CVA Recipe for a Replicating Strategy

HJM

1) Define martingale

$$V_t = \mathbb{E}[B_S^{-1} X \mid \mathcal{F}_t].$$

2) Use Martingale Representation Thm.

$$dV_t = \phi_t \, dZ_t^T$$

3) Build strategy

$$\Pi_t = \phi_t P_t^T + \psi_t B_t$$

with $\psi_t = V_t - \phi_t Z_t^T$.

4) Verify Π_t replicates contingent claim with $\mathcal{F}_S$-measurable payoff X.

(absent possibility of default / CSA)

CVA

1) Define martingale

$$M_t = \mathbb{E}[\tilde{B}_S^{-1} X \mid \mathcal{F}_t].$$

2) Use M.R.T.

$$dM_t = \eta_t \, dZ_t^T$$

3) Build strategy

$$\tilde{\Pi}_t = \phi_t P_t^T + \psi_t B_t + \tilde{\psi}_t \tilde{B}_t$$

with $\phi_t = B_t^{-1} \tilde{B}_t \eta_t$, $\psi_t = -\phi_t Z_t^T$,

and $\tilde{\psi}_t = M_t$.

CVA Replicating Strategy

Verify $\tilde{\Pi}_t$ replicates the risky claim on $\mathcal{F}_s$-measurable payoff X, including exact replication of bilateral counterparty default risks!

$$\tilde{\Pi}_t = \phi_t P_t^T + \underbrace{\psi_t}_{=-\phi_t Z_t^T} B_t + \underbrace{\tilde{\psi}_t}_{=M_t} \tilde{B}_t = \phi_t P_t^T - \phi_t \underbrace{Z_t^T}_{=B_t^{-1} P_t^T} B_t + M_t \tilde{B}_t$$

$$= \cancel{\phi_t P_t^T} - \cancel{\phi_t B_t^{-1} P_t^T B_t} + M_t \tilde{B}_t = M_t \tilde{B}_t$$

$$\tilde{\Pi}_s = \underbrace{M_s}_{=\mathbb{E}[\tilde{B}_s^{-1} X \mid \mathcal{F}_s] = \tilde{B}_s^{-1} X} \tilde{B}_s = \cancel{\tilde{B}_s^{-1}} X \cancel{\tilde{B}_s} = X$$

$\boxed{\text{replicates the } \mathcal{F}_s\text{-measurable \underline{payoff} } X}$

CVA Replicating Strategy

$$d\tilde{\Pi}_t = d(M_t \tilde{B}_t)$$

$$= \tilde{B}_t \underbrace{dM_t}_{=\eta_t dZ_t^T} + M_t d\tilde{B}_t$$

$$= \tilde{B}_t \underbrace{\eta_t}_{=\tilde{B}_t^{-1} B_t \phi_t} dZ_t^T + M_t d\tilde{B}_t$$

$$= \cancel{\tilde{B}_t \tilde{B}_t^{-1}} B_t \phi_t \, dZ_t^T + M_t \, d\tilde{B}_t = \phi_t \Big(d(\overbrace{B_t Z_t^T}^{=P_t^T}) - Z_t^T dB_t \Big) + M_t d\tilde{B}_t$$

$$= \phi_t dP_t^T \underbrace{- \phi_t Z_t^T}_{=\psi_t} dB_t + \underbrace{M_t}_{=\tilde{\psi}_t} d\tilde{B}_t = \phi_t dP_t^T + \psi_t dB_t + \tilde{\psi}_t d\tilde{B}_t$$

remember:

$$d(Y_t B_t) = B_t dY_t + Y_t dB_t$$

and likewise

$$d(Y_t \tilde{B}_t) = \tilde{B}_t dY_t + Y_t d\tilde{B}_t$$

for any Y_t

Strategy is self-financing!

CVA Replicating Strategy — Bilateral Counterparty Default Risks

What about counterparty default exposures? Does it replicate those?

$$\tilde{\Pi}_t = \phi_t P_t^T + \psi_t B_t + \tilde{\psi}_t \tilde{B}_t$$
$$= \phi_t P_t^T - \phi_t P_t^T + \tilde{\Pi}_t$$

$$d\tilde{\Pi}_t = \phi_t \, dP_t^T + \psi_t \, dB_t + \tilde{\psi}_t \, d\tilde{B}_t$$
$$= \phi_t \, dP_t^T + r_t \psi_t B_t \, dt + \tilde{r}_t \tilde{\psi}_t \tilde{B}_t \, dt$$
$$= \phi_t \, dP_t^T - r_t \phi_t P_t^T \, dt + \tilde{r}_t \tilde{\Pi}_t \, dt$$

The bond position is used to secure a portion of the strategy's funding (funded at the secured rate r_t). The balance is funded unsecured at rate $\tilde{r}_t$.

Upon default secured lender seizes bond collateral if necessary, avoiding losses on secured loan. Counterparty loses strategy value (if positive) less recovery. Yes, it replicates default exposure too!!

Forward CVA

$$\tilde{F}_t^S(X) = \tilde{\mathbb{E}}^S\left[X \mid \mathcal{F}_t\right] = \mathbb{E}^S\left[\frac{\Gamma_s^S}{\Gamma_t^S} X \mid \mathcal{F}_t\right]$$

where $\Gamma_t^S = \dfrac{\tilde{P}_t^S B_t}{P_t^S \tilde{B}_t} = \dfrac{\cancel{\tilde{B}_t}\, \mathbb{E}\left[\tilde{B}_s^{-1} \mid \mathcal{F}_t\right] B_t}{P_t^S \cancel{\tilde{B}_t}} = \dfrac{\cancel{P_t^S}\, \mathbb{E}^S\left[B_s \tilde{B}_s^{-1} \mid \mathcal{F}_t\right]}{\cancel{P_t^S}}$

is a P^S-Martingale.

$$\mathbb{E}^S\left[\frac{\Gamma_s^S}{\Gamma_t^S} \,\Big|\, \mathcal{F}_t\right] = \mathbb{E}^S\left[\frac{\Gamma_s^S}{\mathbb{E}^S\left[\Gamma_s^S \mid \mathcal{F}_t\right]} \,\Big|\, \mathcal{F}_t\right] = \frac{\mathbb{E}^S\left[\Gamma_s^S \mid \mathcal{F}_t\right]}{\mathbb{E}^S\left[\Gamma_s^S \mid \mathcal{F}_t\right]} = 1$$

Futures Prices are $\mathbb{Q}$-Martingales

Let A_t denote the time t value of a futures margin account corresponding to long futures contract) with futures price $\mathcal{F}_t^S(X)$.

Then

all assets grow at rate r_t under $\mathbb{Q}$ measure.

$$dA_t = r_t A_t\,dt + d\mathcal{F}_t^S(X)$$

$$d(B_t^{-1}A_t) = B_t^{-1}dA_t - r_t B_t^{-1}A_t\,dt = B_t^{-1}\,d\mathcal{F}_t^S(X)$$

$$d\mathcal{F}_t^S(X) = B_t\,d(B_t^{-1}A_t)$$

$\boxed{B_t^{-1}A_t \text{ is a } \mathbb{Q}\text{-Martingale}}$

CDS Pricing

Portfolio CVA price is $V_t = \mathbb{E}\left[\int_t^T e^{-\int_t^S \tilde{r}_u \, du} \, dX_S \,\middle|\, \mathcal{F}_t\right]$.

$$\mathbb{E}\left[\int_t^T e^{-\int_t^S r_u \, du} \delta_S V_S \lambda_S e^{-\int_t^S \lambda_u \, du} \, dS \,\middle|\, \mathcal{F}_t\right]$$

$$= \mathbb{E}\left[\int_t^T e^{-\int_t^S r_u \, du} \delta_S \left(\int_S^T e^{-\int_S^U \tilde{r}_u \, du} \, dX_U\right) \lambda_S e^{-\int_t^S \lambda_u \, du} \, dS \,\middle|\, \mathcal{F}_t\right]$$

$$= \mathbb{E}\left[\int_t^T \int_t^U e^{-\int_t^S r_u \, du - \int_S^U \tilde{r}_u \, du} e^{-\int_t^S \lambda_u \, du - \int_S^U \lambda_u \, du} \delta_S \lambda_S e^{\int_S^U \delta_u \lambda_u \, du} \, dS \, dX_U \,\middle|\, \mathcal{F}_t\right]$$

$$= \mathbb{E}\left[\int_t^T e^{-\int_t^U r_u \, du} e^{-\int_t^U \lambda_u \, du} \left(\int_t^U \delta_S \lambda_S e^{\int_S^U \delta_u \lambda_u \, du} \, dS\right) dX_U \,\middle|\, \mathcal{F}_t\right]$$

$$= \mathbb{E}\left[\int_t^T e^{-\int_t^U r_u \, du} e^{-\int_t^U \lambda_u \, du} \left(e^{\int_t^U \delta_u \lambda_u} - 1\right) dX_U \,\middle|\, \mathcal{F}_t\right]$$