Pricing Collateralized Interest Rates Derivatives, FVA

Fixed Income Derivatives Workshop

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Outline

- Classical framework
  1. The role of risk-free interest rate
  2. Where does the risk-free interest rate come from
- Mechanics of Collateral
- Concept of CVA, DVA and FVA
- CSA and Perfect CSA
- Pricing under Perfect CSA
- Forward measure
- Effect on rates
- Discussion on FVA

The Mysterious Risk-Free Interest Rate

Let $X(t)$ be a single asset (stock) price and $W^{\mathbb{P}}(t)$ a standard Brownian motion in the real world measure $\mathbb{P}$. Then

$$dX(t) = \mu^{\mathbb{P}}(t, X)\, X(t)\, dt + \sigma(t, X)\, dW^{\mathbb{P}}(t).$$

The process for the derivative price $\Pi(t, X)$ is

$$\begin{aligned}
\mathfrak{D}_r \Pi(t, X) &= r(t)\, \Pi(t, X), \\
\mathfrak{D}_r &= \frac{\partial}{\partial t} + r(t)\, X(t) \frac{\partial}{\partial X} + \frac{1}{2}\sigma^2(t)\, X^2(t) \frac{\partial^2}{\partial X^2}, \\
\Pi(t, X) &= \mathbb{E}_t^{\mathbb{Q}}\left[D(t, T)\, \Pi(T, X)\right], \\
D(t, T) &= \exp\left\{-\int_t^T r(u)\, du\right\},
\end{aligned}$$

where $\mathbb{Q}$ is the risk-neutral probability measure such that

$$dX(t) = r(t)\, X(t)\, dt + \sigma(t, X)\, dW^{\mathbb{Q}}(t).$$

What is risk-free rate $r(t)$?

What is Risk-Free rate?

Candidates [2, Page 77]:

1. The safest asset: Treasury, but for tax reasons Treasury rate is too low;
2. The most liquid and convenient index: LIBOR, a short term borrowing rate between AA-rated financial institutions (banks); LIBOR curve is smooth and liquid. Several serious issues with Libor as a candidate for a risk-free rate have been realized in the last several years.
3. The shortest exposure: Overnight Indexed Swap (OIS) is more correct candidate, since collateral is typically in the form of cash and OIS rate is the return that it has; Collateralized agreement has no risk;

- Prior to 2007 market convention made LIBOR the risk-free rate.
- Basis spread (LIBOR-OIS) was small and stable, but after 2007 it widened out from 2 b.p. to 200 b.p.
- Collateralization and central clearing of swaps is becoming a standard (CSA). This makes OIS the best candidate in the post-crisis era.

How Collateral Works

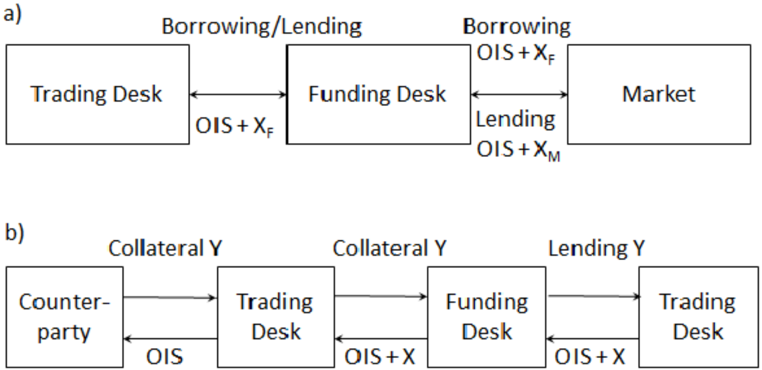

Figure: a) Unsecured Borrowing/Lending from/to Funding Desk; b) Processing of Collateral

How Collateral Works

- Counterparty has a choice of collateral, such as cash, sovereign debt securities, the underlying asset, etc.
- Collateral in the form of cash generates X spread for Trading Desk
- Unsecured borrowing internally costs Trading Desk X spread
- Assume that at the date of initial trading $t_0 = 0$ the value V_0 of the contract is zero and at time $t > 0$ it can become either positive V^+ or negative V^- from the point of view of Trading Desk. Then for the predefined thresholds H and H':

   1. Mark to market $V_t = V^+ > 0$ results in Counterparty posting collateral $C = \max\left(V^+ - H', 0\right)$ to Trading Desk
   2. Mark to market $V_t = V^- < 0$ results in Trading Desk posting collateral $C = -\max\left(-V^- - H, 0\right)$ to Counterparty

   After collateral is posted at t there is funding cost for $[t, t + \Delta t]$ of approximately (not accounting for possibility of default) $C \times D_t \times X \times \Delta t$; D_t means discounting to t.

From the point of view of Trading Desk we can define the following credit situations and simplified CVA and DVA:

1. Mark to market $V_t = V^+ > 0$ creates the credit situation:
   Amount $\min\left(V^+, H'\right)$ is at risk to Trading Desk;
   $CVA = \min\left(V^+, H'\right) \times P'_d \times (1 - R')$, where with P'_d and R' we denote probability of default and recovery rate by Counterparty, correspondingly
2. Mark to market $V_t = V^- < 0$ creates the credit situation:
   Amount $\min\left(-V^-, H\right)$ is at risk to Counterparty;
   $DVA = \min\left(-V^-, H\right) \times P_d \times (1 - R)$, where with P_d and R we denote probability of default and recovery rate by Trading Desk, correspondingly.

Trading Desk borrows \$1 from Funding Desk, lends \$1 to Counterparty and receives \$1 as collateral from Counterparty.
All cash flows are shown in the following table:

Transaction	$t = 0$	$t = T$
Borrow \$1 from Funding Desk	$-\$1$	$-\$e^{r_f T}$
Collateral from Counterparty	$-\$1$	$-\$e^{r_c T}$
Deposit Collateral at Funding Desk	$\$1$	$\$e^{r_f T}$

where r_c is the rate earned by collateral (OIS rate) and r_f is the funding rate at which Trading Desk lends and borrows from Funding Desk.
At time T Counterparty returns Trading Desk the amount $\$e^{r_c T}$, i.e. the discounting rate for collateralized cash flow is r_c.
The value of holding the collateral is $V_c^T = e^{r_f T} - e^{r_c T}$;
$V_c^0 = e^{(r_f - r_c)T} - 1$, or for short time interval

$$dV_c = (r_f - r_c)\, dt$$

Trading Desk borrows \$1 from Funding Desk, lends \$1 to Counterparty and receives $\$\delta$ as collateral from Counterparty.
All cash flows are shown in the following table:

Transaction	$t=0$	$t=T$
Borrow \$1 from Funding Desk	$-\$1$	$-\$e^{r_f T}$
Collateral from Counterparty	$-\$\delta$	$-\$\delta e^{r_c T}$
Deposit Collateral at Funding Desk	$\$\delta$	$\$\delta e^{r_f T}$

At time T the cash flow from Counterparty to Trading Desk is

$$V^T = \delta e^{r_c T} - \delta e^{r_f T} + e^{r_f T} = \delta e^{r_c T} + (1-\delta)\, e^{r_f T},$$

i.e. the collateralized part δ grows at the rate r_c and non-collateralized part $(1-\delta)$ grows at the rate r_f.

Examples

Lending to Counterparty (Buying Bond), Partially Collateralized, Credit Consideration

The difference from the previous example is that Counterparty may default.

Transaction	$t = 0$	$t = T$
Borrow \$1 from Funding Desk	$-\$1$	$-\$e^{r_f T}$
Collateral from Counterparty	$-\$\delta$	$-\$\delta e^{r_c T}$
Deposit Collateral at Funding Desk	$\$1$	$\$\delta e^{r_f T}$
Lost from Counterparty default		$-\left(1-\delta\right) e^{r_f T}\left(1-e^{-uT}\right)$

where u is the Counterparty credit spread, $\left(1-e^{-uT}\right) = \left(1-R'\right) P'_d$, R' is the recovery rate and P'_d is the default probability by Counterparty. Then the final cash flow of Counterparty is

$$
\begin{aligned}
V^T &= \delta e^{r_c T} + \left(1-\delta\right) e^{r_f T} - \left(1-\delta\right) e^{r_f T}\left(1-e^{-uT}\right) \\
&= \delta e^{r_c T} + \left(1-\delta\right) e^{\left(r_f+u\right) T},
\end{aligned}
$$

i.e. the collateralized part δ grows at the rate r_c and non-collateralized part $\left(1-\delta\right)$ grows at the rate $\left(r_f+u\right)$.

FVA Definition

FVA stands for Funding Value Adjustment.
It is defined as the difference between the derivative product value and the same value under the assumption of full collateralization

$$FVA = V - V^{FC},$$

where V is the value of the derivative with partial or no collateral and V^{FC} is the value of the same derivative under full collateralization.
From the examples on previous slides we can find FVA in case of lending \$1 to Counterparty when the amount of collateral is $\$\delta$:

$$\begin{aligned} FVA &= -\delta e^{r_c T} - (1-\delta)\, e^{r_f T} + e^{r_c T} \\ &= -(1-\delta)\left(e^{r_f T} - e^{r_c T}\right) \end{aligned}$$

For a small time increment

$$FVA = -(1-\delta)\,(f_f - r_c)\, dt$$

Funding Account

There are different ways of borrowing funds for trading.
Let $B_\alpha(t)$ be a funding account, where α indicates the source of funding.
Let $R_\alpha(T_1)$ be the funding rate fixed at T_1 and accrued during $[T_1, T_2]$.
The funding interest paid at T_2 is then

$$B_\alpha(T_2) - B_\alpha(T_1) = B_\alpha(T_1)\, R_\alpha(T_1)\, \tau(T_1, T_2, dc_\alpha),$$

where dc_α is the day count convention of the funding source α.
Assume the funding account dynamics based on short funding rate r_α

$$\begin{aligned} dB_\alpha(t) &= r_\alpha(t)\, B_\alpha(t)\, dt,\; B_\alpha(0) = 1 \\ B_\alpha(t) &= \exp\left\{ \int_0^t r_\alpha(u)\, du \right\} \end{aligned}$$

Two main funding sources are:

- Generic account B_f for unsecured borrowing at rate r_f, typically LIBOR plus spread.
- Collateral account B_c with collateral rate r_c, typically overnight.

Perfect CSA

The rules for posting and maintaining of collateral are defined by CSA: Credit Support Annex of the ISDA Standard Master Agreement.

Definition

Define "Perfect CSA" as an ideal collateral agreement with:

- Zero initial margin or deposit;
- Fully symmetric;
- Cash collateral;
- Zero threshold;
- Zero minimum transfer amount;
- Continuous margination;
- Instantaneous margination rate $r_c(t)$;
- Instantaneous settlement.

For perfect CSA $B_c(t) = \Pi(t)$ for all $t \leq T$

Perfect CSA

Perfect CSA, although an idealized version, is a reasonably good approximation to the new ISDA Standard Credit Support Annex (SCSA). The main features of SCSA are:

- Daily margination frequency
- Flat overnight margination rate
- Zero threshold
- Zero minimum transfer amount
- Cash collateral in the same currency as the trade
- Collateral asymmetry

The major financial institutions participating in OTC interest rate derivatives markets are covered by mutual collateral agreements. Thus, market quotations of derivatives, such as FRAs, swaps, basis swaps and OIS reflect collateralized transactions.

Pricing Under Perfect Collateral

Following [1, pp. 113-152] let Π be a derivative with maturity T written on a single asset X following the process under the real world measure $\mathbb{P}$

$$dX(t) = \mu^{\mathbb{P}}(t, X) X(t)\, dt + \sigma(t, X)\, dW^{\mathbb{P}}(t).$$

Assuming "Perfect Collateral" the price $\Pi(t, X)$ at $t < T$ follows

$$\begin{aligned}
\mathfrak{D}_{r_f} \Pi(t, X) &= r_c(t)\, \Pi(t, X), \\
\mathfrak{D}_{r_f} &= \frac{\partial}{\partial t} + r_f(t) X(t) \frac{\partial}{\partial X} + \frac{1}{2} \sigma^2(t) X^2(t) \frac{\partial^2}{\partial X^2}, \\
\Pi(t, X) &= \mathbb{E}_t^{\mathbb{Q}} \left[D_c(t, T)\, \Pi(T, X) \right], \\
D_c(t, T) &= \exp\left\{ -\int_t^T r_c(u)\, du \right\},
\end{aligned}$$

where $\mathbb{Q}_f$ is such that

$$dX(t) = r_f(t) X(t)\, dt + \sigma(t, X)\, dW^{\mathbb{Q}_f}(t)$$

Forward Measure

The "Perfect Collateral" assumption changes the Black-Scholes-Merton framework in how the cash used to replicate the derivative payoff is split between the sources of funding: the cash in the collateral account B_c provides secured funding of the derivatives position, $\Pi(t, X) = B_c(t)$, while the hedge $\left(\frac{\partial \Pi}{\partial X}\right) X(t)$ is funded (unsecured) by B_f.

Discounting and forward measure are associated with collateral rate $r_c(t)$ "Perfect Collateral" [1, p. 123]:

Theorem

The pricing expression

$$
\begin{aligned}
\Pi(t, X) &= P_c(t, T)\, \mathbb{E}_t^{\mathbb{Q}_f^T}\left[\Pi(T, X)\right] \\
P_c(t, T) &= \mathbb{E}_t^{\mathbb{Q}_f}\left[D_c(t, T)\right]
\end{aligned}
$$

holds, where $\mathbb{Q}_f^T$ is the probability measure associated with collateral zero-coupon bond $P_c(t, T)$.

Effect of Collateral on Rates

Denote by $L_m\left(T_{i-1}, T_i\right) \triangleq L_{m,i}$ the spot LIBOR rate fixed at T_{i-1} and covering the period $\left[T_{i-1}, T_i\right]$, where m indexes the tenor of the rate.
Consider "standard" or "textbook" FRA with payoff

$$FRA_{Std}\left(T_i, L_{m,i}, K, \omega\right) = \omega\left[L_{m,i} - K\right]\tau_L\left(T_{i-1}, T_i\right)$$

where $\omega = \pm 1$ for a payer/receiver FRA and we assume that both LIBOR and fixed rate K have same simply compounded annual convention and year fraction τ_L.
Under "Perfect Collateral" the FRA is priced

$$\begin{aligned}
FRA_{Std}\left(t, L_{m,i}, K, \omega\right) &= P_c\left(t, T_i\right)\mathbb{E}_t^{\mathbb{Q}_f^{T_i}}\left[FRA_{Std}\left(T_i, L_{m,i}, K, \omega\right)\right] \\
&= \omega P_c\left(t, T_i\right)\left\{\mathbb{E}_t^{\mathbb{Q}_f^{T_i}}\left[L_{m,i}\right] - K\right\}\tau_L\left(T_{i-1}, T_i\right) \\
&= \omega P_c\left(t, T_i\right)\left[F_{m,i}(t) - K\right]\tau_L\left(T_{i-1}, T_i\right)
\end{aligned}$$

where $F_{m,i}(t) \triangleq \mathbb{E}_t^{\mathbb{Q}_f^{T_i}}\left[L_{m,i}\right]$ is the equilibrium FRA rate.

Properties of FRA rate

The FRA rate $F_{m,i}(t) \triangleq \mathbb{E}_t^{\mathbb{Q}_f^{T_i}}[L_{m,i}]$ has the following properties:

1. At fixing date T_{i-1} $F_{m,i}(t)$ collapses onto the spot LIBOR rate

$$F_{m,i}(T_{i-1}) = \mathbb{E}_{T_{i-1}}^{\mathbb{Q}_f^{T_i}}[L_m(T_{i-1}, T_i)] = L_m(T_{i-1}, T_i)$$

2. By definition $F_{m,i}(t)$ is a martingale under $\mathbb{Q}_f^{T_i}$

$$F_{m,i}(t) = \mathbb{E}_t^{\mathbb{Q}_f^{T_i}}[L_m(T_{i-1}, T_i)] = \mathbb{E}_t^{\mathbb{Q}_f^{T_i}}[F_{m,i}(T_{i-1})]$$

3. In the single-curve limit $F_{m,i}(t)$ turns into the classical single-curve forward LIBOR rate

$$\begin{aligned} F_{m,i}(t) &\triangleq \mathbb{E}_t^{\mathbb{Q}_f^{T_i}}[L_m(T_{i-1}, T_i)] \longrightarrow \mathbb{E}_t^{\mathbb{Q}^{T_i}}[L_m(T_{i-1}, T_i)] \\ &= \mathbb{E}_t^{\mathbb{Q}^{T_i}}[F(T_{i-1}, T_{i-1}, T_i)] = F(t, T_{i-1}, T_i) \end{aligned}$$

The Controversy of FVA

There is a broad argument going on in the industry regarding the role of FVA and its relation to CVA and DVA.
For example, the authors in [5] argue that only CVA and FVA need to be applied to prices of derivatives for the following reasons:

1. Interpretation of DVA as a "gain from self-default" is artificial
2. Hedging DVA requires trading the your own debt which may be problematic
3. The effect of DVA is already included in FVA, so applying both is not necessary

To counter these arguments, the authors of [6] make point that FVA should only be used internally and not included in quoting prices to the market.
We analyze the arguments of [6] in more details below.

The approach used by a bank to value derivatives is to first calculate the price assuming no risk of default by either side and then make adjustments for expected gain or loss due to default.

These adjustments are referred to as CVA and DVA.

The CVA for the counterparty is loss to the bank arising from the possibility of the counterparty default.

The DVA is the gain to the bank (loss to the counterparty) from the possibility that the bank defaults.

The basis of the argument is that the expected return on an asset should reflect the risk of the asset.

If an asset is riskless, it should earn the risk-free return of return, regardless of how it is financed.

Let F be the derivative on the stock X, $dX = \mu X dt + \sigma X dW$.
By Ito's lemma

$$dF = \mu_F F dt + \sigma X \frac{\partial F}{\partial X} dW, \mu_F = \frac{1}{F}\left[\frac{\partial F}{\partial t} + \mu X \frac{\partial F}{\partial X} + \frac{1}{2}\sigma^2 X^2 \frac{\partial^2 F}{\partial X^2}\right]$$

For riskless rate r, market portfolio drift μ_M and stock's beta β_X by CAPM

$$\begin{aligned}
\mu &= r + \beta_X \left(\mu_M - r\right), \\
\mu_F &= r + \beta_F \left(\mu_M - r\right) = r + \frac{X}{F}\frac{\partial F}{\partial X}\beta_X \left(\mu_M - r\right), \beta_F = \frac{X}{F}\frac{\partial F}{\partial X}\beta_X, \\
rF &= \frac{\partial F}{\partial t} + rX \frac{\partial F}{\partial X} + \frac{1}{2}\sigma^2 X^2 \frac{\partial^2 F}{\partial X^2}, F_{nd}\left(X_t, t\right) = e^{-r(T-t)}\mathbb{E}_r\left[F\left(X_T, T\right)\right],
\end{aligned}$$

where $F_{nd}\left(X_t, t\right)$ is the no-default solution to the Black-Scholes differential equation and $\mathbb{E}_r$ corresponds to the risk-neutral measure.

Assume that the derivative is an asset to the bank, i.e. it always has positive value.

$$dF = \mu_F F dt + \sigma X \frac{\partial F}{\partial X} dW - \gamma F dN,$$

dN is a jump default process. The probability of jump in the next period dt is $\lambda_c dt$, λ_c is the counterparty's hazard rate. The size of the jump is 1 and γ is the reduction in the value of the derivative in the event of default.

$$\mathbb{E}\left[\frac{dF}{F}\right] = \left(\mu_F - \gamma\lambda_c\right) dt.$$

This results in the modified Black-Scholes equation

$$\begin{aligned} r_c F &= \frac{\partial F}{\partial t} + rX\frac{\partial F}{\partial X} + \frac{1}{2}\sigma^2 X^2 \frac{\partial^2 F}{\partial X^2} = \left(r + \gamma\lambda_c\right) F \\ F\left(X_t, t\right) &= e^{-r_c(T-t)}\mathbb{E}_r\left[F\left(X_T, T\right)\right] \end{aligned}$$

Define $CVA \triangleq F_{nd}\left(X_t, t\right) - F\left(X_t, t\right)$; $F\left(X_t, t\right) = F_{nd}\left(X_t, t\right) - CVA$.

In this case the derivative has always negative value to the bank.
Let λ_b be the hazard rate of the bank, r_b - the borrowing rate of the bank.
Then the Black-Scholes equation is

$$\begin{aligned} r_b F &= \frac{\partial F}{\partial t} + rX\frac{\partial F}{\partial X} + \frac{1}{2}\sigma^2 X^2 \frac{\partial^2 F}{\partial X^2} \\ F(X_t, t) &= e^{-r_b(T-t)}\mathbb{E}_r\left[F(X_T, T)\right] \end{aligned}$$

Define DVA as

$$\begin{aligned} DVA &\circeq F(X_t, t) - F_{nd}(X_t, t) \\ F(X_t, t) &= F_{nd}(X_t, t) + DVA \end{aligned}$$

In this case the derivative may be positive or negative and no collateral is required.
The differential equation is generalized to

$$\frac{\partial F}{\partial t} + rX\frac{\partial F}{\partial X} + \frac{1}{2}\sigma^2 X^2 \frac{\partial^2 F}{\partial X^2} = \left(r_c \max\left(F, 0\right) + r_b \min\left(F, 0\right)\right) F$$

Fully collateralized derivative corresponds to the case $r_c = r_b = l$.
Arguments similar to the previous cases lead to the result

$$F\left(X_t, t\right) = F_{nd}\left(X_t, t\right) - CVA + DVA$$

Let r_X be the rate at which the equity position in the hedge portfolio is financed (unsecured), and r_F is the rate of funding the derivative transaction.

The total portfolio derivative + hedge is expected to cost

$$d\Pi = \left[-r_F + r_X X \frac{\partial F}{dX}\right] dt$$

and the differential equation is

$$\begin{aligned} r_D F &= \frac{\partial F}{\partial t} + r_X X \frac{\partial F}{\partial X} + \frac{1}{2}\sigma^2 X^2 \frac{\partial^2 F}{\partial X^2}, \\ F\left(X_t, t\right) &= e^{-r_F(T-t)} \mathbb{E}_{r_X}\left[F\left(X_T, T\right)\right] \end{aligned}$$

where expectation is taken with respect to measure that makes r_X the risk free drift.

Define FVA as the difference $FVA = F\left(X_t, t\right) - F_{nd}\left(X_t, t\right)$.

Discussion on FVA: Hull and White Argument

Criticism of FVA I

- FVA ignores economic principle: investments that have the same risk should earn the same expected rate of return.
- Instead, FVA is based on the principle: hedged derivative should earn the bank's cost of debt funding.
- A key criticism: the bank's funding cost reflects the risk of the bank's existing portfolio of investments. If the risk taken by the derivatives desk is the same as average risk of all bank investments then the derivatives portfolio should earn the bank's weighted average cost of debt and equity funding. However, in all other cases the rate should be different.
- Merton approach: Abstracting from counterparty credit risk, the hedged derivative has very little risk and the appropriate discount rate should be the riskless rate.

Discussion on FVA: Hull and White Argument

Criticism of FVA II

- CAPM approach: If a derivative is not hedged the rate charged should reflect the unhedged risk.
- Both these principles lead to the same Black-Scholes price.
- Failing to reflect the risk level in pricing will likely lead to riskier positions of the trading desk.
- FVA is a private, bank specific valuation. The FVA-adjusted price is not an estimate of the fair market value.
- For reporting purposes the bank has to report not an internal value, but the best estimate of the fair market value.

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