FINM 33601: Homework 8
Forward Measure and Change of Numeraire

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June 2, 2016

1.

a)

$$E[X] = 0.5 \times -0.5 + 0.2 \times 0.3 + 0.2 \times 1 = 0.01$$
$$Var[X] = 0.5 \times (0.01 + 0.5)^2 + 0.3 \times (0.01 - 0.2)^2 + 0.2 \times (0.01 - 1)^2 = 0.3369$$

b)

$$0.05 = 0.5 \times (-0.5 - \mu) + 0.3 \times (0.2 - \mu) + 0.2 \times (1 - \mu)$$
$$\therefore \mu = -0.04$$

c)

The variance stays the same.

d)

Let

$$P(X = 1) = a, \quad P(X = -0.5) = 1 - a - b, \quad P(X = 0.2) = b$$

Substituting these in for the expected value and variance above, we have:

$$1.5a = 0.55 - 0.7b$$

$$0.75a = 0.0894 + 0.21b$$

Hence, $a = 0.212, b = 0.3314$ and:

$$P(X = 1) = 0.212,, \qquad P(X = -0.5) = 0.4566, \qquad P(X = 0.2) = 0.3314$$

e)

Yes

2.

$$\begin{aligned}
\widetilde{E}[X] &= \int X \mathrm{d}\widetilde{Q} \\
&= \int X \frac{\mathrm{d}\widetilde{Q}_T}{\mathrm{d}Q_T} \mathrm{d}Q \\
&= \int X Z_t \mathrm{d}Q \\
&= E[Z_T X]
\end{aligned}$$

And if X is $\mathscr{F}_t$ measurable then:

$$\widetilde{E}[X] = E[E[Z_T X|\mathscr{F}_t]] = E[XE[Z_T|\mathscr{F}_t]] = E[XZ_t]$$

3.

Given that $Z_t = E[_T|\mathscr{F}_t]$ is a P-martingale:

$$\begin{aligned}
E[1_A Z_t \widetilde{E}[\xi|\mathscr{F}_t]] &= \widetilde{E}[1_A \widetilde{E}[\xi|\mathscr{F}_t]] \\
&= \widetilde{E}[1_A \xi] = E[1_A \xi Z_T] \\
&= \widetilde{E}[\xi|\mathscr{F}_t]
\end{aligned}$$

Therefore $\widetilde{E}[\xi|\mathscr{F}_t] = Z_t^{-1} E[Z_T \xi|\mathscr{F}_t]$

4.

Using the hint:

$$\begin{aligned}
\left.\frac{\mathrm{d}Q^T}{\mathrm{d}Q}\right| \mathscr{F}_t &= Z_t \\
&= \frac{P(t,T)B_0}{P(0,T)B_t} \\
&= \frac{B_0}{P(0,T)B_T}
\end{aligned}$$

Therefore,

$$
\begin{aligned}
\pi_t(X) &= E[P(0,T)Z_T X|\mathscr{F}_t] \\
&= B_t E^T[X|\mathscr{F}_t] \\
&= P(t,T)E^T[X|\mathscr{F}_t]
\end{aligned}
$$

5.

a) and b)

Based the the B numeraire:

$$
\begin{aligned}
\frac{P(t,T)}{B_t} &= E^{Q_B}[P(T,T)/B_T|\mathscr{F}_t] \\
\therefore P(t,T) &= E^{Q_B}[B_t/B_T|\mathscr{F}_t] \\
&= E^{Q_B}[e^{-\int_t^T r_u \mathrm{d}u} r_T|\mathscr{F}_t] \\
\therefore -\frac{\partial P(t,T)}{\partial T} &= E^{Q_T}[e^{-\int_t^T r_u \mathrm{d}u} r_T \frac{P(t,T)}{e^{-\int_t^T r_u \mathrm{d}u}}|\mathscr{F}_t] \\
&= P(t,T)E^{Q_T}[r_T|\mathscr{F}_t]
\end{aligned}
$$

If we take the log:

$$
\begin{aligned}
F_A(t,T) =\therefore -\frac{\partial log(P(t,T))}{\partial T} &= E^{Q_T}[r_T|\mathscr{F}_t] \\
&= E^{Q_T}[f(T,T)|\mathscr{F}_t] \\
&= E^{Q_T}[A_T|\mathscr{F}_t]
\end{aligned}
$$

Hence, the forward price process is also a martingale under Q^T

6.

Given that $\mathrm{d}W_t^T = \mathrm{d}W_t - \Sigma(t,T)\mathrm{d}t$:

$$
\begin{aligned}
\mathrm{d}P(t,T) &= r(t)P(t,S)\mathrm{d}t + P(t,S)\Sigma(t,S)(\mathrm{d}W_t^T + \Sigma(t,T)\mathrm{d}t) \\
&= (r(t) + \Sigma(t,T)\Sigma(t,S))P(t,S)\mathrm{d}t + P(t,S)\Sigma(t,S)\mathrm{d}W_t^T \\
\therefore \mathrm{d}f(t,T) &= \Sigma(t,T)\sigma(t,T)\mathrm{d}t + \sigma(t,T)\mathrm{d}W_t \\
&= (\Sigma(t,S)\sigma(t,S) - \sigma(t,S)\Sigma(t,T))\mathrm{d}t + \sigma(t,S)\mathrm{d}W_t^T \\
&= (\Sigma(t,S) - \Sigma(t,T))\sigma(t,S)\mathrm{d}t + \sigma(t,S)\mathrm{d}W_t^T
\end{aligned}
$$

7.

a)

$$\begin{aligned}S_t &= S_0 e^{(r-f-0.5\sigma^2)t+\sigma W_t}\\ \therefore \mathrm{d}S_t &= S_t(r-f-0.5\sigma^2)\mathrm{d}t + S_t\sigma \mathrm{d}W_t + 0.5 S_t\sigma^2 \mathrm{d}t\\ &= S_t(r-f)\mathrm{d}t + S_t\sigma \mathrm{d}W_t\end{aligned}$$

Therefore S_t is a solution of the SDE.

b)

$$\begin{aligned}Y_t &= e^{\sigma W_t - 0.5\sigma^2 t}\\ \therefore \mathrm{d}Y_t &= Y_t(-0.5\sigma^2)\mathrm{d}t + Y_t\sigma \mathrm{d}W_t + 0.5\sigma^2 Y_t \mathrm{d}t\\ &= Y_t\sigma \mathrm{d}W_t\end{aligned}$$

This process has no drift, therefore it is martingale under the P measure.

c)

Under the $\widetilde{P}$ measure there is no drift in $W_T - \sigma t$