The Case of One LIBOR Rate

$$L(T) \equiv L(T, T+\delta) = L(T, T, T+\delta)$$

A caplet on this LIBOR rate has payoff

$$X = N\delta \max(L(T) - K, 0) = N\delta (L(T) - K)_+$$

at $T+\delta$.

Caplet value :

$$c(t) = B_t \mathbb{E}[B_{T+\delta}^{-1} X \mid \mathcal{F}_t] \quad \text{Risk-neutral measure}$$

$$c(t) = P(t, T+\delta) \mathbb{E}^{T+\delta}[X \mid \mathcal{F}_t] \quad (T+\delta)\text{-forward measure}$$

$$L(t) \equiv L(t, T, T+\delta) = \frac{P(t,T) - P(t,T+\delta)}{\delta\, P(t,T+\delta)}$$

traded asset

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$\Rightarrow$ $L(t)$ is a Martingale under $\mathbb{Q}^{T+\delta}$ $\Rightarrow$

$dL(t) = \lambda L(t)\, dW_t^{T+\delta}$, λ - non-random

$$L(t) = L(0) \exp\left(\lambda W_t^{T+\delta} - \frac{1}{2}\lambda^2 t\right)$$

SDE for $L(t)$ under $\mathbb{Q}^{T+\delta}$

$$L(t) = \frac{P(t,T) - P(t,T+\delta)}{\delta P(t,T+\delta)} = \frac{1}{\delta}\left(F(t,T+\delta,T) - 1\right),$$

$$F(t,T+\delta,T) = \frac{P(t,T)}{P(t,T+\delta)}$$

$$dF(t,T+\delta,T) = \gamma(t,T+\delta,T)\,F(t,T+\delta,T)\,dW_t^{T+\delta}, \text{ where}$$

$$\gamma(t,T+\delta,T) = \Sigma(t,T) - \Sigma(t,T+\delta)$$

$$dL(t) = \frac{1}{\delta}\,dF(t,T+\delta,T)$$

$$= \frac{1}{\delta}\left[\Sigma(t,T) - \Sigma(t,T+\delta)\right]F(t,T+\delta,T)\,dW_t^{T+\delta}$$

$$dL(t) = \boxed{\frac{\delta L(t)+1}{\delta L(t)}\left[\Sigma(t,T) - \Sigma(t,T+\delta)\right]} \underbrace{L(t)\,dW_t^{T+\delta}}_{}$$

$$dL(t) = \lambda \underbrace{L(t)\,dW_t^{T+\delta}}_{}$$

$$\frac{\delta L(t)+1}{\delta L(t)} \gamma(t, T+\delta, T) = \lambda \implies$$

$$\Sigma(t,T) - \Sigma(t,T+\delta) \equiv \gamma(t,T+\delta,T) = \lambda \frac{\delta L(t)}{\delta L(t)+1}$$