Assignment 8: Forward Measure and Change of Numeraire

Course: Fixed Income Derivatives
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Due Date: June 2^{nd}, 2015

This is an individual assignment.

Where applicable, assume the setup from the lecture notes "Forward Measure and Change of Numeraire".

1. Consider a random variable X with the following values and corresponding probabilities:

$$X = 1,\ p(X = 1) = 0.2$$
$$X = 0.2,\ p(X = 0.2) = 0.3$$
$$X = -0.5,\ p(X = -0.5) = 0.5$$

a) Calculate the mean and the variance of this random variable.

b) Change the mean of this random variable to 0.05 by subtracting an appropriate constant from X, that is, calculate μ such that $Y = X - \mu$ has mean 0.05.

c) Has the variance changed?

d) Now do the same transformation by changing the probabilities, so that the variance remains constant.

e) Have the values of X changed?

2. **Dynamic Measure Change - Change of Expectation**

Let Z_t be a positive martingale with initial value 1. For all times T define a measure $\widetilde{\mathbb{Q}}_T$ on the σ-algebra $\mathcal{F}_T$ by the standard formula

$$\widetilde{\mathbb{Q}}_T(A) \triangleq \mathbb{E}\left[Z_T \times 1_A\right].$$

Show that the expected value of a claim X under the measure $\widetilde{\mathbb{Q}}_T$ is given by:

$$\widetilde{\mathbb{E}}\left[X\right] = \mathbb{E}\left[Z_T X\right].$$

Moreover, show that if X *is* $\mathcal{F}_t$*-measurable with* $0 \leq t \leq T$, then its expected value is given by:

$$\widetilde{\mathbb{E}}\left[X\right] = \mathbb{E}\left[Z_t X\right].$$

3. **Bayes' Rule**

For the measure $\widetilde{\mathbb{Q}}$ defined in the previous problem, prove that if $0 \leq t \leq T$ and ξ is an $\mathcal{F}_T$-measurable random variable satisfying $\widetilde{\mathbb{E}}\left[|\xi|\right] < +\infty$, then

$$\widetilde{\mathbb{E}}\left[\xi|\mathcal{F}_t\right] = Z_t^{-1}\mathbb{E}\left[Z_T\xi|\mathcal{F}_t\right]. \tag{1}$$

Hint: by definition of conditional expectation, $Y = \widetilde{\mathbb{E}}[\xi|\mathcal{F}_t]$ is the $\mathcal{F}_t$-measurable random variable that satisfies

$$\widetilde{\mathbb{E}}[1_A Y] = \widetilde{\mathbb{E}}[1_A \xi]$$

for all $A \in \mathcal{F}_t$. Show that the expression on the right-hand side of (1) satisfies this relation.

4. Forward Measure Pricing Formula

Prove that if X is an $\mathcal{F}_T$-measurable payoff, then

$$\pi_t(X) = P(t,T)\,\mathbb{E}^T[X|\mathcal{F}_t]$$

where $\mathbb{E}^T$ is the expectation operator with respect to the T-forward measure $\mathbb{Q}^T$.

Hint: apply Bayes' Rule.

5. Forward Prices are Martingales Under Forward Measure

Let A_t be the time t price of a traded asset. Let $F_A(t,T)$ denote its forward price at time t for settlement at time T. Show the following:

1. The forward price can be computed as $F_A(t,T) = \mathbb{E}^T[A_T|\mathcal{F}_t]$.
2. The forward price process $\{F_A(t,T)\}_{t=0}^T$ is a martingale under the T-forward measure $\mathbb{Q}^T$.

6. Forward Measure in Gaussian HJM

Show that a bond with maturity time $S < T$ follows the SDE

$$dP(t,S) = (r(t) + \Sigma(t,T)\Sigma(t,S))\,P(t,S)\,dt + P(t,S)\,\Sigma(t,S)\,dW_t^T$$

and the instantaneous forward rate with maturity S follows the SDE

$$df(t,S) = (\Sigma(t,T) - \Sigma(t,S))\,\sigma(t,S)\,dt + \sigma(t,S)\,dW_t^T$$

under the T-forward measure $\mathbb{Q}^T$.

7. Consider the Black-Scholes setting applied to foreign currency denominated assets. Let r and f denote the domestic and foreign risk-free rates, respectively. Let S_t be the exchange rate, that is, the price of 1 unit of foreign currency in terms of domestic currency. Assume a geometric process for the dynamics of S_t under probablity measure $\mathbb{P}$:

$$dS_t = (r - f)\,S_t dt + \sigma S_t dW_t, \quad \sigma = \text{ const.}$$

a) Show that

$$S_t = S_0 e^{\left(r-f-\frac{1}{2}\sigma^2\right)t+\sigma W_t},$$

where W_t is a Brownian motion under probablity measure $\mathbb{P}$, is a solution of the above SDE.

b) Is the process

$$\frac{S_t e^{ft}}{S_0 e^{rt}} = e^{\sigma W_t - \frac{1}{2}\sigma^2 t}$$

a martingale under measure $\mathbb{P}$?

c) Let $\widetilde{\mathbb{P}}$ be a new probability measure defined by:

$$\widetilde{\mathbb{P}}(A) = \int_A e^{\sigma W_T - \frac{1}{2}\sigma^2 T} d\mathbb{P}$$

What does Girsanov's theorem imply about the process $(W_t - \sigma t)$ under $\widetilde{\mathbb{P}}$?