NO FREE LUNCH (NFL)

$$\Pi_T = \Gamma_T \Rightarrow \Pi_t = \Gamma_t \quad \text{for all } t \leq T$$

for any 2 self-financing strategies Π_t and Γ_t.

Suppose NFL was violated at some time $t \leq T$.

$$\Pi_t < \Gamma_t \quad \text{with} \quad \Pi_T = \Gamma_T$$

Then going long the Π strategy and simultaneously short the Γ strategy at t would generate an immediate profit $\Gamma_t - \Pi_t > 0$, would be self-financing, and have zero value $\Pi_T - \Gamma_T = 0$ at T.

I.e., it has no more cashflows and hence is an ARBITRAGE.

MARTINGALE REPRESENTATION THEOREM (MRT)

W_t - Brownian Motion under measure $\mathbb{Q}$

N_t - $\mathbb{Q}$-Martingale (plus predictable & square integrable)

There must exist a predictable process v_t with $\mathbb{E}^{\mathbb{Q}}\left[\int_0^T v_t^2 \, dt\right] < +\infty$

such that

$$N_t = N_0 + \int_0^t v_s \, dW_s \quad \Big| \quad dN_t = v_t \, dW_t$$

(drift-free Ito integral)

If M_t is another $\mathbb{Q}$-Martingale (plus predictable & square integrable)

then

$$dM_t = y_t \, dW_t \quad \text{hence} \quad dM_t = \underbrace{\frac{y_t}{v_t}}_{=\phi_t} dN_t = \phi_t \, dN_t$$

$\boxed{\text{One Martingale is Ito integral of another!}}$

GIRSANOV'S THEOREM

Brownian motion plus a drift is still Brownian, but under a different measure.

Let θ_t be $\mathcal{F}_t$-adapted and well-behaved

$$\mathbb{E}\left[e^{\frac{1}{2}\int_0^T (\theta_t)^2 dt}\right] < +\infty$$

Expectation taken under measure $\mathbb{P}$.

Novikov Condition

and W_t be Brownian under $\mathbb{P}$.

New drift adjusted process $\tilde{W}_t$

$$d\tilde{W}_t = \theta_t \, dt + dW_t, \qquad 0 \leq t \leq T < +\infty$$

is Brownian too, but under new measure $\mathbb{Q}$.

GIRSANOV'S THEOREM

Recipe for new measure $\mathbb{Q}$ given by

$$\mathbb{Q}(A) = \int_A \xi_T \, d\mathbb{P}, \quad \text{for all } A \in \mathcal{F}$$

where

$$\xi_t = \exp\left\{ -\int_0^t \theta_s \, dW_s - \frac{1}{2}\int_0^t (\theta_s)^2 ds \right\}.$$

Notice ξ_t is a $\mathbb{P}$-martingale:

$$\text{Ito} \Rightarrow d(e^{X_t}) = e^{X_t} dX_t + \frac{1}{2} e^{X_t} (dX_t)^2$$

$$d\xi_t / \xi_t = \left(-\theta_t \, dW_t - \cancel{\frac{1}{2}(\theta_t)^2 dt} \right) + \cancel{\frac{1}{2}\left(\theta_t^2 \underbrace{(dW_t)^2}_{=dt} \right)}$$

$$= -\theta_t \, dW_t$$

Homework: prove $\mathbb{Q}$ is a measure equivalent to $\mathbb{P}$.

HJM

$$f_t^T = f_0^T + \int_0^t \sigma_s^T \, dW_s + \int_0^t \alpha_s^T \, ds$$

Integral form

$$df_t^T = \alpha_t^T \, dt + \sigma_t^T \, dW_t$$

differential form

Expressed in terms of (instantaneous) forward rates.

Volatility σ_t^T and drift α_t^T are quite general processes!!

BOND DYNAMICS

$$P_t^T = e^{-\int_t^T f_t^u\,du} = \exp\left\{-\int_t^T \left(f_0^u + \int_0^t \sigma_s^u\,dW_s + \int_0^t \alpha_s^u\,ds\right)du\right\}$$

$$= \exp\left\{-\int_t^T \int_0^t \sigma_s^u\,dW_s\,du \quad - \int_t^T f_0^u\,du\right.$$

$$\left. - \int_t^T \int_0^t \alpha_s^u\,ds\,du\right\}$$

$$= \exp\left\{-\int_0^t \int_t^T \sigma_s^u\,du\,dW_s - \int_t^T f_0^u\,du - \int_0^t \int_t^T \alpha_s^u\,du\,ds\right\}$$

MONEY MARKET DYNAMICS

$$dB_t = r_t B_t \, dt, \quad B_0 = 1$$

Check $B_t = e^{\int_0^t r_u \, du}$ is the solution.

$$dB_t = d\left(e^{\int_0^t r_u \, du}\right) = \underbrace{e^{\int_0^t r_u \, du}}_{= B_t} \cdot r_t \, dt + \frac{1}{2} e^{\int_0^t r_u \, du} \, (r_t)^2 \underbrace{dt^2}_{=0}$$

$$= r_t B_t \, dt$$

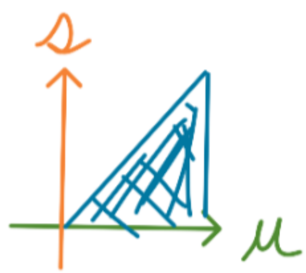
$$B_0 = e^0 = 1$$

$$B_t = e^{\int_0^t r_u \, du} = e^{\int_0^t f_u^u \, du}$$

MONEY MARKET DYNAMICS

$$B_t = e^{\int_0^t f_u^u \, du} = \exp\left\{\int_0^t \left(f_0^u + \int_0^u \sigma_s^u \, dW_s + \int_0^u \alpha_s^u \, ds\right) du\right\}$$

$$= \exp\left\{\int_0^t \int_0^u \sigma_s^u \, dW_s \, du + \int_0^t f_0^u \, du + \int_0^t \int_0^u \alpha_s^u \, ds \, du\right\}$$



$$0 \le s \le u \le t$$

$$= \exp\left\{\int_0^t \int_s^t \sigma_s^u \, du \, dW_s + \int_0^t f_0^u \, du + \int_0^t \int_s^t \alpha_s^u \, du \, ds\right\}$$

MARTINGALES NEEDED TO BUILD REPLICATING STRATEGY

We want to use MRT to represent one bond P^S in terms of another P^T as a recipe for a ~~replicating strategy~~

$$\cancel{dP_t^S = \phi_t \, dP_t^T}$$

This would require P_t^S and P_t^T to both be martingales under some measure. NO HOPE FOR THIS!

How about two assets discounted by the money market account B_t? These would be good candidates. Let's explore $Z_t^T = P_t^T / B_t$.

DISCOUNTED BOND DYNAMICS

$$Z_t^T = P_t^T / B_t$$

$$= \exp\left\{ -\int_0^t \int_t^T \sigma_s^u \, du \, dW_s - \int_t^T f_0^u \, du - \int_0^t \int_t^T \alpha_s^u \, du \, ds \right\}$$

$$\times \exp\left\{ -\int_0^t \int_s^t \sigma_s^u \, du \, dW_s - \int_0^t f_0^u \, du - \int_0^t \int_s^t \alpha_s^u \, du \, ds \right\}$$

$$= \exp\left\{ -\int_0^t \underbrace{\int_s^T \sigma_s^u \, du}_{= -\Sigma_s^T} \, dW_s - \int_0^T f_0^u \, du - \int_0^t \underbrace{\int_s^T \alpha_s^u \, du}_{= A_s^T} \, ds \right\}$$

$$= \exp\left\{ \int_0^t \Sigma_s^T \, dW_s - \int_0^T f_0^u \, du - \int_0^t A_s^T \, ds \right\}$$

DISCOUNTED BOND DYNAMICS

$$Z_t^T = \exp\left\{\int_0^t \Sigma_s^T dW_s - \int_0^T f_0^u du - \int_0^t A_s^T ds\right\}$$

Use Ito's lemma: $d(e^{X_t}) = e^{X_t}\left(dX_t + \frac{1}{2}(dX_t)^2\right)$

$$\begin{aligned}
dZ_t^T / Z_t^T &= \Sigma_t^T dW_t - A_t^T dt + \frac{1}{2}(\Sigma_t^T)^2 \underbrace{(dW_t)^2}_{=dt} \\
&= \Sigma_t^T dW_t + \left(\frac{1}{2}(\Sigma_t^T)^2 - A_t^T\right) dt \\
&= \Sigma_t^T \left(dW_t + \underbrace{\left(\frac{1}{2}\Sigma_t^T - A_t^T / \Sigma_t^T\right)}_{=\gamma_t} dt \right) \\
&= \Sigma_t^T \underbrace{\left(dW_t + \gamma_t dt \right)}_{=d\tilde{W}_t} \\
&= \Sigma_t^T d\tilde{W}_t
\end{aligned}$$

REPLICATING STRATEGY: STEP 1

$dZ_t^T = Z_t^T \Sigma_t^T d\tilde{W}_t$ where $d\tilde{W}_t = dW_t + \gamma_t dt = dW_t + \left(\frac{1}{2}\Sigma_t^T - \frac{A_t^T}{\Sigma_t^T}\right)dt$

W_t is Brownian under measure $\mathbb{P}$

Use Girsanov's Theorem to construct a new measure $\mathbb{Q}$ (equivalent to $\mathbb{P}$) for which $d\tilde{W}_t$ is Brownian (using γ_t as drift θ_t in recipe).

This is the risk-neutral measure.

Then Z_t^T is a $\mathbb{Q}$-martingale.
(no drift term in $dZ_t^T = Z_t^T \Sigma_t^T d\tilde{W}_t$)

REPLICATING STRATEGY: STEP 2

Define

$$V_t = \mathbb{E}^{\mathbb{Q}}\left[B_S^{-1} X \mid \mathcal{F}_t\right]$$

① It is another $\mathbb{Q}$-martingale, by construction.

② $V_S = B_S^{-1} X$, by definition of conditional expectation.

STEP 3 Since both Z_t^T and V_t are $\mathbb{Q}$-martingales, we may apply the Martingale Representation Theorem:

$$dV_t = \phi_t \, dZ_t^T$$

(where MRT guarantees ϕ_t exists) if $\Sigma_t^T > 0$ a.s.

STEP 4 Set $\psi_t = V_t - \phi_t Z_t^T$, to complete the strategy

VERIFY STRATEGY IS REPLICATING

$$\Pi_t = \phi_t P_t^T + \underbrace{\psi_t}_{=V_t - \phi_t Z_t^T} B_t = \phi_t P_t^T + (V_t - \phi_t \underbrace{Z_t^T}_{=B_t^{-1} P_t^T}) B_t = \cancel{\phi_t P_t^T} + V_t B_t - \cancel{\phi_t \cancel{B_t^{-1}} P_t^T \cancel{B_t}}$$

$$= V_t B_t$$

In particular, $\Pi_S = \underbrace{V_S}_{=B_S^{-1} X} B_S = \cancel{B_S^{-1}} X \cancel{B_S} = X$

This proves it replicates the time S payoff $X \in \mathcal{F}_S$.

Is it also self-financing?

VERIFY STRATEGY IS SELF-FINANCING

Apply Ito's lemma:

$$d\Pi_t = d(V_t B_t) = \underbrace{dV_t}_{=\phi_t dZ_t^T} \cdot B_t + V_t \cdot dB_t + \frac{1}{2}\left(\cancelto{0}{0 \cdot (dV_t)^2} + \cancelto{0}{0 \cdot (dB_t)^2}\right) + \cancelto{0}{1 \cdot dV_t\, dB_t}$$

0 (since $dB_t = r_t B_t\, dt$ has no "dW_t" term)

$$= B_t \phi_t\, dZ_t^T + V_t\, dB_t$$

$$\boxed{\begin{aligned} d(B_t Z_t^T) &= B_t\, dZ_t^T + Z_t^T\, dB_t \\ \Rightarrow B_t\, dZ_t^T &= d(B_t Z_t^T) - Z_t^T\, dB_t \end{aligned}}$$

$$= \phi_t \left(d(B_t \underbrace{Z_t^T}_{=B_t^{-1} P_t^T}) - Z_t^T\, dB_t \right) + V_t\, dB_t$$

$$= \phi_t\, d(\cancel{B_t B_t^{-1}} P_t^T) + \underbrace{(V_t - \phi_t Z_t^T)}_{=\psi_t} dB_t$$

$$= \phi_t\, dP_t^T + \psi_t\, dB_t$$

RISK-NEUTRAL PRICING FORMULA

Π_t is a self-financing strategy replicating the payoff $X \in \mathcal{F}_S$ $(\Pi_S = X)$

In the absence of arbitrage, via NFL,

all self-financing strategies replicating X must share the same time t value

$$\Pi_t = V_t B_t = \mathbb{E}^{\mathbb{Q}}\left[B_S^{-1} X \,\middle|\, \mathcal{F}_t\right] \cdot B_t$$

which we denote

$$\pi_t(X) = B_t \, \mathbb{E}^{\mathbb{Q}}\left[B_S^{-1} X \,\middle|\, \mathcal{F}_t\right]$$

REPLICATING ONE BOND WITH ANOTHER

We had set out originally to replicate the P^S bond with a strategy holding the P^T bond and money market account B. Instead we replicated the more general payoff $X \in \mathcal{F}_S$.

Let's see what happens with the P^S bond, setting $X = P^S_S = 1$.

$$P^S_t = B_t \mathbb{E}^{\mathbb{Q}}\left[B_S^{-1} P^S_S \mid \mathcal{F}_t \right]$$

Divide by B_t

$$\underbrace{B_t^{-1} P^S_t}_{= Z^S_t} = \mathbb{E}^{\mathbb{Q}}\Big[\underbrace{B_S^{-1} P^S_S}_{= Z^S_S} \mid \mathcal{F}_t \Big]$$

$\boxed{\text{Both } Z^S_t \text{ and } Z^T_t \text{ are } \mathbb{Q}\text{-martingales!}}$

Z_t^S IS $\mathbb{Q}$-MARTINGALE

$$Z_t^S = \mathbb{E}^{\mathbb{Q}}\left[Z_S^S \mid \mathcal{F}_t \right]$$

The SDE for Z_t^S we already derived

$$dZ_t^S = Z_t^S \Sigma_t^S \left(dW_t + \left(\frac{1}{2}\Sigma_t^S - \frac{A_t^S}{\Sigma_t^S} \right) dt \right)$$

$$= d\tilde{W}_t - \gamma_t \, dt$$

$$= Z_t^S \Sigma_t^S \left(d\tilde{W}_t + \left\{ \left(\frac{1}{2}\Sigma_t^S - \frac{A_t^S}{\Sigma_t^S} \right) - \gamma_t \right\} dt \right)$$

But Z_t^S is a $\mathbb{Q}$-martingale and therefore it must have no drift.

Thus $\gamma_t = \frac{1}{2}\Sigma_t^T - \frac{A_t^T}{\Sigma_t^T} = \frac{1}{2}\Sigma_t^S - \frac{A_t^S}{\Sigma_t^S}$ for all S!

DRIFT CONSTRAINT

$$\gamma_t = \frac{1}{2}\Sigma_t^T + A_t^T/\Sigma_t^T$$ must be independent of T

Rearrange in terms of A_t^T

$$A_t^T = \frac{1}{2}(\Sigma_t^T)^2 - \Sigma_t^T \gamma_t$$

differentiate with respect to T

$$\alpha_t^T = -\Sigma_t^T \sigma_t^T + \sigma_t^T \gamma_t = \sigma_t^T(\gamma_t - \Sigma_t^T)$$

This restricts the HJM drift α_t^T!

MARKET "PRICE" OF RISK

$$dP_t^T / P_t^T = r_t\,dt + \Sigma_t^T d\tilde{W}_t \quad = (r_t + \gamma_t \Sigma_t^T)dt + \Sigma_t^T dW_t$$

$= dW_t + \gamma_t\,dt$

Therefore γ_t is the excess drift (under $\mathbb{P}$) per unit of risk (Σ_t^T).

Drift restriction

$$\alpha_t^T = \sigma_t^T \left(\gamma_t - \Sigma_t^T \right)$$

is equivalent to restricting all assets to have the same excess drift per unit of risk taken.