Homework Assignment:
The Heath-Jarrow-Morton Framework

Course: Fixed Income Derivatives

Instructor: Jeff Greco

Due Date: May 26, 2016

1 One-Factor HJM Computations

Assume we have a one-factor HJM model with deterministic volatility. Compute the following quantities with respect to the risk-neutral measure $\mathbb{Q}$.

1. The expectation of the short rate, i.e.
$$\mathbb{E}^{\mathbb{Q}}\left[r_t\right].$$
2. The expectation of the bond, i.e.
$$\mathbb{E}^{\mathbb{Q}}\left[P_t^T\right].$$
3. The variance of the bond, i.e.
$$\mathrm{Var}\left[P_t^T\right].$$

2 Girsanov's Theorem

Show that the mapping $\mathbb{Q}$ as defined in Girsanov's Theorem in the lecture notes satisfies the following properties:

1. $\mathbb{Q}$ is a measure on $(\Omega, \mathcal{F})$.
2. $\mathbb{Q}$ is equivalent to $\mathbb{P}$.

3 Replicating Strategy

Assume the single-factor Heath-Jarrow-Morton framework with the same notations as the lecture notes. Fix the times $S > T > 0$ and consider the strategy

$$\Pi_t = \phi_t P_t^S + \psi_t B_t$$

where

$$\phi_t = \frac{\Sigma_t^T}{\Sigma_t^S} \cdot \frac{P_t^T}{P_t^S} \qquad \text{and} \qquad \psi_t = \frac{\Sigma_t^S - \Sigma_t^T}{\Sigma_t^S} \cdot \frac{P_t^T}{B_t}$$

for $0 \leq t \leq T$.

Show that the portfolio is a replicating strategy for the T-maturity bond P_t^T, provided there are no arbitrage opportunities.

Hint: to check whether it is self-financing, remember that

$$dZ_t^T = Z_t^T \Sigma_t^T \left(dW_t + \left(\frac{1}{2}\Sigma_t^T - \frac{A_t^T}{\Sigma_t^T} \right) dt \right)$$

where W_t is a Brownian motion under the real world measure $\mathbb{P}$. Then apply Ito's lemma to $P_t^T = B_t Z_t^T$ and $P_t^S = B_t Z_t^S$.