HEATH-JARROW-MORTON FRAMEWORK

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1. Introduction

We are going to explore change of measure and its effect on pricing in the HJM framework. Our intention is to derive useful results for pricing derivatives.

Setup. We assume a *probability space*[1] consisting of the triple

$$(\Omega, \mathcal{F}, \mathbb{P})$$

where Ω is a *sample space*, $\mathcal{F}$ is a *σ-algebra*, and $\mathbb{P} : \mathcal{F} \to [0, 1]$ is a *probability measure*. We also assume the existence of a *filtration*

$$\{\mathcal{F}_t \subset \mathcal{F} \,|\, t \in [0, +\infty)\,\}$$

of σ-algebras such that

$$\mathcal{F}_s \subset \mathcal{F}_t \subset \mathcal{F}, \qquad \text{for} \quad 0 \le s \le t < +\infty.$$

Each *random variable* we encounter will be *$\mathcal{F}$-measurable*. We will consider only those *stochastic processes* that have continuous trajectories $\mathbb{P}$ a.s. In addition we will only deal with *$\mathcal{F}_t$-adapted processes*. These are known as *predictable* or *previsible processes*.

2. Girsanov's Theorem and Changes of Measure

Recall Girsanov's Theorem (see [10, p. 290]), as it will be required to obtain the risk-neutral measure. The transition from real world measure to risk-neutral measure will enable us to recover a straight forward method to value contingent claims.

[1] You should already be familiar with any emphasized words in this document. If not, definitions can be found in the glossary.

Theorem 1. *(Girsanov's Theorem) Let W_t, $0 \le t \le T$ be a Brownian motion on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with $\mathcal{F}_t$, $0 \le t \le T$ the accompanying filtration. Let θ_t, $0 \le t \le T$ be an $\mathcal{F}_t$-adapted process satisfying the Novikov condition*

$$\mathbb{E}\left[\exp\left\{\frac{1}{2}\int_0^T (\theta_t)^2\, dt\right\}\right] < +\infty.$$

Then the process

$$\widetilde{W_t} = W_t + \int_0^t \theta_s ds, \qquad 0 \le t \le T,$$

is a Brownian motion under the new probability measure

$$\mathbb{Q}\,(A) = \int_A \xi_T d\mathbb{P}, \qquad \forall A \in \mathcal{F}$$

where

$$\xi_t = \exp\left\{-\int_0^t \theta_s dW_s - \frac{1}{2}\int_0^t (\theta_s)^2\, ds\right\}.$$

It is worth noting that the process ξ_t is a martingale and that the measure $\mathbb{Q}$ is equivalent to $\mathbb{P}$.

3. Replicating Strategies

The concept of a *replicating strategy* is very important. Let's recall the definition. Suppose we have an economy with two primary instruments I and J. Their prices at time t are denoted by I_t and J_t, respectively.

Definition 2. A *strategy* is a pair of predictable processes (ϕ_t, ψ_t) representing the amounts of instruments I and J we hold at time t, respectively. The value of this portfolio at time t is

$$\Pi_t = \phi_t I_t + \psi_t J_t.$$

A strategy is called *self-financing* if the change in value of the associated portfolio arises solely from the changes in the prices of the underlying instruments, i.e.

$$d\Pi_t = \phi_t dI_t + \psi_t dJ_t.$$

This does not imply that ϕ_t and ψ_t are constant.

A self-financing strategy is a *replicating* strategy for the claim with (random) payoff X at time T if

$$\Pi_T = X, \quad \mathbb{P}\text{ a.s.}$$

Our approach will be to construct replicating strategies to price derivative securities. Of course, this is only possible provided the following condition is satisfied.

Condition 3. (No Free Lunch) Any two self-financing strategies with the same payoff in the future must also have the same value today.

Under this assumption, if the value of a replicating strategy at expiration is equal to the payoff of a contingent claim, then the value of the claim today is equal to the initial investment required for the replicating strategy. Otherwise there would be an *arbitrage* opportunity! We will denote the value at time t of the derivative that pays X at some future time T by

$$\pi_t(X).$$

It follows from the No Free Lunch Condition that $\pi_t(X)$ must be equal to Π_t.

Still the question remains: given two traded instruments, how do we replicate one with the other? The following theorem provides a partial answer, see [9, p. 182].

Theorem 4. *(Martingale Representation Theorem) Let the filtration $\{\mathcal{F}_t\}_{t=0}^{+\infty}$ be generated by a Brownian motion W_t, $t \geq 0$ under some measure $\mathbb{Q}$ on $(\Omega, \mathcal{F})$. Let the process N_t be an $\mathcal{F}_t$-adapted continuous square-integrable martingale under the same measure. Then there exists a predictable process v_t, $t \geq 0$ such that*

$$\mathbb{E}^{\mathbb{Q}}\left[\int_0^T v_t^2 dt\right] < +\infty$$

and

$$N_t = N_0 + \int_0^t v_s dW_s.$$

Corollary 5. *If $v_s > 0$, $\mathbb{Q}$a.s. and M_t is another $\mathcal{F}_t$-adapted continuous square-integrable martingale (under $\mathbb{Q}$), then there exists another predictable process ϕ_t such that*

$$M_t = M_0 + \int_0^t \phi_s dN_s.$$

We can interpret this corollary as being able to "replicate" one martingale M by holding ϕ_t units of the martingale N at time t. Our goal is to find a measure under which the values of traded instruments are martingales. Then we can use the corollary to replicate them.

4. Comments About Term Structure Modelling

As we saw last quarter, the price at time t of a zero-coupon bond which pays \$1 at time T is denoted by[2]

$$P_t^T.$$

The instantaneous forward rate at time t for the forward accrual period $[T, T+0]$ is denoted by

$$f_t^T.$$

And the short rate at time t is denoted by

$$r_t = f_t^t.$$

Recall that by definition

$$P_t^T = \exp\left\{-\int_t^T f_t^u du\right\}.$$

[2]This notation may be used interchangeably with the alternate $P(t,T)$ as a matter of convenience.

This allows us to recover the bond prices from the instantaneous forward rates and vice versa. Thus it is equivalent to specify the random dynamics of the bond prices or of the forward rates.

Note that for bonds P_t^T and forward rates f_t^T, the superscript or second argument T functions as an index which differentiates various traded instruments. The subscript or first argument t functions as an index representing the time at which the random variable is observed. For instance were t to vary and T to remain fixed, we would have the stochastic process of a single bond[3] $\{P_t^T\}_{t=0}^T$. We will restrict our consideration to the case where this process is predictable. The same must clearly be true for the process $\{f_t^T\}_{t=0}^T$.

5. What Does Arbitrage Free Mean?

Suppose we were to create a term structure model by imposing dynamics on the random processes for the bond prices[4] P_t^T. How would we know if our model was a sensible one?

In the case of a one-factor model, we will see that we can replicate an S-maturity bond using another T-maturity bond, as long as $T > S$. This will produce two prices for the S-maturity bond:

- one given by the original model specification;
- another obtained from the replicating strategy.

The Heath-Jarrow-Morton approach provides a framework for constructing models where these prices coincide. Thus automatically preventing any[5] "model arbitrage" non-sense.

6. Single-Factor HJM

6.1. Assumptions of HJM. Start by fixing a Wiener process W_s, $s \geq 0$ under the (original) measure $\mathbb{P}$. Assume the filtration $\{\mathcal{F}_t\}_{t=0}^{+\infty}$ is generated by W. The HJM framework specifies the form

$$f_t^T = f_0^T + \int_0^t \sigma_s^T dW_s + \int_0^t \alpha_s^T ds, \qquad 0 \leq t \leq T \tag{6.1}$$

for the dynamics of the forward rates. Here σ_s^T and α_s^T are very general processes. The only technical assumptions placed on them are that they are

- predictable processes of s;
- square-integrable in s;
- square-integrable in both arguments combined.

Specifying the forward rates' evolution by (6.1) completely determines the term structure dynamics. We derive the processes for other quantities below.

[3]We generally assume that t cannot exceed T as it would not make much economic sense.

[4]Obviously we would need to do this for each maturity time T.

[5]By this we mean cases where the same instrument can have two distinct prices at the same time.

6.1.1. *Bond Price Dynamics.*

$$
\begin{aligned}
(6.2)\qquad P_t^T &= \exp\left\{-\int_t^T f_t^u du\right\} \\
&= \exp\left\{-\int_t^T \left(f_0^u + \int_0^t \sigma_s^u dW_s + \int_0^t \alpha_s^u ds\right) du\right\} \\
&= \exp\left\{-\int_0^t \int_t^T \sigma_s^u du dW_s - \int_t^T f_0^u du - \int_0^t \int_t^T \alpha_s^u du ds\right\}
\end{aligned}
$$

6.1.2. *Money Market Dynamics.* The money market account B_t is the value of \$1 invested at time zero and rolled up every instant at the prevailing short rate and is defined by

$$
(6.3)\qquad dB_t = r_t B_t dt, \qquad B_0 = 1.
$$

This can be rewritten as

$$
B_t = \exp\left\{\int_0^t r_s ds\right\}.
$$

Recall that $r_t = f_t^t$ so that

$$
r_u = f_0^u + \int_0^u \sigma_s^u dW_s + \int_0^u \alpha_s^u ds
$$

and

$$
\begin{aligned}
(6.4)\qquad B_t &= \exp\left\{\int_0^t \left(f_0^u + \int_0^u \sigma_s^u dW_s + \int_0^u \alpha_s^u ds\right) du\right\} \\
&= \exp\left\{\int_0^t \int_s^t \sigma_s^u du dW_s + \int_0^t f_0^u du + \int_0^t \int_s^t \alpha_s^u du ds\right\}.
\end{aligned}
$$

6.1.3. *Dynamics of Discounted Bonds.* We want to compare values of instruments at different points in time. In order to do this we need to somehow express them in the same units, in this case, "time 0 dollars."

We define the discounted T-maturity bond by

$$
(6.5)\qquad Z_t^T = B_t^{-1} P_t^T.
$$

This is the value in today's ($t = 0$) dollars of a bond at time t.

Combining (6.2) and (6.4) yields

$$
\begin{aligned}
Z_t^T &= B_t^{-1} P_t^T \\
&= \exp\left\{-\int_0^t \int_s^t \sigma_s^u du dW_s - \int_0^t f_0^u du - \int_0^t \int_s^t \alpha_s^u du ds\right\} \\
&\quad \times \exp\left\{-\int_0^t \int_t^T \sigma_s^u du dW_s - \int_t^T f_0^u du - \int_0^t \int_t^T \alpha_s^u du ds\right\} \\
&= \exp\left\{-\int_0^t \int_s^T \sigma_s^u du dW_s - \int_0^T f_0^u du - \int_0^t \int_s^T \alpha_s^u du ds\right\} \\
&= \exp\left\{\int_0^t \Sigma_s^T dW_s - \int_0^T f_0^u du - \int_0^t A_s^T ds\right\}
\end{aligned}
$$

where

$$\Sigma_s^T = -\int_s^T \sigma_s^u du$$
$$A_s^T = \int_s^T \alpha_s^u du.$$

In particular, using Ito's lemma,

$$dZ_t^T = Z_t^T \left(\Sigma_t^T dW_t - A_t^T dt\right) + Z_t^T \times \frac{1}{2}\left(\Sigma_t^T\right)^2 dt. \tag{6.6}$$

6.2. **Replicating an Arbitrary Payoff.** As mentioned earlier, we wish to construct a replicating strategy for an S-maturity bond using the T-maturity bond. We will do one better by replicating an arbitrary payoff using only the T-maturity bond and the money market account.

Let the payoff X be an $\mathcal{F}_S$-measurable random variable, i.e. its value is "known" at time S. If all of the relevant processes were martingales, then we could apply the Martingale Representation Theorem to construct a replicating strategy for this payoff. This is not the case, so we will instead take these steps:

(1) Find a measure $\mathbb{Q}$ under which the discounted bond Z_t^T is a martingale;
(2) Find a $\mathbb{Q}$-martingale V_t which recovers the discounted payoff $V_S = B_S^{-1}X$;
(3) Find a predictable process ϕ_t such that $V_t = V_0 + \int_0^t \phi_s dZ_s^T$;
(4) Use ϕ_t to build a replicating strategy.

6.2.1. *Step 1.* By rearranging (6.6) we have

$$dZ_t^T = Z_t^T \Sigma_t^T \left(dW_t + \left(\frac{1}{2}\Sigma_t^T - \frac{A_t^T}{\Sigma_t^T}\right) dt\right). \tag{6.7}$$

If we define

$$\gamma_t = \frac{1}{2}\Sigma_t^T - \frac{A_t^T}{\Sigma_t^T} \tag{6.8}$$

then

$$dZ_t^T = Z_t^T \Sigma_t^T \left(dW_t + \gamma_t dt\right).$$

The process Z_t^T will be a martingale if the drift can be eliminated from the SDE. From Girsanov's Theorem we know that changes in drift correspond to changes in measure. In particular, there must be a measure $\mathbb{Q}$ under which

$$d\widetilde{W}_t = dW_t + \gamma_t dt \tag{6.9}$$

is a Brownian motion. Thus

$$dZ_t^T = Z_t^T \Sigma_t^T d\widetilde{W}_t \tag{6.10}$$

defines a martingale under $\mathbb{Q}$.

6.2.2. *Step 2.* The process V_t defined by

$$V_t = \mathbb{E}^{\mathbb{Q}}\left[B_S^{-1} X \middle| \mathcal{F}_t \right]$$

is automatically a martingale such that

$$V_S = B_S^{-1} X$$

as required.

6.2.3. *Step 3.* By the Martingale Representation Theorem there exists a predictable process ϕ_t such that

$$V_t = V_0 + \int_0^t \phi_s dZ_s^T.$$

6.2.4. *Step 4.* Define

$$\psi_t = V_t - \phi_t Z_t^T$$

as part of the strategy (ϕ_t, ψ_t) which holds

- ϕ_t units of the bond P_t^T; and
- ψ_t units of the money market account B_t.

Regardless of the quantity ϕ_t, the associated portfolio value is

$$\begin{aligned}
\Pi_t &= \phi_t P_t^T + \psi_t B_t \\
&= \phi_t P_t^T + \left(V_t - \phi_t Z_t^T \right) B_t \\
&= \phi_t P_t^T + \left(V_t - \phi_t B_t^{-1} P_t^T \right) B_t \\
&= \phi_t P_t^T + V_t B_t - \phi_t P_t^T \\
&= V_t B_t.
\end{aligned}$$

In particular

$$\Pi_S = V_S B_S = B_S^{-1} X B_S = X.$$

So, it replicates the payoff X at time S.

We must also check that the strategy is self-financing[6]

$$\begin{aligned}
d\Pi_t &= B_t dV_t + V_t dB_t \\
&= B_t \phi_t dZ_t^T + V_t dB_t \\
&= \left(\phi_t d\left(B_t Z_t^T \right) - \phi_t Z_t^T dB_t \right) + V_t dB_t \\
&= \phi_t dP_t^T + \left(V_t - \phi_t Z_t^T \right) dB_t \\
&= \phi_t dP_t^T + \psi_t dB_t.
\end{aligned}$$

6.2.5. *Results.* We have constructed a replicating strategy for an arbitrary payoff X using the T-maturity bond P_t^T and the money market account B_t. Therefore, by the No Free Lunch Condition, the value of the claim must coincide with Π_t and is given by

$$\pi_t(X) = \Pi_t = V_t B_t = B_t \mathbb{E}^{\mathbb{Q}}\left[B_S^{-1} X \middle| \mathcal{F}_t \right]. \tag{6.11}$$

The importance of this formula cannot be overemphasized. It shows that the value of any derivative depends on the dynamics of the bond only under the measure $\mathbb{Q}$, not the original measure $\mathbb{P}$. The measure $\mathbb{Q}$ has a special name, the *risk-neutral measure*.

[6]The logic employed here makes use of the fact that B_t has no stochastic driver.

6.3. **Replicating One Bond with Another.** Let us go back to the problem of replicating the bond P_t^S using P_t^T and B_t. The bond P_t^S is just a "derivative security" with the constant payoff $X \equiv 1$ paid at time S. Hence by (6.11)

$$P_t^S = B_t \mathbb{E}^{\mathbb{Q}} \left[B_S^{-1} \middle| \mathcal{F}_t \right].$$

In other words

$$Z_t^S = B_t^{-1} P_t^S = \mathbb{E}^{\mathbb{Q}} \left[B_S^{-1} P_S^S \middle| \mathcal{F}_t \right] = \mathbb{E}^{\mathbb{Q}} \left[Z_S^S \middle| \mathcal{F}_t \right]$$

must be a martingale under the same measure $\mathbb{Q}$.

From (6.7), (6.8), and (6.9),

$$\begin{aligned} dZ_t^S &= Z_t^S \Sigma_t^S \left(dW_t + \left(\frac{1}{2} \Sigma_t^S - \frac{A_t^S}{\Sigma_t^S} \right) dt \right) \\ &= Z_t^S \Sigma_t^S \left(d\widetilde{W}_t - \gamma_t dt + \left(\frac{1}{2} \Sigma_t^S - \frac{A_t^S}{\Sigma_t^S} \right) dt \right) \\ &= Z_t^S \Sigma_t^S \left(d\widetilde{W}_t + \left[\left(\frac{1}{2} \Sigma_t^S - \frac{A_t^S}{\Sigma_t^S} \right) - \left(\frac{1}{2} \Sigma_t^T - \frac{A_t^T}{\Sigma_t^T} \right) \right] dt \right). \end{aligned}$$

But in order for Z_t^S to be a martingale, the "dt" term must vanish. This requires

$$\gamma_t = \frac{1}{2} \Sigma_t^T - \frac{A_t^T}{\Sigma_t^T} = \frac{1}{2} \Sigma_t^S - \frac{A_t^S}{\Sigma_t^S}$$

for all $0 \le S \le T$. Therefore γ_t must be independent of T. Multiply by Σ_t^T and rearrange

$$A_t^T = \frac{1}{2} \left(\Sigma_t^T \right)^2 - \Sigma_t^T \gamma_t,$$

differentiate with respect to T

$$\begin{aligned} \alpha_t^T &= -\Sigma_t^T \sigma_t^T + \gamma_t \sigma_t^T \\ &= \sigma_t^T \left(\gamma_t - \Sigma_t^T \right). \end{aligned} \tag{6.12}$$

This is a restriction imposed on the drift in the real world measure $\mathbb{P}$! Otherwise arbitrage between bonds is possible. This is quite a different situation than what is found for the Black-Scholes model where the drift term in the real world measure is unrestricted and unobservable.

Yuri will show that his statistical model does not satisfy (6.12). He will modify the model slightly so that it is fulfilled.

6.4. **Properties Under the Risk-Neutral Measure.**

6.4.1. *Forward Rate Dynamics Under* $\mathbb{Q}$. Consider the forward rate dynamics under the risk neutral measure. From (6.1), (6.9), and (6.12)

$$\begin{aligned} df_t^T &= \alpha_t^T dt + \sigma_t^T dW_t \\ &= \sigma_t^T \left(\gamma_t - \Sigma_t^T \right) dt + \sigma_t^T \left(d\widetilde{W}_t - \gamma_t dt \right). \end{aligned}$$

So expressed in differential form

$$df_t^T = -\sigma_t^T \Sigma_t^T dt + \sigma_t^T d\widetilde{W}_t. \tag{6.13}$$

The drift term under the risk-neutral measure

$$-\sigma_t^T \Sigma_t^T = \sigma_t^T \int_t^T \sigma_t^u du \tag{6.14}$$

is completely determined by the volatility σ_t^T. Any model satisfying (6.13) and (6.14) is called a *one-factor Heath-Jarrow-Morton model*.

6.4.2. *Bond Price Dynamics Under Risk-Neutral Measure.* Using (6.3), (6.5), and (6.10)

$$\begin{aligned} dP_t^T &= d\left(B_t Z_t^T\right) \\ &= Z_t^T dB_t + B_t dZ_t^T \\ &= Z_t^T r_t B_t dt + B_t Z_t^T \Sigma_t^T d\widetilde{W}_t. \end{aligned}$$

Thus the SDE for bond prices under $\mathbb{Q}$ is

$$dP_t^T = r_t P_t^T dt + P_t^T \Sigma_t^T d\widetilde{W}_t.$$

So all bonds (indeed all traded instruments) have the same rate of return r_t under $\mathbb{Q}$. This is why it is called the risk-neutral measure.

7. Multi-Factor HJM

In the model we just constructed, all rates are instantaneously perfectly correlated. This is not very realistic – a move in the 30 year forward rate cannot usually be completely predicted from a move in the overnight rate. While one-factor models can be very useful, multi-factor models are needed when dealing with complex instruments or situations. By multi-factor models we mean models that have more than one Brownian motion as stochastic drivers.

7.1. **Construction.** We could choose to go through the same process as we did for the single-factor version. Practitioners rarely bother to do so more than once. They use a shortcut which we will now illustrate. It relies on a theorem with which you should already be familiar.

Theorem 6. *(Fundamental Theorem of Arbitrage Pricing) Absence of arbitrage is equivalent to the existence of a risk-neutral measure under which all traded assets must have the same rate of return.*

Remember that the money market account is governed by the SDE

$$dB_t = r_t B_t dt.$$

Because there are no stochastic terms, this is valid under any measure equivalent to $\mathbb{P}$. Hence the money market account has rate of return r_t under the risk-neutral measure (call it $\mathbb{Q}$ again), whose existence is guaranteed by the Fundamental Theorem of Arbitrage Pricing. This rate of return is shared by all traded assets under $\mathbb{Q}$.

In particular for bonds we have[7]

$$dP(t,T) = r(t)\,P(t,T)\,dt + P(t,T)\left(\Sigma_1(t,T)\,d\widetilde{W}_1(t) + \cdots + \Sigma_N(t,T)\,d\widetilde{W}_N(t)\right)$$

[7] We have employed the alternate notation, i.e. $a_t^T \mapsto a(t,T)$ and $b_t \mapsto b(t)$, in order to avoid confusion with dimensional subscripts.

where $\left(\widetilde{W}_1(t), \ldots, \widetilde{W}_N(t)\right)$ is an N-dimensional Wiener process with independent components. Solving, we obtain

$$\begin{aligned} P(t,T) &= P(0,T)\exp\left\{\int_0^t \left(r(s) - \frac{1}{2}\left(\Sigma_1(s,T)^2 + \cdots + \Sigma_N(s,T)^2\right)\right) ds \right. \\ &\quad \left. + \int_0^t \Sigma_1(s,T)\, d\widetilde{W}_1(s) + \cdots + \int_0^t \Sigma_N(s,T)\, d\widetilde{W}_N(s)\right\}. \end{aligned}$$

To find the equation for the forward rates we need to take the logarithm and differentiate

$$\begin{aligned} f(t,T) &= -\frac{\partial}{\partial T}\log P(t,T) \\ &= -\frac{\partial}{\partial T}\log P(0,T) \\ &\quad +\frac{1}{2}\int_0^t \frac{\partial}{\partial T}\left(\Sigma_1(s,T)^2 + \cdots + \Sigma_N(s,T)^2\right) ds \\ &\quad -\int_0^t \frac{\partial}{\partial T}\Sigma_1(s,T)\, d\widetilde{W}_1(t) - \cdots - \int_0^t \frac{\partial}{\partial T}\Sigma_N(s,T)\, d\widetilde{W}_N(t). \end{aligned}$$

Thus

$$\begin{aligned} f(t,T) &= f(0,T) \\ &\quad -\int_0^t \left(\Sigma_1(s,T)\sigma_1(s,T) + \cdots + \Sigma_N(s,T)\sigma_N(s,T)\right) ds \\ &\quad +\int_0^t \sigma_1(s,T)\, d\widetilde{W}_1(t) + \cdots + \int_0^t \sigma_N(s,T)\, d\widetilde{W}_N(t) \end{aligned} \tag{7.1}$$

where

$$\sigma_n(t,T) = -\frac{\partial}{\partial T}\Sigma_n(t,T), \qquad n = 1, \ldots, N. \tag{7.2}$$

That is it! A multi-factor HJM model is any model that satisfies (7.1) and (7.2). It is automatically an arbitrage free model by the Fundamental Theorem of Arbitrage Pricing.

This is certainly the shorter way around. But, by taking the circuitous route in the prior section, we gained some important insight into the connection between the real-world and risk-neutral measures. We employed a constructive approach to find the risk-neutral measure and saw the connection between replicating strategies and pricing.

7.2. **Pricing.** Well we are not quite finished. We still need a formula for valuing arbitrary traded assets in multi-factor HJM.

If A_t is the price at time t of some traded asset A, it must satisfy the SDE

$$dA_t = r_t A_t dt + \text{stochastic terms}$$

where

$$\text{stochastic terms} = a_1(t)\, d\widetilde{W}_1(t) + \cdots + a_N(t)\, d\widetilde{W}_N(t)$$

for some stochastic processes $(a_1(t), \ldots, a_N(t))_{t\geq 0}$. Thus the discounted value $B_t^{-1}A_t$ satisfies

$$\begin{aligned} d\left(B_t^{-1}A_t\right) &= d\left(B_t^{-1}\right)A_t + B_t^{-1}dA_t \\ &= -r_tB_t^{-1}A_tdt + B_t^{-1}\left[r_tA_tdt + \text{stochastic terms}\right] \\ &= B_t^{-1} \times [\text{stochastic terms}]. \end{aligned}$$

In other words $B_t^{-1}A_t$ is a $\mathbb{Q}$-martingale.

In particular, if

$$A_S = X$$

is an $\mathcal{F}_S$-measurable payoff, then

$$B_t^{-1}\pi_t(X) = B_t^{-1}A_t = \mathbb{E}^{\mathbb{Q}}\left[B_S^{-1}A_S\middle|\mathcal{F}_t\right] = \mathbb{E}^{\mathbb{Q}}\left[B_S^{-1}X\middle|\mathcal{F}_t\right].$$

So the pricing formula is simply

$$\pi_t(X) = B_t\mathbb{E}^{\mathbb{Q}}\left[B_S^{-1}X\middle|\mathcal{F}_t\right]. \tag{7.3}$$

Not surprisingly, the pricing formula is the same as for the one-factor case.

In fact, specifying some measure $\mathbb{Q}$, a law for the money market process B_t, and the previous formula is (almost) all that is needed to construct any arbitrage free interest rate model. This approach covers a wider range of models than does the HJM framework. It includes models with jumps or even cases where instantaneous forward rates do not exist! This is a different story altogether. But, if you are interested, start with [5] and [8].

Glossary

adapted process: If $\{\mathcal{F}_t\}_{t\geq 0}$ is a *filtration*, then a *stochastic process* $\{S_t\}_{t\geq 0}$ is $\mathcal{F}_t$-*adapted* provided $\sigma(S_t) \subset \mathcal{F}_t$ for all $t \geq 0$.

arbitrage: Establishment of a market position in traded assets with no current net commitment which yields a positive profit with non-zero probability.

Brownian motion: A *Wiener process*.

filtration: An increasing collection of *σ-algebras* indexed by time.

martingale: A *stochastic process* M with the property $M_t = \mathbb{E}[X_T|\mathcal{F}_t]$ for all $0 \leq t \leq T$.

measurable: A *random variable* X is *measurable* with respect to the *σ-algebra* $\mathcal{F}$ if $\sigma(X) \subset \mathcal{F}$.

predictable process: A *adapted process* with almost surely continuous trajectories.

previsible process: A *predictable process*.

probability measure: Let $\mathcal{F}$ be a *σ-algebra* with *sample space* Ω. A mapping $\mathbb{P} : \mathcal{F} \to [0,1]$ is a *probability measure* provided $\mathbb{P}(\Omega) = 1$ and $\mathbb{P}\left(\bigcup_{k=1}^{+\infty} A_k\right) = \sum_{k=1}^{+\infty}\mathbb{P}(A_k)$, for all disjoint countable sequences $A_1, A_2, \ldots \in \mathcal{F}$.

probability space: A triple consisting of a *sample space*, a *σ-algebra*, and a *probability measure*.

random variable: A function with the *sample space* as its domain and the set of real numbers $\mathbb{R}$ as its codomain.

sample space: The set of all possible states which may be assumed by all relevant events.

σ-algebra: A collection of subsets of the *sample space* which contains the empty set $\emptyset$ and the sample space itself as elements. Furthermore, it is closed with respect to set complement and countable union operations applied to its members.

σ-algebra generated by . . . : If X is a *random variable*, then the *σ-algebra generated by* X, denoted $\sigma(X)$, is the collection $\left\{ X^{-1}(A) \middle| A \subset \mathbb{R} \right\}$.

stochastic process: A function $S : [0, +\infty) \times \Omega \to \mathbb{R}$ where Ω is the *sample space*. It is interpreted as a continuous sequence of random variables $S_t : \Omega \to \mathbb{R}$ indexed by time $t \geq 0$.

Wiener process: A *stochastic process* W with $W_0 = 0$ such that the increments $W_t - W_s$ are Gaussian with zero mean and variance $t - s$ for all $0 \leq s < t$. Furthermore, it must have continuous trajectories and independent increments for non-overlapping time intervals.

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