HULL-WHITE MODEL

JEFF GRECO

1. HJM FRAMEWORK

We begin by recalling the (one-factor) HJM framework.

$$df_t^T = -\Sigma_t^T \sigma_t^T \, dt + \sigma_t^T \, dW_t, \qquad T \geq t \geq 0, \tag{1.1}$$

where

$$\begin{aligned} \sigma_t^T &= -\tfrac{\partial}{\partial T} \Sigma_t^T \\ \Sigma_T^T &= 0, \end{aligned} \tag{1.2}$$

and W_t is a Brownian motion under the risk-neutral measure.

As we have seen, the term structure is completely determined by the instantaneous forward rate volatility function σ_t^T. HJM allows for a great freedom of choice for its structure. We can make it very complicated: it may depend on the whole curve f, it may be random, it may be non-Markovian, etc. We will focus now on some simple forms for σ_t^T.

2. THE HO-LEE MODEL

2.1. **Introduction.** The simplest non-trivial HJM model that we can choose is one with a constant volatility structure

$$\sigma_t^T \equiv \sigma.$$

This is the Ho-Lee model. Solving for Σ_t^T using the relationship (1.2), we find

$$\Sigma_t^T = -\sigma \times (T - t) \, .$$

Substituting into (1.1)

$$df_t^T = \sigma^2 (T - t) \, dt + \sigma dW_t$$

or

$$\begin{aligned} f_t^T &= f_0^T + \sigma^2 \int_0^t (T-s)\, ds + \sigma \int_0^t dW_s \\ &= f_0^T + \sigma^2 \cdot \left(Tt - \frac{1}{2} t^2 \right) + \sigma W_t. \end{aligned}$$

This also provides the solution for the short rate

$$r_t = f_t^t = f_0^t + \frac{1}{2}\sigma^2 t^2 + \sigma W_t.$$

2.2. **Short Rate State.** We define the *short rate state*

$$x_t \triangleq \sigma W_t.$$

The short rate state represents the dynamic part of our model. It is independent of the initial term structure and $x_0 = 0$. We may express r_t in terms of the short rate state

$$r_t = f_0^t + \frac{1}{2}\sigma^2 t^2 + x_t.$$

This conveniently separates the short rate into the initial term structure f_0^t, a deterministic arbitrage free drift term $\frac{1}{2}\sigma^2 t^2$, and a drift free dynamic part x_t.

The instantaneous forward rates can also be expressed in terms of the short rate state

$$f_t^T = f_0^T + \sigma^2 \cdot \left(Tt - \frac{1}{2} t^2 \right) + x_t.$$

Thus, the dynamics of the term structure are entirely determined by a single source of random movement: the short rate state. Also x_t is Markovian, since it is a function of a Wiener process. I.e., if we know the present value of x_t, then its future evolution is independent of its past:

$$\mathbb{E}\left[x_T \left| \mathcal{F}_t \right. \right] = \mathbb{E}\left[x_T \left| x_t \right. \right], \quad \forall 0 \le t \le T.$$

This is a very important property, as it allows us to numerically price derivatives using recombining tree methods.

2.3. **Bond Prices.** As shown in the last section, all term structure dynamics derive from a single simple process, the short rate state. Therefore, bond prices must be expressible in terms of x_t.

$$\begin{aligned} P_t^T &= \exp\left(-\int_t^T f_t^S dS \right) \\ &= \exp\left(-\int_t^T \left\{ f_0^S + \sigma^2 \cdot \left(St - \frac{1}{2}t^2 \right) + x_t \right\} dS \right) \\ &= \exp\left(-\int_t^T f_0^S dS \right) \times \exp\left(-\sigma^2 \int_t^T \left(St - \frac{1}{2}t^2 \right) dS \right) \\ &\quad \times \exp\left(-\left(T-t\right) x_t \right) \end{aligned}$$

The first exponential is just the forward bond price[1] at time 0, $P_0^{t,T} = \frac{P_0^T}{P_0^t}$. The second exponential is a deterministic drift

$$\begin{aligned} A(t,T) &\triangleq \exp\left(-\sigma^2 \int_t^T \left[St - \frac{1}{2}t^2\right] dS\right) \\ &= \exp\left(-\frac{1}{2}\sigma^2 Tt (T-t)\right). \end{aligned}$$

For the third we define

$$b(t,T) \triangleq T - t.$$

So we have

$$P_t^T = P_0^{t,T} A(t,T) e^{-b(t,T)x_t}$$

and we have succeeded in expressing all bond prices as functions of the short rate state.

3. The Hull-White Model

3.1. **Introduction.** The Hull-White model has an HJM volatility specification given by

$$\sigma_t^T = \sigma e^{-a(T-t)}$$

with constant volatility $\sigma > 0$ and mean reversion $a > 0$ coefficients. This is a generalization of the Ho-Lee model. One can follow the same steps[2] as we have for the Ho-Lee model. This produces the following results.

3.2. **Formulas.**

Volatility functions.

$$\begin{aligned} b(t,T) &\triangleq \frac{1}{a}\left(1 - e^{-a(T-t)}\right) \\ \Sigma_t^T &= -\sigma \cdot b(t,T) \end{aligned}$$

Short rate state.

$$\begin{aligned} x_t &= \sigma \int_0^t e^{-a(t-s)} dW_s \\ r_t &= f_0^t + \frac{1}{2}\sigma^2 b(0,t)^2 + x_t \\ f_t^T &= f_0^T + \frac{1}{2}\sigma^2 \left(b(0,T)^2 - b(t,T)^2\right) + \sigma \int_0^t e^{-a(T-s)} dW_s \\ &= f_0^T + \frac{1}{2}\sigma^2 \left(b(0,T)^2 - b(t,T)^2\right) + e^{-a(T-t)} x_t \end{aligned}$$

Bond prices.

$$\begin{aligned} A(t,T) &\triangleq \exp\left\{-\frac{1}{2}\sigma^2 b(t,T)\left(b(t,T)\frac{1-e^{-2at}}{2a} + b(0,t)^2\right)\right\} \\ P_t^T &= P_0^{t,T} A(t,T) e^{-b(t,T)x_t} \end{aligned}$$

[1] This is known from the initial term structure.

[2] In fact, you will for homework!

3.3. **Distributional Properties.** The short rate state is Gaussian. Its mean is zero and its variance[3] is

$$\mathrm{Var}\left[x_t\right] = \sigma^2 \frac{1-e^{-2at}}{2a}.$$

Bond prices are log-normal[4]. The variance is

$$\mathrm{Var}\left[\log P_t^T\right] = \sigma^2 b\left(t,T\right)^2 \frac{1-e^{-2at}}{2a}.$$

3.4. **Markov Property.** Just as we saw for the Ho-Lee model, all market quantities can be expressed in terms of the short rate state x_t. At first glance

$$x_t = \sigma \int_0^t e^{-a(t-s)} dW_s$$

appears to depend on the entire history or increments $\{W_s\}_{s=0}^t$. However, by computing dx_t, we shall see this is not the case,

$$\begin{aligned}
dx_t &= d\left(\sigma \int_0^t e^{-a(t-s)} dW_s\right) \\
&= d\left(\sigma e^{-at} \int_0^t e^{as} dW_s\right) \\
&= -a\sigma e^{-at}\left(\int_0^t e^{as} dW_s\right) dt + \sigma e^{-at} e^{at} dW_t \\
&= -a\sigma \int_0^t e^{-a(t-s)} dW_s dt + \sigma dW_t.
\end{aligned}$$

After a simple substitution, we find

$$dx_t = -a x_t dt + \sigma dW_t, \quad x_0 = 0. \tag{3.1}$$

The increments dx_t depends only on quantities known at time t. Therefore x_t is a Markov process after all. In fact, it is the familiar Ornstein-Uhlenbeck process.

3.5. **Mean Reversion.** The formula (3.1) makes clear why a is referred to as the mean reversion coefficient. Without the stochastic portion, the ODE

$$dx_t = -a x_t dt$$

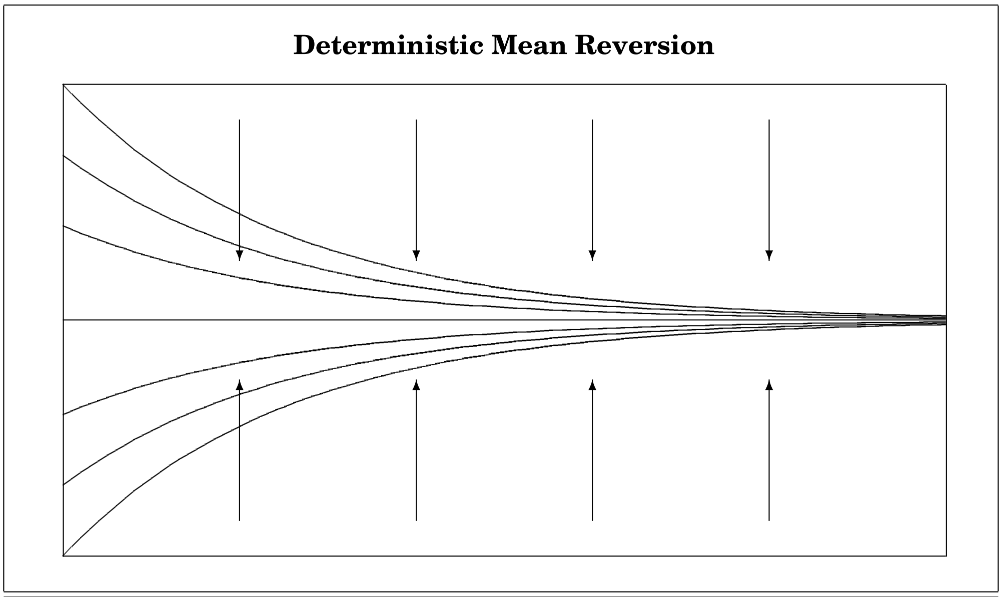
has solution

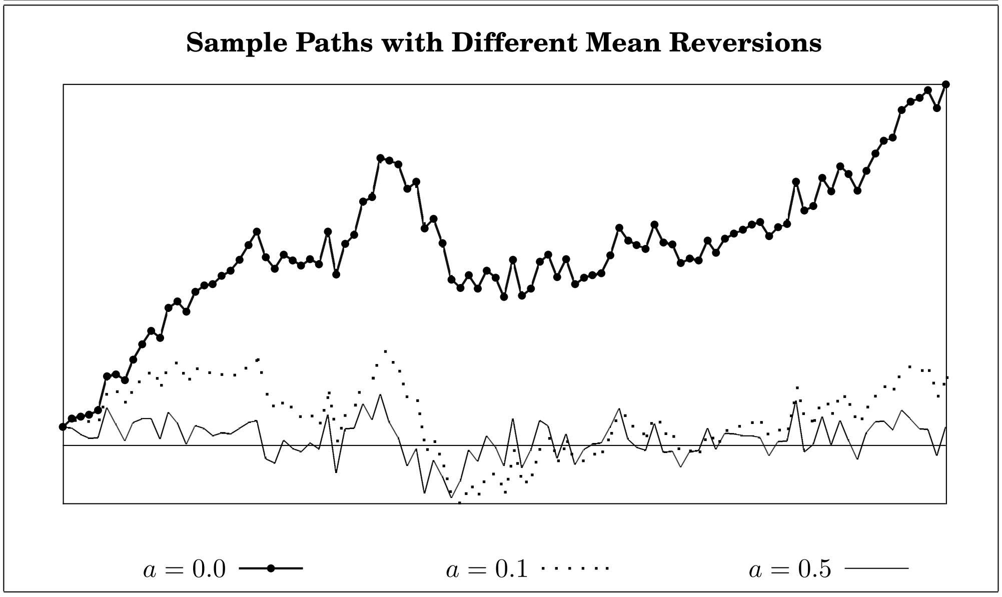
$$x_t = x_0 e^{-at}.$$

So, x_t is pulled toward zero. When the stochastic term is present, random shocks can move x_t up or down, but it is still "pulled" toward zero.

[3] You will have to derive these variance formulas for homework.

[4] I.e., the logarithm of a bond's price is Gaussian.





Interpretation From the Statistical Model. Recall the relationships

$$\begin{aligned} r_t &= \cdots + x_t \\ f_t^T &= \cdots + e^{-a(T-t)} x_t. \end{aligned}$$

Inspired by the statistical model, we can interpret x_t as a factor and $e^{-a(T-t)}$ as a loading. Note that the longer the tenor $T - t$, the smaller the response to changes in the factor.

For zero rates,

$$R_t^T = -\frac{1}{T-t} \log P_t^T = \cdots + \frac{b(t,T)}{T-t} x_t. \tag{3.2}$$

The loading function $\frac{b(t,T)}{T-t}$ is also monotone decreasing. So we see again that longer tenor rates have lower sensitivity to the factor.

The same is true for forward rates

$$
\begin{aligned}
f_t^{T,T+\tau} &= -\frac{1}{\tau}\left(\log P_t^{T+\tau} - \log P_t^{T}\right) \\
&= \cdots + \frac{1}{\tau}\left(b\left(t, T+\tau\right) - b\left(t, T\right)\right) x_t \\
&= \cdots + \frac{1}{\tau}\left(\frac{e^{-a(T-t)} - e^{-a(T+\tau-t)}}{a}\right) x_t \\
&= \cdots + \frac{b\left(T, T+\tau\right)}{\tau} e^{-a(T-t)} x_t.
\end{aligned}
$$

The loading $\frac{b(T,T+\tau)}{\tau}e^{-a(T-t)}$ is smaller for longer tenors τ.

Lower sensitivities for longer tenors imply lower volatilities for longer tenors.

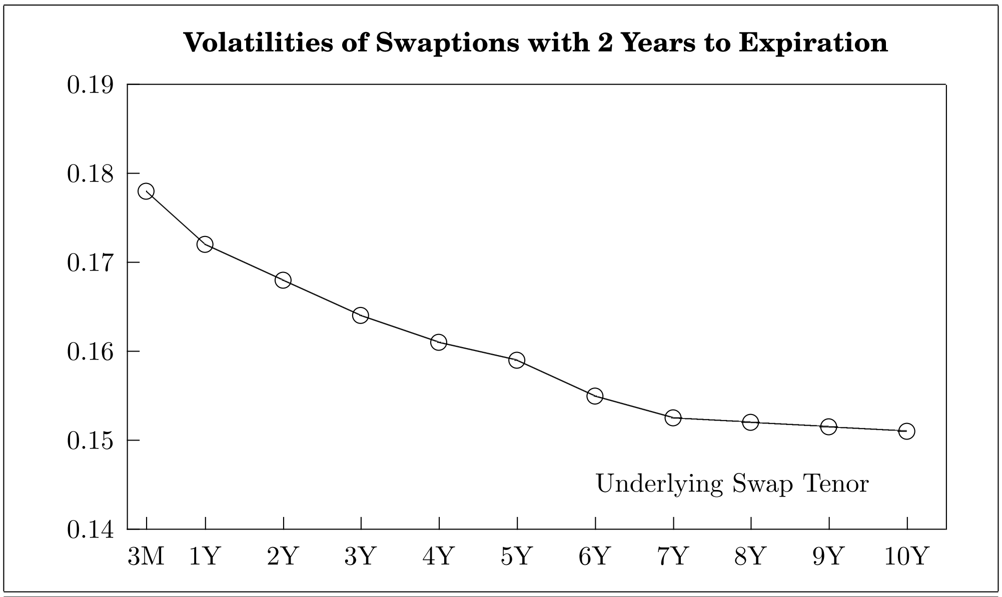
Importance of Mean Reversion. The topic of mean reversion is discussed at length in Rebonato. Term structure models that exhibit mean reversion are generally preferred to those that do not. Many arguments have been given for this. Empirically, real world rates do indeed exhibit mean reversion. This is not very compelling, as this is a condition on drift. When switching to risk neutral measure, the real world drift is simply discarded.

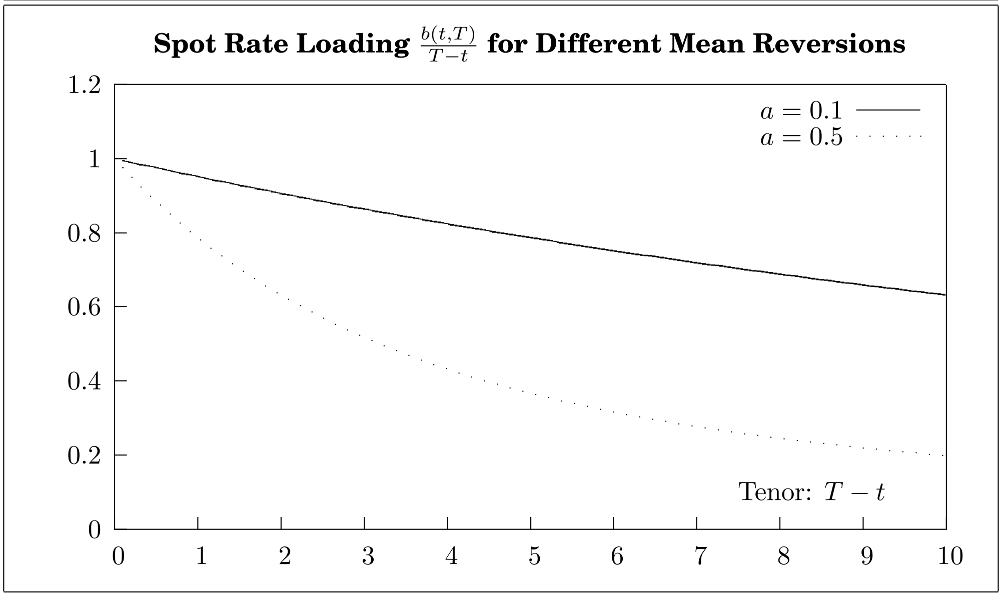
However, as pointed out by Rebonato, the drift of the short rate (non-traded asset) affects the volatility of traded assets, e.g. bonds. This is clear from (3.2)

$$
\operatorname{vol}\left(R_t^T\right) \propto \left(\frac{1 - e^{-a(T-t)}}{a\left(T-t\right)}\right).
$$

Thus, mean reversion allows us to control how volatilities of different bonds relate to one another.

A shortcoming of the Ho-Lee model is that the volatilities of all zero rates are exactly the same. Market observations typically show that longer tenor rates have lower implied volatilities than shorter tenor rates. With Hull-White this is possible! By varying the mean reversion parameter a we can choose the proper volatility decay. It is this broader sense of mean reversion which makes it relevant to modeling interest rates. A model does not need to follow the Ornstein-Uhlenbeck process (3.1) in order to exhibit mean reversion. But, it does need to have loadings which decay.





3.6. **Calibration.** In order to use Hull-White or any term structure model for pricing, the model must be calibrated[5] to the prices of actively traded instruments. Only then can it be used to compute prices of inactively traded instruments and portfolio risk factors.

In most practical circumstances, the only actively traded instruments available for calibration are caps and European swaptions. The swaptions are usually represented as a grid with option expiration dates forming the rows and underlying swap tenors forming the columns. The grid usually contains over 100 swaptions and caps.

[5]More specifically, one must imply the internal model parameters from the prices of market instruments.

The procedure that must be followed is very similar to smooth yield curve building. The goal is the same – find internal parameters of the model (forward rates or curves) which match external market prices.

It is important for the calibration to match as many swaptions[6] in the grid as possible. The Ho-Lee model has only one parameter σ and therefore can generally only match a single swaption. The Hull-White model adds another parameter a, so it can generally match two swaption prices. However, a more useful way to "spend" the extra parameter a is to set a proper loading shape.

Many instruments, notably Bermudan swaptions, require calibration to many European swaptions. The straight Hull-White model does not permit this amount of flexibility. In order to accomplish this, the Hull-White model can be extended by introducing time-dependent volatility $\sigma(t)$ and even time-dependent mean reversion $a(t)$.

Formally, the extended Hull-White model is characterized by the HJM volatility structure

$$\sigma_t^T = \sigma(t) \exp\left(-\int_t^T a(s)\, ds\right).$$

Most of the formulas we have derived can be generalized in the extended Hull-White model. While we lose time-homogeneity, we keep the most important features of the model. It is still a one-dimensional mean-reverting Gaussian Markov model.

The introduction of the time-dependent parameters $\sigma(t)$ and $a(t)$ provides additional flexibility to fit more instrument prices. But, if not handled carefully, it will result in over-fitting. The symptoms of an over-fitted model are parameters which defy intuition, are time-inhomogeneous and unstable in day-to-day recalibrations. *It is very easy to over-fit the extended Hull-White model*.

We can stabilize the calibration of the extended Hull-White model using numerical smoothing techniques. However, the goal of fitting the whole swaption grid is too ambitious for any one factor model. It is more appropriate to fit a subset of the swaption grid whose prices are relevant to the price of a more complex instrument of interest. The appropriate application of Hull-White and other low dimensional models to the pricing of complex derivative instruments is an ongoing field of research.

References

[1] M. Baxter, A. Rennie. *Financial Calculus: An Introduction to Derivative Pricing*. Cambridge University Press, 1996.

[2] John C. Hull, "Options, Futures, and Other Derivatives," 5th edition, **Prentice Hall College Division**.

[3] Riccardo Rebonato, "Interest-Rate Option Models," 2nd edition, **Wiley**.

E-mail address: jgreco@uchicago.edu

[6]Or as least a subset of swaptions which are pertinent to the price of the more complex instrument of interest.