FINM 33601: Homework 9 – Hull-White Model

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1 Hull-White Formulas

$$
\begin{aligned}
\Sigma_t^T &= -\int_t^T \sigma_t^S \mathrm{d}s \\
&= -\int_t^T \sigma e^{-a(s-t)} \mathrm{d}s \\
&= -\sigma \int_t^T e^{-a(s-t)} \mathrm{d}s \\
\therefore \Sigma_t^T &= -\sigma \left[-\frac{1}{a} e^{-a(s-t)} \right]_t^T \\
&= -\sigma \frac{1}{a} (1 - e^{-a(T-t)}) \\
\therefore \Sigma_t^T &= -\sigma b(t,T)
\end{aligned}
$$

, where $b(t,T) = \frac{1}{a}(1 - e^{-a(T-t)})$

Therefore, for the short-rate:

$$
\begin{aligned}
\mathrm{d}f_t^T &= -\Sigma_t^T \sigma_t^T \mathrm{d}t + \sigma_t^T \mathrm{d}W_t \\
X_t &= \sigma \int_0^t e^{-a(t-s)} \mathrm{d}W_s \\
\therefore \mathrm{d}f_t^T &= -]sigmab(t,T)\sigma e^{-a(T-t)} \mathrm{d}t + \sigma e^{-a(T-t)} \mathrm{d}W_t \\
&= -\frac{\sigma^2}{a}(e^{-a(T-t)} - e^{-2a(T-t)}) \mathrm{d}t + \sigma e^{-a(T-t)} \mathrm{d}W_t \\
\therefore f_t^T &= f_0^T - \frac{\sigma^2}{a} \int_0^t (e^{-a(T-s)} - e^{-2a(T-s)}) \mathrm{d}s + \sigma \int_0^t e^{-a(T-s)} \mathrm{d}W_s \\
&= f_0^T + \frac{\sigma^2}{a^2}(e^{-a(T-t)} - e^{-aT}) - \frac{\sigma^2}{2a^2}(e^{-2a(T-t)} - e^{-2aT}) + \sigma \int_0^t e^{-a(T-s)} \mathrm{d}W_s \\
&= f_0^T + \frac{\sigma^2}{2}(b(0,T)^2 - b(t,T)^2) + e^{-a(T-t)} X_t
\end{aligned}
$$

$$\therefore r_t = f_t^t = f_0^t + \frac{\sigma^2}{2} b(0,t)^2 + X_t$$

For the bond price:

$$\begin{aligned}
P_t^T &= e^{-\int_t^T f_t^s \mathrm{d}s} \\
&= e^{-\int_t^T f_0^s + \frac{\sigma^2}{2}(b(0,s)^2 - b(t,s)^2) + e^{-a(s-t)} X_t \mathrm{d}s} \\
&= e^{-\int_t^T f_0^s \mathrm{d}s} e^{-\frac{\sigma^2}{2}\int_t^T (b(0,s)^2 - b(t,s)^2)\mathrm{d}s} e^{-X_t \int_t^T e^{-a(s-t)} ds}
\end{aligned}$$

Now, we must solve the second exponential term:

$$\begin{aligned}
e^{-\frac{\sigma^2}{2}\int_t^T (b(0,s)^2 - b(t,s)^2)\mathrm{d}s} &= -\frac{\sigma^2}{2}\int_t^T \frac{1}{a^2}(e^{-2as} - 2e^{-as} - e^{-2a(s-t)} + 2e^{-a(s-t)})\mathrm{d}s \\
&= -\frac{\sigma^2}{2}\frac{1}{a^3}(0.5e^{-2at} - 0.5e^{-2aT} + 2e^{-aT} - 2e^{-at} - 0.5 + 0.5e^{-2a(T-t)} - 2e^{-a(T-t)} + 2) \\
&= -\frac{\sigma^2}{2} b(t,T)(b(t,T)\frac{1-e^{-2at}}{2a} + b(0,t)^2)
\end{aligned}$$

If we let $A(t,T)$ equal this result, then $P_t^T = P_0^{t,T} A(t,T) e^{-b(t,T)X_t}$

2 Hull-White Variance Calculations

1.

Given that $X_t = \sigma \int_0^t e^{-a(t-s)} \mathrm{d}W_s$, $X_0 = 0$,

$$\begin{aligned}
Var[X_t] &= E[X_t^2] - 0 \\
&= E[X_t^2] \\
&= \int_0^t \sigma^2 e^{-2a(t-s)} \mathrm{d}s \\
&= \sigma^2 \int_0^t e^{-2a(t-s)} \mathrm{d}s \\
&= \sigma^2 \frac{1-e^{-2at}}{2a}
\end{aligned}$$

2.

We have:

$$\begin{aligned}
P_t^T &= P_0^{t,T} A(t,T) e^{-b(t,T)X_t} \\
\therefore log(P_t^T) &= -\int_t^T f_0^s \mathrm{d}s + log A(t,T) - b(t,T)X_t
\end{aligned}$$

The variance here is only going to be based on $b(t,T)X_t$, hence:

$$
\begin{aligned}
Var[log(P_t^T)] &= Var[b(t,T)X_t] \\
&= b^2(t,T)Var[X_t] \\
&= \sigma^2 b(t,T)^2 \frac{1-e^{-2at}}{2a}
\end{aligned}
$$