# Black's Model

## Fixed Income Derivatives

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# Introduction

- We will be concerned with the valuation of interest rate options.
- A financial derivative is defined in terms of some underlying asset(s) which already exists on the market.
- Interest rate derivatives are instruments whose payoffs are dependent in some way on the level of interest rates.
- Unlike other derivatives, the underlying whose value determines the payoff of an interest rate derivative is not always the price of a traded security (!)
- Valuation of derivatives:
  - A derivative cannot be priced arbitrarily in relation to the underlying, but has to be priced in a way that is consistent with the underlying price (level) given by the market.
  - Otherwise, we would have mispricings between the derivative and the underlying.
- What models did traders have at their disposal for the valuation of options in the 1990's?
  - The Black-Scholes model for valuing stock options, published in 1973.

**Louis Jean-Baptiste Alphonse Bachelier**

French mathematician

Born: March 11, 1870 - Le Havre, France

Died: April 28, 1946 - Saint-Servan-sur-Mer, France

- The first to analyze mathematically the stochastic process now called Brownian motion, in order to develop a theory of option pricing
- PhD thesis: *The Theory of Speculation* (published in 1900)
- His thesis was the first paper to use advanced mathematics in the study of finance.
- Bachelier is considered a pioneer in the study of financial mathematics and stochastic processes.

# The Black-Scholes Model

- <u>**Black-Scholes formula**</u> (1973) for pricing European call and put options on a stock with strike $K$ and expiration $T$:

$$
\begin{aligned}
c(0) &= S(0)\,N(d_1) - e^{-rT} K N(d_2)\,, \\
p(0) &= e^{-rT} K N(-d_2) - S(0)\,N(-d_1)\,,
\end{aligned}
$$

where and $S(0)$ is the stock price at $t = 0$, and

$$
d_{1,2} = \frac{\log \frac{S(0)}{K} + \left(r \pm \frac{1}{2}\sigma^2\right) T}{\sigma \sqrt{T}}.
$$

- <u>Question:</u>

- Is the Black-Scholes model restricted to pricing stock options?

# Basic Black-Scholes Assumptions

- Stock prices have a lognormal distribution (the process that stock prices follow is a geometric Brownian motion);
- Volatility is constant during the lifetime of the option;
- Interest rates are constant.

⧫ However, the Black-Scholes model can be and has been extended so that it can be used to value options on foreign exchange, options on stock indices, and options on futures contracts.

⧫ Fischer Black adapted in 1976 the Black-Scholes-Merton setting to the pricing of European options on commodity futures [Black '76 Model]: *Black, F., The Pricing of Commodity Contracts, Journal of Financial Economics 3 (1976) 167-179.*

- <u>**Black's Assumptions**</u>:
  - The forward price of the asset equals its futures price;
  - Futures prices have the same lognormal distribution at expiration that we assumed for stock prices.

# Black's Formula for European Options on Futures

$$
\begin{aligned}
c(0) &= e^{-rT}\left[F(0,T)\,N(d_1) - KN(d_2)\right], \\
p(0) &= e^{-rT}\left[KN(-d_2) - F(0,T)\,N(-d_1)\right],
\end{aligned}
$$

where

$$
d_{1,2} = \frac{\log \frac{F(0,T)}{K} \pm \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}},
$$

- $F(0,T)$ is the forward price of the underlying asset with delivery date $T$;
- $\sigma$ is the volatility of the forward price of the underlying asset.

**Question**:
- Can we extend the Black-Scholes model to the pricing of interest rate derivatives?

# Black-Scholes and Interest Rate Derivatives

- When interest rates are not constant and the underlying asset is sensitive to interest rate movements, the Black-Scholes model is inappropriate.
- In the case of interest rate derivatives, interest rates are the **underlying**, so they must be treated as stochastic variables.
- However, practitioners started applying a modification of Black's formula for valuing caplets, floorlets, swaptions and bond options, which quickly became a market standard.
- This modification of Black's formula for pricing options on assets that are sensitive to interest rate movements was structurally similar to the model suggested by Fischer Black for valuing options on commodity futures.

# Black's Formula For Interest Rate Derivatives

Consider a European call option on a variable whose value is $V$. The variable $V$ does not have to be the price of a traded security. Define:

- $T$ : Time to expiration of the option
- $F(t, T)$ : Forward price of $V$ which sets at time $t$ for delivery date $T$
- $F(0, T)$ : Value of $F(t, T)$ at time $t = 0$
- $V_T$ : Value of $V$ at time $T$
- $\sigma$ : Volatility of $F$
- $K$ : Strike price of the option
- $P(0, T)$ : Price at time 0 of a zero-coupon bond paying \$1 at maturity $T$
- We assume that $V_T$ is lognormally distributed at time $T$ and

$$F(0, T) = \mathbb{E}[V_T].$$

- Traders applied the following formulas to value call and put options at $t = 0$ on an interest rate-sensitive underlying:

$$c(0) = P(0, T)\left[F(0, T)\, N(d_1) - KN(d_2)\right],$$

$$p(0) = P(0, T)\left[KN(-d_2) - F(0, T)\, N(-d_1)\right],$$

where

$$d_{1,2} = \frac{\ln\left(F(0, T)/K\right) \pm \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}},$$

and $N(x)$ is the standard normal cumulative distribution.

- This is the model we will refer to as **Black's model**.
- The academic community did not agree initially with its application to the valuation of interest rate derivatives.

# Features of Black's Model

- We don't have to assume geometric Brownian motion for the evolution of $V$ or $F$. We only require that $V_T$ be lognormal at $T$.
- The parameter $\sigma$ is the volatility of $F$ or the forward volatility of $V$. However, its only role is to define the standard deviation of $\ln V_T$ by mean of the relationship $StdDev(\ln V_T) = \sigma\sqrt{T}$.
- The volatility parameter $\sigma$ does not say anything about the standard deviation of $\ln V_t$ at times other than $T$.
- Black's model is NOT an interest rate model! It can be used to value contracts that depend on the level of interest rates and not the shape of the term curve.
- Black's formula can only correctly price different options independently of each other, one at a time.
- The application of Black's formula may lead to pricing allowing arbitrage, since it is inappropriate to model just the dynamics of the underlying security.
- Broadly speaking, the entire term curve is the "underlying asset" for fixed income derivatives.

**Black-Scholes formula:**

$$c = S(0)\,N(d_1) - e^{-rT} K N(d_2),$$

$$d_{1,2} = \frac{\log \frac{S(0)}{K} + \left(r \pm \frac{1}{2}\sigma^2\right) T}{\sigma\sqrt{T}}$$

**Black's formula:**

$$c = P(0,T)\left[F(0,T)\,N(d_+) - KN(d_-)\right],$$

$$d_{+,-} = \frac{\log \frac{F(0,T)}{K} \pm \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}$$

- If the interest rate is constant, these formulas agree because:

- $P(0,T) = e^{-rT}$,
- $F(0,T) = e^{rT} S(0)$.
  - $r$ is the risk-free rate.

# Role of Interest Rates in the Valuation of Derivatives

- They define the expected return from stocks, bonds, and other underlying assets in a risk-neutral world so that the expected payoff on the derivative can be calculated.
- They also define the rate used for discounting the expected payoff.

- For interest rate derivatives, they actually represent the underlying itself.
- Standard derivatives pricing theory is based on the assumption that one can borrow and lend at a unique risk-free or riskless rate.

# Risk-Free Interest Rate

- The rate of interest that can be earned without assuming any risks:
  - No risk due to market price movements
  - Money is borrowed and lent with no credit risk
- The "risk-free" term structure of interest rates is a key input to the pricing of derivatives.

**Before the Credit Crisis**

- Academic research assumed that the government borrowing rate is the risk-free rate (AAA credit rating) -> Treasury term structure
- Derivatives dealers have traditionally used LIBOR and LIBOR swap rates as proxies for the risk-free rate (AA credit rating) -> LIBOR term structure

**After the Credit Crisis**

- LIBOR is no longer the benchmark index.
- Overnight Indexed Swap (OIS) rates are the new proxy for the risk-free interest rates (shortest exposure) -> OIS term structure

# Collateral Requirements

- To reduce credit risk, collateral posting agreements have been put into place to collateralize mutual exposure between dealers.
- Such agreements are based on the credit support annex (CSA) to the International Swaps and Derivatives Association (ISDA) master agreement.
- Collateralized trades are often referred to as CSA trades.

# Asset Pricing Methods in Modern Finance

- The Black-Scholes Approach
  - Create a riskless portfolio
  - Find the PDE that the option value must satisfy
  - Solve the PDE either analytically or numerically
- The Martingale Method
  - Find an equivalent probability measure under which discounted asset prices are martingales
  - Evaluate the expected value of the payoff under this equivalent measure

• Only after the second asset pricing method was developed, academics realized that the application of Black's model to interest rate derivatives, the way traders had done it from the beginning, was actually justified, but under special conditions.

• In order to describe the Martingale Method, we will need to introduce some prerequisites in the following slides.

# Martingale Method Prerequisites: Money Market Account

- Money deposited in a money market account is invested every day at that day's overnight rate. We only know how much we will have tomorrow, but not afterwards.
- We denote by $B_t$ the value of a money market account at time $t$ in which we deposited \$1 at $t = 0$:

$$\begin{aligned} \frac{dB_t}{dt} &= r_t B_t, \qquad B_0 = 1. \\ dB_t &= r_t B_t dt. \end{aligned}$$

- The time-$t$ value of the money market account is: The time-$t$ value of the money market account is:

$$B_t = e^{\int_0^t r_u du}$$

- $r_u$ is the time-$u$ instantaneous spot rate of interest or short rate (the risk-free rate).

# Martingale Method Prerequisites: Zero-Coupon Bond

- A zero coupon bond pays \$1 (in the absence of default) at a future prearranged time $T$. We denote the value at time $t$ of this bond by $P(t, T)$.
- Consider a zero-coupon bond maturing at time $T$, and a money market account $B$.
  - If interest rates are non-random, then:

$$P(0, T) = \frac{1}{B_T}.$$

  - In particular, if $r$ is constant, then:

$$P(0, T) = e^{-rT}.$$

# Forward Contract

Consider an asset $V_t$ denominated in units of domestic currency.

- A forward contract on $V_t$ with delivery date $T$ and non-random delivery price $K$ obligates the holder to pay $K$ and receive $V_T$ at time $T$.
- The payoff at time $T$ of the forward contract is therefore:

$$\text{Payoff} = V_T - K.$$

**Valuation of a Forward Contract**

- The $t = 0$ value of a forward contract is:

$$V_0 - KP(0, T).$$

# Forward Price

- The forward price $F(0,T)$ set at $t=0$ for delivery at time $T$ is the price that makes the forward contract have zero value at $t=0$.
- A forward contract on an asset (variable) $V$ has time-0 value:

$$V_0 - KP(0,T).$$

- The choice of $K$ that makes the value equal to zero is $K = F(0,T)$:

$$0 = V_0 - F(0,T)\,P(0,T) \qquad \Longrightarrow \qquad F(0,T) = \frac{V_0}{P(0,T)}.$$

- If $r = \text{const.}$, and the asset $V$ is a non-dividend-paying stock $S$:

$$F(0,T) = S_0 e^{rT}.$$

- A forward price $F(0,T) = \frac{V_0}{P(0,T)}$ is not the same thing as the value of a forward contract.
- Today's forward price for delivery at time $T$ is exactly equal to today's value of the asset expressed in units of the zero-coupon bond maturing at time $T$, $P(0,T)$.

# Numéraires

- We don't necessarily have to express the values of assets in units of domestic currency.
- Actually, it is preferable that we express them in units of (or normalize them by) some other assets that don't lose value.
- Such assets that we use to "value" other assets are called **numéraires**.
- A numéraire $N_t$ plays the role of a currency that does not lose its value with time.
- Any asset that a) is a traded asset and b) has a non-zero value at all times - can be chosen as a numéraire.
- Examples of numéraires:
  - Money market account (bank account)
  - Zero-coupon bond (discount bond)
  - A non-dividend-paying stock

## The First Fundamental Theorem of Asset Pricing

No arbitrage $\Leftrightarrow$ For any numéraire $N_t$, there exists a probability measure $\mathbb{Q}^N$, equivalent to $\mathbb{P}$, under which the prices of all tradeable assets normalized by $N_t$ are martingales.

- $\mathbb{P}$ is the real-world probability measure.
- $\mathbb{Q}^N$ equivalent to $\mathbb{P}$ means that they agree on what is possible, i.e., for any event $A$, $\mathbb{Q}^N(A) > 0 \iff \mathbb{P}(A) > 0$.
- In this one-period model, $M_t$ is a martingale means that

$$M_0 = \mathbb{E}^{\mathbb{Q}^N}[M_T].$$

- Thus, to say that the discounted price $X/N$ is a martingale here means that

$$\frac{X_0}{N_0} = \mathbb{E}^{\mathbb{Q}^N}\left[\frac{X_T}{N_T}\right].$$

- Today's price equals today's expectation of tomorrow's price (the law of conservation of expectation).
- The expected value is not computed under the real-world measure, but under a probability measure that is specific to the numéraire.

- For any asset $V$, its present value is given by:

$$V_0 = \mathbb{E}^B \left[ \frac{V_T}{B_T} \right]$$

- The expected value is taken under the probability measure associated with the money market account: the **risk-neutral measure**.
- If we assume interest rates to be non-random, then:

$$\text{Non-random } r_t \Longrightarrow V_0 = B_T^{-1} \mathbb{E}^B \left[ V_T \right].$$

- If we assume the interest rate to be constant, then:

$$r_t = const. \Longrightarrow V_0 = e^{-rT} \mathbb{E}^B \left[ V_T \right].$$

- In general, we cannot do this. So, if we are trying to model $V_t$, we will need to know its probabilistic behavior with respect to $B_t$ (the joint distribution between $V_t$ and $B_t$).
- Therefore, the risk-neutral measure is not the most comfortable measure to work with in fixed income.

# Zero-Coupon Bond as Numéraire

- The bond price at time $T$ is known today. The money market account value at time $T$ is only known at time $T$. Is there any relationship between them?
- Yes. Under the risk-neutral measure, compare the present value $P(0,T)$ of the zero coupon bond with its value at $t=T$:

$$\frac{P(0,T)}{B_0} = \mathbb{E}^B\left[\frac{P(T,T)}{B_T}\right] \Longrightarrow P(0,T) = \mathbb{E}^B\left[B_T^{-1}\right]$$

- If interest rates are <u>non-random</u>, then $P(0,T) = B_T^{-1}$.
- Using the zero-coupon bond as numéraire, the value of any asset $V$ is:

$$\frac{V_0}{P(0,T)} = \mathbb{E}^T\left[\frac{V_T}{P(T,T)}\right] \Longrightarrow V_0 = P(0,T)\mathbb{E}^T\left[V_T\right].$$

- The expected value is taken with respect to the measure associated with the zero coupon bond $P(\cdot,T)$: this is the **T-forward measure**.
- Advantage of this measure: We only need to specify the distribution of $V_T$. Magically, the discounting appears outside of the expectation.

# The Forward Price Under the T-Forward Measure

- Consider a forward contract on $V_t$, with delivery price $K$ and delivery date $T$. Payoff to holder is $V_T - K$. Denote its value by $\mathcal{V}_t$.
- At $t = 0$, the forward price $F(0, T)$ is the value of $K$ that makes the value of the forward contract be zero: $\mathcal{V}_0 = 0$
- At time $t = T$, the payoff to holder is: $\mathcal{V}_T = \text{payoff} = V_T - F(0, T)$.
- Apply the First Fundamental Theorem to the value of the forward contract under the $T$-forward measure:

$$\frac{\mathcal{V}_0}{P(0, T)} = \mathbb{E}^T \left[ \frac{\mathcal{V}_T}{P(T, T)} \right]$$

$$\frac{0}{P(0, T)} = \mathbb{E}^T \left[ V_T - F(0, T) \right] \Rightarrow \quad 0 = P(0, T)\, \mathbb{E}^T \left[ V_T - F(0, T) \right]$$

$$F(0, T) = \mathbb{E}^T \left[ V_T \right]$$

- The forward price of any variable $V_t$ is the expected value of the future value $V_T$ under the $T$-forward measure $\mathbb{Q}^T$.
- This is the only probability measure that has this property!

# Forward Contract with Collateral (CSA)

- Consider a forward contract covered by CSA, where the collateral posted $C(t)$ is always equal to the value of the contract $\mathcal{V}(t)$.
- As the collateral is used to offset liabilities in case of default $=>$ this is a riskless investment
- In this case we can use the risk-free term curve for lending.
- Denote the risk-free short rate at time $t$ by $r_C(t)$, and the corresponding risk-free discount factor by $P_C(t, T), 0 \leq t \leq T$.
- Let the CSA forward price fixed at $t = 0$ be $F_{CSA}(0, T)$.Then:

$$F_{CSA}(0, T) = \mathbb{E}^T[V_T],$$

where the expected value is taken under the standard (risk-free) $T$-forward measure $\mathbb{Q}^T$, i.e., a measure defined by the risk-free bond:

$$P_C(0, T) = \mathbb{E}^B\left[e^{-\int_0^T r_C t dt}\right]$$

- The value at time $t = 0$ of an asset under CSA with payout $V_T$ is:

$$V_0 = P_C(0, T)\mathbb{E}^T[V_T]$$

# Forward Contract without Collateral (No CSA)

- Consider a forward contract with no CSA, where the collateral posted is $C(t) = 0$.
- Denote the short rate for unsecured funding by $r_F(t)$. This is not a risk-free rate, but a risky rate => $r_C(t) \leq r_F(t)$.
- Let the non-CSA forward price or delivery at $T$ be $F_{noCSA}$. Then:

$$F_{noCSA}(0, T) = \widetilde{\mathbb{E}}^T [V_T],$$

where the expected value is taken under the measure $\widetilde{\mathbb{Q}}^T$ defined by the numéraire:

$$P_F(0, T) = \mathbb{E}^B \left[ e^{-\int_0^T r_F tdt} \right].$$

- Note that this is a credit-risky bond. Then we can call the $\widetilde{\mathbb{Q}}^T$ measure - the risky $T$-forward measure.
- The value at time $t = 0$ of an asset under no CSA with payout $V_T$ is:

$$V_0 = P_F(0, T) \widetilde{\mathbb{E}}^T [V_T].$$

# CSA or No CSA

- We note that in general:

$$F_{noCSA}(0,T) \neq F_{CSA}(0,T).$$

- The idea is that different discounting rates should be used for CSA (collateralized) and non-CSA (non-collateralized) versions of the same derivative (Vladimir Piterbarg 2010).

- The difference between the risk-free $T$-forward measure $\mathbb{Q}^T$ and the risky $T$-forward measure $\widetilde{\mathbb{Q}}^T$ reflects the fact that the underlying processes obey different probabilities. The two measures have different means, as well as different variances and higher moments.

# Black's Model and the Martingale Approach

- Consider a European call option on a variable whose value is $V$ in units of domestic currency. Denote by $c(0)$ the price of the call option at $t = 0$.
- We are going to use two results:

1. The current value of any contingent claim normalized by the price of the zero-coupon bond maturing at $T$ (numéraire) is equal to the expected value of its future value at time $T$ under the $T$-forward measure (the martingale property). This derives from the first Fundamental Theorem of Asset Pricing:

$$\frac{c(0)}{P(0,T)} = \mathbb{E}^T\left[\frac{\max(V_T - K, 0)}{P(T,T)}\right]$$

2. The expected value of any variable (except an interest rate) at time $T$ under the $T$-forward measure equals its forward value:

$$\mathbb{E}^T[V_T] = F(0,T)$$

# Log-Normal Version of Black's Model

$$\begin{aligned} \frac{c(0)}{P(0,T)} &= \mathbb{E}^T\left[\frac{\max(V_T - K, 0)}{P(T,T)}\right] \\ c(0) &= P(0,T)\,\mathbb{E}^T\left[\max(V_T - K, 0)\right] \end{aligned}$$

- All we have to do now is to calculate the expected value by specifying a probability distribution for $V_T$.
- Assume that $\ln(V_T)$ is normally distributed at expiration $T$ with a standard deviation equal to $\sigma\sqrt{T}$. Then we can write:

$$V_T = \mathbb{E}^T\left[V_T\right] e^{\sigma\sqrt{T}\eta - \frac{1}{2}\sigma^2 T} = F(0,T)\, e^{\sigma\sqrt{T}\eta - \frac{1}{2}\sigma^2 T}$$

  where $\eta$ is a standard normal variable, $\eta \sim$Normal$(0,1)$.
- If we compute $\mathbb{E}^T\left[\max(V_T - K, 0)\right]$ and plug in $c(0)$, we obtain:

$$c(0) = P(0,T)\left[F(0,T)\,N(d_1) - KN(d_2)\right],$$

  where $N(x)$ is the standard normal cumulative distribution function.

# Calculate the Expected Value

- $V_T$ is log-normally distributed
- Then $\ln V_T$ is normally distributed: $\ln V_T \sim N\left(\mu T, \sigma^2 T\right)$
- Let $Y_T \sim N\left(\mu T, \sigma^2 T\right)$, and $A = const$.

$$\ln V_T = Y_T + \ln A$$

- $$e^{\ln V_T} = e^{Y_T + \ln A} \Longrightarrow V_T = Ae^{Y_T}$$
- Let $\eta \sim N(0,1)$ be a standard normal random variable
- $$Y_T = \sigma\sqrt{T}\eta + \mu T$$
- $$V_T = Ae^{\sigma\sqrt{T}\eta + \mu T}$$
- Determine the constant $A$ from the condition $\mathbb{E}^T\left[V_T\right] = F(0, T)$.

# Calculate the Expected Value

- $$\mathbb{E}^T[V_T] = \mathbb{E}^T\left[Ae^{\sigma\sqrt{T}\eta+\mu T}\right] = A\mathbb{E}^T\left[e^{\sigma\sqrt{T}\eta+\mu T}\right]$$
- Use the moment generating function of the process $Y_T$ : $\mathbb{E}^T\left[e^{\sigma\sqrt{T}\eta+\mu T}\right] = e^{\frac{1}{2}\sigma^2 T+\mu T}$
- $$\mathbb{E}^T[V_T] = Ae^{\frac{1}{2}\sigma^2 T+\mu T}$$
- $$Ae^{\frac{1}{2}\sigma^2 T+\mu T} = F(0, T).$$
- This condition can be satisfied if:
$$\begin{cases} A = F(0, T) \\ \frac{1}{2}\sigma^2 T + \mu T = 0 \Longrightarrow \mu = -\frac{1}{2}\sigma^2 \end{cases}$$
- Therefore:
$$V_T = F(0, T)e^{\sigma\sqrt{T}\eta-\frac{1}{2}\sigma^2 T}$$

# Calculate the Expected Value

- The standard normal probability function is:

$$f(X) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}X^2}$$

- Proceed to calculate the expected value:

$$\begin{aligned} c(0) &= P(0,T)\,\mathbb{E}^T\left[\max\left(F(0,T)e^{\sigma\sqrt{T}\eta-\frac{1}{2}\sigma^2 T} - K, 0\right)\right] \\ &= P(0,T)\int_{-\infty}^{\infty}\max\left(F(0,T)e^{\sigma\sqrt{T}X-\frac{1}{2}\sigma^2 T} - K, 0\right) f(X)\,dX \end{aligned}$$

- Eliminate the max function:

$$F(0,T)e^{\sigma\sqrt{T}X-\frac{1}{2}\sigma^2 T} - K \geq 0 \Longrightarrow X \geq \frac{\ln\left(\frac{K}{F(0,T)}\right) + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}$$

- Denote

$$\frac{\ln\left(\frac{F(0,T)}{K}\right) - \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}} = d_2 \Longrightarrow X \geq -d_2$$

# Calculate the Expected Value

- $$c(0) = P(0,T)\int_{-d_2}^{\infty}\left(F(0,T)e^{\sigma\sqrt{T}X-\frac{1}{2}\sigma^2 T} - K\right) f(X)\,dX$$

- $$\begin{aligned} c(0) &= P(0,T)\int_{-d_2}^{\infty} F(0,T)e^{\sigma\sqrt{T}X-\frac{1}{2}\sigma^2 T} f(X)\,dX \\ &\quad -P(0,T)\int_{-d2}^{\infty} Kf(X)\,dX \end{aligned}$$

- $$c(0) = P(0,T)(I_1 - I_2).$$

- $$I_2 = K\int_{-d_2}^{\infty} f(X)\,dX = K\int_{-\infty}^{d_2} f(X)\,dX = KN(d_2),$$

- where $N(x)$ is the standard normal cumulative distribution function.

- $$I_1 = \int_{-d_2}^{\infty} F(0,T) e^{\sigma\sqrt{T}X - \frac{1}{2}\sigma^2 T} f(X)\, dX$$

- $$I_1 = F(0,T) \int_{-d_2}^{\infty} e^{\sigma\sqrt{T}X - \frac{1}{2}\sigma^2 T} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}X^2} dX$$

- $$I_1 = \frac{1}{\sqrt{2\pi}} F(0,T) \int_{-d_2}^{\infty} e^{-\frac{1}{2}\left(X^2 - 2\sigma\sqrt{T}X + \sigma^2 T\right)} dX$$

- $$I_1 = \frac{1}{\sqrt{2\pi}} F(0,T) \int_{-d_2}^{\infty} e^{-\frac{1}{2}\left(X - \sigma\sqrt{T}\right)^2} dX$$

- Change the variable:

$$\begin{aligned} Y &= X - \sigma\sqrt{T} \Longrightarrow dY = dX \\ X &= -d_2 \Longrightarrow Y = -d_2 - \sigma\sqrt{T} = -\frac{\ln\left(\frac{F(0,T)}{K}\right) + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}} \equiv -d_1 \end{aligned}$$

- $$I_1 = \frac{1}{\sqrt{2\pi}} F(0,T) \int_{-d_1}^{\infty} e^{-\frac{1}{2}Y^2} dY$$

- $$I_1 = F(0,T) \int_{-d_1}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}Y^2} dY = F(0,T) \int_{-d_1}^{\infty} f(Y)\, dY$$

- $$I_1 = F(0,T) \int_{-\infty}^{d_1} f(Y)\, dY = F(0,T)\, N(d_1)$$

- $$c(0) = P(0,T)\,(I_1 - I_2)$$

- $$c(0) = P(0,T)\left[F(0,T)\, N(d_1) - KN(d_2)\right].$$

- This is Black's formula!

# Log-Normal Version of Black's Model

- Similarly for a put:

$$
\begin{aligned}
\frac{p(0)}{P(0,T)} &= \mathbb{E}^{T}\left[\frac{\max(K - V_T, 0)}{P(T,T)}\right] \\
p(0) &= P(0,T)\,\mathbb{E}^{T}\left[\max(K - V_T, 0)\right] \\
p(0) &= P(0,T)\left[KN(-d_2) - F(0,T)\,N(-d_1)\right], \text{ where} \\
d_{1,2} &= \frac{\ln(F(0,T)/K) \pm \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}
\end{aligned}
$$

**Delayed Payoff**

- The payoff becomes known at time $T$, but the payment is actually made at some later time $T^* > T$.
- Then the expected payoff is discounted from time $T^*$ instead of $T$:

$$
\begin{aligned}
c(0) &= P(0,T^*)\left[F(0,T)\,N(d_1) - KN(d_2)\right], \\
p(0) &= P(0,T^*)\left[KN(-d_2) - F(0,T)\,N(-d_1)\right].
\end{aligned}
$$

- $d_1$ and $d_2$ are the same as above.

# Normal Version of Black's Model

- Assume that $V_T$ is normally distributed at expiration $T$ with a standard deviation equal to $\sigma_n\sqrt{T}$. Then we can write:

$$V_T = \mathbb{E}^T\left[V_T\right] + \sigma_n\sqrt{T}\eta = F\left(0, T\right) + \sigma_n\sqrt{T}\eta,$$

  where $\eta$ is a standard normal variable, $\eta \sim$Normal$(0, 1)$.
- If we compute $\mathbb{E}^T\left[\max\left(V_T - K, 0\right)\right]$ and plug in $c\left(0\right)$ and $p\left(0\right)$, we obtain:

$$\begin{aligned} c\left(0\right) &= P\left(0, T\right)\sigma_n\sqrt{T}\left[hN\left(h\right) + N'\left(h\right)\right], \\ p\left(0\right) &= P\left(0, T\right)\sigma_n\sqrt{T}\left[N'\left(h\right) - hN\left(-h\right)\right], \end{aligned}$$

  where $N\left(x\right)$ is the standard normal cumulative distribution function and

$$h = \frac{F\left(0, T\right) - K}{\sigma_n\sqrt{T}}.$$

# Converting Normal and Log-Normal Vols I

- The volatilities in the two versions of Black's model are different. Is there a relationship between them?
- In the log-normal version of Black's model we have:

$$\begin{aligned} V_T &= F(0,T)e^{Y_T}, \quad \text{where} \\ Y_T &\sim N\left(-\frac{1}{2}\sigma^2 T, \sigma^2 T\right). \end{aligned}$$

The moment-generating function of $Y_T$ is:

$$M(\lambda) = \mathbb{E}^T\left[e^{\lambda Y_T}\right] = e^{\lambda \mu T + \frac{1}{2}\lambda^2\sigma^2 T}.$$

- If we compute the variance of $V_T$ assuming that it is log-normally distributed we obtain:

# Converting Normal and Log-Normal Vols II

$$
\begin{aligned}
var^T\left[V_T\right] &= \mathbb{E}^T\left[V_T^2\right] - \mathbb{E}^T\left[V_T\right]^2 \\
&= \mathbb{E}^T\left[F(0,T)^2 e^{2Y_T}\right] - \mathbb{E}^T\left[F(0,T)\, e^{Y_T}\right]^2 \\
&= F(0,T)^2\, \mathbb{E}^T\left[e^{2Y_T}\right] - F(0,T)^2\, \mathbb{E}^T\left[e^{Y_T}\right]^2 \\
&= F(0,T)^2\, e^{2\left(-\frac{1}{2}\sigma^2\right)T + \frac{1}{2}4\sigma^2 T} - F(0,T)^2\left(e^{-\frac{1}{2}\sigma^2 T + \frac{1}{2}\sigma^2 T}\right)^2 \\
&= F(0,T)^2\left(e^{\sigma^2_{\text{lognormal}} T} - 1\right).
\end{aligned}
$$

- In the normal version of Black's model we have:

$$
\begin{aligned}
V_T &= F(0,T) + Y_T, \text{ where} \\
Y_T &\sim N\left(0, \sigma_n^2 T\right).
\end{aligned}
$$

# Converting Normal and Log-Normal Vols III

- If we compute the variance of $V_T$ assuming that it is normally distributed we obtain:

$$\begin{aligned} var^T [V_T] &= var^T [F(0,T) + Y_T] = var^T [Y_T] \\ &= \sigma^2_{\text{normal}} T. \end{aligned}$$

- By equating these two variances we obtain:

$$\sigma^2_{\text{normal}} = \frac{F(0,T)^2}{T} \left( e^{\sigma^2_{\text{lognormal}} T} - 1 \right).$$

- This can be approximated by

$$\sigma_{\text{normal}} \simeq F(0,T)\, \sigma_{\text{lognormal}}.$$

- Since option prices are quoted in terms of volatilities in the market place, this is a useful relation for translating log-normal into normal volatilities and vice-versa.

# Black's Model with CSA or with No CSA

## Black's Model in the Presence of CSA

We assume that the collateral is always equal to the option value:

$$
\begin{aligned}
c_{CSA}(0) &= P_C(0,T)\mathbb{E}^T\left[\max\left(V_T - K, 0\right)\right], \\
p_{CSA}(0) &= P_C(0,T)\mathbb{E}^T\left[\max\left(K - V_T, 0\right)\right],
\end{aligned}
$$

where the expected value is taken under the **risk-free $T$-forward measure** $\mathbb{Q}^T$ defined by the risk-free $P_C$ bond as numéraire.

## Black's Model in the Absence of CSA

Collateral $= 0$

$$
\begin{aligned}
c_{noCSA}(0) &= P_F(0,T)\widetilde{\mathbb{E}}^T\left[\max\left(V_T - K, 0\right)\right], \\
p_{noCSA}(0) &= P_F(0,T)\widetilde{\mathbb{E}}^T\left[\max\left(K - V_T, 0\right)\right],
\end{aligned}
$$

where the expected value is taken under the **risky $T$-forward measure** $\widetilde{\mathbb{Q}}^T$ defined by the risky bond $P_F$ as numéraire.

# Theoretical Justification for Black's Model with Random IR

Since academics could not persuade traders that it is not theoretically justified to apply the Black-Scholes framework to the pricing of interest rate derivatives, they gave up and asked themselves this **question** instead:

**Problem**

*Can we build models in which Black's Formula is correct for pricing options with random interest rates?*

After struggling for about 20 years, they came up with this **answer**:

**Solution**

*Yes, and to do it we use the bond price as numéraire.*

Changing numeraire to a zero-coupon bond price simplifies option pricing in the presence of random interest rates and leads to a surprising result: **Black's model is correct after all**!

# Black's Model for Bond Options

- A bond option is an option to buy or sell a particular bond by a specified date for a specified price.
- Bond options trade in the over-the-counter market
- Bond options are also frequently embedded in bonds when they are issued to make them more attractive to either the issuer or the buyer.

  Examples of Embedded Bond Options:
- **Callable bond**: Contains provisions allowing the issuer to buy back the bond at a predetermined price at certain times in the future. The holder of such a bond has sold a call option to the issuer. The strike price is the predetermined price that must be paid by the issuer to the holder.
- **Puttable bond**: Contains provisions that allow the holder to demand early redemption at a predetermined price at certain times in the future.The holder of such a bond has purchased a put option on the bond as well as the bond itself.

# Valuation of European Bond Options

- In the beginning, the most straightforward approach was taken for valuing bond options, i.e., the Black-Scholes formula was used with the spot bond price as the underlying.
- The Black-Scholes formula requires a constant percentage volatility.
- The volatility of a bond is certainly not constant, since the price has to converge to par at maturity (the "pull-to-par" phenomenon).
- The easy-fix solution was to consider a non-traded quantity (the bond yield) as the underlying. The main advantage of this approach is that the yield does not exhibit a deterministic pull to par.
- The main drawback was that the yield is not a traded asset and, therefore, the Black-Scholes reasoning behind the self-financing replicating strategy that reproduces the final payoff of the option could not be easily adapted.
- The 'correct' solution was, of course, to use the Black – rather than the Black-Scholes – formula, with the forward price as underlying (as opposed to the spot price).

# A Price-Based Bond Option I

- Assume that the option expires at time $T > 0$ and has strike price $K$.
- The stochastic variable (the underlying) is $V_T = P(T, S)$, the price of the bond maturing at $S > T$:
- In the log-normal version of Black's model we assume that the underlying has a lognormal distribution at expiration under the $T$-forward measure, with mean

$$\mathbb{E}^T [P(T, S)] = P(0, T, S)$$

and standard deviation $\sigma_B \sqrt{T}$:

$$P(T, S) = \mathbb{E}^T [P(T, S)] \, e^{\sigma_B \sqrt{T} \eta - \frac{1}{2}\sigma^2 T} = P(0, T, S) \, e^{\sigma_B \sqrt{T} \eta - \frac{1}{2}\sigma^2 T}$$

where $\eta$ is a standard normal variable, $\eta \sim$Normal$(0, 1)$, and $P(0, T, S)$ is the forward bond price:

$$P(0, T, S) = \frac{P(0, S)}{P(0, T)}.$$

# A Price-Based Bond Option II

- Apply the numeraire (pricing) equation under the $T$-forward measure for the value of a call/put bond option:

$$\begin{aligned} c(0) &= P(0,T)\,\mathbb{E}^T\left[\max\left(P(T,S)-K\right)\right] \\ p(0) &= P(0,T)\,\mathbb{E}^T\left[\max\left(K-P(T,S)\right)\right]. \end{aligned}$$

- Calculate the expected values. We get the following formulas for the values of European bond call/put options, where in the post-crisis, multi-curve world, the discounting will be most likely OIS discounting:

$$c(0) = P_{OIS}(0,T)\left[P(0,T,S)\,N(d_1) - KN(d_2)\right]$$

$$p(0) = P_{OIS}(0,T)\left[KN(-d_2) - P(0,T,S)\,N(-d_1)\right]$$

where

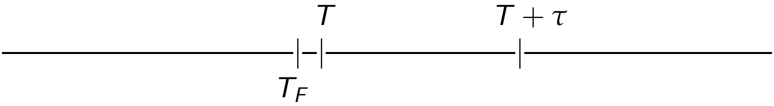
$$d_{1,2} = \frac{\ln\left(P(0,T,S)/K\right) \pm \frac{1}{2}\sigma_B^2 T}{\sigma_B\sqrt{T}}$$

# Caps and Floors

- A company that has liabilities funded at a floating (i.e., Libor) rate is naturally concerned with the possibility that interest rates, and thus its interest rate payments, may increase in the future. An **interest rate cap** is a security that allows one to benefit from low floating rates, yet be protected against high rates.
- Similarly, for an investor with assets earning a floating rate, a low-rate scenario is unfavorable. An **interest rate floor** is an instrument designed to protect against low interest rates, yet allow the holder to benefit from high rates.
- A **cap** is a strip of **caplets**, i.e., *call* options on successive LIBOR rates (corresponding to the floating cashflows of a swap).
- A **floor** is a strip of **floorlets**, i.e., *put* options on successive LIBOR rates.

# Caps and Floors

- Let's consider one floating leg of a swap:




- At time $T_F$: the underlying floating index (LIBOR) fixes
- At time $T$: interest starts accruing over the interval $[T, T + \tau]$. We will assume $T_F = T$.
- The caplet or floorlet expires at time $T_F = T$, when the underlying floating index (the forward LIBOR rate spanning the interval $[T, T + \tau]$) is fixed and decides the payoff of the option.

## Black's Model For Caplets I

- A *caplet* written on the LIBOR rate $L(T, T+\tau)$ has payoff:

$$\text{Payoff}_{caplet}(T+\tau) = N\tau \left[L(T, T+\tau) - K\right]_+$$

occuring at time $T+\tau$, the end of the period, where $N$ is the notional of the swap, $\tau$ is the day basis corresponding to the time interval $[T, T+\tau]$, $K$ is the cap strike rate. The caplet expires at time $T$.

- If it is payed at time $T$, the caplet payoff is instead:

$$\text{Payoff}_{caplet}(T) = N\tau \left[L(T, T+\tau) - K\right]_+ P(T, T+\tau).$$

- The underlying for the caplet is the LIBOR rate $L(T, T+\tau)$ spanning the period from $T$ to $T+\tau$ and set at time $T$ (technically it is set two business days before, but we are ignoring that).

- Log-normal version of Black's model: we assume that the underlying $L(T, T+\tau)$ is lognormally distributed at the expiration $T$ of the caplet under the risk-free $(T+\tau)$-forward measure with mean

$$\mathbb{E}^{T+\tau}\left[L(T, T+\tau)\right] = L(0, T, T+\tau)$$

and standard deviation $\sigma\sqrt{T}$:

$$L(T, T+\tau) = \mathbb{E}^{T+\tau}\left[L(T, T+\tau)\right] e^{\sigma\sqrt{T}\eta - \frac{1}{2}\sigma^2 T}$$

$$L(T, T+\tau) = L(0, T, T+\tau)\, e^{\sigma\sqrt{T}\eta - \frac{1}{2}\sigma^2 T}$$

where $\eta$ is a standard normal variable, $\eta \sim$Normal$(0, 1)$ , and $L(0, T, T+\tau)$ is today's value of the forward LIBOR.

# Black's Model For Caplets III

- According to the First Fundamental Theorem of Asset Pricing, the caplet value normalized by the zero-coupon bond maturing at time $T+\tau$ is a martingale under the risk-free $(T+\tau)$-forward measure in the absence of arbitrage:

$$\begin{aligned}
\frac{c(0)}{P(0,T+\tau)} &= \mathbb{E}^{T+\tau}\left[\frac{N\tau\left(L(T,T+\tau)-K\right)_+}{P(T+\tau,T+\tau)}\right] \\
c(0) &= N\tau P(0,T+\tau)\,\mathbb{E}^{T+\tau}\left[\left(L(T,T+\tau)-K\right)_+\right]
\end{aligned}$$

- The value of the caplet at $t=0$ is:

$$c(0) = N\tau P(0,T+\tau)\left[L(0,T,T+\tau)\,N(d_1) - KN(d_2)\right]$$

and a floorlet struck at $K$ with payoff

$$\text{Payoff}_{floorlet}(T+\tau) = N\tau\left[K - L(T,T+\tau)\right]_+$$

has value

$$
\begin{aligned}
p(0) &= N\tau P(0, T+\tau)\,\mathbb{E}^{T+\tau}\left[\left(K - L(T, T+\tau)\right)_+\right] \\
p(0) &= N\tau P(0, T+\tau)\left[KN(-d_2) - L(0, T, T+\tau)\,N(-d_1)\right]
\end{aligned}
$$

where

$$
d_{1,2} = \frac{\ln\left(L(0, T, T+\tau)/K\right) \pm \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}
$$

- The LIBOR curve will be used for the projection of the payoff. The discounting factors $P(0, T+\tau)$ will be provided either by the LIBOR curve or the OIS curve according to the specific funding rate.
- Each caplet has to be valued separately. One approach is to use a different volatility for each caplet. The volatilities are then referred to as *spot volatilities*.

- A European receiver (payer) swaption gives the holder the right to receive (pay) fixed on a forward starting swap. The fixed rate $c$ and tenor of the swap are established as part of the contract.

• Let the fixed payment times are $t_1 < t_2 < ... < t_n$, and $\{\tau_k\}_{k=1}^n$ are the lengths of the accrual periods in fractions of a year:

$$\tau_k = (t_k - t_{k-1}) \quad \text{expressed with a 30/360 day count convention}$$

• The swaption expires at $t_0$, the begin date of the first accrual period. The fixed rate of the swap is $c$ and the notional is $N = 1$.
• A payer's swaption has a payoff that depends on the value of the swap at expiration $t_0$:

$$\text{Payoff (swaption)} = \left( P(t_0, t_0) - P(t_0, t_n) - c \sum_{k=1}^{n} \tau_k P(t_0, t_k) \right)$$

# European Swaptions II

## Libor Swaps with Matched Discounting

- The swaption value at $t = 0$ under the $t_0$-forward measure is:

$$c(0) = P(0, t_0)\, \mathbb{E}^{t_0}\left[\left(P(t_0, t_0) - P(t_0, t_n) - c \sum_{k=1}^{n} \tau_k P(t_0, t_k)\right)_+\right].$$

- There are many quantities involved: exactly $n$ random variables.
- The $t_0$-forward measure was nice in the case of one single payment date, but it does not help in this case. Here we are going to use another numeraire to value this option.
- Let $A_t$ be the annuity

$$A_t = \sum_{k=1}^{n} \tau_k P(t, t_k).$$

- The annuity $A_t$ is often called the *monetizer*. It is an asset with positive price process, so it can be used as a numeraire.

# European Swaptions III

## Libor Swaps with Matched Discounting

- Using $A_t$ as our new numeraire, the swaption value can be rewritten as:

$$c(0) = A_0 \mathbb{E}^A \left[ \frac{\left(P(t_0, t_0) - P(t_0, t_n) - c \sum_{k=1}^{n} \tau_k P(t_0, t_k)\right)_+}{A_{t_0}} \right]$$

- The expected value is taken under the probability measure associated with the numeraire $A_t$. It can be simplified to be:

$$c(0) = A_0 \mathbb{E}^A \left[ (s_{t_0} - c)_+ \right], \text{ where}$$

$$s_{t_0} = \frac{P(t_0, t_0) - P(t_0, t_n)}{\sum_{k=1}^{n} \tau_k P(t_0, t_k)}$$

is the **break-even swap rate**.

**Fact**

*We have reduced the swaption to a call on a swap rate!*

# European Swaptions IV

## Libor Swaps with Matched Discounting

• We can now assume that under this measure the break-even swap rate $s_{t_0}$ has a log-normal or normal distribution.

• Log-normal version of Black's model: We assume that the break-even swap rate $s_{t_0}$ is log-normally distributed at $t_0$ with mean:

$$s_0 = \mathbb{E}^A \left[ s_{t_0} \right] = \frac{P(0, t_0) - P(0, t_n)}{\sum_{k=1}^{n} \tau_k P(0, t_k)},$$

and standard deviation $\sigma\sqrt{t_0}$. Then we can represent $s_{t_0}$ as :

$$s_{t_0} = s_0 e^{\sigma\sqrt{t_0}\eta - \frac{1}{2}\sigma^2 t_0}$$

• Next we derive the formulas for the option values like we did before by calculating the expected value. The formulas will look very similar, the only difference will be the discounting by the annuity, not by the zero-coupon bond.

- The value of the payer's swaption at $t = 0$ is:

$$c(0) = A_0 \left[ s_0 N(d_1) - c N(d_2) \right]$$

- The value of the receiver's swaption at $t = 0$ is:

$$p(0) = A_0 \left[ c N(-d_2) - s_0 N(-d_1) \right],$$

where

$$d_{1,2} = \frac{\ln(s_0 / c) \pm \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}$$

• The simplifications we benefited from in the case of matched discounting are no longer applicable here, and the differences in fixed and floating frequencies and number of payments become visible.
• Let $m$ be the number of floating payment periods, $t_1^{'} < t_2^{'} < ... < t_m^{'}$ be the floating payment times, and $\left\{\tau_j^{'}\right\}_{j=1}^{m}$ be the lengths of the accrual periods in fractions of a year.

$$\text{Payoff}\,(\text{swaption}) = \left(\sum_{j=1}^{m} \tau_j^{'} L\left(t_0, t_{j-1}^{'}, t_j^{'}\right) P\left(t_0, t_j^{'}\right) - c \sum_{k=1}^{n} \tau_k P\left(t_0, t_k\right)\right)_+$$

• Choose the annuity $A_t$ as numeraire again:

$$A_t = \sum_{k=1}^{n} \tau_k P\left(t, t_k\right).$$

# European Swaptions II

## LIBOR Swaps with OIS Discounting

- The swaption value at $t = 0$ is:

$$c(0) = A_0 \mathbb{E}^A \left[ \frac{\left( \sum_{j=1}^{m} \tau_j' L\left(t_0, t_{j-1}', t_j'\right) P\left(t_0, t_j'\right) - c \sum_{k=1}^{n} \tau_k P\left(t_0, t_k\right) \right)_+}{A_{t_0}} \right]$$

- Denote by $s_{t_0}$ the **break-even swap rate**:

$$s_{t_0} = \frac{\sum_{j=1}^{m} \tau_j' L\left(t_0, t_{j-1}', t_j'\right) P\left(t_0, t_j'\right)}{\sum_{k=1}^{n} \tau_k P\left(t_0, t_k\right)}.$$

- Rewrite:

$$c(0) = A_0 \mathbb{E}^A \left[ \left(s_{t_0} - c\right)_+ \right].$$

- We can now specify a distribution for $s_{t_0}$.

• In the log-normal version of Black's model assume that $s_{t_0}$ is log-normally distributed at expiration $t_0$ of the swaption with mean

$$s_0 = \mathbb{E}^A\left[s_{t_0}\right] = \frac{\sum_{j=1}^{m} \tau_j^{'} L\left(t, t_{j-1}^{'}, t_j^{'}\right) P\left(0, t_j^{'}\right)}{\sum_{k=1}^{n} \tau_k P\left(0, t_k\right)}$$

and standard deviation $\sigma\sqrt{t_0}$. Then:

$$s_{t_0} = s_0 e^{\sigma\sqrt{t_0}\eta - \frac{1}{2}\sigma^2 t_0}.$$

• Calculate the expected values $\mathbb{E}^A\left[\left(s_{t_0} - c\right)_+\right]$ and $\mathbb{E}^A\left[\left(c - s_{t_0}\right)_+\right]$. We get the same formulas as before for the values of $c\left(0\right)$ and $p\left(0\right)$, but with a different value for $s_0$.

# Interest Rate Exposure and Hedging

- The value of a fixed-income portfolio changes as interest rates change. The rate of this change is called *exposure*.
- When discussing options, we are concerned with the changes in options value as the underlying changes. For equity options the underlying is the value of a stock, for FX options, it is the FX rate.
- For interest-rate options the underlying is the *whole term curve*. So, unlike other markets, we have a multidimensional exposure. We have risk to the whole curve, but different "points" (forward rates, zero rates, discount factors) on the term curve affect the value of an interest-rate option in different ways.
- Hedging interest rate derivatives can be tricky because in some cases one cannot trade the underlying!
- To hedge a portfolio we first need to understand how the value of the portfolio changes under different interest-rate scenarios. Then we create a hedging portfolio that responds to the same scenarios in the opposite way such as to minimize the change in value of the aggregation of these two portfolios.

# Why Study Exposure?

- To avoid losses when markets go against us.
- To make a profit if we have a view of where the markets are headed (i.e., we want to bet on a particular change in interest rates) — then we need to know what instruments will rise in value if such a view is correct. Here are some examples of views:
  - Interest rates will go up.
  - The curve will become steeper (e.g., 10y rates will rise more than 5y rates). In such a case, one would like the portfolio to be hedged against parallel moves since we don't have a view on that.
  - Volatility at the short end will decrease if long rates rise.

# First-order, second-order exposure I

- *First-order exposure*: how the value of the portfolio will change if the underlying changes. It corresponds to the first derivative with respect to the underlying, or *delta*.
- *Second-order* exposure:how the first-order exposure will change if the underlying changes. It corresponds to the second derivative or *gamma*.
- *Second-order cross* exposure: how the exposure to one underlying will change if another underlying changes. This corresponds to a second cross derivative or *cross gamma*.

- The delta risk is the risk associated with a shift in the zero curve. Because there are many ways in which the zero curve can shift, many deltas can be calculated. Some choices are:

▶ Calculate the impact of a 1 basis-point parallel shift in the zero curve. This is sometimes called DV01.

▶ Calculate the impact of small changes in the quotes of each of the instruments used to construct the zero curve.

# First-order, second-order exposure II

- Divide the zero curve (or the forward curve) into a number of sections (or buckets). Calculate the impact of shifting the rates in one bucket by 1 basis point, keeping the rest of the initial term structure unchanged.
- Carry out a principal components analysis (see Yuri's lecture on the Statistical Model). Calculate a delta with respect to the changes in each of the first few factors. The first delta then measures the impact of a small, approximately parallel, shift in the zero curve; the second delta measures the impact of a small steepening in the zero curve; and so on.

# Numerical Delta Hedging

- We can describe a term curve in a number of equivalent ways: in terms of discount factors, zero rates, or forward rates. Different definitions of the term curve would imply different (but similar) notions of hedging.
- Suppose the value $B$ of a book depends on some forward rate $f$, i.e., $B = B(f)$. Then we define the delta of $B$ with respect to $f$ as $\frac{dB(f)}{df}$. This is the rate of change of the portfolio value when this particular forward rate changes (first-order exposure). We can create similar expressions for deltas with respect to discount factors and zero rates.
- We usually take numerical derivatives (actual change versus change in rate) instead of analytic, like

$$\frac{dB(f)}{df} = \frac{B(f + 1\text{bp}) - B(f)}{1\text{bp}}, \text{ where } 1 \text{ bp} = 10^{-4} \text{ (1 basis point)}$$

- A typical book depends on all forward rates and it is very impractical to show deltas for all forward rates. We present here two approaches: **bucket hedging** and **factor hedging**.

# Bucket Hedging I

- This approach is mainly used by individual trading desks for computing sensitivities.
- Divide the continuous time axis into buckets, for example 3 month buckets corresponding to Eurodollar contracts. The forward rate for each bucket will be an independent variable and so we will have a vector variable $f$ made of forward rates $\overrightarrow{f} = (f_1, ..., f_n)$. Assume we can value the book given a vector $\overrightarrow{f}$ as $B = B(\overrightarrow{f})$. Note that we will probably have to interpolate other forward rates to value the book. The bucket-delta vector is then defined to be

$$\left( \frac{\partial B(f)}{\partial f_1}, ..., \frac{\partial B(f)}{\partial f_n} \right).$$

- Rates go out to 30 years, so it amounts to 120 buckets (and deltas). But individual buckets do not usually move independently. A combined move is more likely. Is then the concept of bucket delta useless? No.

# Bucket Hedging II

- If bucket deltas are computed, it is easy to calculate the delta exposure to a realistic move. Suppose that after a market move, each $i$-th forward rate moves by $\alpha_i$ basis points. Then the change in the value of the portfolio is approximately

$$\sum_i \frac{\partial B(f)}{\partial f_i} \frac{\alpha_i}{10000}.$$

- That is, the delta of any realistic move can be expressed as a linear combination of bucket deltas. Borrowing the concept from linear algebra, we call this decomposition a basis.
- A portfolio is said to be delta-neutral if the vector of deltas is 0. This is a local condition, in the sense that if the portfolio is delta-neutral today, it might be exposed tomorrow. To be delta neutral one should hedge against all the moves in the underlying term curve.

# Factor Hedging

- This approach is used for scenario analysis (VaR) at the firm level.
- We will define a different basis. Recall Yuri's lecture on factor analysis. Statistically a generic change of an instantaneous forward rate $f(t)$ is approximately equal to $L_1(T)\,x_1(t) + L_2(T)\,x_2(t) + L_3(T)\,x_3(t)$, where $L_1(T)$, $L_2(T)$, $L_3(T)$ are the loadings and $x_1$, $x_2$ and $x_3$ are changes in the factors. So, we reduce the infinite dimensional space of possible curve moves to a three-dimensional one.
- The risk can then be expressed as a combination of three factor deltas

$$\frac{\partial B(f + L_i x)}{\partial x},$$

where $f$ is the forward curve as a whole and $L_i x$ is the perturbation of the $i$-th loading of the curve. If these three factor deltas are zeroed, then we are hedged to (almost) all changes in the term curve. It seems like a great improvement over bucket hedging - instead of 120 deltas we get to use only three. Unfortunately, these three deltas do not tell nearly as much as bucket deltas about the risk of the portfolio.

# Bucket-Hedging a Swap Analytically I

A swap value can be written as a linear function of discount factors. For that reason, deltas with respect to discount factors are very easy to work with. For the rest of this section, delta will mean discount factor delta and bucket hedging will mean computing delta with respect to all discount factors to which an instrument is sensitive.

Consider a tenor structure

$$0 < t_0 < t_1 < \ldots < t_n$$

with $\tau_i = t_i - t_{i-1}$.

A payer's vanilla swap with fixed rate $c$, notional $N$, forward starting at $t_0$ and matching dates for both legs has value $S$ equal to

$$S = N\left(P\left(t_0, t_0\right) - P\left(t_0, t_n\right)\right) - Nc\sum_{i=1}^{n} \tau_i P\left(t_0, t_i\right).$$

# Bucket-Hedging a Swap Analytically II

The deltas with respect to each discount factor are:

$$\begin{aligned}\frac{\partial S}{\partial P(t_0, t_0)} &= N, \\ \frac{\partial S}{\partial P(t_0, t_i)} &= -Nc\tau_i,\end{aligned}$$

for $i = 1, ..., n-1$, and

$$\frac{\partial S}{\partial P(t_0, t_n)} = -N - Nc\tau_n.$$

# Bucket-Hedging a Swaption Analytically in Black's Model I

Consider an option on this swap (payer's swaption) expiring at time $t_0$. The log-normal Black's value $C_0$ of the swaption is

$$C_0 = NA_0 \left( s_0 \Phi \left( d_1 \right) - c \Phi \left( d_2 \right) \right)$$

where $N$ is the notional, $\Phi \left( x \right)$ is the normal cumulative distribution function, $c$ is the fixed coupon rate and $A_0$ is the corresponding annuity

$$A_0 = \sum_{i=1}^{n} \tau_i P \left( 0, t_i \right).$$

Clearly, $s_0$ and $A_0$ are functions of the discount factors $P \left( 0, t_0 \right), \ldots, P \left( 0, t_n \right)$, and so will be $C_0$.

# Bucket-Hedging a Swaption Analytically in Black's Model II

Let's compute $\frac{\partial C_0}{\partial P(0,t_i)}$, $i = 1, ..., n$ :

$$\begin{aligned} \frac{\partial C_0}{\partial P(0,t_i)} &= N\left(s_0\Phi\left(d_1\right) - K\Phi\left(d_2\right)\right)\frac{\partial A_0}{\partial P(0,t_i)} \\ &\quad + NA_0\frac{\partial\left(s_0\Phi\left(d_1\right) - K\Phi\left(d_2\right)\right)}{\partial s_0}\frac{\partial s_0}{\partial P(0,t_i)} \end{aligned}$$

Observe that

$$\begin{aligned} \frac{\partial\left(s_0\Phi\left(d_1\right) - K\Phi\left(d_2\right)\right)}{\partial s_0} &= \Phi\left(d_1\right) \\ \frac{\partial A_0}{\partial P(0,t_i)} &= \tau_i \end{aligned}$$

# Bucket-Hedging a Swaption Analytically in Black's Model III

Remember from the previous section that

$$s_0 = E^A\left[s_{t_0}\right] = \frac{P\left(0, t_0\right) - P\left(0, t_n\right)}{\sum_{k=1}^{n} \tau_k P\left(0, t_k\right)} = \frac{P\left(0, t_0\right) - P\left(0, t_n\right)}{A_0}.$$

Then from $s_0 A_0 = P\left(0, t_0\right) - P\left(0, t_n\right)$ we obtain

$$\begin{aligned}
A_0 \frac{\partial s_0}{\partial P\left(0, t_0\right)} + s_0 \frac{\partial A_0}{\partial P\left(0, t_0\right)} &= A_0 \frac{\partial s_0}{\partial P\left(0, t_0\right)} + s_0 \tau_0 = 1 \\
A_0 \frac{\partial s_0}{\partial P\left(0, t_n\right)} + s_0 \frac{\partial A_0}{\partial P\left(0, t_n\right)} &= A_0 \frac{\partial s_0}{\partial P\left(0, t_n\right)} + s_0 \tau_n = -1 \\
A_0 \frac{\partial s_0}{\partial P\left(0, t_i\right)} + s_0 \frac{\partial A_0}{\partial P\left(0, t_i\right)} &= A_0 \frac{\partial s_0}{\partial P\left(0, t_i\right)} + s_0 \tau_i = 0, \;\; i \neq 0, n
\end{aligned}$$

# Bucket-Hedging a Swaption Analytically in Black's Model IV

Putting everything together, we have

$$
\begin{aligned}
\frac{\partial C_0}{\partial P(0, t_0)} &= \frac{C_0}{A_0}\tau_0 + N\Phi(d_1)(1 - s_0\tau_0) \\
\frac{\partial C_0}{\partial P(0, t_n)} &= \frac{C_0}{A_0}\tau_n - N\Phi(d_1)(1 + s_0\tau_n) \\
\frac{\partial C_0}{\partial P(0, t_i)} &= \frac{C_0}{A_0}\tau_i - N\Phi(d_1)\, s_0\tau_i, \;\; i \neq 0, n.
\end{aligned}
$$

# Conclusion I