CREDIT VALUE ADJUSTMENT

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ABSTRACT. The topic of counterparty risk is ever important in light of recent events. The Heath-Jarrow-Morton (HJM) family of models applies in a setting that is essentially without default risk. This is not generally a realistic assumption. The following article extends the HJM framework to derivatives exposed to counterparty default risk and connects it with reduced form credit models (CDS pricing). This supplies a practical means to apply standard risk management methods to pricing derivatives with counterparty default exposure.

1. COLLATERAL, COUNTERPARTY RISK AND FUNDING

Most derivatives dealers have instituted agreements to collateralize mutual exposures. These agreements are based on the credit support annex (CSA) to the International Swaps and Derivatives Association (ISDA) master agreement. When implemented, this reduces inter-dealer counterparty exposure to an essentially default-free setting. However, the market contains an ecosystem of trades ranging from fully collateralized to completely unsecured. Standard HJM pricing theory applies only to those trades on the fully collateralized end of the spectrum.

For all that follows, assume the setup, notation and results as found in the Heath-Jarrow-Morton Framework (HJM) lecture notes.

1.1. **Fully Collateralized Contingent Claims.**

The value of a contingent claim with the $\mathcal{F}_S$-measurable payoff X satisfies the pricing formula[1]

$$\pi_t(X) = B_t \mathbb{E}\left[B_S^{-1} X \,|\mathcal{F}_t\right]$$

under the risk-neutral measure $\mathbb{Q}$, only in the absence of default risk.

Definition 1. We will refer to the standard HJM value $\pi_t(X)$ as the *fully collateralized* or *CSA value*.

[1] Expectation is taken with respect to the risk-neutral measure $\mathbb{Q}$. We drop the superscript notation giving explicit reference to this measure since its existence has already been established.

1.2. **Risky Contingent Claims.**

A contingent claim without collateral backing is exposed to counterparty default risk.

Definition 2. We will refer to its (yet unknown) value as the *risky* or *unsecured value* and adorn it with a tilde to distinguish it from the corresponding fully collateralized claim, e.g.

$$\widetilde{\pi}_t(X)$$

for the risky claim with payoff X.

Definition 3. The *credit value adjustment (CVA)* is the reduction to the CSA value needed to obtain the risky value, e.g.

$$\pi_t(X) - \widetilde{\pi}_t(X).$$

Note that due to the (possibly) bilateral nature of the default risk, the CVA may be either positive or negative.

1.3. **The Unsecured Funding Rate.**

The risk-free rate r_t only applies when funding fully collateralized borrowing, e.g. repo agreements.

Definition 4. We define the *unsecured funding rate* $\widetilde{r}_t$ as the short rate available for inter-counterparty unsecured funding.

The unsecured rate is determined by the two specific counterparties involved in the lending agreement. As we shall see later, this does not imply that the borrowing and lending rates will be the same.

Definition 5. The associated *risky money market account* is denoted by $\widetilde{B}_t$ and satisfies the SDE

$$d\widetilde{B}_t = \widetilde{r}_t \widetilde{B}_t dt, \qquad \widetilde{B}_0 = 1.$$

2. RISKY REPLICATING STRATEGIES

Here we shadow the steps as in the HJM notes to build a strategy which replicates the risky claim with $\mathcal{F}_S$-measurable payoff X.

(1) Define the martingale $M_t = \mathbb{E}\left[\widetilde{B}_S^{-1} X \,|\mathcal{F}_t\right]$.
(2) Fix $T \geq S$ and use the Martingale Representation Theorem to find $dM_t = \eta_t dZ_t^T$. (Recall $Z_t^T = B_t^{-1} P_t^T$.)
(3) Define the processes

$$\phi_t = B_t^{-1}\widetilde{B}_t \eta_t, \qquad \psi_t = -\phi_t Z_t^T, \quad \text{and} \quad \widetilde{\psi}_t = M_t$$

and the corresponding strategy

$$\widetilde{\Pi}_t = \phi_t P_t^T + \psi_t B_t + \widetilde{\psi}_t \widetilde{B}_t.$$

(4) Show that it replicates the risky claim.

We now demonstrate that this successfully replicates the risky claim and hence gives a recipe for its arbitrage-free value $\widetilde{\pi}_t(X)$.

2.1. **Verification of the Replicating Strategy.**

2.1.1. *Does it Replicate the Payoff?*

$$\begin{aligned}
\widetilde{\Pi}_t &= \phi_t P_t^T + \psi_t B_t + \widetilde{\psi}_t \widetilde{B}_t \\
&= \phi_t P_t^T - \phi_t Z_t^T B_t + M_t \widetilde{B}_t \\
&= \phi_t P_t^T - \phi_t B_t^{-1} P_t^T B_t + M_t \widetilde{B}_t \\
&= M_t \widetilde{B}_t
\end{aligned}$$

$$\begin{aligned}
\widetilde{\Pi}_S &= M_S \widetilde{B}_S \\
&= \widetilde{B}_S^{-1} X \widetilde{B}_S \\
&= X
\end{aligned}$$

2.1.2. *Is it Self-Financing?*

$$
\begin{aligned}
d\widetilde{\Pi}_t &= d\left(M_t \widetilde{B}_t\right) \\
&= \widetilde{B}_t dM_t + M_t d\widetilde{B}_t \\
&= \widetilde{B}_t \eta_t dZ_t^T + M_t d\widetilde{B}_t \\
&= B_t \phi_t dZ_t^T + M_t d\widetilde{B}_t \\
&= \phi_t \left(d\left(B_t Z_t^T\right) - Z_t^T dB_t\right) + M_t d\widetilde{B}_t \\
&= \phi_t dP_t^T - \phi_t Z_t^T dB_t + M_t d\widetilde{B}_t \\
&= \phi_t dP_t^T + \psi_t dB_t + \widetilde{\psi}_t d\widetilde{B}_t
\end{aligned}
$$

2.1.3. *Does it Replicate the Funding and Default Exposure?* It is not enough to check that the strategy simply replicates the contingent claim payoff. It must furthermore reproduce the bilateral counterparty default exposures. This distinguishes it from the strategy which replicates the CSA value.

$$
\begin{aligned}
\widetilde{\Pi}_t &= \phi_t P_t^T + \psi_t B_t + \widetilde{\psi}_t \widetilde{B}_t \\
&= \phi_t P_t^T - \phi_t P_t^T + \widetilde{\Pi}_t \\
d\widetilde{\Pi}_t &= \phi_t dP_t^T + \psi_t dB_t + \widetilde{\psi}_t d\widetilde{B}_t \\
&= \phi_t dP_t^T + r_t \psi_t B_t dt + \widetilde{r}_t \widetilde{\psi}_t \widetilde{B}_t dt \\
&= \phi_t dP_t^T - r_t \phi_t P_t^T dt + \widetilde{r}_t \widetilde{\Pi}_t dt
\end{aligned}
$$

Looking at the above equations:

- The bonds (first term) secure a portion of the financing (second term), with the counterparties financing the total strategy value as unsecured funding (third term).
- The secured lender seizes the bond collateral at default, if necessary, avoiding loss.
- The counterparty to the defaulting party losses the value of the strategy less recovery, if it is a positive amount.

2.2. **Risky Risk-Neutral Pricing Formula.** By the No Free Lunch condition we can now write the pricing formula for the risky contingent claim[2]

$$
\widetilde{\pi}_t(X) = \widetilde{\Pi}_t = M_t \widetilde{B}_t = \widetilde{B}_t \mathbb{E}\left[\widetilde{B}_S^{-1} X \,|\mathcal{F}_t\right]
$$

or simply

$$
\widetilde{\pi}_t(X) = \mathbb{E}\left[e^{-\int_t^S \widetilde{r}_u du} X \,|\mathcal{F}_t\right].
$$

As with the CSA value, the expectation is taken under the risk-neutral measure. The change is that discounting is done using the unsecured funding rate in place of the risk-free rate.

From the prior subsection we see the growth rate

$$
\mathbb{E}\left[d\widetilde{\Pi}_t \,|\mathcal{F}_t\right] = \widetilde{r}_t \widetilde{\Pi}_t dt
$$

equals the unsecured funding rate. Thus we may instead choose the risky bond $\widetilde{P}^S$ as numeraire

$$
\widetilde{\pi}_t(X) = \widetilde{P}_t^S \widetilde{\mathbb{E}}^S \left[X \,|\mathcal{F}_t\right].
$$

[2] Strictly speaking, the replicating strategy and associated price $\widetilde{\pi}_t(X)$ represent the "pre-default" value of the contingent claim, i.e. the value conditioned on neither counterparty having yet defaulted. Cf. [1] for technical conditions and rigorous proofs relating $\widetilde{\pi}_t(X)$ to the contingent claim value.

3. Risky Forward Contract

Let $\widetilde{F}_t^S(X)$ denote the unsecured S-forward price for the $\mathcal{F}_S$-measureable payoff X.

The forward contract is cost-free, therefore

$$
\mathbb{E}\left[e^{-\int_t^S \widetilde{r}_u du}\left(X - \widetilde{F}_t^S(X)\right) |\mathcal{F}_t\right] = 0
$$

and

$$
\begin{aligned}
\widetilde{F}_t^S(X) &= \mathbb{E}\left[e^{-\int_t^S \widetilde{r}_u du} X \,|\mathcal{F}_t\right] \Big/ \mathbb{E}\left[e^{-\int_t^S \widetilde{r}_u du} \,|\mathcal{F}_t\right] \\
&= \mathbb{E}\left[e^{-\int_t^S \widetilde{r}_u du} X \,|\mathcal{F}_t\right] \Big/ \widetilde{P}_t^S \\
&= \widetilde{\mathbb{E}}^S \left[X \,|\mathcal{F}_t\right].
\end{aligned}
$$

The risky forward price $\widetilde{F}_t^S(X)$ is a $\widetilde{P}^S$-martingale.

3.1. **Forward CVA.** First express the risky forward price using P^S as numeraire

$$
\begin{aligned}
\widetilde{F}_t^S(X) &= \mathbb{E}\left[e^{-\int_t^S \widetilde{r}_u du} X \,|\mathcal{F}_t\right] \Big/ \widetilde{P}_t^S \\
&= \mathbb{E}\left[e^{-\int_t^S r_u du} e^{-\int_t^S (\widetilde{r}_u - r_u) du} X \,|\mathcal{F}_t\right] \Big/ \widetilde{P}_t^S \\
&= \frac{P_t^S}{\widetilde{P}_t^S} \mathbb{E}^S \left[e^{-\int_t^S (\widetilde{r}_u - r_u) du} X \,|\mathcal{F}_t\right] \\
&= \mathbb{E}^S \left[\frac{\Gamma_S^S}{\Gamma_t^S} X \,|\mathcal{F}_t\right]
\end{aligned}
$$

where

$$
\Gamma_t^S = \frac{\widetilde{P}_t^S}{P_t^S} e^{-\int_0^t (\widetilde{r}_u - r_u) du} = \mathbb{E}^S \left[e^{-\int_0^S (\widetilde{r}_u - r_u) du} \,|\mathcal{F}_t\right]
$$

is a P^S-martingale with

$$
\mathbb{E}^S \left[\Gamma_S^S / \Gamma_t^S \,|\mathcal{F}_t\right] = 1.
$$

Then the forward CVA is simply

$$
\begin{aligned}
& F_t^S(X) - \widetilde{F}_t^S(X) \\
&= \mathbb{E}^S \left[\left(1 - \Gamma_S^S / \Gamma_t^S\right) X \,|\mathcal{F}_t\right] \\
&= \mathbb{E}^S \left[\left(\mathbb{E}^S \left[\Gamma_S^S / \Gamma_t^S \,|\mathcal{F}_t\right] - \Gamma_S^S / \Gamma_t^S\right)\left(X - F_t^S(X)\right) |\mathcal{F}_t\right] \\
&= -\frac{P_t^S}{\widetilde{P}_t^S} \mathrm{Cov}^S \left[e^{-\int_t^S (\widetilde{r}_u - r_u) du}, X \,|\mathcal{F}_t\right].
\end{aligned}
$$

This should be familiar to you as a type of convexity adjustment.

To obtain the actual value of the adjustment requires specifying the joint dynamics of the funding spread

$$
\widetilde{r}_u - r_u
$$

and the forward price

$$
F_u^S(X).
$$

3.2. **Futures Convexity Adjustment Versus Forward CVA.** Let $\mathfrak{F}_t^S(X)$ denote the futures price with $\mathcal{F}_S$-measurable final settlement X. The futures is marked-to-market much like collateral is exchanged for the CSA forward contract. However, there is an important difference: posted collateral earns interest.

The futures is a martingale under the risk-neutral measure. Therefore the futures convexity adjustment is

$$\begin{aligned}\mathfrak{F}_t^S(X) - \widetilde{F}_t^S(X) &= \mathbb{E}\left[X \,|\mathcal{F}_t\right] - \mathbb{E}\left[e^{-\int_t^S \widetilde{r}_u du} X \,|\mathcal{F}_t\right] \Big/ \widetilde{P}_t^S \\ &= \mathbb{E}\left[\left(1 - e^{-\int_t^S \widetilde{r}_u du}/\widetilde{P}_t^S\right) X \,|\mathcal{F}_t\right] \\ &= \mathbb{E}\left[\left(1 - e^{-\int_t^S \widetilde{r}_u du}/\widetilde{P}_t^S\right)\left(X - \mathfrak{F}_t^S(X)\right) |\mathcal{F}_t\right] \\ &= -\frac{1}{\widetilde{P}_t^S} \mathrm{Cov}\left[e^{-\int_t^S \widetilde{r}_u du}, X \,|\mathcal{F}_t\right],\end{aligned}$$

which depends on the joint dynamics of the unsecured funding rate $\widetilde{r}_u$ (instead of the spread) and the futures price $\mathfrak{F}_u^S(X)$.

4. Portfolio Based CVA

Consider a portfolio of trades between two counterparties which we denote, respectively, as party A and party B. Assume there is no CSA agreement between them.

Let V_t denote the value of the portfolio from the perspective of party A. The process X_t represents the cumulative cash flow stream from party B to party A. It follows that

$$V_t = \mathbb{E}\left[\int_t^T e^{-\int_t^S \widetilde{r}_u du} dX_S \,|\mathcal{F}_t\right]. \tag{4.1}$$

For example, if $\{\theta_{t_i}\}_{i=1}^n$ represents the sequence of payments for a swap traded between parties A and B, netting the fixed and floating payments, then

$$X_t = \sum_{t_i \le t} \theta_{t_i}.$$

4.1. Mechanics of the Unsecured Funding Rate.

At any point in time, either party A is borrowing and B is lending or vice versa. Assuming parties A and B are not of matching credit quality, the unsecured funding rate $\widetilde{r}_u$ will depend on which counterparty is doing the borrowing:

$$\widetilde{r}_u = r_u + s_u^A 1_{\{V_u<0\}} + s_u^B 1_{\{V_u \ge 0\}} \tag{4.2}$$

where s_u^A and s_u^B are the unsecured funding spreads for which parties A and B are able to borrow from parties B and A, respectively.

4.2. Connection with Reduced Form Credit Models (CDS Pricing).

Let the stopping times τ^A and τ^B denote the time of default for parties A and B, respectively. The stopping time $\tau = \tau^A \wedge \tau^B$ is the time of first default between parties A and B. The default intensity is given by the expression

$$\lambda_t = \mathbb{Q}\left[\tau \le t + dt \,|\tau > t\right].$$

This implies the survival probability up to time t is

$$\mathbb{Q}\left[\tau > t\right] = \mathbb{E}\left[e^{-\int_0^t \lambda_u du}\right]$$

and the default probability density equals

$$\begin{aligned}\mathbb{Q}\left[t < \tau \le t + dt\right] &= \mathbb{Q}\left[\tau \le t + dt \,|\tau > t\right] \mathbb{Q}\left[\tau > t\right] \\ &= \lambda_t \mathbb{E}\left[e^{-\int_0^t \lambda_u du}\right].\end{aligned}$$

We use δ_t to denote the portfolio recovery fraction given $\tau = t$.

Here we attempt to relate the portfolio value V_t to pricing under a reduced form credit model.

We seek to show

$$\begin{aligned}V_t = &\ \mathbb{E}\left[\int_t^T e^{-\int_t^S r_u du} e^{-\int_t^S \lambda_u du} dX_S \,|\mathcal{F}_t\right] \\ &+ \mathbb{E}\left[\int_t^T e^{-\int_t^S r_u du} \delta_S V_S \lambda_S e^{-\int_t^S \lambda_u du} dS \,|\mathcal{F}_t\right]\end{aligned} \tag{4.3}$$

where the first term's integrand represents each discounted risk-free portfolio cash flow times the probability it is received, and the second term is the expected recovery.

We anticipate the triangle relationship

$$\widetilde{r}_u - r_u = (1 - \delta_u) \lambda_u,$$

i.e. the funding spread matches the expected rate of loss (gain) due to counterparty default.

We seek to demonstrate that the portfolio value from formula (4.1) is the solution to the reduced form credit model specification in (4.3).

First substitute the portfolio value from (4.1) into the second term on the right hand side of (4.3).

$$\begin{aligned}&\mathbb{E}\left[\int_t^T e^{-\int_t^S r_u du} \delta_S V_S \lambda_S e^{-\int_t^S \lambda_u du} dS \,|\mathcal{F}_t\right] \\ &= \mathbb{E}\left[\int_t^T e^{-\int_t^S r_u du} \delta_S \left(\int_S^T e^{-\int_S^U \widetilde{r}_u du} dX_U\right) \lambda_S e^{-\int_t^S \lambda_u du} dS \,|\mathcal{F}_t\right] \\ &= \mathbb{E}\left[\int_t^T e^{-\int_t^U r_u du} e^{-\int_t^U \lambda_u du} \int_t^U \delta_S \lambda_S e^{\int_S^U \delta_u \lambda_u du} dS dX_U \,|\mathcal{F}_t\right] \\ &= \mathbb{E}\left[\int_t^T e^{-\int_t^U r_u du} e^{-\int_t^U \lambda_u du} \left(e^{\int_t^U \delta_u \lambda_u du} - 1\right) dX_U \,|\mathcal{F}_t\right]\end{aligned}$$

Show that this solves the reduced form credit model specification.

$$\begin{aligned}V_t &= \mathbb{E}\left[\int_t^T e^{-\int_t^S r_u du} e^{-\int_t^S \lambda_u du} dX_S \,|\mathcal{F}_t\right] \\ &\quad + \mathbb{E}\left[\int_t^T e^{-\int_t^U r_u du} e^{-\int_t^U \lambda_u du} \left(e^{\int_t^U \delta_u \lambda_u du} - 1\right) dX_U \,|\mathcal{F}_t\right] \\ &= \mathbb{E}\left[\int_t^T e^{-\int_t^S (r_u + (1-\delta_u)\lambda_u) du} dX_S \,|\mathcal{F}_t\right] \\ &= \mathbb{E}\left[\int_t^T e^{-\int_t^S \widetilde{r}_u du} dX_S \,|\mathcal{F}_t\right]\end{aligned}$$

4.3. Connection to Pricing American Options.

Notice the recursive nature of the credit model integral equation (4.3). The portfolio value V_t appears on both sides of the equation. This is similar to an American option price recursion relation

$$V_t = \mathbb{E}\left[e^{-\int_t^T r_u du} V_T \,|\mathcal{F}_t\right]^+$$

with the analogue to the maximization expressed through the recovery fraction

$$\delta_S(\omega) = \begin{cases} 1, & \text{if } \ V_S(\omega) \le 0,\ \tau(\omega) = \tau^B(\omega) \\ & \text{or } \ V_S(\omega) \ge 0,\ \tau(\omega) = \tau^A(\omega); \\ \delta_S^A(\omega), & \text{if } \ V_S(\omega) \le 0,\ \tau(\omega) = \tau^A(\omega); \\ \delta_S^B(\omega), & \text{if } \ V_S(\omega) \ge 0,\ \tau(\omega) = \tau^B(\omega). \end{cases}$$

The same can be said of the risk-neutral pricing formula (4.1) with the recursion and dependence on the sign of V expressed through the funding rate specification (4.2).

5. Cautions on Hedging and Arbitrage-Free Pricing

In practice, an unsecured bilateral lending market based on inter-counterparty netted payments does not exist. Counterparties may engage in fully collateralized trading based on the CSA, completely unsecured trading, or something in between. There are no opportunities to hedge between these alternatives.

Thus the risky replicating strategy is a theoretical construct. The connection with reduced form credit modeling should be emphasized. CDS, corporate bond, and bank loan markets provide worthwhile information, though they only incorporate unilateral default exposure. The risk department still needs to assess credit quality and supply counterparty risk estimates.

Counterparties do not typically share information with full transparency. Therefore they will not always agree on funding levels and hence CVA amounts.

6. Conclusion

We introduced the concept of an inter-counterparty unsecured funding rate and incorporated it into replicating strategies. This allowed us to extend the HJM framework to incorporate counterparty default exposure, i.e. CVA estimation. Next we established the connection between funding spreads, default intensity, and recovery assumptions. This linked CVA estimation to reduced form credit modeling as a risk management framework.

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