Heath-Jarrow-Morton Framework

Jeff Greco

jgreco@uchicago.edu

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Probability Space Refresher

Probability space triple $(\Omega, \mathcal{F}, \mathbb{P})$.

- Ω is the *sample space* (a set).
- $\mathcal{F}$ is the σ*-algebra* over Ω.

$$A \in \mathcal{F} \implies A \subset \Omega$$
$$\emptyset, \Omega \in \mathcal{F}$$
$$A \in \mathcal{F} \implies \Omega \setminus A \in \mathcal{F}$$
$$\{A_i\}_{i=1}^{\infty} \subset \mathcal{F} \implies \bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$$

- $\mathbb{P} : \mathcal{F} \longrightarrow [0, 1]$ is the *probability measure.*

$$\mathbb{P}(\Omega) = 1$$
$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(A_i)$$

(for all disjoint countably infinite sequences $\{A_i\}_{i=1}^{\infty} \subset \mathcal{F}$)

Stochastic Process Refresher

- $A \subset \Omega$ is $\mathcal{F}$*-measurable* if $A \in \mathcal{F}$.
- A *random variable* $X : \Omega \longrightarrow \mathbb{R}^N$ is $\mathcal{F}$*-measurable* if $X^{-1}(J) \in \mathcal{F}$, for all measurable subsets $J \subset \mathbb{R}^N$.
- A *filration* $\{\mathcal{F}_t \subset \mathcal{F} \,|\, t \in [0, +\infty)\}$ is an increasing sequence of σ-algebras, i.e. $s \leq t \implies \mathcal{F}_s \subset \mathcal{F}_t$.
- A *stochastic process* $\left\{X_t : \Omega \longrightarrow \mathbb{R}^N \,|\, t \in [0, +\infty)\right\}$ is a sequence of random variables.
- It is $\mathcal{F}_t$*-adapted* if X_t is $\mathcal{F}_t$-measurable $\forall t \geq 0$.
- It is *predictable* if in addition its trajectories $t \mapsto X_t$ are continuous $\mathbb{P}$ a.s.

Expectation Refresher

- *Expectation* of a random variable X is given by the integral

$$\mathbb{E}\left[X\right] = \int X\left(\omega\right) d\mathbb{P}\left(\omega\right).$$

- *Conditional expectation* $\mathbb{E}\left[X \left| \mathcal{F}_t\right.\right]$ is the (unique) $\mathcal{F}_t$-measurable random variable such that

$$\mathbb{E}\left[\mathbb{E}\left[X \left| \mathcal{F}_t\right.\right] \cdot 1_A\right] = \mathbb{E}\left[X \cdot 1_A\right]$$

  for all $A \in \mathcal{F}_t$.
  - $\mathbb{E}\left[X \left| \mathcal{F}_t\right.\right]$ and X are indistinguishable for all events in $\mathcal{F}_t$.
  - X is $\mathcal{F}$-measurable and $\mathbb{E}\left[X \left| \mathcal{F}_t\right.\right]$ is $\mathcal{F}_t$-measurable.
  - If X is also $\mathcal{F}_t$-measurable then $X = \mathbb{E}\left[X \left| \mathcal{F}_t\right.\right]$ $\mathbb{P}$ a.s.

Replicating Strategies

Let I_t and J_t be the prices for two traded securities.

- A *strategy* is a pair of processes (ϕ_t, ψ_t) giving the amounts of I_t and J_t held in a portfolio

$$\Pi_t = \phi_t I_t + \psi_t J_t$$

- It is *self-financing* if

$$d\Pi_t = \phi_t dI_t + \psi_t dJ_t$$

- It *replicates* the time T payoff X if

$$\Pi_T = X, \quad \mathbb{P} \text{ a.s.}$$

Arbitrage Free Pricing

Replicating an Asset's Payoff Yields its Price

Condition

NFL (No Free Lunch): Any two self-financing strategies with the same future payoff must have the same value today.

$$\Pi_T = \Gamma_T \implies \Pi_t = \Gamma_t, \quad \forall t \leq T$$

This condition ensures that the contingent claim has a unique value which we'll denote

$$\pi_t(X) = \Pi_t.$$

Theorem (Martingale Representation)

If W_t is a Brownian motion under the measure $\mathbb{Q}$ and N_t is a predictable square-integrable $\mathbb{Q}$-martingale, then there exists a predictable process v_t such that

$$\mathbb{E}^{\mathbb{Q}}\left[\int_0^T v_t^2 dt\right] < +\infty$$

and

$$N_t = N_0 + \int_0^t v_s dW_s.$$

Martingales from Martingales

Corollary

If $v_s > 0$, $\mathbb{Q}$ a.s. and M_t is another square-integrable predictable martingale (under $\mathbb{Q}$), then there exists another predictable process ϕ_t such that

$$M_t = M_0 + \int_0^t \phi_s dN_s.$$

In other words, if we can somehow make martingales from assets then we can use the Martingale Representation Theorem to build a replicating strategies for contingent claims.

Girsanov's Theorem

Change of Measure Equals Change of Drift

Theorem (Girsanov's)

Let W_t be a Brownian motion under $\mathbb{P}$ and θ_t an $\mathcal{F}_t$-adapted process satisfying the Novikov condition

$$\mathbb{E}\left[\exp\left\{\frac{1}{2}\int_0^T (\theta_t)^2\, dt\right\}\right] < +\infty$$

Then

$$\widetilde{W}_t = W_t + \int_0^t \theta_s ds, \qquad 0 \leq t \leq T,$$

is a Brownian motion under the new measure $\mathbb{Q}$

$$\mathbb{Q}(A) = \int_A \xi_T d\mathbb{P}, \qquad \forall A \in \mathcal{F}$$

$$\xi_t = \exp\left\{-\int_0^t \theta_s dW_s - \frac{1}{2}\int_0^t (\theta_s)^2\, ds\right\}$$

Girsanov's Theorem

Note

ξ_t is a $\mathbb{P}$-martingale and $\mathbb{Q}$ is equivalent to $\mathbb{P}$.

$$d\left(e^{X_t}\right) = e^{X_t}\left(dX_t + \frac{1}{2}\left(dX_t\right)^2\right)$$

$$\xi_t = \exp\left\{-\int_0^t \theta_s dW_s - \frac{1}{2}\int_0^t \left(\theta_s\right)^2 ds\right\}$$

$$d\xi_t = \xi_t\left(\left(-\theta_t dW_t - \frac{1}{2}\left(\theta_t\right)^2 dt\right) + \frac{1}{2}\left(\theta_t\right)^2 dt\right)$$

$$d\xi_t = -\theta_t \xi_t dW_t$$

The HJM Framework

f_t^T	instantaneous forward rate
$r_t = f_t^t$	spot/short/risk-free rate
$P_t^T = \exp\left\{-\int_t^T f_t^S dS\right\}$	zero coupon bond price

The price of a single bond as time flows is $t \mapsto P_t^T$.
The processes P_t^T and P_t^S represent the prices of two separate securities.

Single Factor HJM

W_t is a Brownian motion under $\mathbb{P}$

$$f_t^T = f_0^T + \int_0^t \sigma_s^T dW_s + \int_0^t \alpha_s^T ds$$

The only restriction on the stochastic processes σ and α are that they are predictable and square integrable.

Bond Dynamics

$$
\begin{aligned}
f_t^u &= f_0^u + \int_0^t \sigma_s^u dW_s + \int_0^t \alpha_s^u ds \\
P_t^T &= \exp\left\{ -\int_t^T f_t^u du \right\} \\
&= \exp\left\{ -\int_t^T \left(f_0^u + \int_0^t \sigma_s^u dW_s + \int_0^t \alpha_s^u ds \right) du \right\} \\
&= \exp\left\{ -\int_0^t \int_t^T \sigma_s^u du dW_s - \int_t^T f_0^u du - \int_0^t \int_t^T \alpha_s^u du ds \right\}
\end{aligned}
$$

Money Market Dynamics

$$
\begin{aligned}
dB_t &= r_t B_t dt, \qquad B_0 = 1 \\
r_u &= f_u^u = f_0^u + \int_0^u \sigma_s^u dW_s + \int_0^u \alpha_s^u ds \\
B_t &= \exp\left\{ \int_0^t r_u du \right\} \\
&= \exp\left\{ \int_0^t \left(f_0^u + \int_0^u \sigma_s^u dW_s + \int_0^u \alpha_s^u ds \right) du \right\} \\
&= \exp\left\{ \int_0^t \int_s^t \sigma_s^u du dW_s + \int_0^t f_0^u du + \int_0^t \int_s^t \alpha_s^u du ds \right\}
\end{aligned}
$$

Dynamics of Discounted Bonds

We want to use the Martingale Representation Theorem to replicate an S-maturity bond with a T-maturity bond.
The problem is bonds are not martingales.

- What do you think could be?

$$
\begin{aligned}
Z_t^T &= B_t^{-1} P_t^T \\
&= \exp\left\{ -\int_0^t \int_s^t \sigma_s^u du dW_s - \int_0^t f_0^u du - \int_0^t \int_s^t \alpha_s^u du ds \right\} \\
&\quad \times \exp\left\{ -\int_0^t \int_t^T \sigma_s^u du dW_s - \int_t^T f_0^u du - \int_0^t \int_t^T \alpha_s^u du ds \right\} \\
&= \exp\left\{ -\int_0^t \int_s^T \sigma_s^u du dW_s - \int_0^T f_0^u du - \int_0^t \int_s^T \alpha_s^u du ds \right\}
\end{aligned}
$$

Dynamics of Discounted Bonds

- Under what measure is it a martingale?

$$Z_t^T = \exp\left\{ \int_0^t \Sigma_s^T dW_s - \int_0^T f_0^u du - \int_0^t A_s^T ds \right\}$$

$$\Sigma_s^T = -\int_s^T \sigma_s^u du \qquad A_s^T = \int_s^T \alpha_s^u du$$

Apply Ito's lemma

$$dZ_t^T = Z_t^T \left(\Sigma_t^T dW_t - A_t^T dt + \frac{1}{2} \left(\Sigma_t^T \right)^2 dt \right).$$

Building a Replicating Strategy

Steps for Replicating an Arbitrary Payoff

1. Find a new measure $\mathbb{Q}$ where Z_t^T is a martingale. *(Girsanov's Theorem)*
2. Find a $\mathbb{Q}$-martingale V_t which recovers the discounted payoff $V_S = B_S^{-1}X$.
3. Find a predictable process ϕ_t such that $V_t = V_0 + \int_0^t \phi_s dZ_s^T$. *(Martingale Representation Theorem)*
4. Use ϕ_t to build a replicating strategy.

Building a Replicating Strategy

Step 1: Change of Measure

Factor the SDE for Z_t^T

$$
\begin{aligned}
dZ_t^T &= Z_t^T \left(\Sigma_t^T dW_t - A_t^T dt + \frac{1}{2} \left(\Sigma_t^T \right)^2 dt \right) \\
&= Z_t^T \Sigma_t^T \left(dW_t + \left(\frac{1}{2} \Sigma_t^T - \frac{A_t^T}{\Sigma_t^T} \right) dt \right).
\end{aligned}
$$

Define the drift $\gamma_t = \frac{1}{2}\Sigma_t^T - \frac{A_t^T}{\Sigma_t^T}$.

Use Girsanov's theorem to change to a new measure $\mathbb{Q}$ for which

$$d\widetilde{W}_t = dW_t + \gamma_t dt$$

is a Brownian motion. We call $\mathbb{Q}$ the *risk-neutral* measure.

Z_t^T is a $\mathbb{Q}$-martingale (no drift term) $dZ_t^T = Z_t^T \Sigma_t^T d\widetilde{W}_t$.

Building a Replicating Strategy

Step 2: Find a $\mathbb{Q}$-Martingale V_t which Recovers the Discounted Payoff $V_S = B_S^{-1}X$

$$V_t = \mathbb{E}^{\mathbb{Q}}\left[B_S^{-1}X \,|\mathcal{F}_t\right]$$

1. It is automatically a martingale.
2. Clearly it recovers the discounted payoff $V_S = B_S^{-1}X$.

Building a Replicating Strategy

Step 3: Representing V_t in Terms of Z_t^T

By the Martingale Representation Theorem

$$V_t = V_0 + \int_0^t \phi_s dZ_s^T \qquad dV_t = \phi_t dZ_t^T.$$

Building a Replicating Strategy

Step 4: Build the Replicating Strategy

Set

$$\psi_t = V_t - \phi_t Z_t^T$$

and verify that (ϕ_t, ψ_t) is a replicating strategy for X

$$\Pi_t = \phi_t P_t^T + \psi_t B_t$$

Verification of Replicating Strategy

Is it Replicating?

Simply substitute the variable definitions

$$
\begin{aligned}
\Pi_t &= \phi_t P_t^T + \psi_t B_t \\
&= \phi_t P_t^T + \left(V_t - \phi_t Z_t^T \right) B_t \\
&= \phi_t P_t^T + V_t B_t - \phi_t B_t^{-1} P_t^T B_t \\
&= V_t B_t \\
\Pi_S &= V_S B_S \\
&= B_S^{-1} X B_S \\
&= X.
\end{aligned}
$$

Verification of Replicating Strategy

Is it Self-Financing?

Apply Ito's lemma

$$\begin{aligned}
d\Pi_t &= d\left(B_t V_t\right) \\
&= B_t dV_t + V_t dB_t \\
&= B_t \phi_t dZ_t^T + V_t dB_t \\
&= \left(\phi_t d\left(B_t Z_t^T\right) - \phi_t Z_t^T dB_t\right) + V_t dB_t \\
&= \phi_t dP_t^T + \left(V_t - \phi_t Z_t^T\right) dB_t \\
&= \phi_t dP_t^T + \psi_t dB_t.
\end{aligned}$$

The Risk-Neutral Pricing Formula

By the No Free Lunch condition we can now write the *risk-neutral pricing formula*

$$\pi_t(X) = \Pi_t = V_t B_t = B_t \mathbb{E}^{\mathbb{Q}}\left[B_S^{-1} X \,|\mathcal{F}_t \right],$$

or simply

$$\pi_t(X) = B_t \mathbb{E}^{\mathbb{Q}}\left[B_S^{-1} X \,|\mathcal{F}_t \right].$$

This means that the value of *any* derivative depends on the risk-neutral measure only. The real-world measure is not involved!

Replicating One Bond with Another

Plug the S-maturity bond into the pricing formula

$$P_t^S = B_t \mathbb{E}^{\mathbb{Q}}\left[B_S^{-1} P_S^S \,|\mathcal{F}_t\right] = B_t \mathbb{E}^{\mathbb{Q}}\left[B_S^{-1} \,|\mathcal{F}_t\right]$$

Divide by B_t

$$Z_t^S = B_t^{-1} P_t^S = \mathbb{E}^{\mathbb{Q}}\left[B_S^{-1} P_S^S \,|\mathcal{F}_t\right] = \mathbb{E}^{\mathbb{Q}}\left[Z_S^S \,|\mathcal{F}_t\right].$$

So Z_t^S is a $\mathbb{Q}$-martingale just like Z_t^T.
Let's look at the dynamics of Z_t^S

$$\begin{aligned} dZ_t^S &= Z_t^S \Sigma_t^S \left(dW_t + \left(\frac{1}{2}\Sigma_t^S - \frac{A_t^S}{\Sigma_t^S} \right) dt \right) \\ &= Z_t^S \Sigma_t^S \left(d\widetilde{W}_t - \gamma_t dt + \left(\frac{1}{2}\Sigma_t^S - \frac{A_t^S}{\Sigma_t^S} \right) dt \right). \end{aligned}$$

Real-World Drift is Constrained

But Z_t^S must be drift free. Thus

$$\gamma_t = \frac{1}{2}\Sigma_t^T - \frac{A_t^T}{\Sigma_t^T} = \frac{1}{2}\Sigma_t^S - \frac{A_t^S}{\Sigma_t^S}, \quad \forall S \geq 0.$$

I.e., γ_t is independent of T!
Multiply by Σ_t^T and differentiate w.r.t. T

$$\begin{aligned} A_t^T &= \frac{1}{2}\left(\Sigma_t^T\right)^2 - \Sigma_t^T \gamma_t \\ \alpha_t^T &= -\Sigma_t^T \sigma_t^T + \gamma_t \sigma_t^T = \sigma_t^T \left(\gamma_t - \Sigma_t^T\right). \end{aligned}$$

THIS PLACES A RESTRICTION ON THE DRIFT UNDER THE REAL-WORLD MEASURE!

HJM Under Risk-Neutral Measure

Forward Rate Dynamics Under $\mathbb{Q}$

$$
\begin{aligned}
df_t^T &= \alpha_t^T dt + \sigma_t^T dW_t \\
&= \sigma_t^T \left(\gamma_t - \Sigma_t^T \right) dt + \sigma_t^T \left(d\widetilde{W}_t - \gamma_t dt \right) \\
&= -\sigma_t^T \Sigma_t^T dt + \sigma_t^T d\widetilde{W}_t
\end{aligned}
$$

The drift term is completely determined by the volatility process.

$$
-\sigma_t^T \Sigma_t^T = \sigma_t^T \int_t^T \sigma_t^u du
$$

HJM Under Risk-Neutral Measure

Bond Prices Under $\mathbb{Q}$

$$\begin{aligned} dP_t^T &= d\left(B_t Z_t^T\right) \\ &= Z_t^T dB_t + B_t dZ_t^T \\ &= Z_t^T r_t B_t dt + B_t Z_t^T \Sigma_t^T d\widetilde{W}_t \\ &= r_t P_t^T dt + P_t^T \Sigma_t^T d\widetilde{W}_t \end{aligned}$$

All assets have the same rate of return r_t. That's why we call it the "risk-neutral" measure.

Market Price of Risk

Let's substitute $d\widetilde{W}_t = dW_t + \gamma_t dt$ and have a look at bond prices under the real-world measure.

$$\begin{aligned} dP_t^T &= r_t P_t^T dt + P_t^T \Sigma_t^T d\widetilde{W}_t \\ &= r_t P_t^T dt + P_t^T \Sigma_t^T \left(dW_t + \gamma_t dt\right) \\ &= P_t^T \left(\left(r_t + \gamma_t \Sigma_t^T\right) dt + \Sigma_t^T dW_t\right) \end{aligned}$$

Or simply

$$dP_t^T / P_t^T = \left(r_t + \gamma_t \Sigma_t^T\right) dt + \Sigma_t^T dW_t.$$

Therefore γ_t is the amount of excess return earned under the real-world measure per unit of risk (Σ_t^T), i.e. the *market price of risk*.

The drift restriction $\alpha_t^T = \sigma_t^T \left(\gamma_t - \Sigma_t^T\right)$ is equivalent to all assets having the same market price of risk.

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