LIBOR Market Models

Fixed Income Derivatives

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Pricing Interest Rate Derivatives

- In the idyllic risk-free model that was used before the market shift of 2007, interest rate derivatives were priced on a single interest rate curve.
- This curve was used for discounting and for all interest rates appearing in the payout, no matter what their tenor was.
- Even in this simple framework, exotic interest rate derivatives required very complex mathematical models for the dynamics of the term structure. Why?

1. The mathematical complexity arises mainly from the fact that **interest rates are not prices of tradable assets**. If they were, we could easily find their no-arbitrage dynamics along the lines of the first Black-Scholes model.
2. Another reason for the mathematical complexity is the fact that **the entire term curve is the underlying for any interest rate derivative**. In addition to pricing one particular derivative, we also have to be able to recover all the bond prices simultaneously.

Black's Model

- We know already that at the beginning, practitioners often applied slightly modified versions of the Black-Scholes-Merton model and option pricing formula to price other derivatives than stock options, including many fixed income securities.
- These modifications were based on Black (1976) [4] who adapted the Black-Scholes-Merton setting to the pricing of European options on commodity futures, hence the name of Black's model.
- And so, interest-rate options (caps, floors and swaptions) were valued using Black's model **without having a term-structure model**, assuming that the underlying forward rate process is lognormally distributed with zero drift.
- Black's model was well understood by all, and quickly became "industry standard" for these instruments.

- However, Black's model can allow arbitrage because it models just the dynamics of the underlying security, it does not model the entire term-structure.
- Therefore, **Black's model is not an interest rate model!**
- Consistent and arbitrage-free pricing of fixed income securities must be based on the dynamics of the entire term structure of interest rates because the entire term curve is the underlying "asset" for all fixed income securities.
- Moreover, Black's model for caps is derived under the assumption that LIBOR rates (corresponding to all caplets in the cap) are log-normally distributed. Likewise, Black's model for swaptions is derived under the assumption that the corresponding swap rate is log-normally distributed.
- However, in an arbitrage-free setting, forward rates over consecutive time intervals are related to one another and cannot all be lognormal under one arbitrage-free measure.

Interest Rate Models

- Academics felt uneasy about models that had not been derived from "first principles", so they applied the no-arbitrage paradigm to the interest-rate markets.
- This resulted in three main families of models for interest rate dynamics, each offering a different solution to the above issues, and making a different choice of fundamental modeling variables.

The First Generation of Interest Rate Models (Short-Rate Models)

- Was introduced in 1977 by Vasicek.
- Here there is only one modeling variable: the instantaneous spot rate (short rate).
- In a single-curve world, all interest rates can be seen as functions of bonds, and each bond can be seen as the expectation of a discount factor.
- The factor for discounting the value of money from T to t is defined as a function of the path of the short rate from t to T.
- Thus all rates are given by different functions of expectations of the future behavior of the short rate.
- These models do not guarantee that the prices of all bonds can be recovered, therefore they are not arbitrage-free.

The Second Generation of Interest Rate Models (HJM)

- Was introduced in 1992 by Heath, Jarrow and Morton.
- Here the modeling variables are instantaneous forward rates, whose no-arbitrage dynamics is in turn deduced from modeling the no-arbitrage dynamics of bonds for all possible maturities in a term structure.
- The difference between the short-rate models and the HJM models is striking: we move from one modeling variable to an infinite number of modeling variables.

The Third Generation of Interest Rate Models (LIBOR/Swap Market Models)

- In contrast to the previous two families, these take as modeling variables real-world rates rather than instantaneous rates.
- In terms of the number of variables, we are intermediate between the single variable of short-rate models and the infinity of variables of HJM.
- More importantly, instead of modeling theoretical instantaneous rates, we model real-world, discrete-tenor rates. This required quants to understand the correct no-arbitrage dynamics of these real-world rates.
- However, market models are just a reparameterization of the dynamics of interest rates in term of different variables. Underneath, there is still an HJM model at work.

Why HJM Was Not Well Received by Traders

- Even after reasearch culminated with the creation of the **HJM** framework, to everyone's dismay, traders kept using their beloved Black's model for valuing caps and swaptions. The reasons were clear:
- The "stochastic drivers" of Black's model (LIBOR and swap rates) were easily observable, and so were their volatilities.
- On the contrary, the stochastic drivers of HJM models (instantaneous forward rates) were not directly observable, and neither were their volatilities. A quant equipped with an HJM model was forced to constantly perform translations between observable quantities and his model's input parameters (a process known as calibration).
- The fact that practitioners kept applying the Black formula to caplets and swaptions despite its then perceived lack of theoretical standing $\Longrightarrow$ turned Black's formulas into a **market standard**.
- This has sparked vigorous research with the aim of identifying an arbitrage-free term structure model in which Black's formulas could be used for pricing caplets and swaptions.

A Model for Black's Formula

- So, academics gave up at some point trying to convince practitioners not to use Black's model and asked themselves this question instead:

Can we build models in which Black's formula is correct? (i.e., create an HJM, no-arbitrage model in which caps/swaptions would be valued using Black-like expressions, and where the input parameters would be directly observable on the market).

The answer they reached was: **Yes, and to do it we use the bond price as numéraire.**

- And so the breed of market models was born. Market models proved that the market practice can be made consistent with an arbitrage-free term structure model: Consecutive quarterly or semi-annual forward rates can all be lognormal while the model remains arbitrage free.

Bond Price As Numéraire I

Let $P(t, T)$ be the numéraire. The T-forward measure $\mathbb{Q}^T$ is given by the Radon-Nikodym derivative:

$$\left.\frac{d\mathbb{Q}^T}{d\mathbb{Q}}\right|_{\mathcal{F}_t} = \frac{P(t, T)/B_t}{P(0, T)/B_0}$$

At $t = T$:

$$\frac{d\mathbb{Q}^T}{d\mathbb{Q}} = \frac{P(T, T)/B_T}{P(0, T)/B_0} = \frac{1}{B_T P(0, T)}$$

- The $t = 0$ price of a call option is:

$$\begin{aligned} C_0 &= \mathbb{E}\left[\frac{(V_T - K)^+}{B_T}\right] = P(0, T)\,\mathbb{E}\left[\frac{(V_T - K)^+}{B_T P(0, T)}\right] \\ &= P(0, T)\,\mathbb{E}\left[\frac{d\mathbb{Q}^T}{d\mathbb{Q}}(V_T - K)^+\right] = P(0, T)\,\mathbb{E}^T\left[(V_T - K)^+\right] \\ &= P(0, T)\,\mathbb{E}^T\left[(F(T, T) - K)^+\right] \end{aligned}$$

Bond Price As Numéraire II

- Let's remember that Black's model assumes that the underlying is log-normally distributed at expiration of the option. Therefore, in order to calculate the value of an option using Black's formula, it would be enough to build a model in which the underlying is log-normally distributed under the T-forward measure.

Brownian Motion for Forward Measure I

• The bond price normalized by the money-market account must be a martingale under the risk-neutral measure $\mathbb{Q}$, and hence there exists a volatility process $\Sigma(t, T)$ for the bond (a process in t; T is fixed; $0 \leq t \leq T$) such that:

$$d\left(\frac{P(t,T)}{B_t}\right) = \Sigma(t,T)\left(\frac{P(t,T)}{B_t}\right) dW_t \Longrightarrow$$

• This SDE is one of the few that has an analytic solution. Its solution is called the Doléans exponential of Brownian motion:

$$\frac{P(t,T)}{B_t} = \frac{P(0,T)}{B_0} \exp\left\{\int_0^t \Sigma(s,T)\, dW_s - \frac{1}{2}\int_0^t (\Sigma(s,T))^2\, ds\right\}.$$

• Extract the Radon-Nikodym derivative of the T-forward measure w.r.t. the risk-neutral measure from above:

$$\frac{P(t,T)/B_t}{P(0,T)/B_0} = \left.\frac{d\mathbb{Q}^T}{d\mathbb{Q}}\right|_{\mathcal{F}_t} = \exp\left\{\int_0^t \Sigma(s,T)\, dW_s - \frac{1}{2}\int_0^t (\Sigma(s,T))^2\, ds\right\}$$

Brownian Motion for Forward Measure II

- Compare with ξ_t in Girsanov's theorem:

$$\Sigma(s, T) = -\theta_s$$

- Therefore, replacing θ_s from above in

$$W_t^T = W_t + \int_0^t \theta_s ds$$

we see that we can use the volatility process of the numeraire bond to adjust the drift of a Brownian motion under the risk-neutral measure $\mathbb{Q}$ so that we obtain a Brownian motion under the T-forward measure $\mathbb{Q}^T$:

$$W_t^T = W_t - \int_0^t \Sigma(s, T)\, ds$$

- We also see why the T-forward measure is identified by the condition that

$$dW_t^T = dW_t - \Sigma(t, T)\, dt$$

is a Brownian motion under $\mathbb{Q}^T$.

Forward Prices Under the Forward Measure

Let V_t be the value at time t of an asset, in units of domestic curency.
1) The value of this asset expressed in units of the numeraire is the forward price set at time t for delivery at time T of the asset V:

$$F(t, T) = \frac{V_t}{P(t, T)}.$$

2) The forward price for delivery at time T is the expected value of the time-T future value of the asset under the T-forward measure:

$$F(t, T) = \mathbb{E}^T [V_T].$$

3) The forward price for delivery at time T of any asset V is a martingale under the T-forward measure.

- We can therefore write its evolution as:

$$dF(t, T) = \gamma(t, T) F(t, T) dW_t^T, \quad 0 \leq t \leq T,$$

where $\gamma(t, T)$ is the volatility of the forward price and W_t^T is the Brownian motion associated with $\mathbb{Q}^T$.

- If $\gamma(t, T)$ is non-random, $F(t, T)$ will be log-normally distributed.

Forward Prices Under the Forward Measure

Example: Forward bond price under Gaussian HJM

• The forward bond price evolution is given under the T-forward measure by:

$$
\begin{aligned}
dF(t, T, M) &= (\Sigma(t, M) - \Sigma(t, T)) F(t, T, M)\, dW_t^T \\
&= \gamma(t, T, M) F(t, T, M)\, dW_t^T, \text{ where}
\end{aligned}
$$

$$\gamma(t, T, M) = \Sigma(t, M) - \Sigma(t, T).$$

• Forward bond prices are indeed log-normally distributed under the T-forward measure in Gaussian HJM (i.e., when $\Sigma(t, T)$ is deterministic).

Option Pricing with Random Interest Rates

Theorem

(Jamshidian (1989) J. Finance 44; Geman, El Karoui, Rochet (1995), J. Appl. Prob. 32)
Assume

$$dF(t,T) = \sigma F(t,T)\, dW_t^T$$

where σ is non-random (but may depend on T). The value at time zero of the call on V expiring at time T is given by:

$$\begin{aligned} C(0) &= P(0,T)\left[F(0,T)N(d_1) - KN(d_2)\right] \\ &= V(0)N(d_1) - KP(0,T)N(d_2), \end{aligned}$$

where

$$d_{1,2} = \frac{\log\frac{F(0,T)}{K} \pm \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}.$$

But this is **Black's formula**!

Option Pricing with Random Interest Rates

• No assumption is made directly on the interest rate. The only assumption is that the forward price of the underlying asset has non-random volatility σ and an SDE of a certain type.

• This is equivalent to forward prices being log-normal under the forward measure!

• The same conclusion holds in the case where the underlying is a forward rate, although it applies under a different forward measure.

• Therefore, in order to adapt the Black-Scholes formula for equity options to fixed income derivatives and obtain the Black caplet formula, e.g., it is necessary to build a model in which forward rates are log-normal.

Term-Structure Evolution Under Risk-Neutral Measure

In a term-structure model satisfying the HJM **no-arbitrage** condition, the forward rates evolve under the risk-neutral measure according to the equation:

$$df(t, T) = -\Sigma(t, T)\,\sigma(t, T)\,dt + \sigma(t, T)\,dW_t,$$

and the zero-coupon bond prices evolve according to the equation:

$$\begin{aligned} dP(t, T) &= r(t)\,P(t, T)\,dt + P(t, T)\,\Sigma(t, T)\,dW_t, \\ \frac{dP(t, T)}{P(t, T)} &= r(t)\,dt + \Sigma(t, T)\,dW_t, \end{aligned}$$

where W_t is a Brownian motion under the risk-neutral measure $\mathbb{Q}$, and

$$\frac{\partial \Sigma(t, T)}{\partial T} = -\sigma(t, T),$$

$\sigma(t, T)$	- the volatility structure of instantaneous forward rates
	- uniquely defines the HJM model
$r(t)$	- the short rate at time t

The Problem with Forward Rates in HJM

• The SDE for instantaneous forward rates in HJM is not of the type required by the previous Theorem, but it is instead an aritmetic process:

$$df\left(t,T\right)=-\Sigma\left(t,T\right)\sigma\left(t,T\right)dt+\sigma\left(t,T\right)dW_t$$

which does not imply a log–normal distribution. Actually, under Gaussian HJM, we know that instantaneous forward rates are normally-distributed.
• The presence of the drift term will sometimes cause forward rates to grow uncontrollably (explode).
• Although under the T-forward measure, instantaneous forward rates follow a driftless stochastic process, the SDE is still not of the right type:

$$\begin{aligned}
dW_t^T &= dW_t-\Sigma\left(t,T\right)dt \\
df\left(t,T\right) &= -\Sigma\left(t,T\right)\sigma\left(t,T\right)dt+\sigma\left(t,T\right)dW_t \\
&= -\Sigma\left(t,T\right)\sigma\left(t,T\right)dt+\sigma\left(t,T\right)\left[dW_t^T+\Sigma\left(t,T\right)dt\right] \\
&= \sigma\left(t,T\right)dW_t^T.
\end{aligned}$$

The Problem with Forward Rates in HJM

- Moreover, working directly with instantaneous forward rates is not particularly attractive in applications for a number of reasons:

1. Instantaneous forward rates are never quoted in the market, nor do they appear directly in the payoff definition of any traded derivative contract. Realistic securities (swaps, caps, futures, etc.) involve instead simply compounded (Libor) rates, which represent integrals of instantaneous forward rates.
2. An infinite set of instantaneous forward rates cannot, in general, be represented on a computer, but will require discretization into a finite set.
3. Prescribing the form of the volatility function of instantaneous forward rates is subject to a number of technical complications, requiring sub-linear growth to prevent explosions in the forward rate dynamics.

The Problem with Forward Rates in HJM

• As discovered in Miltersen et al. [1], Brace et al. [2], and Jamshidian [3], these three complications can be addressed simultaneously by simply reformulating the model in terms of a non-overlapping set of simply compounded Libor rates.

• Not only do we then conveniently work with a finite set of directly observable rates that can be represented on a computer, but an explosion-free log-normal forward rate model also becomes possible.

• Consecutive quarterly or semi-annual forward rates can all be lognormal while the model will remain arbitrage free.

• This is possible because each rate is lognormal under the corresponding forward (to the settlement date) measure, rather than under one unique measure.

• Lognormality under the appropriate forward, and not spot, measure is needed to justify the use of Black's futures formula for caplet pricing.

• Despite the change to simply compounded rates, the Libor market model will still be a special case of an HJM model, albeit one where we only *indirectly* specify the volatility function of the instantaneous forward rates.

LIBOR and Forward LIBOR in a Single-Curve World

Let $0 \leq t \leq T$ and $\delta > 0$ be given. At time t, one can lock in an interest rate for investing over the interval $[T, T+\delta]$.

$$L(t, T, T+\delta) \equiv L(t) = \frac{P(t,T) - P(t, T+\delta)}{\delta P(t, T+\delta)}$$

$$L(T, T, T+\delta) \equiv L(T) = \frac{P(T,T) - P(T, T+\delta)}{\delta P(T, T+\delta)} = \frac{1 - P(T, T+\delta)}{\delta P(T, T+\delta)}$$

<u>New notations</u>:

- $L(t)$ is called forward LIBOR.
- $L(T)$ is the spot δ-LIBOR (e.g., $\delta = 3$ months) set at time T.

• The forward LIBOR for the interval $[T, T+\delta]$ set at time t is the expected value of the spot LIBOR rate that will fix at time T under the $(T+\delta)$-forward measure:

$$L(t, T, T+\delta) \equiv L(t) = \mathbb{E}^{T+\delta}\left[L(T, T+\delta) \middle| \mathcal{F}_t \right].$$

HJM Forward Rate vs. Forward LIBOR

- HJM forward rate is an instantaneous forward interest rate.
- Forward LIBOR is a discrete, simply-compounded, forward interest rate applied over a time interval.
- It is not possible for the HJM forward rate to be log-normal.
- We will build forward LIBOR models in which forward LIBOR is log-normal $\Longrightarrow$ we will choose the volatility structure in such a way that forward LIBOR is log-normal under some measure. This measure will be the forward measure.

The Case of One LIBOR Rate

- Let us consider a single LIBOR rate, and try to choose the volatility structure in such a way that the option on this LIBOR rate (caplet) is priced using Black's formula.
- The fixing date of the caplet is set to T, and the tenor to δ.
- A caplet (call option on L) pays

$$X = \delta \left(L \left(T \right) - K \right)_+$$

at time $T + \delta$ (assuming a notional $N = 1$). Note that the payoff $\delta \left(L \left(T \right) - K \right)_+$ is determined at time T, so that it is $\mathcal{F}_T$-measurable, yet it is paid later at $\left(T + \delta \right)$.
- Then the caplet's value at time t is:

$$\begin{aligned} c\left(t\right) &= B_t \mathbb{E}\left[B_{T+\delta}^{-1} X | \mathcal{F}_t \right] \\ &= B_t \mathbb{E}\left[B_{T+\delta}^{-1} \delta \left(L \left(T \right) - K \right)_+ | \mathcal{F}_t \right]. \end{aligned}$$

The Case of One LIBOR Rate

- Let us change the measure to $(T+\delta)$-forward. We have,

$$c(t) = \delta P(t, T+\delta)\, \mathbb{E}^{T+\delta}\left[(L(T)-K)_+ \,|\mathcal{F}_t\right].$$

- Note that $L(t)$ is the value of a traded asset

$$\frac{P(t,T) - P(t,T+\delta)}{\delta}$$

divided by the numeraire $P(t, T+\delta)$.

$$\Downarrow$$

$L(t)$ is a **martingale** under $\mathbb{Q}^{T+\delta}$

- We can therefore write it as (for some non-random volatility λ):

$$dL(t) = \lambda L(t)\, dW_t^{T+\delta}$$

The Case of One LIBOR Rate

- If we can calibrate the model so that λ is not random, then we know we will get a log-normal distribution for $L(t)$ under the $(T+\delta)$-forward measure:

$$L(t) = L(0) \exp\left(\lambda W_t^{T+\delta} - \frac{1}{2}\lambda^2 t\right)$$

- If $L(t)$ has a log-normal distribution, then we are in the conditions of Black's model.
- This is the idea behind the forward LIBOR model (Brace-Gatarek-Musiela).
- We have removed the possibility of explosion by switching to a probability measure under which drift is zero.

The Case of One LIBOR Rate

- What is the SDE for $L(t)$ under $\mathbb{Q}^{T+\delta}$ in our HJM model?

$$L(t) = \frac{P(t, T) - P(t, T+\delta)}{\delta P(t, T+\delta)}$$

- For a forward bond price $F(t, S, M) = \frac{P(t,M)}{P(t,S)}$ we have under the S-forward measure:

$$\begin{aligned} dF(t, S, M) &= \gamma(t, S, M) F(t, S, M) \, dW_t^S, \\ \gamma(t, S, M) &= \Sigma(t, M) - \Sigma(t, S). \end{aligned}$$

- We can represent the LIBOR rate as:

$$L(t) = \delta^{-1} \left(F(t, T+\delta, T) - 1 \right)$$

and

$$dF(t, T+\delta, T) = \gamma(t, T+\delta, T) F(t, T+\delta, T) \, dW_t^{T+\delta},$$

The Case of One LIBOR Rate

where

$$\gamma\left(t, T+\delta, T\right) = \Sigma\left(t, T\right) - \Sigma\left(t, T+\delta\right),$$

so that

$$\begin{aligned} dL\left(t\right) &= \delta^{-1} dF\left(t, T+\delta, T\right) \\ &= \delta^{-1}\left[\Sigma\left(t, T\right) - \Sigma\left(t, T+\delta\right)\right] F\left(t, T+\delta, T\right) dW_t^{T+\delta} \\ &= \left(L\left(t\right) + \delta^{-1}\right)\left[\Sigma\left(t, T\right) - \Sigma\left(t, T+\delta\right)\right] dW_t^{T+\delta}. \end{aligned}$$

- Then under the $\left(T+\delta\right)$-forward measure,

$$dL\left(t\right) = \frac{\delta L\left(t\right) + 1}{\delta L\left(t\right)}\left[\Sigma\left(t, T\right) - \Sigma\left(t, T+\delta\right)\right] L\left(t\right) dW_t^{T+\delta}.$$

- Let us compare this with what we want:

$$dL\left(t\right) = \lambda L\left(t\right) dW_t^{T+\delta}$$

- Therefore, as long as we choose $\Sigma(t, T)$ and $\Sigma(t, T+\delta)$ such that:

$$\frac{\delta L(t)+1}{\delta L(t)} \gamma(t, T+\delta, T) = \lambda$$

$$\gamma(t, T+\delta, T) = \lambda \frac{\delta L(t)}{\delta L(t)+1}$$

we are assured that

$$dL(t) = \lambda L(t)\, dW_t^{T+\delta},$$

and the caplet on $L(t)$ is priced using Black's formula (with volatility λ):

$$\begin{aligned} c(0) &= P(0, T+\delta)\, \mathbb{E}^{T+\delta}\left[\delta (L(T)-K)_+\right] \\ &= \delta P(0, T+\delta)\, \mathbb{E}^{T+\delta}\left[\left(L(0)\, e^{\lambda W_T^{T+\delta} - \lambda^2 T/2} - K\right)_+\right]. \end{aligned}$$

The Case of One LIBOR Rate

Summary

1. Specify (observe on the market) a caplet's volatility λ.
2. Specify the dynamics of the LIBOR rate $L(t)$ under the $(T+\delta)$-forward measure $\mathbb{Q}^{T+\delta}$ by

$$dL(t) = \lambda L(t)\, dW_t^{T+\delta},$$

so that

$$L(t) = L(0)\, e^{\lambda W_t^{T+\delta} - \lambda^2 t/2}.$$

3. Define the HJM volatility of the forward bond price by:

$$\gamma(t, T+\delta, T) = \lambda \frac{\delta L(t)}{\delta L(t) + 1}.$$

The Case of One LIBOR Rate

4. Define

$$\begin{aligned}\gamma(t, T+\delta, T) &= \Sigma(t, T) - \Sigma(t, T+\delta) \\ &= \int_T^{T+\delta} \sigma(t, u)\, du.\end{aligned}$$

Choose $\sigma(t, u)$ constant for $u \in [T, T+\delta]$ so that the equation

$$\lambda \frac{\delta L(t)}{\delta L(t) + 1} = \int_T^{T+\delta} \sigma(t, u)\, du$$

is satisfied. Namely, take

$$\begin{aligned}\sigma(t, u) &= \frac{1}{\delta} \lambda \frac{\delta L(t)}{\delta L(t) + 1} \\ &= \frac{\lambda L(t)}{\delta L(t) + 1}.\end{aligned}$$

5. For $u \notin [T, T+\delta]$ choose $\sigma(t,u)$ arbitrarily. For example, set

$$\sigma(t,u) = \frac{\lambda L(t)}{\delta L(t) + 1}$$

for **all** u, or even

$$\sigma(t,u) \equiv 0$$

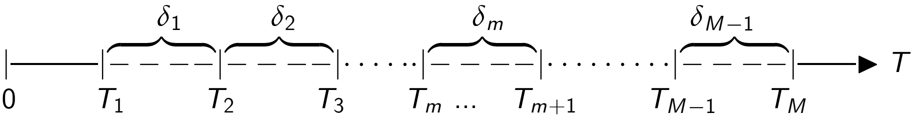
for $u \notin [T, T+\delta]$. The choice we make will not affect the value of this caplet.

6. Now the model is completely specified under the $(T+\delta)$-forward measure $\mathbb{Q}^{T+\delta}$. Change to risk-neutral measure, and you have the specification of an HJM model. Remember that a change of measure leaves the volatility (variance) unchanged, so the volatility is the same under any measure.

Market Model of Forward LIBOR

- We constructed an HJM model where a single LIBOR rate followed a log-normal process.
- It is possible to extend that to a collection of LIBOR rates. Such a model is called BGM/J (Brace-Gatarek-Musiela/Jamshidian).
- Fix a tenor structure

$$
\begin{aligned}
T_0 &= 0 < T_1 < \cdots < T_M, \\
\delta_m &= T_{m+1} - T_m, \quad \forall m = 1, ..., M-1
\end{aligned}
$$



• The modeling variables of a standard, single-curve LIBOR market model are forward rates with fixings at T_m and maturities at T_{m+1}, $m = 1, 2, ..., M-1$:

$$\{L_m(t)\}_{m=1}^{M-1},$$

where for $\forall m$, $L_m(t)$ is a LIBOR rate that:

- resets at T_m
- has tenor δ_m
- the corresponding cashflow pays at T_{m+1}

- Suppose a collection of caplet volatilities $\{\lambda_m\}_{m=1}^{M-1}$ that we want to match is observed on the market.

Market Model of Forward LIBOR

• What allowed quants to understand the correct no-arbitrage dynamics of forward rates was the particular representation of forward rates in terms of risk-free bonds, and the fact that, under a probability measure associated to a numeraire N_t, all tradable asset prices divided by N_t are martingales:

$$\begin{aligned} L_m(t) &= \frac{P(t, T_m) - P(t, T_m + \delta)}{\delta_m P(t, T_m + \delta)} = \frac{P(t, T_m) - P(t, T_{m+1})}{\delta_m P(t, T_{m+1})} \\ &= \frac{1}{\delta_m} \left(\frac{P(t, T_m)}{P(t, T_{m+1})} - 1 \right). \end{aligned}$$

• But this relationship between forward rates and zero-coupon bonds is valid only in a single-curve market.

• In a multi-curve market, this representation is lost. This is what makes it challenging to extend any interest rate model to the multi-curve setting.

Market Model of Forward LIBOR - Single-Curve World

• Jamshidian and Geman proved in the mid 1990s that such rates are martingales, each under the probability measure $\mathbf{Q}^{T_{m+1}}$ associated to the bond numeraire $P(t, T_{m+1})$ having the same maturity as the rate.

Lognormal Model of Forward LIBORs

Theorem

(LIBOR market model). There exists an HJM model on a probability space $(\Omega, \mathcal{F})$ with risk-neutral measure $\mathbb{Q}$ such that for every m, $m = 1, \ldots, M-1$,

$$dL_m(t) = \lambda_m L_m(t)\, dW_t^{T_{m+1}},$$

where $W_t^{T_{m+1}}$ is a Brownian motion under the T_{m+1}-forward measure $\mathbb{Q}^{T_{m+1}}$.

Proof.

Goes pretty much like in our simple example for a single LIBOR rate. □

Market Model of Forward LIBOR - Single-Curve World

- Each LIBOR rate L_m follows a generalized geometric Brownian motion (log-normal process) *under its own forward measure* $\mathbb{Q}^{T_m+1}$.
- This guarantees that the assumptions of Black's model are satisfied for each of them. This is what is needed to justify the use of Black's futures formula for caplet pricing.
- However, it makes the valuation of instruments that depend on *more than one* LIBOR rate quite difficult.
- It would be much more convenient if we knew the simultaneous dynamics of *all* LIBOR rates under a *single* measure.
- Jamshidian [3] was probably the first one to construct such a universal measure. He called it *spot LIBOR measure*.
- Recall that the risk-neutral measure corresponds to the choice of money-market account as a numeraire. In a money market account, the money is constantly reinvested at the short rate (continuous compounding).

• Spot LIBOR measure is a probability measure that corresponds to a "discretely compounded numeraire" (obtained by rolling over one-period bonds).

- We start at time T_0 with \$1 invested in a discount bond of maturity T_1. At each reset date T_m, $m = 1, 2, ..., M$, the wealth is reinvested in a discount bond maturing at the next reset date, i.e., at time T_{m+1}:

$$
\begin{aligned}
G_{T_0} &= 1, \\
G_{T_1} &= \frac{G_{T_0}}{P\left(T_0, T_1\right)} = G_{T_0}\left(1 + \delta_0 L_0\left(T_0\right)\right), \\
G_{T_2} &= \frac{G_{T_1}}{P\left(T_1, T_2\right)} = G_{T_1}\left(1 + \delta_1 L_1\left(T_1\right)\right), \\
&\vdots \\
G_{T_M} &= \frac{G_{T_{M-1}}}{P\left(T_{M-1}, T_M\right)} = G_{T_{M-1}}\left(1 + \delta_{M-1} L_{M-1}\left(T_{M-1}\right)\right)
\end{aligned}
$$

$$1 + \delta_j L_j (T_j) = \frac{1}{P(T_j, T_{j+1})}$$

so that

$$\begin{aligned} G_{T_m} &= \prod_{j=0}^{m-1} (1 + \delta_j L_j (T_j)) \\ &= \prod_{j=1}^{m} P^{-1} (T_{j-1}, T_j), \text{ for } \forall m = 1, 2, ..., M \end{aligned}$$

- In between "rollover" dates $\{T_m\}_{m=0}^{M}$, G_t is uniquely specified by no-arbitrage arguments. If

$$T_{m-1} < t < T_m, \text{ then}$$

$$G_t = P(t, T_m) \cdot G_{T_m}.$$

- This is because at time T_{m-1} we already know that we are going to receive G_{T_m} at time T_m, so the price has to be discounted accordingly.

- Define a deterministic function ("index of a first rollover date after t")

$$m(t) = \inf\{m \in \mathbb{Z} : T_m \geq t\}, \text{ for } \forall t > 0.$$

$$G_t = P\left(t, T_{m(t)}\right) \prod_{j=1}^{m(t)} P^{-1}\left(T_{j-1}, T_j\right).$$

- We choose the LIBOR savings account G as a numeraire asset, and we introduce the corresponding martingale measure.

Definition

A spot LIBOR measure $\overline{\mathbb{Q}}^L$ is a probability measure that corresponds to G_t being a numeraire, namely the measure under which the bond price discounted by the numeraire G_t

$$\frac{P(t, T_m)}{G_t}$$

is a martingale for each T_m, $m = 1, ..., M$.

Market Model of Forward LIBOR - Single-Curve World

Theorem

(On spot LIBOR measure) The dynamics of any j-th forward LIBOR under the spot LIBOR measure $\overline{\mathbb{Q}}^L$ are given by

$$dL_j(t) = \sum_{k=m(t)}^{j} \frac{\delta_k \lambda_k \lambda_j L_k(t) L_j(t)}{1 + \delta_k L_k(t)} dt + \lambda_j L_j(t)\, d\overline{W}_t^L, \qquad j = 1, \ldots, M-1, \tag{1}$$

where $\overline{W}_t^L$ is a Brownian motion under $\overline{\mathbb{Q}}^L$.

Lemma

For every $m \geq 0$ and every $\mathcal{F}_{T_m}$-measurable payoff X, and for $\forall t$, $T_{m-1} \leq t \leq T_m$:

1) The spot LIBOR measure $\overline{\mathbb{Q}}^L$ satisfies:

$$\overline{\mathbb{E}}^L \left[X \middle| \mathcal{F}_t \right] = \mathbb{E}^{T_m} \left[X \middle| \mathcal{F}_t \right]$$

2) The Brownian motion under $\overline{\mathbb{Q}}^L$ is given by:

$$d\overline{W}_t^L = dW_t - \Sigma \left(t, T_m \right) dt.$$

Proof. 1) Under $\mathbb{Q}^{T_m}$

$$\pi_t \left(X \right) = P \left(t, T_m \right) \mathbb{E}^{T_m} \left[X \middle| \mathcal{F}_t \right]$$

Under $\overline{\mathbb{Q}}^L$

$$\pi_t \left(X \right) = G_t \overline{\mathbb{E}}^L \left[\frac{X}{G_{T_m}} \middle| \mathcal{F}_t \right]$$

$$\begin{aligned}
\pi_t(X) &= G_t \overline{\mathbb{E}}^L\left[\left.\frac{X}{G_{T_m}}\right| \mathcal{F}_t\right] = P(t, T_m)\, G_{T_m} \overline{\mathbb{E}}^L\left[\left.\frac{X}{G_{T_m}}\right| \mathcal{F}_t\right] \\
&= P(t, T_m)\, \frac{G_{T_{m-1}}}{P(T_{m-1}, T_m)} \overline{\mathbb{E}}^L\left[\left.\frac{X}{G_{T_{m-1}} P^{-1}(T_{m-1}, T_m)}\right| \mathcal{F}_t\right] \\
&= P(t, T_m)\, \overline{\mathbb{E}}^L\left[X \middle| \mathcal{F}_t\right]
\end{aligned}$$

Equating the two expressions:

$$P(t, T_m)\, \mathbb{E}^{T_m}\left[X \middle| \mathcal{F}_t\right] = P(t, T_m)\, \overline{\mathbb{E}}^L\left[X \middle| \mathcal{F}_t\right]$$

$$\overline{\mathbb{E}}^L\left[X \middle| \mathcal{F}_t\right] = \mathbb{E}^{T_m}\left[X \middle| \mathcal{F}_t\right]$$

$$\overline{\mathbb{Q}}^L \sim \mathbb{Q}^{T_m}, \text{ for } T_{m-1} \leq t \leq T_m$$

2)

$$T_{m-1} \leq t \leq T_m \Longrightarrow m(t) = m$$

Let W_t be the Brownian motion under the risk-neutral measure. Then the Brownian motion under $\mathbb{Q}^{T_m}$ is given by:

$$dW_t^{T_m} = dW_t - \Sigma(t, T_m)\, dt$$

(the characterization of forward measure under Gaussian HJM).
Since the spot LIBOR measure behaves exacty like each of these different forward measures simultaneously at each reset time,

$$\overline{\mathbb{Q}}^L \sim \mathbb{Q}^{T_m}\text{ , for } T_{m-1} \leq t \leq T_m$$

then

$$d\overline{W}_t^L = dW_t - \Sigma(t, T_m)\, dt.$$

Proof for Theorem on Spot LIBOR Measure:
We know that each forward LIBOR follows under the T_{j+1}-forward measure:

$$dL_j(t) = \lambda_j L_j(t)\, dW_t^{T_{j+1}}$$

Assume $T_{m-1} < t < T_m$, $m < j$. Then $m(t) = m$.

$$dW_t^{T_{j+1}} = dW_t - \Sigma(t, T_{j+1})\, dt$$

$$d\overline{W}_t^L = dW_t - \Sigma(t, T_m)\, dt$$

$$dW_t^{T_{j+1}} = d\overline{W}_t^L + (\Sigma(t, T_m) - \Sigma(t, T_{j+1}))\, dt$$

$$
\begin{aligned}
dL_j(t) &= \lambda_j L_j(t)\left[d\overline{W}_t^L + \left(\Sigma(t, T_m) - \Sigma(t, T_{j+1})\right)dt\right] \\
&= \lambda_j L_j(t)\, d\overline{W}_t^L + \lambda_j L_j(t)\left[\Sigma(t, T_m) - \Sigma(t, T_{j+1})\right]dt \\
&= \lambda_j L_j(t)\, d\overline{W}_t^L + \lambda_j L_j(t)\left[\left(\Sigma(t, T_m) - \Sigma(t, T_{m+1})\right)\right. \\
&\quad \left. + \left(\Sigma(t, T_{m+1}) - \Sigma(t, T_{m+2})\right) + \ldots + \left(\Sigma(t, T_j) - \Sigma(t, T_{j+1})\right)\right]
\end{aligned}
$$

$$
\Sigma(t, T_k) - \Sigma(t, T_{k+1}) = \gamma(t, T_{k+1}, T_k) = \lambda_k \frac{\delta_k L_k(t)}{1 + \delta_k L_k(t)}
$$

$$
dL_j(t) = \lambda_j L_j(t)\, d\overline{W}_t^L + \lambda_j L_j(t) \sum_{k=m(t)}^{j} \gamma(t, T_{k+1}, T_k)\, dt
$$

$$
dL_j(t) = \lambda_j L_j(t) \sum_{k=m(t)}^{j} \lambda_k \frac{\delta_k L_k(t)}{1 + \delta_k L_k(t)} dt + \lambda_j L_j(t)\, d\overline{W}_t^L .
$$

Market Model of Forward LIBOR - Single-Curve World

- LIBOR rates are of course no longer martingales under $\overline{\mathbb{Q}}^L$. However, the drift terms are still expressed in terms of "observables".
- We can use $\{\lambda_m\}_{m=1}^{M-1}$ and

$$dL_j(t) = \sum_{k=m(t)}^{j} \frac{\delta_k \lambda_k \lambda_j L_k(t) L_j(t)}{1 + \delta_k L_k(t)} dt + \lambda_j L_j(t)\, d\overline{W}_t^L, \qquad j = 1, \ldots, M-1$$

to completely and consistently specify the evolution of LIBOR rates $\{L_m(t)\}_{m=1}^{M-1}$ under the same measure.
- Also note that the numeraire G_t is specified in terms of observables, so we can evolve forward LIBOR rates and the numeraire simultaneously (say in Monte-Carlo), only knowing caplet volatilities.

Summary

- We have introduced a model for forward LIBOR. One can build this model so that forward LIBOR $L(t)$ is log-normal under the forward measure $\mathbb{Q}^{T+\delta}$, and this allows a rigorous derivation of the *Black caplet formula*. This formula is similar to the Black-Scholes-Merton formula for equities, but is used in fixed income markets in which the essence of the market is that the interest rate is random, in contrast to the Black-Scholes-Merton assumption.

Model or Parameterization?

- In market models, volatilities of LIBOR rates need not be constant, nor the number of factors need be one.
- The problem with the equation

$$dL_j(t) = \sum_{k=m(t)}^{j} \frac{\delta_k \lambda_k \lambda_j L_k(t) L_j(t)}{1+\delta_k L_k(t)} dt + \lambda_j L_j(t)\, d\overline{W}_t^L, \qquad j = 1, ..., M-1$$

is that we have only one source of randomness.
- In the most general form we can write the following joint system for the stochastic evolution of LIBOR rates $\{L_j(t)\}_{j=1}^{M-1}$,

$$\frac{dL_j(t)}{L_j(t)} = \mu_j(t)\, dt + \sum_{n=1}^{N} a_{jn}(t)\, dZ_n(t), \qquad j = 1, ..., M-1.$$

- It does not really matter what measure we use; this general form will hold under any measure.

Model or Parameterization?

- The difference between different measures will be in the different drifts $\left\{\mu_j(t)\right\}_{j=1}^{M-1}$, that in general will not be deterministic functions, but some rather complex expressions involving rates, volatilities, correlations, etc.
- In the equation above,

$$\begin{cases} N & \text{- is the number of factors} \\ \{Z_n\}_{n=1}^{N} & \text{- are independent Brownian motions} \\ & \text{(under whatever measure we are working in)} \\ a_{jn}(t) & \text{- are instantaneous factor volatilities} \\ & \text{(deterministic, time-dependent)} \end{cases}$$

- The LIBOR rate instantaneous volatility $v_j(t)$ for the j-th rate $L_j(\cdot)$ is defined by

$$v_j^2(t) = a_{j1}^2(t) + \cdots + a_{jN}^2(t).$$

Model or Parameterization?

Calibration is the process of specifying the time-dependent matrix of instantaneous volatilities $\{a_{jn}(t);\ j = 1, ..., M-1;\ n = 1, ..., N;\ t \geq 0\}$ so that the market prices of some instruments are recovered by the model.
- The number of degrees of freedom in the model is enormous!
- Without loss of generality, the instantaneous volatilities $a_{jn}(\cdot)$ can be taken to be constant on time intervals $[T_m, T_{m+1}]$.
- So for each $a_{jn}(\cdot)$ we have as many degrees of freedom as there are intervals before the fixing of the j-th rate.
- For the LIBOR rate $L_j(t)$, there are j periods between today (T_0) and the LIBOR fixing date T_j.

Model or Parameterization?

- So there are $j \times N$ degrees of freedom (N is the number of factors) associated with the rate $L_j(\cdot)$.
- For the total of $(M-1)$ LIBOR rates there are

$$N \sum_{j=1}^{M-1} j = \frac{1}{2} NM (M-1)$$

degrees of freedom in total. Assuming a 30 years span that the model has to cover and the LIBOR tenor of 3 months, we have $M = 120$. Given, say, three factors, we have about 21420 degrees of freedom!!!
- The number of (actively enough) traded instruments is much smaller. It is clear that in a calibration, a huge number of model parameters will be left "unfixed", given a relatively sparse universe of traded instruments. This is another classic case of potential OVERFITTING!

Model or Parameterization?

- At this point we should realize that what we have is not really a new model. At least not a model in the sense we understand it.
- We should regard it as merely a new and (arguably) better **parameterization** of the dynamics of the interest rates in terms of other (namely LIBOR) rates.
- The original parameterization of HJM was done in terms of instantaneous forward rates.

Model or Parameterization?

- The new parameterization is considered better because it allows traders to express their opinion using familiar notions of caplet volatilities, etc. and not in the much less intuitive concepts related to instantaneous forward rates.
- Still, an opinion MUST be expressed one way or another, as the model itself does not impose any.
- A parameterization becomes a model when we add assumptions (or opinions, or views) to it.
- The parameterization has to be coupled with something else, something that provides a sensible and stable description of the term structure of caplet volatilities/correlations.
- This something has to come from an *external* source.

Model or Parameterization?

- The assumptions reduce the dimensionality of the space of model parameters (from 21420 to a more manageable number) and as such allow us:

- ▶ To express a view on the dynamics of rates;
- ▶ To compare the actual evolution with the predicted one and identify mispricings and trading opportunities;
- ▶ To have a stable set of input parameters that does not change unpredictably day-to-day; when it does change, it sends a powerful and important signal (change of regime, etc.)

- The issue of calibrating market models is very complex. Reading Rebonato's books [1] and [2] is mandatory for anyone interested in the subject.

Conclusion

- The development of market models was a major advance in the modelling of interest rates. They allow using market-observed and intuitive quantities in specifying the dynamics of the term structure of interest rates, and thus make it possible to price all Bermudan swaptions in addition to all European swaptions, caps and floors with one model.
- The switch from continuously compounded forward rates to simply compounded forward rates in order to remove the explosion problem for instantaneous forward rates was proposed by Sandmann and Sondermann [5, 6].
- The use of log-normal simple interest rates to price caps and floors was worked out by Miltersen, Sandmann and Sondermann [1].
- This idea was embedded in a full forward LIBOR term-structure model by Brace, Gatarek, Musiela [2].
- This was the first full term-structure model consistent with the heuristic formula provided by Black [4] in 1976 and in common use since then.
- Later, a variation on forward LIBOR models has been developed for swap markets; see Jamshidian [3].

- Whereas the name “market model” suggests a specific set of dynamical equations for interest rates, it is in fact more of a framework than an actual model, whose only difference from any other HJM model is in choosing a different parameterization of the term curve (market rates instead of instantaneous forward ones).
- Merely rewriting the equations for market (LIBOR or swap) rates, does not constitute a model. To have a model, a set of exogenous constraints should be specified.
- The market models are convenient to calibrate since prices of caplets, swaptions and other related quantities can readily be expressed in terms of the model parameters (time-dependent instantaneous caplet volatilities). Path-dependent instruments can be valued via Monte-Carlo.

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