FINM 33601: Homework 7 – The HJM Framework

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1 One-Factor HJM Computations

1.

We have that:

$$\begin{aligned} \mathrm{d}f_t^T &= -\sigma_t^T \Sigma_t^T \mathrm{d}t + \sigma_t^T \mathrm{d}W_t \\ \therefore f_t^T &= f_0^T - \int_0^t \sigma_s^T \Sigma_s^T \mathrm{d}s + \int_0^t \sigma_s^T \mathrm{d}W_s \end{aligned}$$

Hence, r_t is given by:

$$r_t = f_t^t = f_0^t - \int_0^t \sigma_s^t \Sigma_s^t \mathrm{d}s + \int_0^t \sigma_s^t \mathrm{d}W_s$$

Now, we just take the expectation, which drops the $\mathrm{d}W$ term:

$$E^Q[r_t] = f_0^t - \int_0^t \sigma_s^t \Sigma_s^t \mathrm{d}s$$

2.

From the notes, we know that:

$$\begin{aligned} P_t^T &= e^{-\int_t^T f_t^s \mathrm{d}s} \\ &= e^{-\int_t^T f_0^s \mathrm{d}s + \int_t^T \int_0^t \sigma_u^s \Sigma_u^s \mathrm{d}u \mathrm{d}s - \int_t^T \int_0^t \sigma_u^s \mathrm{d}W_u \mathrm{d}s} \end{aligned}$$

If we take the expectation of this w.r.t. Q:

$$
\begin{aligned}
E^Q[P_t^T] &= e^{-\int_t^T f_0^s \mathrm{d}s + \int_t^T \int_0^t \sigma_u^s \Sigma_u^s \mathrm{d}u\mathrm{d}s} E^Q\left[-\int_t^T \int_0^t \sigma_u^s \mathrm{d}W_u \mathrm{d}s\right] \\
&= e^{-\int_t^T f_0^s \mathrm{d}s + \int_t^T \int_0^t \sigma_u^s \Sigma_u^s \mathrm{d}u\mathrm{d}s} e^{0.5\int_0^t \left(\int_t^T -\sigma_u^s \mathrm{d}s\right)^2 \mathrm{d}u} \\
&= e^{-\int_t^T f_0^s \mathrm{d}s + \int_t^T \int_0^t \sigma_u^s \Sigma_u^s \mathrm{d}u\mathrm{d}s + 0.5\int_0^t \left(\int_t^T -\sigma_u^s \mathrm{d}s\right)^2 \mathrm{d}u}
\end{aligned}
$$

3.

$$
\begin{aligned}
Var[P_t^T] &= E^Q\left[(P_t^T)^2\right] - \left(E^Q[P_t^T]\right)^2 \\
&= e^{-2\int_t^T f_0^s \mathrm{d}s + 2\int_t^T \int_0^t \sigma_u^s \Sigma_u^s \mathrm{d}u\mathrm{d}s} e^{2\int_0^t \left(\int_t^T -\sigma_u^s \mathrm{d}s\right)^2 \mathrm{d}u + \int_0^t \left(\int_t^T -\sigma_u^s \mathrm{d}s\right)^2 \mathrm{d}u} \\
&= e^{-2\int_t^T f_0^s \mathrm{d}s + 2\int_t^T \int_0^t \sigma_u^s \Sigma_u^s \mathrm{d}u\mathrm{d}s + 3\int_0^t \left(\int_t^T -\sigma_u^s \mathrm{d}s\right)^2 \mathrm{d}u}
\end{aligned}
$$

2 Girsanov's Theorem

1.

- $Q(\Omega) = \int_\Omega \xi_T \mathrm{d}P = E^P(\xi_T 1_\Omega) = E^P[E^P(\xi_T|\mathscr{F}_t)1_\Omega] = E^P(1_\Omega) = \int_\Omega 1\mathrm{d}P = 1$
- $Q(\bigcup_{i=1}^\infty A_i) = \sum_{i=1}^\infty \int_{A_i} \xi_T \mathrm{d}P = \sum_{i=1}^\infty Q(A_i)$, hence, the A_i are disjoint.
- We know that $\xi_T \geq 0$ and $P(A) = \int_A 1\mathrm{d}P$ is a probability measure. Now, if $A \in \mathscr{F}$, then $Q(A) = \int_A \xi_T \mathrm{d}P$. Hence, $\forall A$ where $\int_A 1\mathrm{d}P \geq 0$, $Q(A) \geq 0$

Hence, Q is a measure on $(\Omega, \mathscr{F})$.

2.

We have

- $Q(\Omega) = 1$
- $Q(\emptyset) = \int_\emptyset \xi_T \mathrm{d}P = 0$
- $Q(A) = 0 \iff P(A) = 0$

Hence, Q is equivalent to P.

3 Replicating Strategy

First, let us design π_t to match the payoff of P_t^T:

$$
\begin{aligned}
\pi_t &= \phi_t P_t^S + \psi_t B_t \\
&= \frac{\Sigma_t^T P_t^T}{\Sigma_t^S P_t^S} P_t^S + \left(1 - \frac{\Sigma_t^T}{\Sigma_t^S}\right) \frac{P_t^T}{B_t} B_t \\
&= P_t^T
\end{aligned}
$$

Next, we show that the portfolio is self-financing. Here we can use the hint gives:

$$\begin{aligned}
\mathrm{d}P_t^T &= \mathrm{d}(B_t Z_t^T) \\
&= Z_t^T \mathrm{d}B_t + B_t \mathrm{d}Z_t^T \\
&= Z_t^T r_t B_t \mathrm{d}t + B_t Z_t^T \Sigma_t^T (\mathrm{d}W_t + (0.5\Sigma_t^T - \frac{A_t^T}{\Sigma_t^T})\mathrm{d}t) \\
&= r_t P_t^T \mathrm{d}t + P_t^T \Sigma_t^T (\mathrm{d}W_t + (0.5\Sigma_t^T - \frac{A_t^T}{\Sigma_t^T})\mathrm{d}t)
\end{aligned}$$

so:

$$\begin{aligned}
\mathrm{d}P_t^T &= r_t P_t^T \mathrm{d}t + P_t^T \Sigma_t^T \mathrm{d}\widetilde{W}_t \\
\mathrm{d}P_t^S &= r_t P_t^S \mathrm{d}t + P_t^S \Sigma_t^S \mathrm{d}\widetilde{W}_t
\end{aligned}$$

, where $\mathrm{d}\widetilde{W}_t = \mathrm{d}W_t + \sigma_t \mathrm{d}_t$. Therefore:

$$\begin{aligned}
\mathrm{d}\pi_t &= \mathrm{d}P_t^T \\
&= r_t P_t^T \mathrm{d}t + P_t^T \Sigma_t^T \mathrm{d}\widetilde{W}_t
\end{aligned}$$

We also have that:

$$\begin{aligned}
\phi_t \mathrm{d}P_t^S + \psi_t \mathrm{d}B_t &= \phi_t (r_t P_t^S \mathrm{d}t + P_t^S \Sigma_t^S \mathrm{d}\widetilde{W}_t) + \psi_t r_t B_t \mathrm{d}t \\
&= \frac{\Sigma_t^T P_t^T}{\Sigma_t^S P_t^S} r_t P_t^S \mathrm{d}t + \frac{\Sigma_t^T}{\Sigma_t^S} \frac{P_t^T}{P_t^S} P_t^S \Sigma_t^S \mathrm{d}\widetilde{W}_t + \frac{\Sigma_t^S - \Sigma_t^T}{\Sigma_t^S} P_t^T r_t \mathrm{d}t \\
&= r_t P_t^T \mathrm{d}t + \Sigma_t^T P_t^T \mathrm{d}\widetilde{W}_t
\end{aligned}$$

, which is exactly the same. Therefore we have that $\mathrm{d}\pi_t = \phi_t \mathrm{d}P_t^S + \psi_t \mathrm{d}B_t$. Hence, the portfolio is self-financing and this is an arbitrage free replicating strategy.