Forward Measure and Change of Numeraire

Fixed Income Derivatives

Lida Doloc

MSFM, University of Chicago

May 2016

Copyright and Disclaimer

Introduction

We will explore the Martingale Method in the HJM framework, the second method developed by modern finance for pricing derivatives.

Setup. We asume a probability space $(\Omega, \mathcal{F})$ equipped with a *risk-neutral* measure $\mathbb{Q}$.

• We also assume the existence of a Brownian motion $\{W_t\}_{t\geq 0}$ which is adapted to the filtration $\{\mathcal{F}_t \subset \mathcal{F} | t \in [0, +\infty)\}$.

• Bond prices are denoted by $P(t, T)$ and evolve according to HJM:

$$dP(t, T) = r(t) P(t, T) dt + P(t, T) \Sigma(t, T) dW_t$$

• The risk free money market account is denoted B_t and is governed by:

$$dB_t = r(t) B_t dt.$$

• Under the risk-neutral measure $\mathbb{Q}$, a security which has payoff X at time T has time-t value:

$$\pi_t(X) = B_t \mathbb{E}\left[B_T^{-1} X | \mathcal{F}_t\right].$$

Change of Measure in the HJM Framework

- It is important to separate process and measure: W_t is not strictly a Brownian motion *per se*, but a Brownian motion with respect to the measure $\mathbb{Q}$.
- The HJM stochastic differential formulation describes the behavior of the bond price process P with respect to the measure $\mathbb{Q}$ that makes W_t a Brownian motion.
- But how do W_t and P change as the measure changes?
- It happens that Brownian motions change in easy and pleasant ways under changes of measure (Girsanov's Theorem).

Motivational Example

- Consider a call option on a discount bond that expires at time $T > 0$ and has strike price K. The underlying bond matures at time $M > T$. The option payoff at $t = T$ is:

$$X = (P(T, M) - K)_+$$

- Under the **risk-neutral** measure, the option premium at $t = 0$ is:

$$\pi_0(C) = \mathbb{E}\left[B_T^{-1} X\right] = \mathbb{E}\left[B_T^{-1}(P(T, M) - K)_+\right]$$

- To compute this expectation, we would need to know the joint distribution of two random variables: B_T^{-1} and $P(T, M)$. This would lead to a two-dimensional integration with respect to the joint density function.
- Recall that when using Black's model we determined the premium as:

$$\pi_0(C) = P(0, T)\,\mathbb{E}^T\left[(P(T, M) - K)_+\right].$$

- Under the T-forward measure we have only a **one**-dimensional integration against a *one*-dimensional density function for $P(T, M)$.

Change of Measure in the HJM Framework

- In the HJM lecture we have already seen a **change of measure** from the real-world measure $\mathbb{P}$ to the risk-neutral measure $\mathbb{Q}$.
- In this lecture we will be concerned with how to actually perform a change of measure from the risk-neutral measure $\mathbb{Q}$ to the T-forward measure $\mathbb{Q}^T$. For this we will use Girsanov's theorem again.

What Does "Change of Measure" Mean?

- By "measure", we mean probability measure.
- By "change of measure" we mean a change of the probability density function.

• When we talk about a certain probability measure we always have in mind a **shape** and a **location** for the density of the random variable.
• It follows from this that we can subject a probability distribution to two types of transformations:

1. We can leave the shape of the distribution the same, but move the density to a different location (different mean).
2. We can also change the shape of the distribution.

• The Martingale Method for pricing derivative assets uses a novel way of transforming the probability measure $d\mathbb{P}$ so that the **mean** of a random process changes, while the shape is preserved.
• The transformation permits treating an asset that carries a positive "risk premium" as if it were risk-free.

Why Do We Change Probability Measures?

- We change the probability measure to justify prices!
- We also change the probability measure to make it easier to calculate an expectation.

- **Note**. These new probabilities do not relate to the "true" odds of the experiment. The "true" probabilities are still given by the original measure.
- **Girsanov's theorem** provides the general framework for transforming one probability measure into another "equivalent" measure in the case of random processes.

How Can We Change Probability Measures?

- Either we change the values assumed by a random process z_t.
- Or we leave the values assumed by z_t unchanged, but instead change the probabilities associated with z_t.
- The first method cannot be used in asset pricing.
- The second method is a very useful tool in asset pricing because:
  - The risk premiums of asset prices can be "eliminated"
  - The volatility structure remains intact.
- The option prices, e.g., do not depend on the mean growth of the underlying asset price, but they depend on the **volatility** in a fundamental way. Therefore, transforming original probability distributions while preserving the shape (variance) would be very convenient.

Measure Change - A Simple 1D Example

- Let X be a random variable with probability density function $f(x)$. Let $\phi(x)$ be a real-valued function. Then the expected value of $\phi(X)$ is:

$$\mathbb{E}_f[\phi(X)] = \int \phi(x) f(x) dx.$$

- Let $g(x)$ be another function such that it is always positive and integrates to 1, i.e.,

$$g(x) > 0, \forall x \in \mathbb{R} \quad and \quad \int g(x) dx = 1.$$

- We can do an apparently trivial transformation:

$$\begin{aligned}\mathbb{E}_f[\phi(X)] &= \int \phi(x) f(x) dx \\ &= \int \phi(x) f(x) \left(\frac{g(x)}{g(x)}\right) dx \\ &= \int \left(\phi(x) \frac{f(x)}{g(x)}\right) g(x) dx.\end{aligned}$$

- Define

$$\psi(x) = \phi(x) \frac{f(x)}{g(x)}.$$

- Then

$$\begin{aligned} \mathbb{E}_f[\phi(X)] &= \int \psi(x) g(x)\, dx \\ &= \mathbb{E}_g[\psi(X)] \\ &= \mathbb{E}_g\left[\phi(X) \frac{f(X)}{g(X)}\right] \end{aligned}$$

- If we read the equation above from right to left, we see that we can simplify the expression under the expectation operator by an appropriate measure change.
- The term $\frac{f(x)}{g(x)}$ represents the density of the measure $df(\cdot)$ with respect to the measure $dg(\cdot)$ and is called the *Radon-Nikodym derivative*.

Girsanov's Theorem

Let W_t, $0 \leq t \leq T$ *be a Brownian motion on the probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *with* $\mathcal{F}_t$, $0 \leq t \leq T$ *the accompanying filtration. Let* θ_t, $0 \leq t \leq T$ *be an* $\mathcal{F}_t$*-adapted process satisfying the Novikov condition*

$$\mathbb{E}^{\mathbb{P}}\left[\exp\left\{\frac{1}{2}\int_0^T (\theta_t)^2\, dt\right\}\right] < +\infty.$$

Then the process

$$\widetilde{W}_t = W_t + \int_0^t \theta_s ds, \quad 0 \leq t \leq T$$

is a Brownian motion under the new probability measure

$$\mathbb{Q}(A) = \mathbb{E}^{\mathbb{P}}\left[1_A \xi_T\right], \quad \forall A \in \mathcal{F}$$

where

$$\xi_t = \exp\left\{-\int_0^t \theta_s dW_s - \frac{1}{2}\int_0^t (\theta_s)^2\, ds\right\}.$$

The process ξ_t *is a* $\mathbb{P}$*-martingale and the measure* $\mathbb{Q}$ *is equivalent to* $\mathbb{P}$.

- $$\mathbb{Q}(A) = \mathbb{E}^{\mathbb{P}}\left[1_A \xi_T\right] = \int_\Omega 1_A \xi_T d\mathbb{P} = \int_A \xi_T d\mathbb{P}$$
- $$d\mathbb{Q} = \xi_T d\mathbb{P}$$
- $$\xi_t = \exp\left\{-\int_0^t \theta_s dW_s - \frac{1}{2}\int_0^t (\theta_s)^2 ds\right\}$$
- Consider the simple case when θ_s is constant: $\boxed{\theta_s = \mu}$
- Then, since $W_0 = 0$:

$$\begin{aligned}\xi_t &= \exp\left\{-\mu W_t - \frac{1}{2}\mu^2 t\right\} \\ &= e^{-\mu W_t - \frac{1}{2}\mu^2 t}.\end{aligned}$$

- Also:

$$\widetilde{W}_t = W_t + \int_0^t \theta_s ds = W_t + \mu t$$

Girsanov's Theorem Demystified

W_t is a Brownian motion under $\mathbb{P}$, i.e.:

$$W_t \sim N(0, t)$$

- Denote the density function by $f(W_t)$, the probability measure by $d\mathbb{P}$:
$$d\mathbb{P}(W_t) = f(W_t)\, dz_t = \frac{1}{\sqrt{2\pi t}} e^{-\frac{1}{2t}W_t^2} dW_t.$$
- Multiply $d\mathbb{P}$ by $\xi_t = e^{-\mu W_t - \frac{1}{2}\mu^2 t}$ from above:
$$\xi_t d\mathbb{P}(W_t) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{1}{2t}W_t^2 - \mu W_t - \frac{1}{2}\mu^2 t} dW_t$$
- $$\xi_t d\mathbb{P}(W_t) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{1}{2t}\left(W_t^2 + 2\mu t W_t + \mu^2 t^2\right)} dW_t$$
- $$\xi_t d\mathbb{P}(W_t) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{1}{2t}(W_t + \mu t)^2} dW_t$$

- Denote

$$dQ\left(W_t\right) = \xi_t d\mathbb{P}\left(W_t\right) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{1}{2t}\left(W_t+\mu t\right)^2} dW_t$$

- $d\mathbb{Q}$ is a new probability also associated with a normally distributed random variable, but with a different mean.
- Denote

$$\widetilde{W}_t = W_t + \mu t$$

- Then, since $\mu = const$:

$$d\mathbb{Q}\left(\widetilde{W}_t\right) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{1}{2t}\widetilde{W}_t^2} d\widetilde{W}_t$$

- This means that $\widetilde{W}_t = W_t + \mu t$ is a Brownian motion under the new probability measure $\mathbb{Q}$.
- The process θ_s measures how much the original mean will be changed (drift)!

Girsanov's Theorem Demystified

Multiplying $d\mathbb{P}\left(W_t\right)$ by the function ξ_t, we succeeded in changing the mean of W_t, but we preserved the shape of the original probability measure.

$$d\mathbb{Q} = \xi_T d\mathbb{P}$$

- $$\frac{d\mathbb{Q}}{d\mathbb{P}} = \xi_T$$
- This expression reads as if the "derivative" of the measure $\mathbb{Q}$ with respect to $\mathbb{P}$ is given by ξ_T.
- Such expressions are called **Radon-Nikodym derivatives**, and ξ_T can be regarded as the *density* of the probability measure $\mathbb{Q}$ with respect to the measure $\mathbb{P}$.

- $\mathbb{P}$ and $\mathbb{Q}$ are different probability measures.
  - Under the measure $\mathbb{P}$: $\begin{cases} \mathbb{E}^{\mathbb{P}}\left[W_t\right] = 0 \\ Var^{\mathbb{P}}\left[W_t\right] = t \end{cases}$; $\begin{cases} \mathbb{E}^{\mathbb{P}}\left[\widetilde{W_t}\right] = \mu t \\ Var^{\mathbb{P}}\left[W_t\right] = t \end{cases}$
  - Under the measure $\mathbb{Q}$: $\begin{cases} \mathbb{E}^{\mathbb{Q}}\left[W_t\right] = -\mu t \\ Var^{\mathbb{Q}}\left[W_t\right] = t \end{cases}$; $\begin{cases} \mathbb{E}^{\mathbb{Q}}\left[\widetilde{W_t}\right] = 0 \\ Var^{\mathbb{Q}}\left[W_t\right] = t \end{cases}$

Discussion of Girsanov's Theorem

- The process $\xi_t = \exp\left\{-\int_0^t \theta_s dW_s - \frac{1}{2}\int_0^t (\theta_s)^2 \, ds\right\}$ is always positive:
$$\xi_t > 0$$
- For $t = 0$:
$$\xi_0 = 1$$
- ξ_t is a $\mathbb{P}$-martingale (a driftless stochastic process):
$$d\xi_t = -\xi_t \theta_t dW_t$$
- If ξ_t is a $\mathbb{P}$-martingale:
$$\mathbb{E}\left[\xi_t\right] = \xi_0 = 1$$
- Therefore, in order to change measure we need to find a Radon-Nikodym derivative that satisfies these conditions:
$$\begin{cases} \xi_t > 0 \\ \xi_t \text{ is a martingale} \\ \mathbb{E}\left[\xi_t\right] = 1 = \xi_0. \end{cases}$$

Static Theory

- Consider a random variable Z such that:

$$Z > 0, \ \mathbb{Q} \text{ a.s.} \qquad \text{and} \qquad \mathbb{E}[Z] = 1.$$

- According to Girsanov's theorem, we can define another measure $\overline{\mathbb{Q}}$ on the same probability space, equivalent to $\mathbb{Q}$, by[1]

$$\overline{\mathbb{Q}}(A) \triangleq \mathbb{E}[1_A \cdot Z] = \int_A Z d\mathbb{Q}, \quad \forall A \in \mathcal{F}$$

-

$$\begin{aligned} d\overline{\mathbb{Q}} &= Z d\mathbb{Q} \\ \frac{d\overline{\mathbb{Q}}}{d\mathbb{Q}} &= Z \end{aligned}$$

- The random variable Z is the Radon-Nikodym derivative of the measure $\overline{\mathbb{Q}}$ with respect to the measure $\mathbb{Q}$.

[1] The quantity 1_A is the indicator function for the subset A.

Lemma

$\overline{\mathbb{Q}}$ has the following properties and is therefore a probability measure:

- $\overline{\mathbb{Q}}\left(\emptyset\right) = 0$
- $\overline{\mathbb{Q}}\left(\Omega\right) = 1$
- $A \in \mathcal{F} \Longrightarrow \overline{\mathbb{Q}}\left(A\right) \geq 0$
- $\{A_i\}_{i=1}^{\infty} \subset \mathcal{F}$ *and* $A_i \cap A_j = \emptyset$, *for* $i \neq j \Longrightarrow \overline{\mathbb{Q}}\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \overline{\mathbb{Q}}\left(A_i\right)$

Lemma

Change of Expectation. The expected value of any random variable X under measure $\overline{\mathbb{Q}}$ is given by:

$$\overline{\mathbb{E}}\left[X\right] = \mathbb{E}\left[ZX\right]$$

Proof.

$\overline{\mathbb{E}}\left[X\right] = \int X\left(\omega\right) d\overline{\mathbb{Q}} = \int X\left(\omega\right) \frac{d\overline{\mathbb{Q}}}{d\mathbb{Q}} d\mathbb{Q} = \int X\left(\omega\right) Z d\mathbb{Q} = \mathbb{E}\left[ZX\right]$ $\square$

Static Theory

- <u>**The static theory**</u> represents just one simple case: $\frac{d\overline{\mathbb{Q}}}{d\mathbb{Q}}$ is defined for a fixed time horizon.
- We specified X at this time and we only wanted an unconditional expectation. The result we actually derived was:

$$\overline{\mathbb{E}}\left[X_T|\mathcal{F}_0\right] = \mathbb{E}\left[\frac{d\overline{\mathbb{Q}}}{d\mathbb{Q}} X_T|\mathcal{F}_0\right],$$

  where T is the time horizon for $\frac{d\overline{\mathbb{Q}}}{d\mathbb{Q}}$ and X_T is known at time T.
- However, in general we need to know

$$\overline{\mathbb{E}}\left[X_t|\mathcal{F}_s\right] \text{ for } t \neq T \text{ and } s \neq 0$$

- For this we need to know $\frac{d\overline{\mathbb{Q}}}{d\mathbb{Q}}$ not just for the ends of the paths, but everywhere.
- $\frac{d\overline{\mathbb{Q}}}{d\mathbb{Q}}$ is a random variable now, but we need a process.

Dynamic Theory

- We would like to have a formula similar to $\overline{\mathbb{E}}[X] = \mathbb{E}[ZX]$ that holds for both *conditional* and unconditional expectations.
- The random variable Z from the previous section cannot be extended to conditional expectations because it is defined for the end of the paths. We need to find a **process**.
- We can do this by letting the time horizon vary.
- For example, we can create a martingale using the conditional expectation process of the random variable Z:

$$Z_t = \mathbb{E}\left[Z|\mathcal{F}_t\right].$$

- **Fact**. *For any claim Z, the process $\mathbb{E}\left[Z|\mathcal{F}_t\right]$ is a martingale.*
- *Proof*. Choose $0 < s < t \leq T \implies$
  - $\mathbb{E}\left[Z_t|\mathcal{F}_s\right] = \mathbb{E}\left[\mathbb{E}\left[Z|\mathcal{F}_t\right]|\mathcal{F}_s\right] = \mathbb{E}\left[Z|\mathcal{F}_s\right] = Z_s$

Dynamic Theory

- The martingale $\{Z_t = \mathbb{E}[Z|\mathcal{F}_t]\}_{t=0}^{\infty}$ satisfies the conditions of Girsanov's theorem:

$$\begin{cases} Z_t > 0 \quad \mathbb{Q} \text{ a.s.} \\ \mathbb{E}[Z_t|\mathcal{F}_s] = Z_s \quad \text{for } t \geq s \Longrightarrow Z_t \text{ is a martingale} \\ Z_0 = \mathbb{E}[Z] = 1 \end{cases}$$

- For each time $T > 0$:

$$\mathbb{E}[Z_T] = 1$$

- Therefore we can define the measure $\widetilde{\mathbb{Q}}_T : \mathcal{F}_T \to \mathbb{R}^+$ the same way as in the static case:

$$\widetilde{\mathbb{Q}}_T(A) \triangleq \mathbb{E}[1_A \cdot Z_T] = \int_A Z_T d\mathbb{Q}, \qquad \forall A \in \mathcal{F}_T$$

  using Z_T as the Radon-Nikodym derivative of the measure $\widetilde{\mathbb{Q}}_T$ w.r.t. $\mathbb{Q}$:

$$\frac{d\widetilde{\mathbb{Q}}_T}{d\mathbb{Q}} = Z_T.$$

Lemma

The Consistency Condition

For $0 \leq t \leq T$, $\mathcal{F}_t \subset \mathcal{F}_T$, *i.e., if* $A \in \mathcal{F}_t \Longrightarrow A \in \mathcal{F}_T$. *The consistency condition*

$$\widetilde{\mathbb{Q}}_t (A) = \widetilde{\mathbb{Q}}_T (A)$$

holds $\forall A \in \mathcal{F}_t$.

Lemma

Change of expectation

The expected value of a claim X *under the measure* $\widetilde{\mathbb{Q}}_T$ *is given by:*

$$\widetilde{\mathbb{E}} [X] = \mathbb{E} [Z_T X] .$$

Moreover, if X *is* $\mathcal{F}_t$*-measurable with* $0 \leq t \leq T$, *then its expected value is given by:*

$$\widetilde{\mathbb{E}} [X] = \mathbb{E} [Z_t X] .$$

Corollary

Because of the consistency condition, we can drop the subscript T and define a new measure $\widetilde{\mathbb{Q}}$ on the whole σ-algebra $\widetilde{\mathbb{Q}} : \mathcal{F} \longrightarrow \mathbb{R}^+$.

Proof.

Define $\widetilde{\mathbb{Q}} : \mathcal{F} \longrightarrow \mathbb{R}^+$. Then for $\forall A \in \mathcal{F}$, $\exists t > 0$ such that $A \in \mathcal{F}_t$.
By virtue of the consistency condition: $\widetilde{\mathbb{Q}}\,(A) = \widetilde{\mathbb{Q}}_t\,(A)$. □

Note. We have actually defined a whole family of measures indexed by time $\left\{\widetilde{\mathbb{Q}}_t\right\}_{t=0}^{T}$, infinitely many, that are all the same measure. This unique measure is defined by the Radon-Nikodym derivative:

$$\left.\frac{d\widetilde{\mathbb{Q}}}{d\mathbb{Q}}\right|_{\mathcal{F}_T} = Z_T .$$

Bayes' Rule

Lemma

Change of conditional expectation or Bayes' Rule

If $0 \leq t \leq T$ *and* X *is an* $\mathcal{F}_T$*-measurable random variable satisfying* $\widetilde{\mathbb{E}}\left[|X|\right] < +\infty$, *(i.e.,* X *is integrable), then*

$$\widetilde{\mathbb{E}}\left[X|\mathcal{F}_t\right] = \frac{1}{Z_t}\mathbb{E}\left[Z_T X|\mathcal{F}_t\right]$$

- Bayes' Rule is the formula that relates conditional expectations under different measures that we've been looking for.
- It is the key theorem that makes the transformation of pricing formula through different measures work.
- It also gives us hope that we can find a measure which will allow us to simplify the expression under the expectation operator.

Forward Measure Construction

- We are trying to simplify the pricing formula for $\mathcal{F}_T$-measurable payoffs X under the risk-neutral measure $\mathbb{Q}$:

$$\pi_t(X) = B_t \mathbb{E}\left[B_T^{-1} X | \mathcal{F}_t\right].$$

- We are looking for a model in which Black's formula is correct for the valuation of interest rate derivatives, i.e., we can "take out" B_T^{-1} from under the expectation operator.
- The right-hand side of the pricing formula is very similar to the right-hand side of Bayes' formula:

$$\widetilde{\mathbb{E}}\left[X | \mathcal{F}_t\right] = \frac{1}{Z_t} \mathbb{E}\left[Z_T X | \mathcal{F}_t\right]$$

- X is $\mathcal{F}_T$-measurable in both. However, we cannot use B_T^{-1} as a Radon-Nikodym derivative to define a new measure based on $\mathbb{Q}$, because $\{B_t^{-1}\}_{t=0}^{+\infty}$ is not a martingale under $\mathbb{Q}$.

Forward Measure Construction

- We remember that the discounted bond price is a $\mathbb{Q}$-martingale and is also positive. It satisfies two of the three requirements from Girsanov's Theorem, but not the third one:
  $Z(t,T) = \frac{P(t,T)}{B_t} > 0$
- $Z(t,T)$ is a $\mathbb{Q}$-martingale
  $Z(0,T) = \frac{P(0,T)}{B_0} = P(0,T) \neq 1$
- Easy fix: we choose the normalized discounted bond price:

$$Z_t = \frac{Z(t,T)}{Z(0,T)} \Longrightarrow Z_0 = 1.$$

- Z_t satisfies all three requirements:

$$\begin{cases} Z_t = \frac{P(t,T)/B_t}{P(0,T)/B_0} > 0 \\ Z_t \text{ is a } \mathbb{Q}\text{-martingale} \\ Z_0 = 1 \end{cases}$$

Forward Measure Construction

Definition

Fix time $T > 0$. The measure $\mathbb{Q}^T$ defined by the following Radon-Nikodym derivative:

$$\left.\frac{d\mathbb{Q}^T}{d\mathbb{Q}}\right|_{\mathcal{F}_t} = Z_t = \frac{P(t,T)/B_t}{P(0,T)/B_0}$$

is the T-forward measure.

Theorem

Forward Measure Pricing Formula If X is an F_T-measurable payoff, then

$$\pi_t(X) = P(t,T)\,\mathbb{E}^T\left[X|\mathcal{F}_t\right],$$

where $\mathbb{E}^T$ is the expectation operator with respect to the T-forward measure $\mathbb{Q}^T$ defined above.

Forward Measure Construction

- The theorem achieves our main goal of decoupling the money market B_T and payoff X in the pricing formula.
- This can be done only under the T-forward measure.
- The measure $\mathbb{Q}^T$ depends on a particular time T. Therefore we actually have an entire family of different forward measures $\{\mathbb{Q}^T\}_{T=0}^{+\infty}$.
- **Remark**. *The successful application of forward measure depends critically upon identification of the proper time T. Usually T is taken to be the expiration time of an option, so that the payoff is measurable with respect to $\mathcal{F}_T$.*

Post-Crisis Complications: Pricing Under Collateral

- Pricing derivatives under collateral implies that we must take into account:
  - the cashflows generated by the derivative
  - the cashflows generated by the margination mechanism provided by the CSA
- **Best Case Scenario**: Pricing under **perfect collateral** differs from the standard Black-Scholes-Merton framework without collateral in how the cash used to replicate the derivative payoff is split among the different sources of funding:
  - Due to the perfect collateral assumption, the cash in the collateral acount, denoted by B_c, provides exact secured funding of the derivative position: $\pi_t(X) = B_c$. The collateral rate, denoted by r_c, is typically the overnight rate (OIS), therefore the riskless rate
  - While the hedge is funded (unsecured) by the generic funding account B_f at the risky rate r_f

Even with these complications, in the case of **perfect collateral**, the correct discount rate is the collateral (riskless) rate r_c.

Theorem

Forward Measure Pricing Under Collateral

The pricing formula

$$\begin{aligned} \pi_t(X) &= P_c(t,T)\,\mathbb{E}^{\mathbb{Q}_c^T}\left[X|\mathcal{F}_t\right], \\ P_c(t,T) &= \mathbb{E}^{\mathbb{Q}}\left[B_c^{-1}(t,T)\right] \end{aligned}$$

holds, where $\mathbb{Q}_c^T$ is the probability measure associated with the collateral (riskless) zero-coupon bond $P_c(t,T)$, and $\mathbb{Q}$ is the risk-neutral measure associated with the collateral money market account B_c (riskless rate r_c).

Example Revisited

- By changing to forward measure, we can express the value of the call option on a bond from our motivational example as:

$$\begin{aligned}\pi_t(C) &= B_t \mathbb{E}\left[B_T^{-1}\left(P(T,M)-K\right)^+ |\mathcal{F}_t\right] \\ &= P(t,T)\,\mathbb{E}^T\left[\left(P(T,M)-K\right)^+ |\mathcal{F}_t\right].\end{aligned}$$

- We know that if we calculate the expected value, we will get exactly Black's formula!
- We succeeded in building a model in which Black's formula is correct for pricing options with random interest rates.
- We could do this under the T-forward measure (i.e., using the bond price as numeraire)!
- We will see later in the course that *by the appropriate change of measure*, Black's formula can be rigorously applied to the valuation of caps, swaptions, etc., within the HJM framework of the third-generation of interest rate models called "Market Models".

Properties of Forward Measure

Lemma

Let $F(t, T, M)$ be the time T forward price of the zero coupon bond which matures at time M, i.e.,

$$F(t, T, M) = \frac{P(t, M)}{P(t, T)}.$$

Then

$$F(t, T, M) = \mathbb{E}^T \left[P(T, M) \,|\mathcal{F}_t \right].$$

Lemma

The process

$$\{F(t, T, M)\}_{t=0}^{T}$$

is a martingale with respect to the T-forward measure $\mathbb{Q}^T$.

• **Note**. There is more than one discount bond maturing at any given time T. Have to work consistently within a **single credit quality**.

Lemma

Let A_t be the price at time t of a traded asset. We know that

$$B_t^{-1} A_t$$

is a martingale under the risk neutral measure $\mathbb{Q}$. Let $F_A(t, T)$ be its forward price at time t for delivery at time T, i.e.,

$$F_A(t, T) = \frac{A_t}{P(t, T)}.$$

- *The forward price can also be computed as*

$$F_A(t, T) = \mathbb{E}^T \left[A_T | \mathcal{F}_t \right],$$

- *The forward price process*

$$\left\{ F_A(t, T) \right\}_{t=0}^{T}$$

is a martingale under the T-forward measure $\mathbb{Q}^T$.

Properties of Forward Measure

- These results explain why we called the measure $\mathbb{Q}^T$ the T-forward measure.
- We know from the Black's Model lecture that forward prices are equal to the expected value of the future prices only under the T-forward measure.
- The T-forward measure is identified by the fact that the forward prices for delivery at time T of all traded instruments are martingales with respect to it.
- In fact, this property uniquely identifies the measure.

Forward Measure for Gaussian HJM

- An HJM model is Gaussian when the forward rate volatilities $\sigma(t, T)$ are deterministic functions: Ho-Lee, Hull-White, and a modified version of Yuri's statistical model.
- This is a widely used subclass of HJM models, appreciated mostly for extensive analytical tractability.
- The reason is that all forward rates are normally distributed and all discount bonds are log-normally distributed.
- **One-factor Gaussian HJM**:

$$
\begin{aligned}
df(t, T) &= -\Sigma(t, T)\,\sigma(t, T)\,dt + \sigma(t, T)\,dW_t \\
dP(t, T) &= r(t)\,P(t, T)\,dt + P(t, T)\,\Sigma(t, T)\,dW_t \\
\sigma(t, T) &= -\frac{\partial}{\partial T}\Sigma(t, T) \\
\Sigma(T, T) &= 0
\end{aligned}
$$

where $\sigma(t, T)$ and $\Sigma(t, T)$ are deterministic.

Forward Rates in Gaussian HJM

- The solution of the SDE for instantaneous forward rates is:

$$f(t,T) = f(0,T) - \int_0^t \Sigma(s,T)\,\sigma(s,T)\,ds + \int_0^t \sigma(s,T)\,dW_s.$$

- The first two terms in the above equation are deterministic
- The third term is the only source of randomness. The increments of Brownian motion dW_s are normally-distributed with zero mean and variance s:

$$dW_s \sim N(0,s)$$

- It means that the **instantaneous forward rates $f(t,T)$ are normally distributed** under the risk-neutral measure with mean and variance:

$$\begin{aligned}\mathbb{E}\left[f(t,T)\right] &= f(0,T) - \int_0^t \Sigma(s,T)\,\sigma(s,T)\,ds \\ Var\left[f(t,T)\right] &= \int_0^t \sigma^2(s,T)\,ds.\end{aligned}$$

Forward Bond Prices in Gaussian HJM

- Consider a bond that expires at time T. Its evolution is governed by:

$$dP(t,T) = r(t)P(t,T)\,dt + P(t,T)\Sigma(t,T)\,dW_t$$

- Let's derive the SDE for forward bond prices under the risk-neutral measure $\mathbb{Q}$:

$$\begin{aligned}
dF(t,T,M) &= d\left(\frac{P(t,M)}{P(t,T)}\right) \\
&= \frac{1}{P(t,T)}dP(t,M) - \frac{P(t,M)}{P(t,T)^2}dP(t,T) \\
&\quad -\frac{1}{P(t,T)^2}dP(t,T)\,dP(t,M) + \frac{P(t,M)}{P(t,T)^3}dP(t,T) \\
&= F(t,T,M)\left(\Sigma(t,T)^2 - \Sigma(t,T)\Sigma(t,M)\right)dt \\
&\quad +F(t,T,M)\left(\Sigma(t,M) - \Sigma(t,T)\right)dW_t.
\end{aligned}$$

Forward Bond Prices in Gaussian HJM

- If we collect terms in the equation above, we find

$$dF(t,T,M) = F(t,T,M)\left(\Sigma(t,M) - \Sigma(t,T)\right)\left[dW_t - \Sigma(t,T)\,dt\right].$$

- This holds under the risk-neutral measure $\mathbb{Q}$. We would like to change measure so that $F(t,T,M)$ becomes a martingale.
- By Girsanov's theorem, measure changes in our setting are equivalent to adding a drift to the Brownian motion.
- We want to "change" the drift so that the dt part in equation for $F(t,T,M)$ disappears.
- But we already know from the previous section that forward bond prices are martingales under the T-forward measure. This yields the required measure change.
- We have proven the following theorem:

Theorem (Forward Measure in Gaussian HJM)

• *The T-forward measure $\mathbb{Q}^T$ is (uniquely) identified by the condition that*

$$dW_t^T \triangleq dW_t - \Sigma(t, T)\, dt$$

is a (driftless) Brownian motion under $\mathbb{Q}^T$.

• *The T-forward bond price evolution is given by*

$$dF(t, T, M) = F(t, T, M)\left(\Sigma(t, M) - \Sigma(t, T)\right) dW_t^T$$

under $\mathbb{Q}^T$. This implies that the T-forward bond price is a martingale.

• *Under $\mathbb{Q}^T$ the forward bond price $F(t, T, M)$ has a log-normal distribution*

$$\log \frac{F(t, T, M)}{F(0, T, M)} \quad \text{is} \quad \text{Gaussian}$$

$$\mathbb{E}^T\left[\log \frac{F(t, T, M)}{F(0, T, M)}\right] = -\frac{1}{2}\mathrm{Var}\left(\log \frac{F(t, T, M)}{F(0, T, M)}\right)$$

$$\mathrm{Var}\left(\log \frac{F(t, T, M)}{F(0, T, M)}\right) = \int_0^t \left(\Sigma(s, M) - \Sigma(s, T)\right)^2 ds.$$

Forward Measure in Gaussian HJM

- The evolution of other quantities under $\mathbb{Q}^T$ can easily be deduced by replacing dW_t with $dW_t^T + \Sigma(t,T)\,dt$. Here is an example:

Corollary

A bond with maturity time S follows the equation

$$dP(t,S) = (r(t) + \Sigma(t,T)\Sigma(t,S))\,P(t,S)\,dt + P(t,S)\,\Sigma(t,S)\,dW_t^T$$

and the instantaneous forward rate with maturity S follows the equation

$$df(t,S) = (\Sigma(t,T) - \Sigma(t,S))\,\sigma(t,S)\,dt + \sigma(t,S)\,dW_t^T$$

under the T-forward measure $\mathbb{Q}^T$.

General Numéraires

- The choice of numeraire is not limited to the money market account or discount bonds.
- **Any traded asset whose value is always strictly positive can be used as a numeraire.**
- Let N_t be a traded asset such that $N_t > 0, \forall t \geq 0 \;\; \mathbb{Q}$ a.s. Then the quantity

$$\frac{N_t}{B_t}$$

is a martingale under $\mathbb{Q}$. Therefore we can construct a measure $\widehat{\mathbb{Q}}^N$ which corresponds to the numeraire N.

Theorem

(General Change of Numéraire)

There exists a measure $\widehat{\mathbb{Q}}^N$, equivalent to $\mathbb{Q}$, such that

- *for any traded asset A:*

$$A_t = N_t \widehat{\mathbb{E}}^N \left[N_T^{-1} A_T | \mathcal{F}_t \right], \quad \forall 0 \leq t \leq T$$

- *the value A_t discounted by N_t,*

$$\frac{A_t}{N_t}$$

is a martingale under $\widehat{\mathbb{Q}}^N$.

Fact

Once we have a numéraire N, the Radon-Nikodym derivative we need to build a new probability measure associated with that numéraire is always provided by the normalized price of the numéraire discounted by the money market account.

Proof. Define $\widehat{\mathbb{Q}}^N$ using the following Radon-Nikodym derivative:

$$\left.\frac{d\widehat{\mathbb{Q}}^N}{d\mathbb{Q}}\right|_{\mathcal{F}_t} = \frac{N_t / B_t}{N_0 / B_0}.$$

Let's denote $\frac{N_t / B_t}{N_0 / B_0} = Z_t \Longrightarrow B_t = \frac{1}{Z_t} \frac{N_t}{N_0 / B_0}$ and $B_T^{-1} = Z_T \frac{N_0 / B_0}{N_T}$. We start from the pricing formula under the risk-neutral measure and apply Bayes' Rule:

$$\begin{aligned} A_t &= B_t \mathbb{E}\left[B_T^{-1} A_T | \mathcal{F}_t\right] \\ &= \frac{1}{Z_t} \frac{N_t}{N_0 / B_0} \mathbb{E}\left[Z_T \frac{N_0 / B_0}{N_T} A_T \middle| \mathcal{F}_t\right] \\ &= N_t \frac{1}{Z_t} \mathbb{E}\left[\left(N_T^{-1} A_T\right) Z_T \middle| \mathcal{F}_t\right] = N_t \widehat{\mathbb{E}}^N \left[N_T^{-1} A_T | \mathcal{F}_t\right]. \end{aligned}$$

The second assertion follows directly if we divide by N_t:

$$\frac{A_t}{N_t} = \widehat{\mathbb{E}}^N \left[\frac{A_T}{N_T} \middle| \mathcal{F}_t\right].$$

European Payer's Swaption on a LIBOR Swap with Matched Discounting

- Suppose we have a tenor structure

$$\begin{aligned} t_0 &< t_1 < \ldots < t_n \\ \tau_i &= t_i - t_{i-1}, \quad i = 1, 2, \ldots, n \end{aligned}$$

- and a time-t_1 forward starting payer's swap with this tenor structure, notional 1, and fixed rate c. The payer's swaption expires at time t_0.
- The value of the swaption at time t is:

$$V_t = B_t \mathbb{E}\left[B_{t_0}^{-1} \left(\left(P\left(t_0, t_0\right) - P\left(t_0, t_n\right)\right) - c \sum_{i=1}^{n} P\left(t_0, t_i\right) \tau_i \right)_+ \middle| \mathcal{F}_t \right].$$

- We will choose the (normalized) value of the fixed leg as numéraire:

$$N_t = \sum_{i=1}^{n} P\left(t, t_i\right) \tau_i .$$

- This is just a linear combination of bonds, therefore it definitely qualifies as a numéraire.

Let's use the General Change of Numéraire Theorem:

$$
\begin{aligned}
V_t &= N_t \widehat{\mathbb{E}}^N \left[N_{t_0}^{-1} \left(\left(P\left(t_0, t_0\right) - P\left(t_0, t_n\right)\right) - c \sum_{i=1}^{n} P\left(t_0, t_i\right) \tau_i \right)_+ \middle| \mathcal{F}_t \right] \\
&= N_t \widehat{\mathbb{E}}^N \left[\left(\frac{\left(P\left(t_0, t_0\right) - P\left(t_0, t_n\right)\right) - c \sum_{i=1}^{n} P\left(t_0, t_i\right) \tau_i}{\sum_{i=1}^{n} P\left(t_0, t_i\right) \tau_i} \right)_+ \middle| \mathcal{F}_t \right] \\
&= N_t \widehat{\mathbb{E}}^N \left[\left(\frac{P\left(t_0, t_0\right) - P\left(t_0, t_n\right)}{\sum_{i=1}^{n} P\left(t_0, t_i\right) \tau_i} - c \right)_+ \middle| \mathcal{F}_t \right] \\
&= N_t \widehat{\mathbb{E}}^N \left[\left(s_{t_0} - c\right)_+ \middle| \mathcal{F}_t \right],
\end{aligned}
$$

Example of a Useful Numéraire

where s_{t_0} is the break-even swap rate:

$$\begin{aligned} s_{t_0} &= \frac{P(t_0, t_0) - P(t_0, t_n)}{\sum_{i=1}^{n} P(t_0, t_i)\, \tau_i} \\ &= \frac{P(t_0, t_0) - P(t_0, t_n)}{N_{t_0}}. \end{aligned}$$

- If we define the forward swap rate s_0 to be:

$$s_0 = \widehat{\mathbb{E}}^N \left[s_{t_0} \right] = \frac{P(0, t_0) - P(0, t_n)}{\sum_{i=1}^{n} P(0, t_i)\, \tau_i},$$

- then we can assume, for example, that s_{t_0} has a log-normal distribution:

$$s_{t_0} = s_0 e^{\sigma \sqrt{t_0} \eta - \frac{1}{2} \sigma^2 t_0}$$

- In this model where the swap rate s_{t_0} follows geometric Brownian motion, the swaption is priced using Black's formula!

Conclusions

- The risk-neutral measure is characterized by the choice of the money market account as numéraire.
- However, almost any traded instrument can be used as a numéraire, giving a wide choice of measures to use when computing contingent claim values.
- The valuation of many instruments can be significantly simplified by the appropriate choice of numéraire.
- Of special importance in Fixed Income Derivatives are measures which correspond to using zero-coupon bonds as numéraires.
- These measures are called *forward measures*.
- They may be expressed very conveniently for Gaussian HJM models.

References

[1] *Interest Rate Modelling after the Financial Crisis*, Editors Massimo Morini and Marco Bianchetti, Incisive Media Investment Ltd (2013).
[2] V. Piterbarg, Funding Beyond Discounting: Collateral Agreements and Derivatives Pricing, Risk 23 (2) 2010
[3] M. Morini, *Solving the Puzzle in the Interest Rate Market*, SSRN Working Paper (2009).
[3] F. Mercurio, *Interest Rates and the Credit Crunch:New Formulas and Market Models*, Working Paper (2008).