_R(t) - r_D(t)\right) X(t)\,dt + \sigma_X(t) X(t)\,dW(t) \tag{7}$$

Repo rate has been considered a very close to risk-free rate when valuing derivatives on $X(t)$ [1, Page 77].
Thus, we can think of the measure in (6) as risk-neutral measure $\mathbb{P}$.

Solution to Extended Black-Scholes-Merton Equation

Another form of solution to (5) is

$$
\begin{aligned}
V(X,t) &= \mathbb{E}_t\left[e^{-\int_t^T r_C(u)du} V(X,T)\right] \qquad (8)\\
&\quad -\mathbb{E}_t\left[\int_t^T e^{-\int_t^u r_C(v)dv}\left(r_F(u)-r_C(u)\right)\left(V(X,u)-C(u)\right)du\right]
\end{aligned}
$$

Since

$$
\begin{aligned}
\mathbb{E}_t\left[dV(t)\right] &= \left(r_F(t)V(t)-\left(r_F(t)-r_C(t)\right)C(t)\right)dt\\
&= \left(r_F(t)V(t)-s_F(t)C(t)\right)dt
\end{aligned}
$$

the growth rate in the derivative security is the funding rate $r_F(t)$ applied to its value, minus credit spread $s_F(t)$ applied to the collateral.

Full Collateral, Zero Collateral

In particular case, if $C(t) = V(X,t)$ then

$$\begin{aligned}
\mathbb{E}_t\left[dV(X,t)\right] &= r_C(t)\,V(X,t)\,dt \\
V(X,t) &= \mathbb{E}_t\left[e^{-\int\limits_t^T r_C(u)du} V(X,T)\right]
\end{aligned}$$

and the derivative grows at the risk-free rate.
If $C(0) = 0$, then

$$\mathbb{E}_t\left[dV(X,t)\right] = r_F(t)\,V(X,t)\,dt, \tag{9}$$

i.e. the growth rate is unsecured funding rate, or risk-free rate $r_C(t)$ adjusted by the bank's credit spread $s_F(t)$.

Zero-Strike Call Option

A contract that promises delivery of the asset at a given future time T can be expressed as Call option with strike 0, expiring at T.
The payoff of the contract is $V_{ZSC}(T) = X(T)$ and without CSA

$$V_{ZSC}(t) = \mathbb{E}_t\left[e^{-\int_t^T r_F(u)du} X(T)\right]$$

If $r_D(t) = 0$, then from (7)

$$X(t) = \mathbb{E}_t\left[e^{-\int_t^T r_R(u)du} X(T)\right]$$

Comparing the two, we see that unlike classical case, $X(t) \neq V_{ZSC}(t)$ (unless $r_F(t) \equiv r_R(t)$). This is because Call option carries the credit risk of the bank, while the asset does not. The asset can be used to secure funding through repo, but the Call option cannot.

Forward Contract. No CSA

In what follows we denote assets covered by CSA with the accent above "$\wedge$" (for example, $\hat{F}$) and use no accent in no-CSA case.

According to a forward contract on $X(t)$, the bank agrees at time t to make time-T delivery of the asset in exchange for cash amount $F(t,T)$, determined at t and paid at T.

In no-CSA case the derivative contract has time-T payoff $X(T) - F(t,T)$. Since the contract has no entry cost (6) gives

$$
\begin{aligned}
0 &= \mathbb{E}_t\left[e^{-\int_t^T r_F(u)du}\left(X(T) - F(t,T)\right)\right] \\
F(t,T) &= P_F^{-1}(t,T)\,\mathbb{E}_t\left[e^{-\int_t^T r_F(u)du} X(T)\right].
\end{aligned}
$$

Here $P_F(t,T) \triangleq \mathbb{E}_t\left[e^{-\int_t^T r_F(u)du}\right]$ is credit-risky bond issued by the bank.

Note that $P_F(t,T)$ can be used as a numeraire. By definition

$$e^{-\int_0^t r_F(u)du} P_F(t,T) = \mathbb{E}_t\left[e^{-\int_0^T r_F(u)du}\right],$$

i.e. the discounted process $e^{-\int_0^t r_F(u)du} P_F(t,T)$ is a $\mathbb{P}$-martingale. Then

$$F(t,T) = P_F^{-1}(t,T)\,\mathbb{E}_t\left[e^{-\int_t^T r_F(u)du} X(T)\right] = \mathbb{E}_{F,t}^T\left[X(T)\right] \qquad (10)$$

where $\mathbb{E}_{F,t}^T$ is with respect to the T-forward risky measure $\mathbb{P}_F^T$ defined by the numeraire $P_F(t,T)$.

General Payoff. No CSA

Recall that in case of a general derivative payoff $V(X,t)$ and no CSA, like in (9) the pricing formulas are:

$$
\begin{aligned}
\mathbb{E}_t\left[dV(X,t)\right] &= r_F(t)\,V(X,t)\,dt \\
V(X,t) &= \mathbb{E}_t\left[e^{-\int_t^T r_F(u)du} V(X,T)\right].
\end{aligned}
$$

Applying again the risky T-forward measure, defined by the risky bond $P_F(t,T)$ as numeraire, we derive under no-CSA assumption:

$$V(X,t) = P_F(t,T)\,\mathbb{E}_t^T\left[V(X,T)\right]$$

Note that in the pre-crisis setup we assumed that unsecured funding (at LIBOR) is risk-free.
We then had the same formula with respect to the T-forward measure.

Forward Contract. With CSA

Now look at the case when at any moment collateral posted under CSA is equal to the value of the derivative asset: $V(X,t) = C(t)$.
At time t of the trade, according to (8) we have for the forward price $\hat{F}(t,T)$

$$\begin{aligned}
\hat{V}(X,t) &= 0 = \mathbb{E}_t\left[e^{-\int_t^T r_C(u)du}\left(X(T) - \hat{F}(t,T)\right)dt\right], \\
\hat{F}(t,T) &= P_C^{-1}(t,T)\,\mathbb{E}_t\left[e^{-\int_t^T r_C(u)du} X(T)\right] \neq F(t,T),
\end{aligned}$$

where $P_C(t,T) \triangleq \mathbb{E}_t\left[e^{-\int_t^T r_C(u)du}\right]$.

Forward Contract. With CSA

By using this time $P_C(t,T)$ as a numeraire we find

$$\hat{F}(t,T) = \mathbb{E}^T_{C,t}\left[X(T)\right],$$

where expectation $\mathbb{E}^T_{C,t}$ is taken with respect to the risk-free T-forward measure $\mathbb{P}^T_C$ defined by $P_C(t,T)$.
Fore more general payoff $V(t,T)$ in the case $C(t) = V(X,t)$ the pricing formula in terms of risk-free T-forward measure $\mathbb{P}^T_C$ is

$$\hat{V}(X,t) = \mathbb{E}_t\left[e^{-\int\limits_t^T r_C(u)du} V(X,T)\,dt\right] = P_C(t,T)\,\mathbb{E}^T_{C,t}\left[V(X,T)\right]$$

FVA

Starting with (10) obtain

$$
\begin{aligned}
F(t,T) &= \mathbb{E}_{F,t}^{T}\left[X(T)\right] = P_F^{-1}(t,T)\,\mathbb{E}_t\left[e^{-\int_t^T r_F(u)du} X(T)\right] \\
&= P_F^{-1}(t,T)\,\mathbb{E}_t\left[e^{-\int_t^T r_C(u)du}\, e^{-\int_t^T (r_F(u)-r_C(u))du} X(T)\right] \\
&= P_C(t,T)\,P_F^{-1}(t,T)\,\mathbb{E}_{C,t}^{T}\left[e^{-\int_t^T s_F(u)du} X(T)\right] \\
&= \mathbb{E}_{C,t}^{T}\left[\frac{M(T,T)}{M(t,T)} X(T)\right],
\end{aligned}
$$

where

$$
M(t,T) \triangleq P_C(t,T)\,P_F^{-1}(t,T)\,e^{-\int_0^t s_F(u)du} = \mathbb{E}_{C,t}^{T}\left[e^{-\int_0^T s_F(u)du}\right]
$$

FVA

Note that

$$\mathbb{E}^{T}_{C,t}\left[\frac{M(T,T)}{M(t,T)}\right]=1$$

So,

$$\begin{aligned}
FVA &= F(t,T)-\hat{F}(t,T)\\
&= \mathbb{E}^{T}_{C,t}\left[\left(\frac{M(T,T)}{M(t,T)}-\mathbb{E}^{T}_{C,t}\left[\frac{M(T,T)}{M(t,T)}\right]\right)\left(X(T)-\hat{F}(t,T)\right)\right]\\
&= \frac{1}{M(t,T)}Cov^{T}_{C,t}\left(M(T,T),X(T)\right)
\end{aligned}$$

Recall that $M(T,T)$ depends on the dynamics of $s_F(u)$, $u \geq t$.
Thus, FVA for forward contract depends on joint dynamics of the credit spread $s_F(u)$ and the underlying asset $X(u)$.

Forward Contract with CSA is not Futures

Like forward contract with perfect CSA, futures contract provides immediate debit/credit to the margin account.
Consider a forward contract at time t', which was traded at time t, $t' > t$; $V(X,t) = 0$.

$$
\begin{aligned}
\hat{V}(X,t') &= \mathbb{E}_{t'}\left[e^{-\int_{t'}^{T} r_C(u)du}\left(X(T) - \hat{F}(t,T)\right)\right] \\
&= \mathbb{E}_{t'}\left[e^{-\int_{t'}^{T} r_C(u)du} X(T)\right] - \mathbb{E}_{t'}\left[e^{-\int_{t'}^{T} r_C(u)du}\right]\hat{F}(t,T) \\
&= P_C(t',T)\left(\hat{F}(t',T) - \hat{F}(t,T)\right).
\end{aligned}
$$

This shows that the difference $\hat{V}(X,t') - \hat{V}(X,t) = \hat{V}(X,t')$ is the difference between the forward prices, discounted by the $P_C(t',T)$.
Adjustment of the futures account is not discounted. There is convexity.

Option Pricing Under Perfect CSA and No CSA

For a European call option on $X(t)$ with strike K with perfect or without CSA we have

$$
\begin{aligned}
V(X,t) &= \mathbb{E}_t\left[e^{-\int_t^T r_F(u)du}\left(X(T)-K\right)^+\right] \\
\hat{V}(X,t) &= \mathbb{E}_t\left[e^{-\int_t^T r_C(u)du}\left(X(T)-K\right)^+\right]
\end{aligned}
$$

Or, after the corresponding measure changes,

$$
\begin{aligned}
V(X,t) &= P_F(t,T)\,\mathbb{E}_{F,t}^T\left[\left(X(T)-K\right)^+\right] \\
\hat{V}(X,t) &= P_C(t,T)\,\mathbb{E}_{C,t}^T\left[\left(X(T)-K\right)^+\right]
\end{aligned}
$$

If for forward contract different measure changes result only in different expectation of the underlying $X(t)$, the same measure changes in the case of options result in different distributions (volatility, skew).

Convexity Adjustment and Change in the Distribution

$$
\begin{aligned}
V(X,t) &= \mathbb{E}_t\left[e^{-\int_t^T r_F(u)du}(X(T)-K)^+\right] \\
&= P_F(t,T)\,\mathbb{E}_{C,t}^T\left[\frac{M(T,T)}{M(t,T)}(X(T)-K)^+\right] \\
&\approx P_F(t,T)\,\mathbb{E}_{C,t}^T\left[\alpha(t,T,X)(X(T)-K)^+\right] \\
\alpha(t,T,x) &= \mathbb{E}_{C,t}^T\left[\frac{M(T,T)}{M(t,T)}\middle| X(T)=x\right] \\
&= \alpha_0(t,T)+\alpha_1(t,T)\,x \\
\alpha_0(t,T) &= 1-\alpha_1(t,T)\,\hat{F}(t,T) \\
\alpha_1(t,T) &= \frac{\mathbb{E}_{C,t}^T\left[\frac{M(T,T)}{M(t,T)}X(T)\right]-\hat{F}(t,T)}{\mathbb{V}_{C,t}^T\left[X(T)\right]} = \frac{F(t,T)-\hat{F}(t,T)}{\mathbb{V}_{C,t}^T\left[X(T)\right]}
\end{aligned}
$$

Convexity Adjustment and Change in the Distribution

The relationship between the two measures is then given by

$$\begin{aligned} V(X,t) &= P_F(t,T)\,\mathbb{E}^T_{F,t}\left[(X(T)-K)^+\right] \\ &\approx P_F(t,T)\,\mathbb{E}^T_{C,t}\left[(\alpha_0(t,T)+\alpha_1(t,T)X(T))(X(T)-K)^+\right] \end{aligned}$$

Differentiating with respect to K twice turns it in the relationship between the probability densities:

$$\mathbb{P}^T_{F,t}(X(T)\in dK) = (\alpha_0(t,T)+\alpha_1(t,T)K)\,\mathbb{P}^T_{C,t}(X(T)\in dK)$$

For illustration we plot the distribution density in log-scale corresponding to the commonly used process

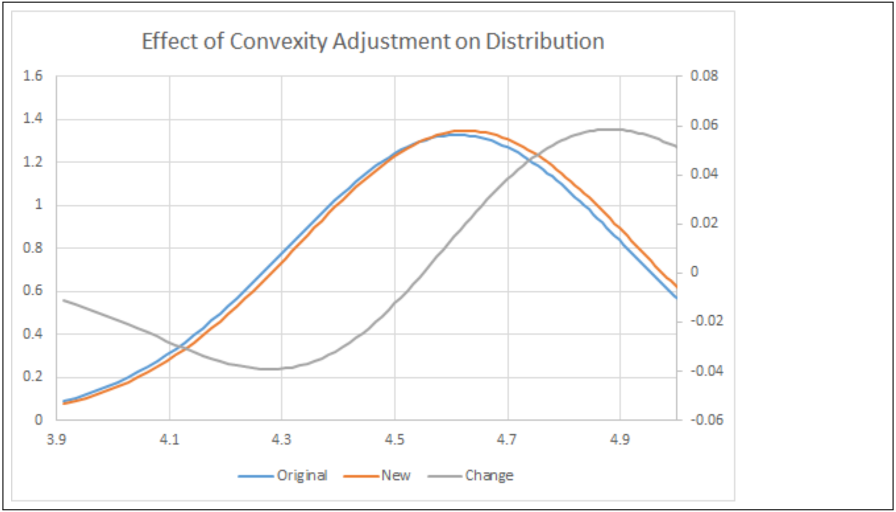
$$dX = \mu X dt + \sigma X dW,$$

from which follows

$$\ln(X(T)) \sim \mathbb{N}\left(\ln\left(X(t) + \left(\mu - \frac{\sigma^2}{2}\right)(T-t)\right), \sigma\sqrt{T-t}\right),$$

where $\mu = 0.05, \sigma = 0.3$, $T = 1, \alpha_0 = 0.09, \alpha_1 = 0.2$.

Convexity Adjustment and Change in the Distribution



Appendix

$$
\begin{aligned}
& V(X,t) \\
= \quad & \mathbb{E}_t\left[e^{-\int\limits_t^T r_F(u)du}(X(T)-K)^+\right] = \mathbb{E}_t\left[e^{-\int\limits_t^T (r_C(u)+s_F(u))du}(Call)\right] \\
= \quad & P_C(t,T)\,\mathbb{E}_{C,t}^T\left[e^{-\int\limits_t^T s_F(u)du}(Call)\right] \\
= \quad & P_F(t,T)\,\mathbb{E}_{C,t}^T\left[\frac{1}{\frac{P_F(t,T)}{P_C(t,T)}}e^{-\int\limits_t^T s_F(u)du}(Call)\right] \\
= \quad & P_F(t,T)\,\mathbb{E}_{C,t}^T\left[\frac{\frac{P_F(T,T)}{P_C(T,T)}e^{-\int\limits_o^T s_F(u)du}}{\frac{P_F(t,T)}{P_C(t,T)}e^{-\int\limits_o^t s_F(u)du}}(Call)\right]
\end{aligned}
$$

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