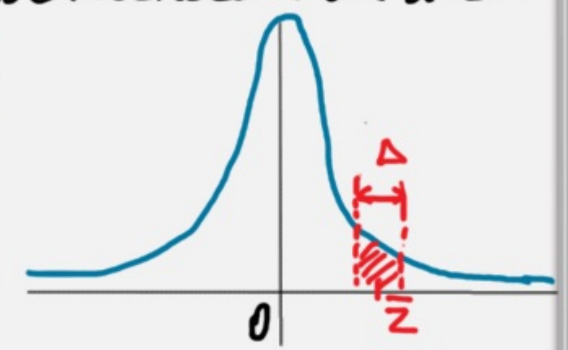
Probability as "Measure"

Consider a normally distributed random variable

$$Z_t \sim N(0,1)$$



Probability density function:

$$f(z_t) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} z_t^2}$$

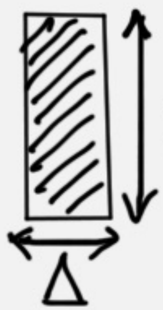
The probability that z_t falls near a specific value $\bar{z}$ is:

$$\mathbb{P}\left(\bar{z} - \tfrac{1}{2}\Delta < z_t < \bar{z} + \tfrac{1}{2}\Delta\right) = \int_{\bar{z} - \frac{1}{2}\Delta}^{\bar{z} + \frac{1}{2}\Delta} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} z_t^2} \, dz_t$$

If the region Δ around $\bar{z}$ is small $\Rightarrow$ $f(z_t) \cong f(\bar{z})$:

$$\int_{\bar{z}-\frac{1}{2}\Delta}^{\bar{z}+\frac{1}{2}\Delta} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z_t^2} dz_t \cong \int_{\bar{z}-\frac{1}{2}\Delta}^{\bar{z}+\frac{1}{2}\Delta} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\bar{z}^2} = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\bar{z}^2} \cdot \Delta$$

This probability is a "mass" represented by a rectangle:



$$f(\bar{z}) = \mathbb{P}\left(\bar{z} - \frac{1}{2}\Delta < z_t < \bar{z} + \frac{1}{2}\Delta\right)$$

That is, probability corresponds to a "measure" associated with possible values of z_t.

For infinitesimal $\Delta = dz_t$, we denote these measures by the symbol $d\mathbb{P}(z_t)$:

$$d\mathbb{P}(z_t) = \mathbb{P}\left(\bar{z} - \tfrac{1}{2}dz_t < z_t < \bar{z} + \tfrac{1}{2}dz_t\right)$$

$$\int_{-\infty}^{\infty} d\mathbb{P}(z_t) = 1$$

The expected value of z_t is the "center of probability mass"

$$\mathbb{E}[z_t] = \int_{-\infty}^{\infty} z_t \, d\mathbb{P}(z_t)$$

The variance indicates how the probability mass spreads around the center:

$$\mathbb{E}\left[z_t - \mathbb{E}[z_t]\right]^2 = \int_{-\infty}^{\infty} \left[z_t - \mathbb{E}[z_t]\right]^2 d\mathbb{P}(z_t)$$

We Change Probability Measures To Justify Prices

Game: You flip a coin, the values of the payoff are:

$$Z = \begin{cases} \$1, & \text{if you flip heads} \\ \$0.1, & \text{if you flip tails} \end{cases}$$

How much would you pay to play this game?

Real-World Measure

$$\begin{tikzcd}[row sep=small] & \$1 \text{ heads} \\ \$0.55 \arrow[ur, "p=\frac{1}{2}"] \arrow[dr, "1-p=\frac{1}{2}"'] & \\ & \$0.1 \text{ tails} \end{tikzcd}$$

$$\frac{1}{2}\cdot \$1 + \frac{1}{2}\,\$0.1 = \$0.55$$

The Game Dealer Asks \$0.60

$$\begin{tikzcd}[row sep=small] & \$1 \text{ heads} \\ \$0.60 \arrow[ur, "q=\frac{5}{9}"] \arrow[dr, "1-q=\frac{4}{9}"'] & \\ & \$0.10 \text{ tails} \end{tikzcd}$$

New Probability Measure

$$\$1\cdot q + \$0.10(1-q) = \$0.60$$

$$0.9\,q = 0.5 \Rightarrow q = \frac{5}{9}$$

Static Theory

$Z > 0$, $\mathbb{Q}$ a.s., $\mathbb{E}[Z] = 1$

Define $\bar{\mathbb{Q}}(A) \triangleq \mathbb{E}[1_A \cdot Z]$

$\bar{\mathbb{Q}}(\emptyset) = 0$: $\bar{\mathbb{Q}}(\emptyset) = \mathbb{E}[1_\emptyset \cdot Z] = \mathbb{E}[0 \cdot Z] = \mathbb{E}[0] = 0$

$\bar{\mathbb{Q}}(\Omega) = 1$: $\bar{\mathbb{Q}}(\Omega) = \mathbb{E}[1_\Omega \cdot Z] = \mathbb{E}[1 \cdot Z] = \mathbb{E}[Z] = 1$

$A \in \mathcal{F} \Rightarrow \bar{\mathbb{Q}}(A) \geqslant 0$: Denote $N = \{\omega \in A \mid Z(\omega) < 0\}$

$$\bar{\mathbb{Q}}(A) = \mathbb{E}[1_A \cdot Z] = \int_A Z \, d\mathbb{Q} = \int_{A \setminus N} Z \, d\mathbb{Q} + \underbrace{\int_N \cancel{Z} \, d\mathbb{Q}}_{\to 0} = \int_{A \setminus N} Z \, d\mathbb{Q} > \int_{A \setminus N} 0 \, d\mathbb{Q} = 0$$

$\forall A, B \in \mathcal{F}$, $A \cap B = \emptyset \Rightarrow \bar{\mathbb{Q}}(A \cup B) = \bar{\mathbb{Q}}(A) + \bar{\mathbb{Q}}(B)$

$$\bar{\mathbb{Q}}(A \cup B) = \int Z \cdot 1_{A \cup B} \, d\mathbb{Q} = \int_\Omega Z (1_A + 1_B) \, d\mathbb{Q} = \int_\Omega Z \cdot 1_A \, d\mathbb{Q}$$
$$+ \int_\Omega Z \cdot 1_B \, d\mathbb{Q} = \int_A Z \, d\mathbb{Q} + \int_B Z \, d\mathbb{Q} = \bar{\mathbb{Q}}(A) + \bar{\mathbb{Q}}(B)$$

Dynamic Measure Change Consistency Condition

For $0 \le t \le T$, $\tilde{\mathcal{F}}_t \subset \tilde{\mathcal{F}}_T$: $A \in \tilde{\mathcal{F}}_t \Rightarrow A \in \tilde{\mathcal{F}}_T$

$$\underline{\tilde{\mathbb{Q}}_t(A) \overset{?}{=} \tilde{\mathbb{Q}}_T(A)}, \quad \forall A \in \tilde{\mathcal{F}}_t$$

$A \in \tilde{\mathcal{F}}_t \Rightarrow A,\ 1_A - \tilde{\mathcal{F}}_t$-measurable

Z_t is a martingale $\Rightarrow Z_t = \mathbb{E}[Z_T / \tilde{\mathcal{F}}_t]$

$$\underline{\tilde{\mathbb{Q}}_t(A)} = \mathbb{E}[1_A \cdot Z_t] = \mathbb{E}[1_A \cdot Z_t / \tilde{\mathcal{F}}_0] = \mathbb{E}[1_A \mathbb{E}[Z_T / \tilde{\mathcal{F}}_t] / \tilde{\mathcal{F}}_0]$$

$$= \mathbb{E}[\mathbb{E}[1_A Z_T / \tilde{\mathcal{F}}_t] / \tilde{\mathcal{F}}_0] = \mathbb{E}[1_A \cdot Z_T / \tilde{\mathcal{F}}_0] = \mathbb{E}[1_A Z_T] = \underline{\tilde{\mathbb{Q}}_T(A)}$$

Tower Rule

Bayes' Rule

if $0 \leqslant t \leq T$, and X is an $\mathcal{F}_T$-measurable r.v. so that $\widetilde{\mathbb{E}}[|X|] < +\infty$, then $\boxed{\widetilde{\mathbb{E}}[X/\mathcal{F}_t] = \frac{1}{Z_t}\mathbb{E}[Z_T X|\mathcal{F}_t]}$

By definition, $Y = \widetilde{\mathbb{E}}[X/\mathcal{F}_t]$ is the conditional expectation of X if $\widetilde{\mathbb{E}}[1_A \cdot Y] = \widetilde{\mathbb{E}}[1_A \cdot X]$.

Set $Y = Z_t^{-1}\mathbb{E}[Z_T X|\mathcal{F}_t]$ and fix $A \in \mathcal{F}_t \implies$

Y and 1_A are $\mathcal{F}_t$-measurable

Z is a martingale $\Rightarrow Z_t = \mathbb{E}[Z_T|\mathcal{F}_t]$

Tower Rule

$$\widetilde{\mathbb{E}}[1_A Y] = \mathbb{E}[Z_T 1_A Y] = \mathbb{E}[Z_T 1_A Y \mid \mathcal{F}_0] = \mathbb{E}\big[\mathbb{E}[Z_T 1_A Y \mid \mathcal{F}_t] \mid \mathcal{F}_0\big]$$

$$= \mathbb{E}\Big[1_A Y \underbrace{\mathbb{E}[Z_T \mid \mathcal{F}_t]}_{Z_t} \mid \mathcal{F}_0\Big] = \mathbb{E}[1_A Y Z_t \mid \mathcal{F}_0]$$

$$= \mathbb{E}\big[1_A \cancel{Z_t^{-1}}\, \mathbb{E}[Z_T X \mid \mathcal{F}_t]\, \cancel{Z_t} \mid \mathcal{F}_0\big]$$

Tower Rule

$$= \mathbb{E}\big[1_A \mathbb{E}[Z_T X \mid \mathcal{F}_t]\big] = \mathbb{E}\big[\mathbb{E}[1_A Z_T X \mid \mathcal{F}_t]\big]$$

$$= \mathbb{E}[1_A Z_T X] = \widetilde{\mathbb{E}}[1_A X] \Longrightarrow$$ Y is conditional expectation of X

$$\frac{1}{Z_t}\mathbb{E}[Z_T X \mid \mathcal{F}_t] = \widetilde{\mathbb{E}}[X \mid \mathcal{F}_t]$$

Forward Measure Pricing Formula

X is an $\mathcal{F}_T$-measurable payoff. Then

$$\Pi_t(X) = P(t,T)\,\mathbb{E}^T[X \mid \mathcal{F}_t]$$

$$B_t = \frac{P(t,T)}{Z(t,T)} \; ; \; B_T^{-1} = Z(T,T)$$

$$\Pi_t(X) = B_t\,\mathbb{E}[B_T^{-1} X \mid \mathcal{F}_t] = \frac{P(t,T)}{Z(t,T)}\,\mathbb{E}[Z(T,T) X \mid \mathcal{F}_t]$$

$$= P(t,T)\frac{Z(0,T)}{Z(t,T)}\,\mathbb{E}\left[\frac{Z(T,T)}{Z(0,T)} X \mid \mathcal{F}_t\right]$$

$$= P(t,T)\frac{1}{Z_t}\,\mathbb{E}[Z_T X \mid \mathcal{F}_t] \rightarrow \text{Bayes' Rule}$$

$$= P(t,T)\,\mathbb{E}^T[X \mid \mathcal{F}_t]$$

Properties of Forward Measure

Set up a forward contract to deliver at T the M-maturity bond

At T: Payoff $= X = P(T,M) - F(t,T,M)$

At t: $\Pi_t(X) = 0$

$$F(t,T,M) = \frac{P(t,M)}{P(t,T)}$$

$$0 = \Pi_t(X) = P(t,T)\, \mathbb{E}^T[X \mid \mathcal{F}_t]$$

$$P(t,T)\, \mathbb{E}^T[P(T,M) - F(t,T,M) \mid \mathcal{F}_t] = 0$$

$$\mathbb{E}^T[P(T,M)] = \mathbb{E}^T[F(t,T,M)] = F(t,T,M)$$

$\mathcal{F}_t$-measurable

$$\boxed{F(t,T,M) = \mathbb{E}^T[P(T,M) \mid \mathcal{F}_t]}$$

The process $\{F(t,T,M)\}_{t=0}^{T}$ is a martingale under $\mathbb{Q}^T$ //

$$F(t,T,M)=\frac{P(t,M)}{P(t,T)}$$

$$t=T:\ F(T,T,M)=\frac{P(T,M)}{P(T,T)}=P(T,M)$$

$$F(t,T,M)=\mathbb{E}^T\left[P(T,M)\,/\,\mathcal{F}_t\right]$$

$$\boxed{F(t,T,M)=\mathbb{E}^T\left[F(T,T,M)\,/\,\mathcal{F}_t\right]}\Rightarrow F(t,T,M) \text{ is a } \mathbb{Q}^T\text{-martingale}$$

Forward Measure in Gaussian HJM

$$Z(t,T) = \frac{P(t,T)}{B_t}$$ – is a martingale under $\mathbb{Q}$

$$dZ(t,T) = \Sigma(t,T)\,Z(t,T)\,dW_t \quad / : Z(t,T)$$

$$\frac{dZ(t,T)}{Z(t,T)} = \Sigma(t,T)\,dW_t \Rightarrow$$ geometric process

Solution of SDE: Doléans exponential of Brownian motion

$$Z(t,T) = Z(0,T)\exp\left\{\int_0^t \Sigma(s,T)\,dW_s - \frac{1}{2}\int_0^t \Sigma(s,T)^2\,ds\right\}$$

$$\frac{P(t,T)}{B_t} = \frac{P(0,T)}{B_0} \exp\left\{\int_0^t \Sigma(s,T)\,dW_s - \frac{1}{2}\int_0^t [\Sigma(s,T)]^2\,ds\right\}$$

$$\frac{dQ^T}{dQ} = \frac{P(t,T)/B_t}{P(0,T)/B_0} = \boxed{\exp\left\{\int_0^t \Sigma(s,T)\,dW_s - \frac{1}{2}\int_0^t [\Sigma(s,T)]^2\,ds\right\}}$$

$$\frac{dQ^T}{dQ} = \frac{P(t,T)/B_t}{P(0,T)/B_0}$$

Radon-Nikodym derivative of Q^T w.r.t. Q

Compare with ξ_t in Girsanov's Theorem $\Rightarrow$

$$\left.\begin{array}{l} \Sigma(s,T) = -\theta_s \\ \text{Girsanov: } dW_t^T = dW_t + \theta_t\,dt \end{array}\right\} \Rightarrow$$

$dW_t^T = dW_t - \Sigma(s,T)\,dt$ is Brownian motion under Q^T!

Forward Bond Prices under Gaussian HJM

$$dF(t,T,M)=F(t,T,M)\left[\Sigma(t,M)-\Sigma(t,T)\right]dW_t^T$$

$$\frac{dF(t,T,M)}{F(t,T,M)}=\left[\Sigma(t,M)-\Sigma(t,T)\right]dW_t^T$$ geometric process

Solution of SDE: Doléans exponential

$$F(t,T,M)=F(0,T,M)\exp\left\{\int_0^t\left[\Sigma(s,M)-\Sigma(s,T)\right]dW_s^T-\frac{1}{2}\int_0^t\left[\Sigma(s,M)-\Sigma(s,T)\right]^2ds\right\}$$

$$\log\left(\frac{F(t,T,M)}{F(0,T,M)}\right) = \int_0^t \left[\Sigma(s,M) - \Sigma(s,T)\right] dW_s^T - \frac{1}{2}\int_0^t \left[\Sigma(s,M) - \Sigma(s,T)\right]^2 ds$$

$$\mathbb{E}^T\left(\log\left(\frac{F(t,T,M)}{F(0,T,M)}\right)\right) = -\frac{1}{2}\int_0^t \left[\Sigma(s,M) - \Sigma(s,T)\right]^2 ds = -\frac{1}{2}\operatorname{Var}\left[\log\left(\frac{F(t,T,M)}{F(0,T,M)}\right)\right]$$

$$\operatorname{Var}\left[\log\left(\frac{F(t,T,M)}{F(0,T,M)}\right)\right] = \int_0^t \left[\Sigma(s,M) - \Sigma(s,T)\right]^2 ds$$