Hull-White Model

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HJM Framework

Start with the one-factor HJM framework

$$df_t^T = -\Sigma_t^T \sigma_t^T dt + \sigma_t^T dW_t$$

$$\sigma_t^T = -\frac{\partial}{\partial T}\Sigma_t^T \qquad\qquad \Sigma_T^T = 0$$

where W_t is a Brownian motion under the risk-neutral measure. The dynamics of the term-structure are completely determined by the choice of the forward rate volatility σ_t^T.

The Ho-Lee Model

The simplest non-trivial choice for the volatility is a constant

$$\sigma_t^T \equiv \sigma$$

This is the Ho-Lee model.

- Bond Price Volatility: $\Sigma_t^T = -\int_t^T \sigma dS = -\sigma \cdot (T - t)$
- Forward Rate SDE:

$$\begin{aligned} df_t^T &= \sigma^2 \cdot (T - t)\, dt + \sigma dW_t \\ f_t^T &= f_0^T + \sigma^2 \int_0^t (T - s)\, ds + \sigma \int_0^t dW_s \\ &= f_0^T + \sigma^2 \cdot \left(Tt - \frac{1}{2} t^2 \right) + \sigma W_t \end{aligned}$$

- Short Rate: $r_t = f_t^t = f_0^t + \frac{1}{2}\sigma^2 t^2 + \sigma W_t$

The Ho-Lee Model

The Short Rate State

Define the short rate state

$$x_t = \sigma W_t$$

which is a single source of randomness driving the dynamics of the entire term structure.

$$f_t^T = f_0^T + \sigma^2 \cdot \left(Tt - \frac{1}{2} t^2 \right) + x_t$$

$$r_t = f_0^t + \frac{1}{2} \sigma^2 t^2 + x_t$$

The Ho-Lee Model

Markov Property

The short rate state x_t is Markovian since it is a function of a Wiener process, i.e.

$$\mathbb{E}\left[x_T \,|\mathcal{F}_t\right] = \mathbb{E}\left[x_T \,|x_t\right]$$

This is very important. It allows numerical pricing with recombining trees.

The Ho-Lee Model

Bond Prices

$$
\begin{aligned}
P_t^T &= \exp\left\{-\int_t^T f_t^S dS\right\} \\
&= \exp\left\{-\int_t^T \left(f_0^S + \sigma^2 \cdot \left(St - \frac{1}{2}t^2\right) + x_t\right) dS\right\} \\
&= \exp\left\{-\int_t^T f_0^S dS\right\} \times \exp\left\{-\sigma^2 \int_t^T \left(St - \frac{1}{2}t^2\right) dS\right\} \\
&\quad \times \exp\left\{-\left(T - t\right) x_t\right\} \\
&= P_0^{t,T} A\left(t, T\right) e^{-b(t,T)x_t}
\end{aligned}
$$

where

$$
b\left(t, T\right) = T - t
$$

$$
A\left(t, T\right) = \exp\left\{-\sigma^2 \int_t^T \left(St - \frac{1}{2}t^2\right) dS\right\} = \exp\left\{-\frac{1}{2}\sigma^2 Tt\left(T - t\right)\right\}
$$

The Hull-White Model

The Hull-White model has HJM volatility

$$\sigma_t^T = \sigma e^{-a(T-t)}$$

with constant volatility $\sigma > 0$ and mean reversion $a > 0$.
If we take $a = 0$ then it is simply the Ho-Lee model.

The Hull-White Model

- Volatility Functions

$$\Sigma_t^T = -\sigma b(t, T)$$

$$b(t, T) \triangleq \frac{1}{a}\left(1 - e^{-a(T-t)}\right)$$

- Short Rate State

$$x_t = \sigma \int_0^t e^{-a(t-s)} dW_s$$

$$r_t = f_0^t + \frac{1}{2}\sigma^2 b(0, t)^2 + x_t$$

$$f_t^T = f_0^T + \frac{1}{2}\sigma^2 \left(b(0, T)^2 - b(t, T)^2\right) + e^{-a(T-t)} x_t$$

- Bond Prices

$$P_t^T = P_0^{t,T} A(t, T) e^{-b(t,T)x_t}$$

$$A(t, T) \triangleq \exp\left\{-\frac{1}{2}\sigma^2 b(t, T)\left(b(t, T)\frac{1 - e^{-2at}}{2a} + b(0, t)^2\right)\right\}$$

The Hull-White Model

Distributional Properties

The short rate state x_t is Gaussian with mean zero and variance

$$\mathrm{Var}\left[x_t\right] = \sigma^2 \frac{1 - e^{-2at}}{2a}$$

Bond prices are log-normal with variance

$$\mathrm{Var}\left[\log P_t^T\right] = \sigma^2 b\left(t, T\right)^2 \frac{1 - e^{-2at}}{2a}$$

The Hull-White Model

The Short Rate State Follows an Ornstein-Uhlenbeck Process

$$
\begin{aligned}
dx_t &= d\left(\sigma \int_0^t e^{-a(t-s)} dW_s\right) \\
&= d\left(\sigma e^{-at} \int_0^t e^{as} dW_s\right) \\
&= -a\sigma e^{-at} \left(\int_0^t e^{as} dW_s\right) dt + \sigma e^{-at} e^{at} dW_t \\
&= -a \cdot \left(\sigma \int_0^t e^{-a(t-s)} dW_s\right) dt + \sigma dW_t
\end{aligned}
$$

This is the familiar Ornstein-Uhlenbeck process

$$
\begin{aligned}
dx_t &= -a x_t dt + \sigma dW_t \\
x_0 &= 0
\end{aligned}
$$

The Hull-White Model

Markov Property

Fix the times $T \geq t \geq 0$

$$
\begin{aligned}
x_T &= \sigma \int_0^T e^{-a(T-s)} dW_s \\
&= \sigma \int_0^t e^{-a(T-s)} dW_s + \sigma \int_t^T e^{-a(T-s)} dW_s \\
&= e^{-a(T-t)} \left(\sigma \int_0^t e^{-a(t-s)} dW_s \right) + \sigma \int_t^T e^{-a(T-s)} dW_s \\
&= e^{-a(T-t)} x_t + \sigma \int_t^T e^{-a(T-s)} dW_s
\end{aligned}
$$

Therefore

$$\mathbb{E}\left[x_T \,|\mathcal{F}_t \right] = e^{-a(T-t)} x_t$$

The Hull-White Model

Mean Reversion

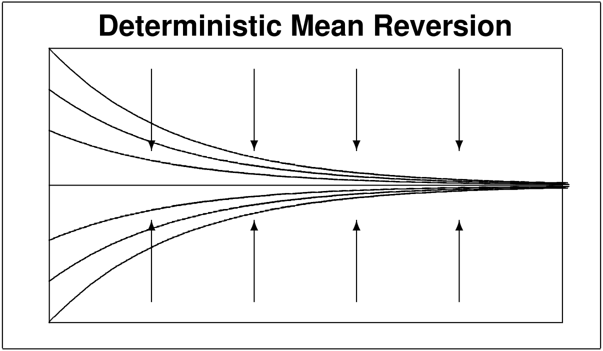
Removing the innovation term from the Ornstein-Uhlenbeck SDE leaves an ODE

$$dx_t = -ax_t dt$$

with solutions

$$x_T = x_t e^{-a(T-t)}$$

which pull toward zero



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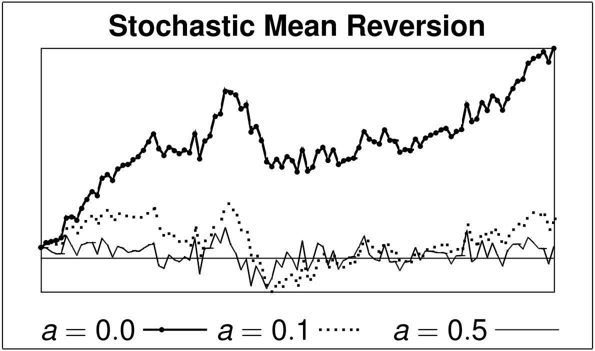
Mean Reversion

We have already seen the full solution to Ornstein-Uhlenbeck

$$x_T = e^{-a(T-t)} x_t + \sigma \int_t^T e^{-a(T-s)} dW_s$$

The trajectories still pull toward zero, but the random driver adds noise



The Hull-White Model

Interpreting Mean Reversion as a Loading

Just as in the *Statistical Model* lecture, we can interpret x_t as a factor and $e^{-a(T-t)}$ as a loading, for various interest rates

$$\begin{aligned} r_t &= \cdots + x_t \\ f_t^T &= \cdots + e^{-a(T-t)} x_t \\ R_t^T &= \cdots + \frac{b(t,T)}{T-t} x_t \end{aligned}$$

Each of these loadings is monotone decreasing as a function of the tenor $T-t$.

Be sure to read the complete notes for a more in depth discussion of mean reversion.